Sliding Mode Control of Time-Delay Systems with Delayed Nonlinear Uncertainties

Adrian E. Onyeka * **Xing-Gang Yan** * **Jianqiu Mu** *

* *Instrumentation,Control and Embedded systems Research Group, School of Engineering and Digital Arts, University of Kent, Canterbury, CT2 7NT,U.K. (e-mail: aeo20@ kent.ac.uk, x.yan@kent.ac.uk, jm838@kent.ac.uk)*

Abstract: This paper considers a class of time delay systems with delayed states and nonlinear uncertainties using sliding mode techniques. In order to improve robustness, matched and mismatched disturbances are considered and assumed to be nonlinear functions of system states and delayed states. A sliding function is designed and a set of sufficient conditions is derived to guarantee the stability of the corresponding sliding motion by using Lyapunov-Razumikhin approach which allows large time varying delay with fast changing rate. A delay dependent sliding mode control is synthesized to drive the system to the sliding surface in finite time and maintain a sliding motion thereafter. Effectiveness of the proposed method is tested via a case study on a continuous stirred tank reactor system.

Keywords: Sliding mode control, Uncertain systems, Time delay, Lyapunov-Razumikhin approach.

1. INTRODUCTION

Over the past decades, time delay systems as often referred to as after-effect systems have been an active area of research in a wide range of natural and social sciences. They belong to a class of functional differential equations, existing widely in the practical world and are mostly encountered in numerous engineering systems such as electrical networks, chemical reactors, and hydraulic, pneumatic and manufacturing processes to mention but a few (Gu et al. (2003); Richard (2003)). Time delay is usually a source of instability and performance degradation in control systems which needs to be considered seriously in design, and as such, has received considerable attention over the past years (Yan et al. (2014)).

Motivated by the development in control theory, various techniques have been developed in dealing with the problem of stabilization and robust control of uncertain time delay systems(see, e.g. Yan et al. (2014); Richard (2003)). One method which has proved very effective in dealing with uncertainties in the system is the sliding mode control due to its strong robustness properties against parametric uncertainties and external disturbances in the input channel, as well as its attractive features such as fast and good transient response (Edwards and Spurgeon (1998); Orlov et al. (2003); Al-Shamali et al. (2003); Yan et al. (2017)).

Due to its complete robustness to matched uncertainties, sliding mode control has been extended to time delay systems with disturbances, and most of the existing results are in combination with other techniques such as fuzzy control (Yang et al. (2009)), optimal control (Dong et al. (2010)), adaptive control (Baek et al. (2016)), where the common goal is to present less conservative conditions to guarantee the robust stability of systems considered.

The problem of sliding mode control for uncertain time delay systems has been a continuous area of interest and development. Recent work carried out in this area recorded slight differences in techniques, most of which the sliding surfaces (Yunhao et al. (2009); Al-Shamali et al. (2003)) are different from the usual or conventional sliding surface in Edwards and Spurgeon (1998); Xia and Jia (2002), or that it only considered matched uncertainty Dong et al. (2010), or the bounds on the uncertainties only satisfy the linear growth condition (Hua et al. (2008); Xu (1997)).

It should be noted that sliding mode control for time delays with nonlinear disturbances has been studied in Yan et al. (2010) where static output feedback was considered, which has strong limitation on the system including the bounds on the uncertainties. Yunhao et al. (2009) proposed the robust sliding mode control of nonlinear uncertain systems by analysing the lump estimated disturbances via a disturbance observer. However, dealing with uncertainty in both state feedback and disturbance estimation may pose greater challenge alongside reducing the reliability of the system.

This work proposes a sliding mode control scheme for a class of time delay systems with nonlinear delayed disturbances. The assumptions for nonlinear terms are imposed on the transformed systems to avoid unnecessary conservatism. Lyapunov-Razumikhin approach is used to derive a set of conditions to guarantee that the derived sliding motion is stable. Then under assumption that all the system states are accessible, sliding mode control is synthesized such that the controlled system is driven to the sliding surface in finite time and maintains sliding motion thereafter. Compared with associated existing work, the proposed approach not only allows the bounds on the uncertainty have more general nonlinear form but all the design parameters can be obtained using linear matrix inequality (LMI) techniques. Case study on a continuous stirred tanked reactor (CSTR) is provided to show the feasibility of the developed results and the effectiveness of the proposed method.

2. PRELIMINARIES

First, recall some basic linear system theory. Consider a linear system

$$\dot{x} = Ax + Bu \tag{1}$$

where $x \in \Re^n, u \in \Re^m$ are states and inputs respectively, with $m < n$. The matrix pair (A, B) is of appropriate dimensions whereas B is of full rank.

Then from basic matrix theory, it can be shown that a coordinate transformation $z = Tx$ exists such that the matrix pair (A, B) in the new coordinates z has the following structure:

$$\tilde{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \tag{2}$$

where $A_{11} \in \Re^{(n-m)\times(n-m)}$, and $B_2 \in \Re^{m\times m}$ is non-singular. From Edwards and Spurgeon (1998), the fact that (A, B) is controllable implies that (A_{11}, A_{12}) is controllable, and thus there exists a matrix $M \in \Re^{m\times n}$ such that

$$A_{11} - A_{12}M$$

is Hurwitz stable. Let

$$S = S_2[M \quad I_m] \tag{3}$$

where $S_2 \in \Re^{m\times m}$ is any non-singular matrix. It is shown in Edwards and Spurgeon (1998) that the invariant zeros of (A, B, S) lie in the open left half plane. The detailed discussion is available in Edwards and Spurgeon (1998).

In order to deal with time delay systems, the following well-known Razumikhin Theorem is required.

Consider a time-delay system

$$\dot{x}(t) = \bar{f}(t, x_t) \tag{4}$$

with an initial condition

$$x(t) = \phi(t), \qquad t \in [-\bar{d}, 0]$$

where $\tilde{f} : \Re^+ \times \mathcal{C}_{[-\bar{d},0]} \mapsto \Re^n$ takes $\Re\times$(bounded sets of $\mathcal{C}_{[-\bar{d},0]}$) into bounded sets in $\Re^n$; $d(t)$ is the time-varying delay and $\bar{d} := \sup_{t\in\Re^+}\{d(t)\} < \infty$.

Lemma 1. (Razumikhin Theorem, Gu et al. (2003)) If there exist class $\mathcal{K}_\infty$ functions $\gamma_i(\cdot)$ with $i = 1, 2$, a class $\mathcal{K}$ function $\gamma_3(\cdot)$ and a continuous function $V_1(\cdot) : [-\bar{d}, \infty] \times \Re^n \mapsto \Re^+$ satisfying

$$\gamma_1(\|x\|) \le V_1(t, x) \le \gamma_2(\|x\|), \qquad t \in [-\bar{d}, \infty], \quad x \in \Re^n$$

such that the time derivative of V_1 along the solution of system (4) satisfies

$$\dot{V}_1(t, x) \le -\gamma_3(\|x\|)$$

whenever

$$V_1(t + \theta, x(t + \theta)) \le V_1(t, x(t)) \tag{5}$$

for any $\theta \in [-\bar{d}, 0]$, then the system (4) is uniformly stable. if, in addition, $\gamma_3(\tau) > 0$ for $\tau > 0$, and there exist a continuous non-decreasing function $\gamma_4(\cdot)$ which satisfies $\gamma_4(\tau) > \tau$ for $\tau > 0$ such that inequality (5) is strengthened to

$$\dot{V}_1(t, x) \le -\gamma_3(\|x\|) \quad \text{whenever}$$
$$V_1(t + \theta, x(t + \theta)) \le \gamma_4(V_1(t, x(t))) \tag{6}$$

for any $\theta \in [-\bar{d}, 0]$, then system (4) is uniformly asymptotically stable.

Lemma 2. (see Yan et al. (2012)) Let the matrix $N_1 \in \Re^{m\times n}$ and vectors $x \in \Re^m$ and $y \in \Re^n$. Then, the inequality

$$x^T N_1 y \le \frac{1}{2\epsilon} x^T N_1 N_2^{-1} N_1^T x + \frac{\epsilon}{2} y^T N_2 y \tag{7}$$

holds for any symmetric positive-definite matrix $N_2 \in \Re^{n\times n}$ and any positive constant ϵ.

The results above will be used in the subsequent analysis.

3. PROBLEM FORMULATION

Consider a time varying delay system with delayed disturbance described by

$$\dot{x} = Ax + A_d x_d + B(u + \bar{G}(t, x, x_d)) + \bar{F}(t, x, x_d) \tag{8}$$

where $x \in \Omega \subset \Re^n$ (Ω is an neighborhood of the origin), $u \in \Re^m$ are the states and inputs respectively; A, $A_d \in \Re^{n\times n}$ and $B \in \Re^{n\times m}$ ($m < n$) are constant matrices with B being of full rank; The vectors $\bar{G}(\cdot)$ and $\bar{F}(\cdot)$ represent the matched and mismatched disturbances affecting the system. The symbol $x_d := x(t - d)$ represent the delayed state where $d := d(t)$ is the time varying delay which is assumed to be known, continuous, non-negative and bounded in $\Re^+ := \{t \mid t \ge 0\}$, that is

$$d := \sup_{t\in\Re^+}\{d(t)\} < \infty$$

The initial condition related to the delay is given by

$$x(t) = \phi(t), \qquad t \in [-d, 0] \tag{9}$$

where $\phi(\cdot)$ is continuous in $[-d, 0]$. It is assumed that all the nonlinear functions are smooth enough for the subsequent analysis, which guarantees that the unforced system has unique continuous solutions.

In this paper, the objective is to design a sliding mode control for the system (8), such that the corresponding closed loop system is asymptotically stable in the presence of time delay and uncertainties, with focus on disturbance tolerability but of convenient parameter design methodology.

4. SLIDING MODE CONTROL ANALYSIS AND DESIGN

In this section, a sliding surface will be designed and the stability of corresponding sliding motion will be analysed.

First, it is necessary to impose the following fundamental assumptions on the system (8).

Assumption 1. The pair (A, B) is controllable.

From Section 2, there exists new coordinates $z = Tx$ such that the system (8) can be described by

$$\dot{z}_1 = A_{11}z_1 + A_{d11}z_1(t-d) + A_{12}z_2 + A_{d12}z_2(t-d) + F_1(t, z(t), z_d) \quad (10)$$

$$\dot{z}_2 = A_{21}z_1 + A_{d21}z_1(t-d) + A_{22}z_2 + A_{d22}z_2(t-d) + B_2u + B_2G(t, z(t), z_d) + F_2(t, z(t), z_d) \quad (11)$$

where $z(t) = (z_1, z_2)^T$ with $z_1 \in \Re^{n-m}$ and $z_2 \in \Re^m$, $z_d = (z_1(t-d), z_2(t-d))^T$ with $z_1(t-d) \in \Re^{n-m}$ and $z_2(t-d) \in \Re^m$, and

$$TAT^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

$$TA_dT^{-1} = \begin{bmatrix} A_{d11} & A_{d12} \\ A_{d21} & A_{d22} \end{bmatrix}, \quad TB = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}$$

where $A_{11}, A_{d11} \in \Re^{(n-m)\times(n-m)}$, $A_{12}, A_{d12} \in \Re^{(n-m)\times m}$, $A_{21}, A_{d21} \in \Re^{m\times(n-m)}$, $A_{22}, A_{d22} \in \Re^{m\times m}$, and $B_2 \in \Re^{m\times m}$ is nonsingular, and

$$\begin{bmatrix} F_1(t, z(t), z_d) \\ F_2(t, z(t), z_d) \end{bmatrix} := T\bar{F}(t, x(t), x_d)|_{x=T^{-1}z}$$

$$G(t, z(t), z_d(t)) = \bar{G}(t, x, x_d)|_{x=T^{-1}z}$$

where $F_1(\cdot) \in \Re^{n-m}$ and $F_2(\cdot) \in \Re^m$.

Assumption 2. The uncertain terms $G(\cdot)$ and $F_2(\cdot)$ satisfy:

$$\|G(t, z(t), z_d)\| \leq \phi(t, z(t), z_d) \quad (12)$$

$$\|F_2(t, z(t), z_d)\| \leq \rho(t, z(t), z_d) \quad (13)$$

where $\phi(\cdot)$ and $\rho(\cdot)$ are known nonnegative continuous functions.

Assumption 3. There exist known positive constants ϖ_1 and ϖ_2 such that

$$F_1^T(t, z(t), z_d)F_1(t, z(t), z_d) \leq \varpi_1^2 z^T(t)z(t) + \varpi_2^2 z_d^T z_d \quad (14)$$

Remark 1. Assumptions 2 and 3 are limitations to the nonlinear uncertainties. The bounds on $G(\cdot)$ and $F_2(\cdot)$ in (12) and (13) have general nonlinear form. In order to use LMI techniques to obtain the design parameters for sliding surface, Assumption 3 is imposed on the mismatched uncertainty $F_1(\cdot)$ to facilitate the sliding motion analysis.

4.1 Stability of sliding motion

Based on the above assumptions, the main aim of this paper is to achieve robust stability in the presence of disturbances and delay in (8) using sliding mode control which generates a sliding motion. From Section 2, it follows that under Assumption 1, the following sliding function is defined as

$$\sigma(z) = Mz_1(t) + z_2(t) \quad (15)$$

where $M \in \Re^{m\times(n-m)}$ is a designed matrix. When the system is limited to the sliding surface

$$\sigma(z) = 0, \quad (16)$$

it follows that $z_2 = -Mz_1$.

From the structure of system (10)-(11), the sliding motion of system (8) associated with the sliding surface (16) is dominated by system (10). When dynamic (10) is limited to the sliding surface (16), it can be described by

$$\dot{z}_1 = (A_{11} - A_{12}M)z_1 + (A_{d11} - A_{d12}M)z_1(t-d) + F_{1\delta}(t, z_1(t), z_{1d}(t)) \quad (17)$$

where

$$F_{1\delta}(t, z_1, z_{1d}) = F_1(t, z, z_d)|_{z_2=-Mz_1} \quad (18)$$

with $z = col(z_1, z_2)$ and $F_1(\cdot)$ defined in (10).

Remark 2. System (17) is the sliding mode of system (10)-(11) corresponding to the sliding surface (16). It should be noted that the mismatched uncertainty $F_{1\delta}(\cdot)$ is the uncertainty $F_1(\cdot)$ when it is limited to the sliding surface (16).

From equation (17) it is clear to see that the mismatched uncertainty $F_{1\delta}$ can affect the sliding mode dynamics and as such it is necessary to impose some constraint on it in order to guarantee asymptomatic stability of the sliding motion.

From (14), (15) and (18), it follows that

$$\begin{aligned} & F_{1\delta}^T(\cdot)\, F_{1\delta}(\cdot) \\ \leq\; & \varpi_1^2 \left(\begin{bmatrix} z_1^T & -(Mz_1)^T \end{bmatrix} \begin{bmatrix} z_1 \\ -Mz_1 \end{bmatrix} \right) + \varpi_2^2 \\ & \cdot \left(\begin{bmatrix} z_1^T(t-d) & -(Mz_1)^T \end{bmatrix} \begin{bmatrix} z_1(t-d) \\ -Mz_1(t-d) \end{bmatrix} \right) \\ =\; & \varpi_1^2[z_1^T z_1 + z_1^T(M^TM)z_1] + \varpi_2^2[z_1^T(t-d) \\ & \cdot z_1(t-d) + z_1(t-d)(M^TM)z_1(t-d)] \\ \leq\; & \varpi_1^2(1 + \lambda_{max}(M^TM))z_1^T z_1 + \varpi_2^2(1 \\ & + \lambda_{max}(M^TM))z_1^T(t-d)z_1(t-d) \\ \leq\; & \psi_1 z_1^T z_1 + \psi_2 z_1^T(t-d)z_1(t-d) \end{aligned} \quad (19)$$

where

$$\psi_1 = \varpi_1^2[1 + \lambda_{max}(M^TM)]$$

$$\psi_2 = \varpi_2^2[1 + \lambda_{max}(M^TM)] \quad (20)$$

The following results is ready to be presented.

Theorem 1. Under Assumptions 1 and 3, the sliding motion of system (8) associated with the sliding surface (15), governed by (17) is asymptotically stable if there exist a scalar $\alpha > 0$ and a real positive definite matrix P_1 such that the following LMI

$$\begin{bmatrix} W & P_1 \\ P_1 & -\alpha I \end{bmatrix} < 0 \quad (21)$$

where

$$W = A_o^T P_1 + P_1 A_o + P_1 + P_1 A_1 P_1^{-1} A_1^T P_1 + \beta\alpha I$$

$$A_o = A_{11} - A_{12}M$$

$$A_1 = A_{d11} - A_{d12}M$$

$$\beta = \psi_1 + \psi_2 \frac{(1+\epsilon)\lambda_{max}(P_1)}{\lambda_{min}(P_1)}$$

where $\epsilon > 0$, ψ_1 and ψ_2 are defined in (19), and M is defined in (15).

Proof. For sliding mode (17), consider the candidate Lyapunov function

$$V_1 = z_1^T(t)P_1z_1(t)$$

Then the time derivative of V_1 along the trajectory of the system (17), is given by

$$\dot{V}_1|_{(17)} = z_1^T(t)(A_o^T P_1 + P_1 A_o)z_1(t) + 2z_1^T P_1 A_1 z_1(t-d) + 2z_1^T(t)P_1 F_{1\delta}(t, z_1(t), z_1(t-d)) \quad (22)$$

where $A_o = A_{11} - A_{12}M$.
From Lemma 2, it follows that

$$2z_1^T P_1 A_1 z_1(t-d) \le z_1^T(t-d)Pz_1(t-d) + z_1^T(t)P_1 A_1 P_1^{-1} A_1^T P_1 z_1(t) \quad (23)$$

From (22) and (23) it can be observed that the derivative V_1 along the trajectory of system (17) can be described by

$$\dot{V}_1|_{(17)} = z_1^T(t)[A_o^T P_1 + P_1 A_o]z_1(t) + z_1^T(t-d)P_1 z_1(t-d) + z_1^T P_1 A_1 P_1^{-1} A_1^T P_1 z_1(t) + 2z_1^T P_1 F_{1\delta}(t, z_1(t), z_1(t-d)) \quad (24)$$

Applying the Razumikhin condition (see Lemma 1), for some positive constant $q = (1+\epsilon)$ with $\epsilon > 0$, the following inequality holds:

$$z_1^T(t-d)P_1 z_1(t-d) \le (1+\epsilon)z_1^T(t)P_1 z_1(t) \quad (25)$$

From (25), it follows that

$$\lambda_{min}(P_1)\|z_1(t-d)\|^2 \le z_1^T(t-d)P_1 z_1(t-d) \le (1+\epsilon)\lambda_{max}(P_1)z_1^T z_1 \quad (26)$$

Thus

$$\|z_1(t-d)\|^2 \le \frac{(1+\epsilon)\lambda_{max}(P_1)}{\lambda_{min}(P_1)} z_1^T(t)z_1(t) \quad (27)$$

Then from (19) and (27)

$$\begin{aligned} & F_{1\delta}^T(\cdot)\, F_{1\delta}(\cdot) \\ \le\ & \psi_1 z_1^T(t)z_1(t) + \psi_2 z_1^T(t-d)z_1(t-d) \\ =\ & \psi_1 z_1^T(t)z_1(t) + \psi_2\|z_1(t-d)\|^2 \\ \le\ & \psi_1 z_1^T(t)z_1(t) + \psi_2\frac{(1+\epsilon)\lambda_{max}(P_1)}{\lambda_{min}(P_1)} z_1^T(t)z_1(t) \\ =\ & (\psi_1 + \psi_2\frac{(1+\epsilon)\lambda_{max}(P_1)}{\lambda_{min}(P_1)})z_1^T(t)z_1(t) \end{aligned} \quad (28)$$

where ψ_1 and ψ_2 are defined in (20).
Substituting (25) into (24) yields

$$\begin{aligned}\dot{V}_1|_{(17)} &= z_1^T(t)[A_o^T P_1 + P_1 A_o + (1+\epsilon)P_1 + P_1 A_1 \cdot P_1^{-1} A_1^T P_1]z_1(t) + 2z_1^T P_1 \bar{F}_1[t, z_1(t), z_1(t-d)] \\ &= \begin{bmatrix} z_1(t) \\ F_{1\delta} \end{bmatrix}^T \begin{bmatrix} W & P_1 \\ P_1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ F_{1\delta} \end{bmatrix}\end{aligned} \quad (29)$$

where

$$W = A_o^T P_1 + P_1 A_o + (1+\epsilon)P_1 + P_1 A_1 P^{-1} A_1^T P_1$$

The inequality (28) can be rewritten as

$$\begin{bmatrix} z_1(t) \\ F_{1\delta} \end{bmatrix}^T \begin{bmatrix} \beta I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} z_1 \\ F_{1\delta} \end{bmatrix} \ge 0 \quad (30)$$

where $\beta = \psi_1 + \psi_2 \frac{(1+\epsilon)\lambda_{max}(P_1)}{\lambda_{min}(P_1)}$.

It can be seen from (30) and (29) that,

$$\begin{aligned}\dot{V}_1|_{(17)} &\le \begin{bmatrix} z_1(t) \\ F_{1\delta} \end{bmatrix}^T \left(\begin{bmatrix} W & P_1 \\ P_1 & 0 \end{bmatrix} + \alpha \begin{bmatrix} \beta I & 0 \\ 0 & -I \end{bmatrix} \right) \begin{bmatrix} z_1(t) \\ F_{1\delta} \end{bmatrix} \\ &= \begin{bmatrix} z_1(t) \\ F_{1\delta} \end{bmatrix}^T \begin{bmatrix} W + \beta\alpha I & P_1 \\ P_1 & -\alpha I \end{bmatrix} \begin{bmatrix} z_1 \\ F_{1\delta} \end{bmatrix}\end{aligned} \quad (31)$$

where α is a positive constant.

As seen from (31) and the inequality (21), $\dot{V}$ is symmetric negative definite. Hence the result follows. ∎

Remark 3. A set of sufficient conditions has been presented in Theorem 1, to guarantee the asymptotic stability of the designed sliding motion. The conditions can be expressed in LMI and thus, the associated design parameters can be obtained systematically using LMI techniques. This is in comparison with the work (Yan et al. (2010, 2012)).

Remark 4. From the proof of Theorem 1, it follows that it is unnecessary to assume that the bound on the uncertainty $F_1(t, z(t), z_d(t))$ has the special form in (14). Actually, it is only required that the bound on $F_{1\delta}(\cdot)$ defined in (19) has the special form in (14). Therefore, in this paper, the requirement on the bound on mismatched uncertainty is relaxed which is allowed to have more general form.

4.2 Sliding mode control design

The objective now is to design a state feedback sliding mode control law such that the system state is driven to the sliding surface (16) in finite time. The following control is proposed:

$$u(t) = -B_2^{-1}(\gamma + \{\|M\|(\varpi_1^2\|z\|^2 + \varpi_2^2\|z_d\|^2)^{1/2} + \|B_2\|\phi(t, z(t), z_d) + \rho(t, z(t), z_d) + \eta\}\text{sgn}(\sigma(z))) \quad (32)$$

where ϖ_1, ϖ_2, $\phi(\cdot)$, $\rho(\cdot)$ are defined in (12)-(14) respectively and η is the reachability constant. The following result is ready to be presented.

Theorem 2. Consider the system(8). Under Assumptions 1-3, the control (32) drives the system (8) to the sliding surface (16) in finite time and maintains a sliding motion on it thereafter.

Proof. From (14), it is observed that

$$\|F_1(\cdot)\|^2 \le (\varpi_1^2\|z\|^2 + \varpi_2^2\|z_d\|^2) \quad (33)$$

Then

$$\|F_1(\cdot)\| = (\varpi_1^2\|z\|^2 + \varpi_2^2\|z_d\|^2)^{1/2} \quad (34)$$

From (34)and (13),

$$\begin{aligned}\|MF_1(\cdot) + F_2(\cdot)\| &\le \|M\|\|F_1(\cdot)\| + \|F_2(\cdot)\| \\ &\le \|M\|(\varpi_1^2\|z\|^2 + \varpi_2^2\|z_d\|^2)^{1/2} \\ &\quad + \rho(t, z(t), z_d)\end{aligned} \quad (35)$$

From (15) and (10)-(11), it can be verified that

$$\begin{aligned}\dot\sigma(z) = & M(A_{11}z_1 + A_{d11}z_1(t-d) + A_{12}z_2 + A_{d12}z_2(t-d) \\ & + F_1(\cdot)) + (A_{21}z_1 + A_{d21}z_1(t-d) + A_{22}z_2 \\ & + A_{d22}z_2(t-d)) + Bu(t) + BG(t, z(t), z_d) \\ & + F_2(t, z(t), z_d)) \\ = & \gamma + Bu(t) + BG(t, z(t), z_d) + MF_1(t, z(t), z_d) \\ & + F_2(t, z(t), z_d) \end{aligned} \tag{36}$$

where γ is defined as

$$\begin{aligned}\gamma = & (MA_{11} + A_{21})z_1(t) + (MA_{d11} + A_{d21})z_1(t-d) + \\ & (MA_{12} + A_{22})z_2(t) + (MA_{d21} + A_{d22})z_2(t-d)\end{aligned}$$

Applying the control u in (32) to system (8), it follows from (12) and (13),

$$\begin{aligned}\sigma^\tau\dot\sigma = & \sigma^\tau(z)[\gamma + MF_1(t, z(t), z_d) + Bu + BG(t, z(t), z_d) \\ & + F_2(t, z(t), z_d)] \\ = & \sigma^\tau(z)\gamma - \sigma^\tau(z)MF_1(t, z(t), z_d) - \sigma^\tau(z)[\gamma + \{\|M\| \\ & (\varpi_1^2\|z\|^2 + \varpi_2^2\|z_d\|^2)^{1/2} + \|B\|\phi(t, z(t), z_d) \\ & + \rho(t, z(t), z_d) + \eta\}\text{sgn}(\sigma)] + BG(t, z(t), z_d) \\ & + F_2(t, z(t), z_d) \\ = & \sigma^\tau(z)MF_1(t, z(t), z_d) - \|\sigma(z)\|\|M\|(\varpi_1^2\|z\|^2 \\ & + \varpi_2^2\|z_d\|^2)^{1/2} + \sigma^\tau(z)BG(t, z(t), z_d) - \|\sigma\|\|B\| \\ & \phi(t, z(t), z_d) + \sigma^\tau(z)F_2(t, z(t), z_d) \\ & - \|\sigma\|\rho(t, z(t), z_d) - \eta\|\sigma(z)\| \\ \le & \|\sigma^\tau(z)MF_1(t, z(t), z_d)\| - \|\sigma(z)\|\|M\|(\varpi_1^2\|z\|^2 \\ & + \varpi_2^2\|z_d\|^2)^{1/2} + \|\sigma^\tau(z)BG(t, z(t), z_d)\| \\ & - \|\sigma\|\|B\|\phi(t, z(t), z_d) + \|\sigma^\tau(z)F_2(t, z(t), z_d)\| \\ & - \|\sigma\|\rho(t, z(t), z_d) \\ \le & -\eta\|\sigma(z)\| \end{aligned} \tag{37}$$

where the fact that $\sigma^\tau(z)\text{sgn}(\sigma(z)) \ge \|\sigma(z)\|$ is used to obtain the inequality above.
This shows that the reachability condition holds and hence the conclusion follows. ■

Theorems 1 and 2 together show that the corresponding closed-loop system is asymptotically stable.

5. APPLICATION AND SIMULATION RESULTS

Consider the cascaded CSTR in fig.1 which is used to illustrate the effectiveness of the developed method in this paper. The compositions C_A and C_B of the produce streams from reactor A and reactor B, represents the system states which are to be controlled. The output of one reactor CSTR determines the flow rate into the second reactor and vice versa. A time delay is added between the output of one reactor and the input (flow rate) of the other reactor such that at a certain time, the state of one reactor is determined by the state of the other reactor at a previous time $t-d(t)$. Refer to Holz and Schneider (1993); Hua et al. (2009) for more information on CSTR. By choosing the same parameters as in Hua et al. (2009), the mathematical model to describe the CSTR is given by

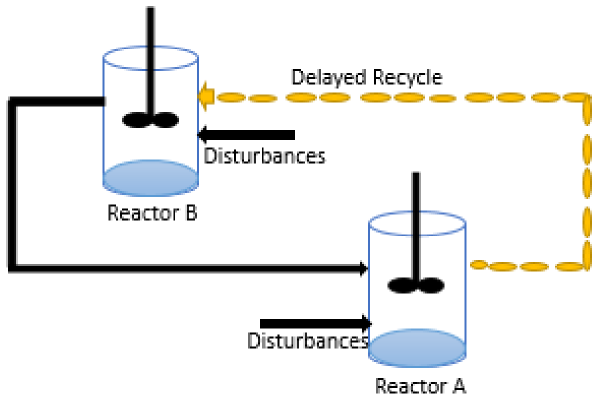

Fig. 1. Cascade chemical reactor system.

$$\dot z_1 = -z_1 - 0.5z_1(t-d) + z_2 + F_1(t, z, z_d) \tag{38}$$

$$\begin{aligned}\dot z_2 = & z_1 + z_1(t-d) - 2.8333z_2 + z_2(t-d) \\ & + (u + G(t, z(t), z_d)) + F_2(t, z, z_d)\end{aligned} \tag{39}$$

where $z_1 := C_A - C_A^*$, $z_2 := C_B - C_B^*$ and $C_A^* = 14/9$ and $C_B^* = 7/3$ (see Hua et al. (2009)).

For system (38)-(39), the uncertainties $G(\cdot)$ and $F_2(\cdot)$ are assumed to satisfy

$$\|G(t, z(t), z_d)\| \le \underbrace{5|\sin(t)||z_1(t)z_2(t-d)|}_{\psi(t,z(t),z_d)}$$

$$\|F_2(t, z(t), z_d)\| \le \underbrace{1.5|\cos(t)||z_2(t-d)|}_{\rho(t,z(t),z_d)}$$

and $F_1(\cdot)$ satisfies

$$F_1^T F_1 \le \underbrace{0.86^2}_{\varpi_1} z^T z_d + \underbrace{0.65^2}_{\varpi_2} z_d^T z_d$$

Choose the sliding function

$$\sigma(z) = [\underbrace{2}_{M} \quad 1]z$$

when the sliding mode takes place,
$A_o = -3, \quad A_1 = -0.5.$
With $Q = I$, and $P_1 = 0.1667$ obtained by solving the LMI (31), $\alpha = 0.07$, and $\beta = 5.5781$ is the maximum boundary which ensures that

$$\begin{bmatrix} -0.3995 & 0.1667 \\ 0.1667 & -0.0700 \end{bmatrix} < 0$$

Thus the matrix (31) is negative definite.
From the result above, it can be verified that all the conditions in Theorem 1 are satisfied. Thus from Theorem 1, the sliding motion associated with the sliding surface is asymptotically stable.
From Theorem 2, the sliding mode control law

$$\begin{aligned}u = & -\gamma - [1.3778\|z(t)\|^2 + 0.845\|z_{d(t)}\|^2 + 1.5\cos(t)|z_{d2}| \\ & + 5\sin(t)|z_1(t)z_{d2}| + 2]\text{sgn}(\sigma(z))\end{aligned}$$

stabilizes the system (38)-(39).

For simulation purposes, assume the initial conditions relating to the time delay is

$$z(t) = \text{col}(\sin(t), \text{e}^t)$$

and the delay is

$$d(t) = 2 - 0.5\text{sin}t$$

The time response of the state variables, control signal and sliding function is shown in Figures 2, 3 and 4 respectively, which demonstrate that the proposed approach is effective.

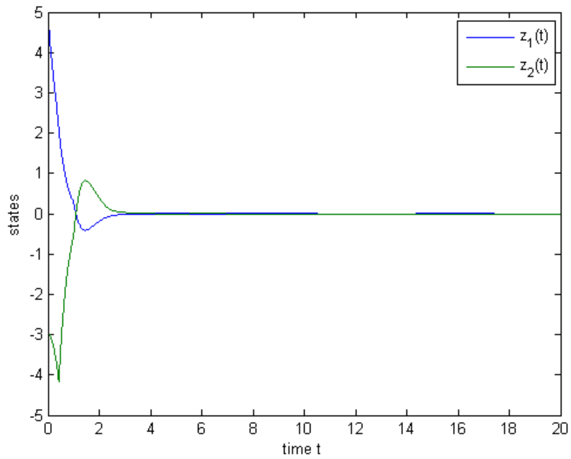

Fig. 2. Time response of system state variables $z_1(t)$ and $z_2(t)$ of system (38)-(39)

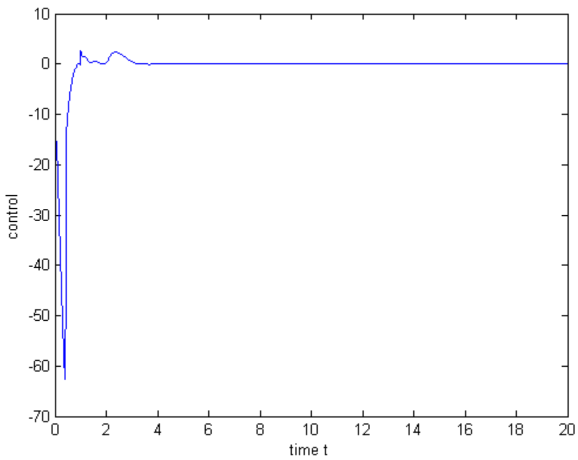

Fig. 3. Time response of the control signal $u(t)$

6. CONCLUSION

In this paper, state feedback sliding mode control for nonlinear uncertain time delay systems has been considered, where time delay exists in both system state and disturbance. Conservatism is reduced by fully using the property that sliding mode control is of reduced order, and the nonlinear bounds on uncertainties have been fully employed in control design. The results of the simulation verify the theoretical analysis and further illustrate the feasibility of the proposed methodology, through application to the control problem of the CSTR system.

REFERENCES

Al-Shamali, S.A., Crisalle, O.D., and Latchman, H.A. (2003). An approach to stabilize linear systems with state and input delay. In *American Control Conference, 2003. Proceedings of the 2003*, volume 1, 875–880. IEEE.

Baek, J., Jin, M., and Han, S. (2016). A new adaptive sliding-mode control scheme for application to robot manipulators. *IEEE Transactions on Industrial Electronics*, 63(6), 3628–3637.

Dong, R., Tang, G., and Guo, Y. (2010). Optimal sliding mode control for uncertain systems with time delay. In *2010 Chinese Control and Decision Conference*, 2814–2819. IEEE.

Edwards, C. and Spurgeon, S. (1998). *Sliding mode control: theory and applications*. CRC Press.

Gu, K., Chen, J., and Kharitonov, V.L. (2003). *Stability of time-delay systems*. Springer Science & Business Media.

Holz, R. and Schneider, F. (1993). Control of dynamic states with time delay between two mutually flow rate coupled reactors. *The Journal of Physical Chemistry*, 97(47), 12239–12243.

Hua, C.C., Wang, Q.G., and Guan, X.P. (2008). Memoryless state feedback controller design for time delay systems with matched uncertain nonlinearities. *IEEE Transactions on Automatic Control*, 53(3), 801–807.

Hua, C., Liu, P.X., and Guan, X. (2009). Backstepping control for nonlinear systems with time delays and applications to chemical reactor systems. *IEEE Transactions on Industrial Electronics*, 56(9), 3723–3732.

Orlov, Y., Perruquettl, W., and Richard, J. (2003). Sliding mode control of uncertain time-delay systems. In *European Control Conference (ECC), 2003*, 53–58. IEEE.

Richard, J.P. (2003). Time-delay systems: an overview of some recent advances and open problems. *automatica*, 39(10), 1667–1694.

Xia, Y. and Jia, Y. (2002). Robust delay-dependent sliding mode control for uncertain time-delay systems. In *American Control Conference*, 2786–2791. Alaska, USA.

Xu, B. (1997). Stability robustness bounds for linear systems with multiple time-varying delayed perturbations. *International Journal of Systems Science*, 28(12), 1311–1317.

Yan, X.G., Spurgeon, S.K., and Edwards, C. (2010). Static output feedback sliding mode control for time-varying delay systems with time-delayed nonlinear disturbances. *International Journal of Robust and Nonlinear Control*, 20(7), 777–788.

Yan, X.G., Spurgeon, S.K., and Edwards, C. (2012). Global decentralised static output feedback slidingmode control for interconnected time-delay systems. *IET control theory & applications*, 6(2), 192–202.

Yan, X.G., Spurgeon, S.K., and Edwards, C. (2014). Memoryless static output feedback sliding mode control for nonlinear systems with delayed disturbances. *IEEE Transactions on Automatic Control*, 59(7), 1906–1912.

Yan, X.G., Spurgeon, S.K., and Edwards, C. (2017). *Variable Structure Control of Complex Systems: Analysis and Design*. Springer International Publishing.

Yang, M., Juntao, L., Wenlin, L., and Yuanwei, J. (2009). A new sliding mode control approach for a class of uncertain takagi-sugeno fuzzy time-delay systems. In *2009 Chinese Control and Decision Conference*, 6117–6121. IEEE.

Yunhao, H., Yang, M., Juntao, L., and Jing, Y. (2009). Robust sliding mode control for nonlinear uncertain time-delay systems based on disturbance observer. In *2009 Chinese Control and Decision Conference*, 5535–5540. IEEE.