Dividend Derivatives

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Dividend derivatives are not simply a by-product of equity derivatives. They constitute a distinct growing market and an entire suite of dividend derivatives are offered to investors. In this paper we look at two potential models for equity index dividends and discuss their theoretical and practical merits. The main results emerge from a downward jump-diffusion model with beta distributed jumps and a stochastic logistic diffusion model, both able to capture the particular dynamics observed for dividends and cum-dividends, respectively, in the market. Smile calibration results are discussed with market data on Dow Jones Euro STOXX50 DVP® dividend index for futures and European call and put options.

Keywords: Dividend futures; Dividend options; Stochastic logistic diffusion; Market price of risk; Smile calibration

JEL Classification: G13; C51

1. Introduction

Dividends played a major role in the development of equity financial products over the years. Lu and Karaban (2009) showed that since 1926, dividends have represented approximately one-third of total returns, the rest coming from capital appreciation. Moreover, total dividend income has increased in U.S. six-fold between 1988 and 2008, reaching almost 800 billion USD. While dividends have grown in proportion to increasing stock market capitalization, evidence shows that dividends have also grown as a proportion of personal income. Furthermore, dividends are considered a good hedge against rising inflation and they have in general lower volatility than equities as discovered by Shiller (1981).

The increasing market activity in equity futures and forwards, options, and structured products implies a growing exposure for banks to dividend payments on balance sheets. Dividend risk is traded through many type of contracts from single-stock and index to swaps, steepeners, yield trades, ETFs, options, knock-out dividend swaps, dividend yield swap and even swaptions. Brennan (1998) suggested to strip off the equity index from its dividends and create a market in the dividend strips which should improve the informational efficiency in the economy. Another financial innovation designed to offer dividend protection is the endowment warrant, although, as discussed by Brown and Davis (2004), the protection is only partial and pricing is not easy since it is a long-term option having a stochastic strike price driven by the cash-flow of dividends.

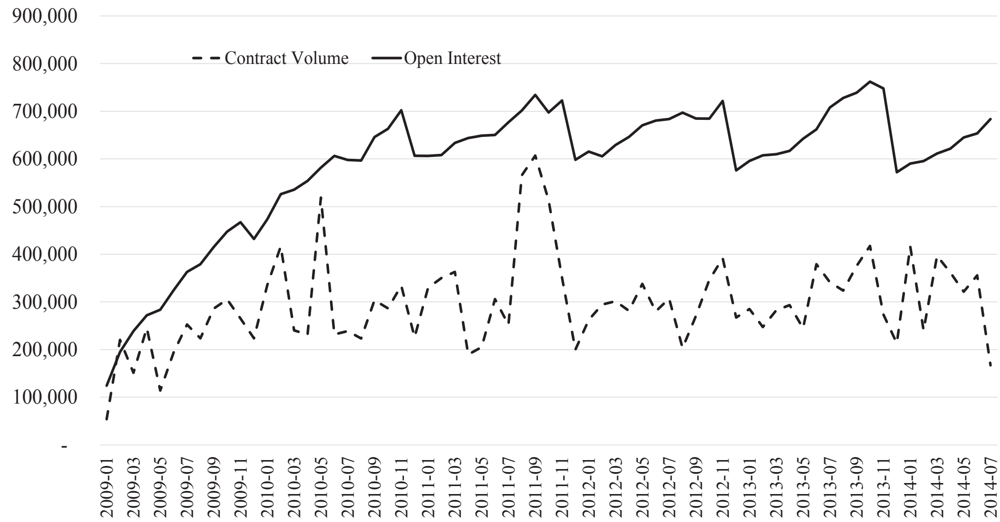
Dividend derivatives have been traded over-the-counter (OTC) since the turn of the millennium, mainly in the form of index dividend swaps. The first time dividend derivatives were traded on an exchange was in 2002 in South-Africa, see Wilkens and Wimschulte (2010), but the success was moderate. NYSE Liffe have launched futures contracts on the FTSE100® dividend index in May 2009. There are also exchange-traded dividend futures on the HSI and HSCEI indexes traded in Hong Kong, on the AEX index in Netherlands. On 25 May 2010 the S&P 500 Dividend Index (DVS) and options on the S&P500 Annual Dividend Index (DIVD) were introduced on CBOE, while options on Euro STOXX50 Index Dividend Futures (FEXD) and Euro STOXX50 Index Dividend Points (DVP) were also introduced on Eurex. The futures contract on the Dow Jones Euro STOXX50 DVP® index introduced on 30 June 2008 by Eurex has experienced a meteoric development. This is hardly surprising since reinvested dividends accounted for almost half of the Dow Jones Euro STOXX50® total returns since the end of December 1991.

There is a buoyant market now driven by these contracts, establishing dividends as an asset class of its own as argued by Manley and Mueller-Glissmann (2008). The volume of contracts and open interest are illustrated in Figure 2(a) for dividend futures and in Figure 2(b) for dividend options.

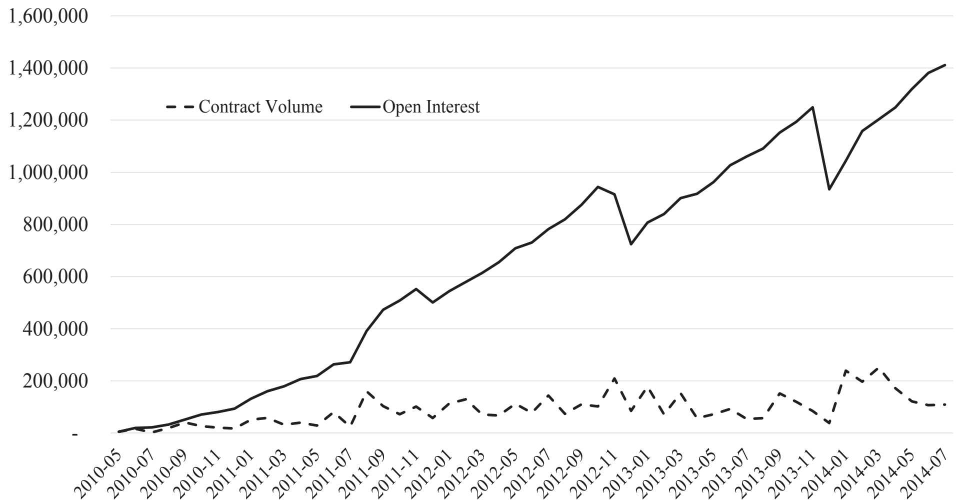
Dividend derivatives have many applications for investors. Equity derivatives traders and structured products engineers must consider their dividend risk and manage its risk. Portfolio managers with convertible bond positions and equity positions have exposure to dividend risk. In some countries investing in dividends offer a degree of tax reduction. Last but not least, carrying equity stock during systemic crises may imply less dividend payments than expected so by taking positions on dividend derivatives the investor on exchanges may help avoiding liquidity pressures.

As with many other emerging asset classes, the modeling for dividend derivatives lags behind in development, although modeling dividends has preoccupied academics and practitioners for many years. The literature on *dividend derivatives* pricing is very sparse. In general the models covering dividends modeling *assume* the dividend payments have either known *size* or *timing* or both. This strong assumption will make almost impossible a dividend derivatives smile calibration. The scope of this paper is to present two models that are flexible enough to provide an overarching calibration to dividend futures and dividend European options. One model focuses on the dividend individual payments as jumps associated with the evolution of the underlying stock price. Both the size and frequency is captured from historical data and then the incomplete market model is completed with the futures prices. The second model aims to capture the dynamics of the cum-dividend value within the year. This is very useful since the dividend derivatives payoffs are functions of the cum-dividend itself. For both models we show how to calibrate the smile across the strike prices and

Figure 1. The number of traded contracts and open interest in the Eurostoxx 50 Dividend Futures and Options traded on Eurex as of September 2014. *Source:* Eurex.



(a) Eurostoxx 50 Dividend Futures.



(b) Eurostoxx 50 Dividend Options.

also across the maturities going up to ten years ahead. This is achieved with our models having only few parameters.

The article is structured as follows. Section 2 provides a literature review of dividend modeling related literature.

Section 3 describes the data used in this research. The main modeling results are contained in Section 4. Numerical results using the available data are provided in Section 5. The last Section discusses the advantages and disadvantages of the models and points out their current applications.

2. The Linkage Between Equity and Dividends

2.1. *Connection with Literature*

Black (1976) and Black (1990) argued that investors value equity by predicting and discounting dividends but that we know very little about what should investors do with their dividends and what should issuing corporations do about dividend policy. Using tests based on volatility, Shiller (1981, 1986) emphasized that stock price movements cannot be explained only from the information on the future dynamics of dividends around a long-run historical trend. In the literature the overwhelming conclusion is that future dividends are uncertain, both in their timing and size.

Denoting with $Div_{t,T}$ the gross dividend paid on the equity index over the period $[t,T]$ the *forward* price at time t on the cumulative dividend stream $\{Div_{t,T}\}_{t\leq u\leq T}$ is given by $FW_t(Div_{t,T}) = PV_t(Div_{t,T})\exp\left(r_{t,T}(T-t)\right)$ where $r_{t,T}$ is the risk-free interest rate and $PV_t(Div_{t,T})$ is the present value of the gross dividend stream for the period $[t,T]$ at time t. For dividend derivatives it would be useful to know the dividends that will be paid at a given horizon. Harvey and Whaley (1992) and Brooks (1994) extracted implied dividends employing the put-call parity but these estimators were too noisy for predicting the next dividend. However, their idea to calculate the implied dividend quantity to the required horizon of dividend derivatives as

$$PV_t(Div_{t,T}) = S_t + p_t^E(K,T) - c_t^E(K,T) - K\exp\left[-r_{t,T}(T-t)\right]. \tag{1}$$

where S is the underlying equity index, c^E and p^E are the call and put European option prices with maturity T and exercise price K, is used by some investors[1] to calculate the price of dividend derivatives, assuming a given risk-free rate. Considering $q_{t,T}$ as the continuously compounded dividend yield for the period $[t,T]$ Golez (2014) suggested reverse engineering *both* the implied risk free rate $r_{t,T}$ and the implied dividend yield $q_{t,T}$ from the futures price formula[2] and the put-call parity from associated equity options

$$F_t(T) = S_t\exp\left[(r_{t,T} - q_{t,T})(T-t)\right] \tag{2}$$

where $F_t(T)$ is the futures price at time t for maturity T,

$$c_t^E(K,T) - p_t^E(K,T) = S_t\exp\left[-q_{t,T}(T-t)\right] - K\exp\left[-r_{t,T}(T-t)\right] \tag{3}$$

The two equations (2) and (3) give

$$r_{t,T} = \frac{1}{T-t}\log\left[\frac{F_t(T) - K}{c_t^E(K,T) - p_t^E(K,T)}\right] \tag{4}$$

and

$$q_{t,T} = -\frac{1}{T-t}\log\left[\left(\frac{c_t^E(K,T) - p_t^E(K,T)}{S_t}\right) + \frac{K}{S_t}\left(\frac{c_t^E(K,T) - p_t^E(K,T)}{F_t(T) - K}\right)\right] \tag{5}$$

[1]Some interesting readings in this area can be found in Bos and Vandermark (2002), Frishling (2002), Lu and Karaban (2009), Manley and Mueller-Glissmann (2008) used to deal with dividend paying stocks by assuming constant and unknown dividends, in cash or as an yield, and then proceed with an option pricing calculator such as Black-Scholes for example, with a deflated stock price resulting from stripping out the presumed known dividends over the life of the option from current stock price.

[2]Remark that in this case, due to the tacit assumption that $r_{t,T}$ is constant, forward prices are equal to futures prices.

Because $PV_t(Div_{t,T}) = \exp\left[-q_{t,T}(T-t)\right]$ the value of the forward price on the dividends on Dow Jones Euro STOXX50® would follow immediately[3].

This model free approach, while easy to implement, has several flaws. First and foremost, it assumes that the future dividends are *unknown but constant* which is not the case in reality. Secondly, another problem is the choice of the exercise price K since the results may be sensitive between the futures price series and interest rate series to see if the futures prices are equal to forward prices.

Since there is an intrinsic relationship between a stock index and the corresponding dividend stream one may consider calibrating both index options and dividend derivatives, *jointly.* The efficient market model (Campbell *et al.* 1997) is described by the two relationships given all information available at time t $R_{t+1} = \frac{S_{t+1}+Div_{t,t+1}}{S_t} - 1$ and $E_t(R_{t+1}) = r$, where r is the risk-free rate. Then, straightforward calculations lead to $S_t = \frac{E_t(S_{t+1}+Div_{t,t+1})}{1+r}$ and applying the law of iterated expectations gives

$$S_t = E_t\left(\sum_{i=0}^{T} \frac{Div_{t+i,t+i+1}}{(1+r)^i}\right) + E_t\left(\frac{S_{t+T}}{(1+r)^T}\right) \tag{7}$$

This relationship may imply a possible calibration of dividend expectation curves to the stock index. However, as Shiller (1981) demonstrated this efficient markets model leads to a mismatch in the volatilities of the stock index and the volatility of the associated dividends.

Implied dividends have been utilised as part of the estimation process for risk-neutral densities by Ait-Sahalia and Lo (1998). The empirical properties of dividends have been amply discussed by van Binsbergen *et al.* (2012). Models that forecast dividends have had mixed results in the literature and the empirical evidence is divided on their usefulness, particularly for long maturities. Chance *et al.* (2000) developed a forecasting model for dividends accounting for seasonality and mean reversion effects, showing that it is possible to produce unbiased estimators of dividend *related* quantities. Recent papers showing how to forecast the dividend yields are van Binsbergen *et al.* (2012), Chen *et al.* (2012), Overhaus *et al.* (2007), Kruchen and Vanini (2008), Buehler *et al.* (2010). A no-arbitrage methodology for estimating discrete dividend payments from market prices of options is proposed in Desmettre *et al.* (2015) on the assumption that dividend cash payments are replicable by market instruments such as forwards on associated equity.

Chance *et al.* (2000) analysed index option prices based on ex-post realized dividend information, with the corresponding options valued using ex-ante dividend forecasts, and they found that the latter does not lead to biased pricing, although the sample error is quite large. On the other hand, the implied dividends from S&P500 options may improve significantly the forecasts of market returns as demonstrated by Golez (2014). Using data between 1994 and December 2009, Golez first shows that the dividend-price ratio gives a poor forecast for future returns and dividend growth. Kragt *et al.* (2014) used dividend futures term structure to extract information about the investors' expectations on equity.

Geske (1978) pointed out that assuming that dividends are known when in fact they are not, has the effect to mis-estimate the volatility. On the other hand, Chance *et al.* (2002) demonstrated that when dividends are stochastic and discrete such that the present value of all future dividends is observable and tradable in a forward contract, Black-Scholes formula still applies for pricing European options. Assuming that the present value of all dividend payments generated by a stock to a given maturity is known may seem quite strong, although the very nascency of dividend futures markets may provide a good mechanism to ascertain this value. The interlink between dividends

[3]Alternatively one can reverse-engineer from put-call parity directly the present value of all gross returns

$$PV(Div_{t,T}) = S_t - \left[c_t^E(K,T) - p_t^E(K,T)\right]\frac{F_t}{F_t - K}. \tag{6}$$

and volatility was emphasized by Broadie *et al.* (2000), who proved that both dividend risk and volatility risk are relevant for pricing American options contingent on an asset with stochastic volatility and an uncertain dividend yield.

Frishling (2002) discussed three different approaches to model the linkage between dividends and stocks modeled stochastically. For the same dividend payment and identical parameters for stock price Frishling (2002) provided an example illustrating that it is possible to get very different distributions for the stock at maturity T when using different models. Hence, the method employed to model dividends can have a great impact on the final results. Lioui (2005) derived analytical formulae for pricing forward and futures on assets with a *stochastic* dividend yield and Lioui (2006) developed European options pricing formula of Black-Scholes type, incorporating stochastic dividend yield and using a stochastic mean-reverting market price of risk. Importantly, Lioui (2006) showed that stochastic dividend yields may lead to a different type of put-call parity for equity from the one that is normally used to reverse engineer the dividend yield from market European option prices, thus invalidating the approach based on the direct formula (1). Recognizing that in practice dividends on stocks are not paid continuously but at discrete times, Korn and Rogers (2005) developed a general approach for *stock* option pricing, where the absolute size of the dividend is random but its relative size is still *constant*. One advantage of their model can be adapted to deal with dividends announced in advance and with changing in dividend policy. Bernhart and Mai (2015) generalized this line of modeling dividends as a discrete cash-flow series and proposed a no-arbitrage methodology capable of embedding many well-known stochastic processes and general dividend specification. Brockhaus (2016) presented a general family of models treating dividends for equity derivatives, encompassing the model in Korn and Rogers (2005). Another interesting approach in Buehler *et al.* (2010) considers an equity stock price model with discrete stochastic proportional dividends. Their model assumes that dividend ratios are a linear combination between the classic known proportional dividends and a stochastic dividend part described by an Ornstein-Uhlenbeck process.

van Binsbergen *et al.* (2013) used the dividend futures to construct equity yields by analogy with bond yields. They showed that the equity yields obtained in this manner can be decomposed into expected dividend growth rates and risk premia. Using the dividend futures term structure facilitates the study of the term structure of risk premia and conclude that the slope of the term structure of expected dividend growth rates is counter-cyclical. Denoting by $Div_{t,T}$ the cum-dividend paid by a stock index over the period $(t, T]$ and by r the risk-free rate for that same period, van Binsbergen *et al.* (2013) price dividend futures with the formula

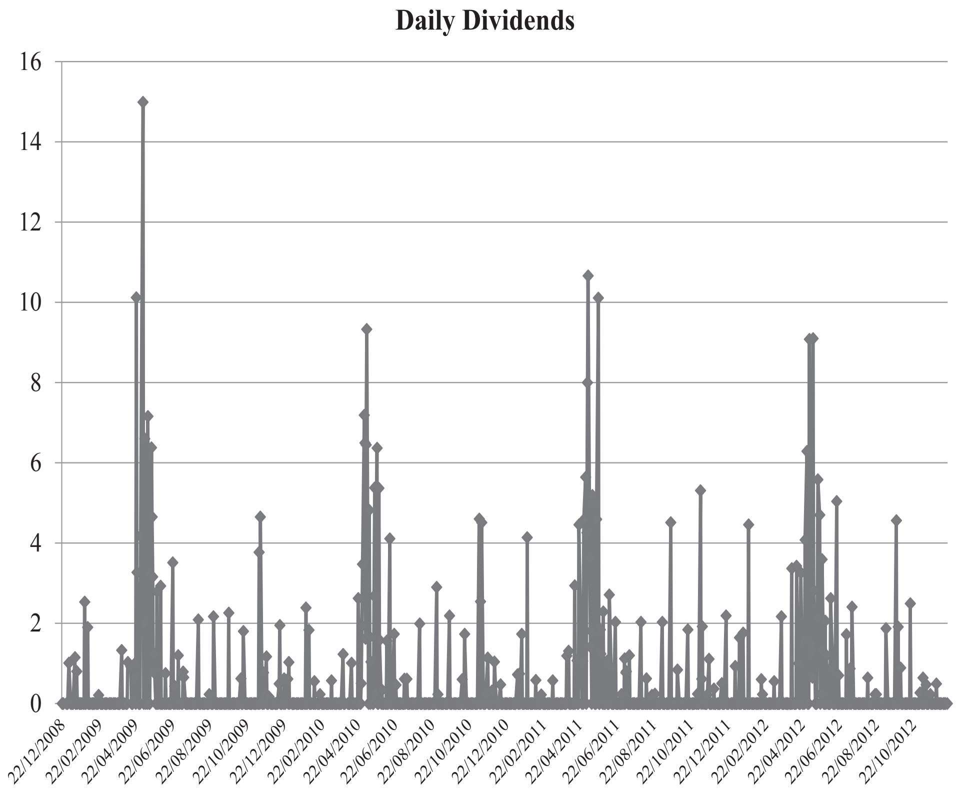
$$F_{t,T}(Div_{t,T}) = PV(Div_{t,T}) \exp(r(T - t)) \tag{8}$$

arguing that this formula is correct because:

> "This no-arbitrage relationship holds for non-dividend paying assets. At first sight this may be confusing,as the focus of the paper is on dividends. The index does indeed pay dividends, and therefore, futures on the index are affected by these dividend payments. However, the futures contracts we study are not index futures, but dividend futures. These dividend futures have the dividend payments as their underlying, not the index value. As dividends themselves do not pay dividends, Eq. (8) is the appropriate formula."

There are many assets that do not pay dividends such as oil, gas, soybean, coffee, freight, real-estate indices and so on. Yet, none of the futures traded on this asset classes has a stock paying no dividend *forward* type of formula such as (8). In other words, the formula (8) is simply wrong, the mistake consisting in van Binsbergen *et al.* (2013) failing to recognise that dividends are not tradable and therefore dividend derivatives should be priced in an incomplete market set-up, as for any non-tradable underlyings– see Bjork (2009). The main scope of this paper is to propose suitable pricing models that can be used to price dividend futures and European options. One should recognise in the case of dividend futures that a) we are dealing with an incomplete market and any pricing model should take this into account and b) one the dividend derivatives market

Figure 2. The daily dividends in index points paid on the Dow Jones Euro STOXX50® index. The series is daily between 22 December 2008 and 17 December 2012. *Source:* Eurex.



is completed with futures any other derivatives such as options should be priced under the same martingale pricing measure.

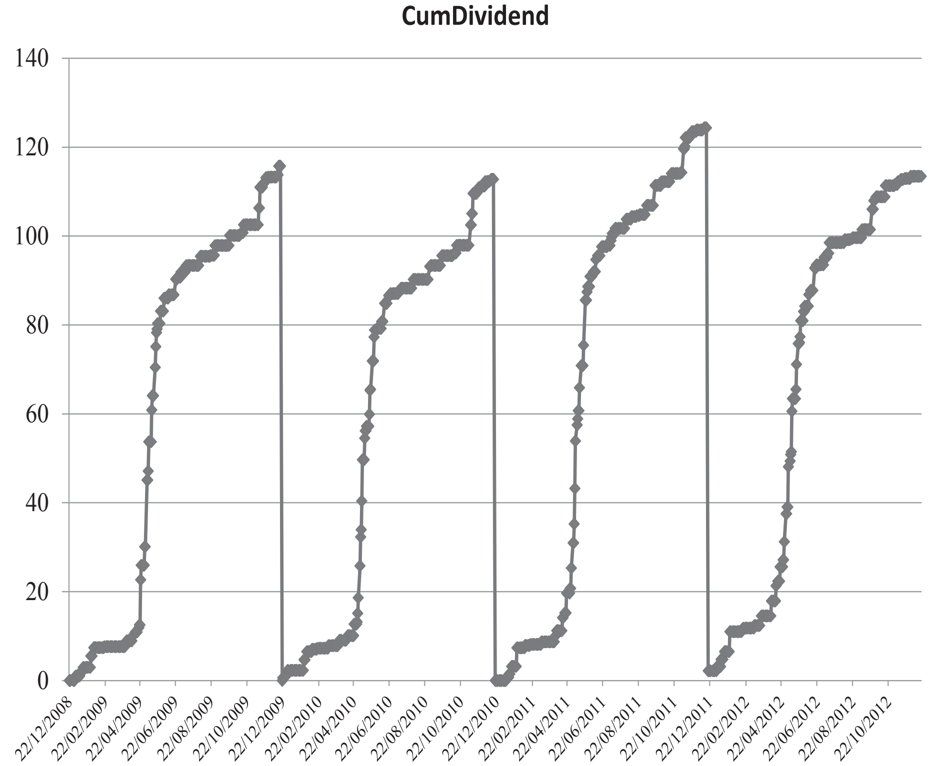
2.2. *Motivation of proposed models*

The literature suggests that for pricing dividend derivatives the focus should be on the dividend cash-flow stream itself and consider dividends evolving in a stochastic manner over time. An insight into how can we capture the randomness in the dividends evolution can be gained from the historical time series of paid dividends for Dow Jones Euro STOXX50® presented in Figure 2. Dividends are also measured in index points. One clear characteristics of this series is that the time series resembles a jump process and that the size of the jumps looks stochastic.

Another way dividends are reported is based on the cum-dividend series within each calendar market year. This is helpful for the dividend futures contracts traded on Eurex or for index dividend swaps contracts traded over-the-counter. The cum-dividend series depicted in Figure 3 display a very interesting regular pattern. The shape is clearly sigmoidal with an inflection point almost half-way in June. Our stochastic logistic diffusion model described in Section 4.2 is capable of producing exactly this type of dynamics pattern.

Whether dividends should be outstripped from equity or modelled on their own is still subject to debate but we offer in this paper in section 4 a model from each category. What has been less emphasized in dividend derivatives pricing literature is that this market is clearly incomplete and therefore the dividend futures prices will help identifying an equivalent martingale pricing measure. Then, pricing of all other derivatives such as European vanilla options on dividends is done under the same pricing measure and the quality of the model can be measured by analysing how well the

Figure 3. The cum-dividend daily time series in index points paid on Dow Jones Euro STOXX50® index. The historical data is daily for the period 22 December 2008 and 17 December 2012. *Source:* Eurex



options prices are calibrated.

3. Data Description

The analysis in this paper is focused on the dividend stream generated by the Dow Jones EURO STOXX50® stock index introduced on 26 February 1998 by Stoxx Ltd. This index covers 50 blue-chip companies from 12 Eurozone countries: Austria, Belgium, Finland, France, Germany, Greece, Ireland, Italy, Luxembourg, the Netherlands, Portugal, and Spain traded on the Eurex. The index has a base value of 1000 points for 31 December 1991. The composition of the index is refreshed every September based on free float market capitalization, with all weightings capped at 10%. The weight is divided by an index divisor. On 16 June 2008 the Dow Jones EURO STOXX50 DVP® Index was introduced, cumulating all dividend payments in index points from the first business day after the third Friday in December to present. Index resets annually on the third day in December. The stock and the dividend indices use the same calculation formula, one with prices and the other with ordinary un-adjusted gross dividends.

The Dow Jones Euro STOXX50® index *dividend futures* contract traded on Eurex was introduced on 30 June 2008 trading initially for seven annual maturities with a value of 100 EUR per one index dividend point as described in Baldwin (2008). The contract is cash settled on the first exchange day after the settlement day which is the third Friday of December of each maturity year[1]. The minimum price change is 0.1 points and since May 4, 2009 there are ten annual con-

[1]If the third Friday is not an exchange day then the settlement day is the exchange day immediately preceding that day.

tracts. In this contract, at maturity, the buyer pays the futures price established when the contract is initiated and the seller pays the euro amount of dividends during a full calendar year determined by the market roll on the third Friday of December. For example, on 9th February 2012 the dividend futures contract on Dow Jones EURO STOXX50 DVP® for the December 2012 maturity was 114.90 EUR while the contract for the December 2021 maturity traded for 93.10 EUR. Hence, on the third Friday of December 2012, the buyer of the futures contract will pay 114.90 EUR while the seller will pay the cash dividend generated from the companies on the Dow Jones EURO STOXX50® index between the third Friday in December 2011 and the third Friday in December of 2012. Likewise, on the third Friday of December 2021, the buyer will pay 93.10 EUR and will receive from the seller the cash dividend between the third Friday in December 2020 and the third Friday in December of 2021.

The final settlement price in this futures contract is determined by the final value of the underlying Dow Jones EURO STOXX50 DVP®, the index of dividends calculated for that annual period. Only gross unadjusted dividends that are declared and paid in the contract period by any of the individual components of the Dow Jones Euro STOXX50® are considered for settlement purposes[1]. The special dividends are excluded from the dividend cash-flow stream underlying the futures contracts but this should not cause concern since the share of special dividends from all dividends has decreased over time as pointed out by DeAngelo *et al.* (2000). The gross ordinary dividends are the unadjusted cash dividends paid between the third Friday of December in preceding year, excluding, and the third Friday of December of current year, inclusive. The futures prices are quoted daily. Hence, index companies paying multiple dividends will contribute on each ex-dividend date based on the free float adjusted share. The descriptive statistics for the Dow Jones Euro STOXX50® , its corresponding logarithmic returns series[2] and its corresponding cum-dividend series in index points are presented in Table 1. The standard deviation of the equity index is equal to 278.08 and this is more than five times larger than the standard deviation of the cum-dividend series equal to 44.43, in line with the conclusions from Shiller (1981). Interestingly, the standard deviations for the futures series presented in Table 2 vary between 6.26 for the nearest maturity contract and 18.50 for the second maturity, even lower than the standard deviation of the dividend time series.

Table 1. Descriptive Statistics for the Dow Jones Euro STOXX 50 index, the STOXX50 logarithmic returns annualized, its corresponding cum-dividend series and dividend series in index points. The historical series covers the daily data for the period 22 December 2008 and 17 December 2012.*Source:* Eurex.

	STOXX50 index	STOXX50 log-return	Cum-Dividend	Dividend
Mean	2579.89	2.25%	64.95	0.45
Median	2592.71	-2.37%	88.27	0
Standard Deviation	278.08	25.39%	44.43	1.39
Kurtosis	2.05	3.0101	1.37	30.27
Skewness	-0.29	0.0049	-0.3823	4.63
Minimum	1809.98	-1592%	0	0.00
Maximum	3068.00	2481%	124.34	14.99
Count	1028	1027	1028	1028

The descriptive statistics of the dividend futures prices are reported in Table 2. The last three maturities of the currently ten contracts traded actively on Eurex from 4 May 2009, and in general these three contracts have very similar prices. The graph in Figure 4 shows the futures settlement prices on Dow Jones Euro STOXX50 DVP® index from Eurex for the first seven maturities, using

[1]The settling at maturity is done versus the weighted sum of the gross cumulative cash dividends paid by each company that is part of the Dow Jones Euro STOXX50® index during that period, multiplied by the number of free-float adjusted shares, and the total is then divided by the index divisor.

[2]The descriptive statistics for the log-returns of the equity index are annualized. It is not straightforward to annualize the skewness and kurtosis and here we used the procedure described in Meucci (2010) based on calculating the cumulants of the aggregated series to the required horizon, in this case one year.

a longer historical data. The nearest maturity contract has had a different evolution compared to the remaining six maturities futures depicted[3], which have a more correlated dynamics. The only time when they *all* seem to converge is at rollover time due to the pull to maturity effect.

Table 2. Descriptive Statistics for the Futures on Dow Jones Euro STOXX50 DVP® index for all maturities. The historical series is daily between 22 December 2008 and 8 February 2012 for the first seven maturities and between 1 May 2009 and 8 February 2012 for the last three yearly maturities. *Source:* Eurex.

	F1	F2	F3	F4	F5	F6	F7	F8	F9	F10
Mean	115.56	102.05	96.42	94.55	94.38	94.96	95.79	100.44	101.37	102.20
Median	113.45	107.70	99.50	98.30	98.30	99.00	100.10	103.60	104.80	106.30
Mode	112.80	113.90	88.60	104.40	114.80	115.20	105.60	105.50	78.60	111.60
Std	6.26	18.50	18.42	17.55	17.35	17.03	17.14	14.45	15.04	15.42
Kurtosis	-0.19	0.49	-0.15	-0.53	-0.72	-0.85	-0.89	-1.02	-0.97	-0.95
Skewness	-0.02	-1.18	-0.84	-0.64	-0.52	-0.44	-0.42	-0.33	-0.35	-0.39
Min	96.10	54.00	51.70	53.70	54.50	55.50	57.20	69.90	69.50	69.20
Max	125.10	125.30	119.90	120.60	122.90	124.30	126.50	128.80	131.10	132.50
Count	806	806	806	806	806	806	806	717	717	717

For pricing purposes discount factors to the required maturity are also needed. In the aftermath of the subprime crisis the role of the funding rate has become prominent. Since the purpose of our paper is to price dividend derivatives traded on an exchange there is no reason for a collateral agreement and therefore for using OIS curve for discounting. Hence, here we work with discount factors calibrated from the Euribor-swaps curve. In order to have a smooth pasting from euro futures implied rates to swap implied rates we use 3-month tenor swaps with a 3-month Euribor reference rate. The risk-free discount curves produced in this way are equivalent to those used by an investment bank on the day where pricing of dividend futures and options is undertaken. The discount curves go to the required maturity, up to ten years ahead.

4. Models for Index Dividend Derivatives

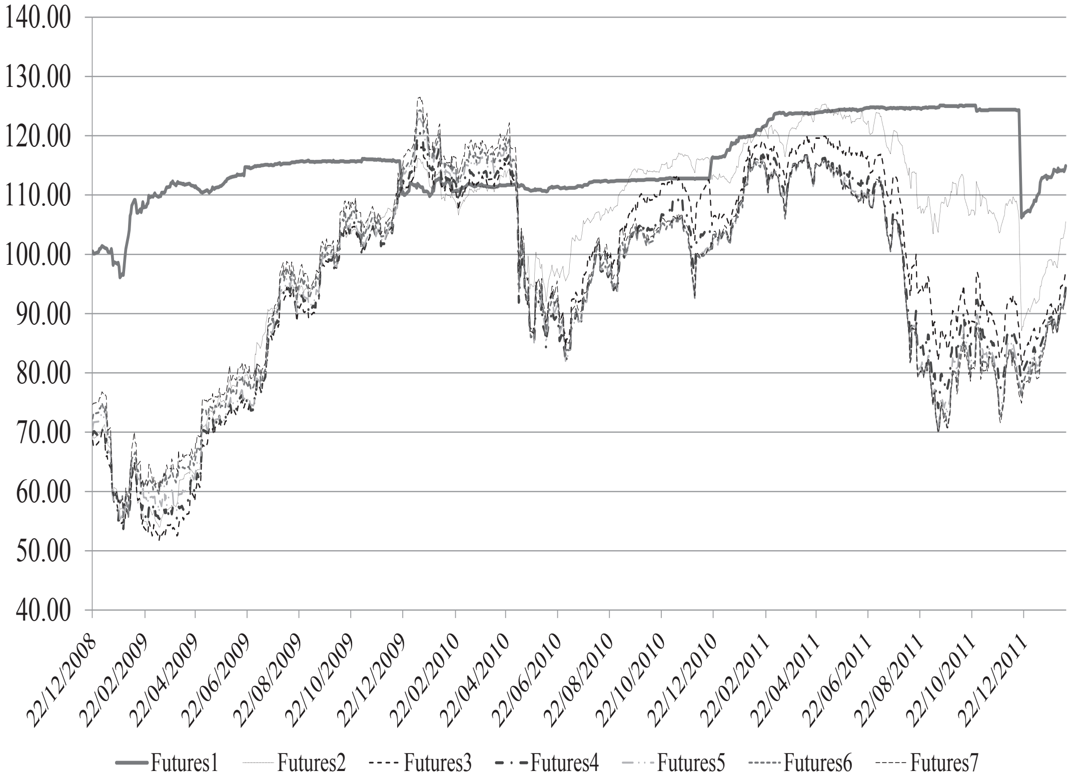
Previous studies, see Baldwin (2008), have hinted that dividend yields implied by the Dow Jones Euro STOXX50® Index dividend swap contracts are uncorrelated to the three-month EURIBOR rates. Here we have tested this analysis for the period 23 December 2008 to 8 February 2012 and for the first six maturities of the Eurex dividend futures contracts. The OLS regression lines depicted on each graph all have very low R^2 values, confirming previous conclusions that interest rates are uncorrelated[1] to dividend futures prices. This empirical artefact supports the idea that futures prices may be congruent with forward prices in the case of Dow Jones Euro STOXX50® dividend index. Thus the futures prices are given by the expectation of the payoff under a suitable risk-neutral measure.
For pricing and calibrating dividend index derivatives we consider the time grid

$$T_0 < t_0 < t_1 \ldots < t_{n_1} < \ldots < T_1 < \ldots < T_2 < \ldots < T_{10} < \ldots < T^*$$

where T^* is a very large but still finite maturity, T_i are yearly December maturities with $i = 1, \ldots, 10$, and t_j are daily times so $t_{j+1} - t_j = \Delta t$ represents one day, for any positive integer j and $T_{i+1} - T_i = 1$ year, for any i.
One may consider that in order to price dividend derivatives the only thing that is needed is the distribution of cum-dividends at the end of the year. This is correct if the only objective

[3]To an extent the second maturity dividend futures contract also departs from the rest.
[1]Remark that it is possible to have a low R^2 value but the explanatory regression variable to be significant. Hence, for each December maturity the null hypothesis that the changes in Euribor rates do not impact upon the changes on implied dividend yields was tested. In all cases, we have failed to reject the null hypothesis.

Figure 4. The futures curves for Dow Jones Euro STOXX50 DVP® index. The daily series for the first seven yearly December maturities are presented for the period 22 December 2008 to 8 December 2012.*Source:* Eurex.



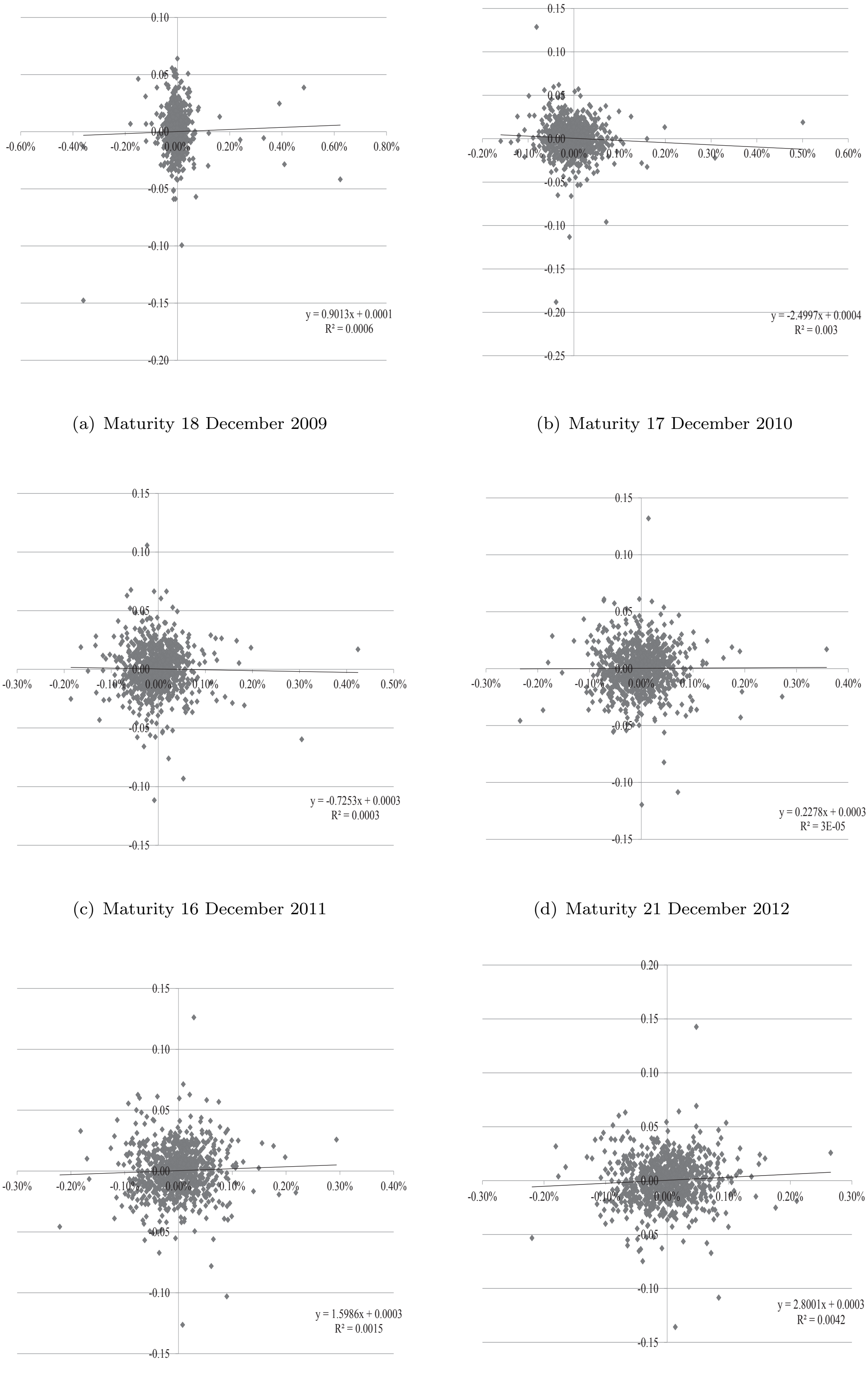
is pricing futures and European options. However, some complex financial instruments traded over-the-counter, see section 6, will require a path modelling approach capable of handling path-dependent payoffs. One case where a distribution rather than a stochastic process seems to be more appropriate for modelling dividends is Japan where dividends are paid usually on the 30 June and 31 December each year, or more exactly at the end of the financial fiscal year for that company.

4.1. *A jump-diffusion model for dividends*

The first model analysed here is a jump-diffusion model with jumps tailored for dividends only. Thus, the jumps can be only downward jumps. The dividend payments are intrinsically linked to the corresponding equity index. The dynamics therefore should follow the equity index. Under the physical measure $\mathbb{P}$

$$\frac{dS_t}{S_t} = [\mu - \theta E(V-1)]dt + \sigma dW_t + d\left(\sum_{i=1}^{i=N_t}(V_i - 1)\right) \tag{9}$$

Figure 5. Scatter plots of daily changes in dividend futures implied yields and the corresponding three month EURIBOR funding rates. Data for the period 23 December 2008 to 8 February 2012. The daily first differences in implied dividend yields are on the vertical axis in index points while the daily changes in 3-month Euribor funding equivalent rate to the maturity of the corresponding futures contract are on the horizontal axis.



(a) Maturity 18 December 2009

(b) Maturity 17 December 2010

(c) Maturity 16 December 2011

(d) Maturity 21 December 2012

(e) Maturity 20 December 2013

(f) Maturity 19 December 2014

where[1] $\{W_t\}_{0\le t\le T^*}$ is a Wiener process, $\{N_t\}_{0\le t\le T^*}$ is a Poisson process with arrival rate θ accounting for the payment times of dividends per unit of time and $\{V\}_{i=1,2,\dots}$ is a sequence of independently and identically distributed random variables, with distribution function H, representing the jump factor size. The three stochastic structures are *assumed* to be mutually independent, with $\mu \in \mathbb{R}$ constant and $\theta, \sigma \in (0,\infty)$. The SDE (9) has the solution

$$S_T = S_t \exp\left\{\sigma(W_T - W_t) + \left(\mu - \theta E(V-1) - \frac{1}{2}\sigma^2\right)(T-t)\right\} \prod_{i=N_t+1}^{N_T} V_i \tag{10}$$

For the first model proposed here the following assumptions are made.

Assumption 1 *All jumps in the equity index dynamics are downward, reflecting dividend adjustments.*

Assumption 2 *All dividends are in index points and are a stochastic proportion of the equity index.*

Hence, this model lies between the usual jump-diffusion models for equity asset pricing due to Merton (1990) and the jump to default credit risk models. Merton's model does not price the jump-risk and it assumes that the extra randomness due to jumps is fully diversifiable[1], furthermore arguing that the distribution of the Poisson jump components does not change under the change of measure. We take a theoretically more robust approach here and account for the market price of the jump-risk. This will allow us to fix the martingale pricing measure by matching exactly the market dividend futures prices.

All dividend derivatives are based on the the cum-dividend process in index points. For a generic period $(t,T]$ the cum-dividend generated by the model can be approximated by

$$Div_{(t,T]} = \sum_{j=1}^{j=m} S_{t_{j+k}} \delta_{t_{j+k}} Y_{t_{j+k}} \tag{11}$$

where $m = \frac{T-t}{\Delta t}$ and $t \equiv t_k$, and $\{Y_{t_j}\}_{j\ge 1}$ are Bernoulli variables taking the value 1 with probability $\theta\Delta t$ and the value zero with probability $1-\theta\Delta t$. The pricing of any contingent claim on $Div_{(t,T]}$ can be carried out by Monte Carlo simulation under a suitable risk-pricing measure.

The model presented so far is quite general[2] and it covers a wide range of specifications that

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[1]Our notation follows Kou (2008). The process in (9) can be defined more formal as follows.

$$dS_t = \mu S_{t^-} dt + \sigma S_{t^-} dW_t + \int_{-\infty}^{\infty} y S_{t^-} (\psi(dy,dt) - H(dy)dt)$$

where W is a standard Brownian motion and ψ is a homogeneous Poisson measure with deterministic compensator $\Upsilon(dy)dt$, with θ the jumps arrival rate per unit of time and Υ the θ-fold of the distribution of the random variable $V_i - 1$. The i.i.d random variables $L_i \equiv V_i - 1$ allow rewriting the Poisson measure as

$$\psi(\omega; dy, dt) = \sum_{i=1,2,\dots} 1_{[T_i(\omega)]} \delta_{(L_i(\omega), T_i(\omega))}(dy, dt)$$

where δ_a is the usual Dirac measure notation in point a and $1_{[\cdot]}$ is the indicator function. One can then show that $\int_{-\infty}^{\infty} y S_{t^-}(\psi(dy,dt) - H(dy)dt) = S_{t^-}(I_t dN_t - \theta E(V-1)dt)$. This process is defined on a probability space $(\Omega, \mathcal{F}_T, P)$, being endowed with the right-continuous, P-complete filtration.

[1]Assuming jumps to be diversifiable helped Merton arrive at a pricing measure that gave the price of a European call as a convex combination of Black-Scholes prices with weights given by the jumps arrival probabilities. This approach is not pursued here and pricing measure will be associated with a separate market price of risk parameter. Furthermore, van Binsbergen *et al.* (2012) found substantial excess returns on dividend contracts so a pure Merton's approach will not be correct.

[2]Buehler *et al.* (2010) proposed a model combining a diffusion part for the equity stock with a random part driven by dividends and modelled with a sum of Dirac measures. However, their model can generate negative dividends. Our model specification

depend further on how jumps are viewed in relation to the underlying index and also on various distributions for the jump sizes such that jumps are only downward. Here we propose a full parametric approach and we make a third assumption about the distribution of the jump sizes.

When a jump occurs at time t the stock moves from S_t to $S_t V_t$. Therefore V_t acts as deflation factor so if $V_t = 0.90$ it implies that a 10% dividend was paid. The jump sizes here are random quantities $V_i \in (0,1)$. The price of the index ex-dividend is $S_t V_t$ so the dividend paid for day t is $S_t(1-V_t)$, and this will be paid with probability $\theta\Delta t$. In order to simplify the notation, $\delta_t \equiv 1-V_t$ henceforth. The model will be referred to henceforth as the Beta Jump Down (BJD) model.

Assumption 3 *$\{V_i\}_{i\geq 1}$ are i.i.d and $V_i \sim c+(1-c)Beta(a,b)$ where $c \in (0,1)$ and $Beta(a,b)$ is the Beta distribution with shape parameters a, b.*

Then the jump sizes V have support[1] in $(c,1)$ which means that $\delta = 1-V$ has support in $(0,1-c)$. Thus $E(V) = \frac{a+bc}{a+b}$ and $E(\delta) = \frac{b(1-c)}{a+b}$.

There have been previous attempts of using the beta distribution for the jumps amplitude in a jump-diffusion model. Ramezani and Zeng (1998) proposed a jump-diffusion model with two jump processes, an upward process with Pareto distributed jumps for good news and a downward process with Beta distributed jumps for bad news. Ramezani and Zeng (2007) showed the link of that model to the double exponential jump diffusion model of Kou (2002). Our model maps dividends strictly with downward jumps and our focus is to generate jumps along the path in order to harvest the actual dividends being paid for a given stock. Using two asymmetric jump processes will lead to identifiability problems regarding the actual size of dividends. For example, at a given ex-dividend time, an upward jump due to positive news will offset the downward jump triggered by a dividend payment. Thus, the total change in stock price at that time will mask the actual dividend payment. In addition, our distribution for jumps is further rescaled such that dividend payments fall within a range similar to what is observed empirically, dividend payments being equal to relatively low percentages of contemporaneous stock prices. Having closed form solution for European options contingent on jump diffusion processes does not help very much because we are only interested in the derivatives contingent on the dividend flow.

In Figure 6 we illustrate in panel 7(a) 20 paths simulated by Monte Carlo from the jump-diffusion model with Beta distributed jumps calibrated on the market data for STOXX50, together with the corresponding market observed path of STOXX50. Furthermore, the histograms depicted in panels 7(b) and 7(c) show the distributions of the dividends paid on the STOXX50 and resulting from the BJD model, respectively. We also report in Table 3 the descriptive statistics for the numbers obtained on three of these paths.

Table 3. Descriptive Statistics for three Monte Carlo of STOXX50 simulated paths of STOXX50 index and the corresponding Cum-dividend paths for one year period, based on the calibrated jump-diffusion model with Beta distributed jumps.

	STOXX50 one	Cum-div one	STOXX50 two	Cum-div two	STOXX50 three	Cum-div three
Mean	2563.1	45.59	2757.6	48.43	2452.3	50.59
Std	222.28	25.96	129.67	26.91	222.34	24.03
Skewness	-0.47	0.0008	0.20	-0.3605	0.69	-0.4104
Kurtosis	2.38	1.68	2.10	1.72	2.39	2.07

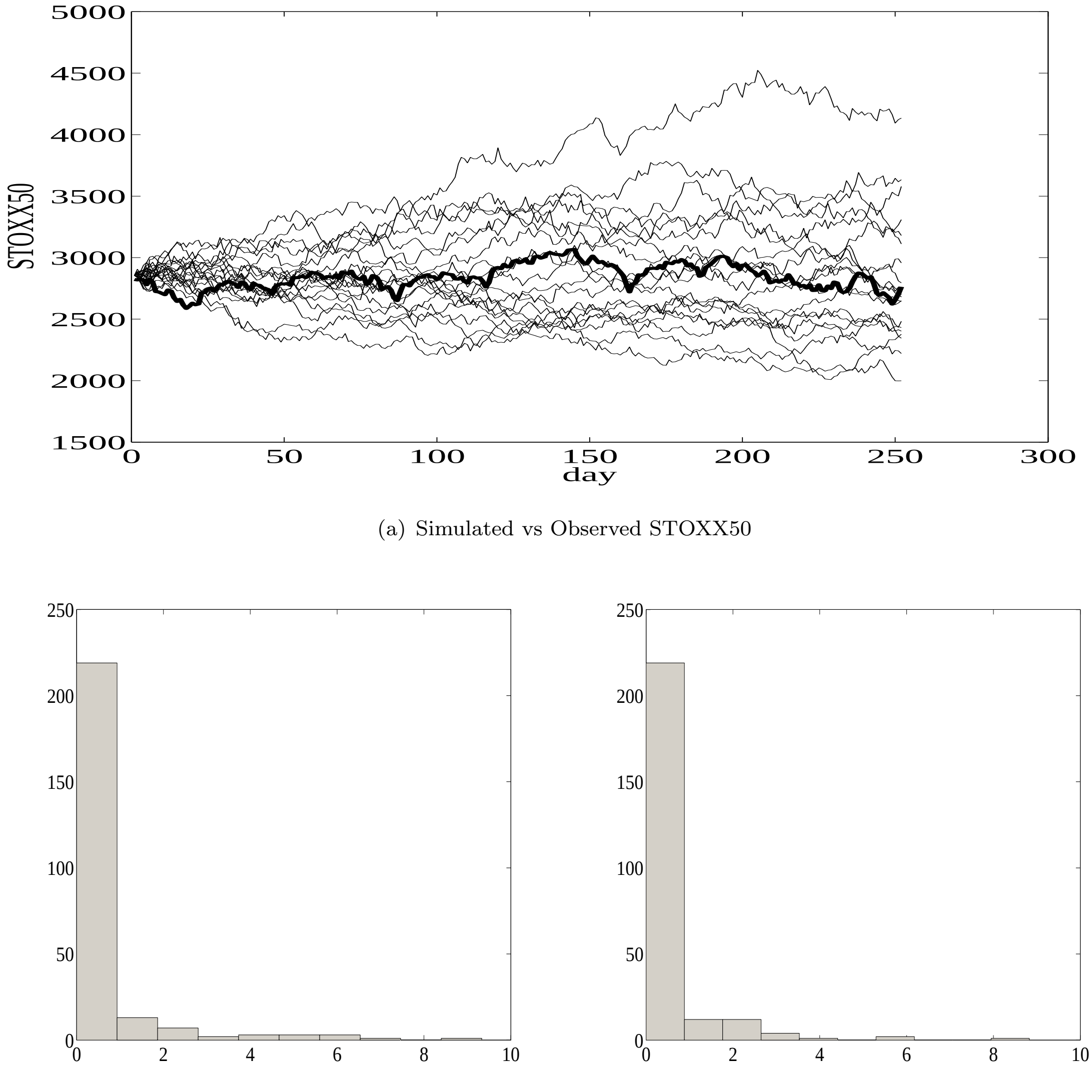
Changing to a risk-neutral measure, denoted by $\tilde{}$, is done over all three sources of stochasticity. We follow the framework described in Chapter 11 of Shreve (2004). The change to a risk-neutral measure is reflected in the market price of risk identity

$$\mu - \theta E(V-1) = r - \tilde{\theta}\tilde{E}(V-1) + \lambda\sigma \tag{12}$$

ensures that dividends are always positive and only a fraction of the current stock value.

[1]The probability density of random variable V is $f_V(x) = \frac{1}{(1-c)B(a,b)}(x-c)^{a-1}(1-x)^{b-1}$.

Figure 6. Simulated paths of STOXX50 index based on the calibrated jump-diffusion model with Beta distributed jumps and the market observed path (thick line) for one year evolution of STOXX50 index.



(a) Simulated vs Observed STOXX50

(b) STOXX50 Dividends Market

(c) STOXX50 Dividends BJD

The parameter λ reflects the market price of risk. Fixing the measure can be done by calibrating the parameters for each maturity, given the way the market is organised with payments being determined only by the cum-dividend over a one year period. Denoting by $\tilde{q} \doteq -\tilde{E}(V-1) = \tilde{E}(\delta)$, under the risk-neutral measure $\mathbb{Q}$ the dynamics relevant for pricing dividend derivatives is the SDE

$$\frac{dS_t}{S_t} = (r + \tilde{\theta}\tilde{q} + \lambda\sigma)dt + \sigma d\tilde{W}_t + d\left(\sum_{i=1}^{i=\tilde{N}_t}(V_i - 1)\right) \tag{13}$$

where r is the risk-free rate.

Proposition 1 *Under the BJD model the dividend futures price for first year maturity is given* *by the formula*

$$F_t^{Div}(T_1) = S_t e^{(r+\lambda\sigma)(T_1-t)}\left[e^{(\tilde{\theta}\tilde{q}_1)(T_1-t)} - 1\right] \tag{14}$$

while for the remaining nine yearly maturities the dividend futures prices are calculated with

$$F_t^{Div}(T_i, T_i+1]) = S_t e^{(r+\lambda\sigma)(T_i+1-t)}\left[e^{(\tilde{\theta}\tilde{q}_{i+1})} - 1\right] \tag{15}$$

for all $i \in \{1, \ldots, 9\}$

Proof. The solution of equation(13) is

$$S_T = S_t \exp\{\sigma\tilde{W}_{T-t} + \left(r + \tilde{\theta}\tilde{q} + \lambda\sigma - \frac{1}{2}\sigma^2\right)(T-t)\} \prod_{j=\tilde{N}_t+1}^{\tilde{N}_T} V_j \tag{16}$$

Therefore, at time t, under the BJD model the cum-dividend that will be produced until the nearest maturity T_1 is given by

$$Div(t, T_1] = S_t \exp\left\{\sigma\tilde{W}_{T_1-t} + \left(r + \tilde{\theta}\tilde{q}_1 + \lambda\sigma - \frac{1}{2}\sigma^2\right)(T_1-t)\right\}\left[1 - \prod_{j=\tilde{N}_t+1}^{\tilde{N}_{T_1}} V_j\right] \tag{17}$$

where evidently $q_1 \equiv q$. For the future $(T_i, T_i+1]$ period[1] the cum dividend that we can project is

$$Div(t, T_i, T_i+1] = S_{T_i} \exp\left\{\sigma\tilde{W}_1 + \left(r + \tilde{\theta}\tilde{q}_{i+1} + \lambda\sigma - \frac{1}{2}\sigma^2\right)\right\}\left[1 - \prod_{j=\tilde{N}_{T_i}+1}^{\tilde{N}_{T_i+1}} V_j\right] \tag{18}$$

Under the BJD model, the dividend futures at time t with maturity T_1 is given by

$$F_t^{Div}(T_1) = \tilde{E}_t(Div(t, T_1]) = S_t e^{(r+\lambda\sigma)(T_1-t)}\left[e^{\tilde{\theta}\tilde{q}_1(T_1-t)} - 1\right] \tag{19}$$

Using (17)

$$\begin{aligned}
\tilde{E}_t(Div(t, T_1]) &= S_t e^{(r+\lambda\sigma-\frac{\sigma^2}{2}+\tilde{\theta}\tilde{q}_1)(T_1-t)}\left(e^{\sigma W_{T_1-t}}\left[1 - \prod_{j=\tilde{N}_{T_i}+1}^{\tilde{N}_{T_i+1}} V_j\right]\right) \\
&= S_t e^{(r+\lambda\sigma-\frac{\sigma^2}{2}+\tilde{\theta}\tilde{q}_1)(T_1-t)} e^{\frac{\sigma^2}{2}(T_1-t)} \tilde{E}_t^{N,V}\left[1 - \prod_{j=\tilde{N}_{T_i}+1}^{\tilde{N}_{T_i+1}} V_j\right] \\
&= S_t e^{(r+\lambda\sigma+\tilde{\theta}\tilde{q}_1)(T_1-t)} \tilde{E}_t^{N,V}\left[1 - \prod_{j=\tilde{N}_{T_i}+1}^{\tilde{N}_{T_i+1}} V_j\right]
\end{aligned}$$

[1]Remark that since the tenor in dividend derivatives markets is annual then $T_{i+1} = T_i + 1$.

Using a standard conditioning argument and taking into account that all thee sources of stochasticity are independent we calculate $\tilde{E}_t\left(\prod_{j=\tilde{N}_t+1}^{\tilde{N}_{T_1}} V_j\right)$ (and similar calculations occur for $\tilde{E}_t\left(\prod_{j=\tilde{N}_{T_i}+1}^{\tilde{N}_{T_i+1}} V_j\right)$). If $\eta_k(\theta t)$ denotes the probability that a Poisson process $\{N_t\}_{t\geq 0}$ with arrival rate θ is equal to the positive integer k, then for any positive real number a it is known that $\sum_{k=0}^{\infty}\eta_k(\theta t)a^k = e^{\theta(a-1)t}$. Thus

$$\begin{aligned}
\tilde{E}_t^{N,V}\left(\prod_{j=\tilde{N}_t+1}^{\tilde{N}_{T_1}} V_j\right) &= \tilde{E}_t^{N}\left[\tilde{E}_t^{V|N}\left(\prod_{j=\tilde{N}_t+1}^{\tilde{N}_{T_1}} V_j\right)\right] \\
&= \sum_{k=0}^{\infty}\eta_k(\tilde{\theta}(T_1-t))\tilde{E}_t^{V}\left(\prod_{j=1}^{k} V_j\right) \\
&\overset{\text{i.i.d}}{=} \sum_{k=0}^{\infty}\eta_k(\tilde{\theta}(T_1-t))[\tilde{E}_t^{V}(V)]^k \\
&= e^{-\tilde{\theta}\tilde{q}(T_1-t)}
\end{aligned}$$

Similarly one can prove that

$$\tilde{E}_t\left(\prod_{j=\tilde{N}_{T_i}+1}^{\tilde{N}_{T_i+1}} V_j\right) = e^{-\tilde{\theta}\tilde{q}_{i+1}} \tag{20}$$

so, under the BJD model, the dividend futures at time t with maturity T_i+1, i.e. for the period $(T_i, T_i+1]$, is given by

$$F_t^{Div}(T_i, T_i+1]) = \tilde{E}_t(Div(t, T_i, T_i+1]) = S_t e^{(r+\lambda\sigma)(T_i+1-t)}\left[e^{\tilde{\theta}\tilde{q}_{i+1}} - 1\right] \tag{21}$$

for all $i \in \{1, \ldots, 9\}$. □

The model can be now calibrated in several ways. One possibility would be to use two derivatives for the stock index and the full term structure of dividend futures. The first two instruments are used for λ and θ while the parameters $\tilde{q}_{i i=1,\ldots,10}$ are used to calibrate the distribution of jump sizes for various maturities, which is changing shape as pointed out by Kruchen and Vanini (2008). Another possibility would be to calibrate the martingale pricing measure using all the ten dividend futures contracts and two dividend options. It is also helpful to assume that the arrival rate is the same under the physical measure and under the risk-neutral measure, since the calendar of dividend payments is more or less the same in a given equity market. Another simplification that could be taken into consideration is that the mean of the cash dividend payments may increase with time.

This allows a direct calibration of futures contracts. Matching the right hand side of (19) with the market futures value $F_t^{Div,mkt}(T_1)$ allows reverse-engineering the values of $\tilde{q}_1$ which represents the mean size of dividends under the risk-neutral measure view. Thus,

$$\tilde{q}_1 = \frac{1}{\tilde{\theta}(T_1-t)}\ln\left(1 + \frac{F_t^{Div,mkt}(T_1)}{S_t e^{(r+\lambda\sigma)(T_1-t)}}\right) \tag{22}$$

The parameter fixing the pricing measure will be fixed by

$$\tilde{q}_{i+1} = \frac{1}{\tilde{\theta}} \ln \left(1 + \frac{F_t^{Div}(T_i, T_i + 1])}{S_t e^{(r+\lambda\sigma)(T_i+1-t)}} \right) \tag{23}$$

While there are analytical formulae for the futures under the BJD model[1], for pricing dividend European options (and possibly other path-dependent options), we must resort to Monte Carlo simulation. The simulation can be done with formulae (17) and (18) or based on discretizing the equation (13).

4.2. *The Stochastic Logistic Diffusion Model*

From the graph in Figure 3 the cumulative dividends time series paid on the Dow Jones Euro STOXX50® index displays an interesting stationarity and yearly periodicity. The most striking characteristic is the sigmoidal shape of the series within each year and the fact that there is an acceleration of dividend payments followed by a change of convexity during the period May-June. It would seem useful if one could model directly the cum-dividends series. In this section we denote by X_t the cum-dividend from the beginning of the year T_{i-1} until the current time t, where $t \leq T_i$, and $i = \{1, \ldots, 10\}$, with $T_0 = 0$.

Under the physical measure $\mathbb{P}$ the main model proposed in this research is characterised by the following SDE

$$dX_t = \nu X_t \left(1 - \frac{X_t}{F} \right) dt + \sigma X_t dW_t^{\mathbb{P}}. \tag{24}$$

with X_0, ν, F and σ all positive. This is the stochastic diffusion version of the Verhulst-Pearl differential model describing constrained growth in biology [2]. This model has been called also the geometric mean reversion model. It appeared in finance literature early on but financial research on it has been sparse so far. Merton (1975) arrived at this process looking at the output-to-capital ratio derived from a growth model with uncertainty based on a Cobb-Douglas production function and assuming that gross savings are a deterministic fraction of output. The general model discussed by Metcalf and Hasset (1995) contains the model given in (24) as a particular case.

In Figure 7 we illustrate a possible path generated under this model. The usual logistic sigmoidal curve is superimposed for illustration purposes as a deterministic shape around which the stochastic process fluctuates. The shape resembles the observed data for cum dividend for VSTOXX presented earlier in Figure 3.

It can be proved, see Appendix, that the solution to the equation (24) is given by

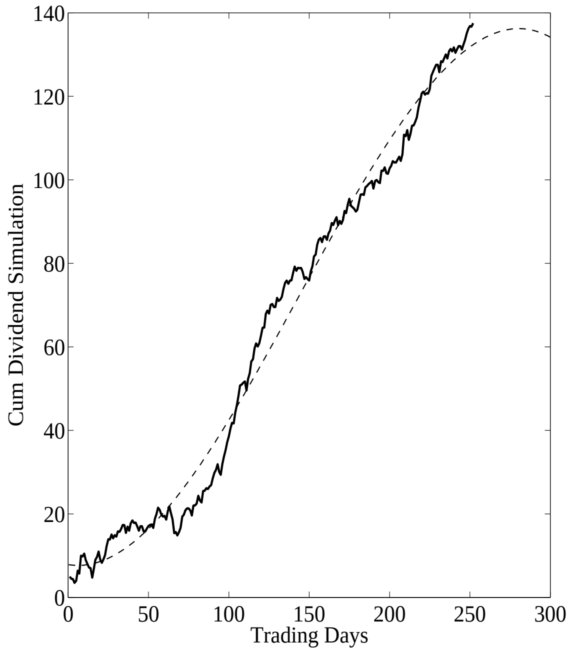
$$X_t = \frac{X_0 \exp\left((\nu - \frac{\sigma^2}{2})t + \sigma W_t \right)}{1 + \frac{\nu X_0}{F} \int_0^t \exp\left((\nu - \frac{\sigma^2}{2})s + \sigma W_s \right) ds} \tag{25}$$

where $X_0 \equiv X_{T_{i-1}}$ is the initial point. The solution shows that $X_t > 0$ at any time $t \in (T_{i-1}, T_i]$ for any parameters ν, F, σ and initial starting point $X_0 > 0$. The interpretation of the parameters is interesting in itself in a dividends market space. The upper limit for the corresponding logistic

[1]The approach finally taken here with the jump-diffusion model requires only the expected jump size, but not its distribution, under a pricing measure. We thank an anonymous referee for pointing out this flexible feature. The distribution of jumps under the physical measure P is still useful for risk management calculations such as value-at-risk, which are outside the scope of this paper.

[2]The model was called the logistic growth model because it gives the dynamics of a population which grows at a geometric rate in an environment with limited feeding resources. In that context b denotes the growth rate per individual, F is the maximal level of population that can be supported by the resources in the environment and σ is a variation parameter.

Figure 7. Simulated path for cum dividends based on the stochastic logistic diffusion model.



process[3] is F while ν is the speed of production of dividends. As pointed out by Merton (1975) and reinforced recently by Yang and Ewald (2010), for the parameter F of the stochastic logistic diffusion model it is not true that

$$\lim_{t\to\infty} \mathbb{E}^{\mathbb{P}}(X_t) = F$$

The model given above is in isolation of any dynamics of the equity index itself. This would solve the equity-dividend puzzle discovered by Shiller (1981) that makes equity dynamics incompatible with the production of future dividends from a volatility perspective. Thus, this model should be more robust for pricing dividend derivatives than the previous jump-diffusion model. The stochastic logistic diffusion model described by (24) implies an incomplete market for dividend payments because the underlying is not a tradable asset. Fortunately the dividend futures contracts traded on Eurex are available to complete the market and determine the martingale pricing measure that can be used for pricing other derivatives such as European call and put options. This can be done period by period. Following Bjork (2009) we can fix the martingale measure $\mathbb{Q}$ by determining the market price of risk $\lambda(t, X_t)$ such that

$$dX_t = X_t\left(\nu - \lambda(t, X_t)\sigma - \frac{X_t}{F}\right)dt + \sigma X_t dW_t^{\mathbb{Q}}. \tag{26}$$

Since at each moment in time t the market will be completed for all 10 years spanned by the running futures contracts, we assume that $\lambda(t, X_t) \equiv \lambda_i$, for all $i = \{1, \ldots, 10\}$. Each parameter λ_i will be identified by exact calibration to dividend futures prices from the model with the dynamics

[3]The logistic process is defined purely by the drift so the equation is the following ODE $\frac{dX_t}{dt} = \nu X_t\left(1 - \frac{X_t}{F}\right)$ which can be solved analytically to give the logistic function with the well-known sigmoidal shape.

given by the SDE for any $T_{i-1} < t \le T_i$

$$dX_t = X_t \left(\nu - \lambda_i \sigma - \frac{X_t}{F} \right) dt + \sigma X_t dW_t^{\mathbb{Q}}. \tag{27}$$

The distribution of the solution in (25) has been derived in closed-form by Yang and Ewald (2010) but it is cumbersome for practical calculations even of vanilla derivatives such as European put and call options.

Nevertheless, the calibration of parameters λ can be easily done from futures market prices because the futures price of the payment $Div_{(T_{i-1},T_i]}$ is equal to $\mathbb{E}_t^{\mathbb{Q}}(X_{T_i})$. Thus, the parameter λ_i can be determined by first discretizing the equation (28) into

$$X_{T_{i-1}+j\Delta t} = X_{T_{i-1}+(j-1)\Delta t} \left[1 + \left(\nu - \lambda_i \sigma - \frac{X_{T_{i-1}+(j-1)\Delta t}}{F} \right) \Delta t + \sigma \sqrt{\Delta t} Z_j \right] \tag{28}$$

where $Z_j \sim N(0,1), \ \forall j$, and then generating M different paths between $X_{T_{i-1}}$ and X_{T_i}, and finally computing the required expectation by Monte Carlo simulation

$$\mathbb{E}_t^{\mathbb{Q}}(X_{T_i}) = \frac{1}{M} \sum_{k=1}^{M} X_{T_i}^{(k)}.$$

There are two major advantages of this Monte Carlo approach: a) the futures curve provided by the dividend futures market on Eurex will be perfectly calibrated, and b) other derivatives, including path-dependent derivatives, can be directly priced since path values are readily available under the correct martingale measure.

While for the maturities 2 to 10 the simulation exercise is more straightforward since the entire year is used for path simulations of cum-dividends, for the current year care must be taken since at any time $t > T_0$ some dividends may have been paid already.

5. Numerical Examples

In this section we shall explore some numerical exemplification of the two dividend models proposed in this paper.

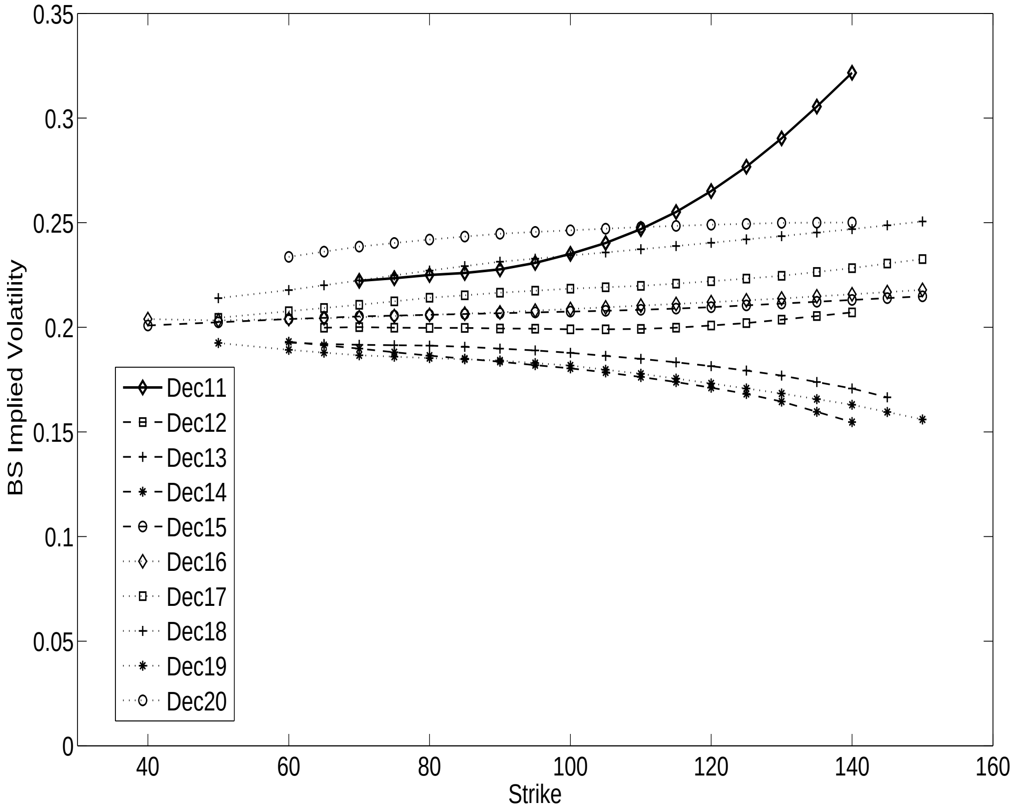
5.1. *Jump-down diffusion model*

For practical purposes we need to calibrate the parameters $\tilde{a}, \tilde{b}, \tilde{c}, \sigma, \tilde{\theta}$ defining the risk-neutral measure driving the dynamics of the BJD model. This process can be done maturity by maturity since the payments at time T_i refers only to the cum-dividend in the year running up to that time. The only constraints among the parameters are those provided by relationships (22) and (23). The risk-free rate is considered here as a constant[1] approximating the cost of funding to the required horizon. Different values are used for different horizons and the risk-free rate is calibrated from Euribor-swap market curve on the day of calculation.

For pricing futures and European options a Monte Carlo approach is followed that simulates daily paths to the required maturity. Each day we simulate possible values from a standard geometric Brownian motion under the risk-neutral pricing measure. Then, we simulate in a binary fashion

[1] A more elaborated approach would involve having a separate short-rate model or market model for the risk-free rate. Given that post subprime-liquidity crisis it is difficult to say which model would be most appropriate for interest free rate concept, we prefer to use a unique number for r.

Figure 8. European Option pricing with the downward jump-diffusion beta dividend model for the first four December maturities, on 20 Dec 2010.



whether a dividend payment is made. The probability of success is equal to $\tilde{\theta}\Delta t$. Conditional on a dividend payment being made a random draw from a $Beta(\tilde{b}, \tilde{a})$ distribution is made for the size of the jump. If a dividend payment is made the value of the simulated equity index is reduced proportionately exactly with the size of the jump.

This methodology has the advantage that once paths are simulated to required maturities, *any* other derivatives, including path dependent derivatives, can be priced accordingly. A similar procedure can be implemented to produce risk measures, such as value-at-risk, derived from the dynamics of the model presented in this section, under the physical measure.

The model ensures matching futures prices exactly and so taking advantage of put-call parity of options on futures, which is a tradeable asset here, we can construct the Black-Scholes implied volatilities. The fitting results for options pricing is exhibited in Figure 8 where the implied volatilities are presented for all ten maturities on 20 December 2010.

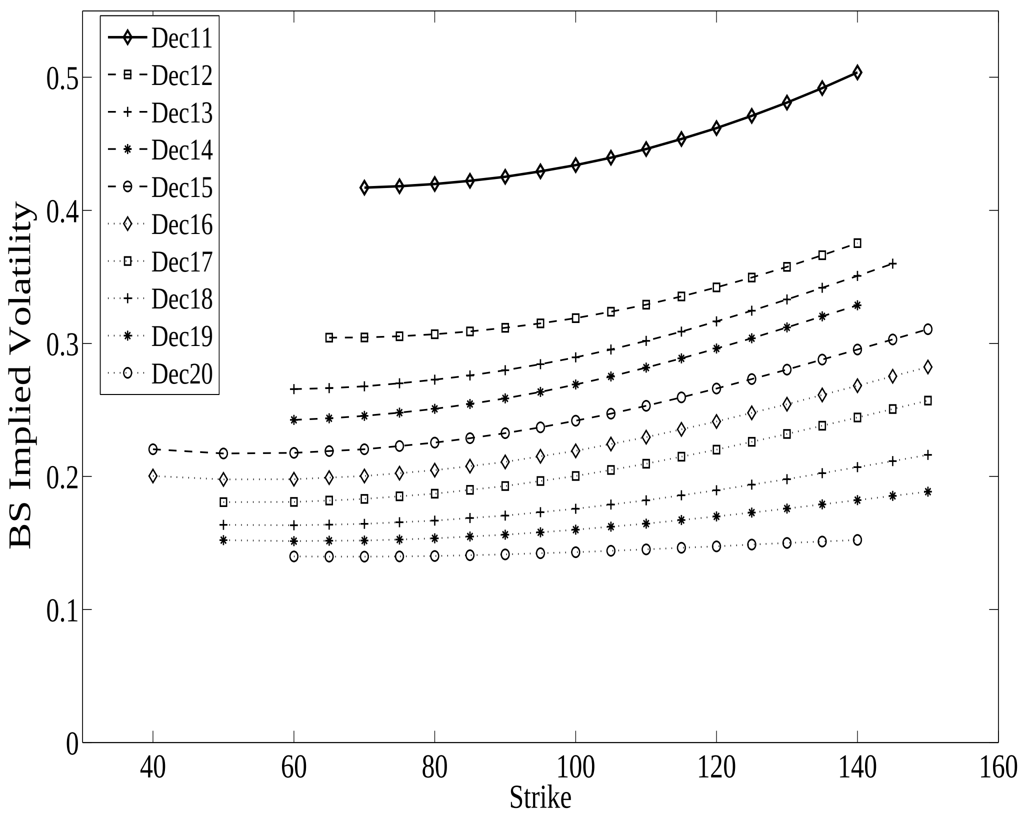
Overall the smile calibration is very good, particularly for longer maturities. For the nearest maturity the fit can be improved. The jump diffusion model works well in terms of producing fatter tails as observed in the empirical literature. However, the correct statistical distribution of jumps amplitude is difficult to get. Even if the dividend proportion is correct, if this is applied to an index that is drawn from a distribution that is not quite correct, it may lead to possible misfits. Future research may replace the geometric Brownian motion for the equity index with other processes such as a Lévy process or a stochastic volatility model.

5.2. *The Stochastic Logistic Diffusion Model*

For calibration purposes we need to determine the parameters $\tilde{\nu}, \tilde{F}, \lambda$ and σ that will give the risk-neutral measure. In order to get an idea where the values of the parameters are under the physical measure, the parameters ν, F and σ can be estimated from the OLS estimates over one year of data. Once the pricing measure is determined by calibrating the futures curves, *all other contingent claims* on the Dow Jones Euro STOXX50® dividend index can be calculated directly. Applying the Monte Carlo methodology described in Section 4.2 it is possible to determine the price of European call and put options, as well as other path dependent derivatives.

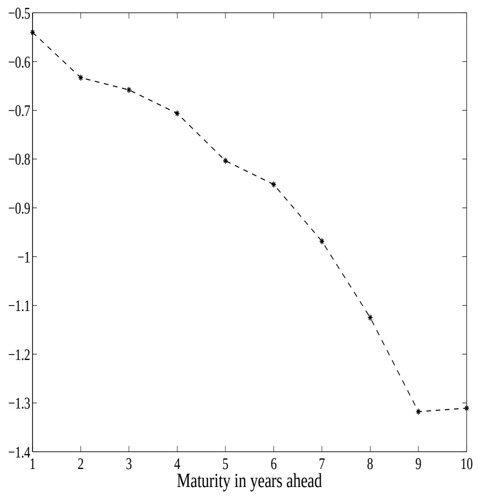
The graph in Figure 9 depicts the smile fit for European options on Dow Jones Euro STOXX50® dividend index on 20 Dec 2010 based on market data from EUREX. The smile fit is an improvement over the smile fit resulting from the jump-diffusion model with Beta distributed jumps. Considering the small number of parameters underpinning the stochastic logistic diffusion model this flexible model looks very promising. The smile curves in this example suggests that the implied volatility decreases with time. Furthermore the smile is flattening at the back end of the futures curve for this model whereas for the jump-diffusion model with Beta distributed jumps the Black-Scholes implied volatility curves are alternating. Both models calibrations suggest that the nearest maturity implied vols are quite different from the remaining nine maturities. This can be explained by the well-known mean-reverting stylized feature of volatility and the fact that the near term volatility implied from the dividend futures options are high, due possibly to a combination of Samuelson effect and increased uncertainty at the time in EU economic area caused by the sovereign crisis that started in 2010.

Figure 9. European Option pricing with the stochastic logistic diffusion model for cum-dividends for the first four December maturities for the indicated maturities, on 20 Dec 2010.

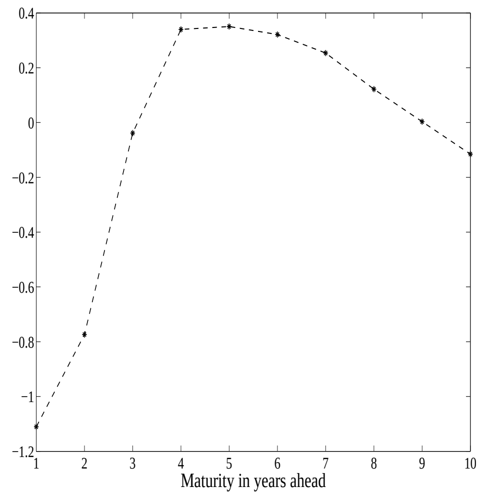


In Figure 10 the term structure of market price of risk parameter λ are illustrated for three different days. These values are calculated at the beginning of the December roll and they are fixing the martingale pricing measure for each of the ten December maturities. The shape of the term structure of market price of risk for Dow Jones Euro STOXX50 DVP® can be inverted, upward trending and upward then downward trending. Overall the curves presented in Figure 10 suggest that the term structure of λ is almost always concave, but this conclusion is more a conjecture at this stage of research in this area.

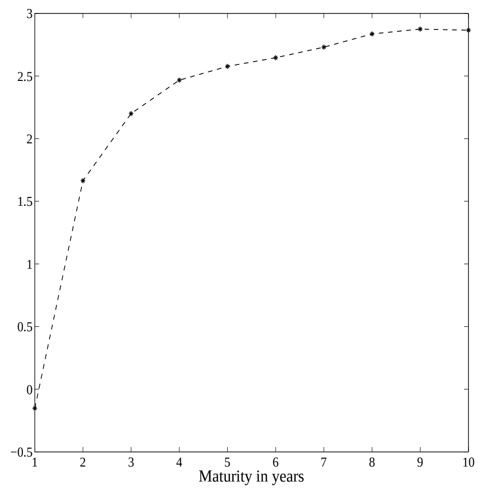
Figure 10. Term structure of market price of risk parameter λ for all ten December maturities calibrated from Eurex market futures prices on three different days. The calibration is done by matching the dividend futures market prices with the theoretical dividend futures given by the stochastic logistic diffusion model



(a) 21 Dec 2009



(b) 20 Dec 2010



(c) 19 Dec 2011

5.3. *Testing the Calibration with the SLD model*

In this section we follow the smile calibration of our best model, the stochastic logistic diffusion model, for more than one day. Thus, we calibrate the smile for put and call options with all exercise prices, for all ten year maturities, daily[1] between 19 Dec 2011 and 9 Feb 2012. Each day there are between 164 and 169 call prices, paired by the same number of put prices. In total we get a sample of 6347 market prices that will be compared with the prices produced by the model, and ideally the two series should be very close. The testing is done each day because new dividends may be added daily to the cum-dividend series and the information filtration set is changing from one-day to another, providing possibly more information about the terminal distribution of dividends.

In order to gauge the matching of prices we employ, each day and for each option series, the two sample Kolmogorov-Smirnov test (KS test) that compares the empirical distribution obtained from market prices with the distribution obtained under the stochastic logistic distribution model. The testing is done daily at 95% confidence level and the null hypothesis is that the two series come from the same distribution. Hence, failing to reject the null is indicative that the SLD model explains well the market data.

Table 4. Summary statistics of the p-values of the two-sample Kolmogorov-Smirnov tests for the stochastic logistic diffusion model applied to all European call and put dividend derivatives daily between 19 Dec 2011 and 9 Feb for all strike prices and all ten yearly December maturities 2012.

KS call		KS put	
Mean	0.8088	Mean	0.7152
Median	0.8115	Median	0.7212
Mode	0.6787	Mode	0.6787
Standard Deviation	0.1278	Standard Deviation	0.1923
Minimum	0.4093	Minimum	0.2709
Maximum	0.9887	Maximum	0.9651

The results reported[1] in Table 4 reflect the quality of calibration under the SLD model. Each day, for the dividend European call series, we fail to reject the two-sample KS test, the minimum p-value calculated being 0.4093. Likewise, we fail to reject the two-sample KS test each day for the dividend European put series, the minimum p-value calculated being 0.2709.

6. Summary Discussion of the Two Models

The literature on pricing dividend derivatives is sparse. From the equity derivatives pricing literature it seems conclusive that dividends are stochastic in nature. Hence, it is important to find models that can be easily implemented but that also preserve the stochastic character of dividends. The two models proposed here can be applied to dividend streams generated by equity stocks and indices worldwide. Our empirical investigation was conducted on STOXX dividend index due to data availability, rather than any other rationale. Since the two models have very different motivations and structure it is worthwhile to discuss comparatively the advantages and disadvantages of both models. Essentially all models are "wrong" and they are only approximations of the reality.

Our first model, the jump-diffusion model, can be accepted based on the following arguments: a) the model must be arbitrage free and ours by design is; b) the jump part will introduce fat tails, leptokurtosis and is helping calibrating the smile observed in options prices; c) once calibrated the model can be used for other derivatives such as path-dependent options, and d) the model must have a clear financial economics interpretations– in our case downward jumps are directly associated with jumps. Last but not least, we are also highly appreciative of *parsimony* and models with less parameters should present less challenges for risk management and sensitivity analysis.

[1]The cum-dividend paid up to each date is taken into consideration when constructing the distribution of cum-dividend to the nearest maturity. For the second to the tenth maturities the model projects a full year cum-dividend.
[1]The individual KS tests for all 76 series can be obtained from the author upon request. Due to lack of space it is not possible to report all of them here.

One clear disadvantage of the BJD model is that it depends on the dynamics specified for the equity index. If this model is not well specified, even if the linkage between equity index and dividend payment generation is correct, it may lead to miscalibrated distributions of dividend payments.

The stochastic logistic diffusion model is a continuous-time finance model that has not been used very often in finance in the past. The model is very easy to interpret and it calibrates very well the dividend options smile. One great advantage of this model is that it considers directly the dynamics of the dividend index itself, in other words it is suited for dividend derivatives as an asset class of its own. The only slight disadvantage is that it must be reset on an annual basis, but since this is driven by the mechanics of the dividend derivatives market is not really a total negative feature.

The two models developed here for pricing dividend derivatives are very different, the first one modeling the dividend payment series while the latter follows the cum-dividend series. Both models rely on the Monte Carlo simulation approach for implementation but there are immediate advantages in doing so since other path dependent derivatives would be priced directly based on the same set of simulations. The BJD model seems to provide not as better fit to the option prices smiles as the SLD model, in spite of having more parameters. Computational time for the Monte Carlo exercise is also longer for BJD model than for the SLD model. The stochastic logistic diffusion model for dividend index has the advantage that it is disconnected from the dynamics of the associated stock index. In this way the well-known puzzle identified by Shiller (1981) that no observed movements in the aggregated dividends were ever correctly forecast by movements in aggregate stock prices is circumvented.

Once the model parameters are estimated and a martingale pricing measure is determined from the futures term structure, the two models can be easily applied to price more advanced dividend derivatives such as path dependent options like Asian, barriers, lookbacks and so on. Given that dividend derivatives have been traded on the exchanges only recently there is limited information about market prices on more advanced dividend derivatives. The first model can be applied directly to the endowment warrant discussed in Brown and Davis (2004) which is a contract giving the buyer a claim on both capital gains and dividends since the final exercise price is reduced based on all dividends paid on the stock. There are other more advanced over-the-counter products on dividends linked to the performance of the equity itself, as described in Buehler *et al.* (2010), where the first model can be very useful due to its intrinsic relationship between equity and dividends. The knock-out dividend swap pays a standard forward-type payoff $Div_{(T_1,T_2]} - K$ if the associated equity index S trades below a given barrier B. Another instrument where our first model can be useful is a dividend yield swap that will pay the sum of realized dividends over the monthly average spot of the equity, or another variant obtained by dividing each dividend by the stock price of the previous trading day.

The second model, the stochastic logistic diffusion model, can be applied to a leveraged dividend yield swap certificate. This is a fixed income instrument paying coupons equal to the difference between interest rates and dividends. If $\{t_i\}_{i\in\{1,2,\ldots\}}$ are monthly fixings and defining the realized dividend yield generated by a stock S between two time points T_1 and T_2 as $dyld(T_1,T_2) = \frac{1}{S_{T_2}} Div_{(T_1,T_2]}$ and letting $Libor_t$ be the 12 months Libor rate at time t then the certificate pays something like

$$\max\left(100\% + 5 \sum_{t=1y,2y,3y} [Libor_t - dyld(t,t+1y)], floor\right)$$

where the floor is typically 30%. The model working directly with the cum-dividends seems to be more useful for dividend derivatives.

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