A novel data-driven approach to optimizing replacement policy

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Abstract

Parallel systems are a commonly used structure in reliability engineering. A common characteristic of such systems is that the failure of a component may not cause its system to fail. As such, the failure may not be immediately detected and the random (disruption) time at which the number of failed components reaches a certain predefined number d may therefore be unknown. For such systems, scheduling maintenance policy is a difficult task, which is tackled in this paper. The paper assumes that times between inspections conform to a modulated Poisson process. This assumption allows the frequency of inspection responds to the variation of the disruption state. The paper then estimates the disruption time on the basis of inspection point process observations in the framework of filtering theorem. The paper develops a unified cost structure to jointly optimise inspection frequency and replacement time for the system when the lifetime distribution of a component follows the Pareto or exponential distribution. Numerical results are provided to show the application of the proposed model.

Keywords: Replacement; Inspection; Renewal-Reward; Optimization; Partial information; Filtering theorem.

Acronyms

MRD: Mean residual disruption time
PM: Preventive maintenance
RMA: Repair and maintenance action

Nomenclature

m	Number of components in the system
t_r	Periodic replacement time
$X_d(t)$	Disruption state
τ_d	Disruption (alert) time at which the process $X_d(t)$ describing the system state jumps from normal state (0) to the degraded state (1)

T_n	Time at which the nth inspection is conducted
$N(t)$	Counting process describing the total number of inspections until time t
$\mathcal{F}_t^N$	Partial filtration generated by the history of the counting process $N(t)$ of inspections
$\mathcal{F}_t$	Complete filtration generated by the history of both the counting process $N(t)$ and the state process $X_d(t)$
$\lambda_{X_d(t)}$	State-dependent inspection intensity adapted to the complete information level $\mathcal{F}_t$
$\hat{\lambda}_t$	Inspection alert function: state-dependent inspection intensity adapted to the partial information level $\mathcal{F}_t^N$
$q_{01}(t)$	Failure rate of components at time t
$F(t)$	Lifetime distribution of a component in the system
$\tilde{q}_{01}(t)$	Transition rate of the state process $X_d(t)$ from normal state (0) to the degraded state (1)
$F_d(t)$	Disruption (alert) time distribution
$\phi_d(t,n)$	Disruption alert function: $\mathcal{F}_t^N$-adapted state process $X_d(t)$ within $(n+1)^{\text{th}}$ inter-inspection time $[T_n, T_{n+1})$
$\bar{\varphi}_n(\cdot)$	$\mathcal{F}^N$-adapted distribution of $(n+1)^{\text{th}}$ inter-inspection time $[T_n, T_{n+1})$ $(n \geq 0)$
c_t	Inspection cost rate at time t
$C_{X_d(t)}$	State-dependent penalty cost per unit time of the system being unavailable due to an undetected failure within inter-inspection times $\big(t \in [T_n, T_{n+1})\big)$
$k_{X_d(t)}$	Replacement cost of the system as the state process at time t is $X_d(t)$
$C_d(t)$	Total cost up to time t
$\mathcal{C}_d(t,n)$	Expected total cost adapted to the complete information level $\mathcal{F}_t$
$c_d(t,n)$	Cost rate adapted to the complete information level $\mathcal{F}_t$
$\hat{c}_d(t,n)$	Cost rate adapted to the partial information level $\mathcal{F}_t^N$

Indexes used in the nomenclature and in the subsequent sections are also defined as follows:

$X_d(t)$	Index of modulated stochastic processes
n	Index of inspection epoch
l	Index of component
i	Index of the disruption state
d	Index of inspection alert parameter

1. Introduction

1.1. Motivation

Parallel systems are a commonly used structure in reliability engineering as they provide a redundant approach to improving system reliability and availability. For this there is strong demand within the two sectors for more analytical approach to decision making (e.g. see [34, 35]). Nuclear reactor safety systems, emergency core cooling systems, fire detectors, and protective devise are good examples of parallel systems used in the real world. A common characteristic of such systems is that the failure of a component may not cause its system to fail. As such, the failure may not be immediately detected and can be regarded as a hidden failure. Consequently, the number of failed components in the system, which is called the state of the system in what follows, may be unknown at a given time. That is, a parallel system may experience hidden failures and their state is not observable. It is reported that 40% of complex industrial systems experience hidden failures [20]. This raises an intriguing question: how can the reliability and the maintenance cost be analysed to handle such a lack of data scenario while providing effective decision making? This paper aims to answer those two questions.

1.2. Related works

For the past several decades, many classical inspection scheduling models for maintaining systems subject to hidden failure have been studied. For example, earlier works on the maintenance problem are given by Barlow et al. [10], Munford and Shahani [21], Keller [15] and Hauge [12]. The problem of inspection for deteriorating systems subject to hidden failure was first proposed by Barlow et al. [10], who minimize the total expected cost associated to the inspection and the elapsed time between system failure and its detection. Applying linear cost functions, Munford and Shahani [21] suggest an optimal inspection policy for the failure detection of a system. Keller [15] addresses an optimal inspection scheduling problem for systems subject to random failure. The model is based on the assumption that the frequency of inspection is driven by a continuous density function.

In preference to previous works (e.g. see [8, 9, 11, 13, 17, 23, 25–29, 31–33]), both the cost structure and the inspection model explored here respond to the variation of the system state. Examples of papers that use particular cost structures can be found in [11, 23, 26, 33]. Using a constant penalty cost rate as the majority of existing relevant literature (e.g. see [11, 23, 26, 27]), Zequeira and Brenguer [33] propose an approach to the determination of optimal inspection policies given three types of inspections: partial, perfect and imperfect. Their model allows to determine the optimal (constant) inter-inspection period and the optimal times of preventive maintenance actions. Using the net cost consisting of deterministic components, Tsai et al. [27] study a trivariate replacement policy for a deteriorating system whose failures can only be detected through inspection. He et al. [13], given a more flexible cost structure, consider a system with periodic inspection and preventive maintenance with the aim of minimizing the expected cost with respect to some maintenance system parameters.

Aperiodic inspection policies proposed in literature (e.g. see [1–3, 14, 27]) is often more useful and realistic than the periodic policy, since it is more adaptive to deteriorating systems and generally results in policies with lower costs. A commonly used approach, however, is periodic inspection policies as the implementation of periodic inspection is much easier than that of aperiodic one.

Maintenance optimization models based on the periodic inspection policy is vast (e.g. see [8, 9, 17, 18, 25, 28, 29]). For example, Lienhardt [17] study both the corrective maintenance and a periodic inspection policy for a repairable system subject to non-self announcing failures. They use the cost rate as a measure of policy to optimize the model with respect to maintenance parameters. Bada and Berrade [9] deal with a bivariate policy for a system subject to non-self announcing failures and periodic inspections. Taghipour and Banjevic [28] propose two optimization models for the periodic inspection of a system with hard-type and soft-type components. Given the basic assumption that, soft-type components failure is non-self announced, their model aims at optimizing the periodic inspection interval. Recently, based on a periodic inspection policy, Liu et al. [18] develop a delay-time-based model to determine the optimal inspection interval for parallel systems. More recently, Bjarnason and Taghipour [8] investigate periodic inspection frequency and inventory policies for a k-out-of-n system where the component's failures are hidden and conform to a non-homogeneous Poisson process.

1.3. Our modelling approach

From the above literature review, we can conclude: Although some maintenance models consider joint inspection and preventive maintenance policies (e.g. see [13, 19, 27, 30]) for systems subject to hidden failures, a unified maintenance model which accounts for all these factors does not exist. To fill up this knowledge gap, this paper assumes that the failure of a component in a parallel system can only be detected through an inspection and then proposes two-step method. The first step is to estimate the unobservable disruption state (system state) based on the inspection point process observations. This problem is solved in the framework of filtering theorem. The second step is to schedule a preventive maintenance policy for an m-component parallel system. More generally, given partial information, the approach explored here can deal with two basic problems: how to inspect and when to stop operating the system and carrying out a replacement in order to detect the system failure and minimize some maintenance cost.
The proposed method has the following characteristics.

- The method assumes that the inspection times follow the modulated Poisson process, which is a Poisson process whose rate varies according to a Markov process (see [4] the application of the modulated Poisson process in reliability engineering). This inspection modelling technique is a convenient and realistic way that allows the system to deliver an alert on approaching the disruption. The crew uses this information (alert) to perform inspections. The alert can be regarded as a signal, upon which an inspection has to be performed.

- Since the state of the system is unobservable, this paper uses the filtering theorem to estimate the unobservable state. This estimation method not only provides (i) an estimate of the disruption model, but also gives (ii) an estimate of the inspection model adapted to the observed information. In the current context, the first result delivering an alert on approaching the disruption and the second result that reflects the reaction of the maintenance crew to perform inspection are respectively remarked as disruption alert function and inspection alert function.

The advantages of the proposed approach can be summarized as follows: the probabilistic modelling of the disruption time as a measure of alertness helps to warn on a possibly approaching failure of system's component(s) (disruption) and therefore avoid system failure by performing PM. Furthermore, the model can adapt itself to the system state in a sense that, changes in the system state ideally induce changes in the maintenance cost and the maintenance system parameters that include inspection frequency, periodic replacement times and the detection probability of the system failure as well.

It is known that changes in the maintenance decision mechanism may induce changes in both the total maintenance costs and the detection of a possible failure. On the one hand, insufficient maintenance may save maintenance cost; but it may result in undetected failures. On the other hand, excessive maintenance results in detecting the system failure more promptly; but it incurs higher maintenance cost. Since both the amount of maintenance and the maintenance cost depends on the maintenance strategy, the problem is to determine an appropriate inspection and replacement policy avoiding the system failure by a possible PM, whilst simultaneously reducing total RMAs cost. To this end, this paper minimises the long-run expected cost per unit time for optimizing maintenance policy. Numerical examples are provided to demonstrate how the proposed degradation model may affect the inspection and maintenance policies.

1.4. Contribution and importance of this work

With above motivation, based on a set of realistic assumptions, this paper proposes a novel data-driven approach to optimizing replacement policy with some characteristics which have not been addressed or previously studied in isolation. More specifically, the novel aspect of our model lies in the proposal of a framework which is capable of handling the lack of information on the state of parallel systems while providing an effective method for decision making. It contributes to not only the joint modelling of inspection and preventive maintenance policy, but also the formulation of a unique and realistic inspection scheduling function incorporating a disruption alert function. The explored disruption alert function and scheduling function respectively help the practitioners to: (i) perform appropriate maintenance actions in response to the system state and (ii) propose a systematic approach to scheduling non-periodic inspection times. This approach is typically appropriate for non-self announcing failure systems particularly safety systems whose state is not directly observed and their failure is detected only by inspections.

1.5. Overview

The paper is organized as follows. Section 2 presents the model for calibration. The model assumptions are detailed in Section 2.1 and further discussions on both the degradation model and the inspection model are made in Section 2.2. Section 2.3 formulates the disruption problem. Since the model is given partial information, using the filtering theorem, Section 2.3 is developed to get an estimate of the disruption time by projection on the observed history. Section 2.4 is assigned to formulate the inspection scheduling model. Section 2.5 examines the explored disruption model. Given some assumptions, Section 3 explores some disruption and inspection scheduling models emerging as special cases. Section 4 introduces the cost model and formulates the average cost rate used as a measure of policy to optimize the model with respect to maintenance parameters. In Section 5, given two underlying degradation models some numerical results are given to illustrate the use of the proposed model in practical application. Section 6 concludes the paper.

2. Assumption and model development

2.1. Model assumptions

The following assumptions are made.

(1) The complex system consists of m components.
(2) Failure of the system can only be detected by inspections ("hidden or non-self announcing failures").
(3) The inspection intensity process is assumed to follow a modulated Poisson process.
(4) The only available information is given by the inspection point process observations (observation filtration).
(5) With the same approach as Zhao et al. [36] the model assumes that the system is replaced at periodic times $\{t_r, 2t_r, \cdots\}$.
(6) Inspections do not impact on the failure characteristics of the system. This implies that upon detection of failure before the replacement time t_r, the system is brought back to its operational state without affecting it's failure rate.

Assumptions (5)-(6) intuitively imply that the action space may include two types of actions: a planned replacement after kt_r $(k = 1, 2, \cdots)$ time units and a minimal repair action at inspection instant if the system experiences a failure within replacement cycles.

2.2. Model Development

This section aims to develop a method based on the above assumptions.

2.2.1. Modelling degradation

Consider a multi-component parallel system consisting of m components whose lifetimes are independent and identically distributed random variables. The system is subject to random failure which can only be detected through inspection. The system state is characterized by a two unobservable states: a normal state and a degraded state. The transition time from the steady state to the degraded state called "disruption time" is defined as the first time the total number of failed components reach a predetermined threshold d:

$$\tau_d = \inf\{t : Y(t) = d\}\,; d = 1, 2, \cdots, m-1 \tag{1}$$

where τ_d is the disruption time and $Y(t)$ is a stochastic process counting the total number of failed components up to time t. Denote

$$X_d(t) = \begin{cases} 0, & \text{if} \quad \tau_d > t, \\ 1, & \text{if} \quad \tau_d \le t. \end{cases} \tag{2}$$

From equations (1) and (2), one can see that the disruption time at which the system state, $X_d(t)$, jumps from 0 to 1 depends on the threshold's value d: a smaller value of d may result in earlier disruption.

2.2.2. Modelling inspections

In the present setting, we suppose that to detect the system failure, the system is inspected according to a modulated Poisson process. Specifically, let $N(t)$ be a modulated Poisson process such that $N(t)$ with the associated time points of inspections, $T_1 < T_2 < \cdots$, depicts the total number of arrivals up to time t. More precisely, according to Aven and Jensen [5], $N(t)$ admits a smooth semi-martingale (SSM) with the $\mathcal{F}$-intensity λ_t and the $\mathcal{F}$-martingale M_t:

$$N(t) = \int_0^t \lambda_s ds + M_t = \int_0^t [\lambda_0 + (\lambda_1 - \lambda_0)X_d(s)]ds + M_t,$$

where $t \in \mathbb{R}^+, M \in \mathcal{M}$ and $\mathcal{M}$ denotes the class of martingales adapted to the filtration $\mathcal{F}_t$. As noted, the inspection intensity process λ_t is modulated by the stochastic process $X_d(t)$ such that

$$\lambda_t = \lambda_{X_d(t)} \quad \forall t \ge 0, \tag{3}$$

where λ_i satisfying

$$\lambda_0 < \lambda_1 < \infty, \tag{4}$$

denotes the rate of arrivals when the state of $X_d(t)$ is i $(i = 0, 1)$. Here, $X_d(t)$ influences and modulates the arrival rate of the Poisson process. The reader is referred to Özekici [22] for more details about the modulated Poisson process.
Equations (1)-(3) indicate that changes in the threshold's value d induce changes in both the disruption time and the inspection intensity: as d decreases, the state process $X_d(t)$ jumps sooner from the normal state to the degraded state which implies an increase in the number of inspections. It is evident that this modeling technique, in an elegant way, not only allows the inspection intensity responds to the variation of the system state, but also ensures (see equation (4)) the system upon approaching disruption is inspected more frequently which properly results in more certain detection of failure.
Since the model including $N(t)$ is driven by the unobservable state process $X_d(t)$, the first main problem investigated here is to get an estimate of the

unobservable state $X_d(t)$ given the inspection point process observations up to time t,

$$\mathcal{F}_t^N = \sigma\left\{N(s) : 0 \le s \le t\right\},$$

that is

$$\phi_d(t) = \mathbf{E}[X_d(t)|\mathcal{F}_t^N] = \mathbf{P}[\tau_d \le t|\mathcal{F}_t^N]. \tag{5}$$

This helps in two ways:

(i) the detection of the disruption time τ_d at which the state process $X_d(t)$ jumps from the normal state to the degraded state, and

(ii) the estimation of the intensity process $\lambda_{X_d(t)}$ by projection on the observed history. That is,

$$\hat{\lambda}_t = \mathbb{E}\big[\lambda_{X_d(t)}|\mathcal{F}_t^N\big] = \mathbb{E}\big[\lambda_0 + (\lambda_1 - \lambda_0)X_d(t)|\mathcal{F}_t^N\big]. \tag{6}$$

The function $\hat{\lambda}_t$ reflecting the reaction of the maintenance crew is called inspection alert function. Indeed, given that the probability of the disruption detection at time t is $\phi_d(t)$, the inspection intensity that is at the discretion of the maintenance crew should be

$$\hat{\lambda}_t = \lambda_0 + (\lambda_1 - \lambda_0)\phi_d(t).$$

The situation may be regarded as a case of condition-based maintenance in a sense that the function (5) provides some information (alert) regarding the state of the disruption. Through the inspection alert function (6), the crew will use the information to perform inspections in order to detect system failures. Intuitively, this alert is the signal that an inspection has to be performed.

2.3. Modelling disruption

This section aims to provide a solution to the disruption alert function $\phi_d(t)$ by setting in the filtering theorem framework [e.g. see 6, 7]. The solution technique used here is similar to that of models proposed by Ahmadi et al. [1] and Ahmadi [3].

2.3.1. Complete information-based disruption model

We have to detect the random time of change of the inspection intensity $\lambda_{X_d(t)}$ i.e. τ_d (see Figure 1) based on the observation of the inspection point process $N(t)$. For this purpose, let $F(t) = \int_0^t f(u)du$ be the cumulative distribution of components lifetime. Using the semi-martingale argument [5], one can note that the increasing right-continuous state process $X_d(t) = I(\tau_d \le t)$ with the associated maintenance parameter d admits the following semi-martingale representation:

$$X_d(t) = \int_0^t \tilde{q}_{01}(s)(1 - X_d(s))ds + m_t,$$

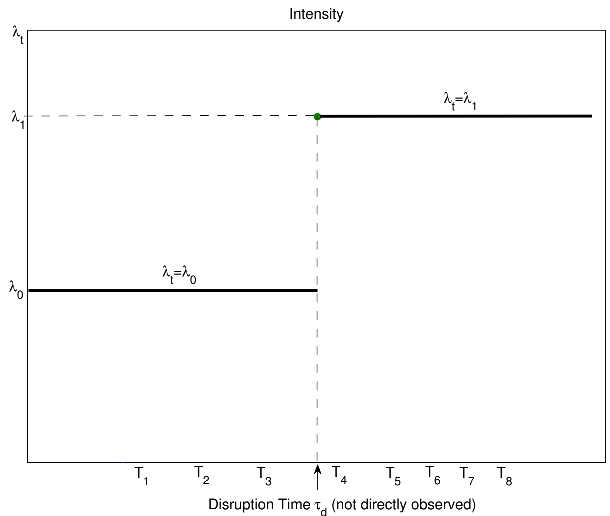

Figure 1: The disruption time τ_d at which the state process $X_d(t)$ jumps from 0 to 1.

where m_t is an $\mathcal{F}_t$-martingale and

$$\tilde{q}_{01}(t) = \begin{cases} \frac{f_d(t)}{1-F_d(t)}, & \text{if } F_d(t) < 1; \\ 0, & \text{otherwise,} \end{cases}$$

denotes the transition rate of the state process $X_d(t)$ from normal state (0) to the degraded state (1) with the disruption distribution function

$$F_d(t) = \mathbf{P}(\tau_d \leq t) = \sum_{k=d}^{m} \binom{m}{k} [F(t)]^k [1 - F(t)]^{m-k}.$$

Using the fact that the density function of the disruption time is

$$f_d(t) = \frac{dF_d(t)}{dt} = \frac{m!}{(d-1)!(m-d)!} [F(t)]^{m-1} [1 - F(t)]^{m-d} f(t),$$

the transition rate of the state process $X_d(t)$ can be formulated as

$$\tilde{q}_{01}(t) = \frac{\binom{m}{d-1,m-d} q_{01}(t)}{\sum_{k=0}^{d-1} \binom{m}{k} [\exp(Q_{01}(t)) - 1]^{k-d+1}}$$

where $q_{01}(t) = \frac{f(t)}{\bar{F}(t)}$ denotes the failure rate of the system and

$$Q_{01}(t) = \int_0^t q_{01}(s) ds.$$

2.3.2. Partial information-based disruption model

Since the disruption time at which the process $X_d(t)$ jumps from 0 to 1 is not directly observed, in the filtering theorem framework [e.g. see 6, 7], we get an estimate of the state process $X_d(t) = I(\tau_d \leq t)$ based on the observed history F_t^N. More precisely, the problem is that of computing $\phi_d(t)$ given in (5) and getting an $\mathcal{F}^N$-adapted estimate of the inspection intensity process

$$\lambda_t = \lambda_{X_d(t)} = \lambda_0 + (\lambda_1 - \lambda_0)X_d(t), \tag{7}$$

that is to say, before the disruption time τ_d, $\lambda_t = \lambda_0$, and after τ_d, $\lambda_t = \lambda_1$ (see Figure 1).

By projection on the observed history $\mathcal{F}_t^N$ one can show that

$$\begin{aligned}\phi_d(t) = &\int_0^t \big[\tilde{q}_{01}(s) + (\lambda_1 - \lambda_0)\phi_d(s)\big](1 - \phi_d(s))ds \\ &+ \sum_{n\geq 1} \frac{(\lambda_1 - \lambda_0)\phi_d(T_n^-)\big(1 - \phi_d(T_n^-)\big)}{\lambda_0 + (\lambda_1 - \lambda_0)\phi_d(T_n^-)} I(T_n \leq t),\end{aligned}$$

or, equivalently, between the jumps $\big(t \in [T_n, T_{n+1})\big)$,

$$\begin{aligned}\phi_d(t,n) &= \phi_d(t)I(T_n \leq t < T_{n+1}) \\ &= \phi_d(T_n) + \int_{T_n}^t \big(\tilde{q}_{01}(s) + \bar{\lambda}\phi_d(s,n)\big)\big(1 - \phi_d(s,n)\big)ds;\end{aligned} \tag{8}$$

at the jumps,

$$\phi_d(T_n) = \frac{\lambda_1 \phi_d(T_n^-)}{\lambda_0 + \bar{\lambda}\phi_d(T_n^-)}, \tag{9}$$

where $\phi_d(T_n^-)$ denote the left limit of $\phi_d(\cdot)$ at time T_n and $\bar{\lambda} = \lambda_1 - \lambda_0$. From (8) one can see that for $t \in [T_n, T_{n+1})$ the disruption time distribution $\phi_d(t,n)$ satisfies the differential equation (7) with the initial condition (9):

$$\phi'_d(t,n) = \frac{d}{dt}\phi_d(t,n) = \big(\tilde{q}_{01}(t) + \bar{\lambda}\phi_d(t,n)\big)(1 - \phi_d(t,n) > 0, \tag{10}$$

where the positive derivative implies increasing trajectories of $\phi_d(t,n)$ between the jumps. Therefore, by projection on the observed history $\mathcal{F}_t^N$ for $t \in [T_n, T_{n+1})$ $(n \geq 0)$ and the use of the disruption time distribution $\phi_d(t,n)$, an $\mathcal{F}_t^N$-adapted estimate of the inspection intensity (7) can be given by

$$\hat{\lambda}_t = \mathbb{E}\big[\lambda_{X_d(t)} | \mathcal{F}_t^N\big] = \lambda_0 + \bar{\lambda}\phi_d(t,n). \tag{11}$$

To get insight to the effect of the disruption alert function $\phi_d(t,n)$ and the inspection alert function $\hat{\lambda}_t$ on both the performance of inspections and failures, let the measure of alertness on approaching a disruption, $\phi_d(t,n)$, tend to 1. From (11) we get $\hat{\lambda}_t \longrightarrow \lambda_1$ $(\lambda_1 > \lambda_0)$ that implies more frequent inspections to

avoid the system failure.
From equations (9) and (10) one can note that the solution of the differential equation (10) based on the initial condition (9) rests on the estimation of inspection times T_n $(n \geq 1)$. In Section 2.4 we devise an inspection scheduling function based on the $\mathcal{F}^N$-adapted inspection intensity. We will see the explored scheduling function incorporating the disruption alert function $\phi_d(t,n)$ enables us (i) to provide a systematic approach to the determination of non-periodic inspection times and (ii) to estimate the disruption alert function $\phi_d(t,n)$ over inter-arrival inspection times $[T_n, T_{n+1})$ $(n \geq 1)$,

2.4. Scheduling inspections

Let $\varphi_n(v)$ for $v \in [0, V_{n+1})$ be the distribution function of the $(n+1)^{\text{th}}$ inter-inspection time $V_{n+1} = T_{n+1} - T_n$ $(n \geq 0)$ adapted to the observed information $\mathcal{F}^N$. Then for $t \in [T_n, T_{n+1})$ we have

$$\bar{\varphi}_n(v) = \mathbf{P}\big(V_{n+1} \geq v|\hat{\lambda}_t\big) = \exp\left(-\int_{T_n}^{T_n+v} \hat{\lambda}_s ds\right), \tag{12}$$

where $\bar{\varphi}_n(v) = 1 - \varphi_n(v)$. Since $\hat{\lambda}(t,n) = \lambda_0 + (\lambda_1 - \lambda_0)\phi_d(t,n)$, $t \in [T_n, T_{n+1})$, equation (12) can be expressed as

$$\bar{\varphi}_n(v) = \exp(-\lambda_0 v) \times \exp\left(-(\lambda_1 - \lambda_0)\int_{T_n}^{T_n+v} \phi_d(s,n)ds\right). \tag{13}$$

If $\mu_n = \mathbb{E}\big[V_{n+1}|\hat{\lambda}_t\big]$ $(n = 0, 1, 2, \cdots)$ denote the $(n+1)^{\text{th}}$ expected time between inspections, using the inter-inspection time distribution (13), an $\mathcal{F}^N$-adapted estimate of inter-inspection times can be given by

$$\mu_n = \int_0^\infty \bar{\varphi}_n(t)dt \tag{14}$$

This provides a sequence of inspection times η_n $(n = 1, 2, \cdots)$:

$$\eta_n = \sum_{k=0}^{n-1} \mu_k.$$

The scheduling function (14) develops the sequence of inspections and alert functions $\phi_d(t,n)$ $(n \geq 0)$ emerging as the solution of the differential equation (10) in the following way: given initial condition at time $T_0 = 0$, i.e. $\phi_d(0) = 1$, the solution of the differential equation (10) provides information regarding the state of disruption, that is, $\phi_d(t,0)$ for $t \in [0, T_1)$. By employing equations (13)-(14), this information is used to schedule the first inspection time $\mu_1 = \mathbb{E}(T_1)$. The scheduled inspection time μ_1 reveals the initial condition $\phi_d(\mu_1)$. Similarly as above, given the initial state $\phi_d(\mu_1)$, the differential equation (10) determines the updated disruption state $\phi_d(t,1)$ for $t \in [\mu_1, T_2)$. By using the scheduling function (14), the updated disruption state characterizes the second inspection epoch $\mu_2 = \mathbb{E}(T_2)$ and this process continues.

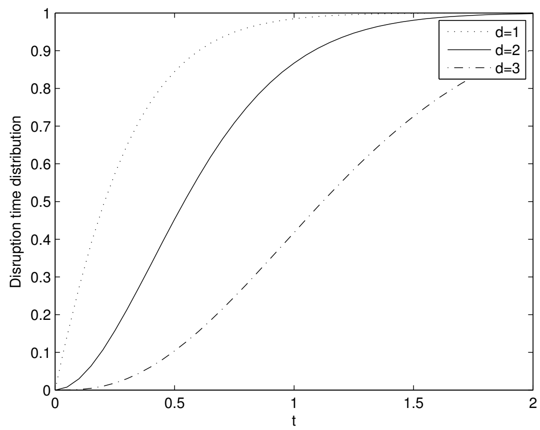

Figure 2: The disruption time distribution $\phi_d(t) = \mathbf{P}[\tau_d \leq t | \mathcal{F}_t]$ for different $d = 1, 2, 3$.

2.5. Examining the model

In this Section, using the mechanism described above, we examine the response of the model to threshold's values d. For this, we consider a 4-component system whose components lifetime conforms to a generalized Pareto distribution with the non-linear failure rate $q_{01}(t) = \frac{a+1}{at+b}$, $t \in [0, \frac{-b}{a})$, and know parameters $(a, b) = (-\frac{1}{4}, 1)$ and $(\lambda_0, \lambda_1) = (2, 3)$. Figure 2 reveals a strong influence of the threshold's value d on the sojourn time distribution in normal state of the state process $X_d(t)$. As the threshold's value d decreases to 1 the state process $X_d(t)$ is more susceptible to depart from normal state ($X_d(t) = 0$) with the $\mathcal{F}^N$-adapted departure rate

$$r_d(t, n) = \frac{\phi'_d(t, n)}{\bar{\phi}_d(t, n)}; \quad t \in [T_n, T_{n+1}) \tag{15}$$

where $\bar{\phi}_d(t, n) = 1 - \phi_d(t, n)$ denotes the sojourn time distribution of $X_d(t)$ in normal state ($X_d(t) = 0$). The response of the inspection intensity (11) to the maintenance alert parameter d is examined for different threshold's values $d \in \{1, 2, 3\}$. As illustrated with decreasing d, the disruption time occurs sooner (see Figures 3-5) making inspections more frequent (see Figures 6-8). As noted, both the inspection alert function and the inspection intensity as a measure of alertness reflect the reaction of the maintenance crew. In the sense that they deliver an alert regarding a possibly approaching disruption and this alert is used to make inspections. Thus, the alert is a signal that an inspection has to be performed to detect a failure.
Section 3.1, given a general degradation model and the threshold's value $d = 1$, provides some results on both the disruption and inspection problem for a two-component parallel system. Section 3.2 examines the results based on an exponential degradation model.

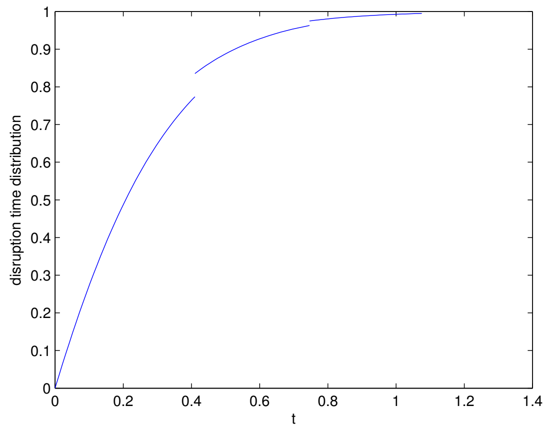

Figure 3: The disruption time distribution $\phi_d(t,n) = \mathbf{P}[\tau_d \le t|\mathcal{F}_t]$ given $d = 1$.

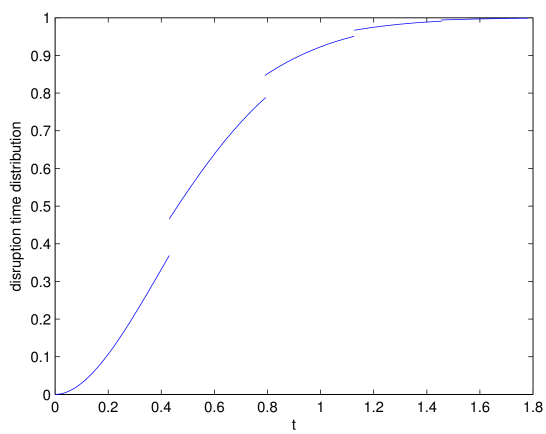

Figure 4: The disruption time distribution $\phi_d(t,n) = \mathbf{P}[\tau_d \le t|\mathcal{F}_t]$ given $d = 2$.

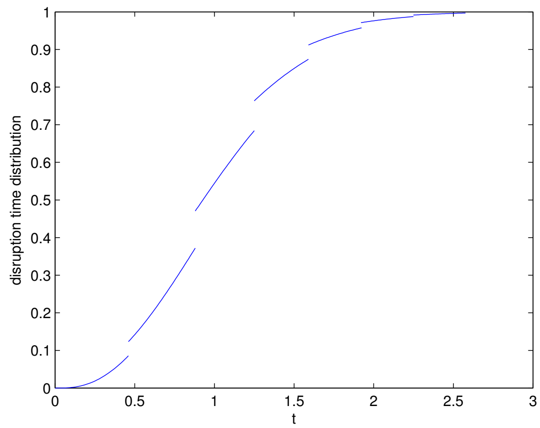

Figure 5: The disruption time distribution $\phi_d(t,n) = \mathbf{P}[\tau_d \leq t|\mathcal{F}_t]$ given $d = 3$.

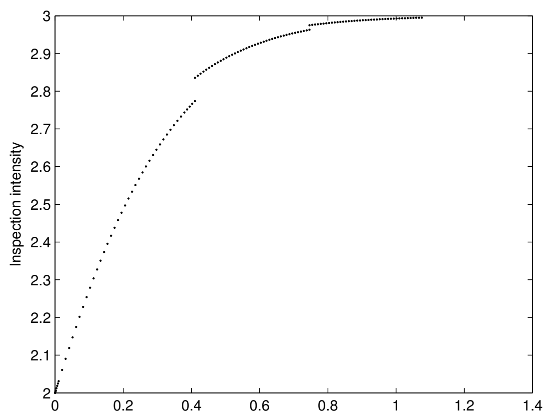

Figure 6: $\mathcal{F}^N$-adapted inspection intensity $\hat{\lambda}_t$ given d=1.

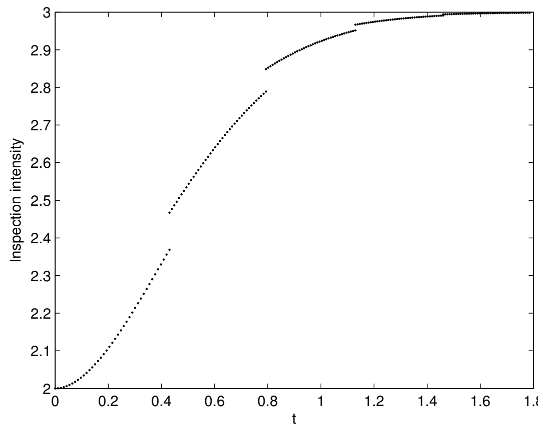

Figure 7: $\mathcal{F}^N$-adapted inspection intensity $\hat{\lambda}_t$ given d=2.

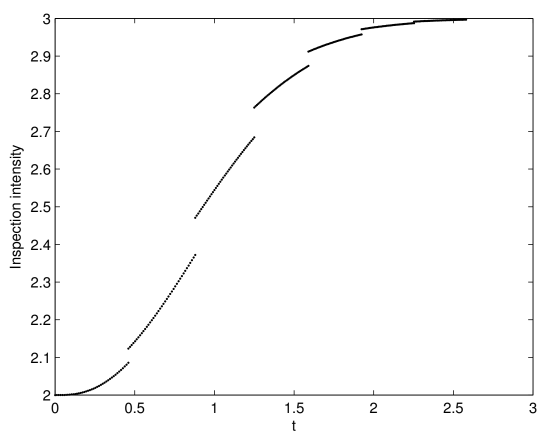

Figure 8: $\mathcal{F}^N$-adapted inspection intensity $\hat{\lambda}_t$ given d=3.

3. The Disruption problem $(d, m) = (1, 2)$

3.1. General degradation model

Let's consider a particular case of the model with $d = 1$ and $m = 2$. Using the differential equation (10), for $t \in [T_n, T_{n+1})$ $(n \geq 0)$, one can see the sojourn time distribution of $X_d(t)$ in normal state can be expressed as

$$\bar{\phi}_d(t, n) = 1 - \phi_d(t, n) = \frac{g(t, n)}{G(t, n)}, \tag{16}$$

where

$$g(t, n) = \bar{\phi}_d(T_n)\Big[\frac{\bar{F}(t)}{\bar{F}(T_n)}\Big]^2 \times \exp\big(-\bar{\lambda}(t - T_n)\big)$$

$$G(t, n) = 1 - \bar{\lambda}\int_{T_n}^{t} g(x, n)dx.$$

Alternatively, the measure of alertness can be characterized by the departure rate $r_d(t, n)$ (see equation (15)):

$$r_d(t, n) = \frac{{\phi'}_d(t, n)}{1 - \phi_d(t, n)} = \bar{\lambda}\phi_d(t, n) + 2q_{01}(t) \tag{17}$$

and the mean residual disruption time $m_d(t, n)$ (MRD function for short):

$$m_d(t, n) = \frac{\ln\big(G(t, n)/G(\infty, n)\big)}{\bar{\lambda}\bar{\phi}_d(t, n)}. \tag{18}$$

More specifically, $m_d(t, n)$ is the conditional expectation of $\tau_d - t$ given that the state process has not departed from the normal state $(\tau_d > t)$. Continuing the argument and using the equation (16), the distribution function of the $(n+1)^{\text{th}}$ inter-inspection time and the inspection scheduling function can be respectively expressed as

$$\bar{\varphi}_n(v) = \frac{\exp(-\lambda_1 v)}{G(T_n + v, n)}, \tag{19}$$

$$\mu_n = \int_0^\infty \frac{\exp(-\lambda_1 t)}{G(T_n + t, n)}dt; \quad (n \geq 0). \tag{20}$$

Remark 3.1. *In a special case, let $n = 0$. Continuing the example, equations (16)-(18) for $t \in [0, T_1)$, are respectively simplified as*

$$\bar{\phi}_d(t, 0) = 1 - \phi_d(t, 0) = \frac{g(t, 0)}{G(t, 0)}, \tag{21a}$$

$$r_d(t, 0) = \frac{{\phi'}_d(t, 0)}{1 - \phi_d(t, 0)} = \bar{\lambda}\phi_d(t, 0) + 2q_{01}(t), \tag{21b}$$

$$m_d(t, 0) = \frac{\ln\big(G(t, 0)/G(\infty, 0)\big)}{\bar{\lambda}\bar{\phi}_d(t, 0)}, \tag{21c}$$

with

$$g(t,0) = \exp(-\bar{\lambda}t) \times \bar{F}(t)^2$$

and

$$G(t,0) = 1 - \bar{\lambda} \int_0^t g(x,n)dx.$$

Examining (21c) *it is clear that as* $t \to 0$ *the mean residual disruption time tends to the mean time to disruption. In other words,*

$$m_d = m_d(0,0) \to \frac{-\ln\big(G(\infty,0)\big)}{\bar{\lambda}}. \tag{22}$$

Furthermore, let $\bar{\lambda} = \lambda_1 - \lambda_0$ *approaches to zero. This condition relaxes the assumption that the frequency of inspections is modulated by the state process* $X_d(t)$. *In this case,* (21a)-(21c) *and* (22) *become*

$$\lim_{\bar{\lambda}\mapsto 0} \phi_d(t,0) = 1 - \frac{g(t,0;0)}{G(t,0;0)} = 1 - \bar{F}^2(t), \tag{23a}$$

$$\lim_{\bar{\lambda}\mapsto 0} r_d(t,0) = \tilde{q}_{01}(t) = 2q_{01}(t). \tag{23b}$$

$$\lim_{\bar{\lambda}\mapsto 0} m_d(t,0) = \frac{\int_t^\infty \bar{F}^2(x)dx}{\bar{F}^2(t)}, \tag{23c}$$

$$\lim_{\bar{\lambda}\mapsto 0} m_d = \int_0^\infty g(x,n)dx. \tag{23d}$$

Using equations (16)-(18), we illustrate how changing the inspection model and the information pattern $\mathcal{F}^N$ determined by the history of inspection point process observations impacts on the reliability characteristics of the disruption time τ_d. Intuitively, we examine the response of the model to the parameter $\bar{\lambda} = \lambda_2 - \lambda_1$: $\bar{\lambda} = 0$ means that the approach towards inspection frequency does not change over time, while $\bar{\lambda} > 0$ allows the inspection intensity responds to the variation of the disruption state. Figures 9-11 clearly show the impact of the alert on detecting the system failure. As the system does not deliver a signal on approaching the potential disruption ($\bar{\lambda} = 0$), the crew may not appropriately react to perform inspections, which leads to a delay on detecting failures, a risky position. This supports our approach to inspection modelling given in Section 2.2.2. That means, considering the problem of modelling the dependence between the inspection intensity and the disruption state in terms of (i) an alter of the system and (ii) a supplementary reaction to take this alter into account.

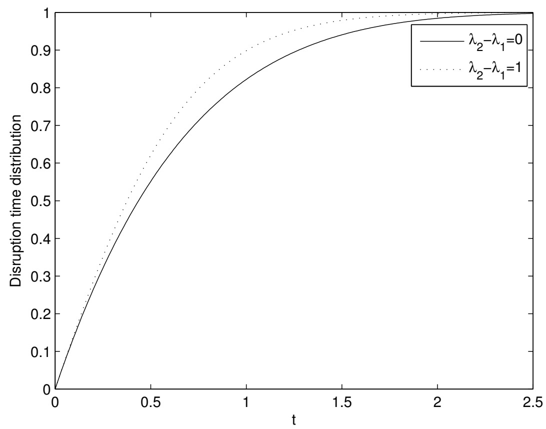

Figure 9: The response of the disruption time distribution $\phi_d(t, 0)$ to $\bar{\lambda}$ given $d = 1$ and $m = 2$.

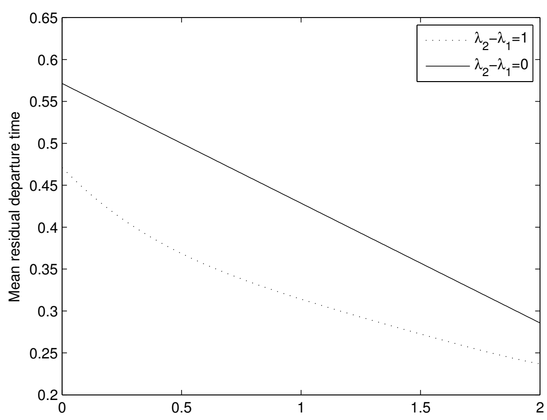

Figure 10: The response of the $\mathcal{F}^N$-adapted mean residual disruption time to $\bar{\lambda}$ given $d = 1$ and $m = 2$.

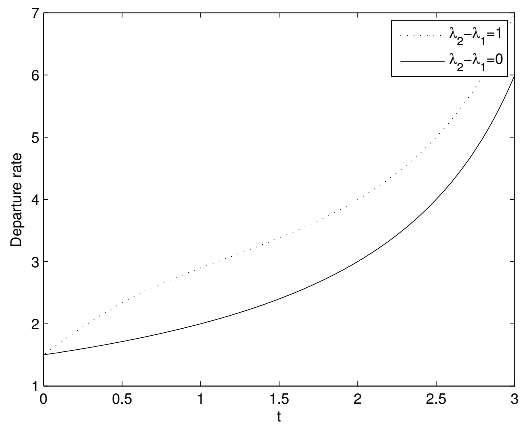

Figure 11: The response of the $\mathcal{F}^N$-departure rate to $\bar{\lambda}$ given $d = 1$ and $m = 2$.

3.2. Exponential degradation model

Letting components lifetime be distributed exponentially with parameter γ, results (16)-(18) for $t \in [T_n, T_{n+1})$ are respectively formulated as

$$\bar{\phi}_d(t,n) = \frac{\bar{\phi}(T_n)}{\mathbf{b}_n + \mathbf{a}_n \times \exp\big[\theta(t-T_n)\big]} \tag{24a}$$

$$r_d(t,n) = \frac{\theta \mathbf{a}_n}{\mathbf{a}_n + \mathbf{b}_n \times \exp\big[-\theta(t-T_n)\big]}, \tag{24b}$$

$$m_d(t,n) = \frac{-\ln(\mathbf{a}_n)}{\theta \mathbf{b}_n}, \tag{24c}$$

where $\theta = (\bar{\lambda} + 2\gamma)$,

$$\mathbf{a}_n = 1 - \bar{\lambda}\bar{\phi}_1(T_n)/\theta, \qquad \mathbf{b}_n = \bar{\lambda}\bar{\phi}(T_n)/\theta.$$

In addition, the inspection time distribution (19) may be rewritten as

$$\bar{\varphi}_n(v) = \frac{\exp(-\lambda_1 v)}{\mathbf{a}_n + \mathbf{b}_n \exp\big(-\theta v\big)}. \tag{25}$$

Remark 3.2. *Continuing to the example, let $\lambda_0 = 2\gamma$. Given this equation* (25) *becomes*

$$\bar{\varphi}_n(v) = \frac{\exp(-\lambda_1 v)}{\boldsymbol{c} + \boldsymbol{d}\exp(-\lambda_1 v)},$$

with

$$\mu_n = -\ln\big[1 - \bar{\lambda}\bar{\phi}(T_n)/\lambda_1\big]/\boldsymbol{d},$$

and

$$\boldsymbol{c} = \frac{\lambda_1 - \bar{\lambda}\bar{\phi}_1(T_n)}{\lambda_1}, \quad \boldsymbol{d} = \bar{\lambda}\bar{\phi}(T_n).$$

Setting $n = 0$, one can get an estimate of the first inspection time:

$$\mu_0 = \mathbb{E}(V_1) = \frac{\ln(\lambda_1/\lambda_0)}{\bar{\lambda}}.$$

4. Cost model

This section aims to jointly determine an optimal replacement policy and an optimal inspection frequency with respect to a state-dependent cost-reward model. The extended approach rests on the identification of an embedded renewal process defined by the periodic replacement epochs. This allows the application of the renewal reward theorem.

4.1. Average cycle costs

A cycle is comprised of a sequence of inspections and repair and maintenance actions that ultimately ends with replacement. Replacement times at periodic times $\{t_r, 2t_r, \cdots\}$ are regenerative epochs, and inter-replacement times form a renewal sequence. Upon detection of failures within replacement cycles, minimal repairs are carried out with negligible costs. Remaining repair and maintenance actions in a cycle incur costs that include: (i) inspection costs to detect the system failures, (ii) a penalty cost associated with undetected failures and (iii) a periodic replacement cost made after every t_r units of time. More precisely, each inspection at time t incurs a time-dependent cost $c_t = ct$ $(c > 0)$. Undetected failures within inter-inspection times generate a state-dependent penalty cost per unit time $C_{X_d(t)}$:

$$C_{X_d(t)} = C_0 + (C_1 - C_0)X_d(t),$$

$(C_1 > C_0)$. This implies that as the system state shifts to more degraded state $X_d(t) : 0 \mapsto 1$, the penalty cost increases from C_0 to C_1. In addition, we assume that the replacement of the system in different states incurs different costs. In other words, the replacement of the system at time t is measured by a state-dependent cost $k_{X_d(t)}$:

$$k_{X_d(t)} = k_0 + (k_1 - k_0)X_d(t), \quad (k_1 > k_0).$$

It is noted, as the state process $X_d(t)$ moves from the normal state $(X_d(t) = 0)$ to the degraded state $(X_d(t) = 1)$ the replacement cost increases $k_0 \mapsto k_1$. With respect to the above cost structure, the total cost up to the periodic replacement time t_r termed by $C_d(t_r)$ can be expressed as

$$C_d(t_r) = \underbrace{\int_0^{t_r} c_s dN(s)}_{\textit{Inspection cost}} + \underbrace{\int_0^{t_r} C_{X_d(s)} ds}_{\textit{Penalty cost}} + \underbrace{k_{X_d(t_r)}}_{\textit{Replacement cost}}. \tag{26}$$

The first and the second term on the right hand side of (26) represents the total inspection cost and the penalty cost associated with undetected failures up to the replacement time t_r respectively. The last cost is incurred as the system is found in the state $X_d(t_r)$ at replacement time t_r. Since $\lambda_{X_d(t)}$ is an $\mathcal{F}$-intensity of $N(t)$ and both the $C_{X_d(s)}$ and $k_{X_d(t)}$ are $\mathcal{F}$-measurable, an $\mathcal{F}$-adapted total cost $\mathcal{C}_d(t,n)$, $t \in [T_n, T_{n+1})$ $(n \geq 0)$, can be expressed as

$$\begin{aligned}
\mathcal{C}_d(t_r,n) = \mathbb{E}\big[C_d(t_r)|\mathcal{F}_{t_r}\big] &= \int_0^{t_r} c_s \lambda_{X_d(s)} ds + \int_0^{t_r} C_{X_d(s)} ds + k_{X_d(t_r)}, \\
&= k_0 + \int_0^{t_r} \big[C_{\lambda_0}(s) + \big(C_{\lambda_1}(s) - C_{\lambda_0}(s)\big)X_d(s)\big] ds + \int_0^{t_r} (k_1 - k_0) dX_d(s) \\
&= k_0 + \int_0^{t_r} c_d(s,n) ds,
\end{aligned}$$

where

$$c_d(t,n) = \big[C_{\lambda_0}(t) + \big(C_{\lambda_1}(t) - C_{\lambda_0}(t)\big)X_d(t)\big] + (k_1 - k_0) dX_d(t),$$

is the total cost rate and $C_{\lambda_i}(t) = C_i + \lambda_i c_t$ for $i = 0, 1$.

4.2. Partial information-based cost model

Since the state indicator $X_d(t)$ and therefore $c_d(t,n)$ cannot be observed, a projection on the observation filtration $\mathcal{F}^N$ is needed. As described in Section 2.3.2 such a projection from the $\mathcal{F}$-level to the $\mathcal{F}^N$-level leads to the following conditional expectation;

$$\begin{aligned}
\hat{c}_d(t,n) &= \mathbb{E}\left[c_d(t,n)|\mathcal{F}_t^N\right] \\
&= \big[C_{\lambda_0}(t) + \big(C_{\lambda_1}(t) - C_{\lambda_0}(t)\big)\phi_d(t,n)\big] + (k_1 - k_0) d\phi_d(t,n).
\end{aligned} \tag{27}$$

Thus, an $\mathcal{F}^N$-adapted estimate of the expected total cost can be given by

$$\hat{\mathcal{C}}_d(t_r,n) = k_0 + \int_0^{t_r} \hat{c}_d(s,n) ds, \tag{28}$$

The integrand $\hat{c}_d(s,n)$ is the conditional expectation of the cost rate at time s given the observations up to time s. In addition, by plugging the derivative of $\phi_d(t,n)$, i.e.,

$$\phi_d'(t,n) = \frac{d\phi_d(t,n)}{dt} = 2\bar{\phi}_d(t,n) q_{01}(t) + \bar{\lambda}\bar{\phi}_d(t,n)\big(1 - \bar{\phi}_d(t,n)\big),$$

(see equation (10)) into equation (27), in terms of the sojourn time distribution $\bar{\phi}_d(t,n)$, an $\mathcal{F}^N$-adapted estimate of the cost rate can be given by

$$\hat{c}_d(t,n) = -k_{10}\bar{\lambda}\bar{\phi}_d^2(t,n) + \Big(\bar{\lambda}(k_{10} - ct) + 2k_{10} q_{01}(t) - C_{10}\Big)\bar{\phi}_d(t,n) + C_{\lambda_1}(t)$$

where $k_{10} = k_1 - k_0$ and $C_{10} = C_1 - C_0$.

4.2.1. Long-run average cost rate

The main objective is to minimize the long-run average cost rate by optimizing the periodic replacement time t_r. To this end, let $\psi_d(n, t_r)$ be the long-run average cost per unit time. Since the sequence of replacement times $\{t_r, 2t_r, \cdots\}$ forms a regenerative process, the time between two consecutive replacements is a renewal cycle. Therefore, by the renewal reward theorem (c.f. Ross [24]), the long-run average cost rate is the average per cycle cost divided by the cycle length t_r given by

$$\psi_d(t_r, n) = \frac{\hat{\mathcal{C}}_d(t_r, n)}{t_r},$$

with expression for $\hat{\mathcal{C}}_d(t_r, n)$ provided in (28). We set out to solve the optimization problem

$$t_r^* = \underset{t_r \in \mathbb{R}_+}{\arg\min} \{\psi_d(t_r, n)\},$$

along with determining the optimal inspection frequency n^* subject to the optimal replacement time t_r^*. In other words, the optimization problem investigated here is a two-step process optimization. In the sense that we first minimize the long-run average cost rate via the renewal-reward theorem by optimizing the periodic replacement time t_r, then given the optimal stopping time t_r^*, a solution to the optimal inspection frequency n^* of the system is obtained.

5. Numerical results: (d,m)=(1,2)

We consider a two-component parallel system (e.g. Nuclear reactor safety systems/Emergency core cooling systems, fire detector and protective devise) whose failure is non-self announcing. The response of the model to two degradation models is examined as the failure rate of components is non-linear $q_{01}(t) = \frac{a+1}{at+b}$ or constant $q_{01}(t) = \gamma$ with expressions for sojourn time distribution given in (16) and (24a). Let $(a, b) = (-0.25, 1)$ and $\gamma = 1$ and the choice for the costs and the maintenance parameters are $d = 1$, $(C_0, C_1) = (1, 3)$, $(k_0, k_1) = (3, 6)$, $c = 0.75$ and $(\lambda_0, \lambda_1) = (2, 3)$. The choices were chosen arbitrarily to show some important features of both the inspection and maintenance policies.
As noted, results summarized in Table 1 reveal an unconvincing influence of the degradation model on the inspection model which leads to the same behaviour (see Figure 12) and thus fairly similar results. Table 2 and Figures 13-14 show that the optimal strategy determined by the periodic replacement time t_r^* gives an optimal solution to the inspection frequency and the average cost rate: the minimum cost for both degradation models is achieved at optimal replacement time $t_r^* = 1.63$ with the optimum inspection frequency $n^* = 4$. As described in Section 3.1, an alternative way of considering the measure of alertness is via the expected remaining time to the potential disruption. The MRD function is meant to reflect the reaction of the maintenance crew. More precisely, $m_d(t, n)$ ought to be small at times t for which disruptions are expected and the alert therefore should be high. This information meaningfully provides insight into the inspection frequency and the detection of the system failure. Figure 15 and

Table 1: Inspection times, optimal replacement time and disruption time for different degradation models.

	η_1	η_2	η_3	η_4	t_r^*	η_5	τ_d	η_6	η_7
Pareto	0.4136	0.77	1.116	1.4526	**1.636**	1.7869	**1.91**	2.1206	2.45
Exponential	0.4055	0.7565	1.0942	1.4285	**1.637**	1.7621	**1.96**	2.0955	2.43

Table 2: Optimal parameters for different degradation models.

	n^*	t_r^*	$m_d(t_r^*, n^*)$	$\psi_d(t_r^*, n^*)$
Pareto	4	1.636	0.28	6.66
Exponential	4	1.637	0.33	6.66

Figure 16 illustrate the behaviour of this function for both degradation models. It is evident that when the underlying degradation process conforms to a Pareto model, at optimal time $t_r^* = 1.63$, the expected remaining time to the disruption (or, departure) at which the state process jumps from the normal state to the degraded state is $m_d(t^*, n^*) = 0.28$ (see Figure 15). This intuitively implies that the time to detect the disruption at which the system's component experiences a failure is $\tau_d = t_r^* + 0.28 = 1.91$. While, the expected remaining time to the disruption time corresponding to an exponential model is to some extent more $m_d : 0.28 \mapsto 0.33$ indicating an estimate of the disruption time $\tau_d = t_r^* + 0.33 = 1.96$. For an illustration purpose, an evolution of the mean residual departure time corresponding to the exponential model is given (see Figure 16). As noted, in contrast to the Pareto model whose MRD function over inter-inspection times is not constant, the MRD associated with the exponential model is a piecewise constant function. In addition, results summarized in Table 1 and Table 2 clearly show that the inter-inspection times and particularly optimal solutions are not sensitive to degradation models and as illustrated in Table 1 and Figures 15-16 only the mean residual departure time responds to the degradation models. As time progresses and the system begins to degrades, inspections for both degradation models occur more frequently. This behaviour which ideally increases the failure detection probability of the system is more pronounced when the system state shifts to the degraded state. Therefore, based on above discussion we would suggest that corresponding to the Pareto (Exponential) degradation model, the process should be stopped with the optimal inspection frequency $n^* = 4$, and the periodic replacement is scheduled at optimal time

$$t_r^* = \arg\min_{t_r \in \mathbb{R}_+} \{\psi_d(t, n)\} = 1.63.$$

Thus, the main results not only provide information regarding the inspection scheme, but also help the crew to detect a disruption and avoid a possibly approaching failure by a preceding preventive replacement.

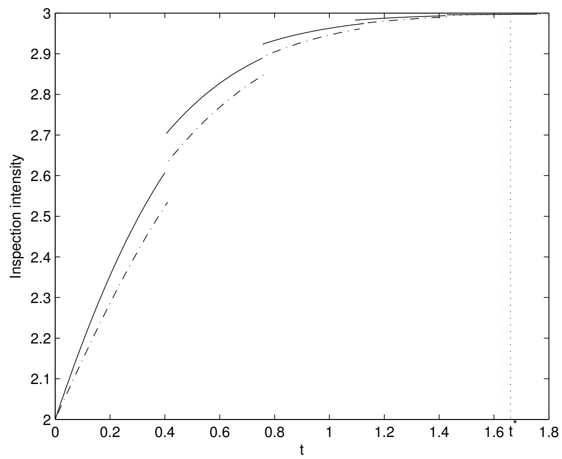

Figure 12: Inspection intensity with optimal parameters $(n^*, t^*) = (4, 1.63)$. The solid line and the dash-dotted line correspond to the exponential and the Pareto model.

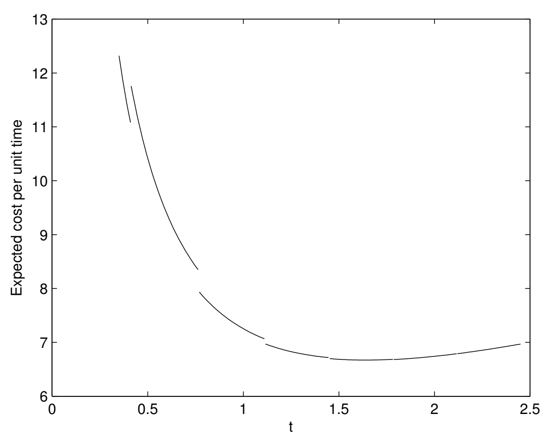

Figure 13: Expected cost per unit time of the Pareto model with $(n^*, t^*) = (4, 1.6366)$ and the optimal cost $\psi_d(t^*, n^*) = 6.66$.

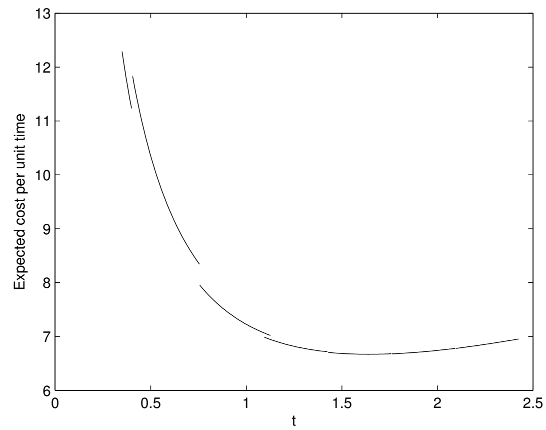

Figure 14: Expected cost per unit time of the exponential model with $(n^*, t^*) = (4, 1.6367)$ and the optimal cost $\psi_d(t^*, n^*) = 6.66$.

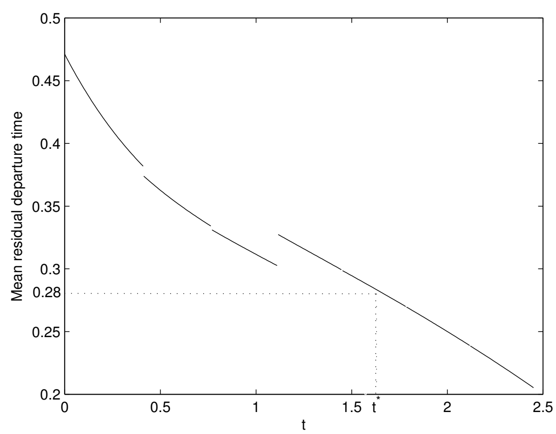

Figure 15: Mean residual departure (disruption) time of the Pareto model.

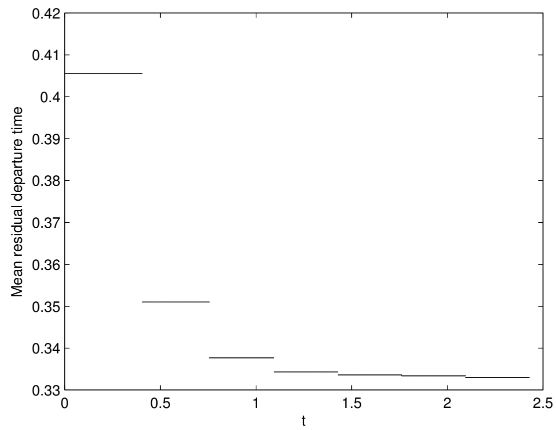

Figure 16: Mean residual departure (disruption) time of the Exponential model.

6. Conclusions

In many applications, particularly safety systems whose state is not directly observed and their failure are detected only by inspections, the intrinsic reliability requires methodologies which are capable of handling such a lack of data while providing effective decision making. The main issue discussed here is to detect the random time of change of the unobservable system state so-called disruption. The disruption problem is given limited and partially observed data including the inspections point process observations, the typical situation in practice. The approach to estimating the disruption problem, which rests on an associated maintenance alert parameter d, helps in two ways: firstly, it delivers an alert or a signal on approaching the system's failure. This information is used to appropriately control the inspection frequency in order to detect the system failures. Secondly, it provides insight to perform a replacement in order to avoid typically more costly system failures. This estimation problem has been solved in the framework of filtering theorem. The second problem investigated here is the construction and the estimation of a unified cost model based on the available information. The estimated cost model as a measure of policy contributes to the joint determination of an optimal inspection frequency and an optimal replacement time for systems with partial information.
The main examples considered are that of a Pareto and an exponential degradation model for a two-component parallel safety system with the alert parameter $d = 1$. Given both the degradation models, the results of the model provide sensible and realistic inspection and replacement policy for safety systems and give insight into the behaviour of the model and the effect of information pattern.
Two main findings of our model include:

- to detect the random time of change of the unobservable state (disruption time) of a non-self announcing failure system on the basis of limited and partially observed data by the use of the filtering theorem;

- construction of both an inspection alert function and an inspection scheduling function based on the estimated disruption model.

Both new findings contribute to diagnosis, prediction of the system failure and maintenance decision making for such systems. The model shows the feasibility of these programmes.
This paper outlines an approach and unified structure which will be developed later by setting the model in an optimal control framework [7]. The use of the control framework provides an extension of modelling techniques from non-repairable systems to repairable systems through incorporating a control process as a repair action into the degradation model.

Acknowledgment

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