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On the Theory of Dissipative Extensions

A thesis presented for the degree of Doctor of Philosophy

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ABSTRACT

We consider the problem of constructing dissipative extensions of given dissipative operators.

Firstly, we discuss the dissipative extensions of symmetric operators and give a sufficient condition for when these extensions are completely non-selfadjoint. Moreover, given a closed and densely defined operator A, we construct its closed extensions which we parametrize by suitable subspaces of $\mathcal{D}(A^*)$.

Then, we consider operators A and $\widetilde{A}$ that form a dual pair, which means that $A \subset \widetilde{A}^*$, respectively $\widetilde{A} \subset A^*$. Assuming that A and $(-\widetilde{A})$ are dissipative, we present a method of determining the proper dissipative extensions $\widehat{A}$ of this dual pair, i.e. we determine all dissipative operators $\widehat{A}$ such that $A \subset \widehat{A} \subset \widetilde{A}^*$ provided that $\mathcal{D}(A) \cap \mathcal{D}(\widetilde{A})$ is dense in $\mathcal{H}$. We discuss applications to symmetric operators, symmetric operators perturbed by a relatively bounded dissipative operator and more singular differential operators. Also, we investigate the stability of the numerical ranges of the various proper dissipative extensions of the dual pair $(A, \widetilde{A})$.

Assuming that zero is in the field of regularity of a given dissipative operator A, we then construct its Kreĭn–von Neumann extension A_K, which we show to be maximally dissipative. If there exists a dissipative operator $(-\widetilde{A})$ such that A and $\widetilde{A}$ form a dual pair, we discuss when A_K is a proper extension of the dual pair $(A, \widetilde{A})$ and if this is not the case, we propose a construction of a dual pair $(A_0, \widetilde{A}_0)$, where $A_0 \subset A$ and $\widetilde{A}_0 \subset \widetilde{A}$ such that A_K is a proper extension of $(A_0, \widetilde{A}_0)$.

After this, we consider dual pairs $(A, \widetilde{A})$ of sectorial operators and construct proper sectorial extensions that satisfy certain conditions on their numerical range. We apply this result to positive symmetric operators, where we recover the theory of non-negative selfadjoint and sectorial extensions of positive symmetric operators as described by Birman, Kreĭn, Vishik and Grubb.

Moreover, for the case of proper extensions of a dual pair $(A, \widetilde{A})$ of sectorial operators, we develop a theory along the lines of the Birman–Kreĭn–Vishik theory and define an order in the imaginary parts of the various proper dissipative extensions of $(A, \widetilde{A})$.

We finish with a discussion of non-proper extensions: Given a dual pair $(A, \widetilde{A})$ that satisfies certain assumptions, we construct all dissipative extensions of A that have domain contained in $\mathcal{D}(\widetilde{A}^*)$. Applying this result, we recover Crandall and Phillip's description of all dissipative extensions of a symmetric operator perturbed by a bounded dissipative operator. Lastly, given a dissipative operator A whose imaginary part induces a strictly positive closable quadratic form, we find a criterion for an arbitrary extension of A to be dissipative.

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Contents

CHAPTER 1

Historical overview and results of the thesis

A densely defined operator A on a Hilbert space is called dissipative if and only if its numerical range is contained in the upper complex plane. [1.1] Moreover, it is called maximally dissipative if it has no non-trivial dissipative operator extensions. Maximally dissipative operators play a very important role in mathematics as well as in physics as they generate C_0-semigroups of contractions and can for example be used to describe physical systems that fail to conserve energy [**35**]. In general, dissipative operators have many interesting applications in physics like (magneto-) hydrodynamics, lasers or nuclear scattering (for details and more examples, see the Pseudospectra Gateway website www.comlab.ox.ac.uk/pseudospectra). Thus, if one starts with a dissipative operator that is not maximally dissipative (like e.g. a Schrödinger operator with a suitable complex potential defined on the set of compactly supported smooth functions), one has to construct suitable maximally dissipative extensions. The purpose of this thesis is to contribute towards the theory of dissipative extensions.

1.1. On the development of extension theory

The study of abstract extension problems for operators on Hilbert spaces goes at least back to von Neumann [**34**, Chapters V-VIII], who considered the problem of characterizing all selfadjoint extensions of a given symmetric operator. His well-known von Neumann formulae provide a full characterization of all selfadjoint extensions of a given closed symmetric operator S with equal defect indices (for a presentation in a more modern terminology, see e.g. [**1**, Vol. II, Sect. 80] or [**44**, Satz 10.9]). The main tool ("Der wesentliche Kunstgriff", [**34**, p. 62]) of his analysis is the Cayley transform of S, formally given by $C := (S-i)(S+i)^{-1}$ with domain $\operatorname{ran}(S+i)$, which is an isometry if S is symmetric. Von Neumann showed that there is a one-to-one correspondence between all selfadjoint extensions of S and all unitary extensions of C, which are parametrized

[1.1] According to Dolph [**16**, p. 30] the name "dissipative operator" was first introduced by Mukminov in [**33**].

by all unitary maps U from $\ker(S^* - i)$ to $\ker(S^* + i)$. Moreover, in the same paper, von Neumann also discussed semibounded symmetric operators S with lower bound $C > -\infty$, where he managed to prove that for any $\varepsilon > 0$, it is possible to construct a selfadjoint extension S_ε of S such that S_ε is bounded from below by $(C - \varepsilon)$ ([**34**, Satz 43]). In particular, if $C > 0$, this proves the existence of positive selfadjoint extensions of symmetric operators with positive semibound. The proof of this result relies on the construction of a non-negative selfadjoint extension S_K of a given positive symmetric operator S, which is commonly known as the Kreĭn–von Neumann extension of S (cf. [**34**, Satz 42]). In a footnote to the statement of [**34**, Satz 43], he also conjectured the existence of a selfadjoint extension with the same lower bound as the initial symmetric operator.

This conjecture was answered in the affirmative by Friedrichs in [**21**], who constructed what is nowadays known as the Friedrichs extension. Its construction exploits the fact that the quadratic form induced by a semibounded symmetric operator S is always closable with its closure being the quadratic form associated to a selfadjoint extension S_F that has the same lower bound as S. See also [**41**, Chapter 2.2] for a textbook presentation of the construction of the Friedrichs extension.

In [**27**], Kreĭn treated the problem of determining all non-negative selfadjoint extensions of a non-negative closed symmetric operator S by considering the fractional linear transformation $F := (S - \mathbb{1})(S + \mathbb{1})^{-1}$ on $\operatorname{ran}(S + \mathbb{1})$, whose compression $(P_{\operatorname{ran}(S+\mathbb{1})}F)$ to $\operatorname{ran}(S + \mathbb{1})$ is selfadjoint ($P_{\operatorname{ran}(S+\mathbb{1})}$ denotes the orthogonal projection onto $\operatorname{ran}(S + \mathbb{1})$). Moreover, if S is non-negative, we have that F is a contraction ($\|F\varphi\| \leq \|\varphi\|$ for all $\varphi \in \mathcal{D}(F)$, resp. $\|(S - \mathbb{1})f\| \leq \|(S + \mathbb{1})f\|$ for all $f \in \mathcal{D}(S)$). He showed that the problem of finding all non-negative selfadjoint extensions of S is equivalent to finding all selfadjoint contractive extensions F' of F that are defined on the entire Hilbert space $\mathcal{H}$. Furthermore, he proved that there exist two special extensions of S, the above mentioned Kreĭn–von Neumann extension S_K and the Friedrichs extension S_F. They are extremal in the sense that any other non-negative selfadjoint extension $\widehat{S}$ satisfies

$$(S_F + \mathbb{1})^{-1} \leq (\widehat{S} + \mathbb{1})^{-1} \leq (S_K + \mathbb{1})^{-1} ,$$

which is equivalent to

$$S_K \leq \widehat{S} \leq S_F \tag{1.1}$$

in the quadratic form sense. Recall that for two non-negative selfadjoint operators A and B on a Hilbert space $\mathcal{H}$, the relation $A \leq B$ is defined as

$$A \leq B :\Leftrightarrow \mathcal{D}(A^{1/2}) \supset \mathcal{D}(B^{1/2}) \text{ and } \|A^{1/2}f\| \leq \|B^{1/2}f\|$$

for all $f \in \mathcal{D}(B^{1/2})$. As done in [**2**], we extend this definition to the case that B is selfadjoint on a closed subspace $\mathcal{K} \subset \mathcal{H}$. For example, let $\mathcal{K}$ be a closed proper subspace of $\mathcal{H}$ and define $0_{\mathcal{K}}$ and $0_{\mathcal{H}}$ to be, respectively, the zero operators on $\mathcal{K}$ and $\mathcal{H}$. According to this definition, we then would get that $0_{\mathcal{H}} \leq 0_{\mathcal{K}}$. In [**2**], the convention $B := \infty$ on $\mathcal{D}(B)^{\perp}$ is introduced to make this more apparent. For a brief introduction into Kreĭn's construction, cf. also [**39**, Sect. 125].

The further investigations of Vishik and Birman [**42, 13**] resulted in the following characterization of all non-negative selfadjoint extensions of a positive closed symmetric operator S:

Proposition 1.1 (Mainly following the notation and presentation of [**2**]). *Let $S > \mathbb{1}$ be a closed symmetric operator. Then, there is a one-to-one correspondence between all non-negative selfadjoint extensions of S and all pairs $(\mathfrak{M}, B)$, where $\mathfrak{M} \subset \ker(S^*)$ is a closed subspace and B is a non-negative selfadjoint auxiliary operator in $\mathfrak{M}$ (in particular, $\overline{\mathcal{D}(B)} = \mathfrak{M}$). These non-negative selfadjoint extensions are given by*

$$\begin{aligned} S_{\mathfrak{M},B} : \ \mathcal{D}(S_{\mathfrak{M},B}) &= \mathcal{D}(S) \dot{+} \{(S_F^{-1}B + \mathbb{1})f : f \in \mathcal{D}(B)\} \dot{+} \{S_F^{-1}g : g \in \mathfrak{M}^{\perp} \cap \ker(S^*)\} \\ S_{\mathfrak{M},B} &= S^* \restriction_{\mathcal{D}(S_{\mathfrak{M},B})} . \end{aligned}$$

These results have also been obtained and extended by Grubb in [**23**, Chapter II §2], who was also able to characterize (maximally) sectorial and (maximally) accretive extensions $\widehat{S}$ of S such that $S \subset \widehat{S} \subset S^*$ by allowing the auxiliary operator B to be (maximally) sectorial and (maximally) accretive (cf. also the addendum acknowledging Grubb's contributions to the field [**3**]). While these approaches predominantly relied on operator methods, the presentation of Alonso and Simon in [**2**] emphasizes form methods. They obtain the following description of the quadratic form induced by the operators $S_{\mathfrak{M},B}$:

$$S_{\mathfrak{M},B}^{1/2}: \qquad \mathcal{D}(S_{\mathfrak{M},B}^{1/2}) = \mathcal{D}(S_F^{1/2}) \dot{+} \mathcal{D}(B^{1/2})$$
$$\|S_{\mathfrak{M},B}^{1/2}(f+\eta)\|^2 = \|S_F^{1/2} f\|^2 + \|B^{1/2}\eta\|^2 ,$$

where $f \in \mathcal{D}(S_F^{1/2})$ and $\eta \in \mathcal{D}(B^{1/2})$. From this, it immediately follows that

$$S_B \leq S_{B'} \quad \Leftrightarrow \quad B \leq B'$$

and in particular, this implies (1.1), where $S_F = S_{\{0\},0}$ and $S_K = S_{\ker(S^*),0}$.

There has been a large number of contributions towards the problem of determining all (maximally) sectorial and (maximally) accretive extensions of a given sectorial operator A with contributions from authors like Arlinskiĭ, Derkach, Kovalev, Malamud, Mogilevskii and Tsekanovskiĭ (cf. the surveys [**8, 11**] and all the references therein). We will focus on just a few main results.

Friedrichs' construction of a selfadjoint extension of a given non-negative symmetric operator can be generalized to the sectorial case. A densely defined operator A is called sectorial (or more precisely "α-sectorial"), if its numerical range is confined to a sector of the complex plane with semi-angle α, i.e. if there exists an $\alpha \in [0, \pi/2)$ such that

$$\mathcal{N}_A \subset \{z \in \mathbb{C} : -\alpha \leq \arg(z) \leq \alpha\} ,$$

where $\mathcal{N}_A := \{\langle f, Af\rangle : f \in \mathcal{D}(A), \|f\| = 1\}$ is the numerical range of A. In this case, the quadratic form $\mathfrak{a}$ induced by A, which is given by

$$\mathfrak{a}: \quad \mathcal{D}(\mathfrak{a}) = \mathcal{D}(A), \quad f \mapsto \langle f, Af\rangle$$

is still closable. This follows from the sectoriality of A, which implies that the quadratic form $\mathfrak{a}$ satisfies

$$|\mathfrak{a}(f)| \leq (1 + \tan(\alpha)) \cdot \mathrm{Re}(\mathfrak{a}(f))$$

for any $f \in \mathcal{D}(\mathfrak{a})$. Moreover, it can be shown that the closure of $\mathfrak{a}$ corresponds to a maximally sectorial operator A_F, which is the Friedrichs extension of A. It is well-known that A_F is the unique maximally sectorial extension of A that has domain contained in the form domain $\mathcal{Q}(A_F)$ of A_F, and the numerical range of A lies dense in the numerical range of A_F. (For the details, cf. [**26**, Chap. VI, §1-2].)

In [**4**, Thm. 1], Ando and Nishio have found a useful description of the Kreĭn–von Neumann extension S_K of a given symmetric non-negative operator S. Arlinskiĭ used this result and generalized it to a description of the Kreĭn–von Neumann extension A_K of a given α-sectorial operator A (cf. the survey [**8**, Thm. 3.6]). In particular, it can be shown that A_K is maximally α-sectorial and that the form domain $\mathcal{Q}(\widehat{S})$ of any maximally α-sectorial extension $\widehat{A}$ of A has to satisfy $\mathcal{Q}(A_F) \subset \mathcal{Q}(\widehat{A}) \subset \mathcal{Q}(A_K)$, cf. [**6**].

Analogously to Kreĭn's construction of non-negative selfadjoint extensions of a given non-negative symmetric operator, Arlinskiĭ considered the fractional linear transformation F given by $F := (A - \mathbb{1})(A + \mathbb{1})^{-1}$ and defined on $\mathrm{ran}(A + \mathbb{1})$. He found that A being α-sectorial (here: $0 < \alpha < \pi/2$) implies that F satisfies

$$\|F \sin\alpha \pm i \cos\alpha\| \leq 1 \,, \tag{1.2}$$

which can be shown to imply that F is a contraction. A contraction F satisfying (1.2) is said to belong to the class $\mathcal{C}(\alpha)$. In [**5**], it was shown that — via the fractional transform and its inverse — there is a one-to-one correspondence between all maximally β-sectorial extensions of A and all everywhere defined contractive extensions of F that belong to the class $\mathcal{C}(\beta)$.

Arlinskiĭ and Popov also solved the problem of determining all (maximally) accretive and (maximally) sectorial extensions of a given densely defined sectorial operator in terms of abstract boundary conditions [**8, 10**]. Arlinskiĭ also constructed parametrizations of m-accretive extensions of a given coercive sectorial operator in the spirit of the Birman–Vishik–Grubb formulas in the symmetric case.

The so called Phillips–Kato problem (cf. [**11**]) in its fullest generality is the problem of determining all (maximally) accretive extensions of a given accretive operator A.[1.1] Unlike in the sectorial case, it is not possible to construct a Friedrichs extension for accretive operators. In [**35**], Phillips was the first to consider this problem in a systematic way, for which he provided a full solution in [**36**] in terms of so called boundary spaces. We follow [**18**, Sect. 2] and [**11**] for a short presentation of his results. Phillips' main idea is to consider the graph $\Gamma(A)$ of a closed accretive operator A in a Hilbert space

[1.1] A densely defined operator is called accretive if its numerical range is contained in the right half-plane $\Pi_+ = \{z \in \mathbb{C} : \mathrm{Re}(z) \geq 0\}$. Of course, this extension problem is equivalent to the problem of finding all the dissipative extensions of a dissipative operator.

$\mathcal{H}$ as a positive closed subspace of the indefinite inner product space $(\mathfrak{H}, Q)$, where $\mathfrak{H} = \mathcal{H} \times \mathcal{H}$ and the inner product Q is given by

$$Q(\vec{u}, \vec{v}) := \langle u_1, v_2 \rangle + \langle u_2, v_1 \rangle ,$$

where $\vec{u} = (u_1, u_2)$ and $\vec{v} = (v_1, v_2)$. Moreover, $(\mathfrak{H}, Q)$ allows for the fundamental decomposition

$$\mathfrak{H} = \mathfrak{H}_+ \oplus_Q \mathfrak{H}_- , \tag{1.3}$$

where $\mathfrak{H}_\pm := \{(\psi, \pm\psi) : \psi \in \mathcal{H}\}$ are maximally positive/negative subspaces of $\mathfrak{H}$ and $\oplus_Q$ denotes a direct sum of spaces that are orthogonal with respect to $Q(\cdot, \cdot)$.[1.2] This means that $(\mathfrak{H}, Q)$ is a Kreĭn space. The graph of any maximally accretive extension $\widehat{A}$ of A is a maximally positive subspace of $\mathfrak{H}$ containing $\Gamma(A)$. Then, $\Gamma(\widehat{A})^{\perp_Q}$ is a maximally negative subspace of $\mathfrak{H}$, where $\Gamma(\widehat{A})^{\perp_Q}$ denotes the Q-orthogonal companion of $\Gamma(\widehat{A})$ in $\mathfrak{H}$. Also, for any densely defined operator T it can be shown that $\Gamma(T)^{\perp_Q} = \Gamma(-T^*)$. The so called Phillips boundary space $\widehat{H}_P$ is now given by $\widehat{H}_P = \mathfrak{M}_+ \oplus_Q \overline{\mathfrak{M}_-}^Q$, where $\mathfrak{M}_+ = \Gamma(-A^*) \cap \mathfrak{H}_+$ and $\mathfrak{M}_-$ is obtained by decomposing the graph of $-A^*$ as follows:

$$\Gamma(-A^*) = \mathfrak{M}_+ \oplus_Q \mathfrak{M}_- .$$

We get that $\mathfrak{M}_+ = \Gamma(-A^*) \cap \mathfrak{H}_+$ is already closed with respect to Q and $\overline{\mathfrak{M}_-}^Q$ is the closure of $\mathfrak{M}_-$ with respect to norm induced by Q on the strictly negative space $\mathfrak{M}_-$. Moreover, it can be shown that $\mathfrak{M}_+ = \{(\psi, \psi) : \psi \in \ker(A^* + \mathbb{1})\}$, which means that $\mathfrak{M}_+$ is finite-dimensional in the case of finite defect index m. In this case, $\widehat{H}_P$ is a Pontryagin space with m positive squares. Phillips showed the following

Proposition 1.2 ([**36**, Thm. 5.2]). *There is a one-to-one correspondence between all maximally negative subspaces $\widehat{N}$ of $\widehat{H}_P$ and the graphs of all maximally accretive restrictions $\widehat{A}^*$ of A^* via*

$$\Gamma(-\widehat{A}^*) = \widehat{N} \cap \Gamma(-A^*) .$$

This result has been used [**18, 19**] to construct the accretive extensions of strictly positive even-order differential operators, however in many applications, it seems to be

[1.2] A subspace $\mathfrak{K}$ of a Kreĭn-space is called positive if $Q(u,u) \geq 0$ for all $u \in \mathfrak{K}$ and negative if $Q(u,u) \leq 0$ for all $u \in \mathfrak{K}$.

quite difficult to construct manageable realizations of the Phillips boundary space (cf. also the remark in [**14**, p. 148]).

Thus, in [**14**], Crandall and Phillips considered a special class of dissipative operators A that were of the form[1.3]

$$A = S + iV ,$$

where S is symmetric and $V \geq 0$ is non-negative and selfadjoint (but possibly unbounded). By non-negativity and selfadjointness of V it follows that the operator $(\mathbb{1}+V)$ is a boundedly invertible bijection from $\mathcal{D}(V)$ onto $\mathcal{H}$. Crandall and Phillips then introduce the weighted Hilbert space $\mathcal{H}_{+1}$ which is the linear space $\mathcal{D}(V^{1/2})$ equipped with the inner product $\langle f, g\rangle_{+1} := \langle (\mathbb{1} + V)^{1/2} f, (\mathbb{1} + V)^{1/2} g\rangle$. Using standard ideas of the construction of Gel'fand triples, they associate every element f of $\mathcal{H}$ with an element ℓ_f of the dual space $\mathcal{H}_{+1}^*$ of $\mathcal{H}_{+1}$ via

$$\ell_f(g) := \langle f, g\rangle \quad \text{for any} \quad g \in \mathcal{H}_{+1} ,$$

which has norm equal to

$$\|\ell_f\| = \|(\mathbb{1} + V)^{-1/2} f\| =: \|f\|_{-1} .$$

The space $\mathcal{H}_{-1}$ is then obtained as the completion of $\mathcal{H}$ in $\mathcal{H}_{+1}^*$ with respect to $\|\cdot\|_{-1}$. Since for any $f \in \mathcal{D}(V^{1/2})$ and for any $g \in \mathcal{H}$ we have that $\|f\| \leq \|(\mathbb{1}+V)^{1/2} f\| = \|f\|_{+1}$ and $\|g\| \geq \|(\mathbb{1} + V)^{-1/2} g\| = \|g\|_{-1}$, we obtain the following inclusions:

$$\mathcal{H}_{+1} \subset \mathcal{H} \subset \mathcal{H}_{-1} .$$

In particular, this implies that V is bounded as an operator from $\mathcal{H}_{+1}$ to $\mathcal{H}_{-1}$ — a feature which Crandall and Phillips use in order to determine all maximally dissipative extensions of A as an operator from $\mathcal{H}_{+1}$ to $\mathcal{H}_{-1}$ [**14**, Thm. 1.1]. Having obtained a maximally dissipative operator $\widehat{A}$ from $\mathcal{H}_{+1}$ to $\mathcal{H}_{-1}$, they then construct a dissipative extension $\widehat{A}^0$ of A (as an operator in $\mathcal{H}$) via

$$\widehat{A}^0 : \quad \mathcal{D}(\widehat{A}^0) = \{f \in \mathcal{D}(\widehat{A}) : \widehat{A} f \in \mathcal{H}\}, \quad \widehat{A}^0 f := \widehat{A} f .$$

[1.3] In [**14**], a densely defined operator is called dissipative if its numerical range is confined to the left half plane $\Pi_- := \{z \in \mathbb{C} : \mathrm{Re}(z) < 0\}$. Since we will call an operator dissipative if its numerical range is confined to the upper complex plane, we have changed the presentation of the results in [**14**] accordingly.

If V is bounded, this provides a full characterization of all maximally dissipative extensions of A, since the spaces $\mathcal{H}_{+1}, \mathcal{H}$ and $\mathcal{H}_{-1}$ are equivalent in this case. For the unbounded case, this construction yields at best dissipative extensions of A that have domain contained in $\mathcal{D}(V^{1/2})$, which does not always provide a full description of all maximally dissipative extensions of A (cf. [**14**, Example 2]). Also, even if $\widehat{A}$ is a maximally dissipative operator from $\mathcal{H}_{+1}$ to $\mathcal{H}_{-1}$, it is possible that $\widehat{A}^0$ is not a maximally dissipative operator in $\mathcal{H}$ [**14**, Example 1]. However, Crandall and Phillips prove a necessary and sufficient condition for when all maximally dissipative extensions $\widehat{A}$ from $\mathcal{H}_{+1}$ to $\mathcal{H}_{-1}$ induce also a maximally dissipative extension $\widehat{A}^0$ in $\mathcal{H}$ [**14**, Thm. 3.3].
Another approach towards the problem of extending a given closed dissipative operator A in a Hilbert space $\mathcal{H}$ makes use of the fact that its Cayley transform $C := (A-i)(A+i)^{-1}$ with domain $\operatorname{ran}(A+i)$ is a contraction if A is dissipative. Moreover, via the inverse Cayley transform, one obtains a one-to-one correspondence between all maximally dissipative extensions of A and all contractive extensions of C that are defined on the entire Hilbert space $\mathcal{H}$. This has been used by Crandall [**15**, Thm. I and Cor. I] to give a full solution to the extension problem. He established that if C is a contraction defined on a closed subspace $\mathcal{C}$ of a Hilbert space $\mathcal{H}$ and mapping to $\mathcal{H}$, all contractive extensions $\widetilde{C}$ of C can be described via

$$\widetilde{C} = CP_{\mathcal{C}} + (\mathbb{1} - CP_{\mathcal{C}}(CP_{\mathcal{C}})^*)^{1/2} B(\mathbb{1} - P_{\mathcal{C}})\,,$$

where $P_{\mathcal{C}}$ is the orthogonal projection onto $\mathcal{C}$ and B is an arbitrary contraction on $\mathcal{H}$. Using the inverse Cayley transform of $\widetilde{C}$, given by $\widetilde{A} := i(\mathbb{1}+\widetilde{C})(\mathbb{1}-\widetilde{C})^{-1}$, we then obtain all maximally dissipative extensions $\widetilde{A}$ of A. However, for concrete applications, the operators involved in this construction are often very difficult to compute. (See also [**12**] for the construction of all possible contractive extensions of a given matrix contraction.)

1.2. Structure and results of the thesis

We will proceed as follows:

In Chapter 2, we introduce dissipative operators and present some important already-known results. We will define the Cayley transform C_A of a dissipative operator A, which is known to be a contraction and show that there is one-to-one correspondence between dissipative extensions of A and certain contractive extensions of C_A. Moreover, we will introduce the notion of a dual pair of operators.

In Chapter 3, we will investigate the dissipative extensions of symmetric operators. We will show that any dissipative extension of a symmetric operator S has to be a restriction of S^*. In addition to that we will give a criterion for a dissipative extension of a symmetric operator to be completely non-selfadjoint.

In Chapter 4, we study a more abstract extension problem. Given a densely defined closed operator A, we construct its closed extensions. To this end, we use certain subsets $\mathfrak{M}$ of $\mathcal{D}(A^*)$ in order to parametrize closable extensions $B_{\mathfrak{M}}$ of A. Furthermore, we construct restrictions $C_{\mathfrak{M}} \subset A^*$ and show that $B_{\mathfrak{M}}^* = C_{\mathfrak{M}}$. Then, we find a necessary and sufficient condition on $\mathfrak{M}$ to ensure that $C_{\mathfrak{M}}$ is densely defined. Using that $C_{\mathfrak{M}}^* = \overline{B_{\mathfrak{M}}}$ is a closed extension of A, this provides a full characterization of the closed extensions of A.

In Chapter 5, we introduce the common core property of a dual pair $(A, \widetilde{A})$, which ensures that the dual pair under consideration provides us with a convenient way of defining an operator V that corresponds to the "imaginary part" of A.
It will be the square root of the selfadjoint Kreĭn–von Neumann extension of V — denoted by $V_K^{1/2}$ — which will play an important role. The description of $V_K^{1/2}$ obtained by Ando and Nishio [**4**] will allow us to give a necessary and sufficient condition (Theorem 5.2.8) for an extension of $(A, \widetilde{A})$ to be dissipative, which we only have to check on the space by which we extend the operator A rather than on the whole domain of the extension. From this result, we proceed to give a description of all dissipative extensions of the dual pair $(A, \widetilde{A})$ in terms of contractions from one "small" auxiliary space to another. We also generalize our results to the case that the common core property is not satisfied by the dual pair as long as $\mathcal{D}(A) \cap \mathcal{D}(\widetilde{A})$ is still dense. As a first application, we start by considering symmetric operators with relatively bounded

dissipative perturbations and after that, we consider more singular dissipative operators — our first examples being such that the associated imaginary part V is already essentially selfadjoint and our last example being such that there is a family of selfadjoint extensions of V. Finally, we find lower bounds for the numerical range of the dissipative extensions we have obtained and apply this result to the examples from the previous section. The results of this chapter were obtained in collaboration with Ian Wood and Sergey Naboko and have been published in [**20**].

In Chapter 6, we define the Kreĭn-von Neumann extension A_K of a dissipative operator A with zero in its field of regularity and show that it is maximally dissipative. We will see that for typical dual pairs $(A, \widetilde{A})$, the Kreĭn-von Neumann extension cannot be a proper extension of $(A, \widetilde{A})$. Thus, we will construct restrictions $A_0 \subset A$ and $\widetilde{A}_0 \subset \widetilde{A}$ such that A_K is a proper extension of the dual pair $(A_0, \widetilde{A}_0)$. Moreover, we will discuss when these restrictions are densely defined.

In Chapter 7, we apply the results of Chapter 5 in order to construct proper sectorial extensions of dual pairs of sectorial operators $(A, \widetilde{A})$. We apply this result to obtain Grubb's description of (maximally) sectorial extensions of positive symmetric operators and we use a similar idea to construct the (maximally) dissipative extensions of symmetric operators with at least one real point in their field of regularity. After this, we introduce the Friedrichs extension A_F of a sectorial operator A and show that for dual pairs $(A, \widetilde{A})$ of sectorial operators that have the common core property, we have that $A_F = \widetilde{A}_F^*$. We finish this chapter with a detailed discussion of the proper sectorial and dissipative extensions of the operator $-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + i\frac{\gamma}{x^2}$, where $\gamma > 0$, and its formal adjoint.

In Chapter 8, we develop a parametrization of dissipative extensions of a dual pair of sectorial operators along the lines of the Birman-Kreĭn-Vishik theory of positive selfadjoint extensions of positive symmetric operators as presented in [**2**]. We use auxiliary operators D that have domain contained in $\ker(\widetilde{A}^*)$ and map into $\ker(A^*)$ in order to describe these extensions, which we will denote by A_D. Moreover, we will find a necessary and sufficient condition for when the quadratic form $\mathfrak{im}_{D,0} : f \mapsto \operatorname{Im}\langle f, A_D f\rangle$ is closable and show that the selfadjoint operator V_D associated to the closure of $\mathfrak{im}_{D,0}$

is an extension of the imaginary part V as defined in Chapter 5. This will allow us to define a partial order in the imaginary parts of the extensions A_D.

Finally, in Chapter 9, we discuss more general dissipative extensions of a given dissipative operator A. We start again by considering dual pairs $(A, \widetilde{A})$ that have the common core property, but we construct extensions of A that have domain contained in $\mathcal{D}(\widetilde{A}^*)$ but do not preserve the action of $\widetilde{A}^*$. As an application, we use our results to provide a full description of all dissipative extensions of dissipative operators with bounded imaginary part — a result that has already been obtained by Crandall and Phillips. Lastly, we consider dissipative operators A for which the quadratic form $f \mapsto \mathrm{Im}\langle f, Af\rangle$ is strictly positive and closable. Under this assumption, we find a necessary and sufficient condition for an extension B of A to be dissipative. We apply this result to a singular differential operator and to the problem of finding accretive extensions of strictly positive symmetric operators.

CHAPTER 2

Introduction

Let us introduce some notation and terminology as well as gather some useful results that we will need later.

2.1. Elementary definitions

Throughout this thesis, we will only consider complex Hilbert spaces $\mathcal{H}$ with scalar product $\langle\cdot,\cdot\rangle: \mathcal{H}\times\mathcal{H}\to\mathbb{C}$. Note that we define our scalar product to be antilinear in the first component and linear in the second component, i.e. for any $f,g\in\mathcal{H}$ and $\lambda\in\mathbb{C}$, we get $\langle\lambda f,g\rangle=\overline{\lambda}\langle f,g\rangle=\langle f,\overline{\lambda}g\rangle$.

Moreover, for an operator A in $\mathcal{H}$, we use $\mathcal{D}(A)$, $\operatorname{ran}(A)$ and $\ker(A)$ to denote its domain, range and kernel respectively.

Also, $\rho(A)$ denotes its resolvent set and $\widehat{\rho}(A)$ is the field of regularity of A, which is given by

$$\widehat{\rho}(A)=\{\lambda\in\mathbb{C}:\exists k(\lambda)>0 \text{ such that } \|(A-\lambda)f\|\geq k(\lambda)\|f\|\ \forall\, f\in\mathcal{D}(A)\}\,.$$

Note that if $\lambda\in\mathbb{C}\setminus\{-1,1\}$, we write $(A-\lambda)$ rather than $(A-\lambda\mathbb{1})$, where $\mathbb{1}$ denotes the identity operator.

An operator B is called an extension of A, which we denote by $A\subset B$, if $\mathcal{D}(A)\subset\mathcal{D}(B)$ and for any $f\in\mathcal{D}(A)$, we have $Af=Bf$. Note that $\mathcal{D}(A)\subset\mathcal{D}(B)$ does not mean that $\mathcal{D}(A)$ is a proper subset of $\mathcal{D}(B)$, i.e. it is in particular true that $A\subset A$. Conversely, if for two operators A and B we have $A\subset B$, then A is called a restriction of B. Let A be an operator in $\mathcal{H}$ and let $\mathcal{D}\subset\mathcal{D}(A)$. The operator $A\restriction_{\mathcal{D}}$ is called "the restriction of A to $\mathcal{D}$" and is given by

$$A\restriction_{\mathcal{D}}:\quad \mathcal{D}(A\restriction_{\mathcal{D}})=\mathcal{D},\quad A\restriction_{\mathcal{D}}f=Af\,,$$

for any $f\in\mathcal{D}$.

Also, we call an operator S *symmetric* if and only if it is densely defined and satisfies $S\subset S^*$. If in addition, it holds that $S=S^*$, then we call S *selfadjoint.*

Moreover, since we will mainly apply our results to closable differential operators, we will use $H^n(\Omega)$ to denote the n^{th} Sobolev space of square-integrable functions over Ω that possess a square-integrable n^{th} weak derivative.

Lastly, $\mathcal{B}(\mathcal{H})$ denotes the set of bounded operators on $\mathcal{H}$.

2.2. Dissipative operators

Let us now give a few basic definitions and results on dissipative operators.

DEFINITION 2.2.1. *An operator A on a Hilbert space $\mathcal{H}$ is said to be* **dissipative** *if and only if it is densely defined and*

$$Im\langle f, Af\rangle \geq 0$$

for all $f \in \mathcal{D}(A)$. An operator $\widetilde{A}$ is called **antidissipative** *if and only if $(-\widetilde{A})$ is dissipative and* **accretive** *if and only if $(i\widetilde{A})$ is dissipative.*

Note that we require A to be densely defined for it to be dissipative. Finally, let us remark that any operator A, which is dissipative in the above sense, is also closable with its closure $\overline{A}$ being dissipative as well [**29**, Proposition 6.9].

EXAMPLE 2.2.2. Consider a bounded operator $A \in \mathcal{B}(\mathcal{H})$. Then, A is dissipative if and only if the operator $\operatorname{Im} A := \frac{1}{2i}(A - A^*)$ is non-negative:

$$\operatorname{Im}\langle f, Af\rangle = \frac{1}{2i}(\langle f, Af\rangle - \langle Af, f\rangle) = \langle f, \frac{1}{2i}(A - A^*)f\rangle = \langle f, (\operatorname{Im} A)f\rangle \,.$$

For instance, let $\mathcal{H} = \mathbb{C}^2$ and consider the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \Rightarrow \operatorname{Im} A = \begin{pmatrix} \operatorname{Im} a & \frac{b-\bar{c}}{2i} \\ \frac{c-\bar{b}}{2i} & \operatorname{Im} d \end{pmatrix} \,,$$

which implies that A is dissipative if and only if

$$\operatorname{Im} a \geq 0 \quad \text{and} \quad (\operatorname{Im} a)(\operatorname{Im} d) \geq \frac{|b - \overline{c}|^2}{4} \,.$$

DEFINITION 2.2.3. *A dissipative operator A is said to be* **maximally dissipative** *if for any dissipative operator extension $A \subset A'$ we get that $A = A'$. Analogously, an operator $\widetilde{A}$ is* **maximally antidissipative** *if and only if $(-\widetilde{A})$ is maximally dissipative and* **maximally accretive** *if and only if $(i\widetilde{A})$ is maximally dissipative.*

Let us remark at this point that the distinction between m-dissipative and maximally dissipative operators as it can be found in the literature (cf. e.g. [**17**, Sec. 3] for accretive operators) is not needed if one only considers densely defined dissipative operators as they coincide for this case.

Example 2.2.4. Let $\mathcal{H} = L^2(\mathbb{R})$ and consider the operator A given by

$$A: \qquad \mathcal{D}(A) = \mathcal{C}_c^\infty(\mathbb{R})$$
$$(Af)(x) = -f''(x) + iV(x)f(x)\,,$$

where we assume that $V \in L^\infty(\mathbb{R})$ and $V(x) \geq 0$ almost everywhere. Using integration by parts, we get

$$\operatorname{Im}\langle f, Af\rangle = \operatorname{Im}\left(\int_{-\infty}^{\infty} \overline{f(x)}(-f''(x) + iV(x)f(x))\mathrm{d}x\right) = \int_{-\infty}^{\infty} V(x)|f(x)|^2\mathrm{d}x \geq 0\,,$$

which shows that A is dissipative. However, A is not maximally dissipative since the operator B given by

$$B: \qquad \mathcal{D}(B) = H^2(\mathbb{R})$$
$$(Bf)(x) = -f''(x) + iV(x)f(x)$$

is a dissipative extension of A. Here, f'' denotes the second weak derivative of f.

The following result is a well-known fact:

Proposition 2.2.5 ([**35**, Theorems 1.1.1, 1.1.2 and 1.1.3]). *Let A be dissipative. Then, the following are equivalent:*

- *A is maximally dissipative.*
- *There exists a $\lambda \in \mathbb{C}$ with Im$(\lambda) < 0$ such that $\lambda \in \rho(A)$.*
- *$\mathbb{C}^- := \{z \in \mathbb{C} : Im(z) < 0\} \subset \rho(A)$.*
- *$(-A^*)$ is dissipative.*
- *iA is the generator of a strongly continuous semigroup of contractions on $\mathcal{H}$.*

Example 2.2.6 (Continuation of Example 2.2.4). Let us use Proposition 2.2.5 to show that the operator B defined in Example 2.2.4 is maximally dissipative. To this end, observe that $B = S + iV$, where S is the selfadjoint Laplace operator on the real axis:

$$S: \qquad \mathcal{D}(S) = H^2(\mathbb{R}), \quad f \mapsto -f''$$

and V is the operator of multiplication by $V(x)$. Since S is selfadjoint, we know that for any $\tau > 0$ we have that $-i\tau \in \rho(S)$. Since V is a bounded operator, we can choose τ large enough such that $\|V(S+i\tau)^{-1}\| \leq \|V\|\|(S+i\tau)^{-1}\| \leq \tau^{-1}\|V\| < 1$ from which we get that

$$\mathrm{ran}(S + iV + i\tau) = \mathrm{ran}((\mathbb{1} + iV(S+i\tau)^{-1})(S+i\tau)) = \mathcal{H}\,,$$

which implies that $-i\tau \in \rho(B)$ and thus by Proposition 2.2.5, we have shown that B is maximally dissipative.

2.3. The Cayley transform

Let us now define the Cayley transform of a dissipative (antidissipative) operator, which is a useful theoretical tool for the study of dissipative extensions of a given dissipative operator.

Definition 2.3.1. *Let A be a closed and dissipative (antidissipative) operator. For any $\lambda \in \mathbb{C}^-$ ($\lambda \in \mathbb{C}^+$), its associated* **Cayley transform** *$C_A(\lambda)$ is given by*

$$\begin{aligned} C_A(\lambda): \qquad &\mathrm{ran}(A-\lambda) \to \mathrm{ran}(A-\overline{\lambda}) \\ &(A-\lambda)f \mapsto (A-\overline{\lambda})f\,. \end{aligned} \tag{2.3.1}$$

Convention 2.3.2. *Let A be dissipative. We adopt the convention that for $\lambda = -i$, we will write C_A instead of $C_A(-i)$.*

We will now prove a few well-known properties of the Cayley transform of a given dissipative (antidissipative) operator. For example, it is a well-known fact that the Cayley transform of a dissipative (antidissipative) operator is a contraction, i.e. it satisfies

$$\|C_A(\lambda)\psi\| \leq \|\psi\|$$

for all $\psi \in \mathcal{D}(C_A(\lambda))$, where either A is dissipative and $\lambda \in \mathbb{C}^-$ or A is antidissipative and $\lambda \in \mathbb{C}^+$. However, in the literature (up to a suitable multiplication by i cf. [**35**, Thm. 1.1.1]) this is often only shown for one specific value of λ and only for the case of A being dissipative. Let us therefore give our own proof of this fact:

THEOREM 2.3.3. *Let A be dissipative (antidissipative). Then, for any $\lambda \in \mathbb{C}^-$ ($\lambda \in \mathbb{C}^+$) we have that $C_A(\lambda)$ is a contraction. Moreover, we have that* $\ker(\mathbb{1} - C_A(\lambda)) = \{0\}$.

PROOF. Let $(A-\lambda)f \in \operatorname{ran}(A-\lambda) = \mathcal{D}(C_A(\lambda))$ and consider

$$\begin{aligned}
\|C_A(\lambda)(A-\lambda)f\|^2 = \|(A-\overline{\lambda})f\|^2 &= \|Af\|^2 + 2\mathrm{Re}\langle Af, \overline{\lambda} f\rangle + |\lambda|^2\|f\|^2 \\
&= \|Af\|^2 + 2\left((\mathrm{Re}\lambda)\mathrm{Re}\langle f, Af\rangle + (\mathrm{Im}\lambda)\mathrm{Im}\langle f, Af\rangle\right) + |\lambda|^2\|f\|^2 \\
&\leq \|Af\|^2 + 2\left((\mathrm{Re}\lambda)\mathrm{Re}\langle f, Af\rangle - (\mathrm{Im}\lambda)\mathrm{Im}\langle f, Af\rangle\right) + |\lambda|^2\|f\|^2 \\
&= \|(A-\lambda)f\|^2 ,
\end{aligned}$$

where we have used that $\mathrm{sgn}((\mathrm{Im}\lambda)\mathrm{Im}\langle f, Af\rangle) \leq 0$ in the dissipative case as well as in the antidissipative case. Let us now assume that there exists a $(A-\lambda)f \in \ker(\mathbb{1} - C_A(\lambda))$. This $(A-\lambda)f$ would satisfy

$$0 = (\mathbb{1} - C_A(\lambda))(A-\lambda)f = (A-\lambda)f - (A-\overline{\lambda})f = (\overline{\lambda}-\lambda)f ,$$

which implies that $f = 0$ and consequently $(A-\lambda)f = 0$ since $\lambda \in \widehat{\rho}(A)$. This finishes the proof. □

Given a contraction C such that $\ker(\mathbb{1} - C) = \{0\}$, we can use it to define a dissipative or an antidissipative operator:

DEFINITION 2.3.4. *Let C be a contraction such that* $\ker(\mathbb{1} - C) = \{0\}$. *For any $\lambda \in \mathbb{C} \setminus \mathbb{R}$, we define its* **inverse Cayley transform** *$A_C(\lambda)$, via*

$$\begin{aligned}
A_C(\lambda) : \quad & \mathcal{D}(A_C(\lambda)) = \operatorname{ran}(\mathbb{1} - C) \\
& (\mathbb{1} - C)f \mapsto (\overline{\lambda} - \lambda C)f .
\end{aligned}$$

Let us now show that $A_C(\lambda)$ is dissipative for $\lambda \in \mathbb{C}^-$ and antidissipative for $\lambda \in \mathbb{C}^+$:

THEOREM 2.3.5. *Let D be a contraction such that* $\ker(\mathbb{1} - D) = \{0\}$. *Then, for $\lambda \in \mathbb{C}^-$ ($\lambda \in \mathbb{C}^+$), we have that its inverse Cayley transform $A_D(\lambda)$ is dissipative (antidissipative). Moreover, we have that $C_{A_D(\lambda)}(\lambda) = D$, i.e. the Cayley transform* *of the inverse Cayley transform of D yields the contraction D. Conversely, if B is dissipative (antidissipative), we get that the inverse Cayley transform applied to the Cayley transform of B yields the operator B, i.e. $A_{C_B(\lambda)}(\lambda) = B$, where $\lambda \in \mathbb{C}^-$ ($\lambda \in \mathbb{C}^+$).*

Proof. Let us start by showing that $A_D(\lambda)$ is densely defined. To see this, we will show that $\ker(\mathbb{1} - D^*) = \{0\}$, from which we get that $\mathcal{H} = \overline{\operatorname{ran}(\mathbb{1} - D)} = \overline{\mathcal{D}(A_D(\lambda))}$ proving that $A_D(\lambda)$ is densely defined. Assume that there exists $f \in \ker(\mathbb{1} - D^*)$. We then get that

$$\begin{aligned}0 &= \|(\mathbb{1} - D^*)f\|^2 = \|f\|^2 + \|D^*f\|^2 - 2\mathrm{Re}\langle f, D^*f\rangle = \|f\|^2 + \|f\|^2 - 2\mathrm{Re}\langle Df, f\rangle \\ &\geq \|f\|^2 + \|Df\|^2 - 2\mathrm{Re}\langle f, Df\rangle = \|(\mathbb{1} - D)f\|^2 \geq 0\,,\end{aligned}$$

from which follows that $f \in \ker(\mathbb{1} - D)$ and thus $f = 0$.

Let us now show that $A_D(\lambda)$ is dissipative (antidissipative) for $\lambda \in \mathbb{C}^-$ ($\lambda \in \mathbb{C}^+$). To this end, take any $(\mathbb{1} - D)\psi \in \mathcal{D}(A_D(\lambda))$ and consider

$$\begin{aligned}&\mathrm{Im}\langle(\mathbb{1} - D)\psi, A_D(\lambda)(\mathbb{1} - D)\psi\rangle = \mathrm{Im}\langle(\mathbb{1} - D)\psi, (\overline{\lambda} - \lambda D)\psi\rangle \\ &= -\,(\mathrm{Im}\lambda)\left(\|\psi\|^2 - \|D\psi\|^2\right) - \mathrm{Im}\left(\langle\psi, \lambda D\psi\rangle + \langle\lambda D\psi, \psi\rangle\right) = -(\mathrm{Im}\lambda)\left(\|\psi\|^2 - \|D\psi\|^2\right)\,,\end{aligned}$$

which is non-negative for $\lambda \in \mathbb{C}^-$ and non-positive for $\lambda \in \mathbb{C}^+$. Thus, $A_D(\lambda)$ is dissipative (antidissipative).

Let us now show that $C_{A_D(\lambda)}(\lambda) = D$. Let us begin with determining $\mathcal{D}(C_{A_D(\lambda)}(\lambda)) = \operatorname{ran}(A_D(\lambda) - \lambda)$. Let $\phi \in \mathcal{D}(D)$ and consider

$$(A_D(\lambda) - \lambda)(\mathbb{1} - D)\phi = (\overline{\lambda} - \lambda D)\phi - \lambda\phi + \lambda D\phi = (\overline{\lambda} - \lambda)\phi\,, \tag{2.3.2}$$

which implies that $(\overline{\lambda} - \lambda)\phi \in \mathcal{D}(C_{A_D(\lambda)}(\lambda))$ and thus $\mathcal{D}(D) \subset \mathcal{D}(C_{A_D(\lambda)}(\lambda))$. Conversely, since both maps $(A_D(\lambda) - \lambda)$ and $(\mathbb{1} - D)$ are injective, we get that for any $\xi \in \mathcal{D}(C_{A_D(\lambda)}(\lambda))$ there exists a unique $\phi_\xi \in \mathcal{D}(D)$ such that $\xi = (A_D(\lambda) - \lambda)(\mathbb{1} - D)\phi_\xi$. By (2.3.2), ϕ_ξ is simply given by $\phi_\xi = (\overline{\lambda} - \lambda)^{-1}\xi$, which implies that $\xi \in \mathcal{D}(D)$. Let us now show that $(C_{A_D(\lambda)}(\lambda))\xi = D\xi$ for any $\xi \in \mathcal{D}(C_{A_D(\lambda)}(\lambda)) = \mathcal{D}(D)$. By (2.3.2), we get that

$$\xi = \frac{(A_D(\lambda) - \lambda)(\mathbb{1} - D)\xi}{\overline{\lambda} - \lambda}$$

and applying $C_{A_D(\lambda)}(\lambda)$ to it, we obtain

$$\begin{aligned} C_{A_D(\lambda)}(\lambda)\xi &= \frac{C_{A_D(\lambda)}(\lambda)(A_D(\lambda)-\lambda)(\mathbb{1}-D)\xi}{\overline{\lambda}-\lambda} = \frac{(A_D(\lambda)-\overline{\lambda})(\mathbb{1}-D)\xi}{\overline{\lambda}-\lambda} \\ &= \frac{\overline{\lambda}\xi-\lambda D\xi-\overline{\lambda}\xi+\overline{\lambda}D\xi}{\overline{\lambda}-\lambda} = D\xi\,. \end{aligned}$$

The fact that $A_{C_B(\lambda)}(\lambda) = B$ follows from completely analogous reasoning as before using that for any $f \in \mathcal{D}(B)$, we have that

$$(\mathbb{1}-C_B(\lambda))(B-\lambda)f = Bf-\lambda f-Bf+\overline{\lambda}f = (\overline{\lambda}-\lambda)f\,,$$

from which we may argue as before that $\mathcal{D}(B) = \mathcal{D}(A_{C_B(\lambda)}(\lambda))$. Moreover, since for any $f \in \mathcal{D}(B)$ we have that

$$f = \frac{(\mathbb{1}-C_B(\lambda))(B-\lambda)f}{\overline{\lambda}-\lambda}$$

to which we apply $A_{C_B(\lambda)}(\lambda)$ to get

$$\begin{aligned} A_{C_B(\lambda)}(\lambda)f &= \frac{A_{C_B(\lambda)}(\lambda)(\mathbb{1}-C_B(\lambda))(B-\lambda)f}{\overline{\lambda}-\lambda} = \frac{(\overline{\lambda}-\lambda C_B(\lambda))(B-\lambda)f}{\overline{\lambda}-\lambda} \\ &= \frac{\overline{\lambda}Bf-|\lambda|^2f-\lambda Bf+|\lambda|^2f}{\overline{\lambda}-\lambda} = Bf\,. \end{aligned}$$

This finishes the proof. □

The previous results allow us to show that the problem of determining dissipative (antidissipative) extensions of dissipative (antidissipative) operators is equivalent to finding contractive extensions of a given contraction (cf. [**35**, Thm. 1.1.1]).

Corollary 2.3.6. *Fix $\lambda \in \mathbb{C}^-$ ($\lambda \in \mathbb{C}^+$). Then, for any dissipative (antidissipative) A, there exists a unique contraction C with $\ker(\mathbb{1}-C) = \{0\}$ such that $A = A_C(\lambda)$. Conversely, for any contraction C with $\ker(\mathbb{1}-C) = \{0\}$, there exists a unique dissipative (antidissipative) operator A such that $C = C_A(\lambda)$. Moreover, B is a dissipative (antidissipative) extension of A if and only if $C_B(\lambda)$ is a contractive extension of $C_A(\lambda)$.*

Proof. The first part of the Corollary follows immediately from Theorem 2.3.5. The fact that B is a dissipative (antidissipative) extension of A if and only if $C_B(\lambda)$ is a contractive extension of $C_A(\lambda)$ follows directly from the definition of the Cayley transform and of the inverse Cayley transform, since $\operatorname{ran}(A-\lambda) \subset \operatorname{ran}(B-\lambda)$ and $\operatorname{ran}(\mathbb{1}-C_A(\lambda)) \subset \operatorname{ran}(\mathbb{1}-C_B(\lambda))$. □

Remark 2.3.7. By virtue of this result and by Proposition 2.2.5, A is maximally dissipative if and only if $C_A(\lambda)$ is a contraction and $\mathcal{D}(C_A(\lambda)) = \mathcal{H}$, since $\lambda \in \rho(A)$ in this case.

Finally, let us state a lemma on by how many linearly independent vectors the domain of a given closed dissipative operator with finite defect index has to increase in order to obtain a maximally dissipative extension.

Lemma 2.3.8. *Let A be a closed and dissipative linear operator on a separable Hilbert space $\mathcal{H}$ such that $\dim \operatorname{ran}(A+i)^{\perp} < \infty$. Moreover, let A' be a dissipative extension of A. Then, A' is maximally dissipative if and only if*

$$\dim \mathcal{D}(A')/\mathcal{D}(A) = \dim[\operatorname{ran}(A+i)]^{\perp} .$$

Proof. "A' maximally dissipative $\Rightarrow \dim \mathcal{D}(A')/\mathcal{D}(A) = \dim \operatorname{ran}(A+i)^{\perp}$":
A' being maximally dissipative implies that the domain of its Cayley transform $C_{A'}$ is given by $\mathcal{D}(C_{A'}) = \mathcal{H}$ (cf. Remark 2.3.7). Moreover, by Theorem 2.3.3, $C_{A'}$ is a contraction with $\ker(\mathbb{1} - C_{A'}) = \{0\}$. Using $\mathcal{D}(A') = \operatorname{ran}(\mathbb{1} - C_{A'})$, we get that

$$\mathcal{D}(A') = \operatorname{ran}(\mathbb{1} - C_{A'}) = (\mathbb{1} - C_{A'})\mathcal{H} = (\mathbb{1} - C_{A'})\left(\operatorname{ran}(A+i) \oplus \operatorname{ran}(A+i)^{\perp}\right) ,$$

where we also made use of the fact that $\overline{\operatorname{ran}(A+i)} = \operatorname{ran}(A+i)$ by closedness and dissipativity of A. Next, we want to show that

$$(\mathbb{1} - C_{A'})\operatorname{ran}(A+i) \cap (\mathbb{1} - C_{A'})\operatorname{ran}(A+i)^{\perp} = \{0\} .$$

Assume this is not true. Then there would exist $0 \neq f \in \operatorname{ran}(A+i)$ and $0 \neq g \in \operatorname{ran}(A+i)^{\perp}$ such that

$$(\mathbb{1} - C_{A'})f = (\mathbb{1} - C_{A'})g$$

or, equivalently,

$$(\mathbb{1} - C_{A'})(f-g) = 0 .$$

As $f \perp g$, this implies that $f - g \neq 0$, which would mean that $f - g \in \ker(\mathbb{1} - C_{A'})$, which is a contradiction. Thus, $\mathcal{D}(A')$ can be expressed as

$$\mathcal{D}(A') = (\mathbb{1} - C_{A'})(\operatorname{ran}(A+i)) \dot{+} (\mathbb{1} - C_{A'})(\operatorname{ran}(A+i)^{\perp}) = \mathcal{D}(A) \dot{+} (\mathbb{1} - C_{A'})(\operatorname{ran}(A+i)^{\perp}),$$

which implies that

$$\dim \mathcal{D}(A')/\mathcal{D}(A) = \dim[(\mathbb{1} - C_{A'})\mathrm{ran}(A+i)^{\perp}] = \dim \mathrm{ran}(A+i)^{\perp} ,$$

where the last equality follows from the injectivity of $(\mathbb{1} - C_{A'})$.

"A' maximally dissipative $\Leftarrow \dim \mathcal{D}(A')/\mathcal{D}(A) = \dim \mathrm{ran}(A+i)^{\perp}$":

Let $\dim \mathcal{D}(A')/\mathcal{D}(A) = \dim \mathrm{ran}(A+i)^{\perp}$, i.e. there exists a subspace $\mathcal{M}$ with $\dim \mathcal{M} = \dim \mathrm{ran}(A+i)^{\perp}$ such that

$$\mathcal{D}(A') = \mathcal{D}(A)\dot{+}\mathcal{M} .$$

We need to show that $\mathrm{ran}(A'+i) = \mathcal{H}$. It holds that

$$\mathrm{ran}(A'+i) = (A'+i)\mathcal{D}(A') = (A'+i)(\mathcal{D}(A)\dot{+}\mathcal{M}) = (A'+i)\mathcal{D}(A) + (A'+i)\mathcal{M} . \tag{2.3.3}$$

Let us show that $(A'+i)\mathcal{D}(A) \cap (A'+i)\mathcal{M} = \{0\}$. To see this, let us assume this is not true. This would mean that there exist $0 \neq f \in \mathcal{D}(A)$ and $0 \neq g \in \mathcal{M}$ such that

$$(A'+i)f = (A'+i)g \quad \Leftrightarrow \quad (A'+i)(f-g) = 0 \quad \Leftrightarrow \quad A'(f-g) = -i(f-g) ,$$

which would contradict the dissipativity of A' as $f - g \neq 0$. Hence, as $(A'+i)\mathcal{D}(A) = \mathrm{ran}(A+i)$, Equation (2.3.3) reads

$$\mathrm{ran}(A'+i) = \mathrm{ran}(A+i)\dot{+}(A'+i)\mathcal{M}.$$

Moreover, as $(A'+i)$ is injective, it holds that

$$\dim[(A'+i)\mathcal{M}] = \dim \mathcal{M} = \dim \mathrm{ran}(A+i)^{\perp} .$$

But if we have a closed infinite-dimensional space $\mathcal{A}$ and a finite-dimensional space $\mathcal{B}$ such that $\mathcal{A} \cap \mathcal{B} = \{0\}$ and $\dim \mathcal{A}^{\perp} = \dim \mathcal{B}$, it holds that

$$\mathcal{A}\dot{+}\mathcal{B} = \mathcal{A} \oplus \mathcal{A}^{\perp} = \mathcal{H} ,$$

which applies to our situation. This proves the lemma. $\square$

2.4. Dual pairs

Let us introduce the notion of a dual pair of operators (see also [**31**] for more details).

Definition 2.4.1. *Let $(A, \widetilde{A})$ be a pair of densely defined and closable operators. We say that they form a* **dual pair** *if*

$$A \subset \widetilde{A}^* \quad \textit{resp.} \quad \widetilde{A} \subset A^* \, .$$

In this case, A is called a **formal adjoint** *of $\widetilde{A}$ and vice versa.*

Given a densely defined closable operator A, it is a well-known fact that another densely defined closable operator $\widetilde{A}$ can always be found such that $(A, \widetilde{A})$ forms a dual pair as can be seen from the trivial choice $\widetilde{A} := A^*$. A dual pair can be considered as a pair consisting of a "maximal" operator (in our notation $\widetilde{A}^*$) and a "minimal" operator (here: A). In this sense, any extension of A that is a restriction of $\widetilde{A}^*$ can be interpreted as preserving the formal action of $\widetilde{A}^*$:

Definition 2.4.2. *Let $(A, \widetilde{A})$ be a dual pair. An operator A' is said to be a* **proper** *extension of the dual pair $(A, \widetilde{A})$ if*

$$A \subset A' \subset \widetilde{A}^* \quad \textit{resp.} \quad \widetilde{A} \subset (A')^* \subset A^* \, .$$

Let us quote two useful results on the existence of proper extensions of certain dual pairs. The first proposition guarantees the existence of a proper extension of a dual pair $(A, \widetilde{A})$ with $\lambda \in \widehat{\rho}(A)$ and $\overline{\lambda} \in \widehat{\rho}(\widetilde{A})$. This applies in particular if A is dissipative, which means that $\mathbb{C}^- \subset \widehat{\rho}(A)$ and if $\widetilde{A}$ is antidissipative, which implies $\mathbb{C}^+ \subset \widehat{\rho}(\widetilde{A})$.

Proposition 2.4.3 ([**23**, Chapter II, Lemma 1.1]). *Let $(A, \widetilde{A})$ be a dual pair with $\lambda \in \widehat{\rho}(A)$ and $\overline{\lambda} \in \widehat{\rho}(\widetilde{A})$. Then there exists a proper extension $\widehat{A}$ of $(A, \widetilde{A})$ such that $\lambda \in \rho(\widehat{A})$ and $\mathcal{D}(\widetilde{A}^*)$ can be expressed as*

$$\mathcal{D}(\widetilde{A}^*) = \mathcal{D}(\overline{A}) \dot{+} (\widehat{A} - \lambda)^{-1} \ker(A^* - \overline{\lambda}) \dot{+} \ker(\widetilde{A}^* - \lambda) \, . \tag{2.4.1}$$

Likewise, we get the following description for $\mathcal{D}(A^*)$:

$$\mathcal{D}(A^*) = \mathcal{D}(\overline{\widetilde{A}}) \dot{+} (\widehat{A}^* - \overline{\lambda})^{-1} \ker(\widetilde{A}^* - \lambda) \dot{+} \ker(A^* - \overline{\lambda}) \, .$$

EXAMPLE 2.4.4. Let $0 < \gamma < 1/2$ and consider the dual pair of operators

$$\begin{aligned} A_0: \quad & \mathcal{D}(A_0) = \mathcal{C}_c^\infty(0,1), \quad (A_0 f)(x) := if'(x) + \frac{i\gamma}{x} f(x) \\ \widetilde{A}_0: \quad & \mathcal{D}(\widetilde{A}_0) = \mathcal{C}_c^\infty(0,1), \quad (\widetilde{A}_0 f)(x) := if'(x) - \frac{i\gamma}{x} f(x) , \end{aligned}$$

where A_0 is dissipative and $\widetilde{A}_0$ is antidissipative. Since $\frac{\gamma}{x} > \gamma$ for $x \in (0,1)$, this implies that

$$\operatorname{Im}\langle f, A_0 f\rangle \geq \gamma \|f\|^2 \quad \text{and} \quad \operatorname{Im}\langle f, \widetilde{A}_0 f\rangle \leq -\gamma \|f\|^2$$

for all $f \in \mathcal{C}_c^\infty(0,1)$. We therefore get that $0 \in \widehat{\rho}(A_0) \cap \widehat{\rho}(\widetilde{A}_0)$ and we will thus use Equation (2.4.1) from Proposition 2.4.3 with the choice $\lambda = 0$ to describe $\mathcal{D}(\widetilde{A}^*)$. Since for any $f \in \mathcal{C}_c^\infty(0,1)$, we get

$$\begin{aligned} \|f\|^2_{\Gamma(A_0)} &= \|f\|^2 + \left\| if' + \frac{i\gamma}{x} f \right\|^2 \\ &= \|f\|^2 + \|f'\|^2 + \left\| \frac{\gamma f}{x} \right\|^2 + \int_0^1 \left(\overline{f'(x)} \frac{\gamma f(x)}{x} + f'(x) \frac{\gamma \overline{f(x)}}{x} \right) \mathrm{d}x \\ &= \|f\|^2 + \|f'\|^2 + (\gamma^2 + \gamma) \left\| \frac{f}{x} \right\|^2 \begin{cases} \geq \|f\|^2 + \|f'\|^2 \\ \leq \|f\|^2 + (1 + 4\gamma^2 + 4\gamma)\|f'\|^2 \end{cases} , \end{aligned}$$

where the estimate from above is obtained using Hardy's inequality. This implies that the graph norm of A_0, $\|\cdot\|_{\Gamma(A_0)}$, is equivalent to the H^1-norm. Thus, we get that $\mathcal{D}(A) = \overline{\mathcal{C}_c^\infty(0,1)}^{\|\cdot\|_{H^1}} = H_0^1(0,1)$. Moreover, it can be shown that $\widetilde{A}^*$ and A^* are given by

$$\begin{aligned} \widetilde{A}^*: \quad & \mathcal{D}(\widetilde{A}^*) = \left\{ f \in H^1_{\mathrm{loc}}(0,1) \cap L^2(0,1) : if' + \frac{i\gamma}{x} f \in L^2(0,1) \right\} \\ & (\widetilde{A}^* f)(x) = if'(x) + \frac{i\gamma}{x} f(x) \\ A^*: \quad & \mathcal{D}(A^*) = \left\{ f \in H^1_{\mathrm{loc}}(0,1) \cap L^2(0,1) : if' - \frac{i\gamma}{x} f \in L^2(0,1) \right\} \end{aligned}$$

$$(A^* f)(x) = if'(x) - \frac{i\gamma}{x} f(x) , \tag{2.4.2}$$

which we use to determine $\ker(\widetilde{A}^*)$ and $\ker(A^*)$:

$$if'(x) \pm \frac{i\gamma}{x} f(x) = 0 \quad \Leftrightarrow \quad f(x) = x^{\mp\gamma} . \tag{2.4.3}$$

Hence, using Equation (2.4.1) for $\lambda = 0$ yields that

$$\mathcal{D}(\widetilde{A}^*) = \mathcal{D}(A) \dot{+} \widehat{A}^{-1}\mathrm{span}\{x^\gamma\} \dot{+} \mathrm{span}\{x^{-\gamma}\} \tag{2.4.4}$$

and since $\widehat{A} \subset \widetilde{A}^*$, we get

$$\widehat{A}^{-1}x^\gamma = \phi(x) \quad \Rightarrow \quad \widetilde{A}^*\left(\widehat{A}^{-1}x^\gamma\right) = x^\gamma = (\widetilde{A}^*\phi)(x) = i\phi'(x) + \frac{i\gamma}{x}\phi(x) ,$$

which has a one-dimensional solution space given by

$$\frac{-ix^{\gamma+1}}{2\gamma+1} \dot{+} \mathrm{span}\{x^{-\gamma}\} .$$

This allows us to rewrite (2.4.4) as follows:

$$\mathcal{D}(\widetilde{A}^*) = \mathcal{D}(A) \dot{+} \mathrm{span}\{x^{\gamma+1}, x^{-\gamma}\} = H_0^1(0,1) \dot{+} \mathrm{span}\{x^{\gamma+1}, x^{-\gamma}\} .$$

The following proposition guarantees the existence of a proper maximally dissipative extension for any dual pair $(A, \widetilde{A})$, where A is dissipative and $\widetilde{A}$ is antidissipative. Up to a suitable multiplication by i, a proof for this can be found in [**37**, Chapter IV, Proposition 4.2].

Proposition 2.4.5. *Let $(A, \widetilde{A})$ be a dual pair, where A is dissipative and $\widetilde{A}$ is antidissipative. Then there exists a maximally dissipative proper extension of $(A, \widetilde{A})$.*

Let us now show that the requirement that A is dissipative and $\widetilde{A}$ is antidissipative is absolutely necessary to make sure that the dual pair $(A, \widetilde{A})$ allows for a proper maximally dissipative extension:

Corollary 2.4.6. *Let $(A, \widetilde{A})$ be a dual pair and assume that A is dissipative. Then, $(A, \widetilde{A})$ admits a proper maximally dissipative extension if and only if $\widetilde{A}$ is antidissipative.*

Proof. The fact that $\widetilde{A}$ being antidissipative is sufficient for the dual pair $(A, \widetilde{A})$ to admit a proper maximally dissipative extension follows from Proposition 2.4.5. To see that it is also necessary, assume that $\widetilde{A}$ is not antidissipative but there still exists a proper maximally dissipative extension $\widehat{A}$ of the dual pair $(A, \widetilde{A})$. This means that

$$A \subset \widehat{A} \subset \widetilde{A}^* \quad \Leftrightarrow \quad \widetilde{A} \subset \widehat{A}^* \subset A^* .$$

However, by Proposition 2.2.5, $\widehat{A}$ being maximally dissipative implies that $\widehat{A}^*$ is antidissipative, which is impossible since $\widetilde{A}$ is not antidissipative and we have $\widetilde{A} \subset \widehat{A}^*$. $\square$

Finally, let us introduce some convenient notation for complementary subspaces:

Definition 2.4.7. *Let $\mathcal{N}$ be a (not necessarily closed) linear space and $\mathcal{M} \subset \mathcal{N}$ be a (not necessarily closed) subspace. With the notation $\mathcal{N}//\mathcal{M}$ we mean any subspace of $\mathcal{N}$, which is complementary to $\mathcal{M}$, i.e.*

$$(\mathcal{N}//\mathcal{M}) + \mathcal{M} = \mathcal{N} \quad \textit{and} \quad (\mathcal{N}//\mathcal{M}) \cap \mathcal{M} = \{0\}\,.$$

CHAPTER 3

Properties of dissipative extensions of symmetric operators

As a warm-up, let us begin with a discussion of the dissipative extensions of symmetric operators. We will start by showing that any dissipative extension of a symmetric operator S has to be a restriction of S^*. After that, we will discuss the complete non-selfadjointness of these extensions.

3.1. Dissipative extensions of symmetric operators

As the Cayley transform C_S of a symmetric operator S is a partially defined isometry, this restricts our choices for extending C_S to a contraction on the whole Hilbert space:

Lemma 3.1.1. *Let S be a closed symmetric operator on a Hilbert space $\mathcal{H}$. Then, for any $\lambda \in \mathbb{C} \setminus \mathbb{R}$, any contractive extension $C \supset C_S(\lambda)$ has to map $\ker(S^* - \overline{\lambda})$ into $\ker(S^* - \lambda)$, where*

$$\|C\phi_\lambda\| \leq \|\phi_\lambda\|$$

for any $\phi_\lambda \in \ker(S^* - \overline{\lambda})$.

Proof. As S is a symmetric operator, $C_S(\lambda)$ is an isometry that maps $\operatorname{ran}(A - \lambda)$ onto $\operatorname{ran}(A - \overline{\lambda})$ ([**44**, Satz 10.5] for $\lambda = -i$; for other $\lambda \in \mathbb{C}^-$, this follows analogously). Now, let C be a contractive extension of $C_S(\lambda)$.

Assume that C does not map $\ker(S^* - \overline{\lambda})$ into $\ker(S^* - \lambda)$, which means that there exist $\lambda, \mu \in \mathbb{C}, \psi \in \ker(S^* - \overline{\lambda})$, $\phi \in \ker(S^* - \lambda)$ and $g \in \operatorname{ran}(S - \overline{\lambda})$ such that

$$C\psi = \lambda g + \mu\phi \,,$$

where we assume that $\|\psi\| = \|\phi\| = \|g\| = 1$ and $\lambda \neq 0$. Moreover, as $C_S(\lambda)$ maps $\operatorname{ran}(S-\lambda)$ isometrically onto $\operatorname{ran}(S-\overline{\lambda})$, there exists a normalized vector $f \in \operatorname{ran}(S-\lambda)$

such that $Cf = C_S(\lambda)f = g$. Now, let $\alpha, \beta \in \mathbb{C}$ and consider

$$\begin{aligned}\|C(\alpha f + \beta\psi)\|^2 &= \|\alpha Cf + \beta C\psi\|^2 = \|\alpha g + \beta(\lambda g + \mu\phi)\|^2 \\ &= |\alpha + \beta\lambda|^2\|g\|^2 + |\beta\mu|^2\|\phi\|^2 = |\alpha|^2 + 2\mathrm{Re}(\alpha\overline{\beta\lambda}) + |\beta|^2(|\lambda|^2 + |\mu|^2)\end{aligned}$$

Since $f \perp \psi$ implies that $\|\alpha f + \beta\psi\|^2 = |\alpha|^2 + |\beta|^2$, we only need to show that $\|C(\alpha f + \beta\psi)\|^2 > |\alpha|^2 + |\beta|^2$ in order to lead this to a contradiction, since C would not be a contraction in this case. As we can always consider $\widehat{\alpha} := \alpha e^{i\varphi}$ instead of α, where φ is chosen such that $2\mathrm{Re}(\widehat{\alpha}\overline{\beta\lambda}) = 2|\alpha||\beta||\lambda|$, we just need to find α and β such that

$$|\alpha|^2 + 2|\alpha||\beta|\lambda| + |\beta|^2(|\lambda|^2 + |\mu|^2) \leq |\alpha|^2 + |\beta|^2 \tag{3.1.1}$$

is violated. However, as C is assumed to be contraction, it is a necessary condition that

$$\|C\phi_+\|^2 = \|\lambda g + \mu\phi_-\|^2 = |\lambda|^2 + |\mu|^2 \leq 1 \, .$$

This, together with the choice

$$|\alpha| > \frac{(1 - |\lambda|^2 - |\mu|^2)|\beta|}{2|\lambda|}$$

shows that (3.1.1) can be violated for $\lambda \neq 0$. □

The previous lemma shows that our choices for extending the Cayley transform C_S of a symmetric operator S to a contraction defined on $\mathcal{H}$ are rather limited. In the first proof of the following theorem, we will use this in order to show that this implies that any dissipative extension of S has to be a restriction of S^*. This idea of proof can also be found in [**35**, Lemma 1.1.5]. The second proof, which we believe to be new, is direct and does not make any explicit use of the Cayley transform.

THEOREM 3.1.2. *Let S be a closed symmetric operator. Then, all dissipative extension of S are restrictions of S^*.*

FIRST PROOF. Let $\mathcal{N}_\pm := \ker(S^* \mp i)$. By the first von Neumann formula (cf. [**44**, Satz 10.9]), we know that

$$\mathcal{D}(S^*) = \mathcal{D}(S) \dot{+} \mathcal{N}_+ \dot{+} \mathcal{N}_-$$

and

$$S^*(f_0 + \psi_+ + \psi_-) = Sf_0 + i\psi_+ - i\psi_- \, ,$$

where f_0, ψ_+ and ψ_- denote arbitrary elements of $\mathcal{D}(S)$, $\mathcal{N}_+$ and $\mathcal{N}_-$ respectively.

Let $\widehat{S}$ be an arbitrary dissipative extension of S. By Theorem 2.3.5, we have that $C_{\widehat{S}}$ is a contractive extension of the isometry C_S. Thus, there exists a subspace $\mathcal{M}_+ \subset \mathcal{N}_+$ such that

$$\mathcal{D}(C_{\widehat{S}}) = \mathcal{D}(C_S) \oplus \mathcal{M}_+, \quad C_{\widehat{S}} \restriction_{\mathcal{D}(C_S)} = C_S \,.$$

Moreover, since S is a closed and symmetric operator, Lemma 3.1.1 implies that $C_{\widehat{S}}\mathcal{M}_+ \subset \mathcal{N}_-$. From this we get that

$$\mathcal{D}(\widehat{S}) = \operatorname{ran}(\mathbb{1} - C_{\widehat{S}}) = \mathcal{D}(S) \dot{+} \{\psi_+ - C_{\widehat{S}}\psi_+ : \psi_+ \in \mathcal{M}_+\} \subset \mathcal{D}(S) \dot{+} \mathcal{N}_+ \dot{+} \mathcal{N}_- = \mathcal{D}(S^*) \,,$$

i.e. it just remains to show that $S^* \restriction_{\mathcal{D}(\widehat{S})} = \widehat{S}$. To this end, note that by the inverse Cayley transform we get $\widehat{S} = i(\mathbb{1} + C_{\widehat{S}})(\mathbb{1} - C_{\widehat{S}})^{-1}$ and therefore for any element in $\mathcal{D}(\widehat{S})$ we see from the first von Neumann formula (cf. [**44**, Satz 10.9])

$$\begin{aligned}\widehat{S}(f_0 + (\mathbb{1} - C_{\widehat{S}})\psi_+) &= Sf_0 + i(\mathbb{1} + C_{\widehat{S}})(\mathbb{1} - C_{\widehat{S}})^{-1}(\mathbb{1} - C_{\widehat{S}})\psi_+ \\ &= Sf_0 + i\psi_+ - i(-C_{\widehat{S}}\psi_+) = S^*(f_0 + (\mathbb{1} - C_{\widehat{S}})\psi_+) \,,\end{aligned}$$

which completes the proof. □

Second proof. Let T be an extension of S. Now, for any $f \in \mathcal{D}(S)$ and $v \in \mathcal{D}(T)//\mathcal{D}(S)$, consider

$$\begin{aligned}\operatorname{Im}\langle f + v, T(f+v)\rangle &= \operatorname{Im}\left(\langle f, Sf\rangle + \langle v, Tv\rangle + \langle v, Sf\rangle + \langle f, Tv\rangle\right) \\ &= \operatorname{Im}\left(\langle v, Tv\rangle + \langle v, Sf\rangle + \langle f, Tv\rangle\right)\end{aligned} \tag{3.1.2}$$

Now, if $v \notin \mathcal{D}(S^*)$, we can pick a normalized sequence $\{f_n\}_n \subset \mathcal{D}(S)$, such that $\lim_{n\to\infty} \operatorname{Im}(\langle v, Sf_n\rangle) = -\infty$, while all other terms in (3.1.2) stay bounded. Thus, for T to be dissipative, it is necessary that $\mathcal{D}(T) \subset \mathcal{D}(S^*)$. Hence, we can rewrite (3.1.2) as

$$\operatorname{Im}\langle f + v, T(f+v)\rangle = \operatorname{Im}\langle v, Tv\rangle + \operatorname{Im}\langle (S^* - T)v, f\rangle \,.$$

Now, if $T \not\subset S^*$, this means that there exists at least one $\widetilde{v} \in \mathcal{D}(S^*)//\mathcal{D}(T)$ such that $(S^* - T)\widetilde{v} \neq 0$. But again this would imply that T cannot be dissipative in this case since by denseness of $\mathcal{D}(S)$, we can always pick an $\widetilde{f} \in \mathcal{D}(S)$ such that $\langle (S^* - T)v, \widetilde{f}\rangle \neq 0$ and by replacing $\widetilde{f} \mapsto \lambda\widetilde{f}$, where $\lambda \in \mathbb{C}$, we would get

$$\operatorname{Im}\langle \lambda\widetilde{f} + \widetilde{v}, T(\lambda\widetilde{f} + \widetilde{v})\rangle = \operatorname{Im}\langle \widetilde{v}, T\widetilde{v}\rangle + \operatorname{Im}\langle (S^* - T)\widetilde{v}, \lambda\widetilde{f}\rangle \,.$$

This will be less than zero, if we choose λ to have a suitable phase and sufficiently large modulus. Thus, for any $v \in \mathcal{D}(T)//\mathcal{D}(S)$, we conclude that $v \in \mathcal{D}(S^*)$ and $Tv = S^*v$, which shows that T being dissipative implies that $T \subset S^*$.

□

3.2. Completely non-selfadjoint extensions of symmetric operators

It is a well-known result ([**37**, Chapter IV, Prop. 4.3] up to a suitable multiplication by a factor of $\pm i$) that any maximally dissipative operator A on a Hilbert space $\mathcal{H}$ can be uniquely decomposed into its selfadjoint and completely non-selfadjoint part. As we will define below, this means that $A = A_{\text{sa}} \oplus A_{\text{cnsa}}$ acting according to the decomposition $\mathcal{H} = \mathcal{H}_{\text{sa}} \oplus \mathcal{H}_{\text{cnsa}}$, where A_{sa} is selfadjoint in $\mathcal{H}_{\text{sa}}$ and A_{cnsa} is completely non-selfadjoint as an operator in $\mathcal{H}_{\text{cnsa}}$. The proof for this relies on the fact that the Cayley transform C_A of a maximally dissipative operator is an everywhere defined contraction (cf. Thm. 2.3.5), which can be uniquely decomposed into its unitary part (corresponding to the selfadjoint part of A) and its completely non-unitary part (corresponding to the completely non-selfadjoint part of A). This was shown in [**30**]. However, for concrete applications, it is often very difficult to compute the Cayley transform of a maximally dissipative operator and determine its unitary and completely non-unitary subspace. In this section, we are going to focus on the special question of whether a maximally dissipative extension of a symmetric operator S is completely non-selfadjoint or if there exists a reducing subspace on which it is selfadjoint. As we shall see, our result will depend on decomposing the symmetric operator S into its selfadjoint part and its completely non-selfadjoint part.

Following [**43**, Exercise 5.39] we define the notion of a reducing subspace:

Definition 3.2.1 (Reducing subspace). *Let S be an operator on a Hilbert space $\mathcal{H}$. A closed subspace $\mathfrak{M} \subset \mathcal{H}$ is called a* **reducing subspace** *of S, or is said to* **reduce** *the operator S, if*

$$\mathcal{D}(S) = \mathcal{D}(S) \cap \mathfrak{M} + \mathcal{D}(S) \cap \mathfrak{M}^\perp$$

and if
$$S(\mathcal{D}(S) \cap \mathfrak{M}) \subset \mathfrak{M} \quad \text{and} \quad S(\mathcal{D}(S) \cap \mathfrak{M}^\perp) \subset \mathfrak{M}^\perp .$$

If $\widehat{S}$ is maximally dissipative, we know by Proposition 2.2.5 that $\mathbb{C}^- \subset \rho(\widehat{S})$. If the resolvent set is non-empty it is often more convenient to deal with the bounded resolvent $(\widehat{S}-\lambda)^{-1}$, where $\lambda \in \rho(\widehat{S})$, rather than with the possibly unbounded operator $\widehat{S}$ itself. The following result is mentioned in [**37**, Chapter IV, after Equation 4.12], however no formal proof is given. For the sake of completeness, we will prove it here.

LEMMA 3.2.2. *Let $\widehat{S}$ be a maximally dissipative operator on a Hilbert space $\mathcal{H}$ and let $\mathfrak{M}$ be a closed subspace of $\mathcal{H}$. Then, $\mathfrak{M}$ reduces $\widehat{S}$ if and only if $\mathfrak{M}$ reduces $C_{\widehat{S}}(\lambda)$ for any $\lambda \in \mathbb{C}^-$.*

PROOF. Firstly, observe that since $C_{\widehat{S}}(\lambda) = (\widehat{S}-\overline{\lambda})(\widehat{S}-\lambda)^{-1} = \mathbb{1}-(\overline{\lambda}-\lambda)(\widehat{S}-\lambda)^{-1}$, this means that $\mathfrak{M}$ reduces $C_{\widehat{S}}(\lambda)$ if and only if it reduces the resolvent $(\widehat{S}-\lambda)^{-1}$. Hence, the lemma is equivalent to showing that $\mathfrak{M}$ reduces $\widehat{S}$ if and only if $\mathfrak{M}$ reduces $(\widehat{S}-\lambda)^{-1}$ for any $\lambda \in \mathbb{C}$.

Now, let us show that $\mathfrak{M}$ reducing $\widehat{S}$ implies that it also reduces $(\widehat{S}-\lambda)^{-1}$ for any $\lambda \in \mathbb{C}^-$. To this end, take any $f_0 \in \mathfrak{M}$. Since $\mathbb{C}^- \subset \rho(\widehat{S})$, we get that for any $\lambda \in \mathbb{C}^-$, there exists a unique $g_\lambda \in \mathcal{D}(\widehat{S})$, such that $f_0 = (\widehat{S}-\lambda)g_\lambda$. We will show that $g_\lambda \in \mathcal{D}(\widehat{S}) \cap \mathfrak{M}$. To this end, decompose $g_\lambda = g_\lambda^{(0)} + g_\lambda^\perp$, where $g_\lambda^{(0)} \in \mathcal{D}(\widehat{S}) \cap \mathfrak{M}$ and $g_\lambda^\perp = \mathcal{D}(\widehat{S}) \cap \mathfrak{M}^\perp$. Since $f_0 \in \mathfrak{M}$ and $(\widehat{S}-\lambda)g_\lambda^{(0)} \in \mathfrak{M}$ as well as $(\widehat{S}-\lambda)g_\lambda^\perp \in \mathfrak{M}^\perp$, this implies that $(\widehat{S}-\lambda)g_\lambda^\perp = 0$. However, since $\lambda \in \rho(\widehat{S})$, it follows that $g_\lambda^\perp = 0$. Thus, we have $f_0 = (\widehat{S}-\lambda)g_\lambda^{(0)}$. Applying $(\widehat{S}-\lambda)^{-1}$ to this yields $(\widehat{S}-\lambda)^{-1}f_0 = g_\lambda^{(0)} \in \mathfrak{M}$. Analogously, we may argue that for any $f^\perp \in \mathfrak{M}^\perp$, we get $(\widehat{S}-\lambda)^{-1}f^\perp \in \mathfrak{M}^\perp$.

Let us now show that if $\mathfrak{M}$ reduces $(\widehat{S}-\lambda)^{-1}$ for some $\lambda \in \mathbb{C}^-$, this implies that $\mathfrak{M}$ reduces $\widehat{S}$.[3.2.1] Since $\lambda \in \rho(\widehat{S})$, we have that for any $g \in \mathcal{D}(\widehat{S})$, there exists a unique $f \in \mathcal{H}$ such that $g = (\widehat{S}-\lambda)^{-1}f$. Decomposing $f = f_0 + f^\perp$, where $f_0 \in \mathfrak{M}$ and $f^\perp \in \mathfrak{M}^\perp$, we obtain $g = (\widehat{S}-\overline{\lambda})^{-1}f_0 + (\widehat{S}-\overline{\lambda})^{-1}f^\perp$, where $(\widehat{S}-\overline{\lambda})^{-1}f_0 \in \mathcal{D}(\widehat{S}) \cap \mathfrak{M}$ and $(\widehat{S}-\overline{\lambda})^{-1}f^\perp \in \mathcal{D}(\widehat{S}) \cap \mathfrak{M}^\perp$, which follows from the fact that $\mathfrak{M}$ reduces $(\widehat{S}-\lambda)^{-1}$. This shows that $\mathcal{D}(\widehat{S}) = \mathcal{D}(\widehat{S}) \cap \mathfrak{M} + \mathcal{D}(\widehat{S}) \cap \mathfrak{M}^\perp$. Next, observe that trivially, $\mathfrak{M}$ reduces $\widehat{S}$ if and only if it reduces $(\widehat{S}-\lambda)$. Let us now argue that any element $g_0 \in \mathcal{D}(\widehat{S}) \cap \mathfrak{M}$ is of the form $g_0 = (\widehat{S}-\lambda)^{-1}f_0$ for a unique $f_0 \in \mathfrak{M}$. Again, since $(\widehat{S}-\lambda)^{-1}$ is a bijection, there exists a unique $f \in \mathcal{H}$, such that $g_0 = (\widehat{S}-\lambda)^{-1}f$. Decomposing $f = f_0 + f^\perp$, where $f_0 \in \mathfrak{M}$ and $f^\perp \in \mathfrak{M}^\perp$, we use that $\mathfrak{M}$ reduces $(\widehat{S}-\lambda)^{-1}$ to conclude that $(\widehat{S}-\lambda)^{-1}f_0 \in \mathfrak{M}$ and $(\widehat{S}-\lambda)^{-1}f^\perp \in \mathfrak{M}^\perp$. But since $\mathfrak{M} \ni g_0 = (\widehat{S}-\lambda)^{-1}f_0 + (\widehat{S}-\lambda)^{-1}f^\perp$,

[3.2.1] This readily implies that $\mathfrak{M}$ reduces $(\widehat{S}-\lambda)^{-1}$ for any $\lambda \in \mathbb{C}^-$ by what has been argued in the first part of the proof of this lemma.

this implies that $(\widehat{S}-\lambda)^{-1}f^{\perp}=0$, from which we get that $f^{\perp}=0$ since $(\widehat{S}-\lambda)^{-1}$ is bijective. This shows that $(\widehat{S}-\lambda)g_0 = f_0 \in \mathfrak{M}$. Analogously, it can be shown that $(\widehat{S}-\lambda)(\mathcal{D}(\widehat{S})\cap\mathfrak{M}^{\perp}) \subset \mathfrak{M}^{\perp}$. Altogether, this shows that $\mathfrak{M}$ reduces $\widehat{S}$ if and only if it reduces $(\widehat{S}-\lambda)^{-1}$ for any $\lambda\in\mathbb{C}$. This shows the lemma. $\square$

Let us now define what it means for an operator to be completely non-selfadjoint:

DEFINITION 3.2.3 (Completely non-selfadjoint operator). *Let S be a densely defined operator. We say that S is* **completely non-selfadjoint** *if the only reducing subspace of S on which S is selfadjoint is the trivial space $\{0\}$.*

REMARK 3.2.4 (On the terminology). Note that according to this definition, a symmetric operator that is completely non-selfadjoint may have selfadjoint extensions. In the literature, completely non-selfadjoint symmetric operators are also referred to as **simple** symmetric operators; for example in [**1**, Vol. II, Sect. 81].

The following result, which is due to Kreĭn [**28**] (for a more recent English version of the proof, see also [**22**, Chapter 1, Thm. 2.1]), implies that any closed symmetric operator S can be uniquely decomposed into its selfadjoint and completely non-selfadjoint part. Firstly, let us introduce some convenient notation:

DEFINITION 3.2.5. *Let $\{V_\sigma\}_{\sigma\in S}$ be a family of closed subspaces of a Hilbert space $\mathcal{H}$. Then, we define*

$$\bigvee_{\sigma\in S} V_\sigma := \overline{\operatorname{span}\{V_\sigma : \sigma\in S\}}\,,$$

i.e. $\bigvee_{\sigma\in S} V_\sigma$ denotes the closure of the linear span of all V_σ's.

Let us now state the main proposition on the decomposition of a closed symmetric operator S into its selfadjoint and completely non-selfadjoint part.

PROPOSITION 3.2.6 ([**28**]). *Let S be a closed symmetric operator and define*

$$\mathfrak{M} := \bigvee_{\lambda\in\mathbb{C}\setminus\mathbb{R}} \ker(S^*-\overline{\lambda})\,.$$

Then, $\mathfrak{M}$ is a reducing subspace for S. Moreover, S is selfadjoint on $\mathfrak{M}^{\perp}$ and completely non-selfadjoint on $\mathfrak{M}$.

Let us apply this result to two examples:

EXAMPLE 3.2.7. Consider the symmetric momentum operator p defined as

$$p: \quad \mathcal{D}(p)$$

$$=\{f \in H^1(-\infty,-1) \oplus H^1(-1,1) \oplus H^1(1,\infty) : f(-1^-) = f(1^+) = 0, f(-1^+) = f(1^-)\}$$

$$f \mapsto if' , \tag{3.2.1}$$

where f' denotes the weak derivative of f taken over the respective segments of the real line. The adjoint p^* is given by

$$p^*: \quad \mathcal{D}(p^*) = \{f \in H^1(-\infty,-1) \oplus H^1(-1,1) \oplus H^1(1,\infty) : f(-1^+) = f(1^-)\} \tag{3.2.2}$$

$$f \mapsto if' . \tag{3.2.3}$$

We claim that

$$\mathfrak{M} := \bigvee_{\lambda \in \mathbb{C} \setminus \mathbb{R}} \ker(p^* - \overline{\lambda}) = L^2(-\infty,-1) \oplus L^2(1,\infty) \tag{3.2.4}$$

and therefore

$$\mathfrak{M}^\perp = L^2(-1,1) .$$

According to Proposition 3.2.6 we have that $\mathfrak{M}$ is a reducing subspace for p and that $p \restriction_{\mathfrak{M}}$ is completely non-selfadjoint and that $p \restriction_{\mathfrak{M}^\perp}$ is selfadjoint. Thus, let us show that (3.2.4) holds. To this end, let us compute $\ker(p^* - \overline{\lambda})$:

$$if'(x) = \overline{\lambda} f(x) \Leftrightarrow f(x) = e^{-i\overline{\lambda}x} .$$

Thus, we get that

$$\ker(p^* - \overline{\lambda}) = \operatorname{span}\{e^{-i\overline{\lambda}x}\chi_{(1,\infty)}(x)\} \qquad \text{if} \quad \operatorname{Im}(\lambda) > 0$$

$$\ker(p^* - \overline{\lambda}) = \operatorname{span}\{e^{-i\overline{\lambda}x}\chi_{(-\infty,-1)}(x)\} \quad \text{if} \quad \operatorname{Im}(\lambda) < 0 ,$$

where the fact that the functions are only supported on the respective half-lines follows from $-\operatorname{sgn}(\operatorname{Re}(-i\overline{\lambda})) = \operatorname{sgn}(\operatorname{Im}(\lambda))$ and the boundary conditions inside the interval $(-1,1)$. For any $\lambda \in \mathbb{C} \setminus \mathbb{R}$ we have that $\dim \ker(p^* - \overline{\lambda}) = 1$. Let us denote the normalized element of $\ker(p^* - \overline{\lambda})$ by ϕ_λ, which is unique up to a phase. Since it holds that

$$\operatorname{supp}(\phi_\lambda) \subset (-\infty, -1) \cup (1, \infty) \quad \text{for all } \lambda \in \mathbb{C} \setminus \mathbb{R}\,,$$

we have that $\mathfrak{M} \subset L^2(-\infty, -1) \oplus L^2(1, \infty)$. To show the other inclusion, let us show that $\mathfrak{M}^\perp \subset L^2(-1, 1)$. Thus, assume that $\psi \in \mathfrak{M}^\perp$, which implies

$$\langle \phi_\lambda, \psi \rangle = 0 \quad \forall \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{3.2.5}$$

For all λ such that $\operatorname{Im} \lambda > 0$, consider the function $g(\lambda)$ given by

$$g(\lambda) := \langle \phi_\lambda, \psi \rangle = \int_1^\infty e^{i\lambda x} \psi(x) \mathrm{d}x = \int_0^\infty e^{i\lambda x} \left[\chi_{(1,\infty)}(x) \cdot \psi(x) \right] \mathrm{d}x\,.$$

Interpreting $\chi_{(1,\infty)}(x) \cdot \psi(x)$ as an element of $L^2(\mathbb{R}^+)$, we have by [**40**, Thm. 5.9.5 a)] that g is an element of the Hardy space $\mathfrak{H}^2(\mathbb{C}^+)$ and since by Equation (3.2.5) we have that $g(\lambda) = 0$ for any $\lambda \in \mathbb{C}^+$, we get

$$\|g\|^2_{\mathfrak{H}^2} = \sup_{0<y<\infty} \left(\int_{-\infty}^\infty |g(x + iy)|^2 \mathrm{d}x \right) = 0\,.$$

Now, again by [**40**, Thm. 5.9.5 a)], we have that $0 = \|g\|_{\mathfrak{H}^2} = \sqrt{2\pi} \|\chi_{(1,\infty)}\psi\|_{L^2}$, which implies that $\chi_{(1,\infty)}(x)\psi(x) = 0$ almost everywhere in $L^2(\mathbb{R}^+)$. Now, let us consider all $-\lambda$, where $\operatorname{Im} \lambda > 0$, which obviously is the same as considering all λ with negative imaginary part. Equation (3.2.5) now reads as

$$0 = \langle \phi_{-\lambda}, \psi \rangle = \int_{-\infty}^{-1} e^{-i\lambda x} \psi(x) \mathrm{d}x = \int_0^\infty e^{i\lambda x} \left[\chi_{(1,\infty)}(x) \psi(-x) \right] \mathrm{d}x \quad \forall\, \lambda \in \mathbb{C}^+\,.$$

By the same reasoning as above — using [**40**, Thm. 5.9.5 a)] — we get that

$$\chi_{(1,\infty)}(x)\psi(-x) = 0$$

almost everywhere in $L^2(\mathbb{R}^+)$. Altogether, this means that $\psi(x)\chi_{(-\infty,-1)\cup(1,\infty)}(x) = 0$ almost everywhere in $L^2(\mathbb{R})$, respectively that $\psi \in L^2(-1, 1)$. This shows that $\mathfrak{M}^\perp \subset L^2(-1, 1)$.

By Proposition 3.2.6 we have shown that $p \restriction_{L^2(-1,1)}$ is selfadjoint. Indeed, it is the momentum operator on the interval with periodic boundary conditions. On the other hand, we have that $p \restriction_{L^2(-\infty,-1)\oplus L^2(1,\infty)}$ is a completely non-selfadjoint symmetric operator.

EXAMPLE 3.2.8. Let $\Omega = (0,\pi) \times (0,\infty)$ be the half-strip and consider the Hilbert space $\mathcal{H} = L^2(\Omega)$. Let the symmetric operator L be defined as

$$L: \quad \mathcal{D}(L) = \{f \in H^2(\Omega) : f \restriction_{\partial\Omega} = 0, \partial_y f \restriction_{y=0} = 0\}$$
$$f \mapsto -\Delta f = -(\partial_x^2 f + \partial_y^2 f)\,,$$

where the derivatives have to be understood in the weak sense. A short calculation shows that its adjoint L^* is given by

$$L^*: \quad \mathcal{D}(L^*) = \{f \in H^2(\Omega) : f \restriction_{x=0} = f \restriction_{x=\pi} = 0\}$$
$$f \mapsto -\Delta f = -(\partial_x^2 f + \partial_y^2 f)\,.$$

For any given $\lambda \in \mathbb{C} \setminus \mathbb{R}$ let us find elements $\zeta \in \ker(L^* - \overline{\lambda})$ using a factorization ansatz of the form $\zeta(x,y) = f(x)g(y)$:

$$-\Delta(f(x)g(y)) = -f''(x)g(y) - f(x)g''(y) = \overline{\lambda} f(x)g(y)\,. \tag{3.2.6}$$

For $n \in \mathbb{N}$ define $\phi_n(x) := sin(nx)$ and for $\mu \in \mathbb{C}^+$ define $\mu(\overline{\lambda}, n)$ to be the solution of $\mu(\overline{\lambda}, n)^2 = n^2 - \overline{\lambda}$, which has negative real part ($\mathrm{Re}\,\mu(\overline{\lambda}, n) < 0$). Note that such a solution always exists for $\overline{\lambda} \notin \mathbb{R}$ and that it is unique. Defining $\psi_{\lambda,n}(y) := e^{\mu(\overline{\lambda},n)y}$, we find that the function $\zeta_{\lambda,n}(x,y) := \phi_n(x)\psi_{\lambda,n}(y)$ is a $L^2(\Omega)$-solution to Equation (3.2.6) satisfying all boundary conditions such that it is an element of $\mathcal{D}(L^*)$. Thus, we have

$$\bigvee_{n\in\mathbb{N}} \mathrm{span}\{\zeta_{\lambda,n}\} \subset \ker(L^* - \overline{\lambda})$$

and therefore

$$\bigvee_{\lambda\in\mathbb{C}\setminus\mathbb{R}} \left(\bigvee_{n\in\mathbb{N}} \mathrm{span}\{\zeta_{\lambda,n}\} \right) \subset \mathfrak{M}\,.$$

Let us now show that $\mathfrak{M} = \mathcal{H}$, i.e. that L is completely non-selfadjoint by Proposition 3.2.6. Assume that $\chi \in \mathfrak{M}^\perp$. Observe that the following two sets are equal:

$$\{(\lambda, n) : \lambda \in \mathbb{C} \setminus \mathbb{R}, n \in \mathbb{N}\} = \{(\lambda - n^2, n) : \lambda \in \mathbb{C} \setminus \mathbb{R}, n \in \mathbb{N}\}\,,$$

which implies that

$$\bigvee_{\lambda\in\mathbb{C}\setminus\mathbb{R}} \left(\bigvee_{n\in\mathbb{N}} \mathrm{span}\{\zeta_{\lambda,n}\} \right) = \bigvee_{\lambda\in\mathbb{C}\setminus\mathbb{R}} \left(\bigvee_{n\in\mathbb{N}} \mathrm{span}\{\zeta_{\lambda-n^2,n}\} \right)\,.$$

Using Fubini's Theorem, for all $\overline{\lambda} \in \mathbb{C} \setminus \mathbb{R}$ and all $n \in \mathbb{N}$, it holds that

$$
\begin{aligned}
0 = \langle \chi, \zeta_{\lambda - n^2, n} \rangle &= \int_0^\infty \psi_{\lambda - n^2, n}(y) \left(\int_0^\pi \phi_n(x) \overline{\chi(x,y)} \mathrm{d}x \right) \mathrm{d}y \\
&= \int_0^\infty e^{y\sqrt{-\overline{\lambda}}} \left(\int_0^\pi \phi_n(x) \overline{\chi(x,y)} \mathrm{d}x \right) \mathrm{d}y \, , \qquad (3.2.7)
\end{aligned}
$$

where $\sqrt{-\overline{\lambda}}$ again denotes the solution of $z^2 = -\overline{\lambda}$ which has negative real part. By this definition of $\sqrt{-\overline{\lambda}}$ we have that $\{-i\sqrt{-\overline{\lambda}} : \lambda \in \mathbb{C} \setminus \mathbb{R}\} = \{\mu \in \mathbb{C}^+ : \operatorname{Re} \mu \neq 0\}$. Thus, Equation (3.2.7) can be rewritten in this form:

(3.2.8)

$$
0 = \int_0^\infty e^{i\mu y} \left(\int_0^\pi \phi_n(x) \overline{\chi(x,y)} \mathrm{d}x \right) \mathrm{d}y = \int_0^\infty e^{i\mu y} G^{(n)}(y) \mathrm{d}y \quad \forall n \in \mathbb{N}, \, \forall \mu \in \mathbb{C}^+ \setminus i\mathbb{R}^+ \, ,
$$

where we have defined $G^{(n)}(y) := \int_0^\pi \phi_n(x) \overline{\chi(x,y)} \mathrm{d}x$ for all $n \in \mathbb{N}$. It is not hard to check that $G^{(n)} \in L^2(0, \infty)$ for all $n \in \mathbb{N}$. If we can extend Equation (3.2.8) to hold for $\mu \in i\mathbb{R}^+$ as well for all $n \in \mathbb{N}$, we may conclude that $G^{(n)}(y) = 0$ almost everywhere in $L^2(0, \infty)$ by the same reasoning as in Example 3.2.7. To show this, pick an arbitrary sequence of non-zero real numbers $\{\sigma_j\}_j$ such that $\lim_{j \to \infty} \sigma_j = 0$. Given an arbitrary $\mu \in i\mathbb{R}^+$ consider the sequence of functions $\{e^{i(\mu + \sigma_j)y}\}_j$, which converges to $e^{i\mu y}$ pointwise. Thus,

$$
\begin{aligned}
\left| \int_0^\infty e^{i\mu y} G^{(n)}(y) \mathrm{d}y \right| &\leq \left| \int_0^\infty \left(e^{i\mu y} - e^{i(\mu + \sigma_j)y} \right) G^{(n)}(y) \mathrm{d}y \right| + \underbrace{\left| \int_0^\infty e^{i(\mu + \sigma_j)y} G^{(n)}(y) \mathrm{d}y \right|}_{\overset{(3.2.8)}{=} 0} \\
&\leq \int_0^\infty |e^{i\mu y}| |1 - e^{i\sigma_j y}| |G^{(n)}(y)| \mathrm{d}y \xrightarrow{j \to \infty} 0 \, ,
\end{aligned}
$$

which follows from dominated convergence since

$$
|e^{i\mu y}| |1 - e^{i\sigma_j y}| |G^{(n)}(y)| \leq 2 |e^{i\mu y}| |G^{(n)}(y)| \in L^1(0, \infty)
$$

for all j and all n. Thus, for almost every y it holds that

$$
G^{(n)}(y) = \int_0^\pi \phi_n(x) \overline{\chi(x,y)} \mathrm{d}x = 0
$$

for all $n \in \mathbb{N}$. Denote the set of these y's by $\mathcal{E}_n$ and define $\mathcal{E} := \bigcap_{n \in \mathbb{N}} \mathcal{E}_n$, which is a full measure set since it is the countable intersection of full measure sets. Now, for any $y \in \mathcal{E}$ we have that

$$
\langle \chi(\cdot, y), \phi_n \rangle = 0 \quad \forall n \in \mathbb{N} \, ,
$$

which implies that $\chi(x,y) = 0$ for almost every x, since $\{\phi_n\}_n = \{\sin(nx)\}_n$ is total in $L^2(0,\pi)$. Since the product of two full measure sets is again of full measure, we have shown that $\chi(x,y) = 0$ almost everywhere with respect to Lebesgue measure on Ω. This shows that $\mathfrak{M}^\perp = \{0\}$, which implies that L is completely non-selfadjoint by Proposition 3.2.6.

LEMMA 3.2.9. *Let S be symmetric and let $\widehat{S}$ be a dissipative (antidissipative) extension of S. Assume that for some subspace $\mathcal{V} \subset \mathcal{D}(\widehat{S})$ that is complementary to $\mathcal{D}(S)$ in $\mathcal{D}(\widehat{S})$, i.e.*

$$\mathcal{D}(S) \cap \mathcal{V} = \{0\} \quad \textit{and} \quad \mathcal{D}(\widehat{S}) = \mathcal{D}(S) + \mathcal{V}$$

we have that

$$Im\langle v, \widehat{S}v\rangle > 0 \qquad (Im\langle v, \widehat{S}v\rangle < 0)$$

for any nonzero $v \in \mathcal{V}$. Then, for any subspace $\mathcal{V}' \subset \mathcal{D}(\widehat{S})$ that is complementary to $\mathcal{D}(S)$ in $\mathcal{D}(\widehat{S})$, we may conclude that

$$Im\langle v', \widehat{S}v'\rangle > 0 \qquad (Im\langle v', \widehat{S}v'\rangle < 0)$$

for any nonzero $v' \in \mathcal{V}'$.

PROOF. Let $\mathcal{V}' \subset \mathcal{D}(\widehat{S})$ be complementary to $\mathcal{D}(S)$ in $\mathcal{D}(\widehat{S})$. Then, for any $v' \in \mathcal{V}'$, there exists a unique $f \in \mathcal{D}(S)$ and a unique $v \in \mathcal{V}$, such that $v' = f + v$. Note that for $v' \neq 0$, we get that $v \neq 0$. Moreover, by Theorem 3.1.2, we get that a dissipative extension $\widehat{S}$ of a symmetric operator S has to be a restriction of S^*. For any $v' \neq 0$, we then get

$$\begin{aligned}\mathrm{Im}\langle v', \widehat{S}v'\rangle &= \mathrm{Im}\langle f+v, \widehat{S}(f+v)\rangle = \mathrm{Im}\langle f, Sf\rangle + \mathrm{Im}(\langle f, S^*v\rangle + \langle v, Sf\rangle) + \mathrm{Im}\langle v, \widehat{S}v\rangle \\ &= \mathrm{Im}(2\mathrm{Re}\langle Sf, v\rangle) + \mathrm{Im}\langle v, \widehat{S}v\rangle = \mathrm{Im}\langle v, \widehat{S}v\rangle > 0\,,\end{aligned}$$

which shows the lemma for the dissipative case. □

Given a symmetric operator S, we have that the Cayley transform of any maximally dissipative (antidissipative) extension $\widehat{S}$ has to be defined on the whole Hilbert space: $\mathcal{D}(C_{\widehat{S}}(\lambda)) = \mathcal{H}$. Since $\mathcal{D}(C_{\widehat{S}}(\lambda))$ coincides with $\mathcal{D}(C_S(\lambda))$ on $\mathrm{ran}(S - \lambda)$, this means that we have to define the action of $C_{\widehat{S}}(\lambda)$ on $\ker(S^* - \overline{\lambda})$ to describe the extension.

By Lemma 3.1.1, we know that $C_{\widehat{S}}(\lambda)$ has to map $\ker(S^* - \overline{\lambda})$ into $\ker(S^* - \lambda)$. Now, since $\mathcal{H} = \operatorname{ran}(S - \lambda) \oplus \ker(S^* - \overline{\lambda})$, we get that

$$\begin{aligned}\mathcal{D}(\widehat{S}) &= (\mathbb{1} - C_{\widehat{S}}(\lambda))\mathcal{H} = (\mathbb{1} - C_S(\lambda))\operatorname{ran}(S - \lambda)\dot{+}(\mathbb{1} - C_{\widehat{S}}(\lambda))\ker(S^* - \overline{\lambda}) \\ &= \mathcal{D}(S)\dot{+}(\mathbb{1} - C_{\widehat{S}}(\lambda))\ker(S^* - \overline{\lambda}) \,.\end{aligned}$$

Thus, if $\widehat{S}$ is a maximally dissipative (antidissipative) extension of S we may define $\mathcal{V}_\lambda := (\mathbb{1} - C_{\widehat{S}}(\lambda))\ker(S^* - \overline{\lambda})$, which is complementary to $\mathcal{D}(S)$ in $\mathcal{D}(\widehat{S})$. We are now prepared to prove the following result:

Lemma 3.2.10. *Let S be symmetric and $\widehat{S}$ be a maximally dissipative (antidissipative) extension of S such that*

$$Im\langle v, \widehat{S}v\rangle > 0 \qquad (Im\langle v, \widehat{S}v\rangle < 0)$$

for all nonzero $v \in \mathcal{V}$, where $\mathcal{V} \subset \mathcal{D}(\widehat{S})$ is complementary to $\mathcal{D}(S)$ in $\mathcal{D}(\widehat{S})$. We then get that for any $\lambda \in \mathbb{C}^-$ ($\lambda \in \mathbb{C}^+$), the Cayley transform $C_{\widehat{S}}(\lambda)$ satisfies

$$\|C_{\widehat{S}}(\lambda)\phi_\lambda\| < \|\phi_\lambda\|$$

for all nonzero $\phi_\lambda \in \ker(S^* - \overline{\lambda})$.

Proof. For simplicity, we restrict ourselves to the dissipative case, since the antidissipative case can be shown completely analogously. By Lemma 3.2.9, we know that for any $\lambda \in \mathbb{C}^-$ we have that

$$\operatorname{Im}\langle v_\lambda, \widehat{S}v_\lambda\rangle > 0$$

for any nonzero $v_\lambda \in \mathcal{V}_\lambda$. Since any such v_λ can be written as $v_\lambda = (\mathbb{1} - C_{\widehat{S}}(\lambda))\phi_\lambda$ for a unique nonzero $\phi_\lambda \in \ker(S^* - \overline{\lambda})$, we then get

$$\begin{aligned}0 &< \operatorname{Im}\langle v_\lambda, \widehat{S}v_\lambda\rangle = \operatorname{Im}\langle(\mathbb{1} - C_{\widehat{S}}(\lambda))\phi_\lambda, \widehat{S}(\mathbb{1} - C_{\widehat{S}}(\lambda))\phi_\lambda\rangle \\ &= \operatorname{Im}\langle(\mathbb{1} - C_{\widehat{S}}(\lambda))\phi_\lambda, (\overline{\lambda} - \lambda C_{\widehat{S}}(\lambda))\phi_\lambda\rangle = -(\operatorname{Im}(\lambda))\left\langle\phi_\lambda, (\mathbb{1} - C_{\widehat{S}}(\lambda)^* C_{\widehat{S}}(\lambda))\phi_\lambda\right\rangle \,,\end{aligned}$$

which is equivalent to

$$\|C_{\widehat{S}}(\lambda)\phi_\lambda\| < \|\phi_\lambda\|$$

for any nonzero $\phi_\lambda \in \ker(S^* - \overline{\lambda})$. This shows the lemma. □

THEOREM 3.2.11. *Let S be symmetric and assume that*

$$\bigvee_{\lambda\in\mathbb{C}\setminus\mathbb{R}} \ker(S^* - \overline{\lambda}) = \mathcal{H} .$$

Moreover, let $\widehat{S}$ be a maximally dissipative extension of S and let $\mathcal{V} \subset \mathcal{D}(\widehat{S})$ be a subspace complementary to $\mathcal{D}(S)$ in $\mathcal{D}(\widehat{S})$. If

$$Im\langle v, \widehat{S}v\rangle > 0$$

for all nonzero $v \in \mathcal{V}$, then $\widehat{S}$ is completely non-selfadjoint.

PROOF. To begin with, let us show that for any $\lambda \in \mathbb{C}^-$, we have that $\ker(S^* - \overline{\lambda})$ is a reducing subpsace for the operator $[\mathbb{1} - C_{\widehat{S}}(\lambda)^* C_{\widehat{S}}(\lambda)]$. Clearly, this is equivalent to showing that for any $\lambda \in \mathbb{C}^-$, we have that $\ker(S^* - \overline{\lambda})$ reduces the operator $C_{\widehat{S}}(\lambda)^* C_{\widehat{S}}(\lambda)$. We start by showing $C_{\widehat{S}}(\lambda)^* C_{\widehat{S}}(\lambda) \ker(S^* - \overline{\lambda}) \subset \ker(S^* - \overline{\lambda})$. To see this, fix $\lambda \in \mathbb{C}^-$ and let ϕ_λ be an arbitrary element of $\ker(S^* - \overline{\lambda})$. Since $\widehat{S}$ is a dissipative extension of the symmetric operator S, we get by Lemma 3.1.1 that $C_{\widehat{S}}(\lambda)\phi_\lambda \in \ker(S^* - \lambda)$. Now, take any $f \in \mathcal{H}$ and decompose it into $f = f_0 + f^\perp$, where $f_0 \in \operatorname{ran}(S - \lambda)$ and $f^\perp \in \ker(S^* - \overline{\lambda})$. Thus, for any $f \in \mathcal{H}$, we get

$$\begin{aligned}\langle C_{\widehat{S}}(\lambda)^* C_{\widehat{S}}(\lambda)\phi_\lambda, f\rangle &= \langle C_{\widehat{S}}(\lambda)\phi_\lambda, C_{\widehat{S}}(\lambda) f\rangle = \langle C_{\widehat{S}}(\lambda)\phi_\lambda, C_{\widehat{S}}(\lambda)(f_0 + f^\perp)\rangle \\ &= \langle C_{\widehat{S}}(\lambda)\phi_\lambda, C_{\widehat{S}}(\lambda) f^\perp\rangle = \langle C_{\widehat{S}}(\lambda)^* C_{\widehat{S}}(\lambda)\phi_\lambda, f^\perp\rangle ,\end{aligned}$$

where we have used that $C_{\widehat{S}}(\lambda) f_0 \in \operatorname{ran}(S - \overline{\lambda})$ by definition of the Cayley transform, which means that it is orthogonal to $C_{\widehat{S}}(\lambda)\phi_\lambda$. This shows that

$$\langle C_{\widehat{S}}(\lambda)^* C_{\widehat{S}}(\lambda)\phi_\lambda, f_0\rangle = 0$$

for any $f_0 \in \operatorname{ran}(S - \lambda)$, which means that $C_{\widehat{S}}(\lambda)^* C_{\widehat{S}}(\lambda)\phi_\lambda \in \ker(S^* - \overline{\lambda})$. This shows that $\ker(S^* - \overline{\lambda})$ is invariant under $[\mathbb{1} - C_{\widehat{S}}(\lambda)^* C_{\widehat{S}}(\lambda)]$.

Let us now argue that $\ker(S^* - \overline{\lambda})^\perp = \operatorname{ran}(S - \lambda)$ is invariant under $C_{\widehat{S}}(\lambda)^* C_{\widehat{S}}(\lambda)$, too. Since S is symmetric, its Cayley transform $C_S(\lambda)$ is an isometry from $\operatorname{ran}(S - \lambda)$ to $\operatorname{ran}(S - \overline{\lambda})$. Thus, we get $C_{\widehat{S}}(\lambda)^* C_{\widehat{S}}(\lambda)\operatorname{ran}(S - \lambda) \subset \operatorname{ran}(S - \lambda)$ since $C_{\widehat{S}}(\lambda)^* C_{\widehat{S}}(\lambda)\psi = \psi$ for any $\psi \in \operatorname{ran}(S - \lambda)$. Altogether, this shows that $\ker(S^* - \overline{\lambda})$ reduces the operator $[\mathbb{1} - C_{\widehat{S}}(\lambda)^* C_{\widehat{S}}(\lambda)]$. Analogously, for any $\lambda \in \mathbb{C}^+$, it can be shown that $\ker(S^* - \overline{\lambda})$ reduces $[\mathbb{1} - C_{\widehat{S}^*}(\lambda)^* C_{\widehat{S}^*}(\lambda)]$.

Next, let us argue that for any $\lambda \in \mathbb{C}^-$, the operator

$$K_{\widehat{S}}(\lambda) := [\mathbb{1} - C_{\widehat{S}}(\lambda)^* C_{\widehat{S}}(\lambda)] \restriction_{\ker(S^*-\overline{\lambda})}$$

has range dense in $\ker(S^* - \overline{\lambda})$. (For $\lambda \in \mathbb{C}^+$, it will follow completely analogously that the range of $K_{\widehat{S}^*}(\lambda) := [\mathbb{1} - C_{\widehat{S}^*}(\lambda)^* C_{\widehat{S}^*}(\lambda)] \restriction_{\ker(S^*-\overline{\lambda})}$ is dense in $\ker(S^* - \overline{\lambda})$.) Since $C_{\widehat{S}}(\lambda)$ is the Cayley transform of a dissipative operator and therefore a contraction by Theorem 2.3.5, we have that $K_{\widehat{S}}(\lambda)$ is a non-negative selfadjoint operator in the Hilbert space $\ker(S^* - \overline{\lambda})$. Moreover, by Lemma 3.2.10, we have for any non-zero $\phi_\lambda \in \ker(S^* - \overline{\lambda})$ that

$$0 < \|\phi_\lambda\|^2 - \|C_{\widehat{S}}(\lambda)\phi_\lambda\|^2 = \langle \phi_\lambda, K_{\widehat{S}}(\lambda)\phi_\lambda \rangle = \|K_{\widehat{S}}^{1/2}(\lambda)\phi_\lambda\|^2 ,$$

which implies that $\ker(K_{\widehat{S}}(\lambda)^{1/2}) = \ker(K_{\widehat{S}}(\lambda)) = \{0\}$. Thus, $K_{\widehat{S}}(\lambda)$ is a selfadjoint operator on the Hilbert space $\ker(S^* - \overline{\lambda}) = \overline{\operatorname{ran}(K_{\widehat{S}}(\lambda))} \oplus \ker(K_{\widehat{S}}(\lambda)) = \overline{\operatorname{ran}(K_{\widehat{S}}(\lambda))}$, which shows that $K_{\widehat{S}}(\lambda)$ has range dense in $\ker(S^* - \overline{\lambda})$.

Now, assume that $\widehat{S}$ is not completely non-selfadjoint, i.e. that there exists a reducing subspace $\mathcal{M}$ for $\widehat{S}$ on which $\widehat{S}$ is selfadjoint. By Lemma 3.2.2, we have that $\mathcal{M}$ reduces $C_{\widehat{S}}(\lambda)$ for any $\lambda \in \mathbb{C}^-$. Moreover, $C_{\widehat{S}}(\lambda) \restriction_{\mathcal{M}}$ is unitary since it is the Cayley transform of a selfadjoint operator. Thus, for any $\lambda \in \mathbb{C}^-$ and any $m \in \mathcal{M}$ we get

$$\langle \phi_\lambda, m \rangle = \langle \phi_\lambda, C_{\widehat{S}}(\lambda)^* C_{\widehat{S}}(\lambda) m \rangle = \langle C_{\widehat{S}}(\lambda)^* C_{\widehat{S}}(\lambda)\phi_\lambda, m \rangle$$

for all $\phi_\lambda \in \ker(S^* - \overline{\lambda})$, which implies that

$$\langle [\mathbb{1} - C_{\widehat{S}}(\lambda)^* C_{\widehat{S}}(\lambda)]\phi_\lambda, m \rangle = 0 \tag{3.2.9}$$

for all $\phi_\lambda \in \ker(S^* - \overline{\lambda})$. But since the range of $[\mathbb{1} - C_{\widehat{S}}(\lambda)^* C_{\widehat{S}}(\lambda)] \restriction_{\ker(S^*-\overline{\lambda})}$ is dense in $\ker(S^* - \overline{\lambda})$, we get from (3.2.9) that $\mathcal{M} \perp \ker(S^* - \overline{\lambda})$ for any $\lambda \in \mathbb{C}^-$.

To finish the proof, observe that if $\mathcal{M}$ is a reducing subspace for $\widehat{S}$ on which it is selfadjoint, $\mathcal{M}$ is also a reducing subspace for the maximally antidissipative operator $\widehat{S}^*$ on which it is selfadjoint. This means that the Cayley transform $C_{\widehat{S}^*}(\lambda)$ is unitary on $\mathcal{M}$. Analogously as before, we therefore may argue that for any $\lambda \in \mathbb{C}^+$ and any $m \in \mathcal{M}$, we get that

$$\langle \phi_\lambda, m \rangle = \langle \phi_\lambda, C_{\widehat{S}^*}(\lambda)^* C_{\widehat{S}^*}(\lambda) m \rangle = \langle C_{\widehat{S}^*}(\lambda)^* C_{\widehat{S}^*}(\lambda)\phi_\lambda, m \rangle$$

for all $\phi_\lambda \in \ker(S^* - \overline{\lambda})$. We therefore get that

$$\langle [\mathbb{1} - C_{\widehat{S}^*}(\lambda)^* C_{\widehat{S}^*}(\lambda)]\phi_\lambda, m\rangle = 0$$

and since the range of $[\mathbb{1} - C_{\widehat{S}^*}(\lambda)^* C_{\widehat{S}^*}(\lambda))] \restriction_{\ker(S^*-\overline{\lambda})}$ is dense in $\ker(S^* - \overline{\lambda})$ we get that $\mathcal{M} \perp \ker(S^* - \overline{\lambda})$ for any $\lambda \in \mathbb{C}^+$. However, since we assumed that

$$\bigvee_{\lambda \in \mathbb{C}\setminus\mathbb{R}} \ker(S^* - \overline{\lambda}) = \mathcal{H}\,,$$

it follows that $\mathcal{M} \perp \mathcal{H}$, i.e. $\mathcal{M} = \{0\}$, which means that $\widehat{S}$ is completely non-selfadjoint.

□

Let us now combine the results of Proposition 3.2.6 and Theorem 3.2.11 to treat dissipative extensions of symmetric operators that may have a reducing selfadjoint part:

Corollary 3.2.12. *Let S be symmetric and assume that*

$$\bigvee_{\lambda \in \mathbb{C}\setminus\mathbb{R}} \ker(S^* - \overline{\lambda}) = \mathfrak{M} \subset \mathcal{H}$$

and let $\widehat{S}$ be a maximally dissipative extension of S. Moreover, assume that for some subspace $\mathcal{V} \subset \mathcal{D}(\widehat{S})$ that is complementary to $\mathcal{D}(S)$ in $\mathcal{D}(\widehat{S})$, it holds that

$$Im\langle v, \widehat{S}v\rangle > 0$$

for all nonzero $v \in \mathcal{V}$. Then, $\mathfrak{M}$ reduces $\widehat{S}$, where $\widehat{S} \restriction_{\mathfrak{M}}$ is completely non-selfadjoint and $\widehat{S} \restriction_{\mathfrak{M}^\perp}$ is selfadjoint.

Proof. We start by showing that $\mathfrak{M}$ reduces the operator $\widehat{S}$. By Proposition 3.2.6, we already know that $\mathfrak{M}$ reduces the symmetric operator S, which is completely non-selfadjoint on $\mathfrak{M}$ and selfadjoint on $\mathfrak{M}^\perp$. We therefore write $S = S \restriction_{\mathfrak{M}} \oplus S \restriction_{\mathfrak{M}^\perp}$ with the understanding that they act according to the decomposition $\mathcal{H} = \mathfrak{M} \oplus \mathfrak{M}^\perp$. We then get that its adjoint S^* is given by $S^* = S^* \restriction_{\mathfrak{M}} \oplus S \restriction_{\mathfrak{M}^\perp}$ and since any dissipative extension $\widehat{S}$ has to be a restriction of S^* by Theorem 3.1.2, we get that $\widehat{S}$ has to be of the form $\widehat{S} = \widehat{S} \restriction_{\mathfrak{M}} \oplus S \restriction_{\mathfrak{M}^\perp}$, where $S \restriction_{\mathfrak{M}} \subset \widehat{S} \restriction_{\mathfrak{M}} \subset S^* \restriction_{\mathfrak{M}}$ as operators in $\mathfrak{M}$. This shows that $\mathfrak{M}$ reduces $\widehat{S}$ and that $\widehat{S} \restriction_{\mathfrak{M}^\perp}$ is selfadjoint.

Let us therefore consider the symmetric operator $S\restriction_{\mathfrak{M}}$ as an operator on the Hilbert space $\mathfrak{M}$ and use Theorem 3.2.11 on $S\restriction_{\mathfrak{M}}$ to show that $\widehat{S}\restriction_{\mathfrak{M}}$ is completely non-selfadjoint. Since $\mathfrak{M}$ reduces $\widehat{S}$, we can uniquely decompose any $v \in \mathcal{V}$ as $v = v_{\mathfrak{M}} + v_{\mathfrak{M}^\perp}$, where $v_{\mathfrak{M}} \in \mathcal{D}(\widehat{S}) \cap \mathfrak{M}$ and $v_{\mathfrak{M}^\perp} \in \mathcal{D}(\widehat{S}) \cap \mathfrak{M}^\perp$ and since $S\restriction_{\mathfrak{M}^\perp} = \widehat{S}\restriction_{\mathfrak{M}^\perp}$, we have that $v_{\mathfrak{M}^\perp} \in \mathcal{D}(S)$. This means that $\mathcal{V}_{\mathfrak{M}} := P_{\mathfrak{M}}\mathcal{V}$ is complementay to $\mathcal{D}(S)$ in $\mathcal{D}(\widehat{S})$, where $P_{\mathfrak{M}}$ denotes the orthogonal projection onto $\mathfrak{M}$. We then get

$$0 < \mathrm{Im}\langle v, \widehat{S}v\rangle = \mathrm{Im}\langle v_{\mathfrak{M}} + v_{\mathfrak{M}^\perp}, \widehat{S}(v_{\mathfrak{M}} + v_{\mathfrak{M}^\perp})\rangle = \mathrm{Im}\langle v_{\mathfrak{M}}, \widehat{S}v_{\mathfrak{M}}\rangle \,,$$

for all $v_{\mathfrak{M}} \in \mathcal{V}_{\mathfrak{M}}$. This means that the operator $S\restriction_{\mathfrak{M}}$ is a symmetric operator in the Hilbert space $\mathfrak{M}$ with the maximally dissipative extension $\widehat{S}\restriction_{\mathfrak{M}}$ with $\mathcal{D}(\widehat{S}\restriction_{\mathfrak{M}}) = \mathcal{D}(S\restriction_{\mathfrak{M}}) \dot{+} \mathcal{V}_{\mathfrak{M}}$ satisfying the assumptions of Theorem 3.2.11. Thus, $\widehat{S}\restriction_{\mathfrak{M}}$ is completely non-selfadjoint. This proves the corollary. $\square$

Example 3.2.13 (Continuation of Example 3.2.7). Previously, we have shown that the symmetric momentum operator p given by (3.2.1) is reduced by $\mathfrak{M} = L^2(-\infty,-1) \oplus L^2(1,\infty)$ on which it is completely non-selfadjoint, while p is selfadjoint on $\mathfrak{M}^\perp = L^2(-1,1)$. Moreover, its adjoint p^* is given by (3.2.2). Using the the von Neumann formula for the description of $\mathcal{D}(p^*)$ (cf. [**44**, Satz 10.9 a)]) we get

$$\mathcal{D}(p^*) = \mathcal{D}(p) \dot{+} \mathrm{span}\{e^x \chi_{(-\infty,-1)}(x)\} \dot{+} \mathrm{span}\{e^{-x}\chi_{(1,\infty)}(x)\} \,,$$

which means that we can parametrize all maximally dissipative extension of p using the complex parameter ρ, where $|\rho| \leq 1$. Defining $\mathcal{V}_\rho := \mathrm{span}\{e^x\chi_{(-\infty,-1)}(x) + \rho e^{-x}\chi_{(1,\infty)}(x)\}$ and

$$p_\rho: \quad \mathcal{D}(p_\rho) = \mathcal{D}(p) \dot{+} \mathcal{V}_\rho, \quad p_\rho = p^*\restriction_{\mathcal{D}(p_\rho)}$$

describes all maximally dissipative extensions of p. Now, since

$$\mathrm{Im}\langle e^x\chi_{(-\infty,-1)}(x) + \rho e^{-x}\chi_{(1,\infty)}(x), p_\rho(e^x\chi_{(-\infty,-1)}(x) + \rho e^{-x}\chi_{(1,\infty)}(x))\rangle = \frac{1-|\rho|^2}{2} \,,$$

we get that p_ρ satisfies the assumptions of Corollary 3.2.12 for $|\rho| < 1$. (The case $|\rho| = 1$ describes the selfadjoint extensions of p.) Hence, for $|\rho| < 1$, the operator p_ρ is a maximally dissipative extension of p that is selfadjoint on $L^2(-1,1)$ and completely non-selfadjoint on $L^2(-\infty,-1) \oplus L^2(1,\infty)$.

CHAPTER 4

The closed extensions of a closed operator

In this chapter, we are going to obtain a description of all closed extensions of a given densely defined closed operator A.

4.1. The general construction

To this end, we will analyze when an extension B of A is the adjoint of a densely defined restriction of A^*. Then, the following lemma will allow us to conclude that B is closed:

LEMMA 4.1.1. *Let A be densely defined and closed. Then, there is a one-to-one correspondence between all closed extensions of A and all densely defined closed restrictions of A^*.*

PROOF. Let $A \subset B$. Then, by [**44**, Satz 4.9 a)], B is closable if and only if B^* is densely defined. In this case, we have that $\overline{B}^{\,*} = B^*$ and that B^* is a closed densely defined restriction of A^*. Let $A \subset B$ and assume that B is closed. By [**44**, Satz 4.9 a)], this implies that B^* is densely defined and since $A \subset B$ implies that $B^* \subset A^*$, we get that B^* is a densely defined closed restriction of A^*. Moreover, since for any closed operator B we have that $B = B^{**}$, this shows that $^* : B \mapsto B^*$ is a bijection between the set of all closed extensions of A and the set of all densely defined restrictions of A^*. $\square$

Let us now construct closable extensions $B_{\mathfrak{M}}$ of A, which we will parametrize using subspaces $\mathfrak{M} \subset \mathcal{D}(A^*)$:

LEMMA 4.1.2. *Let A be densely defined and closed. Moreover, let $\mathfrak{M} \subset \mathcal{D}(A^*)$ be such that*

$$\ker A^* \cap \overline{\mathfrak{M}}^{\|\cdot\|_{\Gamma(A^*)}} = \{0\} \quad \textit{and} \quad \{A^*\phi : \phi \in \overline{\mathfrak{M}}^{\|\cdot\|_{\Gamma(A^*)}}\} \cap \mathcal{D}(A) = \{0\}\,, \tag{4.1.1}$$

where $\overline{\mathfrak{M}}^{\|\cdot\|_{\Gamma(A^*)}}$ denotes the closure of $\mathfrak{M}$ with respect to the graph norm of A^*. (Recall that for any $f \in \mathcal{D}(A^*)$ its graph norm $\|f\|_{\Gamma(A^*)}$ is given by $\|f\|^2_{\Gamma(A^*)} = \|f\|^2 + \|A^*f\|^2$.) Then, the operator

$$\begin{aligned} B_{\mathfrak{M}} : \qquad & \mathcal{D}(B_{\mathfrak{M}}) = \mathcal{D}(A) \dot{+} \{A^*\phi : \phi \in \mathfrak{M}\} \\ & f + A^*\phi \mapsto Af - \phi \end{aligned}$$

is closable and its closure is given by

$$\begin{aligned} \overline{B_{\mathfrak{M}}} : \qquad & \mathcal{D}(\overline{B_{\mathfrak{M}}}) = \mathcal{D}(A) \dot{+} \{A^*\phi : \phi \in \overline{\mathfrak{M}}^{\|\cdot\|_{\Gamma(A^*)}}\} \\ & f + A^*\phi \mapsto Af - \phi \, . \end{aligned} \tag{4.1.2}$$

PROOF. Firstly, observe that the operators $B_{\mathfrak{M}}$ and $\overline{B_{\mathfrak{M}}}$ as described in the statement of the lemma are well defined, which follows from the assumptions made on $\mathfrak{M}$. Next, let us show that $\mathcal{D}(A)$ and $\{A^*\phi : \phi \in \mathfrak{M}\}$ are orthogonal with respect to the inner product induced by the graph norm of $B_{\mathfrak{M}}$:

$$\langle f, A^*\phi\rangle_{\Gamma(B_{\mathfrak{M}})} = \langle f, A^*\phi\rangle + \langle B_{\mathfrak{M}} f, B_{\mathfrak{M}} A^*\phi\rangle = \langle f, A^*\phi\rangle + \langle Af, -\phi\rangle = 0 \, .$$

This implies that

$$\Gamma(B_{\mathfrak{M}}) = \Gamma(A) \oplus' \{(A^*\phi, -\phi) : \phi \in \mathfrak{M}\} \, ,$$

where $\oplus'$ denotes the orthogonal sum in $\mathcal{H} \oplus \mathcal{H}$. Closing with respect to the norm of $\mathcal{H} \oplus \mathcal{H}$ therefore yields

$$\overline{\Gamma(B_{\mathfrak{M}})} = \overline{\Gamma(A)} \oplus' \overline{\{(A^*\phi, -\phi) : \phi \in \mathfrak{M}\}} \, .$$

Since A is closed by assumption, we get that $\overline{\Gamma(A)} = \Gamma(A)$. Let us now show that

$$\overline{\{(A^*\phi, -\phi) : \phi \in \mathfrak{M}\}} = \{(A^*\phi, -\phi) : \phi \in \overline{\mathfrak{M}}^{\|\cdot\|_{\Gamma(A^*)}}\} \, .$$

We begin by showing the " $\subset$ " inclusion:

Let $(\psi, -\chi) \in \overline{\{(A^*\phi, -\phi) : \phi \in \mathfrak{M}\}}$, which means that there exists a sequence $\{(A^*\phi_n, -\phi_n)\}_n$, where $\{\phi_n\}_n \subset \mathfrak{M}$, such that

$$\|\psi - A^*\phi_n\|^2 + \|\chi - \phi_n\|^2 \overset{n\to\infty}{\longrightarrow} 0 \, ,$$

which means in particular that $\phi_n \to \chi$ and $A^*\phi_n \to \psi$. Since A^* is closed, this implies that $\chi \in \mathcal{D}(A^*)$ and $\psi = A^*\chi$. Hence, any element of $\overline{\{(A^*\phi, -\phi) : \phi \in \mathfrak{M}\}}$ is actually of the form $(A^*\chi, -\chi)$ where $\chi \in \mathcal{D}(A^*)$. Furthermore, there exists a sequence $\{\phi_n\}_n \subset \mathfrak{M}$ such that

$$\|A^*(\chi - \phi_n)\|^2 + \|\chi - \phi_n\|^2 = \|\chi - \phi_n\|^2_{\Gamma(A^*)} \stackrel{n\to\infty}{\longrightarrow} 0 \,,$$

which means that $\chi \in \overline{\mathfrak{M}}^{\|\cdot\|_{\Gamma(A^*)}}$.

Next, let us show the " $\supset$ " inclusion:

To see this, we need to show that if $\phi \in \overline{\mathfrak{M}}^{\|\cdot\|_{\Gamma(A^*)}}$, this implies that $(A^*\phi, -\phi) \in \overline{\{(A^*\phi, -\phi) : \phi \in \mathfrak{M}\}}$. But if $\phi \in \overline{\mathfrak{M}}^{\|\cdot\|_{\Gamma(A^*)}}$, there exists a sequence $\{\phi_n\}_n \subset \mathfrak{M}$ such that

$$\|\phi - \phi_n\|^2_{\Gamma(A^*)} \stackrel{n\to\infty}{\longrightarrow} 0$$

and since

$$\|\phi - \phi_n\|^2_{\Gamma(A^*)} = \|\phi - \phi_n\|^2 + \|A^*(\phi - \phi_n)\|^2 = \|(A^*\phi, -\phi) - (A^*\phi_n, -\phi_n)\|^2_{\mathcal{H}\oplus\mathcal{H}} \,,$$

this shows that $(A^*\phi, -\phi) \in \overline{\{(A^*\phi, -\phi) : \phi \in \mathfrak{M}\}}$. We therefore have shown that

$$\overline{\Gamma(B_{\mathfrak{M}})} = \Gamma(A) \oplus' \{A^*\phi : \phi \in \overline{\mathfrak{M}}^{\|\cdot\|_{\Gamma(A^*)}}\} \,.$$

Let us finish by arguing that $\overline{\Gamma(B_{\mathfrak{M}})}$ is the graph of an operator, which means that we need to show that $(0, g) \in \overline{\Gamma(B_{\mathfrak{M}})}$ implies that $g = 0$. But any element of $\overline{\Gamma(B_{\mathfrak{M}})}$ is of the form $(f + A^*\phi, Af - \phi)$, where $f \in \mathcal{D}(A)$ and $\phi \in \overline{\mathfrak{M}}^{\|\cdot\|_{\Gamma(A^*)}}$. Moreover, by (4.1.1), we have that $f + A^*\phi = 0$ if and only if $f = 0$ and $A^*\phi = 0$. Since — again by (4.1.1) — we have that $A^*\phi = 0$ if and only if $\phi = 0$, this yields that $(f + A^*\phi, Af - \phi) = (0, Af - \phi) = (0, 0)$, which implies that $\overline{\Gamma(B_{\mathfrak{M}})}$ is the graph of the closure $\overline{B_{\mathfrak{M}}}$ of $B_{\mathfrak{M}}$, which is given by (4.1.2). This implies in particular that $B_{\mathfrak{M}}$ is closable and thus the lemma. □

The following equivalent description of $B_{\mathfrak{M}}$ will be useful later:

Corollary 4.1.3. *Let the operator $D_{\mathfrak{M}}$ be given by:*

$$D_{\mathfrak{M}} : \qquad \mathcal{D}(D_{\mathfrak{M}}) = \{f \in \mathcal{H} : \exists \phi \in \mathfrak{M} \text{ such that } f - A^*\phi \in \mathcal{D}(A)\}$$

$$D_{\mathfrak{M}} f = A(f - A^*\phi) - \phi \,.$$

Then, $D_{\mathfrak{M}} = B_{\mathfrak{M}}$.

PROOF. "$B_{\mathfrak{M}} \subset D_{\mathfrak{M}}$": Any $f_0 + A^*\phi$ with $f_0 \in \mathcal{D}(A)$ and $\phi \in \mathfrak{M}$ is also in $\mathcal{D}(D_{\mathfrak{M}})$ as $(f_0 + A^*\phi - A^*\phi) \in \mathcal{D}(A)$. Now, consider

$$D_{\mathfrak{M}}(f_0 + A^*\phi) = A(f_0 + A^*\phi - A^*\phi) - \phi = Af_0 - \phi = B_{\mathfrak{M}}(f_0 + A^*\phi) ,$$

which shows the first inclusion.
"$B_{\mathfrak{M}} \supset D_{\mathfrak{M}}$": Observe that for any $f \in \mathcal{D}(D_{\mathfrak{M}})$, there exists a $\phi \in \mathfrak{M}$ such that f can be written as $f = (f - A^*\phi) + A^*\phi$, where $(f - A^*\phi) \in \mathcal{D}(A)$. This implies that $f \in \mathcal{D}(B_{\mathfrak{M}})$ as well. To finish the proof, consider

$$B_{\mathfrak{M}}f = B_{\mathfrak{M}}(f - A^*\phi + A^*\phi) = A(f - A^*\phi) - \phi = D_{\mathfrak{M}}f .$$

□

Next, let us construct the adjoint of $B_{\mathfrak{M}}$:

LEMMA 4.1.4. *Let $\mathfrak{M} \subset \mathcal{D}(A^*)$ and $B_{\mathfrak{M}}$ be defined as in Lemma 4.1.2. Moreover, let the operator $C_{\mathfrak{M}}$ be defined as:*

$$C_{\mathfrak{M}} : \qquad \mathcal{D}(C_{\mathfrak{M}}) = \{f \in \mathcal{D}(A^*) : \langle f, \phi\rangle + \langle A^*f, A^*\phi\rangle = 0 \text{ for all } \phi \in \mathfrak{M}\}$$
$$C_{\mathfrak{M}} = A^* \restriction_{\mathcal{D}(C_{\mathfrak{M}})} .$$

Then, $B_{\mathfrak{M}}^* = C_{\mathfrak{M}}$.

PROOF. "$C_{\mathfrak{M}} \subset B_{\mathfrak{M}}^*$": Let $g \in \mathcal{D}(C_{\mathfrak{M}})$, $f \in \mathcal{D}(A)$ and $\phi \in \mathfrak{M}$ and consider

$$\langle g, B_{\mathfrak{M}}(f + A^*\phi)\rangle = \langle g, Af - \phi\rangle = \langle A^*g, f + A^*\phi\rangle ,$$

where we have used that $g \in \mathcal{D}(A^*)$ and $-\langle g, \phi\rangle = \langle A^*g, A^*\phi\rangle$. This shows that $g \in \mathcal{D}(B_{\mathfrak{M}}^*)$ and $B_{\mathfrak{M}}^*g = A^*g = C_{\mathfrak{M}}g$.
"$C_{\mathfrak{M}} \supset B_{\mathfrak{M}}^*$": Let $g \in \mathcal{D}(B_{\mathfrak{M}}^*)$, which means that there exists a $\widetilde{g} \in \mathcal{H}$ such that

$$\langle \widetilde{g}, f + A^*\phi\rangle = \langle g, B_{\mathfrak{M}}(f + A^*\phi)\rangle = \langle g, Af - \phi\rangle \tag{4.1.3}$$

for all $f \in \mathcal{D}(A)$ and all $\phi \in \mathfrak{M}$. This holds in particular for the choice $\phi = 0$, from which we get that

$$\langle \widetilde{g}, f\rangle = \langle g, B_{\mathfrak{M}}f\rangle = \langle g, Af\rangle$$

for all $f \in \mathcal{D}(A)$. This implies that $g \in \mathcal{D}(A^*)$ and that $\widetilde{g} = A^* g$. Now, consider again Equation (4.1.3):

$$\langle A^* g, f + A^* \phi \rangle = \langle \widetilde{g}, f + A^* \phi \rangle = \langle g, Af - \phi \rangle = \langle A^* g, f \rangle - \langle g, \phi \rangle \,,$$

which implies that

$$\langle g, \phi \rangle + \langle A^* g, A^* \phi \rangle = 0$$

for all $\phi \in \mathfrak{M}$. This shows that $g \in \mathcal{D}(C_{\mathfrak{M}})$ and $B_{\mathfrak{M}}^* g = A^* g = C_{\mathfrak{M}} g$, from which the lemma follows. $\square$

Let us now analyze when the operator $C_{\mathfrak{M}}$ is a densely defined restriction of A^*:

Theorem 4.1.5. *The operator $C_{\mathfrak{M}}$ as defined in Lemma 4.1.4 is a closed restriction of A^*. Moreover, $C_{\mathfrak{M}}$ is densely defined if and only if*

$$\ker A^* \cap \overline{\mathfrak{M}}^{\|\cdot\|_{\Gamma(A^*)}} = \{0\} \quad \textit{and} \quad \{A^* \phi : \phi \in \overline{\mathfrak{M}}^{\|\cdot\|_{\Gamma(A^*)}}\} \cap \mathcal{D}(A) = \{0\} \tag{4.1.4}$$

Proof. The fact that $C_{\mathfrak{M}}$ is a restriction of A^* follows immediately from its definition. Since A is densely defined and $A \subset B_{\mathfrak{M}}$, it trivially follows that $B_{\mathfrak{M}}$ is densely defined as well. Thus, $C_{\mathfrak{M}}$ is closed because it is the adjoint of the densely defined operator $B_{\mathfrak{M}}$.

Let us now show that Condition (4.1.4) is necessary for $C_{\mathfrak{M}}$ to be densely defined. Assume that there exists a $0 \neq \phi \in \overline{\mathfrak{M}}^{\|\cdot\|_{\Gamma(A^*)}}$ such that $A^* \phi \in \mathcal{D}(A)$. This would mean that there exists a sequence $\{\phi_n\}_n \subset \mathfrak{M}$ such that

$$\lim_{n \to \infty} \left(\|\phi_n - \phi\|^2 + \|A^* \phi_n - A^* \phi\|^2 \right) = 0 \,.$$

Since for any $n \in \mathbb{N}$ and any $f \in \mathcal{D}(C_{\mathfrak{M}})$ we have

$$\langle f, \phi_n \rangle + \langle A^* f, A^* \phi_n \rangle = 0$$

and

$$\langle f, \phi \rangle + \langle A^* f, A^* \phi \rangle = \lim_{n \to \infty} \left(\langle f, \phi_n \rangle + \langle A^* f, A^* \phi_n \rangle \right) = 0 \,,$$

we obtain the condition

$$\langle f, (\mathbb{1} + AA^*) \phi \rangle = 0$$

for all $f \in \mathcal{D}(C_{\mathfrak{M}})$. This means that $\mathcal{D}(C_{\mathfrak{M}}) \perp \operatorname{span}\{(\mathbb{1} + AA^*)\phi\}$, which implies that $C_{\mathfrak{M}}$ is not densely defined.
Let us now show that Condition (4.1.4) is sufficient for $C_{\mathfrak{M}}$ to be densely defined. By Lemma 4.1.2, $B_{\mathfrak{M}}$ is closable and by Lemma 4.1.4, $C_{\mathfrak{M}} = B_{\mathfrak{M}}^*$. Thus, since $C_{\mathfrak{M}}$ is the adjoint of a closable operator, it is densely defined by [**44**, Satz 4.9 a)]. $\square$

Let us summarize all the previous results with the following:

THEOREM 4.1.6. *Let A be a densely defined and closed operator. Then, there is a one-to-one correspondence between all closed extensions of $B_{\mathfrak{M}}$ of A and all subspaces $\mathfrak{M} \subset \mathcal{D}(A^*)$ that are closed with respect to the graph norm $\|\cdot\|_{\Gamma(A^*)}$ and that satisfy the conditions given in* (4.1.1)*. The operator $B_{\mathfrak{M}}$ is given by*

$$B_{\mathfrak{M}}: \qquad \mathcal{D}(B_{\mathfrak{M}}) = \mathcal{D}(A) \dot{+} \{A^*\phi : \phi \in \mathfrak{M}\}$$

$$f + A^*\phi \mapsto Af - \phi\,. \tag{4.1.5}$$

PROOF. Let B be any closed extension of A. By [**44**, Satz 4.9 a)], this implies that B^* is densely defined and since $B^* \subset A^*$, this means that B^* is a closed densely defined restriction of A^*. Thus,

$$\Gamma := \Gamma(A^*) \ominus \Gamma(B^*)$$

is a closed subspace of $\Gamma(A^*)$ and moreover we have $\Gamma(B^*) = \Gamma(A^*) \ominus \Gamma = \Gamma(A^*) \cap \Gamma^{\perp}$. Defining $\mathfrak{M} := \{\phi \in \mathcal{D}(A^*) : (\phi, A^*\phi) \in \Gamma\}$, we then may write

$$B^*: \quad \mathcal{D}(B^*) = \{f \in \mathcal{D}(A^*) : \langle f, \phi\rangle + \langle A^*f, A^*\phi\rangle = 0 \text{ for all } \phi \in \mathfrak{M}\}$$

$$B^* = A^* \restriction_{\mathcal{D}(B^*)}\,. \tag{4.1.6}$$

Moreover, since Γ is closed in $\mathcal{H} \oplus \mathcal{H}$, observe that $\mathfrak{M}$ is closed with respect to the graph norm $\|\cdot\|_{\Gamma(A^*)}$, since for any $\phi \in \mathfrak{M}$ we have

$$\|\phi\|^2_{\Gamma(A^*)} = \|\phi\|^2 + \|A^*\phi\|^2 = \|(\phi, A^*\phi)\|^2_{\mathcal{H}\oplus\mathcal{H}}\,.$$

Now, (4.1.6) means that $B^* \equiv C_{\mathfrak{M}}$, where $C_{\mathfrak{M}}$ is defined as in Lemma 4.1.4. By Theorem 4.1.5, $B^* = C_{\mathfrak{M}}$ being densely defined implies that $\mathfrak{M}$ satisfies the conditions from (4.1.1). Moreover, by Lemma 4.1.4, we have that $B_{\mathfrak{M}}^* = C_{\mathfrak{M}}$, where $B_{\mathfrak{M}}$ is given by (4.1.5). Also, since $\mathfrak{M}$ is closed with respect to the graph norm $\|\cdot\|_{\Gamma(A^*)}$, we have by Lemma 4.1.2 that $B_{\mathfrak{M}}$ is closed. Finally, since $C_{\mathfrak{M}} = B_{\mathfrak{M}}^* = B^*$ and B as well as $B_{\mathfrak{M}}$ are closed, we get that $B \equiv B_{\mathfrak{M}}$, i.e. any closed extension B of A is of the form $B = B_{\mathfrak{M}}$, where $\mathfrak{M}$ is a subspace of $\mathcal{D}(A^*)$ that is closed with respect to the graph norm $\|\cdot\|_{\Gamma(A^*)}$ and satisfies the conditions given by (4.1.1). This finishes the proof. $\square$

4.2. The finite-dimensional case

In the finite-dimensional case, we have also a direct proof that $C_{\mathfrak{M}}$ is densely defined. It is an abstract generalization of a result shown in [**38**, Hilfssatz 1].

THEOREM 4.2.1. *Let A^* be a densely defined closed operator on a Hilbert space $\mathcal{H}$. Moreover, for some non-zero $\phi \in \mathcal{D}(A^*)$ let the set $\mathcal{D}$ be defined as*

$$\mathcal{D} := \{f \in \mathcal{D}(A^*) : \langle f, \phi\rangle + \langle A^* f, A^*\phi\rangle = 0\} .$$

Then $\mathcal{D}$ is dense if and only if $A^*\phi \notin \mathcal{D}(A)$.

PROOF. As in the proof of Theorem 4.1.5, it is easy to show that $A^*\phi \notin \mathcal{D}(A)$ is necessary for $\mathcal{D}$ to be dense. Since $A^*\phi \in \mathcal{D}(A)$ would imply that the condition $\langle f, \phi\rangle + \langle A^* f, A^*\phi\rangle = 0$ could be rewritten as $\langle f, (\mathbb{1} + AA^*)\phi\rangle = 0$, we would get that $\mathcal{D} \perp \operatorname{span}\{(\mathbb{1} + AA^*)\phi\}$, which would mean that $\mathcal{D}$ would not be dense.
To show that it is sufficient, assume that $A^*\phi \notin \mathcal{D}(A)$ and that $\mathcal{D}$ is not dense, i.e. that there exists a $\psi \in \mathcal{H}$ such that $\langle \psi, f\rangle = 0$ for all $f \in \mathcal{D}$. Now, fix any $g \in \mathcal{D}(A^*)$. Moreover, since $A^*\phi \notin \mathcal{D}(A)$, there exists a sequence $\{w_n\}_n \subset \mathcal{D}(A^*)$ with $\|w_n\| = 1$ such that

$$\lim_{n\to\infty} |\langle A^* w_n, A^*\phi\rangle| = \infty . \tag{4.2.1}$$

Define the numbers z_n by

$$\overline{z_n} := -\frac{\langle g, \phi\rangle + \langle A^* g, A^*\phi\rangle}{\langle w_n, \phi\rangle + \langle A^* w_n, A^*\phi\rangle} , \tag{4.2.2}$$

which implies that $g + z_n w_n \in \mathcal{D}$ for all n. (Observe that this expression is certainly well-defined for sufficiently large n as the second term in the denominator goes to infinity.) Using Equation (4.2.1), we get that

$$\lim_{n\to\infty} |z_n| = 0 . \tag{4.2.3}$$

Now, since $g + z_n w_n \in \mathcal{D}$ and $\psi \perp \mathcal{D}$, we get $\langle \psi, g + z_n w_n\rangle = 0$, which implies that

$$|\langle \psi, g\rangle| = |\langle \psi, z_n w_n\rangle| \le |z_n|\|\psi\|\|w_n\| = |z_n|\|\psi\| \overset{n\to\infty}{\longrightarrow} 0 .$$

However, since $g \in \mathcal{D}(A^*)$ was arbitrary, this implies that $\psi \perp \mathcal{D}(A^*)$, i.e. $\psi = 0$ since $\mathcal{D}(A^*)$ is dense. $\square$

Let us now generalize our result to arbitrary finite-dimensional restrictions of A^*:

COROLLARY 4.2.2. *Let $\mathfrak{M} \subset \mathcal{D}(A^*)$ be finite-dimensional. The set*

$$\mathcal{D}_{\mathfrak{M}} = \{f \in \mathcal{D}(A^*) : \langle f, \phi \rangle + \langle A^* f, A^* \phi \rangle = 0 \ \forall \phi \in \mathfrak{M}\}$$

is dense if and only if for all non-zero $\phi \in \mathfrak{M}$, we have $A^*\phi \notin \mathcal{D}(A)$.

PROOF. Again, it is obvious that the condition $A^*\phi \notin \mathcal{D}(A)$ for all $\phi \in \mathfrak{M}$ is necessary for $\mathcal{D}_{\mathfrak{M}}$ to be dense. We use induction over the dimension of $\mathfrak{M}$ to show that it is also sufficient. The base case corresponding to $\dim \mathfrak{M} = 1$ has been shown in Theorem 4.2.1. Let us now show that if for a subspace $\mathfrak{N} \subset \mathfrak{M}$ with $\dim(\mathfrak{M} \ominus \mathfrak{N}) = 1$, we have that $\mathcal{D}_{\mathfrak{N}}$ is dense, this also implies that $\mathcal{D}_{\mathfrak{M}}$ is dense. To this end, observe firstly that $\mathcal{D}_{\mathfrak{M}}$ can be rewritten as

$$\mathcal{D}_{\mathfrak{M}} = \{f \in \mathcal{D}_{\mathfrak{N}} : \langle f, \xi \rangle + \langle A^* f, A^* \xi \rangle = 0\} \,,$$

where ξ spans the one-dimensional space $\mathfrak{M} \ominus \mathfrak{N}$. We may now mimic the proof of Theorem 4.2.1, where we only have to take care of the fact that g must be chosen to be an element of $\mathcal{D}_{\mathfrak{N}}$ and the normalized sequence $\{w_n\}_n$ has to lie in $\mathcal{D}_{\mathfrak{N}}$ as well. It is still possible to choose $\{w_n\}_n$ such that $\lim_{n\to\infty} |\langle A^* w_n, A^* \xi \rangle| = \infty$ since otherwise we would have that the functional $w \mapsto \langle A^* w, A^* \xi \rangle$ would be bounded on a dense set, which would contradict the assumption that $A^*\xi \notin \mathcal{D}(A)$. Again, we would get that any ψ orthogonal to $\mathcal{D}_{\mathfrak{M}}$ would have to be orthogonal to all $g \in \mathcal{D}_{\mathfrak{N}}$, from which we would get that $\psi = 0$ and thus the corollary. $\square$

4.3. A few examples

Let us illustrate the results of the previous two sections with a few examples.

EXAMPLE 4.3.1 (One dimensional restrictions of a selfadjoint operator). Let $A = A^*$ be selfadjoint. From Theorem 4.1.6, it follows that all restrictions $C_\phi \subset A^*$ with

$\dim(\mathcal{D}(A^*)/\mathcal{D}(C_\phi)) = 1$ can be described via

$$\begin{aligned} C_\phi : \quad \mathcal{D}(C_\phi) &= \{f \in \mathcal{D}(A) : \langle \phi, f\rangle + \langle A^*\phi, A^* f\rangle = 0\} \\ &= \{f \in \mathcal{D}(A) : \langle \phi, f\rangle + \langle A\phi, A f\rangle = 0\} \\ C_\phi &= A \restriction_{\mathcal{D}(C_\phi)} , \end{aligned}$$

where $0 \neq \phi \in \mathcal{D}(A) = \mathcal{D}(A^*)$ has to be such that $A^*\phi = A\phi \notin \mathcal{D}(A)$, in order to ensure that C_ϕ is densely defined (cf. Theorem 4.1.5 or 4.2.1). Moreover, since A is a selfadjoint extension of C_ϕ, it is clear that C_ϕ has to be symmetric, which implies that $C_\phi \subset C_\phi^*$, where C_ϕ^* is given by

$$\begin{aligned} C_\phi^* : \qquad \mathcal{D}(C_\phi^*) &= \mathcal{D}(A) \dot{+} \operatorname{span}\{A\phi\} \\ f + \lambda A\phi &\mapsto Af - \lambda\phi . \end{aligned}$$

In order to determine all selfadjoint and maximally dissipative extensions of C_ϕ, let us firstly compute the defect spaces $\ker(C_\phi^* \mp i)$:

$$\begin{aligned} &0 = (C_\phi^* \mp i)(f + \lambda A\phi) = (A \mp i)f + \lambda(-\phi \mp iA\phi) \Leftrightarrow \\ &f = \lambda(A \mp i)^{-1}(\phi \pm iA\phi) , \end{aligned}$$

which implies that

$$\ker(C_\phi^* \mp i) = \operatorname{span}\{(A \mp i)^{-1}(\phi \pm iA\phi) + A\phi\} = \operatorname{span}\{(A \pm i)\phi\} .$$

By Lemma 3.1.1, we know that all maximally dissipative extensions of C_ϕ can be parametrized by contractions from $\ker(C_\phi^* - i)$ into $\ker(C_\phi^* + i)$ and are therefore given by

$$\begin{aligned} C_{\phi,\rho} : \qquad & \mathcal{D}(C_{\phi,\rho}) = \mathcal{D}(C_\phi) \dot{+} \operatorname{span}\{(A+i)\phi + \rho(A-i)\phi\} \\ & f + \lambda((A+i)\phi + \rho(A-i)\phi) \mapsto Af + i\lambda((A+i)\phi - \rho(A-i)\phi) , \end{aligned} \tag{4.3.1}$$

where $|\rho| \leq 1$ describes maximally dissipative extensions of C_ϕ and — more precisely — $|\rho| = 1$ selfadjoint ones. Note that, independently to the choice of ϕ, we have $C_{\phi,-1} = A$. This follows from the fact that $\phi \notin \mathcal{D}(C_\phi)$ but $\mathcal{D}(C_{\phi,-1}) = \mathcal{D}(C_\phi) \dot{+} \operatorname{span}\{\phi\} \subset \mathcal{D}(A)$, from which we get equality by a dimension counting argument. Let us now determine the resolvents of the extensions $C_{\phi,\rho}$, which have to coincide on $\operatorname{ran}(C_\phi+i) = \operatorname{span}\{(A+i)\phi\}^\perp$. Moreover, since we have

$$(C_{\phi,\rho}+i)\,[(A+i)\phi+\rho(A-i)\phi] = 2i(A+i)\phi \in \ker(C_\phi^*-i)\;,$$

we get that

$$(C_{\phi,\rho}+i)^{-1}(A+i)\phi = \frac{1}{2i}\,[(A+i)\phi+\rho(A-i)\phi]\;. \tag{4.3.2}$$

Hence, since $(C_{\phi,\rho}+i)^{-1}\restriction_{\operatorname{ran}(C_\phi+i)} = (A+i)^{-1}\restriction_{\operatorname{ran}(C_\phi+i)}$ and by (4.3.2), we get

$$\left[(C_{\phi,\rho}+i)^{-1}-(A+i)^{-1}\right](A+i)\phi = \frac{1+\rho}{2i}(A-i)\phi\,,$$

which implies that — as an identity of operators — we have

$$(C_{\phi,\rho}+i)^{-1} = (A+i)^{-1} + \frac{1+\rho}{2i}|(A-i)\phi\rangle\langle(A+i)\phi|\;. \tag{4.3.3}$$

Altogether, this shows the following:

THEOREM 4.3.2. *Let $A=A^*$ be selfadjoint. Then, any maximally dissipative (selfadjoint) operator $C_{\phi,\rho}$ whose resolvent $(C_{\phi,\rho}+i)^{-1}$ differs from $(A+i)^{-1}$ by a rank-one operator is given by* (4.3.1)*, where $|\rho|\leq 1$ ($|\rho|=1$) and $0\neq\phi\in\mathcal{D}(A)$ is such that $A\phi\notin\mathcal{D}(A)$. Moreover, the resolvents $(C_{\phi,\rho}+i)^{-1}$ are given by* (4.3.3).

REMARK 4.3.3. Clearly, this idea can be generalized to resolvent differences of higher rank. We only consider the rank-one case for simplicity of presentation.

EXAMPLE 4.3.4 (A Friedrichs model operator). Let $A=A^*$ be the selfadjoint maximal multiplication operator by x:

$$A: \qquad \mathcal{D}(A) = \left\{f\in L^2(\mathbb{R}) : \int_{\mathbb{R}} x^2|f(x)|^2\mathrm{d}x<\infty\right\}$$
$$(Af)(x) = xf(x)\,.$$

Now, let us choose $\phi(x):=\frac{1}{1+x^2}$, where the function $\phi(x)=\frac{1}{1+x^2}\in\mathcal{D}(A^*)$ but $(A^*\phi)(x)=(A\phi)(x)=\frac{x}{1+x^2}\notin\mathcal{D}(A)=\mathcal{D}(A^*)$, which means that $\mathfrak{M}:=\operatorname{span}\{\phi\}$ satisfies the assumptions of Theorem 4.1.6. (Obviously, since $\mathfrak{M}$ is one-dimensional, it is closed with respect to the graph norm $\|\cdot\|_{\Gamma(A^*)}$.) Hence, the operator given by

$$B_\phi : \qquad \mathcal{D}(B_\phi) = \mathcal{D}(A) \dot{+} \mathrm{span}\left\{\frac{x}{1+x^2}\right\}$$
$$f(x) + \lambda\left(\frac{x}{1+x^2}\right) \mapsto xf(x) - \lambda\left(\frac{1}{1+x^2}\right)$$

is a closed extension of A. Next, let us compute the adjoint of B_ϕ, which we know is densely defined. Firstly, let us use Lemma 4.1.4 in order to determine

$$\mathcal{D}(B_\phi^*) = \left\{f \in \mathcal{D}(A) : \left\langle f, \frac{1}{1+x^2}\right\rangle + \left\langle xf, \frac{x}{1+x^2}\right\rangle = 0\right\} .$$

However, since

$$\left\langle f, \frac{1}{1+x^2}\right\rangle + \left\langle xf, \frac{x}{1+x^2}\right\rangle = \int_{\mathbb{R}} \frac{\overline{f(x)}}{1+x^2}\mathrm{d}x + \int_{\mathbb{R}} \frac{x^2\overline{f(x)}}{1+x^2}\mathrm{d}x = \int_{\mathbb{R}} \overline{f(x)}\mathrm{d}x ,$$

we can describe B_ϕ^* as

$$B_\phi^* : \qquad \mathcal{D}(B_\phi^*) = \left\{f \in L^2(\mathbb{R}) : xf(x) \in L^2(\mathbb{R}) \text{ and } \int_{\mathbb{R}} f(x)\mathrm{d}x = 0\right\}$$
$$f(x) \mapsto xf(x) .$$

Moreover, in order to be in accordance with the notation of Theorem 4.3.2, define $C_\phi := B_\phi^*$. Then, by the same theorem, all maximally dissipative extensions of $C_\phi \subset C_{\phi,\rho}$ are given by

$$\begin{aligned} C_{\phi,\rho} : \quad & \mathcal{D}(C_{\phi,\rho}) \\ &= \left\{f \in L^2(\mathbb{R}) : xf(x) \in L^2(\mathbb{R}), \int_{\mathbb{R}} f(x) = 0\right\} \dot{+} \mathrm{span}\left\{\frac{(x+i)+\rho(x-i)}{1+x^2}\right\} \\ f(x) + \lambda & \frac{(x+i)+\rho(x-i)}{1+x^2} \mapsto xf(x) + i\lambda\frac{(x+i)-\rho(x-i)}{1+x^2} , \end{aligned}$$

where $|\rho| \leq 1$. For any $\psi \in L^2(\mathbb{R})$, the resolvents $(C_{\phi,\rho}+i)^{-1}$ act as follows

$$\left((C_{\phi,\rho}+i)^{-1}\psi\right)(x) = \frac{\psi(x)}{x+i} + \frac{1+\rho}{2i}\frac{1}{x+i}\int_{\mathbb{R}} \frac{\psi(t)}{t+i}\mathrm{d}t .$$

Example 4.3.5 (An infinite-dimensional example). Let us now construct two examples, where $\mathfrak{M}$ has infinite dimension and where for any $0 \neq \phi \in \mathfrak{M}$ we have that $A^*\phi \notin \mathcal{D}(A)$, but only one of them describes a closable extension of A, respectively a densely defined restriction of A^*. Firstly, for any $\lambda \in \mathbb{R}$, let us define the function $\phi_\lambda(x) := \exp(-|x-\lambda|)$. Now, define

$$\mathfrak{M}_{\mathbb{Z}} = \operatorname{span}\{\phi_\lambda : \lambda \in \mathbb{Z}\}$$
$$\mathfrak{M}_{\mathbb{Q}} = \operatorname{span}\{\phi_\lambda : \lambda \in \mathbb{Q}\}$$

i.e. the set of finite linear combinations of vectors ϕ_λ, where $\lambda \in \mathbb{Z}, \mathbb{Q}$. Consider the selfadjoint momentum operator $A = A^*$ on the real axis:

$$A : \qquad \mathcal{D}(A) = H^1(\mathbb{R})$$
$$f \mapsto if' \,.$$

Observe that $(A^*\phi_\lambda)(x) = -\operatorname{sgn}(x-\lambda)\exp(-|x-\lambda|) \notin \mathcal{D}(A) = H^1(\mathbb{R})$ for any $\lambda \in \mathbb{R}$. Now, let us show that $\mathfrak{M}_{\mathbb{R}} := \operatorname{span}\{\phi_\lambda : \lambda \in \mathbb{R}\} \subset \overline{\mathfrak{M}_{\mathbb{Q}}}^{\|\cdot\|_{\Gamma(A^*)}}$. To see this let $\lambda \in \mathbb{R}$ and just pick any sequence $\{\lambda_n\}_n \subset \mathbb{Q}$ such that $\lambda_n \to \lambda$ and consider

$$\begin{aligned}&\lim_{n\to\infty} \|\phi_\lambda - \phi_{\lambda_n}\|^2_{\Gamma(A^*)}\\ &= \lim_{n\to\infty} \left(\|e^{-|x-\lambda|} - e^{-|x-\lambda_n|}\|^2 + \|\operatorname{sgn}(x-\lambda)e^{-|x-\lambda|} - \operatorname{sgn}(x-\lambda_n)e^{-|x-\lambda_n|}\|^2\right) = 0\,,\end{aligned}$$

which follows from dominated convergence. Next, let us determine $\mathfrak{M}_{\mathbb{R}}^{\perp_{\Gamma(A^*)}}$, i.e. we want to determine all $f \in \mathcal{D}(A^*) = H^1(\mathbb{R})$ such that

$$\langle f, \phi\rangle + \langle f', \phi'\rangle = 0 \quad \text{for all } \phi \in \mathfrak{M}_{\mathbb{R}}\,, \tag{4.3.4}$$

and hence in particular for all ϕ_λ, where $\lambda \in \mathbb{R}$. Integration by parts shows that Condition (4.3.4) implies

$$0 = \langle f, \phi_\lambda\rangle + \langle f', \phi'_\lambda\rangle = \overline{f(\lambda)} \quad \text{for all } \lambda \in \mathbb{R}\,,$$

meaning that

$$\left(\overline{\mathfrak{M}_{\mathbb{Q}}}^{\|\cdot\|_{\Gamma(A^*)}}\right)^{\perp_{\Gamma(A^*)}} \subset \mathfrak{M}_{\mathbb{R}}^{\perp_{\Gamma(A^*)}} = \{0\}\,.$$

From this, we get that $\overline{\mathfrak{M}_{\mathbb{Q}}}^{\|\cdot\|_{\Gamma(A^*)}} = L^2(\mathbb{R})$. Hence,

$$\overline{\mathfrak{M}_{\mathbb{Q}}}^{\|\cdot\|_{\Gamma(A^*)}} \cap \mathcal{D}(A) = L^2(\mathbb{R}) \cap H^1(\mathbb{R}) = H^1(\mathbb{R}) \neq \{0\}\,,$$

which means that $\mathfrak{M}_{\mathbb{Q}}$ does not satisfy the assumptions of Theorem 4.1.5 and thus the operator $C_{\mathfrak{M}_{\mathbb{Q}}}$ given by

$$C_{\mathfrak{M}_{\mathbb{Q}}}: \qquad \mathcal{D}(C_{\mathfrak{M}_{\mathbb{Q}}}) = \{f \in H^1(\mathbb{R}) : \langle f, \phi\rangle + \langle A^* f, A^*\phi\rangle = 0 \text{ for all } \phi \in \mathfrak{M}_{\mathbb{Q}}\}$$
$$f \mapsto if'$$

is not densely defined.

Let us now consider the case $\mathfrak{M}_{\mathbb{Z}}$. It is not difficult to see that

$$\mathfrak{M}_{\mathbb{Z}}^{\perp_{\Gamma(A^*)}} = \{f \in H^1(\mathbb{R}) : f(z) = 0 \quad \text{for all } z \in \mathbb{Z}\}\,.$$

In particular, we have that $\mathcal{C}_c^\infty(\mathbb{R}\backslash\mathbb{Z}) \subset \mathfrak{M}_{\mathbb{Z}}^{\perp_{\Gamma(A^*)}}$ and thus, any function $g \in \overline{\mathfrak{M}_{\mathbb{Z}}}^{\|\cdot\|_{\Gamma(A^*)}} = \left(\mathfrak{M}_{\mathbb{Z}}^{\perp_{\Gamma(A^*)}}\right)^{\perp_{\Gamma(A^*)}}$ has to satisfy

$$\langle f, g\rangle + \langle f', g'\rangle = 0\,, \tag{4.3.5}$$

for all $f \in \mathcal{C}_c^\infty(\mathbb{R}\setminus\mathbb{Z})$. Now, assume that the condition $\{A^*\phi : \phi \in \overline{\mathfrak{M}_{\mathbb{Z}}}^{\|\cdot\|_{\Gamma(A^*)}}\} \cap \mathcal{D}(A) = \{0\}$ is not satisfied, i.e. that there exists a $\widetilde{g} \in \mathcal{D}(A)$ such that $\widetilde{g}' \in \overline{\mathfrak{M}_{\mathbb{Z}}}^{\|\cdot\|_{\Gamma(A^*)}} \cap \mathcal{D}(A)$ (observe that $\ker A = \{0\}$). Then, we could perform another integration by parts in (4.3.5) and we would obtain

$$\langle f, \widetilde{g}\rangle + \langle f', \widetilde{g}'\rangle = \langle f, \widetilde{g} - \widetilde{g}''\rangle = 0 \quad \text{for all } f \in \mathcal{C}_c^\infty(\mathbb{R}\setminus\mathbb{Z})\,.$$

However, this implies that $\widetilde{g} - \widetilde{g}'' = 0$ since f is an arbitrary element of the dense set $\mathcal{C}_c^\infty(\mathbb{R}\setminus\mathbb{Z})$. Moreover, since there is no $L^2(\mathbb{R})$-solution to the equation $\widetilde{g} = \widetilde{g}''$, we get that $\widetilde{g} = 0$. Thus, by Lemma 4.1.2, the operator $B_{\mathfrak{M}_{\mathbb{Z}}}$ is closable.

CHAPTER 5

The proper dissipative extensions of a dual pair

In this chapter, we will consider dual pairs $(A, \widetilde{A})$, where A is dissipative and $\widetilde{A}$ is antidissipative. Under the assumption that $\mathcal{D}(A) \cap \mathcal{D}(\widetilde{A})$ is dense in $\mathcal{H}$, we will show a necessary and sufficient condition for a proper extension $\widehat{A}$ of $(A, \widetilde{A})$ to be dissipative.

This criterion can be written in a particularly nice way, if $\mathcal{D}(A) \cap \mathcal{D}(\widetilde{A})$ is a core for A as well as for $\widetilde{A}$, in which case we will say that $(A, \widetilde{A})$ has the common core property. If $(A, \widetilde{A})$ has the common core property, we can define the "imaginary part" of A by

$$V := (2i)^{-1}(A - \widetilde{A}) \restriction_{\mathcal{D}(A)\cap\mathcal{D}(\widetilde{A})},$$

which is a symmetric operator. It turns out that the square root of the Kreĭn-von Neumann extension of V, which we denote by $V_K^{1/2}$, plays an important role for the main theorem of this chapter (Theorem 5.2.8). For the proof of this theorem, we will use a description of $V_K^{1/2}$ that was obtained by Ando and Nishio (Proposition 5.2.3).

As a first application, we start by considering symmetric operators with relatively bounded dissipative perturbations and after that, we consider more singular dissipative operators — our first examples being such that the associated imaginary part V is already essentially selfadjoint and our last example being such that there is a family of selfadjoint extensions of V.

Finally, we find lower bounds for the numerical range of the dissipative extensions and apply this result to the examples from the previous section.

The results of this chapter were obtained in collaboration with Sergey Naboko and Ian Wood and have been published in [**20**].

5.1. The common core property

In many situations one considers dual pairs of operators, which are constructed by firstly defining them on a common core and then taking closures:

Definition 5.1.1. *Let $(A, \widetilde{A})$ be a dual pair of closed operators. We say that it has the* **common core property** *if $\overline{A \restriction_{\mathcal{D}(A)\cap\mathcal{D}(\widetilde{A})}} = A$ and $\widetilde{A} = \overline{\widetilde{A} \restriction_{\mathcal{D}(A)\cap\mathcal{D}(\widetilde{A})}}$.*

Example 5.1.2. Consider the dissipative momentum operator T given by

$$T: \qquad \mathcal{D}(T) = \{f \in H^1(0,1), f(0) = \rho f(1)\}, \quad f \mapsto if',$$

where $|\rho| < 1$. Here, f' denotes the weak derivative of f. Its adjoint T^* is given by

$$T^*: \qquad \mathcal{D}(T^*) = \{f \in H^1(0,1), \overline{\rho} f(0) = f(1)\}, \quad f \mapsto if'.$$

Clearly, (T, T^*) is a dual pair. However, since

$$\mathcal{D} := \mathcal{D}(T) \cap \mathcal{D}(T^*) = \{f \in H^1(0,1), f(0) = f(1) = 0\},$$

this dual pair does not have the common core property, as $S := \overline{T \restriction_{\mathcal{D}}}$ is symmetric and a proper restriction of T.

More generally, let S be a closed and symmetric (in particular densely defined) operator. Moreover, let S' be any closed (not necessarily symmetric) extension of S such that $S \subset S' \subset S^*$. This readily implies that (S, S') is a dual pair. However, since $\mathcal{D}(S) \cap \mathcal{D}(S') = \mathcal{D}(S)$, we get $S = \overline{S' \restriction_{\mathcal{D}(S)\cap\mathcal{D}(S')}}$. Thus, the only dual pair of this form, which has the common core property is (S, S). Furthermore, let $V \geq 0$ be S^*-bounded with S^*-bound less than 1. In particular, this implies that V is S'-bounded with S'-bound less than 1 (for a definition of relative boundedness, see e.g. [**26**]). By the Hess–Kato Theorem [**25**, Corollary 1], we have that $(S' + iV)^* = S'^* - iV \subset S^* - iV$. This implies again that any pair of the form $(S + iV, S' - iV)$ is a dual pair. However, again observe that the only dual pair with the common core property is $(S + iV, S - iV)$.

The following lemma shows in particular that if we have a dual pair $(A, \widetilde{A})$ with the common core property, then A being dissipative, implies that $\widetilde{A}$ is antidissipative.

Lemma 5.1.3. *Let $(A, \widetilde{A})$ be a dual pair of closed operators, which has the common core property. Moreover, let $\mathcal{N}_A := \{\langle f, Af\rangle : f \in \mathcal{D}(A), \|f\| = 1\}$ be the numerical range of A and let $\mathcal{N}^*_{\widetilde{A}} := \{\overline{\langle f, \widetilde{A}f\rangle} : f \in \mathcal{D}(\widetilde{A}), \|f\| = 1\}$ be the complex conjugate of the numerical range of $\widetilde{A}$. Then, the closures of the numerical range of A and the complex conjugate of the numerical range of $\widetilde{A}$ coincide:*

$$\overline{\mathcal{N}_A} = \overline{\mathcal{N}^*_{\widetilde{A}}} \, .$$

PROOF. Let $f \in \mathcal{D}(A)$ be normalized. Since $\mathcal{D}(A) \cap \mathcal{D}(\widetilde{A})$ is a core for A, there exists a normalized sequence $\{f_n\}_n \subset \mathcal{D}(A) \cap \mathcal{D}(\widetilde{A})$ such that $f_n \to f$ and $Af_n \to Af$ for $n \to \infty$. Using that $\langle f_n, Af_n \rangle = \overline{\langle f_n, \widetilde{A} f_n \rangle}$, we get that

$$\lim_{n\to\infty} \overline{\langle f_n, \widetilde{A} f_n \rangle} = \lim_{n\to\infty} \langle f_n, Af_n \rangle = \langle f, Af \rangle \, .$$

Since $\{\overline{\langle f_n, \widetilde{A} f_n \rangle}\}_n$ is a sequence of elements in $\mathcal{N}^*_{\widetilde{A}}$, we get that $\langle f, Af \rangle$ is a limit point of $\mathcal{N}^*_{\widetilde{A}}$, which means that

$$\mathcal{N}_A \subset \overline{\mathcal{N}^*_{\widetilde{A}}} \, .$$

By similar reasoning, we get that

$$\mathcal{N}^*_{\widetilde{A}} \subset \overline{\mathcal{N}_A},$$

which — after taking closures — yields the lemma. ☐

REMARK 5.1.4. If A is closed and dissipative and $\mathcal{D}(A) \cap \mathcal{D}(A^*)$ is a core for A, i.e. $A = \overline{A \restriction_{\mathcal{D}(A)\cap\mathcal{D}(A^*)}}$, we can define $\widetilde{A} := \overline{A^* \restriction_{\mathcal{D}(A)\cap\mathcal{D}(A^*)}}$, to construct a dual pair $(A, \widetilde{A})$, which has the common core property. This is in particular possible for the case that $\mathcal{D}(A) \subset \mathcal{D}(A^*)$ (cf. [**37**, Corollary to Proposition IV, 4.2]).

5.2. The main theorem

For a dual pair $(A, \widetilde{A})$ that has the common core property, let us now give a necessary and sufficient condition for a proper extension to be dissipative.

As any dissipative operator is closable with its closure being dissipative as well [**29**, Proposition 6.9], it is necessary and sufficient to check dissipativity of an operator restricted to a core.

LEMMA 5.2.1. *Let A be a closed, densely defined operator and let $\mathcal{C} \subset \mathcal{H}$ be a core for A. Moreover, assume that B is an extension of A, i.e. $A \subset B$ and $\mathcal{D}(B) = \mathcal{D}(A) \dot{+} \mathcal{M}$. Then, $\mathcal{C} \dot{+} \mathcal{M}$ is a core for B.*

PROOF. Since $\mathcal{C}$ is a core for A, this means that for every $f \in \mathcal{D}(A)$ there exists a sequence $\{f_n\}_n \subset \mathcal{C}$ such that $f_n \to f$ and $Af_n \to Af$ and therefore for any element of $\mathcal{D}(B) \ni (f+m)$, where $f \in \mathcal{D}(A)$ and $m \in \mathcal{M}$ we get

$$(f_n + m) \to (f+m) \quad \text{and} \quad B(f_n+m) = (Af_n + Bm) \to (Af + Bm) = B(f+m)\,,$$

which is the desired result. $\square$

For the following result we need the Kreĭn-von Neumann extension of a symmetric non-negative operator, which is defined as follows.

DEFINITION 5.2.2. *Let V be symmetric and non-negative operator, i.e. $\langle f, Vf\rangle \geq 0$ for all $f \in \mathcal{D}(V)$. Then, the* **Kreĭn–von Neumann extension** *of V, which we denote by V_K, is the smallest non-negative selfadjoint extension of V, i.e. for any $\widehat{V} = \widehat{V}^*$ with $V \subset \widehat{V}$ and $\widehat{V} \geq 0$ we have that*

$$0 \leq V_K \leq \widehat{V}\,.$$

*It is a well-known fact that such an extension V_K always exists and that it is unique (cf. [**27**]).*

*For the special case that V is strictly positive, i.e. there exists an $\varepsilon > 0$ such $\langle f, Vf\rangle \geq \varepsilon \|f\|^2$ for all $f \in \mathcal{D}(V)$, we have the following characterization of V_K [**2**]:*

$$V_K: \quad \mathcal{D}(V_K) = \mathcal{D}(V) \dot{+} \ker V^*, \quad V_K = V^* \restriction_{\mathcal{D}(V_K)}$$

and for $V_K^{1/2}$ we get

$$\begin{aligned} V_K^{1/2}: \qquad & \mathcal{D}(V_K^{1/2}) = \mathcal{D}(V_F^{1/2}) \dot{+} \ker V^* \\ & \langle V_K^{1/2}(f+k), V_K^{1/2}(f+k)\rangle = \langle V_F^{1/2} f, V_F^{1/2} f\rangle\,, \end{aligned} \tag{5.2.1}$$

with $f \in \mathcal{D}(V_F^{1/2})$, $k \in \ker V^*$, where V_F is the Friedrichs extension of V.

For the proof of the main theorem without having to assume that the imaginary part is strictly positive, we will make use of an equivalent description for non-negative $V_K^{1/2}$ proved by Ando and Nishio.

Proposition 5.2.3 (T. Ando, K. Nishio, [**4**, Thm. 1]). *Let V be a non-negative closed symmetric operator. The selfadjoint and non-negative square root of the Kreĭn–von Neumann extension of V, which we denote by $V_K^{1/2}$, can be characterized as follows:*

$$\mathcal{D}(V_K^{1/2}) = \left\{ h \in \mathcal{H} : \sup_{f\in\mathcal{D}(V):Vf\neq 0} \frac{|\langle h, Vf\rangle|^2}{\langle f, Vf\rangle} < \infty \right\} ,$$

$$\textit{for any } h \in \mathcal{D}(V_K^{1/2}) : \quad \|V_K^{1/2}h\|^2 = \sup_{f\in\mathcal{D}(V):Vf\neq 0} \frac{|\langle h, Vf\rangle|^2}{\langle f, Vf\rangle} .$$

Remark 5.2.4. Let us point out a slight difference in the manner Proposition 5.2.3 was stated in [**4**], where the supremum is taken over all $f \in \mathcal{D}(V)$ (without the extra condition that $Vf \neq 0$). This only makes sense if one assumes that $\ker V = \{0\}$. The extra condition $Vf \neq 0$ is a remedy for this problem and is a direct result from the reasoning of [**4**].

For our purposes, it will be more convenient to use the following characterization of $\mathcal{D}(V_K^{1/2})$ and $\|V_K^{1/2}h\|$ for any $h \in \mathcal{D}(V_K^{1/2})$:

Corollary 5.2.5. *Let V be a non-negative closed symmetric operator on a Hilbert space $\mathcal{H}$. Then, the square root of its Kreĭn–von Neumann extension can be characterized as follows*

$$\mathcal{D}(V_K^{1/2}) = \left\{ h \in \mathcal{H} : \sup_{g\in\operatorname{ran}(V_F^{1/2}\restriction_{\mathcal{D}(V)}):\|g\|=1} |\langle h, V_F^{1/2}g\rangle| < \infty \right\} , \tag{5.2.2}$$

$$\textit{for any } h \in \mathcal{D}(V_K^{1/2}) : \quad \|V_K^{1/2}h\|^2 = \sup_{g\in\operatorname{ran}(V_F^{1/2}\restriction_{\mathcal{D}(V)}):\|g\|=1} |\langle h, V_F^{1/2}g\rangle|^2 . \tag{5.2.3}$$

Proof. Let us consider any $f \in \mathcal{D}(V)$ such that $Vf \neq 0$. We then get

$$\frac{|\langle h, Vf\rangle|^2}{\langle f, Vf\rangle} = \frac{|\langle h, V_F^{1/2}V_F^{1/2}f\rangle|^2}{\|V_F^{1/2}f\|^2} = \left|\left\langle h, V_F^{1/2}\left(\frac{V_F^{1/2}f}{\|V_F^{1/2}f\|}\right)\right\rangle\right|^2 .$$

Now, observe that $\frac{V_F^{1/2}f}{\|V_F^{1/2}f\|}$ is a normalized element of $\operatorname{ran}(V_F^{1/2}\restriction_{\mathcal{D}(V)})$. Conversely, for any normalized $g \in \operatorname{ran}(V_F^{1/2}\restriction_{\mathcal{D}(V)})$, there exists a $f \in \mathcal{D}(V)$ with $Vf \neq 0$ such that

$g = \frac{V_F^{1/2} f}{\|V_F^{1/2} f\|}$. This implies that

$$\sup_{f \in \mathcal{D}(V): Vf \neq 0} \left| \left\langle h, V_F^{1/2} \left(\frac{V_F^{1/2} f}{\|V_F^{1/2} f\|} \right) \right\rangle \right|^2 = \sup_{g \in \operatorname{ran}(V_F^{1/2} \restriction_{\mathcal{D}(V)}): \|g\|=1} |\langle h, V_F^{1/2} g \rangle|^2 ,$$

which — together with (5.2.2) — implies the corollary. $\square$

For the main theorem, we will make use of the fact that the dual pair under consideration has a common core $\mathcal{D}$, allowing us to define an "imaginary part" on $\mathcal{D}$. It will therefore be helpful to show that the supremum in Proposition 5.2.3 has need only be taken over $\mathcal{D}$.

LEMMA 5.2.6. *Let V be a non-negative closed symmetric operator and $\mathcal{C}$ be a core for V. Then, for any $h \in \mathcal{H}$ we have that*

$$\sup_{f \in \mathcal{D}(V): Vf \neq 0} \frac{|\langle h, Vf \rangle|^2}{\langle f, Vf \rangle} = \sup_{f \in \mathcal{C}: Vf \neq 0} \frac{|\langle h, Vf \rangle|^2}{\langle f, Vf \rangle} .$$

Moreover, for any $h \in \mathcal{H}$, we have

$$\sup_{g \in \operatorname{ran}(V_F^{1/2} \restriction_{\mathcal{D}(V)}): \|g\|=1} |\langle h, V_F^{1/2} g \rangle| = \sup_{g \in \operatorname{ran}(V_F^{1/2} \restriction_{\mathcal{C}}): \|g\|=1} |\langle h, V_F^{1/2} g \rangle| . \tag{5.2.4}$$

PROOF. Let $s \in \mathbb{R}^+ \cup \{\infty\}$ be defined as

$$s := \sup_{f \in \mathcal{D}(V): Vf \neq 0} \frac{|\langle h, Vf \rangle|^2}{\langle f, Vf \rangle} .$$

This means that there exists a sequence $\{f_n\}_n \subset \mathcal{D}(V)$ with $Vf_n \neq 0$ such that

$$\lim_{n \to \infty} \frac{|\langle h, Vf_n \rangle|^2}{\langle f_n, Vf_n \rangle} = s .$$

On the other hand, since $\mathcal{C}$ is a core for V, for any $f_n \in \mathcal{D}(V)$, there exists a sequence $\{f_{n,m}\}_m \subset \mathcal{C}$ such that

$$\lim_{m \to \infty} f_{n,m} = f_n \quad \text{and} \quad \lim_{m \to \infty} Vf_{n,m} = Vf_n .$$

Thus, for any fixed $h \in \mathcal{H}$ and $f_n \in \mathcal{D}(V)$ such that $Vf_n \neq 0$, we have also $\langle f_n, Vf_n \rangle \neq 0$ and therefore

$$\lim_{m \to \infty} \frac{|\langle h, Vf_{n,m} \rangle|}{\langle f_{n,m}, Vf_{n,m} \rangle} = \frac{|\langle h, Vf_n \rangle|}{\langle f_n, Vf_n \rangle} .$$

Hence, a diagonal sequence argument yields the first part of the lemma.

Equation (5.2.4) follows from a similar reasoning as in the proof of Corollary 5.2.5, using that for any $f \in \mathcal{C}$ with $Vf \neq 0$, we have that $V_F^{1/2} f / \|V_F^{1/2} f\|$ is a normalized element of $\operatorname{ran}(V_F^{1/2} \restriction_{\mathcal{C}})$ and that for any normalized $g \in \operatorname{ran}(V_F^{1/2} \restriction_{\mathcal{C}})$ there exists an $f \in \mathcal{C}$ with $Vf \neq 0$ such that $g = V_F^{1/2} f / \|V_F^{1/2} f\|$. $\square$

DEFINITION 5.2.7. *Let $\mathcal{V} \subset \mathcal{D}(\widetilde{A}^*)//\mathcal{D}(A)$ be a subspace. Then, the operator $A_{\mathcal{V}}$ is defined as*

$$A_{\mathcal{V}}: \qquad \mathcal{D}(A_{\mathcal{V}}) = \mathcal{D}(A) \dot{+} \mathcal{V}, \quad A_{\mathcal{V}} = \widetilde{A}^* \restriction_{\mathcal{D}(A_{\mathcal{V}})} .$$

THEOREM 5.2.8. *Let $(A, \widetilde{A})$ be a dual pair of operators having the common core property, where A is dissipative and let $\mathcal{D} \subset (\mathcal{D}(A) \cap \mathcal{D}(\widetilde{A}))$ be a common core for A and for $\widetilde{A}$. Then, the operator $V := \frac{A - \widetilde{A}}{2i}$ defined on $\mathcal{D}$ is a non-negative symmetric operator. Moreover, let $\mathcal{V} \subset \mathcal{D}(\widetilde{A}^*)//\mathcal{D}(A)$ be a linear space. Then, the operator $A_{\mathcal{V}}$ is dissipative if and only if $\mathcal{V} \subset \mathcal{D}(V_K^{1/2})$ and*

$$Im\langle v, \widetilde{A}^* v\rangle \geq \|V_K^{1/2} v\|^2 \quad \text{for all} \quad v \in \mathcal{V} .$$

PROOF. Since $\operatorname{Im}\langle f, Af\rangle \geq 0$ for all $f \in \mathcal{D}(A)$, this implies by Lemma 5.1.3 that $\operatorname{Im}\langle f, \widetilde{A} f\rangle \leq 0$ for all $f \in \mathcal{D}(\widetilde{A})$ and hence, $\widetilde{A}$ is antidissipative. Next, let us show that V is symmetric and non-negative. For any $f \in \mathcal{D}$ we get

$$\langle f, Vf\rangle = \frac{1}{2i}\left(\langle f, Af\rangle - \langle f, \widetilde{A} f\rangle\right) = \frac{1}{2i}\left(\langle f, Af\rangle - \langle Af, f\rangle\right) = \operatorname{Im}\langle f, Af\rangle \geq 0 \tag{5.2.5}$$

by assumption. Let us now prove the criterion for dissipativity. By Lemma 5.2.1, it is sufficient to check dissipativity for all elements of $\mathcal{D}(A_{\mathcal{V}})$ of the form $f + v$, where $f \in \mathcal{D}$ and $v \in \mathcal{V}$. Thus, it suffices to show that

$$\operatorname{Im}\langle f + v, \widetilde{A}^*(f + v)\rangle \geq 0 \quad \text{for all} \quad f \in \mathcal{D},\ v \in \mathcal{V}$$

if $\mathcal{V} \subset \mathcal{D}(V_K^{1/2})$ and $\operatorname{Im}\langle v, \widetilde{A}^* v\rangle \geq \|V_K^{1/2} v\|^2$ for all $v \in \mathcal{V}$. Then by (5.2.5):

$$\begin{aligned}\operatorname{Im}\langle f + v, \widetilde{A}^*(f + v)\rangle &= \operatorname{Im}\langle f, Af\rangle + \operatorname{Im}\langle v, \widetilde{A}^* v\rangle + \operatorname{Im}\left(\langle f, \widetilde{A}^* v\rangle + \langle v, Af\rangle\right) \\ &= \langle f, Vf\rangle + \operatorname{Im}\langle v, \widetilde{A}^* v\rangle - \operatorname{Im}\langle (A - \widetilde{A}) f, v\rangle = \langle f, Vf\rangle + \operatorname{Im}\langle v, \widetilde{A}^* v\rangle - \operatorname{Im}\langle 2iVf, v\rangle .\end{aligned}$$

Observe that for any given v, one can always consider $e^{i\vartheta}v$ instead of v, where $\vartheta \in [0, 2\pi)$ is chosen such that $\operatorname{Im}\langle 2iVf, e^{i\vartheta}v\rangle = -2\,|\langle Vf, v\rangle|$ without changing the other two terms, which means that showing

$$\langle f, Vf\rangle + \operatorname{Im}\langle v, \widetilde{A}^* v\rangle - 2|\langle Vf, v\rangle| \geq 0 \quad \text{for all } f \in \mathcal{D},\ v \in \mathcal{V} \tag{5.2.6}$$

is necessary and sufficient for $A_{\mathcal{V}}$ to be dissipative.

Let us begin by showing that $\mathcal{V} \subset \mathcal{D}(V_K^{1/2})$ and $\operatorname{Im}\langle v, \widetilde{A}^* v\rangle \geq \|V_K^{1/2}v\|^2$ is sufficient for $A_{\mathcal{V}}$ to be dissipative. Thus, let us now assume that these two assumptions are satisfied. Since $V \subset \overline{V} \subset V_K$ and $\mathcal{D}(V) \subset \mathcal{D}(V_K) \subset \mathcal{D}(V_K^{1/2})$, this means that we can write $Vf = V_K f = \left(V_K^{1/2}\right)\left(V_K^{1/2}f\right)$. We therefore get that

$$\begin{aligned}
\langle f, &Vf\rangle + \operatorname{Im}\langle v, \widetilde{A}^* v\rangle - 2|\langle Vf, v\rangle| = \|V_K^{1/2}f\|^2 + \operatorname{Im}\langle v, \widetilde{A}^* v\rangle - 2|\langle V_K^{1/2}f, V_K^{1/2}v\rangle| \\
&\geq \|V_K^{1/2}f\|^2 + \operatorname{Im}\langle v, \widetilde{A}^* v\rangle - 2\|V_K^{1/2}f\|\|V_K^{1/2}v\| \\
&\geq \|V_K^{1/2}f\|^2 + \|V_K^{1/2}v\|^2 - 2\|V_K^{1/2}f\|\|V_K^{1/2}v\| \\
&= \left(\|V_K^{1/2}f\| - \|V_K^{1/2}v\|\right)^2 \geq 0\,.
\end{aligned}$$

Next, let us show that the condition $\mathcal{V} \subset \mathcal{D}(V_K^{1/2})$ is necessary for $A_{\mathcal{V}}$ to be dissipative. Thus, let us assume that $\mathcal{V} \not\subset \mathcal{D}(V_K^{1/2})$, i.e. that there exists a $v \in \mathcal{V}$ such that $v \notin \mathcal{D}(V_K^{1/2})$. Using that $\mathcal{D}(V) = \mathcal{D}$ is a core for $\overline{V}$, we have by Proposition 5.2.3 and by Lemma 5.2.6 that there exists a sequence $\{f_n\}_n \subset \mathcal{D}(V)$ with $Vf_n \neq 0$ and therefore $\langle f_n, Vf_n\rangle \neq 0$, such that

$$\lim_{n\to\infty} \frac{|\langle v, Vf_n\rangle|}{\sqrt{\langle f_n, Vf_n\rangle}} = +\infty\,.$$

Define the sequence $\{h_n\}_n \subset \mathcal{D}(V)$ by $h_n := f_n/\sqrt{\langle f_n, Vf_n\rangle}$ and observe that

$$\frac{|\langle v, Vf_n\rangle|}{\sqrt{\langle f_n, Vf_n\rangle}} = \frac{|\langle v, Vh_n\rangle|}{\sqrt{\langle h_n, Vh_n\rangle}} \quad \text{and} \quad \sqrt{\langle h_n, Vh_n\rangle} = 1 \quad \text{for all} \quad n \in \mathbb{N}\,.$$

From this we get that

$$\begin{aligned}
\lim_{n\to\infty} &\left(\operatorname{Im}\langle v, \widetilde{A}^* v\rangle + \langle h_n, Vh_n\rangle - 2|\langle Vh_n, v\rangle|\right) \\
&= \operatorname{Im}\langle v, \widetilde{A}^* v\rangle + 1 - 2\lim_{n\to\infty} \frac{|\langle v, Vh_n\rangle|}{\sqrt{\langle h_n, Vh_n\rangle}} = -\infty\,,
\end{aligned}$$

which shows that Condition (5.2.6) can never be satisfied in this case.

Let us finish the proof by showing that $\operatorname{Im}\langle v, \widetilde{A}^* v\rangle \geq \|V_K^{1/2}v\|^2$ for all $v \in \mathcal{V}$ is necessary for $A_{\mathcal{V}}$ to be dissipative. By (5.2.6), it suffices to show that for any $v \in \mathcal{D}(V_K^{1/2})$, there exists a sequence $\{g_n\}_n \subset \mathcal{D}(V)$ such that

$$2|\langle Vg_n, v\rangle| - \langle g_n, Vg_n\rangle \overset{n\to\infty}{\longrightarrow} \|V_K^{1/2}v\|^2 \,. \tag{5.2.7}$$

For the case $V_K^{1/2}v = 0$, this sequence would just be given by $f_n = 0$ for all n, therefore let us assume $V_K^{1/2}v \neq 0$ from now on. By Proposition 5.2.3, we know that there exists a sequence $\{f_n\}_n \subset \mathcal{D}(V)$ with $Vf_n \neq 0$ such that

$$\frac{|\langle v, Vf_n\rangle|^2}{\langle f_n, Vf_n\rangle} \overset{n\to\infty}{\longrightarrow} \|V_K^{1/2}v\|^2 \,.$$

Define the positive numbers μ_n by $\mu_n := |\langle v, Vf_n\rangle|/\langle f_n, Vf_n\rangle$ and observe that the sequence $\{g_n\}_n$, where $g_n := \mu_n f_n$, is exactly as required for (5.2.7):

$$\begin{aligned}2|\langle \mu_n Vf_n, v\rangle| - \langle \mu_n f_n, \mu_n Vf_n\rangle &= 2|\langle Vf_n, v\rangle|\frac{|\langle Vf_n, v\rangle|}{\langle f_n, Vf_n\rangle} - \frac{|\langle Vf_n, v\rangle|^2}{\langle f_n, Vf_n\rangle^2}\langle f_n, Vf_n\rangle \\ &= \frac{|\langle Vf_n, v\rangle|^2}{\langle f_n, Vf_n\rangle} \overset{n\to\infty}{\longrightarrow} \|V_K^{1/2}v\|^2 \,.\end{aligned}$$

This finishes the proof. □

Corollary 5.2.9. *Let $(A, \widetilde{A})$ be a dual pair satisfying the assumptions of Theorem 5.2.8. If for some $\lambda \in \mathbb{C}^-$ we have that*

$$\ker(\widetilde{A}^* - \lambda) \cap \mathcal{D}(V_K^{1/2}) = \{0\} \,, \tag{5.2.8}$$

then there exists exactly one proper maximally dissipative extension of the dual pair $(A, \widetilde{A})$.

Proof. By Proposition 2.4.5, we know that there exists a proper maximally dissipative extension $\widehat{A}$ and by Proposition 2.2.5, we know that $\mathbb{C}^- \in \rho(\widehat{A})$. Moreover, by [**23**] we have that

$$\mathcal{D}(\widehat{A}) = \mathcal{D}(A) \dot{+} (\widehat{A} - \lambda)^{-1} \ker(A^* - \overline{\lambda})$$

as well as

$$\mathcal{D}(\widetilde{A}^*) = \mathcal{D}(A) \dot{+} (\widehat{A} - \lambda)^{-1} \ker(A^* - \overline{\lambda}) \dot{+} \ker(\widetilde{A}^* - \lambda) \,.$$

By Theorem 5.2.8, we know that $(\widehat{A} - \lambda)^{-1} \ker(A^* - \overline{\lambda}) \subset \mathcal{D}(V_K^{1/2})$. Note that any other proper extension $A_{\mathcal{V}}$ of $(A, \widetilde{A})$ that is not a restriction of $\widehat{A}$ can be characterized by a subspace $\mathcal{V}$ that without loss of generality we can assume to be contained in

$(\widehat{A}-\lambda)^{-1}\ker(A^*-\overline{\lambda})\dot{+}\ker(\widetilde{A}^*-\lambda)$, where $\mathcal{V}\not\subset(\widehat{A}-\lambda)^{-1}\ker(A^*-\overline{\lambda})$. Thus, there needs to exist at least one element in $v\in\mathcal{V}$, which is of the form $v=(\widehat{A}-\lambda)^{-1}k_{\overline{\lambda}}+\widetilde{k}_\lambda$, where $k_{\overline{\lambda}}\in\ker(A^*-\overline{\lambda})$ and $\widetilde{k}_\lambda\in\ker(\widetilde{A}^*-\lambda)$ with $\widetilde{k}_\lambda\neq 0$. However, by (5.2.8), we have that $v\notin\mathcal{D}(V_K^{1/2})$ which implies that $A_{\mathcal{V}}$ cannot be dissipative. □

REMARK 5.2.10. A corresponding result for sectorial operators was shown in [**8**, Thm. 3.6.5].

REMARK 5.2.11. In Example 5.4.7 below, we will discuss an operator, for which Corollary 5.2.9 applies.

REMARK 5.2.12. It is not necessary that (5.2.8) hold in order for a dual pair to have only one proper maximally dissipative extension as we will see in Example 5.4.6 below.

THEOREM 5.2.13. *In addition to the assumptions of Theorem 5.2.8, assume that*

$$\dim\mathcal{D}(\widetilde{A}^*)/\mathcal{D}(A)<\infty\,.$$

Moreover, let $\mathcal{W}:=(\mathcal{D}(\widetilde{A}^*)//\mathcal{D}(A))\cap\mathcal{D}(V_K^{1/2})$. *Let the quadratic form* $q(\cdot)$ *be defined as*

$$q(w):=Im\langle w,\widetilde{A}^*w\rangle-\|V_K^{1/2}w\|^2\,, \tag{5.2.9}$$

which has domain $\mathcal{W}$ *and let* M *be the selfadjoint operator associated to the unique sesquilinear form induced by* $q(\cdot)$ *by polarization. Let us decompose* $\mathcal{W}=\mathcal{W}_+\oplus\mathcal{W}_0\oplus\mathcal{W}_-$, *where* $\mathcal{W}_+$ *denotes the positive spectral subspace,* $\mathcal{W}_0$ *denotes* $\ker M$ *and* $\mathcal{W}_-$ *denotes the negative spectral subspace of* M. *Furthermore, define*

$$M_\pm:=\pm MP_{\mathcal{W}_\pm}\,,$$

which allows us to write $M=M_+-M_-$. *Note that* $M_\pm>0$ *and that* M_+ *and* M_- *are invertible on* $\mathcal{W}_+$, *resp. on* $\mathcal{W}_-$. *Let* C *be a contraction (*$\|C\|\leq 1$*) from* $\mathcal{W}_+\oplus\mathcal{W}_0$ *into* $\mathcal{W}_-$. *Then, there is a one-to-one correspondence between all proper dissipative extensions of* $(A,\widetilde{A})$ *and all pairs* $(\mathfrak{M},C)$, *where* $\mathfrak{M}$ *is a subspace of* $\mathcal{W}_+\oplus\mathcal{W}_0$ *and* C *is a contraction from* $\mathcal{W}_+$ *into* $\mathcal{W}_-$ *with* $\mathcal{D}(C)=P_{\mathcal{W}_+}\mathfrak{M}$. *These extensions and the* *correspondence are given by*

$$\mathcal{D}(A_{\mathfrak{M},C}) = \mathcal{D}(A)\dot{+}\{w + \sqrt{M_-}^{-1}C\sqrt{M_+}w, w \in \mathfrak{M}\}$$

$$A_{\mathfrak{M},C} = \widetilde{A}^* \restriction_{\mathcal{D}(A_{\mathfrak{M},C})} \ . \tag{5.2.10}$$

Moreover, for an extension $\mathcal{D}(A_{\mathfrak{M},C})$ *to be maximally dissipative, it is necessary that* $\mathfrak{M} = \mathcal{W}_+ \oplus \mathcal{W}_0$.

Proof. By virtue of Theorem 5.2.8, we firstly need to show that

$$q(v) \geq 0 \quad \text{for all} \quad v \in \{w + \sqrt{M_-}^{-1}C\sqrt{M_+}w, w \in \mathfrak{M}\}$$

if C is a contraction. By definition of M and $M_\pm$, we have that

$$\begin{aligned} q(v) = \langle v, Mv\rangle &= \left\langle w + \sqrt{M_-}^{-1}C\sqrt{M_+}w, M\left(w + \sqrt{M_-}^{-1}C\sqrt{M_+}w\right)\right\rangle \\ &= \langle w, M_+w\rangle - \langle w, \sqrt{M_+}C^*\sqrt{M_-}^{-1}M_-\sqrt{M_-}^{-1}C\sqrt{M_+}w\rangle \\ &= \langle w, \sqrt{M_+}(\mathbb{1} - C^*C)\sqrt{M_+}w\rangle = \langle\sqrt{M_+}w, (\mathbb{1} - C^*C)\sqrt{M_+}w\rangle \ , \end{aligned} \tag{5.2.11}$$

which is non-negative if C is a contraction on $\sqrt{M_+}\mathfrak{M} = P_{\mathcal{W}_+}\mathfrak{M} = \mathcal{D}(C)$.
Let us now show that any proper dissipative extension has to be of this form. To this end, let A' be a proper dissipative extension of $(A, \widetilde{A})$ and let $\mathfrak{M}' \subset \mathcal{W}$ be such that $\mathcal{D}(A')//\mathcal{D}(A) = \mathfrak{M}'$. Clearly, $\mathcal{W}_- \cap \mathfrak{M}' = \{0\}$, since otherwise we would have that

$$q(w) = \langle w, Mw\rangle = -\langle w, M_-w\rangle < 0$$

for some non-zero $w \in \mathcal{W}_- \cap \mathfrak{M}'$, violating the necessary condition obtained from Theorem 5.2.8 for A' to be dissipative. This means that any $w \in \mathfrak{M}'$ can be written as $w = w_-^\perp + w_-$ where $w_-^\perp \in \mathcal{W}_+ \oplus \mathcal{W}_0$, $w_-^\perp \neq 0$ and $w_- \in \mathcal{W}_-$ is possibly zero. Since $\mathcal{W}_- \cap \mathfrak{M} = \{0\}$, it is easy to see that w_- is uniquely determined by $w_-^\perp$. Therefore, there exists a linear operator $B : P_{\mathfrak{M}'}(\mathcal{W}_+ \oplus \mathcal{W}_0) \to \mathcal{W}_-$ such that $w = w_-^\perp + Bw_-^\perp$ for any $w \in \mathfrak{M}'$. Next observe that if for any such $w_-^\perp$ we have that $w_-^\perp \in \mathcal{W}_0$, it follows that $Bw_-^\perp = 0$. If this were not true, we would get

$$q(w_-^\perp + Bw_-^\perp) = \underbrace{\langle w_-^\perp, M_+w_-^\perp\rangle}_{=0} - \langle Bw_-^\perp, M_-Bw_-^\perp\rangle = -\langle Bw_-^\perp, M_-Bw_-^\perp\rangle \ ,$$

which again would violate the necessary condition from Theorem 5.2.8 for A' to be dissipative. Plugging this into the quadratic form q yields:

$$
\begin{aligned}
q(w_-^\perp + Bw_-^\perp) &= \langle w_-^\perp, M_+ w_-^\perp\rangle - \langle Bw_-^\perp, M_- Bw_-^\perp\rangle = \langle w_-^\perp, (M_+ - B^* M_- B) w_-^\perp\rangle \\
&= \langle \sqrt{M_+} w_-^\perp, \left(\mathbb{1} - \sqrt{M_+}^{-1} B^* \sqrt{M_-}\sqrt{M_-} B \sqrt{M_+}^{-1}\right) \sqrt{M_+} w_-^\perp\rangle \,,
\end{aligned}
$$

with the understanding that $\sqrt{M_+}^{-1}$ is defined only on $\mathrm{ran}\sqrt{M_+} = \mathrm{ran} M_+$. This is equivalent to saying that the operator $C := \sqrt{M_-} B \sqrt{M_+}^{-1}$ is a contraction on $\sqrt{M_+} P_{\mathcal{W}_+}\mathfrak{M}' = P_{\mathcal{W}_+}\mathfrak{M}'$, or equivalently, $B = \sqrt{M_-}^{-1} C \sqrt{M_+}$, with C being a contraction from $P_{\mathcal{W}_+}\mathfrak{M}'$ to $\mathcal{W}_-$. The condition that $\mathfrak{M} = \mathcal{W}_+ \oplus \mathcal{W}_0$ for $A_{\mathfrak{M},C}$ to be maximally dissipative follows from the fact that one could always extend the operator $A_{\mathfrak{M},C}$ to $A_{\mathcal{W}_+\oplus\mathcal{W}_0,\widehat{C}}$, where $\widehat{C}$ is an extension of C which is just set equal to zero on $(\mathcal{W}_+ \oplus \mathcal{W}_0) \ominus \mathfrak{M}$. □

Remark 5.2.14. For the case that the dual pair $(A, \widetilde{A})$ has only one unique maximally dissipative proper extension $\widehat{A}$, this means that $\widehat{A} = A_{\mathcal{W}_+\oplus\mathcal{W}_0,0}$. In particular, when the assumptions of Corollary 5.2.9 are satisfied, we get that $\mathcal{W}_- = \{0\}$ since $(\mathcal{D}(\widetilde{A}^*)//\mathcal{D}(A)) \cap \mathcal{D}(V_K^{1/2}) = \mathcal{W}_+ \oplus \mathcal{W}_0$.

Remark 5.2.15. Let us show that for a very special situation, the spaces $\mathcal{W}_\pm$ coincide with the defect spaces of a symmetric operator S. (As an example, take the momentum operator $i\frac{d}{dx}$ with domain $\{f \in H^1(\mathbb{R}), f(0) = 0\}$, whose defect spaces are one-dimensional and spanned by exponential functions supported on different half-lines.) Assume that S has finite-dimensional defect spaces $\mathcal{N}_\pm := \ker(S^* \mp i)$. It is a well-known fact [**43**] that

$$
\mathcal{D}(S^*) = \mathcal{D}(S) \dot{+} \mathcal{N}_+ \dot{+} \mathcal{N}_- \,,
$$

where $\mathcal{N}_\pm := \ker(S^* \mp i)$ are the defect spaces. Assume in addition the rather restrictive condition that $\mathcal{N}_+ \perp \mathcal{N}_-$ (orthogonal with respect to the Hilbert space inner product). Choosing the dual pair (S, S), which trivially has the common core property, we find that $V_K = 0_{\mathcal{H}}$, with V_K being defined as in Theorem 5.2.8. Define

$$
q(v) := \mathrm{Im}\langle v, S^* v\rangle \quad \text{with} \quad v \in \mathcal{N}_+ \oplus \mathcal{N}_- \,.
$$

A calculation shows that the operator M associated to $q(\cdot)$ is given by $M = P_{\mathcal{N}_+} - P_{\mathcal{N}_-}$, i.e. $M_\pm = P_{\mathcal{N}_\pm}$, $\mathcal{W}_\pm = \mathcal{N}_\pm$ and $\mathcal{W}_0 = \{0\}$. Thus, by Theorem 5.2.13, all maximally dissipative extensions of such an operator S are given by

$$\mathcal{D}(S_C) = \mathcal{D}(S)\dot{+}\{n_+ + Cn_+, n_+ \in \mathcal{N}_+\}, \quad S_C = S^* \restriction_{\mathcal{D}(S_C)} ,$$

where C is any contraction into $\mathcal{N}_-$ such that $\mathcal{D}(C) = \mathcal{N}_+$. Thus, for the very special case $\mathcal{N}_+ \perp \mathcal{N}_-$, this readily implies the von Neumann theory of selfadjoint/maximally dissipative extensions of symmetric operators. (cf. e.g. [**43**, Thm. 8.12], for the selfadjoint and [**9**, Theorem 2.4], for the more general maximally dissipative case)

Remark 5.2.16. For concrete problems, it can be impractical to construct $\mathcal{W}_+, \mathcal{W}_0$ and $\mathcal{W}_-$ as well as M_+ and M_-. However, this result allows us to calculate the number of independent complex parameters required to describe all proper maximally dissipative extensions of a dual pair, which is given by the number of parameters that describe all contractions C from $\mathcal{W}_+$ into $\mathcal{W}_-$, which is in turn equal to $\dim \mathcal{W}_+ \cdot \dim \mathcal{W}_-$.
See also the operators considered in Section 5.4.3 for a discussion of the spaces $\mathcal{W}_+, \mathcal{W}_-$ and $\mathcal{W}_0$ for a few concrete examples.

Remark 5.2.17. As a reference to [**32**], let us point out that this result means that we can characterize all proper dissipative extensions of such a dual pair using the terminology of *operator balls*. For any three operators $Z, R_l, R_r \in \mathcal{B}(\mathcal{E})$, where $\mathcal{E}$ is an arbitrary Hilbert space, recall that the set of all operators $K \in \mathcal{B}(\mathcal{E})$ such that there exists a contraction C from $\operatorname{ran}(R_r)$ to $\mathcal{D}(R_l)$ such that

$$K = Z + R_l C R_r ,$$

is called an operator ball $\mathfrak{B}(Z, R_l, R_r)$ with center point Z, left radius R_l and right radius R_r. With the identification $\mathcal{E} = \mathcal{W}$, $Z = P_{\mathcal{W}_+} + P_{\mathcal{W}_0}$, $R_l = \sqrt{M_-}^{-1}$ and $R_r = \sqrt{M_+}$ defined on $\mathcal{W}_-$, respectively on $\mathcal{W}_+$, and the result from Theorem 5.2.13, we can characterize all proper dissipative extensions of a dual pair $(A, \widetilde{A})$ satisfying the assumptions of Theorem 5.2.13. This is achieved via:

$$A_K : \quad \mathcal{D}(A_K) = \mathcal{D}(A)\dot{+}\{Kw : w \in \mathcal{W}\}, \quad A_K = \widetilde{A}^* \restriction_{\mathcal{D}(A_K)} , \tag{5.2.12}$$

where $K \in \mathfrak{B}(P_{\mathcal{W}_+} + P_{\mathcal{W}_0}, \sqrt{M_-}^{-1}, \sqrt{M_+})$.

5.3. The non-common core case

Let us now extend this idea to the case where the dual pair $(A, \widetilde{A})$ does not have the common core property. If we assume $\mathcal{D}(A) \cap \mathcal{D}(\widetilde{A})$ still to be dense, we can restrict A and $\widetilde{A}$ to $\mathcal{D}(A) \cap \mathcal{D}(\widetilde{A})$ to obtain a dual pair of operators which has the common core property:

COROLLARY 5.3.1. *Let A and $\widetilde{A}$ be a dual pair of operators, where A is dissipative. Moreover, let $\mathcal{D}(A) \cap \mathcal{D}(\widetilde{A})$ be dense in $\mathcal{H}$. Define the operators A' and $\widetilde{A}'$ as follows:*

$$A' := \overline{A \restriction_{\mathcal{D}(A)\cap\mathcal{D}(\widetilde{A})}} \quad \textit{and} \quad \widetilde{A}' := \overline{\widetilde{A} \restriction_{\mathcal{D}(A)\cap\mathcal{D}(\widetilde{A})}} \; .$$

Furthermore, let V_0' denote the operator $\frac{1}{2i}(A' - \widetilde{A}')$ on $\mathcal{D}(A) \cap \mathcal{D}(\widetilde{A})$ and V_K' its corresponding Kreĭn extension.
Now, let $\mathcal{V} \subset \mathcal{D}(\widetilde{A}'^{*})//\mathcal{D}(A')$ be a subspace. The operator $A'_{\mathcal{V}}$ is a proper dissipative extension of the dual pair A and $\widetilde{A}$ if and only if all of the following conditions are satisfied

- $\mathcal{V} \subset \mathcal{D}({V_K'}^{1/2})$
- $Im\langle v, \widetilde{A}^{*} v\rangle \geq \|{V_K'}^{1/2} v\|^2$ *for all* $v \in \mathcal{V}$
- $\mathcal{D}(A) \subset \mathcal{D}(A'_{\mathcal{V}})$
- $\mathcal{V} \subset \mathcal{D}(\widetilde{A}^{*})$.

PROOF. Since $\mathcal{D}(A) \cap \mathcal{D}(\widetilde{A})$ is dense, the operator $A \restriction_{\mathcal{D}(A)\cap\mathcal{D}(\widetilde{A})}$ is a densely defined dissipative operator and thus closable. Moreover, since

$$\mathrm{Im}\langle \psi, A\psi\rangle = \mathrm{Im}\langle \widetilde{A}\psi, \psi\rangle = -\mathrm{Im}\langle \psi, \widetilde{A}\psi\rangle \geq 0 \quad \text{for all} \quad \psi \in \mathcal{D}(A) \cap \mathcal{D}(\widetilde{A}) \; ,$$

this shows that $\widetilde{A} \restriction_{\mathcal{D}(A)\cap\mathcal{D}(\widetilde{A})}$ is a densely defined antidissipative operator. Thus, by construction, the operators A' and $\widetilde{A}'$ are closed operators, which have the common core property. Moreover,

$$A' \subset A \subset \widetilde{A}^{*} \subset \widetilde{A}'^{*} \; ,$$

from which it follows that any proper dissipative extension of the dual pair $(A, \widetilde{A})$ is a proper extension of the dual pair $(A', \widetilde{A}')$ as well. The corollary now follows from the observation that its first two conditions simply correspond to an application of Theorem 5.2.8 for the dual pair $(A', \widetilde{A}')$ (which has the common core property) to ensure that

$A'_{\mathcal{V}}$ is a dissipative extension of A'. The latter two conditions ensure that $A'_{\mathcal{V}}$ is not just a proper extension of the dual pair $(A', \widetilde{A}')$ but also of $(A, \widetilde{A})$. $\square$

REMARK 5.3.2. Since the dual pair $(A', \widetilde{A}')$ has the common core property and A is a proper dissipative extension of this dual pair, Theorem 5.2.13 implies that there exists a contraction C from $\mathcal{W}'_+$ into $\mathcal{W}'_-$ and a subspace $\mathfrak{M}' \subset \mathcal{W}'_+ \oplus \mathcal{W}'_0$ such that $A = A'_{\mathfrak{M}',C}$, where the notation is the same as employed in (5.2.10). As any proper dissipative extension of the dual pair $(A, \widetilde{A})$ has to be a proper dissipative extension of $(A', \widetilde{A}')$ as well, to which Theorem 5.2.13 applies, this means that the problem of finding the proper dissipative extensions of $(A, \widetilde{A})$ is equivalent to determining $(\mathfrak{N}, \widehat{C})$, where $\mathfrak{M}' \subset \mathfrak{N}$ and $\widehat{C}$ is a contractive extension of C with the additional constraint that $A_{\mathfrak{N},\widehat{C}} \subset \widetilde{A}^*$. For a full discussion of determining the contractive extensions of a given contraction, see [**12**].

5.4. Illustrating examples

In the following, we are going to apply our results to various ODE examples, which we have chosen to illustrate our results without having to worry too much about technicalities.

5.4.1. Weakly perturbed symmetric operators. As a first application of Theorem 5.2.8, let us consider dual pairs of operators of the form $A = S+iV$ and $\widetilde{A} = S-iV$, where S is closed and symmetric and V is a positive symmetric operator, which has S^*-bound less than one.[5.4.1] For convenience, let us recall the definition of relative boundedness:

DEFINITION 5.4.1. *Let A and B be two operators on a Hilbert space $\mathcal{H}$. We say that B is relatively bounded with respect to A if $\mathcal{D}(A) \subset \mathcal{D}(B)$ and there exists numbers $a, b \geq 0$ such that*

$$\|Bf\| \leq a\|f\| + b\|Af\| \tag{5.4.1}$$

for all $f \in \mathcal{D}(A)$. The infimum over all possible b such that there still exists an a such that (5.4.1) *is still satisfied is called the A-bound of B.*

[5.4.1] Actually, we could consider dual pairs of the form $(S+D, S+\widetilde{D})$, where $(D, \widetilde{D})$ is a dual pair of dissipative/antidissipative perturbations, which are both relatively bounded with respect to S^* with relative bound less than 1.

THEOREM 5.4.2. *Let S be a closed symmetric operator and V be a non-negative symmetric operator with S^*-bound less than 1. Moreover, let $\mathfrak{d}(A,\widetilde{A})$ denote the set of proper dissipative extensions of the dual pair $(A,\widetilde{A})$. Then, the set of all proper dissipative extensions of the dual pair $S+iV$ and $S-iV$ is given by*

$$\mathfrak{d}(S+iV,S-iV)=\{\widehat{S}+iV:\widehat{S}\in\mathfrak{d}(S,S)\}\,.$$

PROOF. Firstly, let us apply Theorem 5.2.8 to the dual pair (S,S), where S is closed and symmetric. In this case, the operator $(S-S)/(2i)$ is identical to the zero operator on $\mathcal{D}(S)$, which has a unique bounded selfadjoint extension to the zero operator $0_{\mathcal{H}}$ on the whole Hilbert space $\mathcal{H}$. Thus, for any extension $S_{\mathcal{V}}$, where $\mathcal{V}\subset\mathcal{D}(S^*)//\mathcal{D}(S)$, we trivially have $\mathcal{V}\subset\mathcal{D}(0_{\mathcal{H}})=\mathcal{H}$. Thus, $\mathcal{V}$ needs only to satisfy the condition

$$\operatorname{Im}\langle v,S^*v\rangle\geq 0\quad\text{for all}\quad v\in\mathcal{V}\,. \tag{5.4.2}$$

Next, let us consider the dual pair $(S+iV,S-iV)$. By the Hess–Kato Theorem [**25**, Corollary 1], we get that $(S-iV)^*=S^*+iV$, which we use together with Theorem 5.2.8. By relative boundedness, we therefore have $\mathcal{D}((S-iV)^*)=\mathcal{D}(S^*)$ as well as $\mathcal{D}(S+iV)=\mathcal{D}(S)$, which means that we can choose $\mathcal{D}((S-iV)^*)//\mathcal{D}(S+iV)=\mathcal{D}(S^*)//(S)$. Now, observe that

$$\operatorname{Im}\langle v,(S-iV)^*v\rangle=\operatorname{Im}\langle v,(S^*+iV)v\rangle=\operatorname{Im}\langle v,S^*v\rangle+\langle v,Vv\rangle$$

and that

$$\langle v,Vv\rangle=\|V_K^{1/2}v\|^2\quad\text{for all}\quad v\in\mathcal{D}(S^*)=\mathcal{D}(S^*+iV)\,,$$

which follows from relative boundedness of V with respect to S^*. Hence, again we have that $\mathcal{V}\subset\mathcal{D}(V_K^{1/2})$ is always satisfied for any $\mathcal{V}\subset\mathcal{D}((S-iV)^*)//\mathcal{D}(S+iV)$. This implies that $\mathcal{V}$ only needs to satisfy

$$\operatorname{Im}\langle v,(S-iV)^*v\rangle\geq\|V_K^{1/2}v\|^2\quad\text{which is equivalent to}\quad\operatorname{Im}\langle v,S^*v\rangle\geq 0\quad\text{for all}\quad v\in\mathcal{V}.$$

However, since this is equivalent to Condition (5.4.2), we get that $(S+iV)_{\mathcal{V}}$ is dissipative if and only if $S_{\mathcal{V}}$ is dissipative. $\square$

Let us start with the elementary example of a first order differential operator.

Example 5.4.3. Consider the closed symmetric operator on $L^2(0,1)$, given by

$$S: \quad \mathcal{D}(S) = \{f \in H^1(0,1) : f(0) = f(1) = 0\}, \quad f \mapsto if' ,$$

where f' denotes the weak derivative of f. Its adjoint S^* is given by

$$S^*: \quad \mathcal{D}(S^*) = H^1(0,1), \quad f \mapsto if' .$$

Since for any $f \in \mathcal{D}(S^*)$, we have that

$$\operatorname{Im}\langle f, S^* f\rangle = \frac{1}{2}\left[|f(1)|^2 - |f(0)|^2\right] ,$$

it follows that all dissipative extensions of S are given by

$$S_c: \quad \mathcal{D}(S_c) := \left\{f \in H^1(0,1) : f(0) = cf(1)\right\}, \quad S_c = S^* \restriction_{\mathcal{D}(S_c)} ,$$

where c is any complex number such that $|c| \leq 1$. Using Lemma 2.3.8, it is in fact not hard to see that these extensions are also maximal.
Moreover, let V be the selfadjoint maximal multiplication operator by a non-negative and non-zero L^2-function $V(x)$:

$$V: \quad \mathcal{D}(V) = \left\{f \in L^2(0,1) : \int_0^1 |V(x)f(x)|^2 \mathrm{d}x < \infty\right\}, \quad (Vf)(x) = V(x)f(x) .$$

For example, one could pick $V(x) = x^{-\alpha}$ with $0 < \alpha < 1/2$. Using that $H^1(0,1)$ compactly embeds into the bounded continuous functions $\mathcal{C}([0,1])$ we may use that by Ehrling's Lemma there exists for any $\varepsilon > 0$ a $C(\varepsilon)$ such that

$$\|f\|_\infty \leq \varepsilon\|f'\| + C(\varepsilon)\|f\| , \tag{5.4.3}$$

for all $f \in H^1(0,1)$. This allows us to show that V is S^*-bounded with S^*-bound equal to zero:

$$\|Vf\|_2 \leq \|V\|_2\|f\|_\infty \overset{(5.4.3)}{\leq} \varepsilon\|V\|_2\|f'\|_2 + C(\varepsilon)\|V\|_2\|f\|_2 ,$$

where $\varepsilon\|V\|_2$ can be made arbitrarily small. Thus, for any non-negative $V \in L^2(0,1)$, we may conclude that all proper dissipative extensions of the dual pair $S + iV$ and $S - iV$ are given by $S_c + iV$ by virtue of Theorem 5.4.2.

REMARK 5.4.4. Using that V is S^*-bounded with relative bound equal to zero, we have in particular that V is S_c-bounded with relative bound equal to zero as well. Thus, by the Hess–Kato Theorem [**25**, Corollary 1]

$$-(S_c + iV)^* = -(S_c)^* + iV \ .$$

By Proposition 2.2.5, we have that $-(S_c)^*$ is dissipative, which makes $-(S_c)^* + iV$ dissipative. By the same proposition, we therefore may conclude that $S_c + iV$ is maximally dissipative.

5.4.2. Differential operators with dissipative potentials. For any $n \in \mathbb{N}$, let p_0^n be the symmetric differential operator defined as follows

$$p_0^n : \quad \mathcal{D}(p_0^n) = \mathcal{C}_0^\infty(0,1), \quad f \mapsto i^n f^{(n)} \ ,$$

where $f^{(n)}$ denotes the n^{th} derivative of f. Moreover, let $W \in L^2_{\text{loc}}(0,1)$ be a locally square-integrable potential function with $W \geq 0$ almost everywhere. This means that the dual pair of operators

$$A_0 : \quad \mathcal{D}(A_0) = \mathcal{C}_0^\infty(0,1), \quad (A_0 f)(x) = i^n f^{(n)}(x) + iW(x)f(x) \tag{5.4.4}$$

and

$$\widetilde{A}_0 : \quad \mathcal{D}(\widetilde{A}_0) = \mathcal{C}_0^\infty(0,1), \quad \left(\widetilde{A}_0 f\right)(x) = i^n f^{(n)}(x) - iW(x)f(x) \tag{5.4.5}$$

is well defined. Moreover, their closures $A := \overline{A_0}$ and $\widetilde{A} := \overline{\widetilde{A}_0}$ have the common core property by construction. In Theorem 5.2.8, the operator V is defined as $\frac{A-\widetilde{A}}{2i}$ on a common core $\mathcal{D} \subset (\mathcal{D}(A) \cap \mathcal{D}(\widetilde{A}))$ and we choose $\mathcal{D} = \mathcal{C}_c^\infty(0,1)$. Since V is already essentially selfadjoint, this implies that the Kreĭn extension of V coincides with its closure $V_K = \overline{V}$ and is given by the maximal multiplication operator by the function $W(x)$. Thus, $V_K^{1/2}$ is given by

$$V_K^{1/2} : \qquad \mathcal{D}(V_K^{1/2}) = \left\{ f \in L^2(0,1) : \int_0^1 W(x)|f(x)|^2 \mathrm{d}x < \infty \right\}$$
$$(V_K f)(x) = \sqrt{W(x)} f(x) \ .$$

Moreover, it can be easily shown that the domains of A^* and $\widetilde{A}^*$ are given by

$$\begin{aligned}\widetilde{A}^*: \quad & \mathcal{D}(\widetilde{A}^*) = \big\{f \in L^2(0,1) : f \in H^n_{\mathrm{loc}}(0,1) \cap L^2(0,1); i^n f^{(n)} + iWf \in L^2\big\} \\ & \qquad f \mapsto i^n f^{(n)} + iWf\,, \\ A^*: \quad & \mathcal{D}(A^*) = \big\{f \in L^2(0,1) : f \in H^n_{\mathrm{loc}}(0,1) \cap L^2(0,1); i^n f^{(n)} - iWf \in L^2\big\} \\ & \qquad f \mapsto i^n f^{(n)} - iWf\,,\end{aligned}$$

with the understanding that $f^{(n)}$ denotes the n^{th} weak derivative of f. By Theorem 5.2.8, the operator $A_{\mathcal{V}}$ (cf. Definition 5.2.7) is only maximally dissipative if $\mathcal{V} \subset \mathcal{D}(V_K^{1/2})$. Thus for any $v \in \mathcal{V}$

$$\int_0^1 |v(x)|^2 W(x) \mathrm{d}x < \infty \tag{5.4.6}$$

and since $v \in \mathcal{D}(\widetilde{A}^*) \subset L^2(0,1)$, which implies that $i^n v^{(n)} + iWv \in L^2(0,1)$, it follows that

$$\overline{v(x)} i^n v^{(n)}(x) + i|v(x)|^2 W(x) \in L^1(0,1)\,. \tag{5.4.7}$$

From the above — together with (5.4.6) and an application of the reverse triangle inequality — it follows that

$$\int_0^1 \left|\overline{v(x)} i^n v^{(n)}(x)\right| \mathrm{d}x < \infty\,,$$

i.e. $\overline{v} v^{(n)} \in L^1(0,1)$. Hence, given that $v \in \mathcal{D}(V_K^{1/2})$ the necessary and sufficient condition for $A_{\mathcal{V}}$ to be dissipative

$$\mathrm{Im}\langle v, \widetilde{A}^* v\rangle \geq \|W^{1/2} v\|^2 \quad \text{for all} \quad v \in \mathcal{V}$$

simplifies to

$$\mathrm{Im}\langle v, i^n v^{(n)}\rangle \geq 0 \quad \text{for all} \quad v \in \mathcal{V}\,. \tag{5.4.8}$$

5.4.3. First order differential operators with singular potentials. Let us apply the result of the previous subsection to the simplest case $n = 1$. For any $\varepsilon > 0$, any $x_0 \in (0,1)$ and any $v \in H^1_{\mathrm{loc}}(0,1) \cap L^2(0,1)$ we have that

$$|v(\varepsilon)|^2 = |v(x_0)|^2 - 2\mathrm{Im} \int_\varepsilon^{x_0} \overline{v(x)} i v'(x) \mathrm{d}x$$

and since $\overline{v}v' \in L^1(0,1)$, we have by an explicit calculation

$$\lim_{\varepsilon\downarrow 0}|v(\varepsilon)|^2 = \lim_{\varepsilon\downarrow 0}\left(|v(x_0)|^2 - 2\mathrm{Im}\int_{\varepsilon}^{x_0}\overline{v(x)}iv'(x)\mathrm{d}x\right) = |v(x_0)|^2 - 2\mathrm{Im}\int_{0}^{x_0}\overline{v(x)}iv'(x)\mathrm{d}x\,.$$

The same reasoning can be applied to show the existence of $\lim_{\varepsilon\downarrow 0}|v(1-\varepsilon)|^2$, which shows that $|v|^2$ is continuous up to the boundary of the interval. Defining, at least formally,

$$|v(0)|^2 := \lim_{\varepsilon\downarrow 0}|v(\varepsilon)|^2 \quad \text{and} \quad |v(1)|^2 := \lim_{\varepsilon\downarrow 0}|v(1-\varepsilon)|^2$$

we get that

$$\text{(5.4.9)} \qquad \mathrm{Im}\langle v, iv'\rangle = \frac{1}{2}\left(|v(1)|^2 - |v(0)|^2\right) \quad \text{for all} \quad v \in H^1_{\mathrm{loc}}(0,1) : \overline{v}v' \in L^1\,.$$

Let us now consider a few different potentials:

EXAMPLE 5.4.5. Let $1/2 \leq \alpha < 1$ and let the potential function be given by $W(x) = \frac{1-\alpha}{x^\alpha}$, where the numerator $(1-\alpha)$ is chosen for convenience (the case $0 < \alpha < 1/2$ has been covered in Example 5.4.3). By an explicit calculation, it can be shown that

$$\mathcal{D}(\widetilde{A}^*) = \mathcal{D}(A)\dot{+}\mathrm{span}\left\{\exp(-x^{1-\alpha}), \exp(-x^{1-\alpha})\int_0^x \exp(2t^{1-\alpha})\mathrm{d}t\right\}$$

and it is easy to see that

$$\mathcal{D}(\widetilde{A}^*)//\mathcal{D}(A) = \mathrm{span}\left\{\exp(-x^{1-\alpha}), \exp(-x^{1-\alpha})\int_0^x \exp(2t^{1-\alpha})\mathrm{d}t\right\}$$
$$\subset \mathcal{D}(V_K^{1/2}) = \mathcal{D}(x^{-\frac{\alpha}{2}})\,,$$

where the last inclusion is guaranteed by the choice $\alpha < 1$. A standard linear transformation shows that it is possible to define two vectors $\phi, \psi \in \mathcal{D}(\widetilde{A}^*)//\mathcal{D}(A)$ such that

$$\mathcal{D}(\widetilde{A}^*) = \mathcal{D}(A)\dot{+}\mathrm{span}\{\phi, \psi\}$$

and ϕ, ψ satisfy the boundary conditions

$$\psi(0) = 1, \psi(1) = 0, \phi(0) = 0, \phi(1) = 1\,.$$

Thus, if we choose two complex numbers $(c_1, c_2) \in \mathbb{C}^2 \setminus \{(0,0)\}$ in order to parametrize all one-dimensional proper extensions of $(A, \widetilde{A})$ as

$$A_{c_1,c_2}: \quad \mathcal{D}(A_{c_1,c_2}) = \mathcal{D}(A)\dot{+}\mathrm{span}\{c_1\phi + c_2\psi\}, \quad A_{c_1,c_2} = \widetilde{A}^* \restriction_{\mathcal{D}(A_{c_1,c_2})}$$

and plug $v_{c_1,c_2} := c_1\phi + c_2\psi$ into (5.4.9), we get the condition that

$$\operatorname{Im}\langle v_{c_1,c_2}, iv'_{c_1,c_2}\rangle = \frac{1}{2}\left(|c_1|^2 - |c_2|^2\right) \geq 0\,,$$

i.e. $|c_1| \geq |c_2|$. Thus, we can parametrize all maximally dissipative proper extensions using only one complex parameter $c = c_2/c_1$ with $|c| \leq 1$ and get $\{A_c : |c| \leq 1\}$, where

$$A_c: \quad \mathcal{D}(A_c) = \mathcal{D}(A)\dot{+}\operatorname{span}\{\phi + c\psi\}, \quad A_c = \widetilde{A}^* \restriction_{\mathcal{D}(A_c)}$$

as a complete description of the set of all proper maximally dissipative extensions.

Let us now consider examples, where the singularity of the potential is of "same strength" as the differential operator ($\alpha = 1$).

Example 5.4.6. Let $0 < \gamma < 1/2$ and consider the potential

$$W(x) = \frac{\gamma}{1-x}.$$

Note that this is equivalent to considering the operator $-i\frac{\mathrm{d}}{\mathrm{d}y} + i\frac{\gamma}{y}$ after the coordinate change $(1-x) \mapsto y$, which leads to a change of sign in front of the differential part of the operator, changing the situation significantly compared to Example 5.4.7.
In this case, a calculation shows that for our range of γ, we have

$$\mathcal{D}(\widetilde{A}^*) = \mathcal{D}(A)\dot{+}\operatorname{span}\{(1-x)^\gamma, (1-x)^{1-\gamma}\}\,.$$

Since $0 < \gamma < 1/2$, it is true that

$$\operatorname{span}\{(1-x)^\gamma, (1-x)^{1-\gamma}\} \subset \mathcal{D}(V_K^{1/2}) = \mathcal{D}\left(\frac{1}{\sqrt{1-x}}\right)$$

and since $\dim\ker A^* = 1$, all proper dissipative extensions of A will be at most one-dimensional extensions, i.e. of the form

$$\mathcal{D}(A_{c_1,c_2}) := \mathcal{D}(A)\dot{+}\operatorname{span}\{c_1(1-x)^\gamma + c_2(1-x)^{1-\gamma}\}\,,$$

where $(c_1, c_2) \in \mathbb{C}^2 \setminus \{(0,0)\}$. Plugging $v_{c_1,c_2} := c_1(1-x)^\gamma + c_2(1-x)^{1-\gamma}$ into Equation (5.4.9), we get the condition

$$\operatorname{Im}\langle v_{c_1,c_2}, iv'_{c_1,c_2}\rangle = -\frac{|c_1+c_2|^2}{2} \geq 0\,, \tag{5.4.10}$$

which is satisfied if and only if $c_1 = -c_2$. Thus, there exists a unique proper maximally dissipative extension of the dual pair $(A, \widetilde{A})$, which is given by

$$A' : \quad \mathcal{D}(A') = \mathcal{D}(A) \dot{+} \mathrm{span}\{(1-x)^\gamma - (1-x)^{1-\gamma}\}, \quad A' = \widetilde{A}^* \restriction_{\mathcal{D}(A')} \ .$$

This is an example of a dual pair $(A, \widetilde{A})$ with a unique proper maximally dissipative extension, which does not satisfy the assumptions of Corollary 5.2.9.

Next, let us compute the spaces $\mathcal{W}_+, \mathcal{W}_0$ and $\mathcal{W}_-$ as defined in Theorem 5.2.13. Since the form q as defined in Equation (5.2.9) is given by

$$q(v) = \mathrm{Im}\langle v, iv'\rangle = \frac{1}{2}(|v(1)|^2 - |v(0)|^2)$$

and is non-positive for $v \in \mathrm{span}\{(1-x)^\gamma, (1-x)^{1-\gamma}\}$ by virtue of Equation (5.4.10), we have found the maximizer of $\langle v, Mv\rangle$ which corresponds to the eigenvalue zero:

$$\mathcal{W}_0 = \ker M = \mathrm{span}\{(1-x)^\gamma - (1-x)^{1-\gamma}\}$$

and — using the Gram-Schmidt procedure — we compute

$$\mathcal{W}_- = \mathrm{span}\{\underbrace{(4\gamma^2 - 8\gamma - 5)(1-x)^\gamma - (4\gamma^2 - 8\gamma + 3)(1-x)^{1-\gamma}}_{=:w_-}\}$$

$$\text{with eigenvalue} \quad \lambda_- = \frac{\langle w_-, Mw_-\rangle}{\langle w_-, w_-\rangle} = \frac{\frac{1}{2}(|w_-(1)|^2 - |w_-(0)|^2)}{\int_0^1 |w_-(x)|^2 \mathrm{d}x} = -\frac{2}{-4\gamma^2 + 4\gamma + 7} \ .$$

Example 5.4.7. Let $0 < \gamma < 1/2$ and consider the potential

$$W(x) = \frac{\gamma}{x} \ .$$

In this case, a calculation shows that $\mathcal{D}(\widetilde{A}^*) = \mathcal{D}(A) \dot{+} \mathrm{span}\{x^{-\gamma}, x^{1+\gamma}\}$. This is an example, for which Corollary 5.2.9 applies, since $\ker \widetilde{A}^* = \mathrm{span}\{x^{-\gamma}\}$ has trivial intersection with $\mathcal{D}(V_K^{1/2}) = \{f \in L^2(0,1), \int_0^1 |f(x)|^2 x^{-1} \mathrm{d}x < \infty\}$. Hence, the only possible candidate for a proper maximally dissipative extension for the dual pair $(A, \widetilde{A})$ is the operator $\widehat{A}$, which is given by

$$\widehat{A} : \quad \mathcal{D}(\widehat{A}) = \mathcal{D}(A) \dot{+} \mathrm{span}\{x^{1+\gamma}\}, \quad \widehat{A} = \widetilde{A}^* \restriction_{\mathcal{D}(\widehat{A})} \ .$$

By Proposition 2.4.5, it is already clear that $\widehat{A}$ has to be a proper maximally dissipative extension. This can also be verified explicitly by by plugging $v(x) := x^{1+\gamma}$ into

Condition (5.4.9).

In this concrete case, we have that $\mathcal{W}_0 = \mathcal{W}_- = \{0\}$ and $\mathcal{W}_+ = \mathrm{span}\{x^{1+\gamma}\}$. A short calculation shows that the corresponding eigenvalue of M is given by

$$\lambda_+ = \frac{\langle x^{1+\gamma}, Mx^{1+\gamma}\rangle}{\langle x^{1+\gamma}, x^{1+\gamma}\rangle} = \frac{3}{2} + \lambda\,.$$

5.4.4. A second order example. Let us now apply our results to an example, where the operator V as defined in the statement of Theorem 5.2.8 is not essentially selfadjoint. To this end, consider the dual pair of operators given by

$$\begin{aligned} A_0: \qquad & \mathcal{D}(A_0) = \mathcal{C}_c^\infty(0,1), \quad (A_0 f)(x) = -if''(x) - \gamma\frac{f(x)}{x^2}, \\ \widetilde{A}_0: \qquad & \mathcal{D}(\widetilde{A}_0) = \mathcal{C}_c^\infty(0,1), \quad \left(\widetilde{A}_0 f\right)(x) = if''(x) - \gamma\frac{f(x)}{x^2}\,. \end{aligned}$$

Since we have

$$\mathrm{Im}\langle f, A_0 f\rangle = \mathrm{Im}\int_0^1 \overline{f(x)}\left(-if''(x) - \gamma\frac{f(x)}{x^2}\right)\mathrm{d}x = \int_0^1 |f'(x)|^2 \mathrm{d}x$$

for all $f \in \mathcal{C}_c^\infty(0,1)$, we can estimate $\mathrm{Im}\langle f, A_0 f\rangle$ from below by the lowest eigenvalue of the Dirichlet-Laplacian on the unit interval, which is π^2, i.e.

$$\mathrm{Im}\langle f, A_0 f\rangle \geq \pi^2\|f\|^2 \quad \text{for all} \quad \psi \in \mathcal{D}(A_0)\,. \tag{5.4.11}$$

Now, define $A := \overline{A_0}$ and $\widetilde{A} := \overline{\widetilde{A}_0}$, which means that the dual pair $(A, \widetilde{A})$ has the common core property by construction. Also, (5.4.11) implies in particular that $0 \in \widehat{\rho}(A)$. By a simple calculation, it can be shown that the operator $\widetilde{A}^*$ is given by:

$$\mathcal{D}(\widetilde{A}^*) = \left\{ f \in H^2_{loc}(0,1) \cap L^2(0,1) : \int_0^1 \left| -if''(x) - \gamma\frac{f(x)}{x^2}\right|^2 \mathrm{d}x < \infty \right\}$$

$$\left(\widetilde{A}^* f\right)(x) = -if''(x) - \gamma\frac{f(x)}{x^2}\,.$$

A calculation, using Formula (2.4.1) for $\lambda = 0$, yields

$$\mathcal{D}(\widetilde{A}^*) = \mathcal{D}(A) \dot{+} \mathrm{span}\left\{x^\omega, x^{\overline{\omega}+2}\right\}\,, \tag{5.4.12}$$

where $\omega := (1 + \sqrt{1 + 4i\gamma})/2$. Here we have assumed that $\gamma \geq \sqrt{3}$. This choice for γ ensures that $\dim\ker\widetilde{A}^* = \dim\ker A^* = 1$, which will make our calculations simpler. Also, observe that $\widetilde{A}^* = JA^*J$, where the conjugation operator J is defined as $(Jf)(x) := \overline{f(x)}$. From this it immediately follows that $\mathcal{D}(A^*) = J\mathcal{D}(\widetilde{A}^*) = \{f : \overline{f} \in$

$\mathcal{D}(\widetilde{A}^*)\}$. Now, let us apply the result of Theorem 5.2.8 in order to construct maximally dissipative extensions of the dual pair $(A, \widetilde{A})$. Let $\mathcal{D} = \mathcal{C}_c^\infty(0,1)$, which is a common core for A and $\widetilde{A}$ and define $V := \frac{1}{2i}(A - \widetilde{A}) \restriction_{\mathcal{D}}$, which is given by

$$V: \qquad \mathcal{D}(V) = \mathcal{C}_c^\infty(0,1), \quad f \mapsto -f''.$$

As the norm induced by $\|\cdot\|_V := \|\cdot\| + \langle \cdot, V \cdot \rangle$ is the H^1-norm, closing $\mathcal{D}(V) = \mathcal{C}_c^\infty(0,1)$ with respect to $\|\cdot\|_V$ yields that $\mathcal{D}(V_F^{1/2}) = H_0^1(0,1)$. Moreover, since $\ker V^* = \operatorname{span}\{1, x\}$ and since by (5.2.1) we have $\mathcal{D}(V_K^{1/2}) = \mathcal{D}(V_F^{1/2}) \dot{+} \ker V^*$, it is clear that $\mathcal{D}(V_K^{1/2}) = H^1(0,1)$ and moreover that

$$\|V_K^{1/2} f\|^2 = \|V_F^{1/2}\left[f(x) - f(0) - x(f(1) - f(0))\right]\|^2 = \|f'\|^2 - |f(1) - f(0)|^2, \tag{5.4.13}$$

where the first equality follows from the decomposition (5.2.1) and the second from an explicit calculation. Using this, we can show that the form $q(v) := \operatorname{Im}\langle v, \widetilde{A}^* v\rangle - \|V_K^{1/2} v\|^2$ defined on $\mathcal{D}(\widetilde{A}^*)//\mathcal{D}(A) = \operatorname{span}\{x^\omega, x^{\overline{\omega}+2}\}$ is given by

$$q(v) = -\operatorname{Re}\left(\overline{v(1)} v'(1)\right) + |v(1)|^2. \tag{5.4.14}$$

By Lemma 2.3.8, any maximally dissipative proper extension of $(A, \widetilde{A})$ can be parametrized by a one-dimensional subspace of $\operatorname{span}\{x^\omega, x^{\overline{\omega}+2}\}$. A convenient basis for this is given by the two functions

$$\psi(x) := \frac{(2 + \overline{\omega_+}) x^{\omega_+} - \omega_+ x^{\overline{\omega_+}+2}}{2 + \overline{\omega_+} - \omega_+} \quad \text{and} \quad \phi(x) := \frac{-x^{\omega_+} + x^{\overline{\omega_+}+2}}{2 + \overline{\omega_+} - \omega_+},$$

which satisfy the boundary conditions $\psi(1) = 1, \psi'(1) = 0, \phi(1) = 0$ and $\phi'(1) = 1$. Now define $\xi_\rho := \rho\psi + \phi$, where $\rho \in \mathbb{C}$ has to be determined such that $q(\xi_\rho) \geq 0$. A short explicit calculation shows that this is the case if and only if

$$\left|\rho - \frac{1}{2}\right| \geq \frac{1}{2},$$

i.e. if and only if ρ lies in the exterior of the open circle with radius and center point $\frac{1}{2}$. Since $q(\psi) = 1 > 0$, we have that $\xi_\infty := \psi$ describes a maximally dissipative extension as well. Thus the set of all proper maximally dissipative extensions of $(A, \widetilde{A})$ is given by

$$A_\rho: \qquad \mathcal{D}(A_\rho) = \mathcal{D}(A) \dot{+} \operatorname{span}\{\xi_\rho\}, \quad A_\rho = \widetilde{A}^* \restriction_{\mathcal{D}(A_\rho)}, \tag{5.4.15}$$

where

$$\rho \in \left\{ z \in \mathbb{C}, \left| z - \frac{1}{2} \right| \geq \frac{1}{2} \right\} \cup \{\infty\} . \tag{5.4.16}$$

5.5. Stability of the numerical range

Let us now prove a useful result that allows us to estimate the lower bound of the imaginary part of the numerical range of the extensions of a dual pair $(A, \widetilde{A})$:

LEMMA 5.5.1. *Let the dual pair $(A, \widetilde{A})$ satisfy the assumptions of Theorem 5.2.8 and let $\mathcal{V}$ be a subspace of $\mathcal{D}(\widetilde{A}^*)//\mathcal{D}(A)$ such that $\mathcal{D}(A_{\mathcal{V}})$ is a proper dissipative extension of the dual pair $(A, \widetilde{A})$. Moreover, for $v \in \mathcal{V}$, let $q(v) := Im\langle v, \widetilde{A}^* v\rangle - \|V_K^{1/2} v\|^2$. Then, for all $f \in \mathcal{D}(A), v \in \mathcal{V}$ it is true that*

$$Im\langle (f+v), A_{\mathcal{V}}(f+v)\rangle = \|V_K^{1/2}(f+v)\|^2 + q(v) \geq \|V_K^{1/2}(f+v)\|^2 .$$

PROOF. Let $f \in \mathcal{D}$ and $v \in \mathcal{V}$. As in the proof of Theorem 5.2.8, we use Lemma 5.2.1, from which we know that it is sufficient to check the assertion for only such f and v. From an explicit calculation, we get

$$\begin{aligned}
\mathrm{Im}\langle (f+v), A_{\mathcal{V}}(f+v)\rangle &= \mathrm{Im}\langle (f+v), \widetilde{A}^*(f+v)\rangle \\
&= \mathrm{Im}\langle f, Af\rangle + \mathrm{Im}\langle v, \widetilde{A}^* v\rangle + \mathrm{Im}(\langle f, \widetilde{A}^* v\rangle + \langle v, \widetilde{A}^* f\rangle) \\
&= \mathrm{Im}\langle f, Af\rangle + q(v) + \|V_K^{1/2} v\|^2 + \mathrm{Im}(\langle f, \widetilde{A}^* v\rangle + \langle v, \widetilde{A}^* f\rangle) . \qquad (5.5.1)
\end{aligned}$$

Now, we can use that $\mathrm{Im}\langle f, Af\rangle = \langle f, Vf\rangle$, which implies in particular that $f \in \mathcal{D} \subset \mathcal{D}(V_K) \subset \mathcal{D}(V_K^{1/2})$ since V_K is a selfadjoint extension of V. Thus, we have that

$$\mathrm{Im}\langle f, Af\rangle = \langle f, Vf\rangle = \|V_K^{1/2} f\|^2$$

and another calculation — similar to that in the proof of Theorem 5.2.8 — shows that

$$\mathrm{Im}(\langle f, \widetilde{A}^* v\rangle + \langle v, \widetilde{A}^* f\rangle) = 2\mathrm{Re}\langle V_K^{1/2} f, V_K^{1/2} v\rangle .$$

Plugging these two identities back into (5.5.1) yields

$$\begin{aligned}
\mathrm{Im}\langle (f+v), \widetilde{A}^*(f+v)\rangle &= \|V_K^{1/2} f\|^2 + 2\mathrm{Re}\langle V_K^{1/2} f, V_K^{1/2} v\rangle + \|V_K^{1/2} v\|^2 + q(v) \\
&= \|V_K^{1/2}(f+v)\|^2 + q(v) .
\end{aligned}$$

Since by Theorem 5.2.8 we have that $q(v) \geq 0$ for all $v \in \mathcal{V}$ it trivially follows that

$$\mathrm{Im}\langle f+v, A_{\mathcal{V}}(f+v)\rangle \geq \|V_K^{1/2}(f+v)\|^2$$

for all $f \in \mathcal{D}(A)$ and $v \in \mathcal{V}$. □

Example 5.5.2. As a first example, consider the dual pair $(A, \widetilde{A})$ from Section 5.4.4, with the maximally dissipative extensions A_ρ as described in (5.4.15) and (5.4.16). Again, it suffices to find a lower bound of $\mathrm{Im}\langle f+v, \widetilde{A}^*(f+v)\rangle$ for all $f \in \mathcal{C}_c^\infty(0,1)$ and all $v \in \mathrm{span}\{\xi_\rho\}$, where ξ_ρ was defined in Section 5.4.4. Observe that

$$\mathrm{Im}\langle f+v, A_\rho(f+v)\rangle = \|f'+v'\|^2 - \frac{\mathrm{Re}(\rho)}{|\rho|^2}|v(1)|^2 =: \mathfrak{a}(f+v)$$

and $\mathcal{C}_c^\infty(0,1)\dot{+}\mathrm{span}\{\xi_\rho\} \subset \mathfrak{C}$, where $\mathfrak{C} := \{f \in H^1(0,1) : f(0) = 0\}$. For the special cases $\rho = 0$ and $\rho = \infty$, we have

$$\mathrm{Im}\langle f+v, A_\rho(f+v)\rangle = \|f'+v'\|^2 =: \mathfrak{a}(f+v)\ .$$

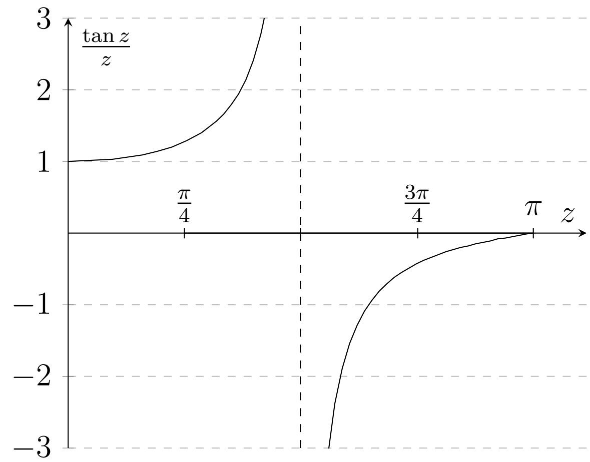
Now, since $\mathfrak{C}$ equipped with the norm induced by $\mathfrak{a}$ is a Hilbert space, this implies that $\mathrm{Im}\langle f+v, A_{\mathcal{V}}(f+v)\rangle \geq \lambda_\rho\|f+v\|^2$, where λ_ρ is the lowest eigenvalue of the selfadjoint operator S_ρ associated to $(\mathfrak{a}, \mathfrak{C})$. This operator is given by

$$S_\rho : \qquad \mathcal{D}(S_\rho) = \left\{f \in H^2(0,1) : f(0) = 0 \text{ and } f'(1) = \frac{\mathrm{Re}(\rho)}{|\rho|^2}f(1)\right\}, \quad f \mapsto -f''\ ,$$

with the understanding that the case $\rho = 0$ corresponds to a Dirichlet boundary condition at one. As it is not difficult to solve the eigenvalue equation $S_\rho f = \lambda_\rho f$, where λ_ρ is the smallest eigenvalue of S_ρ, one finds that λ_ρ is given by $\lambda_\rho = z^2$, where z is the smallest positive solution of the transcendental equation

$$\frac{\tan z}{z} = \frac{|\rho|^2}{\mathrm{Re}(\rho)}\ ,$$

where $\rho \in \{z \in \mathbb{C} : z \neq 0, \mathrm{Re}(z) = 0\}$ corresponds to the singularity of $\frac{\tan z}{z}$ at $z = \frac{\pi}{2}$. For $\mathrm{Re}(\rho) < 0$, this means in particular that $\mathrm{Im}\langle f+v, A_\rho(f+v)\rangle \geq \frac{\pi^2}{4}\|f+v\|^2$ as can easily be seen from the fact that $(\tan z)/z$ is positive in $[0, \pi/2)$ and non-positive in $(\pi/2, \pi]$. See also the graph of $\tan(z)/z$:



REMARK 5.5.3. In this example, the estimate on the lower bound of the imaginary parts is also sharp. This follows from the fact that closing $\mathcal{C}_0^\infty(0,1)\dot{+}\mathrm{span}\{\xi_\rho\}$ with respect to the norm induced by $\mathfrak{a}$ yields $\mathfrak{C}$ for $\rho \neq 0$ and closing $\mathcal{C}_0^\infty(0,1)\dot{+}\mathrm{span}\{\xi_0\}$ with respect to the H^1-norm yields $H_0^1(0,1)$.

THEOREM 5.5.4. *Let the dual pair $(A, \widetilde{A})$ satisfy the same conditions as in Theorem 5.2.8. If in addition we have that $\mathcal{V} \subset \mathcal{D}(V_F^{1/2})$, we get that the imaginary part of the numerical range stays stable, i.e.*

$$\inf_{z\in\mathcal{N}_A} \mathrm{Im} z = \inf_{z\in\mathcal{N}_{A_\mathcal{V}}} \mathrm{Im} z \, ,$$

where $\mathcal{N}_C$ denotes the numerical range of an operator C and $A_\mathcal{V}$ is the extension of A as described in Definition 5.2.7. This is true in particular for any dissipative extension of a dual pair of operators $(A, \widetilde{A})$, where the associated operator V is essentially selfadjoint.

PROOF. For $f \in \mathcal{D}(A) \cap \mathcal{D}(\widetilde{A})$, we have that $f \in \mathcal{D}(V) \subset \mathcal{D}(V_F^{1/2})$. Now, since by assumption $\mathcal{V} \subset \mathcal{D}(V_F^{1/2})$, we get by virtue of Lemma 5.5.1 that

$$\mathrm{Im}\langle (f+v), \widetilde{A}^*(f+v)\rangle \geq \|V_K^{1/2}(f+v)\|^2 = \|V_F^{1/2}(f+v)\|^2 \, , \tag{5.5.2}$$

for all $f \in \mathcal{D}(A) \cap \mathcal{D}(\widetilde{A})$ and for all $v \in \mathcal{V}$. Using that for all $f \in \mathcal{D}(A) \cap \mathcal{D}(\widetilde{A})$ we have that

$$\mathrm{Im}\langle f, Af\rangle = \langle f, Vf\rangle \, ,$$

which implies that

$$\inf_{z\in\mathcal{N}_A} \mathrm{Im} z = \inf_{x\in\mathcal{N}_V} x = \inf_{x\in\mathcal{N}_{V_F}} x \, ,$$

where the last equality follows from the fact that the numerical range of the Friedrichs extension of a semibounded operator stays stable. Using Inequality (5.5.2), we therefore get

$$\inf_{z\in\mathcal{N}_{A_{\mathcal{V}}}} \mathrm{Im}z \geq \inf_{x\in\mathcal{N}_{V_F}} x = \inf_{z\in\mathcal{N}_{A}} \mathrm{Im}z \,,$$

which together with the trivial estimate for taking the infimum over a larger set

$$\inf_{z\in\mathcal{N}_{A_{\mathcal{V}}}} \mathrm{Im}z \leq \inf_{z\in\mathcal{N}_{A}} \mathrm{Im}z$$

yields the theorem. □

EXAMPLE 5.5.5. As an example, consider the operators $(A_0, \widetilde{A}_0)$ as described in Section 5.4.2, (5.4.4) and (5.4.5). Since the operator $V = \frac{1}{2i}(A_0 - \widetilde{A}_0)$ is given by

$$V: \qquad \mathcal{D}(V) = \mathcal{C}_c^\infty(0,1), \quad (Vf)(x) = W(x)f(x)\,,$$

which is essentially selfadjoint, we get that $V_K^{1/2} = V_F^{1/2} = \overline{V}^{1/2}$ is the maximal multiplication operator by $\sqrt{W(x)}$. Hence by virtue of Theorem 5.5.4, we get that for any proper maximally dissipative extension $A_{\mathcal{V}}$, we have

$$\mathrm{Im}\langle f+v, A_{\mathcal{V}}(f+v)\rangle \geq w\|f+v\|^2\,,$$

where $w := \text{ess}\inf_{x\in(0,1)} W(x) = \inf_{f\in\mathcal{D}(A):\|f\|=1} \mathrm{Im}\langle f, Af\rangle$.

CHAPTER 6

A construction to obtain proper Kreĭn–von Neumann extensions

In this chapter, we will introduce the Kreĭn–von Neumann extension A_K of a dissipative operator A with zero in its field of regularity $\widehat{\rho}(A)$. After showing that it is a maximally dissipative extension of A, we discuss the condition under which A_K is a proper extension of a dual pair $(A, \widetilde{A})$. After that, we propose a construction that yields restrictions $A_0 \subset A$ and $\widetilde{A}_0 \subset \widetilde{A}$ such that A_K is proper extension of $(A_0, \widetilde{A}_0)$.

6.1. The Kreĭn–von Neumann extension

For strictly positive closed symmetric operators S ($S \geq \varepsilon > 0$), it was already established by von Neumann in [**34**, Satz 42] that the extension S_K given by

$$S_K : \quad \mathcal{D}(S_K) = \mathcal{D}(S) \dot{+} \ker(S^* f), \quad S_K = S^* \restriction_{\mathcal{D}(S_K)}$$

is a non-negative selfadjoint extension of S. The analysis of Kreĭn ([**27**]) showed that S_K is the "smallest" non-negative extension of S. For a closed dissipative operator A with $0 \in \widehat{\rho}(A)$, one can still define its Kreĭn–von Neumann extension A_K via

$$A_K : \quad \mathcal{D}(A_K) = \mathcal{D}(A) \dot{+} \ker(A^*), \quad (f + k) \mapsto Af \,, \tag{6.1.1}$$

where $f \in \mathcal{D}(A)$ and $k \in \ker(A^*)$. (See also [**8, 10**] for a definition of the Kreĭn–von Neumann extension of a sectorial operator A even without the requirement that $0 \in \widehat{\rho}(A)$.) In order to prove that A_K is well-defined in the dissipative case, we will need the following lemma in order to justify the use of the direct sum "$\dot{+}$" in the definition of $\mathcal{D}(A_K)$:

Lemma 6.1.1. *Let A be closed and dissipative such that $0 \in \widehat{\rho}(A)$. This implies that*

$$\mathcal{D}(A) \cap \ker(A^*) = \{0\} \,.$$

Proof. Assume that this is not true, i.e. that there exists at least one $f \in \mathcal{D}(A) \cap \ker(A^*)$ with $f \neq 0$. For any $g \in \mathcal{D}(A)$, we then get

$$\begin{aligned}\operatorname{Im}\langle f+g, A(f+g)\rangle &= \operatorname{Im}\langle f, Af\rangle + \operatorname{Im}\langle f, Ag\rangle + \operatorname{Im}\langle g, Af\rangle + \operatorname{Im}\langle g, Ag\rangle \\ &= \operatorname{Im}\langle A^*f, f\rangle + \operatorname{Im}\langle A^*f, g\rangle + \operatorname{Im}\langle g, Af\rangle + \operatorname{Im}\langle g, Ag\rangle = \operatorname{Im}\langle g, Af\rangle + \operatorname{Im}\langle g, Ag\rangle\,.\end{aligned}$$

Next, observe that there exists at least one $\widehat{g} \in \mathcal{D}(A)$ such that $\langle \widehat{g}, Af\rangle \neq 0$, since otherwise we would get that $Af = 0$ by density of $\mathcal{D}(A)$. But $Af = 0$ is a contradiction since $0 \in \widehat{\rho}(A)$ implies that $\ker(A) = \{0\}$. This means that we can choose a suitable $\lambda \in \mathbb{C}$ with appropriate phase and sufficiently large modulus such that

$$\operatorname{Im}\langle \lambda f + \widehat{g}, A(\lambda f + \widehat{g})\rangle = \operatorname{Im}\langle \widehat{g}, A\widehat{g}\rangle - |\lambda||\langle \widehat{g}, Af\rangle| < 0\,,$$

which contradicts the dissipativity of A. This shows the lemma. □

Even though we will not need it in the remainder of this thesis, let us also prove the following corollary:

Corollary 6.1.2. *Let A be a closed dissipative operator such that $0 \in \widehat{\rho}(A)$. Then there exists a boundedly invertible extension $\widehat{A}$, i.e. $A \subset \widehat{A}$ and $0 \in \rho(\widehat{A})$.*

Remark 6.1.3. Note that we do not claim that $\widehat{A}$ is dissipative.

Proof. As $0 \in \widehat{\rho}(A)$, the inverse of A on its range is well defined and bounded:

$$A^{-1}: \quad \operatorname{ran}(A) \to \mathcal{D}(A), \quad Af \mapsto f\,.$$

We claim that the extension $A^{-1} \subset T^{-1}$, with

$$T^{-1}: \quad \mathcal{D}(T^{-1}) = \mathcal{H}, \quad T^{-1}(Af + k) = f + k\,,$$

where $Af \in \operatorname{ran}(A)$ and $k \in \ker A^*$, is bounded and has trivial kernel $\ker(T^{-1}) = \{0\}$. This would imply that the operator

$$T: \quad \mathcal{D}(T) = \mathcal{D}(A) \dot{+} \ker(A^*), \quad T(f+k) = Af + k$$

is an extension of A with the desired properties.

Firstly, let us show that T^{-1} is bounded:

$$\begin{aligned}\|T^{-1}(Af+k)\|^2 = \|f+k\|^2 \leq 2(\|f\|^2+\|k\|^2) \leq 2\varepsilon^{-2}\|Af\|^2 + 2\|k\|^2 \\ \leq C_\varepsilon(\|Af\|^2+\|k\|^2) = C_\varepsilon\|Af+k\|^2\,,\end{aligned}$$

where we have used that from the fact that $0 \in \widehat{\rho}(A)$, there exists a $\varepsilon > 0$ such that $\|Af\| \geq \varepsilon\|f\|$ for all $f \in \mathcal{D}(A)$.

Let us now show that $\ker(T^{-1}) = \{0\}$. As $(Af+k) \in \ker(T^{-1})$ means $T^{-1}(Af+k) = f+k = 0$, we would have $f = -k$, where $f \in \mathcal{D}(A)$ and $k \in \ker(A^*)$. This would imply that $f,k \in \mathcal{D}(A)\cap\ker(A^*)$. But from Lemma 6.1.1, we know that $\mathcal{D}(A)\cap\ker(A^*) = \{0\}$, which implies $f = k = 0$ and therefore $(Af+k) = 0$. This shows that $\ker(T^{-1}) = \{0\}$, which finishes the proof. □

Let us now show that A_K is a maximally dissipative extension of A:

THEOREM 6.1.4. *Let A be closed and dissipative and assume that $0 \in \widehat{\rho}(A)$. Then the operator A_K given by* (6.1.1) *is a maximally dissipative extension of A.*

PROOF. By Lemma 6.1.1, we know that $\mathcal{D}(A) \cap \ker(A^*) = \{0\}$, which means that A_K is well-defined. Moreover, for any $f \in \mathcal{D}(A)$ and any $k \in \ker(A^*)$, we have

$$\begin{aligned}\operatorname{Im}\langle f+k, A_K(f+k)\rangle &= \operatorname{Im}\langle f, Af\rangle + \operatorname{Im}\langle k, Af\rangle \\ &= \operatorname{Im}\langle f, Af\rangle + \operatorname{Im}\langle A^*k, f\rangle = \operatorname{Im}\langle f, Af\rangle \geq 0\,,\end{aligned}$$

which implies that A_K is dissipative. Let us now show that A_K is maximally dissipative. Assume it is not, i.e. that $\widehat{A}$ is a non-trivial dissipative extension of A_K. Hence, there exists a $0 \neq v \in \mathcal{D}(\widehat{A})$ such that $v \notin \mathcal{D}(A_K)$. In order for $\widehat{A}$ to be dissipative, in particular it must satisfy

$$\operatorname{Im}\langle v+k, \widehat{A}(v+k)\rangle = \operatorname{Im}\langle v, \widehat{A}v\rangle + \operatorname{Im}\langle k, \widehat{A}v\rangle \geq 0\,, \tag{6.1.2}$$

for any $k \in \ker(A^*)$. This implies that $\widehat{A}v \perp \ker(A^*)$, since otherwise, there would exist a $\widetilde{k} \in \ker(A^*)$ such that $\langle \widetilde{k}, \widehat{A}v\rangle = 1$. With a suitable choice of $\tau > 0$ large enough, we then would get

$$\operatorname{Im}\langle v + i\tau\widetilde{k}, \widehat{A}(v+i\tau\widetilde{k})\rangle = \operatorname{Im}\langle v, \widehat{A}v\rangle - \tau < 0\,,$$

which means that (6.1.2) would be violated in this case. Thus, for $\widehat{A}$ to be dissipative, it is necessary that $\widehat{A}v \perp \ker(A^*)$, or equivalently that $\widehat{A}v \in \operatorname{ran}(A)$. Here we have used that A is closed and that $0 \in \widehat{\rho}(A)$, which implies that $\operatorname{ran}(A)$ is closed. But $\widehat{A}v \in \operatorname{ran}(A)$ means that there exists a unique $\ell \in \mathcal{D}(A)$ such that

$$\widehat{A}v = A\ell\,. \tag{6.1.3}$$

Now, for any $f \in \mathcal{D}(A)$ consider

$$\begin{aligned}
&\operatorname{Im}\langle f+v-\ell, \widehat{A}(f+v-\ell)\rangle = \operatorname{Im}\langle f+v-\ell, Af+\widehat{A}v-A\ell\rangle \\
\stackrel{(6.1.3)}{=}&\operatorname{Im}\langle f+v-\ell, Af+A\ell-A\ell\rangle = \operatorname{Im}\langle f+v-\ell, Af\rangle \\
=&\operatorname{Im}\langle f, Af\rangle + \operatorname{Im}\langle v-\ell, Af\rangle\,.
\end{aligned}$$

Next, let us show that $(v-\ell) \notin \ker(A^*)$. Assume this is not true, i.e. that there exists a $k \in \ker(A^*)$ such that $(v-\ell) = k$. Since $\ell \in \mathcal{D}(A)$, this would mean that $v = (\ell + k) \in \mathcal{D}(A_K)$, which is impossible. Thus, there exists a $\widetilde{f} \in \mathcal{D}(A)$ such that $\langle v-\ell, A\widetilde{f}\rangle = 1$. Mimicking the argument from before, considering

$$\operatorname{Im}\langle \widetilde{f} + i\tau(v-\ell), \widehat{A}[\widetilde{f} + i\tau(v-\ell)]\rangle = \operatorname{Im}\langle \widetilde{f}, A\widetilde{f}\rangle - \tau < 0\,,$$

where $\tau > \operatorname{Im}\langle \widetilde{f}, A\widetilde{f}\rangle$ is chosen suitably large, shows that $\widehat{A}$ cannot be dissipative. Thus we conclude that there exists no dissipative extension of A_K, which therefore is maximally dissipative. This finishes the proof. $\square$

Now, consider a dual pair $(A, \widetilde{A})$, where A is dissipative and $\widetilde{A}$ is antidissipative such that $0 \in \widehat{\rho}(A) \cap \widehat{\rho}(\widetilde{A})$. The purpose of the next result is to describe when the Kreĭn–von Neumann extension A_K is a proper extension of $(A, \widetilde{A})$:

THEOREM 6.1.5. *Let $(A, \widetilde{A})$ be a dual pair of densely defined operators, where A is dissipative and $\widetilde{A}$ is antidissipative. Moreover, assume that $0 \in \widehat{\rho}(A) \cap \widehat{\rho}(\widetilde{A})$. Then, the Kreĭn–von Neumann extension A_K as defined in* (6.1.1) *satisfies $A \subset A_K \subset \widetilde{A}^*$ if and only if $\ker(A^*) \subset \ker(\widetilde{A}^*)$.*

PROOF. Since $(A, \widetilde{A})$ is a dual pair, we have that $\mathcal{D}(A) \subset \mathcal{D}(\widetilde{A}^*)$. Moreover, it trivially holds that $\ker(\widetilde{A}^*) \subset \mathcal{D}(\widetilde{A}^*)$. Thus, if $\ker(A^*) \subset \ker(\widetilde{A}^*)$, this implies that

$\mathcal{D}(A_K) = \mathcal{D}(A) \dot{+} \ker(A^*) \subset \mathcal{D}(\widetilde{A}^*)$. Now, let $f \in \mathcal{D}(A)$ and $k \in \ker(A^*) \subset \ker(\widetilde{A}^*)$. We then get

$$\widetilde{A}^*(f+k) = Af + \widetilde{A}^*k = Af = Af + A_K k = A_K(f+k) \, ,$$

which shows that A_K is a proper extension of $(A, \widetilde{A})$.

Now, assume that $\ker(A^*) \not\subset \ker(\widetilde{A}^*)$, but A_K is still a restriction of $\widetilde{A}^*$, which we want to lead to a contradiction. The assumptions $\ker(A^*) \not\subset \ker(\widetilde{A}^*)$ and $A_K \subset \widetilde{A}^*$ imply that there exists at least one non-zero $\widehat{k} \in \ker(A^*)$ such that $\widehat{k} \notin \ker(\widetilde{A}^*)$. Moreover, since we assumed that $A_K \subset \widetilde{A}^*$ this implies in particular that $\widehat{k} \in \mathcal{D}(\widetilde{A}^*)$. We therefore get

$$\widetilde{A}^*\widehat{k} = A_K\widehat{k} = 0 \, ,$$

i.e. $\widehat{k} \in \ker(\widetilde{A}^*)$, which is a contradiction. This shows the theorem. $\square$

In the following example, we will discuss dual pairs of the form $(S+iV, S-iV)$, where S is symmetric and $V \geq 0$ is a bounded non-negative operator in order to demonstrate that $\ker(A^*) \subset \ker(\widetilde{A}^*)$ is a rather restrictive condition:

Example 6.1.6. Consider the dual pair $(A, \widetilde{A}) := (S + iV, S - iV)$, where S is a closed symmetric (but not maximally symmetric) operator and $V \geq 0$ is a bounded and non-negative operator. Since $(S \pm iV)^* = (S^* \mp iV)$, the condition $\ker(A^*) \subset \ker(\widetilde{A}^*)$ reads as $\ker(S^* - iV) \subset \ker(S^* + iV)$. Thus, if $\ker(S^* - iV) \subset \ker(S^* + iV)$, any $k \in \ker(S^* - iV)$ has to satisfy

$$S^*k = iVk \quad \text{and} \quad S^*k = -iVk \, ,$$

which implies that $iVk = -iVk$, which is only satisfied if $k \in \ker(V)$. This implies that for $\ker(S^* - iV) \subset \ker(S^* + iV)$ to hold, it is necessary that $\ker(S^* - iV) \subset \ker(V)$, which in turn is equivalent to $\ker(S^* - iV) \subset \ker(S^*)$. Thus, for $\ker(S^* - iV) \subset \ker(S^* + iV)$ to be satisfied, it is necessary that $\ker(S^* - iV) \subset (\ker(V) \cap \ker(S^*))$. Since it is also easy to check that this is sufficient, we have that

$$\ker(S^* - iV) \subset \ker(S^* + iV) \quad \text{if and only if} \quad \ker(S^* - iV) \subset (\ker(V) \cap \ker(S^*)) \, .$$

6.2. Construction of suitable restrictions

Now, given a dual pair $(A, \widetilde{A})$ of closed operators, where A is dissipative and $\widetilde{A}$ is antidissipative with the additional assumption that $0 \in \widehat{\rho}(A) \cap \widehat{\rho}(\widetilde{A})$, we construct a restriction of $\widetilde{A}$, which we denote by $\widetilde{A}_0$ such that A_K is a proper extension of the dual pair $(A, \widetilde{A}_0)$.

Theorem 6.2.1. *Let $(A, \widetilde{A})$ be a dual pair of closed operators, such that A is dissipative and $\widetilde{A}$ is antidissipative. Moreover, assume that $0 \in \widehat{\rho}(A) \cap \widehat{\rho}(\widetilde{A})$ and that the preimage $\widetilde{A}^{-1}(\operatorname{ran}(A) \cap \operatorname{ran}(\widetilde{A})) = \{f \in \mathcal{D}(\widetilde{A}) : \widetilde{A}f \in \operatorname{ran}(A) \cap \operatorname{ran}(\widetilde{A})\}$ is dense. Define the operator $\widetilde{A}_0$ as follows:*

$$\widetilde{A}_0 : \quad \mathcal{D}(\widetilde{A}_0) = \widetilde{A}^{-1}(\operatorname{ran}(A) \cap \operatorname{ran}(\widetilde{A})), \quad \widetilde{A}_0 = \widetilde{A} \restriction_{\mathcal{D}(\widetilde{A}_0)} . \tag{6.2.1}$$

Then, $\widetilde{A}_0$ is a closed and antidissipative restriction of $\widetilde{A}$. Moreover, $(A, \widetilde{A}_0)$ is a dual pair and A_K — the Kreĭn–von Neumann extension of A — satisfies $A \subset A_K \subset \widetilde{A}_0^*$, i.e. it is a proper maximally dissipative extension of the dual pair $(A, \widetilde{A}_0)$.

Proof. By assumption, $\widetilde{A}_0$ is a densely defined restriction of $\widetilde{A}$, from which we get that $A \subset \widetilde{A}^* \subset \widetilde{A}_0^*$, which means that $(A, \widetilde{A}_0)$ is a dual pair. Moreover, $\widetilde{A}_0 \subset \widetilde{A}$ implies in particular that $\widetilde{A}_0$ is antidissipative and that $0 \in \widehat{\rho}(\widetilde{A}_0)$. Now, since $0 \in \widehat{\rho}(A) \cap \widehat{\rho}(\widetilde{A})$ and since A and $\widetilde{A}$ are closed by assumption, this implies that $\operatorname{ran}(A)$ and $\operatorname{ran}(\widetilde{A})$ are closed. From this, we get that $\operatorname{ran}(\widetilde{A}_0) = \operatorname{ran}(A) \cap \operatorname{ran}(\widetilde{A})$ is the intersection of two closed subspaces and therefore closed itself. This, together with $0 \in \widehat{\rho}(\widetilde{A}_0)$ implies that $\widetilde{A}_0$ is closed as well. Moreover, since $\operatorname{ran}(\widetilde{A}_0) = \operatorname{ran}(A) \cap \operatorname{ran}(\widetilde{A})$, we get

$$\ker(\widetilde{A}_0^*) = \operatorname{ran}(\widetilde{A}_0)^\perp = \Big(\operatorname{ran}(A) \cap \operatorname{ran}(\widetilde{A})\Big)^\perp = \overline{\ker(A^*) + \ker(\widetilde{A}^*)} ,$$

where the last equality follows from Lemma 9.3.7 which is proved in the Appendix. This implies that $\ker(A^*) \subset \ker(\widetilde{A}_0^*)$, which means that the dual pair $(A, \widetilde{A}_0)$ satisfies the assumptions of Theorem 6.1.5. Hence, $A \subset A_K \subset \widetilde{A}_0^*$, which means that A_K is a proper maximally dissipative extension of the dual pair $(A, \widetilde{A}_0)$. □

Remark 6.2.2. Later, we will also use the operator A_0 given by

$$A_0 : \quad \mathcal{D}(A_0) = A^{-1}(\operatorname{ran}(A) \cap \operatorname{ran}(\widetilde{A})), \quad A_0 = A \restriction_{\mathcal{D}(A_0)} . \tag{6.2.2}$$

By a completely analogous reasoning as in the proof of Theorem 6.2.1, A_0 is a closed dissipative restriction with $0 \in \widehat{\rho}(A_0)$. Moreover, since $A_0 \subset A \subset \widetilde{A}^* \subset \widetilde{A}_0^*$ we have that $(A_0, \widetilde{A}_0)$ is a dual pair.

Let us also describe the action of $\widetilde{A}_0^*$:

Corollary 6.2.3. *Let $A, \widetilde{A}$ and $\widetilde{A}_0$ be defined as in Theorem 6.2.1. Moreover, let $\widehat{A}$ be a proper extension of $(A, \widetilde{A})$ such that $0 \in \rho(\widehat{A})$, which we know by Proposition 2.4.3 to always exist. Then, the operator $\widetilde{A}_0^*$ is given by*

$$\widetilde{A}_0^* : \quad \mathcal{D}(\widetilde{A}_0^*) = \mathcal{D}(\widehat{A}) \dot{+} \overline{\ker(A^*) + \ker(\widetilde{A}^*)}, \quad (f+k) \mapsto \widehat{A}f\,, \tag{6.2.3}$$

where $f \in \mathcal{D}(\widehat{A})$ and $k \in \overline{\ker(A^*) + \ker(\widetilde{A}^*)}$. Moreover, if

$$\overline{\ker(A^*) + \ker(\widetilde{A}^*)} = \ker(\widetilde{A}^*) \dot{+} \ker(A^*)\,,$$

this implies that

$$\widetilde{A}_0^* : \quad \mathcal{D}(\widetilde{A}_0^*) = \mathcal{D}(\widetilde{A}^*) \dot{+} \ker(A^*)\,, \quad f + k \mapsto \widetilde{A}^* f\,, \tag{6.2.4}$$

Proof. The description of $\widetilde{A}_0^*$ as given in (6.2.3) follows from an application of Proposition 2.4.3 to the dual pair $(A, \widetilde{A}_0)$ with the choice $\lambda = 0$ using that $\widehat{A} \subset \widetilde{A}^*$ implies that $\widehat{A} \subset \widetilde{A}_0^*$. Under the additional assumption that $\overline{\ker(\widetilde{A}^*) + \ker(A^*)} = \ker(\widetilde{A}^*) \dot{+} \ker(A^*)$, we may use that $\mathcal{D}(\widetilde{A}^*) = \mathcal{D}(\widehat{A}) \dot{+} \ker(\widetilde{A}^*)$, which can again be seen from an application of Proposition 2.4.3 to the dual pair $(A, \widetilde{A})$. We then get that

$$\mathcal{D}(\widetilde{A}_0^*) = \mathcal{D}(\widehat{A}) \dot{+} \overline{\ker(\widetilde{A}^*) + \ker(A^*)} = \mathcal{D}(\widehat{A}) \dot{+} \ker(\widetilde{A}^*) \dot{+} \ker(A^*) = \mathcal{D}(\widetilde{A}^*) \dot{+} \ker(A^*)\,.$$

Since $\widetilde{A}^* \subset \widetilde{A}_0^*$ and $\ker(A^*) \subset \ker(\widetilde{A}_0^*)$, this also proves that $\widetilde{A}_0^*(f+k) = \widetilde{A}^* f$ for any $f \in \mathcal{D}(\widetilde{A}^*)$ and any $k \in \ker(A^*)$. $\square$

In the statement of Theorem 6.2.1, we have assumed that $\widetilde{A}_0$ is densely defined. Under the assumption that $\overline{\ker(A^*) + \ker(\widetilde{A}^*)} = \ker(A^*) + \ker(\widetilde{A}^*)$, which is always satisfied in the finite-dimensional case, let us give a necessary and sufficient condition for A_0 and $\widetilde{A}_0$ to be densely defined:

Lemma 6.2.4. *Let $(A, \widetilde{A})$ be a dual pair of closed operators such that $0 \in \widehat{\rho}(A) \cap \widehat{\rho}(\widetilde{A})$. Moreover, assume that $\overline{\ker(A^*) + \ker(\widetilde{A}^*)} = \ker(A^*) + \ker(\widetilde{A}^*)$ and let $\mathcal{D}(A_0) :=$*

$A^{-1}(ran(A) \cap ran(\widetilde{A}))$ *and* $\mathcal{D}(\widetilde{A}_0) = \widetilde{A}^{-1}(ran(A) \cap ran(\widetilde{A}))$ *. Then,* $\mathcal{D}(A_0)$ *is dense in* $\mathcal{H}$ *if and only if*

$$\mathcal{D}(A^*) \cap \ker(\widetilde{A}^*) \subset \ker(A^*)$$

and $\mathcal{D}(\widetilde{A}_0)$ *is dense in* $\mathcal{H}$ *if and only if*

$$\mathcal{D}(\widetilde{A}^*) \cap \ker(A^*) \subset \ker(\widetilde{A}^*) \,.$$

PROOF. We will only show that $\mathcal{D}(A_0)$ is dense if and only if

$$\mathcal{D}(A^*) \cap \ker(\widetilde{A}^*) \subset \ker(A^*) \,,$$

since the condition for $\mathcal{D}(\widetilde{A}_0)$ being dense follows from completely analogous reasoning.

We start by showing that if there exists a $\psi \in \mathcal{D}(A^*) \cap \ker(\widetilde{A}^*)$ such that $\psi \notin \ker(A^*)$, we have that $\mathcal{D}(A_0)$ is not dense. Since $\psi \in \ker(\widetilde{A}^*) = \operatorname{ran}(\widetilde{A})^\perp$, we get for all $f \in (\operatorname{ran}(A) \cap \operatorname{ran}(\widetilde{A}))$

$$0 = \langle \psi, f \rangle = \langle \psi, AA^{-1}f \rangle = \langle A^*\psi, A^{-1}f \rangle \,,$$

which means that $A^*\psi \perp A^{-1}(\operatorname{ran}(A) \cap \operatorname{ran}(\widetilde{A})) = \mathcal{D}(A_0)$. Since $\psi \notin \ker(A^*)$, we have $A^*\psi \neq 0$, which implies that $A^*\psi \perp \mathcal{D}(A_0)$, i.e. $\mathcal{D}(A_0)$ is not dense.

Conversely, let us now show that if $\mathcal{D}(A^*) \cap \ker(\widetilde{A}^*) \subset \ker(A^*)$, this implies that $\mathcal{D}(A_0)$ is dense. Let $\psi \perp \mathcal{D}(A_0)$, which means that

$$\langle \psi, A^{-1}f \rangle = 0 \tag{6.2.5}$$

for all $f \in (\operatorname{ran}(A) \cap \operatorname{ran}(\widetilde{A}))$. Since $0 \in \widehat{\rho}(A) \cap \widehat{\rho}(\widetilde{A})$, Proposition 2.4.3 implies that there exists an extension $A \subset \widehat{A} \subset \widetilde{A}^*$ such that $0 \in \rho(\widehat{A})$ and moreover, $A^{-1} \subset \widehat{A}^{-1}$, where $\mathcal{D}(\widehat{A}^{-1}) = \mathcal{H}$. Then, for any $f \in \operatorname{ran}(A) \cap \operatorname{ran}(\widetilde{A})$, (6.2.5) reads as

$$0 = \langle \psi, A^{-1}f \rangle = \langle \psi, \widehat{A}^{-1}f \rangle = \langle (\widehat{A}^{-1})^*\psi, f \rangle = \langle (\widehat{A}^*)^{-1}\psi, f \rangle \,,$$

which implies that $(\widehat{A}^*)^{-1}\psi \in (\operatorname{ran}(A) \cap \operatorname{ran}(\widetilde{A}))^\perp = \overline{\ker(A^*) + \ker(\widetilde{A}^*)}$, where the last identity follows from (9.3.13). Note that $(\widehat{A}^{-1})^* = (\widehat{A}^*)^{-1}$ follows from the fact that $\widehat{A}$ is boundedly invertible (see e.g. [**44**, Satz 2.49 b)]). Since we have assumed that $\overline{\ker(A^*) + \ker(\widetilde{A}^*)} = \ker(A^*) + \ker(\widetilde{A}^*)$, there exists a (not necessarily unique) $k \in \ker(A^*)$ and a (not necessarily unique) $\widetilde{k} \in \ker(\widetilde{A}^*)$ such that $(\widehat{A}^*)^{-1}\psi = k + \widetilde{k}$.

Moreover, since $A \subset \widehat{A} \subset \widetilde{A}^*$, which implies that $\widetilde{A} \subset \widehat{A}^* \subset A^*$, we get that $(\widehat{A}^*)^{-1}\psi \in \mathcal{D}(A^*)$ and since $k \in \mathcal{D}(A^*)$, this implies that $\widetilde{k} \in \mathcal{D}(A^*)$, too. Thus, $\widetilde{k} \in \mathcal{D}(A^*) \cap \ker(\widetilde{A}^*)$ and if $\mathcal{D}(A^*) \cap \ker(\widetilde{A}^*) \subset \ker(A^*)$, this implies that $\widetilde{k} \in \ker(A^*)$. We therefore conclude that $k + \widetilde{k} = (\widehat{A}^*)^{-1}\psi \in \ker(A^*)$, which implies that $0 = A^*((\widehat{A}^*)^{-1}\psi) = \widehat{A}^*(\widehat{A}^*)^{-1}\psi = \psi$, i.e. that $\psi = 0$. Hence, $\mathcal{D}(A^*) \cap \ker(\widetilde{A}^*) \subset \ker(A^*)$ and $\psi \perp \mathcal{D}(A_0)$ imply that $\psi = 0$, which means that $\mathcal{D}(A_0)$ is dense in $\mathcal{H}$. This shows the lemma. $\square$

Remark 6.2.5. Obviously, $\mathcal{D}(A^*) \cap \ker \widetilde{A}^* = \{0\}$ is a sufficient condition for $\mathcal{D}(A_0)$ being dense.

Next, assume that $\mathcal{D}(\widetilde{A}_0)$ is dense and define the Kreĭn–von Neumann extension of $\widetilde{A}_0$ — denoted by $\widetilde{A}_{0,K}$ — as

$$\widetilde{A}_{0,K}: \quad \mathcal{D}(\widetilde{A}_{0,K}) = \mathcal{D}(\widetilde{A}_0) \dot{+} \ker(\widetilde{A}_0^*), \quad (\widetilde{f} + \widetilde{k}) \mapsto \widetilde{A}_0 \widetilde{f}\,,$$

where $\widetilde{f} \in \mathcal{D}(\widetilde{A}_0)$ and $\widetilde{k} \in \ker(\widetilde{A}_0^*)$. By a reasoning similar to the proof of Theorem 6.1.4 (e.g. by considering the dissipative operator $(-\widetilde{A}_0)$), we have that $\widetilde{A}_{0,K}$ is maximally antidissipative. In the following theorem, we will show that $A_{0,K}^* = \widetilde{A}_{0,K}$:

Theorem 6.2.6. *Let $(A, \widetilde{A})$ satisfy the assumptions of Theorem 6.2.1 and moreover, assume that the operator A_0 as defined in Remark 6.2.2 is densely defined. We then get that $A_{0,K}$ is a maximally dissipative proper extension of $(A_0, \widetilde{A}_0)$ and moreover that*

$$A_{0,K} = \widetilde{A}_{0,K}^*\,.$$

Proof. Firstly, observe that

$$\ker(A_0^*) = \operatorname{ran}(A_0)^\perp = (\operatorname{ran}(A) \cap \operatorname{ran}(\widetilde{A}))^\perp = \overline{\ker(A^*) + \ker(\widetilde{A}^*)} = \operatorname{ran}(\widetilde{A}_0)^\perp = \ker(\widetilde{A}_0^*)\,,$$

which means by Theorems 6.1.4 and 6.1.5 that the Kreĭn–von Neumann extension $A_0 \subset A_{0,K}$ is a proper maximally dissipative extension of the dual pair $(A_0, \widetilde{A}_0)$. Likewise, the Kreĭn–von Neumann extension $\widetilde{A}_0 \subset \widetilde{A}_{0,K}$ is a proper maximally antidissipative extension of the dual pair $(\widetilde{A}_0, A_0)$ and therefore $\widetilde{A}_{0,K}^*$ is a proper maximally dissipative extension of the dual pair $(A_0, \widetilde{A}_0)$. Let us now show that $A_{0,K} = \widetilde{A}_{0,K}^*$. Since $A_{0,K}$ as well as $\widetilde{A}_{0,K}^*$ are both maximally dissipative extensions of A_0, it suffices to show that

$A_{0,K} \subset \widetilde{A}_{0,K}^*$. To this end, let $f \in \mathcal{D}(A_0)$, $k_1, k_2 \in \ker(A_0^*) = \ker(\widetilde{A}_0^*)$ and $\widetilde{f} \in \mathcal{D}(\widetilde{A}_0)$ and consider

$$\begin{aligned}\langle f + k_1, \widetilde{A}_{0,K}(\widetilde{f} + k_2)\rangle &= \langle f, \widetilde{A}_0 \widetilde{f}\rangle + \langle k_1, \widetilde{A}_0 \widetilde{f}\rangle \\ &= \langle A_0 f, \widetilde{f}\rangle = \langle A_0 f, \widetilde{f} + k_2\rangle = \langle A_{0,K}(f + k_1), \widetilde{f} + k_2\rangle \,,\end{aligned}$$

which shows that $A_{0,K} \subset \widetilde{A}_{0,K}^*$. This finishes the proof. $\square$

EXAMPLE 6.2.7. Let $\mathcal{H} = L^2(0,1)$. As in Example 2.4.4, let $0 < \gamma < 1/2$ and consider the dual pair $(A, \widetilde{A}^*)$, where $A := \overline{A_{00}}$ and $\widetilde{A} := \overline{\widetilde{A}_{00}}$, where A_{00} and $\widetilde{A}_{00}$ are given by

$$\begin{aligned} A_{00}: &\quad \mathcal{D}(A_{00}) = \mathcal{C}_c^\infty(0,1), \quad (A_{00}f)(x) = if'(x) + \frac{i\gamma}{x} f(x) \\ \widetilde{A}_{00}: &\quad \mathcal{D}(\widetilde{A}_{00}) = \mathcal{C}_c^\infty(0,1), \quad (\widetilde{A}_{00}f)(x) = if'(x) - \frac{i\gamma}{x} f(x) \,. \end{aligned}$$

REMARK 6.2.8. Note that we have slightly changed the notation for the preminimal operators A_{00} and $\widetilde{A}_{00}$ as compared to previous sections. This is to avoid any confusion with the operators A_0 and $\widetilde{A}_0$ as defined in (6.2.1) and (6.2.2).

In (2.4.3), we have already computed $\ker(\widetilde{A}^*) = \operatorname{span}\{x^{-\gamma}\}$ and $\ker(A^*) = \operatorname{span}\{x^{\gamma}\}$, which implies that $\operatorname{ran}(A) = \operatorname{span}\{x^{\gamma}\}^\perp$ and $\operatorname{ran}(\widetilde{A}) = \operatorname{span}\{x^{-\gamma}\}^\perp$. Now, consider the operators $\widehat{A}$ and $\widehat{\widetilde{A}}$ given by

$$\begin{aligned} \widehat{A}: &\quad \mathcal{D}(\widehat{A}) = \left\{ f \in L^2(0,1) : \exists \psi \in L^2(0,1) : f(x) = -ix^{-\gamma} \int_0^x y^{\gamma} \psi(y) \mathrm{d}y \right\} \\ &\quad (\widehat{A}f)(x) = if'(x) + \frac{i\gamma}{x} f(x) = \psi(x) \\ \widehat{\widetilde{A}}: &\quad \mathcal{D}(\widehat{\widetilde{A}}) = \left\{ f \in L^2(0,1) : \exists \psi \in L^2(0,1) : f(x) = -ix^{\gamma} \int_0^x y^{-\gamma} \psi(y) \mathrm{d}y \right\} \\ &\quad (\widehat{\widetilde{A}}f)(x) = if'(x) - \frac{i\gamma}{x} f(x) = \psi(x) \,. \end{aligned}$$

Clearly, $\ker(\widehat{A}) = \ker(\widehat{\widetilde{A}}) = \{0\}$ as well as $\operatorname{ran}(\widehat{A}) = \operatorname{ran}(\widehat{\widetilde{A}}) = \mathcal{H}$. Moreover, the inverse operators $\widehat{A}^{-1}$ and $\widehat{\widetilde{A}}^{-1}$ can be read off immediately from the definition of $\widehat{A}$ and $\widehat{\widetilde{A}}$:

$$\begin{aligned} \widehat{A}^{-1}: &\qquad (\widehat{A}^{-1}\psi)(x) = -ix^{-\gamma} \int_0^x y^{\gamma} \psi(y) \mathrm{d}y \\ \widehat{\widetilde{A}}^{-1}: &\qquad (\widehat{\widetilde{A}}^{-1}\psi)(x) = -ix^{\gamma} \int_0^x y^{-\gamma} \psi(y) \mathrm{d}y \,. \end{aligned}$$

From a direct calculation, it follows that the Hilbert–Schmidt norms of $\widehat{A}^{-1}$ and $\widehat{\widetilde{A}}^{-1}$ are given by

$$\|\widehat{A}^{-1}\|_{\mathrm{HS}}^2 = \frac{1}{2+4\gamma} \quad \text{and} \quad \left\|\widehat{\widetilde{A}}^{-1}\right\|_{\mathrm{HS}}^2 = \frac{1}{2-4\gamma}\,,$$

which means in particular that $\widehat{A}^{-1}$ and $\widehat{\widetilde{A}}^{-1}$ are bounded. Hence, we have that $0 \in \rho(\widehat{A}) \cap \rho(\widehat{\widetilde{A}})$. Next, let us argue that $A_{00} \subset \widehat{A} \subset \widetilde{A}^*$ and $\widetilde{A}_{00} \subset \widehat{\widetilde{A}} \subset A^*$. Firstly, observe that $\mathcal{C}_c^\infty(0,1) \subset \mathcal{D}(\widehat{A})$ and $\mathcal{C}_c^\infty(0,1) \subset \mathcal{D}(\widehat{\widetilde{A}})$, which can be seen from the fact that for any $f \in \mathcal{C}_c^\infty(0,1)$, we can choose $\psi_\pm(x) := if'(x) \pm \frac{i\gamma}{x} f(x)$. Thus, $A_{00} \subset \widehat{A}$ and $\widetilde{A}_{00} \subset \widehat{\widetilde{A}}$.], it is obvious that $\mathcal{D}(\widehat{A}) \subset H^1_{\mathrm{loc}}(0,1)$ and $\mathcal{D}(\widehat{\widetilde{A}}) \subset H^1_{\mathrm{loc}}(0,1)$ and by a direct calculation, it can be checked that for $f \in \mathcal{D}(\widehat{A})$ and $\widetilde{f} \in \mathcal{D}(\widehat{\widetilde{A}})$, we have $if'(x) + \frac{i\gamma}{x} f(x) = \psi(x) \in L^2(0,1)$ and $i\widetilde{f}'(x) - \frac{i\gamma}{x}\widetilde{f}(x) = \widetilde{\psi}(x) \in L^2(0,1)$, which implies that $\widehat{A} \subset \widetilde{A}^*$ and $\widehat{\widetilde{A}} \subset A^*$ (see (2.4.2) for domain and action of $\widetilde{A}^*$ and A^*). Moreover, since $0 \in \rho(\widehat{A}) \cap \rho(\widehat{\widetilde{A}})$, we get that $\widehat{A}$ and $\widehat{\widetilde{A}}$ are closed, which therefore implies that $A \subset \widehat{A}$ and $\widetilde{A} \subset \widehat{\widetilde{A}}$. Using that $\mathcal{D}(A) = \widehat{A}^{-1}\mathrm{ran}(A)$ and $\mathcal{D}(\widetilde{A}) = \widehat{\widetilde{A}}^{-1}\mathrm{ran}(\widetilde{A})$, we obtain the following characterization of A and $\widetilde{A}$:

$$\begin{aligned}
A: \quad & \mathcal{D}(A) = \left\{ f \in L^2(0,1) : \exists \psi \perp x^\gamma : f(x) = -ix^{-\gamma} \int_0^x y^\gamma \psi(y)\mathrm{d}y \right\} \\
& (Af)(x) = if'(x) + \frac{i\gamma}{x} f(x) = \psi(x) \\
\widetilde{A}: \quad & \mathcal{D}(\widetilde{A}) = \left\{ f \in L^2(0,1) : \exists \psi \perp x^{-\gamma} : f(x) = -ix^{\gamma} \int_0^x y^{-\gamma} \psi(y)\mathrm{d}y \right\} \\
& (\widetilde{A}f)(x) = if'(x) - \frac{i\gamma}{x} f(x) = \psi(x)\,.
\end{aligned}$$

Now, observe that since $\ker(A^*) \not\subset \ker(\widetilde{A}^*)$, we have by Theorem 6.1.5 that the Kreĭn–von Neumann extension of $A \subset A_K$ would not be a proper extension of the dual pair $(A, \widetilde{A})$. Following the construction of the restrictions $A_0 \subset A$ and $\widetilde{A}_0 \subset \widetilde{A}$ as presented in (6.2.2) and (6.2.1), we define the domains

$$\mathcal{D}(A_0) := \widehat{A}^{-1}(\mathrm{ran}(A) \cap \mathrm{ran}(\widetilde{A})) \quad \text{and} \quad \mathcal{D}(A_0) := \widehat{\widetilde{A}}^{-1}(\mathrm{ran}(A) \cap \mathrm{ran}(\widetilde{A}))\,,$$

where $\mathrm{ran}(A) \cap \mathrm{ran}(\widetilde{A}) = \mathrm{span}\{x^\gamma, x^{-\gamma}\}^\perp$. Moreover, observe that

$$\left(i\frac{\mathrm{d}}{\mathrm{d}x} \pm i\frac{\gamma}{x}\right) x^{\pm\gamma} = \pm 2i\gamma x^{\pm\gamma-1} \notin L^2(0,1)\,,$$

which implies that $\ker(\widetilde{A}^*) \cap \mathcal{D}(A^*) = \ker(A^*) \cap \mathcal{D}(\widetilde{A}^*) = \{0\}$. Thus, by Lemma 6.2.4, we get that $\mathcal{D}(A_0)$ and $\mathcal{D}(\widetilde{A}_0)$ are dense. We therefore get

$$\begin{aligned}
A_0 : \quad & \mathcal{D}(A_0) = \left\{ f \in L^2(0,1) : \exists \psi \perp \operatorname{span}\{x^\gamma, x^{-\gamma}\} : f(x) = -ix^{-\gamma} \int_0^x y^\gamma \psi(y) \mathrm{d}y \right\} \\
& (A_0 f)(x) = if'(x) + \frac{i\gamma}{x} f(x) = \psi(x) \\
\widetilde{A}_0 : \quad & \mathcal{D}(\widetilde{A}_0) = \left\{ f \in L^2(0,1) : \exists \psi \perp \operatorname{span}\{x^\gamma, x^{-\gamma}\} : f(x) = -ix^{\gamma} \int_0^x y^{-\gamma} \psi(y) \mathrm{d}y \right\} \\
& (\widetilde{A}_0 f)(x) = if'(x) - \frac{i\gamma}{x} f(x) = \psi(x) \, .
\end{aligned}$$

Moreover, by Corollary 6.2.3, Equation (6.2.4), the operators A_0^* and $\widetilde{A}_0^*$ are given by

$$\begin{aligned}
\widetilde{A}_0^* : \quad & \mathcal{D}(\widetilde{A}_0^*) = \mathcal{D}(\widetilde{A}^*) \dot{+} \operatorname{span}\{x^\gamma\}, \qquad \widetilde{f} + \lambda x^\gamma \mapsto \widetilde{A}^* \widetilde{f} \\
A_0^* : \quad & \mathcal{D}(A_0^*) = \mathcal{D}(A^*) \dot{+} \operatorname{span}\{x^{-\gamma}\}, \quad f + \mu x^{-\gamma} \mapsto A^* f \, ,
\end{aligned}$$

where $\widetilde{f} \in \mathcal{D}(\widetilde{A}^*)$, $f \in \mathcal{D}(A^*)$ and $\lambda, \mu \in \mathbb{C}$. The Kreĭn–von Neumann extension of $A_0 \subset A_{0,K}$ is given by

$$A_{0,K} : \quad \mathcal{D}(A_{0,K}) = \mathcal{D}(A_0) \dot{+} \operatorname{span}\{x^\gamma, x^{-\gamma}\}, \quad f + \lambda x^\gamma + \mu x^{-\gamma} \mapsto A_0 f \, ,$$

where $f \in \mathcal{D}(A_0)$ and $\lambda, \mu \in \mathbb{C}$. By Theorem 6.2.6, we know that $A_{0,K}$ is a proper maximally dissipative extension of $(A_0, \widetilde{A}_0)$ and that $A_{0,K} = \widetilde{A}_{0,K}^*$, where $\widetilde{A}_{0,K}$ is given by

$$\widetilde{A}_{0,K} : \quad \mathcal{D}(A_{0,K}) = \mathcal{D}(\widetilde{A}_0) \dot{+} \operatorname{span}\{x^\gamma, x^{-\gamma}\}, \quad \widetilde{f} + \lambda x^\gamma + \mu x^{-\gamma} \mapsto \widetilde{A}_0 \widetilde{f} \, ,$$

where $\widetilde{f} \in \mathcal{D}(\widetilde{A}_0)$ and $\lambda, \mu \in \mathbb{C}$.

CHAPTER 7

Sectorial operators and the Friedrichs extension

In this chapter, we will apply the results of Chapter 5 in order to construct proper dissipative and sectorial extensions of a given dual pair of sectorial operators.

Moreover, we will introduce the Friedrichs extension of sectorial operators and discuss some of its properties.

7.1. Sectorial operators

Let us introduce the class of operators whose numerical range is contained in a sector:

Definition 7.1.1. *Let $\alpha, \beta \in [-\pi, \pi)$ and $\alpha \leq \beta$. A densely defined operator A is said to belong to the class $\mathfrak{S}_{\alpha,\beta}$ if and only if its numerical range $\mathcal{N}_A$ is contained in the sector $\{z \in \mathbb{C} : \alpha \leq \arg(z) \leq \beta\}$, i.e.*

$$A \in \mathfrak{S}_{\alpha,\beta} :\Leftrightarrow \mathcal{N}_A \subset \{z \in \mathbb{C} : \alpha \leq \arg(z) \leq \beta\} .$$

If an operator A is an element of the class $\mathfrak{S}_{\alpha,\beta}$ and it has no non-trivial operator extensions that are in the class $\mathfrak{S}_{\alpha,\beta}$ as well, we say that A is a **maximal element** *of the class $\mathfrak{S}_{\alpha,\beta}$.*

Remark 7.1.2. Note that this definition is only reasonable if $(\beta - \alpha) \leq \pi$, since $\mathcal{N}_A$ is a convex set by the Toeplitz-Hausdorff Theorem.

Example 7.1.3. According to this definition, $\mathfrak{S}_{0,\pi}$ is the set of all dissipative operators, $\mathfrak{S}_{-\frac{\pi}{2},\frac{\pi}{2}}$ the set of all accretive operators and for $0 \leq \eta < \frac{\pi}{2}$, $\mathfrak{S}_{-\eta,\eta}$ is the set of all sectorial operators with semi-angle η as defined in [**26**], p. 280.

Later, we will introduce the Friedrichs extension of operators of class $\mathfrak{S}_{\alpha,\beta}$, where $\beta - \alpha < \pi$ and discuss some of its properties. In [**26**], this is only done for operators that belong to the class $\mathfrak{S}_{-\eta,\eta}$, where $0 \leq \eta < \pi/2$. For technical reasons, let us therefore introduce the following terminology:

DEFINITION 7.1.4. *If $A \in \mathfrak{S}_{\alpha,\beta}$ such that $\beta - \alpha < \pi$, then A is called* **sectorial**. *If in addition, there exists an $\eta \in [0, \pi/2)$ such that $A \in \mathfrak{S}_{-\eta,\eta}$, we call A* **sectorial in the sense of Kato**.

Our first result is obtained by a repeated application of Theorem 5.2.8 to an operator of class $\mathfrak{S}_{\alpha,\beta}$, for $\alpha, \beta \in [0, \pi]$ and a suitably rotated version:

THEOREM 7.1.5. *Let $(A, \widetilde{A})$ be a dual pair of operators that has the common core property. Moreover, assume that $A \in \mathfrak{S}_{\alpha,\beta}$ and let $0 \le \alpha' \le \alpha$ and $\beta \le \beta' \le \pi$ such that $\alpha' \neq \beta'$. For $\varphi \in [-\alpha, \pi - \beta]$ define*

$$V_\varphi : \quad \mathcal{D}(V_\varphi) = \mathcal{D}(A) \cap \mathcal{D}(\widetilde{A})$$
$$V_\varphi = \frac{e^{i\varphi} A - e^{-i\varphi} \widetilde{A}}{2i} \restriction_{\mathcal{D}(A) \cap \mathcal{D}(\widetilde{A})} ,$$

which is a symmetric and non-negative operator. Denote its Kreĭn–von Neumann extension by $V_{\varphi,K}$. Moreover, let $\mathcal{V} \subset \mathcal{D}(\widetilde{A}^*)//\mathcal{D}(A)$ be a linear space and let $A_{\mathcal{V}}$ be defined as in Theorem 5.2.8. Then, $A_{\mathcal{V}} \in \mathfrak{S}_{\alpha',\beta'}$ if and only if

$$\mathcal{V} \subset \mathcal{D}(V_{-\alpha',K}^{1/2}) \cap \mathcal{D}(V_{\pi-\beta',K}^{1/2}) \tag{7.1.1}$$

and the following two inequalities are satisfied for all $v \in \mathcal{V}$:

$$Im\langle v, e^{-i\alpha'} \widetilde{A}^* v\rangle \ge \|V_{-\alpha',K}^{1/2} v\|^2$$
$$Im\langle v, e^{i(\pi-\beta')} \widetilde{A}^* v\rangle \ge \|V_{\pi-\beta',K}^{1/2} v\|^2 . \tag{7.1.2}$$

In the case $\alpha' = \beta' \notin \{0, \pi\}$, which of course is only possible if $\alpha = \beta = \alpha' = \beta'$ in the first place, the conditions that $A_{\mathcal{V}} \in \mathfrak{S}_{\alpha',\beta'} = \mathfrak{S}_{\alpha,\alpha}$ read as

$$\mathcal{V} \subset \mathcal{D}(V_{0,K}^{1/2}), \quad Im\langle v, e^{-i\alpha} \widetilde{A}^* v\rangle = 0 \quad and$$
$$Re\langle v, e^{-i\alpha} \widetilde{A}^* v\rangle \ge \frac{1}{\sin \alpha} \|V_K^{1/2} v\|^2 \quad \forall v \in \mathcal{V} .$$

PROOF. Since $\varphi \in [-\alpha, \pi - \beta]$ and $A \in \mathfrak{S}_{\alpha,\beta}$ we have that $e^{i\varphi} A$ is dissipative. Moreover, $e^{-i\varphi} \widetilde{A}$ is its formal adjoint and $(e^{i\varphi} A, e^{-i\varphi} \widetilde{A})$ is a dual pair, which has the common core property. Thus we may copy the reasoning of Theorem 5.2.8, where we showed that $V = \frac{A - \widetilde{A}}{2i} \restriction_{\mathcal{D}(A) \cap \mathcal{D}(\widetilde{A})}$ is a non-negative symmetric operator. The proposition now follows from the observation that the numerical range of $A_{\mathcal{V}}$ will be contained in the sector $\{z \in \mathbb{C} : \alpha' \leq \arg(z) \leq \beta'\}$ if and only if $e^{-i\alpha'}A_{\mathcal{V}}$ and $e^{i(\pi-\beta')}A_{\mathcal{V}}$ are both dissipative at the same time. Thus, Condition (7.1.1) and Equations (7.1.2) are just a rephrasing of the necessary and sufficient condition given in Theorem 5.2.8 for this case.

In the special case, where $\alpha = \beta = \alpha' = \beta'$ observe that the dual pair $(A, \widetilde{A})$ must be of the form $A = e^{i\alpha}S$ and $\widetilde{A} = e^{-i\alpha}S$, where S is a non-negative symmetric operator. Moreover, for any $\varphi \in [-\alpha, \pi - \alpha]$, the operator V_φ is given by $V_\varphi = \sin(\alpha + \varphi)S$ and since $\mathcal{D}(S^*) \subset \mathcal{D}(S_K^{1/2})$, all elements of $\mathcal{D}(\widetilde{A}^*)//\mathcal{D}(A) = \mathcal{D}(S^*)//\mathcal{D}(S)$ are in $\mathcal{D}(V_{\varphi,K}^{1/2})$, which means that Condition (7.1.1) is automatically satisfied if $\mathcal{V} \subset \mathcal{D}(V_{0,K}^{1/2}) = \mathcal{D}(S_K^{1/2})$. Now, impose the two conditions that

$$\begin{aligned}
&\mathrm{Im}\langle v, e^{-i\alpha}\widetilde{A}^*v\rangle \geq \|V_{-\alpha,K}^{1/2}v\|^2 = 0 \\
&\mathrm{Im}\langle v, e^{i(\pi-\alpha)}\widetilde{A}^*v\rangle = -\mathrm{Im}\langle v, e^{-i\alpha}\widetilde{A}^*v\rangle \geq \|V_{\pi-\alpha,K}^{1/2}v\|^2 = 0 \quad \forall v \in \mathcal{V}\,,
\end{aligned}$$

which is equivalent to $\mathrm{Im}\langle v, e^{-i\alpha}\widetilde{A}^*v\rangle = \mathrm{Im}\langle v, S^*v\rangle = 0$ for all $v \in \mathcal{V}$. This ensures that the numerical range of $A_{\mathcal{V}}$ is contained in the ray $\{z \in \mathbb{C}, \arg(\pm z) = \alpha\}$. To exclude the possibility that $\{z \in \mathbb{C} : \arg(-z) = \alpha\} \subset \mathcal{N}_{A_{\mathcal{V}}}$ observe that $A_{\mathcal{V}} \in \mathfrak{S}_{\alpha,\alpha}$ if and only if $A_{\mathcal{V}} \in \mathfrak{S}_{\alpha-\varepsilon,\alpha+\varepsilon}$ for all $\varepsilon > 0$. In terms of our previous result, this means that for all $\varepsilon > 0$, it needs to be true that

$$\begin{aligned}
\mathrm{Im}\langle v, e^{i\varepsilon}S^*v\rangle &\geq \sin\varepsilon \cdot \|S_K^{1/2}v\|^2 \\
-\mathrm{Im}\langle v, e^{-i\varepsilon}S^*v\rangle &\geq \sin\varepsilon \cdot \|S_K^{1/2}v\|^2 \quad \forall v \in \mathcal{V}\,.
\end{aligned}$$

Plugging $e^{i\varepsilon} = \cos\varepsilon + i\sin\varepsilon$ into this equation and using that $\mathrm{Im}\langle v, \cos\varepsilon\, S^*v\rangle = \cos\varepsilon \cdot \mathrm{Im}\langle v, S^*v\rangle = 0$ by the previous reasoning yields the condition that

$$\mathrm{Re}\langle v, S^*v\rangle = \mathrm{Re}\langle v, e^{-i\alpha}\widetilde{A}^*v\rangle \geq \|S_K^{1/2}v\|^2 = \frac{1}{\sin\alpha}\|V_K^{1/2}v\|^2 \quad \forall v \in \mathcal{V}\,,$$

which finishes the proof. □

Remark 7.1.6 (Continuation of Remark 5.2.17). [**32**] describes all proper sectorial extensions of a given dual pair of operator using intersections of operator balls. In Remark 5.2.17, we pointed out how all proper dissipative extensions of a dual pair could be described using an operator $K \in \mathfrak{B}(P_{\mathcal{W}_+} + P_{\mathcal{W}_0}, \sqrt{M_-}^{-1}, \sqrt{M_+})$ (cf. (5.2.12)). Let us introduce $q_\varphi(v) := \mathrm{Im}\langle v, e^{i\varphi}\widetilde{A}^*v\rangle - \|V_{\varphi,K}^{1/2}v\|^2$ and let M^φ denote the corresponding self-adjoint operator on $\mathcal{W}^\varphi = (\mathcal{D}(\widetilde{A}^*)//\mathcal{D}(A))\cap\mathcal{D}(V_{\varphi,K}^{1/2})$. Moreover, let us assume that $\mathcal{W}^\varphi = \mathcal{W}$, i.e. that it is independent of φ and let M^φ denote the selfadjoint operator associated to q_φ. Moreover, let $\mathcal{W}_\pm^\varphi$ denote its positive/negative spectral subspace, $\mathcal{W}_0^\varphi$ its kernel and $M_\pm^\varphi$ the corresponding positive and negative part of M^φ. Characterizing a proper extension of class $\mathfrak{S}_{\alpha',\beta'}$ using an operator $K \in \mathcal{B}(\mathcal{W})$ as done in Equation (5.2.12), we may apply Theorems 5.2.13 and 7.1.5 in order to get that

$$K \in \mathfrak{B}\left(P_{\mathcal{W}_+^{-\alpha'}} + P_{\mathcal{W}_0^{-\alpha'}}, \sqrt{M_-^{-\alpha'}}^{-1}, \sqrt{M_+^{-\alpha'}}\right) \cap \mathfrak{B}\left(P_{\mathcal{W}_+^{\pi-\beta'}} + P_{\mathcal{W}_0^{\pi-\beta'}}, \sqrt{M_-^{\pi-\beta'}}^{-1}, \sqrt{M_+^{\pi-\beta'}}\right) .$$

7.2. Extensions of strictly positive symmetric operators and symmetric operators with at least one real regular point

Firstly, let us apply our results in order to determine all proper sectorial and accretive extensions of a positive symmetric operator. While this is a well-known result (see e.g. [**23**]), we want to show that it can also be obtained from the above shown results.

Example 7.2.1 (Proper sectorial and accretive extensions of a positive symmetric operator). Let S be a non-negative symmetric operator, where we assume that there exists an $\varepsilon > 0$ such that $\langle f, Sf\rangle \geq \varepsilon\|f\|^2$ for all $f \in \mathcal{D}(S)$. Clearly, finding all proper sectorial and accretive extensions of (S, S) is equivalent to finding all proper extensions of the dual pair $iS \in \mathfrak{S}_{\frac{\pi}{2},\frac{\pi}{2}}$ and $-iS$ that lie in the classes $\mathfrak{S}_{\alpha,\beta}$, where $\alpha \in [0, \frac{\pi}{2}]$ and $\beta \in [\frac{\pi}{2}, \pi]$. Clearly, iS and $-iS$ have the common core property and for $\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ we have that

$$V_\varphi = \frac{e^{i\varphi}iS + e^{-i\varphi}iS}{2i} = \cos\varphi \cdot S ,$$

which implies that $V_{\varphi,K}^{1/2} = \sqrt{\cos\varphi}\cdot S_K^{1/2}$. Moreover, it is true that [**2**, Lemma 2.5, Lemma 2.7]

$$\mathcal{D}(S^*) = \mathcal{D}(S)\dot{+}S_F^{-1}\ker(S^*)\dot{+}\ker(S^*) = \mathcal{D}(S_F)\dot{+}\ker(S^*)$$

and

$$\mathcal{D}(S_K^{1/2}) = \mathcal{D}(S_F^{1/2})\dot{+}\ker(S^*) \supset \mathcal{D}(S_F)\dot{+}\ker(S^*) = \mathcal{D}(S^*) .$$

This means that any $v \in S_F^{-1}\ker(S^*) \dot{+} \ker(S^*)$ can be written as $v = k_1 + S_F^{-1}k_2$, where $k_1, k_2 \in \ker(S^*)$. Thus, for any $\mathcal{V} \subset S_F^{-1}\ker(S^*) \dot{+} \ker(S^*)$, observe that the set

$$\mathbf{B} := \{(k_1, k_2) \in \ker(S^*) \times \ker(S^*) : k_1 + S_F^{-1}k_2 \in \mathcal{V}\}$$

defines a linear relation. Moreover, since $\mathcal{D}(S^*) \subset \mathcal{D}(S_K^{1/2})$, Condition (7.1.1) will always be satisfied for any α, β. Thus, for any $\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ we get

$$\begin{aligned}\mathrm{Im}\langle v, ie^{i\varphi}S^*v\rangle &= \mathrm{Im}\langle k_1 + S_F^{-1}k_2, ie^{i\varphi}S^*(k_1 + S_F^{-1}k_2)\rangle = \mathrm{Im}\langle k_1 + S_F^{-1}k_2, ie^{i\varphi}k_2\rangle \\ &= \cos\varphi \cdot \langle S_F^{-1}k_2, k_2\rangle + \mathrm{Im}\langle k_1, ie^{i\varphi}k_2\rangle\,.\end{aligned}$$

On the other hand, we get

$$\|V_{\varphi,K}^{1/2}(k_1 + S_F^{-1}k_2)\|^2 = \|\sqrt{\cos(\varphi)}S_F^{1/2}S_F^{-1}k_2\|^2 = \cos\varphi \cdot \langle S_F^{-1}k_2, k_2\rangle\,.$$

Thus, the condition

$$\mathrm{Im}\langle(k_1 + S_F^{-1}k_2, ie^{i\varphi}S^*(k_1 + S_F^{-1}k_2)\rangle \geq \|V_{\varphi,K}^{1/2}(k_1 + S_F^{-1}k_2)\|^2 \quad \forall (k_1, k_2) \in \mathbf{B}$$

for $k_1, k_2 \in \ker(S^*)$ is equivalent to the condition that

$$\mathrm{Im}\langle k_1, ie^{i\varphi}k_2\rangle \geq 0 \quad \forall (k_1, k_2) \in \mathbf{B}\,. \tag{7.2.1}$$

Let us introduce the subspace $\mathbf{B}(0) := \{k_2 : (0, k_2) \in \mathbf{B}\} \subset \ker S^*$ and observe that for any $(0, k_2) \in (0, \mathbf{B}(0))$, Condition (7.2.1) is automatically satisfied. Thus it suffices to show that Condition (7.2.1) is satisfied for all $(k_1, k_2) \in \mathbf{B} \ominus (0, \mathbf{B}(0))$, which is the graph of an operator B:

$$\mathrm{Im}\langle k_1, ie^{i\varphi}k_2\rangle = \mathrm{Im}\langle k_1, ie^{i\varphi}Bk_1\rangle \geq 0 \quad \forall (k_1, k_2) = (k_1, Bk_1) \in \mathbf{B} \ominus (0, \mathbf{B}(0)) = \Gamma(B)\,.$$

Note that $\mathbf{B} \ominus (0, \mathbf{B}(0))$ denotes the orthogonal complement of $(0, \mathbf{B}(0))$ in $\mathbf{B}$, with respect to the inner product

$$\langle(k_1, k_2), (l_1, l_2)\rangle = \langle k_1, l_1\rangle + \langle k_2, l_2\rangle\,,$$

for (k_1, k_2) and (l_1, l_2) being elements of $\mathcal{H} \times \mathcal{H}$. This implies that $\mathbf{B}(0)$ is orthogonal to the range of the operator B:

$$\mathbf{B}(0) \perp \mathrm{ran}(B)\,. \tag{7.2.2}$$

Moreover, in order that Condition (7.2.1) be satisfied it is necessary that $\mathcal{D}(B) \perp \mathbf{B}(0)$. To see this, assume it is not true, i.e. assume that there exists a $(k_1, k_2) \in \mathbf{B}$ with $k_1 \neq 0$ (which means that $k_1 \in \mathcal{D}(B)$) and $(0, \widetilde{k}_2) \in \mathbf{B}$ such that $k_1 \not\perp \widetilde{k}_2$. Clearly, $(k_1, k_2 + \lambda\widetilde{k}_2) \in \mathbf{B}$ as well for any $\lambda \in \mathbb{C}$. Now consider

$$\operatorname{Im}\langle k_1, ie^{i\varphi}(k_2 + \lambda\widetilde{k}_2)\rangle = \operatorname{Im}\langle k_1, ie^{i\varphi}k_2\rangle + \operatorname{Im}(\lambda\langle k_1, ie^{i\varphi}\widetilde{k}_2\rangle) ,$$

which can be made an arbitrary negative number by a suitable choice of λ and thus violates Condition (7.2.1). This means that B has to be an operator on $\mathbf{B}(0)^\perp$ (orthogonal complement in $\ker S^*$). Thus by Theorem 7.1.5, for any $\alpha \in [0, \frac{\pi}{2}]$ and $\beta \in [\frac{\pi}{2}, \pi]$, a necessary and sufficient condition for $A_\mathcal{V}$ being an element of $\mathfrak{S}_{\alpha,\beta}$ is given by

$$\begin{aligned} &\operatorname{Im}\langle k, ie^{-i\alpha}Bk\rangle \geq 0 \\ &\operatorname{Im}\langle k, ie^{i(\pi-\beta)}Bk\rangle \geq 0 \quad \forall k \in \mathcal{D}(B) , \end{aligned}$$

which means that $iB \in \mathfrak{S}_{\alpha,\beta}$. This is formally also correct for the special case that $\alpha = \beta = \frac{\pi}{2}$, i.e. for the case that we want to determine all non-negative symmetric extensions of S. This follows from the special result as proved in the second part of Theorem 7.1.5. Let us now show that

$$\begin{aligned} S_\mathbf{B} : \quad &\mathcal{D}(S_\mathbf{B}) = \mathcal{D}(S)\dot{+}\{k_1 + S_F^{-1}k_2 : (k_1, k_2) \in \mathbf{B}\} \\ &S_\mathbf{B} = S^* \restriction_{\mathcal{D}(S_\mathbf{B})} \end{aligned}$$

is a maximal element of $\mathfrak{S}_{\alpha,\beta}$ if and only if $\overline{\mathcal{D}(B)} \oplus \mathbf{B}(0) = \ker S^*$ and B is a maximal element of $\mathfrak{S}_{\alpha,\beta}$. Thus, firstly assume that $\overline{\mathcal{D}(B)} \oplus \mathbf{B}(0) = \ker(S^*)$ and that B is maximal. Then, we need to show that

$$\operatorname{ran}(iS_\mathbf{B} + i) = \operatorname{ran}(S_\mathbf{B} + \mathbb{1}) = \mathcal{H} ,$$

since this means that $iS_\mathbf{B}$ is maximally dissipative by Proposition 2.2.5 which in turn implies that $iS_\mathbf{B}$ is a maximal element of $\mathfrak{S}_{\alpha,\beta}$. Let $\phi \in \mathcal{H}$ such that $\phi \perp \operatorname{ran}(S_\mathbf{B} + \mathbb{1})$. Since

$$\begin{aligned} \operatorname{ran}(S_\mathbf{B} + \mathbb{1}) &= (S_\mathbf{B} + \mathbb{1})(\mathcal{D}(S)\dot{+}\{k_1 + S_F^{-1}k_2 : (k_1, k_2) \in \mathbf{B}\}) \\ &= \operatorname{ran}(S + \mathbb{1})\dot{+}\{k_1 + (\mathbb{1} + S_F^{-1})k_2 : (k_1, k_2) \in \mathbf{B}\} , \end{aligned} \tag{7.2.3}$$

this means in particular that $\phi \in \ker(S^*+\mathbb{1})$. Moreover, since the operator $\mathbb{1}-(S_F+\mathbb{1})^{-1}$ is a bijection between $\ker(S^*)$ and $\ker(S^*+\mathbb{1})$ (cf. [**22**, Chapter 1.2.1, right before Theorem 2.1]), there exists a unique $\psi \in \ker(S^*)$ such that

$$\phi = [\mathbb{1} - (S_F+\mathbb{1})^{-1}]\psi \,. \tag{7.2.4}$$

Now, decompose $\psi = \psi_1 + \psi_2$, where $\psi_1 \in \overline{\mathcal{D}(B)}$ and $\psi_2 \in \mathbf{B}(0)$. Firstly, let us show that $\psi_2 = 0$. As $\psi_2 \in \mathbf{B}(0)$ means that $(0,\psi_2) \in \mathbf{B}$, we get by Equation (7.2.3) that $(\mathbb{1}+S_F^{-1})\psi_2 \in \operatorname{ran}(S_{\mathbf{B}}+\mathbb{1})$. Hence, since $\phi \perp \operatorname{ran}(S_{\mathbf{B}}+\mathbb{1})$, this means that

$$\begin{aligned}
0 &= \langle \phi, (\mathbb{1}+S_F^{-1})\psi_2\rangle \overset{(7.2.4)}{=} \langle (\mathbb{1}-(S_F+\mathbb{1})^{-1})(\psi_1+\psi_2), (\mathbb{1}+S_F^{-1})\psi_2\rangle \\
&= \langle \psi_1+\psi_2, (\mathbb{1}-(S_F+\mathbb{1})^{-1})(\mathbb{1}+S_F^{-1})\psi_2\rangle \\
&= \langle \psi_1+\psi_2, (\mathbb{1}-(S_F+\mathbb{1})^{-1}+S_F^{-1}-(S_F+\mathbb{1})^{-1}S_F^{-1})\psi_2\rangle \\
&= \langle \psi_1,\psi_2\rangle + \|\psi_2\|^2 = \|\psi_2\|^2 \,,
\end{aligned}$$

where we have used the first resolvent identity $-(S_F+\mathbb{1})^{-1}+S_F^{-1} = (S_F+\mathbb{1})^{-1}S_F^{-1}$ for the last step. Next, let us show that $\psi_1 = 0$ as well. Since it is true that $\psi_1 \in \overline{\mathcal{D}(B)}$ and by the above reasoning, we get that $\mathcal{D}(B) \perp \mathbf{B}(0)$ as well as $\operatorname{ran}(B) \perp \mathbf{B}(0)$. Moreover, since we assume that $\overline{\mathcal{D}(B)} \oplus \mathbf{B}(0) = \ker S^*$, this implies that $\operatorname{ran}(B) \subset \overline{\mathcal{D}(B)}$. Therefore, B is a densely defined operator on the Hilbert space $\overline{\mathcal{D}(B)}$, i.e. from $\mathcal{D}(B)$ into $\overline{\mathcal{D}(B)}$. Furthermore, iB is of class $\mathfrak{S}_{\alpha,\beta}$ and maximal by assumption, we have that $\operatorname{ran}(iB+i) = \operatorname{ran}(B+\mathbb{1}) = \overline{\mathcal{D}(B)}$. Thus for all $k \in \mathcal{D}(B)$, we get

$$\begin{aligned}
0 &= \langle (\mathbb{1}-(S_F+\mathbb{1})^{-1})\psi_1, k + (\mathbb{1}+S_F^{-1})Bk\rangle = \langle \psi_1, (\mathbb{1}-(S_F+\mathbb{1})^{-1})k + Bk\rangle \\
&= \langle \psi_1, (B+\mathbb{1})k - (S_F+\mathbb{1})^{-1}k\rangle \,.
\end{aligned}$$

Since $\operatorname{ran}(B+\mathbb{1}) = \overline{\mathcal{D}(B)}$ this means that there exists $k \in \mathcal{D}(B)$ such that $(B+\mathbb{1})k = \psi_1$:

$$\begin{aligned}
0 &= \langle \psi_1,\psi_1\rangle - \langle \psi_1, (S_F+\mathbb{1})^{-1}(B+\mathbb{1})^{-1}\psi_1\rangle \geq \|\psi_1\|^2 - \|(S_F+\mathbb{1})^{-1}\psi_1\|\|(B+\mathbb{1})^{-1}\psi_1\| \\
&\geq \left(1 - \frac{1}{1+\varepsilon}\right)\|\psi_1\|^2 \,,
\end{aligned}$$

where we have used Cauchy–Schwarz for the first estimate. For the second estimate, we use $\langle f, S_F f\rangle \geq \varepsilon\|f\|^2$ for all $f \in \mathcal{D}(S_F)$ by assumption, implying that $\|(S_F+\mathbb{1})^{-1}\| \leq \frac{1}{1+\varepsilon}$ and $\|(B+\mathbb{1})^{-1}\| \leq 1$. This implies that $\psi_1 = 0$ from which we get that $\phi = 0$, i.e. $\operatorname{ran}(iS_{\mathbf{B}}+i) = \mathcal{H}$. Moreover, observe that the conditions that $\overline{\mathcal{D}(B)}^{\perp} \cap \ker S^* = \mathbf{B}(0)$

and iB being a maximal operator of class $\mathfrak{S}_{\alpha,\beta}$ are optimal in the sense that they characterize all maximal extensions of the dual pair $(iS, -iS)$ that are of class $\mathfrak{S}_{\alpha,\beta}$: For the case that $\overline{\mathcal{D}(B)} \oplus \mathbf{B}(0)$ is a proper subset of $\ker S^*$, we could always extend the multivalued part $\mathbf{B}(0)$ of the linear relation $\mathbf{B}$ by $(\overline{\mathcal{D}(B)} \oplus \mathbf{B}(0))^\perp$ (orthogonal complement in $\ker S^*$), which we have shown to correspond to an operator of class $\mathfrak{S}_{\alpha,\beta}$, which obviously would be a proper extension of $S_{\mathbf{B}}$.
If on the other hand we have that iB is not a maximal operator of class $\mathfrak{S}_{\alpha,\beta}$, we can take a maximal extension of iB, denoted by iB', which is of class $\mathfrak{S}_{\alpha,\beta}$ and the linear relation

$$\mathbf{B}' = \Gamma(B') \oplus (0, \mathbf{B}(0))$$

corresponds to an operator $S_{\mathbf{B}'} \in \mathfrak{S}_{\alpha,\beta}$, which is maximal again by the above reasoning. The existence of such an operator B' follows from Proposition 7.3.3 for the sectorial case and is obvious in the dissipative case (since otherwise, B would have already been maximally dissipative). We thus have shown the following result:

Theorem 7.2.2. *Let S be symmetric and semibounded with semibound $\varepsilon > 0$. Then, for $\alpha, \beta \in [0, \pi]$, there is a one-to-one correspondence between all maximal proper extensions of iS that are of class $\mathfrak{S}_{\alpha,\beta}$ and all maximal operators iB of class $\mathfrak{S}_{\alpha,\beta}$, that are densely defined on an arbitrary closed subspace of $\ker S^*$. This correspondence is given by*

$$\begin{aligned}
\mathcal{D}(S_B) &= \mathcal{D}(S) \dot{+} \{S_F^{-1}k : k \in \ker S^* \cap \mathcal{D}(B)^\perp\} \dot{+} \{k + S_F^{-1}Bk : k \in \mathcal{D}(B)\} \\
iS_B &= iS^* \restriction_{\mathcal{D}(S_B)} .
\end{aligned}$$

Example 7.2.3 (Dissipative extensions of a symmetric operator). Very similar to the above statement, we can show a result on maximally dissipative extensions of symmetric operators that are boundedly invertible on their range. This could for example be used to analyze the dissipative extensions of periodic operators or massive Dirac operators:

THEOREM 7.2.4. *Let S be a symmetric operator and let $0 \in \widehat{\rho}(S)$*[7.2.1]*. Then, there is a one-to-one correspondence between all maximally dissipative extensions of S and all maximally dissipative operators B that are densely defined on an arbitrary closed subspace of* $\ker S^*$*. This correspondence is given by*

$$\mathcal{D}(S_B) = \mathcal{D}(S)\dot{+}\{\widehat{S}^{-1}k : k \in \ker S^* \cap \mathcal{D}(B)^\perp\}\dot{+}\{k + \widehat{S}^{-1}Bk : k \in \mathcal{D}(B)\}$$

$$S_B = S^* \restriction_{\mathcal{D}(S_B)} ,$$

where $\widehat{S}$ is any selfadjoint extension of S such that $0 \in \rho(\widehat{S})$.

PROOF. By [**44**, Satz 2.67], there always exists a selfadjoint extension $\widehat{S}$ of S such that $0 \in \rho(\widehat{S})$. Again, (S,S) has the common core property with $S = \widetilde{S}$ being its own formal adjoint. From this, we find that $V = \frac{S-S}{2i} \restriction_{\mathcal{D}(S)} = 0 \restriction_{\mathcal{D}(S)}$ on $\mathcal{D}(S)$, which is essentially selfadjoint. Hence, the Kreĭn–von Neumann extension of V, is given by the zero operator $V_K = 0_{\mathcal{H}}$ defined on the entire Hilbert space and its square root is given by the zero-operator too: $V_K^{1/2} = 0_{\mathcal{H}}$. Thus, we have for all $v \in \mathcal{D}(S^*)//\mathcal{D}(S)$ that $v \in \mathcal{D}(V_K^{1/2}) = \mathcal{H}$ and clearly it is true that $\|V_K^{1/2}v\|^2 = 0$ for all $v \in \mathcal{D}(S^*)//\mathcal{D}(S)$. Using Proposition 2.4.3 for $\lambda = 0$, we get

$$\mathcal{D}(S^*) = \mathcal{D}(S)\dot{+}\widehat{S}^{-1}\ker(S^*)\dot{+}\ker S^* ,$$

which means that there exists a one-to-one correspondence between all subspaces $\mathcal{V} \subset \widetilde{S}^{-1}\ker S^*\dot{+}\ker S^*$ and linear relations $\mathbf{B} = \{(k_1,k_2) \in \ker S^* \times \ker S^* : k_1 + \widehat{S}^{-1}k_2 \in \mathcal{V}\}$. Thus, the condition from Theorem 5.2.8 for $S_{\mathcal{V}}$ being a dissipative extension reads as:

$$\operatorname{Im}\langle k_1 + \widehat{S}^{-1}k_2, S^*(k_1 + \widehat{S}^{-1}k_2)\rangle = \operatorname{Im}\langle k_1 + \widehat{S}^{-1}k_2, k_2\rangle = \operatorname{Im}\langle k_1, k_2\rangle \geq 0 \quad \forall (k_1,k_2) \in \mathbf{B} .$$

The condition for S_B being maximally dissipative now follows from completely analogous reasoning to that in the previous example. □

EXAMPLE 7.2.5 (Taken from [**2**]). Let $\mathcal{H} = L^2(\mathbb{R}^+)$ and consider the operator

$$A : \qquad \mathcal{D}(A) = \{f \in H^2(\mathbb{R}^+), f(0) = f'(0) = 0\}, \quad f \mapsto -f'' + f .$$

The adjoint of this operator is given by

$$A^* : \qquad \mathcal{D}(A^*) = \{f \in H^2(\mathbb{R}^+)\}, \quad f \mapsto -f'' + f$$

[7.2.1] This can easily be generalized to any symmetric operator with at least one real point in is field of regularity and the Friedrichs extension A_F corresponds to a Dirichlet boundary condition at the origin:

$$A_F: \qquad \mathcal{D}(A_F) = \{f \in H^2(\mathbb{R}^+), f(0) = 0\}, \quad f \mapsto -f'' + f\,.$$

It is easy to check that $\ker A^* = \operatorname{span}\{e^{-x}\}$ and $A_F^{-1}(e^{-x}) = \frac{1}{2}xe^{-x}$. Since $\ker A^*$ has dimension 1, there are only two possible choices for $\mathcal{D}(B)$: either $\mathcal{D}(B) = \{0\}$, which corresponds to the Friedrichs extension or $\mathcal{D}(B) = \ker A^*$. In the latter case, all dissipative operators from $\mathcal{D}(B)$ to $\ker A^*$ are given by the multiplication by b, where $\operatorname{Im} b \geq 0$. Thus, all maximally dissipative extensions of A, which are different from the Friedrichs extension, are given by

$$A_b: \quad \mathcal{D}(A_b) = \mathcal{D}(A) \dot{+} \operatorname{span}\left\{\frac{b}{2}xe^{-x} + e^{-x}\right\} = \left\{f \in H^2(\mathbb{R}^+), f'(0) = \left(\frac{b}{2} - 1\right) f(0)\right\}$$
$$f \mapsto -f'' + f\,,$$

where $\operatorname{Im} b \geq 0$. Finally, let us check the dissipativity of A_b by a direct calculation:

$$\begin{aligned}\operatorname{Im}\langle f, A_b f\rangle &= \frac{1}{2i}\left[-\int_0^\infty \left(\overline{f(x)}f''(x) - \overline{f''(x)}f(x)\right)\mathrm{d}x\right] = \frac{1}{2i}\left[\overline{f(0)}f'(0) - \overline{f'(0)}f(0)\right] \\ &= \frac{1}{2i}\left[\left(\frac{b}{2} - 1\right)|f(0)|^2 - \left(\frac{\overline{b}}{2} - 1\right)|f(0)|^2\right] = \left(\operatorname{Im}\frac{b}{2}\right)|f(0)|^2 \geq 0\,.\end{aligned}$$

Example 7.2.6. *(The Dirac-operator on the half-line; following the notation and definitions of* [**45**, Chapter 15]*).* This example is supposed to show that our results work also in the case that the operator is not semibounded but has a real number in its regularity domain.

Let $\mathcal{H} = L^2(\mathbb{R}^+; \mathbb{C}^2)$ and let

$$\tau f = \tau \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} := \begin{pmatrix} -1 & \frac{\mathrm{d}}{\mathrm{d}x} \\ -\frac{\mathrm{d}}{\mathrm{d}x} & 1 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} -f_1 + f_2' \\ -f_1' + f_2 \end{pmatrix}.$$

Then, we define the maximal operator T on $\mathcal{H}$ as follows.

$$\begin{aligned} T: \quad \mathcal{D}(T) = \{f = (f_1, f_2)^t \in L^2(\mathbb{R}^+; \mathbb{C}^2) :& \\ & f \text{ is absolutely continuous in } \mathbb{R}^+, \tau f \in L^2(\mathbb{R}^+; \mathbb{C}^2)\} \end{aligned}$$
$$f \mapsto \tau f\,.$$

The minimal operator T_0 is given by

$$T_0 : \qquad \mathcal{D}(T_0) = \left\{ f \in \mathcal{D}(T) : \begin{pmatrix} f_1(0) \\ f_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$
$$f \mapsto \tau f \, .$$

It holds that T_0 is symmetric and that $(-1, 1) \subset \widehat{\rho}(T_0)$. Moreover, it holds that $T_0^* = T$. Finally, we need one self-adjoint extension of T_0 with zero in its resolvent set. To this end, consider

$$\widehat{T} : \qquad \mathcal{D}(\widehat{T}) = \left\{ f = (f_1, f_2)^t \in \mathcal{D}(T) : f_2(0) = 0 \right\}$$
$$f \mapsto \tau f \, ,$$

which has the desired properties. A short calculation shows that

$$\ker T_0^* = \ker T = \operatorname{span} \left\{ \begin{pmatrix} e^{-x} \\ -e^{-x} \end{pmatrix} \right\}$$

and

$$\widehat{T}^{-1} \begin{pmatrix} e^{-x} \\ -e^{-x} \end{pmatrix} = \begin{pmatrix} -e^{-x} \\ 0 \end{pmatrix} .$$

As in Example 7.2.5, we have that $\dim \ker T_0^* = 1$. Therefore, there are only two possibilities for the choice of $\mathcal{D}(B)$: either $\mathcal{D}(B) = \{0\}$, which corresponds to $\widehat{T}$ or $\mathcal{D}(B) = \ker T_0^*$. Thus, all maximally dissipative extensions of T_0 that are different from $\widehat{T}$ are given by

$$\widehat{T}_b : \qquad \mathcal{D}(\widehat{T}_b) = \mathcal{D}(T_0) \dot{+} \operatorname{span} \left\{ \begin{pmatrix} (1-b)e^{-x} \\ -e^{-x} \end{pmatrix} \right\}$$
$$= \{ f = (f_1, f_2)^t \in \mathcal{D}(T) : f_1(0) = (b-1) f_2(0) \}$$
$$f \mapsto \tau f \, ,$$

where $\operatorname{Im} b \geq 0$. Again, let us verify that these operators are dissipative, i.e. for all $f \in \mathcal{D}(\widehat{T}_b)$, consider

$$\operatorname{Im}\left\langle \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} -1 & \frac{\mathrm{d}}{\mathrm{d}x} \\ -\frac{\mathrm{d}}{\mathrm{d}x} & 1 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \right\rangle = \frac{1}{2i}\left[\langle f_1, f_2'\rangle - \langle f_2', f_1\rangle + \langle f_1', f_2\rangle - \langle f_2, f_1'\rangle\right]$$

$$= \frac{1}{2i}[f_1(0)\overline{f_2(0)} - \overline{f_1(0)}f_2(0)] = (\operatorname{Im} b)|f_2(0)|^2 \geq 0\,.$$

7.3. The Friedrichs extension in the common core case

For convenience, let us recall the definition of a closable quadratic form:

Definition 7.3.1 (Closable quadratic form, cf. [**26**, VI, §1, Sec. 4]). *Let q be a quadratic form. Then, q is called closable if and only if for any sequence $\{f_n\}_n \subset \mathcal{D}(q)$, we have that if*

$$\|f_n\| \overset{n\to\infty}{\longrightarrow} 0 \quad \textit{and} \quad q(f_n - f_m) \overset{n,m\to\infty}{\longrightarrow} 0\,,$$

then this implies that

$$q(f_n) \overset{n\to\infty}{\longrightarrow} 0\,.$$

Remark 7.3.2. If q is closable, its closure q' is given by [**26**, VI, Thm. 1.17]

$$\begin{aligned} q': \quad & \mathcal{D}(q') = \{f \in \mathcal{H} : \exists \{f_n\}_n \subset \mathcal{D}(q) \ \textit{s.t.}\ \|f_n - f\| \overset{n\to\infty}{\longrightarrow} 0 \text{ and } q(f_n - f_m) \overset{n,m\to\infty}{\longrightarrow} 0\} \\ & q'(f) = \lim_{n\to\infty} q(f_n)\,. \end{aligned}$$

For an operator A which is of class $\mathfrak{S}_{\alpha,\beta}$ with $\beta - \alpha < \pi$, we can define its Friedrichs extension A_F. In the literature (e.g. in [**26**]), this is usually done for sectorial operators with angle η, i.e. for operators which have numerical range contained in the set $\{z \in \mathbb{C} : -\eta \le \arg(z) \le \eta\}$ for some $0 \le \eta < \frac{\pi}{2}$:

Proposition 7.3.3. *Let T be sectorial in the sense of Kato and let s_T be the sesquilinear form induced by T, i.e.*

$$\begin{aligned} s_T: \quad & \mathcal{D}(s_T) = \mathcal{D}(T) \times \mathcal{D}(T) \\ & (\varphi, \psi) \mapsto s_T(\varphi, \psi) := \langle \varphi, T\psi \rangle\,. \end{aligned}$$

Then, s_T is closable, where we denote its closure by s_{T_F}. The form domain $\mathcal{Q}(T)$ of s_{T_F} is defined as $\mathcal{Q}(T) := \overline{\mathcal{D}(T)}^{\|\cdot\|_T}$, where the norm $\|\cdot\|_T$ is given by

$$\|\psi\|_T^2 := \|\psi\|^2 + Re\langle \psi, T\psi \rangle\,. \tag{7.3.1}$$

The Friedrichs extension of T — denoted by T_F — is the operator associated to s_{T_F}, i.e. it is given by

$$T_F: \quad \mathcal{D}(T_F) = \{f \in \mathcal{Q}(T) : \exists w \in \mathcal{H} \text{ s.t. } s_{T_F}(f,g) = \langle w, g\rangle \ \forall g \in \mathcal{Q}(T)\}$$
$$f \mapsto w\,.$$

Here, $s_{T_F}(\cdot,\cdot)$ denotes the sesquilinear form associated to s_{T_F} that can be obtained by polarization.

The operator T_F is maximally sectorial and the closures of the numerical ranges of T and T_F coincide.

Moreover, we have the following description of T_F^*:

$$T_F^*: \quad \mathcal{D}(T_F^*) = \mathcal{Q}(T) \cap \mathcal{D}(T^*)$$
$$T_F^* = T^* \restriction_{\mathcal{D}(T_F^*)} \ . \tag{7.3.2}$$

Proof. For the construction of the Friedrichs extension, we refer to [**26**, VI, Theorem 1.27, Theorem 2.1, Corollary 2.4 and VI, §2.3]. For (7.3.2), cf. [**7**, Remarks right after Thm. 1]. □

Let us now define the Friedrichs extension of an arbitrary sectorial operator T. The (mathematically almost trivial) idea is to rotate such an operator by multiplying it by a suitable phase $e^{i\varphi}$ such that one obtains an operator $e^{i\varphi}T$ that is sectorial in the sense of Kato.

Definition 7.3.4. *Let $T \in \mathfrak{S}_{\alpha,\beta}$ with $\beta - \alpha < \pi$ and let $e^{i\varphi}$ be such that $e^{i\varphi}T$ is sectorial in the sense of Kato. The Friedrichs extension of T is defined as*

$$T_F := e^{-i\varphi}\left(e^{i\varphi}T\right)_F\,,$$

where $(e^{i\varphi}T)_F$ denotes the Friedrichs extension of the operator $e^{i\varphi}T$ that is sectorial in the sense of Kato as it is defined in [**26**, p. 280].

The following lemma guarantees that the Friedrichs extension does not depend on the specific choice of φ as long as $e^{i\varphi}T$ is sectorial in the sense of Kato:

Lemma 7.3.5. *Let S be an operator which is sectorial in the sense of Kato with semi-angle $\eta < \frac{\pi}{2}$, i.e.*

$$|Im\langle\psi, S\psi\rangle| \leq \tan\eta \cdot Re\langle\psi, S\psi\rangle \quad \text{for all}\, \psi \in \mathcal{D}(S)\,. \tag{7.3.3}$$

Moreover, let φ be such that $|\pm\eta + \varphi| < \frac{\pi}{2}$, which means that $e^{i\varphi}S$ is still sectorial. We then get that

$$\left(e^{i\varphi}S\right)_F = e^{i\varphi}S_F\,. \tag{7.3.4}$$

Proof. This follows from the fact that the norms induced by the real parts of S and $e^{i\varphi}S$ as defined in (7.3.1) are equivalent. For simplicity, assume that $\varphi \geq 0$. Since

$$\begin{aligned}
\mathrm{Re}\langle\psi, e^{i\varphi}S\psi\rangle &= \cos\varphi \cdot \mathrm{Re}\langle\psi, S\psi\rangle - \sin\varphi \cdot \mathrm{Im}\langle\psi, S\psi\rangle \\
&\geq \cos\varphi \cdot \mathrm{Re}\langle\psi, S\psi\rangle - \sin\varphi \cdot |\mathrm{Im}\langle\psi, S\psi\rangle| \\
&\overset{(7.3.3)}{\geq} \frac{\cos(\eta+\varphi)}{\cos\eta}\mathrm{Re}\langle\psi, S\psi\rangle
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Re}\langle\psi, e^{i\varphi}S\psi\rangle &= \cos\varphi \cdot \mathrm{Re}\langle\psi, S\psi\rangle - \sin\varphi \cdot \mathrm{Im}\langle\psi, S\psi\rangle \\
&\leq \cos\varphi \cdot \mathrm{Re}\langle\psi, S\psi\rangle + \sin\varphi \cdot |\mathrm{Im}\langle\psi, S\psi\rangle| \\
&\overset{(7.3.3)}{\leq} \frac{\cos(\eta-\varphi)}{\cos\eta}\mathrm{Re}\langle\psi, S\psi\rangle
\end{aligned}$$

we get that $\mathcal{Q}(S) = \mathcal{Q}(e^{i\varphi}S)$, using that $\eta+\varphi < \frac{\pi}{2}$. Moreover, since $\mathcal{D}(S^*) = \mathcal{D}((e^{i\varphi}S)^*)$, the lemma follows from (7.3.2). $\square$

Remark 7.3.6. With this generalization of the Friedrichs extension for any operator $T \in \mathfrak{S}_{\alpha,\beta}$ with $\beta - \alpha < \pi$ and the previous lemma, we get that

$$(e^{i\varphi}T)_F = e^{i\varphi}T_F\,, \tag{7.3.5}$$

where $\varphi \in [0, 2\pi)$ is arbitrary.

For the main theorem of this section, we will need the following result:

Lemma 7.3.7. *Let S_0 be sectorial and let S be its closure: $S = \overline{S_0}$. Then, the Friedrichs extension of S_0, which we denote by $S_{0,F}$ and the Friedrichs extension of S, denoted by S_F coincide: $S_{0,F} = S_F$.*

PROOF. The fact that $S_0 \subset S$ implies that $S_{0,F} \subset S_F$ by construction of the Friedrichs extensions of both operators. On the other hand $S_0 \subset S_{0,F}$ implies that $S_{0,F}^* \subset S_0^* = S^*$, from which we conclude that $S \subset S_{0,F}$. But again we may argue that this implies $S_F \subset (S_{0,F})_F$, where $(S_{0,F})_F$ denotes the Friedrichs extension of $S_{0,F}$. However, by [**26**, VI, Thm. 2.9], we know that $S_{0,F} = (S_{0,F})_F$, from which it follows that $S_F \subset S_{0,F}$ and thus the lemma. $\square$

Let us now show that for a dual pair of sectorial operators $(T, \widetilde{T})$ that has the common core property we have that $T_F = \widetilde{T}_F^*$.

THEOREM 7.3.8. *Let $(T, \widetilde{T})$ be a dual pair of operators, which has the common core property. Moreover, assume that T is of class $\mathfrak{S}_{\alpha,\beta}$ such that $\beta - \alpha < \pi$ and let T_0 and $\widetilde{T}_0$ denote the corresponding restrictions of T and $\widetilde{T}$ to a common core $\mathcal{D} \subset \mathcal{D}(T) \cap \mathcal{D}(\widetilde{T})$. Then we have $T_F = \widetilde{T}_F^*$. In particular T_F is a proper maximal class $\mathfrak{S}_{\alpha,\beta}$ extension of the dual pair $(T, \widetilde{T})$.*

PROOF. Since we have that $\beta - \alpha < \pi$, there always exists a complex phase $e^{i\vartheta}$ such that $S := e^{i\vartheta} T$ and $\widetilde{S} := e^{-i\vartheta} \widetilde{T}$ are sectorial in the sense of Kato, i.e. of class $\mathfrak{S}_{-\eta,\eta}$ for some $\eta < \frac{\pi}{2}$. Let S_0 and $\widetilde{S}_0$ denote the corresponding restrictions of S and $\widetilde{S}$ to a common core $\mathcal{D} \subset (\mathcal{D}(T) \cap \mathcal{D}(\widetilde{T}))$. Since T has the common core property, it is true that

$$\overline{S_0} = S \quad \text{and} \quad \overline{\widetilde{S}_0} = \widetilde{S}\,.$$

The sesquilinear forms induced by S_0 and $\widetilde{S}_0$ are given by

$$\begin{aligned}
s_{S_0}: \quad & \mathcal{D}(s_{S_0}) = \mathcal{D}(S_0) \times \mathcal{D}(S_0) \\
& s_{S_0}(\varphi, \psi) = \langle \varphi, S_0 \psi \rangle \\
s_{\widetilde{S}_0}: \quad & \mathcal{D}(s_{\widetilde{S}_0}) = \mathcal{D}(\widetilde{S}_0) \times \mathcal{D}(\widetilde{S}_0) = \mathcal{D}(S_0) \times \mathcal{D}(S_0) = \mathcal{D}(s_{S_0}) \\
& s_{\widetilde{S}_0}(\varphi, \psi) = \langle \varphi, \widetilde{S}_0 \psi \rangle\,.
\end{aligned}$$

and their real parts as defined in [**26**, VI, §1.2], can be shown to be equal:

$$\begin{aligned}
s^{Re}_{S_0}(\varphi,\psi) &= \frac{1}{2}(s_{S_0}(\varphi,\psi) + s^*_{S_0}(\varphi,\psi)) = \frac{1}{2}(s_{S_0}(\varphi,\psi) + \overline{s_{S_0}(\psi,\varphi)}) \\
&= \frac{1}{2}(\langle\varphi, S_0\psi\rangle + \langle S_0\varphi,\psi\rangle) = \frac{1}{2}(\langle\varphi, \widetilde{S}_0\psi\rangle + \langle \widetilde{S}_0\varphi,\psi\rangle) \\
&= \frac{1}{2}(s_{\widetilde{S}_0}(\varphi,\psi) + \overline{s_{\widetilde{S}_0}(\psi,\varphi)}) = \frac{1}{2}(s_{\widetilde{S}_0}(\varphi,\psi) + s^*_{\widetilde{S}_0}(\varphi,\psi)) \\
&= s^{Re}_{\widetilde{S}_0}(\varphi,\psi)\,.
\end{aligned}$$

Note that the adjoint form $s^*_{S_0}$ is defined via $s^*_{S_0}(\varphi,\psi) = \overline{s_{S_0}(\psi,\varphi)}$. Now, let $\|\cdot\|_{S_0}$ and $\|\cdot\|_{\widetilde{S}_0}$ denote the norms induced by the real parts of the sesquilinear forms s_{S_0} and $s_{\widetilde{S}_0}$:

$$\|\psi\|^2_{S_0} = \|\psi\|^2 + s^{Re}_{S_0}(\psi,\psi) = \|\psi\|^2 + s^{Re}_{\widetilde{S}_0}(\psi,\psi) = \|\psi\|^2_{\widetilde{S}_0}\,,$$

which are equal by the above reasoning. Then

$$\mathcal{Q}(S_0) := \overline{\mathcal{D}(S_0)}^{\|\cdot\|_{S_0}}$$

and

$$\mathcal{Q}(\widetilde{S}_0) := \overline{\mathcal{D}(\widetilde{S}_0)}^{\|\cdot\|_{\widetilde{S}_0}}$$

are equal, i.e. $\mathcal{Q}(S_0) = \mathcal{Q}(\widetilde{S}_0)$. Let $s_{S_{0,F}}$ and $s_{\widetilde{S}_{0,F}}$ denote the sesquilinear forms associated to the closure of the quadratic forms induced by s_{S_0} and $s_{\widetilde{S}_0}$. By Proposition 7.3.3, we know that they give rise to two maximally sectorial operators $S_{0,F}$ and $\widetilde{S}_{0,F}$. By construction, we know that for any $\varphi \in \mathcal{Q}(S_0)$ and for any $\psi \in \mathcal{Q}(S_0)$ there exist two sequences $\{\varphi_n\}_n \subset \mathcal{D}(S_0)$ and $\{\psi_n\}_n \subset \mathcal{D}(S_0)$ such that

$$\varphi_n \to \varphi \quad \text{and} \quad \psi_n \to \psi \quad \text{and} \quad \langle\varphi_n, S_0\psi_n\rangle \to s_{S_{0,F}}(\varphi,\psi)\,.$$

Hence, for all $\varphi,\psi \in \mathcal{Q}(S_0) = \mathcal{Q}(\widetilde{S}_0)$

$$\begin{aligned}
s_{S_{0,F}}(\varphi,\psi) = \lim_{n\to\infty}\langle\varphi_n, S_0\psi_n\rangle &= \lim_{n\to\infty}\langle\widetilde{S}_0\varphi_n,\psi_n\rangle \\
&= \overline{\lim_{n\to\infty}\langle\psi_n, \widetilde{S}_0\varphi_n\rangle} = \overline{s_{\widetilde{S}_{0,F}}(\psi,\varphi)} = s^*_{\widetilde{S}_{0,F}}(\varphi,\psi)\,.
\end{aligned}$$

But, from [**26**, VI, Thm. 2.5] it follows that $s^*_{\widetilde{S}_{0,F}} = s_{\widetilde{S}^*_{0,F}}$, which implies that $S_{0,F} = \widetilde{S}^*_{0,F}$ and by Lemma 7.3.7, this implies that

$$S_F = \widetilde{S}^*_F\,. \tag{7.3.6}$$

Since $S \subset S_F$ and $\widetilde{S} \subset \widetilde{S}_F$, respectively $\widetilde{S}_F^* \subset \widetilde{S}^*$, this yields $S \subset S_F \overset{(7.3.6)}{=} \widetilde{S}_F^* \subset \widetilde{S}^*$, which — after a suitable multiplication by $e^{-i\vartheta}$ is the desired result. $\square$

THEOREM 7.3.9. *Let $(T, \widetilde{T})$ be a dual pair of sectorial operators with the common core property. Let T_F be the Friedrichs extension of T. Then for all $\varphi \in [0, 2\pi)$ such that $e^{i\varphi}T$ is dissipative and for all $v \in \mathcal{D}(T_F)$ it is true that*

$$Im\langle v, e^{i\varphi}\widetilde{T}^* v\rangle = Im\langle v, e^{i\varphi}T_F v\rangle = \|V_{\varphi,K}^{1/2} v\|^2 = \|V_{\varphi,F}^{1/2} v\|^2 ,$$

where $V_{\varphi,F}$ and $V_{\varphi,K}$ are the Friedrichs and the Kreĭn extension of

$$V_\varphi := \frac{1}{2i}(e^{i\varphi}T - e^{-i\varphi}\widetilde{T}) \restriction_{\mathcal{D}(T)\cap\mathcal{D}(\widetilde{T})} .$$

This implies in particular that $\mathcal{D}(T_F) \subset \mathcal{D}(V_{\varphi,F}^{1/2})$. Moreover, this is equivalent to saying that the quadratic form q_φ as defined by

$$q_\varphi(v) := Im\langle v, e^{i\varphi}\widetilde{T}^* v\rangle - \|V_{\varphi,K}^{1/2} v\|^2$$

(cf. Equation (5.2.9)) vanishes identically on $\mathcal{D}(T_F)$, i.e.

$$q_\varphi \restriction_{\mathcal{D}(T_F)} \equiv 0 .$$

PROOF. Firstly, observe that by Lemma 7.3.5, we only have to consider the case $\varphi = 0$, which is why will drop the index φ from now on. Moreover, as we will show in Lemma 9.1.2, we have that $\|V_K^{1/2} v\|^2 = \|V_F^{1/2} v\|^2$ for all $v \in \mathcal{D}(V_F^{1/2})$. Now, let us define $T_0 := T \restriction_{\mathcal{D}(T)\cap\mathcal{D}(\widetilde{T})}$ and $\widetilde{T}_0 := \widetilde{T} \restriction_{\mathcal{D}(T)\cap\mathcal{D}(\widetilde{T})}$ and let $v \in \mathcal{D}(T_F)$. This means that there exists a sequence $\{v_n\}_n \subset \mathcal{D}(T)$ such that

$$v_n \to v \quad \text{and} \quad \langle v_n, Tv_n\rangle \to \langle v, T_F v\rangle$$

and thus in particular

$$\mathrm{Im}\langle v_n, Tv_n\rangle \to \mathrm{Im}\langle v, T_F v\rangle .$$

By Lemma 7.3.7, we may choose $\{v_n\}_n$ even such that $\{v_n\}_n \subset \mathcal{D}(T_0)$. By Theorem 7.3.8, we know that $T_F \subset \widetilde{T}^*$ and thus

$$\begin{aligned}\mathrm{Im}\langle v, \widetilde{T}^* v\rangle = \mathrm{Im}\langle v, T_F v\rangle = \lim_{n\to\infty} \mathrm{Im}\langle v_n, T_0 v_n\rangle = \lim_{n\to\infty} \frac{1}{2i}\left(\langle v_n, T_0 v_n\rangle - \langle T_0 v_n, v_n\rangle\right) \\ = \lim_{n\to\infty} \frac{1}{2i}\left(\langle v_n, T_0 v_n\rangle - \langle v_n, \widetilde{T}_0 v_n\rangle\right) = \lim_{n\to\infty} \langle v_n, V_0 v_n\rangle .\end{aligned}$$

Since $v_n \to v$ and we know that $\langle v_n, V_0 v_n\rangle$ converges, this implies that $v \in \mathcal{D}(V_F^{1/2})$ and that $\lim_{n\to\infty}\langle v_n, V_0 v_n\rangle = \|V_F^{1/2} v\|^2$ from which follows that

$$\operatorname{Im}\langle v, \widetilde{T}^* v\rangle = \|V_F^{1/2} v\|^2 \quad \text{for all} \quad v \in \mathcal{D}(T_F)\,,$$

which finishes the proof. $\square$

COROLLARY 7.3.10. *Let T and $\widetilde{T}$ satisfy the assumptions of Theorem 7.3.9 and let $\varphi \in [0, 2\pi)$ be such that $e^{i\varphi}T$ is still dissipative. In addition, assume that*

$$\dim \mathcal{D}(\widetilde{T}^*)//\mathcal{D}(T) < \infty\,.$$

Then, for $\mathcal{W}_+^\varphi, \mathcal{W}_0^\varphi$ and $\mathcal{W}_-^\varphi$ as defined in Theorem 5.2.13 for q_φ, we get

$$\dim \mathcal{W}_+^\varphi \le \dim \mathcal{W}_-^\varphi \tag{7.3.7}$$

as well as

$$\dim \mathcal{W}_+^\varphi + \dim \mathcal{W}_0^\varphi = \dim \ker(T^* - i)\,. \tag{7.3.8}$$

PROOF. Since $(e^{i\varphi}T, e^{-i\varphi}\widetilde{T})$ is a dual pair satisfying the conditions of this corollary, and by (7.3.5), it again suffices to only consider the case $\varphi = 0$. By Theorem 7.3.8, we know that the Friedrichs extension T_F is a proper maximally dissipative extension of the dual pair T and $\widetilde{T}$. Moreover, from Theorem 5.2.13 it follows that there exists a contraction from $\mathcal{W}_+$ into $\mathcal{W}_-$ such that

$$\mathcal{D}(T_F) = \mathcal{D}(T) \dot{+} \{(\mathbb{1} + \sqrt{M_-}^{-1} C \sqrt{M_+}) w_+, w_+ \in \mathcal{W}_+\} \dot{+} \mathcal{W}_0\,.$$

For any $(\mathbb{1} + \sqrt{M_-}^{-1} C \sqrt{M_+}) w_+ + w_0$, with $w_+ \in \mathcal{W}_+$ and $w_0 \in \mathcal{W}_0$, we have shown that

$$q((\mathbb{1} + \sqrt{M_-}^{-1} C \sqrt{M_+}) w_+ + w_0) \overset{(5.2.11)}{=} \langle \sqrt{M_+} w_+, (\mathbb{1} - C^* C)\sqrt{M_+} w_+\rangle\,.$$

On the other hand, by Theorem 7.3.9, it is true that $q \restriction_{\mathcal{D}(T_F)} \equiv 0$, which means that the contraction C has to be an isometry from $\mathcal{W}_+$ into $\mathcal{W}_-$ and since isometries are injective, it immediately follows that $\dim \mathcal{W}_+ \le \dim \mathcal{W}_-$.

Equation (7.3.8) now follows from the fact that T_F is a maximally dissipative proper extension of the dual pair T and $\widetilde{T}$. Using Lemma 2.3.8, we get that

$$\dim \mathcal{D}(T_F)/\mathcal{D}(T) = \dim \ker(T^* - i)$$

and since

$$\begin{aligned}\dim \mathcal{D}(T_F)/\mathcal{D}(T) &= \dim \left(\{(\mathbb{1} + \sqrt{M_-}^{-1} C \sqrt{M_+}) w_+, w_+ \in \mathcal{W}_+\} \dot{+} \mathcal{W}_0 \right) \\ &= \dim \mathcal{W}_+ + \dim \mathcal{W}_0\end{aligned}$$

the result follows. □

Example 7.3.11. For $\gamma > 0$, consider the dual pair of differential operators $(A_0, \widetilde{A}_0)$:

$$\begin{aligned} A_0 : \qquad & \mathcal{D}(A_0) = \mathcal{C}_0^\infty(0,1) \\ & (A_0 f)(x) = -f''(x) + \frac{i\gamma}{x^2} f(x) \\ \widetilde{A}_0 : \qquad & \mathcal{D}(\widetilde{A}_0) = \mathcal{C}_0^\infty(0,1) \\ & (\widetilde{A}_0 f)(x) = -f''(x) - \frac{i\gamma}{x^2} f(x)\,. \end{aligned}$$

Moreover, define $A := \overline{A_0}$ and $\widetilde{A} := \overline{\widetilde{A}_0}$, which means that the dual pair $(A, \widetilde{A})$ has the common core property by construction. Since

$$\langle f, A_0 f\rangle = \int_0^1 \overline{f(x)} \left(-f''(x) + \frac{i\gamma}{x^2} f(x) \right) \mathrm{d}x = \int_0^1 |f'(x)|^2 \mathrm{d}x + i\gamma \int_0^1 \frac{|f(x)|^2}{x^2} \mathrm{d}x$$

for all $f \in \mathcal{C}_c^\infty(0,1)$, we have that $\mathcal{N}_A \subset \{z \in \mathbb{C} : \mathrm{Re} z \geq \pi^2, \mathrm{Im} z \geq \gamma\}$, where the lower bound for the real part of the numerical range is estimated by the first Dirichlet eigenvalue of the Laplacian on the interval and the estimate for the imaginary part is an immediate consequence of $\frac{1}{x} > 1$ for $x \in (0,1)$. This implies that A is of class $S_{0,\frac{\pi}{2}}$ and that $0 \in \widehat{\rho}(A)$. Let us now determine the Friedrichs extension of A. To this end, consider the real part of the form induced by $B_0 := e^{-i\frac{\pi}{4}} A_0$:

$$\|f\|_{B_0}^2 = \|f\|^2 + \mathrm{Re}\langle f, e^{-i\frac{\pi}{4}} A_0 f\rangle = \|f\|^2 + \frac{\sqrt{2}}{2} \left(\|f'\|^2 + \gamma \int_0^1 \frac{|f(x)|^2}{x^2} \mathrm{d}x \right) . \tag{7.3.9}$$

This can be shown to be equivalent to the first Sobolev norm:

$$\frac{\sqrt{2}}{2} (\|f\|^2 + \|f'\|^2) \leq \|\psi\|_{B_0}^2 \leq \max\left\{1, \frac{\sqrt{2}}{2} + 2\sqrt{2}\gamma\right\} (\|f\|^2 + \|f'\|^2)\,, \tag{7.3.10}$$

where the second inequality follows from an application of Hardy's inequality in order to estimate $\int_0^1 \frac{|f(x)|^2}{x^2}\mathrm{d}x$. Thus, we have that the form domain of the Friedrichs extension A_F is given by

$$\mathcal{Q}(A_F) = \overline{\mathcal{C}_0^\infty(0,1)}^{\|\cdot\|_{B_0}} = H_0^1(0,1)\,, \tag{7.3.11}$$

i.e. the first Sobolev space with Dirichlet boundary conditions at 0 and 1. A calculation — using Formula (2.4.1) for $\lambda = 0$ — shows that we have to distinguish two cases for $\mathcal{D}(\widetilde{A}^*)$:

- The case $\gamma < \sqrt{3}$. Define the numbers $\omega_\pm := \frac{1\pm\sqrt{1+4i\gamma}}{2}$. Then we get

  $$\mathcal{D}(\widetilde{A}^*) = \mathcal{D}(A)\dot{+}\mathrm{span}\{x^{\omega_+}, x^{\omega_-}\}\dot{+}\mathrm{span}\{x^{\overline{\omega_+}+2}, x^{\overline{\omega_-}+2}\}\,.$$

- For the case $\gamma \geq \sqrt{3}$, we get

  $$\mathcal{D}(\widetilde{A}^*) = \mathcal{D}(A)\dot{+}\mathrm{span}\{x^{\omega_+}\}\dot{+}\mathrm{span}\{x^{\overline{\omega_+}+2}\}\,,$$

  because $\gamma \geq \sqrt{3}$ implies that $\mathrm{Re}\,\omega_- \leq -\frac{1}{2}$, from which follows that $x^{\omega_-} \notin L^2(0,1)$.

Also, observe that $\widetilde{A}^* = JA^*J$, where the conjugation J is defined as $(Jf)(x) := \overline{f(x)}$. From this it immediately follows that $\mathcal{D}(A^*) = J\mathcal{D}(\widetilde{A}^*) = \{\overline{f} : f \in \mathcal{D}(\widetilde{A}^*)\}$. Using Equation (7.3.11) and that $\mathcal{D}(A_F) = \mathcal{D}(\widetilde{A}_F^*) = \mathcal{Q}(A_F)\cap\mathcal{D}(\widetilde{A}^*)$, where the first equality follows from Theorem 7.3.8 and the second from (7.3.2), we get

- For $\gamma < \sqrt{3}$:

  $$\mathcal{D}(A_F) = \mathcal{D}(A)\dot{+}\mathrm{span}\{x^{\omega_+} - x^{\overline{\omega_+}+2}, x^{\omega_+} - x^{\overline{\omega_-}+2}\}\,.$$

- For $\gamma \geq \sqrt{3}$:

  $$\mathcal{D}(A_F) = \mathcal{D}(A)\dot{+}\mathrm{span}\{x^{\omega_+} - x^{\overline{\omega_+}+2}\}\,.$$

Now, let us apply the results of Theorem 5.2.8 and Theorem 7.1.5 in order to construct sectorial extensions of the dual pair $(A,\widetilde{A})$. To this end, define $A_\varphi := e^{i\varphi}A$ and $\widetilde{A}_\varphi := e^{-i\varphi}\widetilde{A}$ for $\varphi \in \left[0,\frac{\pi}{2}\right]$. As in Theorem 7.1.5, define the operators $V_\varphi := \overline{\frac{1}{2i}(A_\varphi - \widetilde{A}_\varphi)\restriction_{\mathcal{D}}}$, where $\mathcal{D}$ is a common core for A_φ and $\widetilde{A}_\varphi$ as its domain (we may pick e.g. $\mathcal{D} = \mathcal{C}_0^\infty(0,1)$). A calculation shows that for any $f \in \mathcal{C}_0^\infty(0,1)$ we have that

$$\left(V_\varphi f\right)(x) = -(\sin\varphi)f''(x) + (\cos\varphi)\frac{\gamma f(x)}{x^2}\,.$$

Observe that for $\varphi = 0$, the operator $V_{\varphi=0}$ is essentially selfadjoint, with its unique selfadjoint extension being the maximal multiplication operator by the function γx^{-2}. For $\varphi \neq 0$, firstly observe that the norm induced by V_φ is equivalent to the first Sobolev norm. This follows from completely analogous reasoning to that in Equations (7.3.9) and (7.3.10). Hence, we get

$$\mathcal{Q}(V_{\varphi,F}) = H_0^1(0,1) \quad \text{for all} \quad \varphi \in (0, \pi/2]\ . \tag{7.3.12}$$

Moreover, it is not very hard to check that $\langle f, V_\varphi f\rangle \geq \gamma \|f\|^2$ for all $\varphi \in [0, \pi/2]$ and for all $f \in \mathcal{C}_0^\infty(0,1)$, which implies that the form domain of the Kreĭn–von Neumann extension of V_φ is given by

$$\mathcal{Q}(V_{\varphi,K}) = \mathcal{Q}(V_{\varphi,F}) \dot{+} \ker V_\varphi^* \quad \text{for all} \quad \varphi \in (0, \pi/2]\,. \tag{7.3.13}$$

Moreover, it can be shown that

$$\ker V_\varphi^* = \begin{cases} \operatorname{span}\left\{x^{\frac{1+\sqrt{1+4\gamma\cot\varphi}}{2}}, x^{\frac{1-\sqrt{1+4\gamma\cot\varphi}}{2}}\right\} & \text{if } \cot\varphi < \frac{3}{4\gamma}, \\ \operatorname{span}\left\{x^{\frac{1+\sqrt{1+4\gamma\cot\varphi}}{2}}\right\} & \text{else}\,. \end{cases}$$

Now, define $k_\varphi(x) := x^{\frac{1+\sqrt{1+4\gamma\cot\varphi}}{2}}$ and observe that for any $f \in (\mathcal{D}(\widetilde{A}^*)//\mathcal{D}(A)) \cap \mathcal{D}(V_{\varphi,K}^{1/2})$, the following is true:

$$\operatorname{Im}\langle f, e^{i\varphi}\widetilde{A}^* f\rangle - \|V_{\varphi,K}^{1/2} f\|^2 = \operatorname{Im}\langle f, e^{i\varphi}\widetilde{A}^* f\rangle - \|V_{\varphi,F}^{1/2}(f - f(1)k_\varphi)\|^2\,,$$

where we have decomposed $f(x) = \underbrace{(f(x) - f(1)k_\varphi(x))}_{\in\mathcal{Q}(V_F)} + \underbrace{f(1)k_\varphi(x)}_{\in\ker V_\varphi^*}$ according to Equation (7.3.13). Note that this decomposition is independent of whether $x^{\frac{1-\sqrt{1+4\gamma\cot\varphi}}{2}}$ is in $\ker V_\varphi^*$ or not, since this function never matches the required boundary conditions at 0. Also, observe that for $\gamma \geq \sqrt{3}$, we have that $\mathcal{D}(\widetilde{A}^*)//\mathcal{D}(A) \subset \mathcal{D}(V_K^{1/2})$, but for $\gamma < \sqrt{3}$ note that $x^{\omega_-} \notin \mathcal{D}(V_{\varphi,K}^{1/2})$, which yields

$$(\mathcal{D}(\widetilde{A}^*)//\mathcal{D}(A)) \cap \mathcal{D}(V_{\varphi,K}^{1/2}) = \mathcal{D}(A)\dot{+}\operatorname{span}\{x^{\omega_+}, x^{\overline{\omega_+}+2}, x^{\overline{\omega_-}+2}\}\,. \tag{7.3.14}$$

A calculation now shows that for all $f \in (\mathcal{D}(\widetilde{A}^*)//\mathcal{D}(A)) \cap \mathcal{D}(V_{\varphi,K}^{1/2})$, we can simplify

$$\begin{aligned} q_\varphi(f) :=& \operatorname{Im}\langle f, e^{i\varphi}\widetilde{A}^* f\rangle - \|V_{\varphi,F}^{1/2}(f - f(1)k_\varphi)\|^2 \\ =& -\operatorname{Im}\left(\overline{f(1)}e^{i\varphi}f'(1)\right) + |f(1)|^2 \sin\varphi \frac{1+\sqrt{1+4\gamma\cot\varphi}}{2}\,. \end{aligned} \tag{7.3.15}$$

By formally letting $\varphi \to 0$, we also obtain the correct expression for the case $\varphi = 0$ (cf. Equation (5.4.8) after an integration by parts), which we will include from now on. Also, note that the sesquilinear form, which is associated to q_φ on $(\mathcal{D}(\widetilde{A}^*)//\mathcal{D}(A)) \cap \mathcal{D}(V_{\varphi,K}^{1/2})$ is given by:

$$s_\varphi(f,g) = \frac{-\overline{f(1)}e^{i\varphi}g'(1) + \overline{f'(1)}e^{-i\varphi}g(1)}{2i} + \overline{f(1)}g(1)\sin\varphi\frac{1+\sqrt{1+4\gamma\cot\varphi}}{2}\,. \tag{7.3.16}$$

Let us now discuss the two cases depending on the value of γ:

- The case $\gamma < \sqrt{3}$: Since $\dim\ker\widetilde{A}^* = 2$, this means that any maximal extension of A needs to be two-dimensional. Also, since A is sectorial, we know by Corollary 7.3.10 that $\dim\mathcal{W}_+^\varphi + \dim\mathcal{W}_0^\varphi = 2$, which together with the fact that

  $$\dim\left(\left(\mathcal{D}(\widetilde{A}^*)/\mathcal{D}(A)\right) \cap \mathcal{D}(V_{\varphi,K}^{1/2})\right) = 3$$

  implies that $\dim\mathcal{W}_-^\varphi = 1$. Here, the additional index φ for $\mathcal{W}_*^\varphi$, where $* \in \{+,0,-\}$, indicates the spectral subspaces as defined above for the selfadjoint operators M^φ associated to the quadratic forms q_φ. Another consequence of Corollary 7.3.10 is that $\dim\mathcal{W}_+^\varphi \leq 1$, which readily implies that $1 \leq \dim\mathcal{W}_0^\varphi \leq 2$. From the expression of the sesquilinear form s_φ given in Equation (7.3.16), it can be directly seen that a function $\chi \in \operatorname{span}\{x^{\omega_+}, x^{\overline{\omega_+}+2}, x^{\overline{\omega_-}+2}\}$ with $\chi(1) = \chi'(1) = 0$ has to lie in $\ker M^\varphi$. It can be easily verified that the function

  $$\chi(x) := (\omega_+ - \overline{\omega_-} - 2)(x^{\omega_+} - x^{\overline{\omega_+}+2}) - (\omega_+ - \overline{\omega_+} - 2)(x^{\omega_+} - x^{\overline{\omega_-}+2})$$

  satisfies these conditions and thus $\chi \in \ker M^\varphi$ for all $\varphi \in [0,\pi/2]$. Moreover, define the two functions

  $$\psi(x) := \frac{(2+\overline{\omega_+})x^{\omega_+} - \omega_+ x^{\overline{\omega_+}+2}}{2+\overline{\omega_+}-\omega_+} \quad \text{and} \quad \phi(x) := \frac{-x^{\omega_+} + x^{\overline{\omega_+}+2}}{2+\overline{\omega_+}-\omega_+}\,, \tag{7.3.17}$$

  which satisfy the boundary conditions

  $$\psi(1) = 1 \qquad \psi'(1) = 0$$
  $$\phi(1) = 0 \qquad \phi'(1) = 1\,.$$

  Now, observe that for any $\varphi \in [0,\pi/2]$, we have

  $$q(\psi - ie^{-i\varphi}\varepsilon\phi) = \varepsilon + \sin\varphi\frac{1+\sqrt{1+4\gamma\cot\varphi}}{2}\,,$$

which certainly is positive, if we choose $\varepsilon > 0$. It follows that $\dim \mathcal{W}_+^\varphi \geq 1$ and consequently $\dim \mathcal{W}_+^\varphi = \dim \mathcal{W}_0^\varphi = 1$. As we already have established that $\chi \in \ker M^\varphi$, this implies that $\mathcal{W}_0^\varphi = \ker M^\varphi = \operatorname{span}\{\chi\}$. By Theorem 5.2.13, this implies that any proper maximally dissipative extension of A_φ must have $\operatorname{span}\{\chi\}$ in its domain and since the extension has to be two-dimensional, it also needs to contain a suitable linear combination of ψ and ϕ, whose structure we will discuss after having treated the case $\gamma \geq \sqrt{3}$.

- The case $\gamma \geq \sqrt{3}$: Since $\dim \ker \widetilde{A}^* = 1$ in this case, this means that any maximal extension of A needs to be one-dimensional. By analogous reasoning to that in the case $\gamma < \sqrt{3}$, we may conclude that $\dim \mathcal{W}_+^\varphi + \dim \mathcal{W}_0^\varphi = 1$. Moreover, the fact that $\dim((\mathcal{D}(\widetilde{A}^*)/\mathcal{D}(A)) \cap \mathcal{D}(V_K^{1/2})) = 2$ implies that $\dim \mathcal{W}_-^\varphi = 1$. In order to determine $\dim \mathcal{W}_+^\varphi$, observe that the functions ψ and ϕ as defined in Equation (7.3.17), can also be defined for the case $\gamma \geq \sqrt{3}$ and they still have the property that $\psi, \phi \in (\mathcal{D}(\widetilde{A}^*)//\mathcal{D}(A)) \cap \mathcal{D}(V_K^{1/2})$. Thus, by mimicking the reasoning for the case $\gamma < \sqrt{3}$, where we have considered $q(\psi - ie^{-i\varphi}\varepsilon\phi) \geq \varepsilon > 0$, we may again conclude that $\dim \mathcal{W}_+^\varphi = 1$ and consequently that $\dim \mathcal{W}_0^\varphi = 0$ for all $\varphi \in [0, \pi/2]$. Since $\ker M^\varphi$ is trivial in this case, all proper maximally dissipative extensions have to be suitable linear combinations of ψ and ϕ, which we will discuss next.

Firstly, let us exclude the case $\varphi = 0$, which needs to be treated separately. To begin with, look at linear combinations of ψ and ϕ that are of the form

$$\xi_\rho := \rho\psi + \phi \,,$$

where $\rho \in \mathbb{C}$ has to be determined. Note that $\rho = 0$ corresponds to an element in the domain of the Friedrichs extension. Plugging ξ_ρ into the Equation for q_φ as given in (7.3.15), we get — after a short calculation — that $q_\varphi(\xi_\rho) \geq 0$ if and only if

$$\left| \rho + \frac{ie^{i\varphi}}{2\kappa(\varphi)} \right| \geq \frac{1}{2\kappa(\varphi)} \,, \tag{7.3.18}$$

where the function $\kappa(\varphi)$ is given by

$$\kappa(\varphi) = \sin\varphi \cdot k_\varphi'(1) = \sin\varphi \frac{1 + \sqrt{1 + 4\gamma \cot\varphi}}{2} \,.$$

Moreover, for the case $\varphi = 0$ we get that $q_0(\xi_\rho) \geq 0$ if and only if

$$\mathrm{Im}\rho \geq 0 \, .$$

Finally, we have to include the case $\xi_\infty := \psi$, for which we get $q_\varphi(\xi_\infty) = \kappa(\varphi) \geq 0$ for all $\varphi \in [0, \pi/2]$.

We thus have found a full description of all proper maximally dissipative and maximally sectorial extensions of the dual pair $(A, \widetilde{A})$:

- For the case $\gamma < \sqrt{3}$ all proper maximally dissipative extensions of $(A, \widetilde{A})$ are given by

$$\begin{aligned} A_\rho : \qquad & \mathcal{D}(A_\rho) = \mathcal{D}(A) \dot{+} \mathrm{span}\{\xi_\rho\} \dot{+} \mathrm{span}\{\chi\} \\ & A_\rho = \widetilde{A}^* \restriction_{\mathcal{D}(A_\rho)} \, , \end{aligned} \tag{7.3.19}$$

  where

$$\rho \in \{z \in \mathbb{C} : \mathrm{Im} z \geq 0\} \cup \{\infty\} \, . \tag{7.3.20}$$

  If in addition, we require that the numerical range of A_ρ is contained in the sector $S_{0,\pi-\varphi} = \{z \in \mathbb{C} : 0 \leq \arg z \leq \pi - \varphi\}$, where $\varphi \in (0, \pi/2]$, the parameter ρ has to satisfy

$$\rho \in \left(\{z \in \mathbb{C} : \mathrm{Im} z \geq 0\} \cap \left\{ z \in \mathbb{C} : \left| z + \frac{ie^{i\varphi}}{2\kappa(\varphi)} \right| \geq \frac{1}{2\kappa(\varphi)} \right\} \right) \cup \{\infty\} \, . \tag{7.3.21}$$

- For the case $\gamma \geq \sqrt{3}$, we can describe all proper maximally dissipative extensions of $(A, \widetilde{A})$ as follows:

$$\begin{aligned} A_\rho : \qquad & \mathcal{D}(A_\rho) = \mathcal{D}(A) \dot{+} \mathrm{span}\{\xi_\rho\} \\ & A_\rho = \widetilde{A}^* \restriction_{\mathcal{D}(A_\rho)} \, , \end{aligned} \tag{7.3.22}$$

  where ρ has to satisfy (7.3.20). If we require in addition that the numerical range of A_φ be contained in $S_{0,\pi-\varphi}$, Condition (7.3.21) has to be satisfied as well.

For the case $\gamma = 1$, Figures 1 and 2 display the sets (upper figure) of $\rho \in \mathbb{C}$ such that the numerical range of the operator A_ρ is contained in the sector $S_{0,\pi-\varphi}$ (lower figure). For any fixed $\varphi \in \left(0, \frac{\pi}{2}\right]$, one obtains this set by intersecting the closed upper half plane with the exterior of the open circle having center point equal to $-\frac{ie^{i\varphi}}{2\kappa(\varphi)}$ and passing through the origin. The set $C := \left\{-\frac{ie^{i\varphi}}{2\kappa(\varphi)} : \varphi \in \left(0, \frac{\pi}{2}\right]\right\}$ is shown in purple on these figures. The center point $-\frac{ie^{i\varphi}}{2\kappa(\varphi)}$ can be obtained by intersecting C with the straight line, which encloses an angle of φ with the negative imaginary axis. Observe that $\rho = 0$, which by construction describes the Friedrichs extension, is the unique point, which is contained in the intersection of the boundaries of the sets

$$\mathfrak{K}_\varphi := \left\{z \in \mathbb{C} : \left|z + \frac{ie^{i\varphi}}{2\kappa(\varphi)}\right| \geq \frac{1}{2\kappa(\varphi)}\right\} \cup \{\infty\}$$

for $\varphi \in (0, \pi/2]$ and

$$\mathfrak{K}_0 := \{z \in \mathbb{C} : \mathrm{Im} z \geq 0\} \cup \{\infty\} .$$

This is a consequence of Theorem 7.3.9, which states that the quadratic form q_φ has to vanish identically for all elements in the domain of the Friedrichs extension for all φ. However, the quadratic form vanishing for ξ_ρ corresponds to ρ being an element of $\partial\mathfrak{K}_\varphi$.

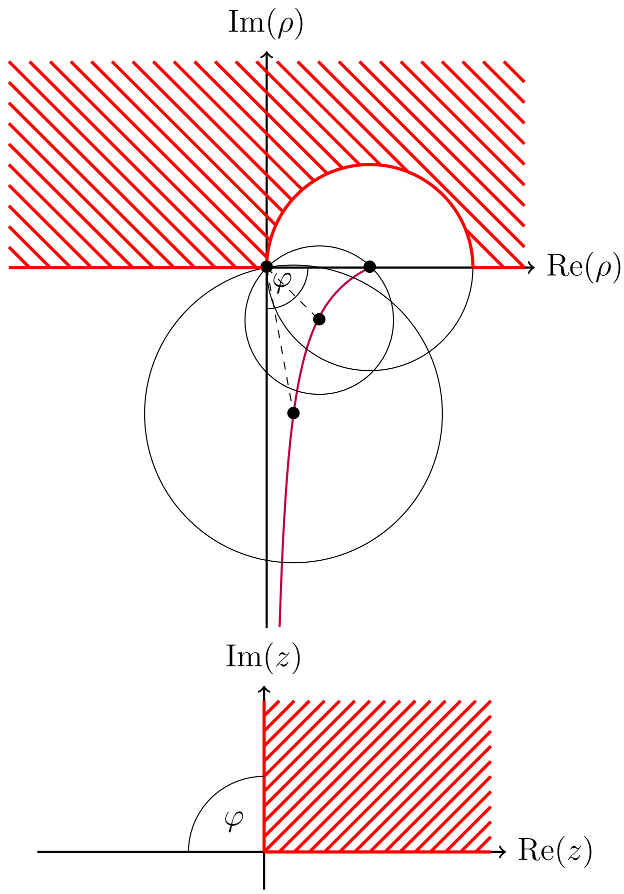


Figure 1. The upper figure depicts the set of all ρ such that A_ρ has numerical range contained in the sector $S_{0,\frac{\pi}{2}}$, which is shown in the lower figure. This means that $\varphi = \frac{\pi}{2}$.

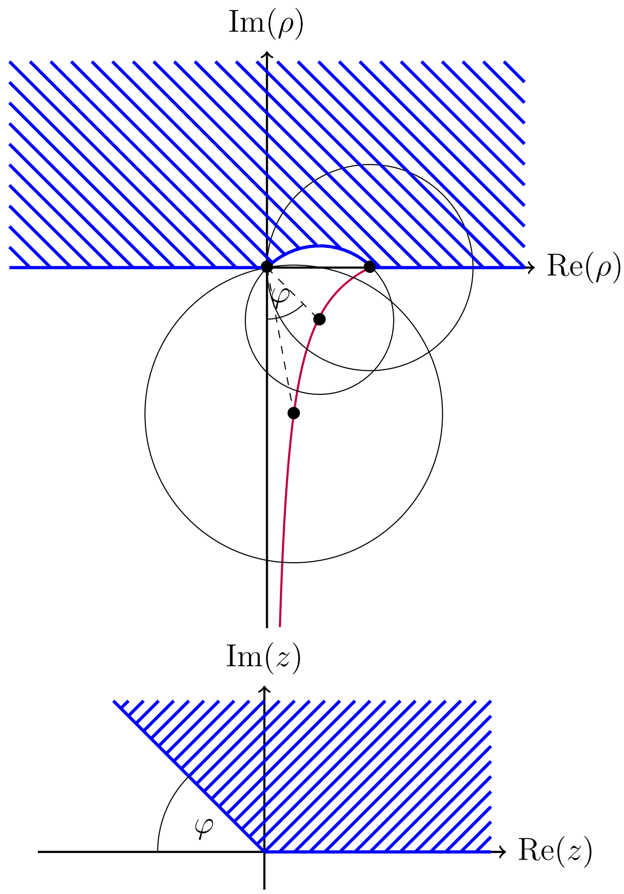


FIGURE 2. The upper figure depicts the set of all ρ such that A_ρ has numerical range contained in the sector $S_{0,\frac{3\pi}{4}}$, which is shown in the lower figure. This means that $\varphi = \frac{\pi}{4}$.

CHAPTER 8

A generalized Birman–Kreĭn–Vishik theory for sectorial operators

In the following, we are going to develop an analog of the Birman–Kreĭn–Vishik theory of selfadjoint extensions, where we want to define a partial order in the imaginary parts of the different extensions of a dual pair of sectorial operators. If the dual pair $(A, \widetilde{A})$ under consideration has the common core property, we have seen that the "imaginary part" of A can be defined as $V := (2i)^{-1}(A - \widetilde{A}) \restriction_{\mathcal{D}}$, where $\mathcal{D}$ is the common core. It turns out that the proper maximally dissipative extensions of $(A, \widetilde{A})$ can be parametrized by auxiliary operators D that map from a subspace of $\ker \widetilde{A}^* \cap \mathcal{D}(V_K^{1/2})$ into $\ker A^*$. We will denote these extensions by A_D. After that, we will show that — provided it is closable — the closure of the quadratic form $f \mapsto \operatorname{Im}\langle f, A_D f\rangle$ corresponds to a non-negative selfadjoint extension of the imaginary part V. This enables us to apply the results of Birman–Kreĭn–Vishik in order to define an order between the imaginary parts of the extensions A_D. In order to present our result a way similar to Proposition 1.1, we need to introduce a modified sesquilinear form.

Definition 8.1. *Let $(A, \widetilde{A})$ be a dual pair which has the common core property, where $0 \in \widehat{\rho}(A)$. Moreover, assume that A is sectorial and dissipative. Let us define the following non-Hermitian sesquilinear form:*

$$[\cdot,\cdot]: \qquad \mathcal{D}(V_K^{1/2}) \times \mathcal{H} \to \mathbb{C}$$
$$[f,g] := \langle f, g\rangle - 2i\langle V_K^{1/2} f, V_K^{1/2} A_F^{-1} g\rangle$$

Since A_F is maximally dissipative, we get $\mathcal{D}(A_F) \subset \mathcal{D}(V_K^{1/2})$, which means that $[\cdot,\cdot]$ is well-defined. Moreover, for any subset $\mathcal{A} \subset \mathcal{D}(V_K^{1/2})$, let us define its **orthogonal companion** $\mathcal{A}^{[\perp]}$ *as*

$$\mathcal{A}^{[\perp]} := \{g \in \mathcal{H} : [f,g] = 0 \text{ for all } f \in \mathcal{A}\}\,.$$

REMARK 8.2. It is not necessary to compute $V_K^{1/2}$ explicitly in order to determine $[f,g]$, as it is sufficient to know the action of the quadratic form $\psi \mapsto \|V_K^{1/2}\psi\|^2$. The value of $\langle V_K^{1/2} f, V_K^{1/2} A_F^{-1} g\rangle$ can then be obtained by polarization.

Since we have chosen to investigate extensions of dissipative sectorial operators, let us make the following convention:

CONVENTION 8.3. *When speaking of a sectorial operator $A \in \mathfrak{S}_{\alpha,\beta}$, where $(\beta-\alpha) < \pi$, let us assume once and for all that $\alpha, \beta \in [0, \pi]$.*

Let us now use auxiliary operators D from $\ker \widetilde{A}^* \cap \mathcal{D}(V_K^{1/2})$ to $\ker A^*$ and subspaces $\mathfrak{M}$ of $(\ker A^* \cap \mathcal{D}(D)^{[\perp]})$ in order to parametrize the proper dissipative extensions of $(A, \widetilde{A})$:

THEOREM 8.4. *Let $(A, \widetilde{A})$ be a dual pair with common core property, where A is sectorial and $0 \in \widehat{\rho}(A)$. Then all proper dissipative extensions of $(A, \widetilde{A})$ can be described by all pairs of the form $(D, \mathfrak{M})$, where*

- *D is an operator from $\ker \widetilde{A}^* \cap \mathcal{D}(V_K^{1/2})$ to $\ker A^*$ that satisfies*

$$Im[\widetilde{k}, D\widetilde{k}] \geq \|V_K^{1/2}\widetilde{k}\|^2 \quad \text{for all } \widetilde{k} \in \mathcal{D}(D) \tag{8.1}$$

- $\mathfrak{M} \subset \left(\ker A^* \cap \mathcal{D}(D)^{[\perp]}\right)$.

The corresponding dissipative extensions can be described by

$$A_{D,\mathfrak{M}}: \qquad \mathcal{D}(A_{D,\mathfrak{M}}) = \mathcal{D}(A) \dot{+} \{A_F^{-1}D\widetilde{k} + \widetilde{k} : \widetilde{k} \in \mathcal{D}(D)\} \dot{+} \{A_F^{-1}k : k \in \mathfrak{M}\}$$

$$A_{D,\mathfrak{M}} = \widetilde{A}^* \restriction_{\mathcal{D}(A_{D,\mathfrak{M}})}$$

Moreover, $A_{D,\mathfrak{M}}$ is maximally dissipative if and only if $\mathfrak{M} = \left(\ker A^* \cap \mathcal{D}(D)^{[\perp]}\right)$ and D is maximal in the sense that there exists no extension of $D \subset D'$ such that $\ker A^* \cap \mathcal{D}(D)^{[\perp]} = \ker A^* \cap \mathcal{D}(D')^{[\perp]}$ and D' still satisfies (8.1).

PROOF. Since A is sectorial and the dual pair $(A, \widetilde{A})$ has the common core property, we have by Theorem 7.3.8 that it allows for the Friedrichs extension A_F, which is proper: $A \subset A_F \subset \widetilde{A}^*$. Moreover, by Proposition 7.3.3, we have that $0 \in \rho(A_F)$, which means by Proposition 2.4.3 that we can write

$$\mathcal{D}(\widetilde{A}^*) = \mathcal{D}(A) \dot{+} A_F^{-1} \ker A^* \dot{+} \ker \widetilde{A}^* \, .$$

Hence, we can choose $\mathcal{D}(\widetilde{A}^*)//\mathcal{D}(A) = A_F^{-1}\ker A^* \dot{+} \ker \widetilde{A}^*$. Now, all possible subspaces of $A_F^{-1}\ker A^* \dot{+} \ker \widetilde{A}^*$ will be of the form

$$\mathcal{V}_{D,\mathfrak{M}} := \{A_F^{-1}D\widetilde{k} + \widetilde{k} : \widetilde{k} \in \mathcal{D}(D)\} \dot{+} \{A_F^{-1}k : k \in \mathfrak{M}\} ,$$

where D is a map from $\mathcal{D}(D) \subset \ker \widetilde{A}^*$ to $\ker A^*$ and $\mathfrak{M} \subset \ker A^*$. We therefore use the pairs $(D, \mathfrak{M})$ to parametrize all proper extensions of the dual pair $(A, \widetilde{A})$ via $A_{\mathcal{V}_{D,\mathfrak{M}}} =: A_{D,\mathfrak{M}}$. Thus, by Theorem 5.2.8, $A_{D,\mathfrak{M}}$ is dissipative if and only if we have $[(A_F^{-1}D\widetilde{k} + \widetilde{k}) + A_F^{-1}k] \in \mathcal{D}(V_K^{1/2})$ and

$$\begin{aligned} q((A_F^{-1}D\widetilde{k} + \widetilde{k}) + A_F^{-1}k) :=& \\ &\operatorname{Im}\langle (A_F^{-1}D\widetilde{k} + \widetilde{k}) + A_F^{-1}k, \widetilde{A}^*[(A_F^{-1}D\widetilde{k} + \widetilde{k}) + A_F^{-1}k]\rangle \\ &\qquad - \|V_K^{1/2}(A_F^{-1}D\widetilde{k} + \widetilde{k} + A_F^{-1}k)\|^2 \geq 0 , \end{aligned} \tag{8.2}$$

for all $\widetilde{k} \in \mathcal{D}(D)$ and $k \in \mathfrak{M}$. Since A_F is a proper maximally dissipative extension of $(A, \widetilde{A})$ we have by Theorem 7.3.9 that $\mathcal{D}(A_F) \subset \mathcal{D}(V_K^{1/2})$, which means that the first condition is satisfied if and only if $\mathcal{D}(D) \subset \mathcal{D}(V_K^{1/2})$. Let us rewrite (8.2):

$$\begin{aligned} &q((A_F^{-1}D\widetilde{k} + \widetilde{k}) + A_F^{-1}k) \\ &= \operatorname{Im}\langle (A_F^{-1}D\widetilde{k} + \widetilde{k}) + A_F^{-1}k, \widetilde{A}^*[(A_F^{-1}D\widetilde{k} + \widetilde{k}) + A_F^{-1}k]\rangle - \|V_K^{1/2}(A_F^{-1}D\widetilde{k} + \widetilde{k} + A_F^{-1}k)\|^2 \\ &= \operatorname{Im}\langle A_F^{-1}D\widetilde{k} + A_F^{-1}k, A_F[A_F^{-1}D\widetilde{k} + A_F^{-1}k]\rangle + \operatorname{Im}\langle \widetilde{k}, D\widetilde{k} + k\rangle \\ &- \|V_K^{1/2}(A_F^{-1}D\widetilde{k} + A_F^{-1}k)\|^2 - \|V_K^{1/2}\widetilde{k}\|^2 - 2\operatorname{Re}\langle V_K^{1/2}\widetilde{k}, V_K^{1/2}(A_F^{-1}D\widetilde{k} + A_F^{-1}k)\rangle \\ &= \operatorname{Im}\left(\langle \widetilde{k}, D\widetilde{k}\rangle - 2i\langle V_K^{1/2}\widetilde{k}, V_K^{1/2}A_F^{-1}D\widetilde{k}\rangle\right) \\ &+ \operatorname{Im}\left(\langle \widetilde{k}, k\rangle - 2i\langle V_K^{1/2}\widetilde{k}, V_K^{1/2}A_F^{-1}k\rangle\right) - \|V_K^{1/2}\widetilde{k}\|^2 \\ &= \operatorname{Im}[\widetilde{k}, D\widetilde{k}] + \operatorname{Im}[\widetilde{k}, k] - \|V_K^{1/2}\widetilde{k}\|^2 \geq 0 , \end{aligned}$$

where we have used that by, Theorem 7.3.9,

$$\operatorname{Im}\langle A_F^{-1}D\widetilde{k} + A_F^{-1}k, A_F[A_F^{-1}D\widetilde{k} + A_F^{-1}k]\rangle = \|V_K^{1/2}(A_F^{-1}D\widetilde{k} + A_F^{-1}k)\|^2 . \tag{8.3}$$

Now, assume that

(8.4)

$$\mathfrak{M} \subset (\ker A^* \cap \mathcal{D}(D)^{[\perp]}) \quad \text{and that} \quad \operatorname{Im}[\widetilde{k}, D\widetilde{k}] - \|V_K^{1/2}\widetilde{k}\|^2 \geq 0 \quad \text{for all} \quad \widetilde{k} \in \mathcal{D}(D) .$$

Hence, we get that

$$q((A_F^{-1}D\widetilde{k}+\widetilde{k})+A_F^{-1}k)=\mathrm{Im}[\widetilde{k},D\widetilde{k}]+\mathrm{Im}[\widetilde{k},k]-\|V_K^{1/2}\widetilde{k}\|^2=\mathrm{Im}[\widetilde{k},D\widetilde{k}]-\|V_K^{1/2}\widetilde{k}\|^2\geq 0\,,$$

for all $\widetilde{k}\in\mathcal{D}(D)$ and all $k\in\mathfrak{M}$. This means that Condition (8.4) being satisfied is sufficient for $A_{D,\mathfrak{M}}$ to be dissipative. Let us now show that it is also necessary. Thus, assume that Condition (8.4) is not satisfied. If there exists a $\widetilde{k}\in\mathcal{D}(D)$ such that $\mathrm{Im}[\widetilde{k},D\widetilde{k}]-\|V_K^{1/2}\widetilde{k}\|^2<0$, this means that (8.2) cannot be satisfied in this case as we can choose $k=0$. Moreover, if there exists a $k\in\mathfrak{M}$ and a $\widetilde{k}\in\mathcal{D}(D)$ such that $[\widetilde{k},k]\neq 0$, this means that we can replace $k\mapsto\lambda k$, where $\lambda\in\mathbb{C}$ is suitably chosen such that

$$q((A_F^{-1}D\widetilde{k}+\widetilde{k})+A_F^{-1}\lambda k)=\mathrm{Im}[\widetilde{k},D\widetilde{k}]+\mathrm{Im}[\widetilde{k},\lambda k]-\|V_K^{1/2}\widetilde{k}\|^2<0\,,$$

which means that $A_{D,\mathfrak{M}}$ cannot be dissipative in this case either.
Let us now prove that $A_{D,\mathfrak{M}}$ is maximally dissipative if and only if D is maximal in the sense as stated in the theorem and $\mathfrak{M}=\ker A^*\cap\mathcal{D}(D)^{[\perp]}$. Clearly, if there exists a $D\subset D'$ such that $\mathrm{Im}[\widetilde{k},D'\widetilde{k}]\geq\|V_K^{1/2}\widetilde{k}\|^2$ for all $\widetilde{k}\in\mathcal{D}(D')$ and $(\ker A^*\cap\mathcal{D}(D)^{[\perp]})=(\ker A^*\cap\mathcal{D}(D')^{[\perp]})$ or a $\mathfrak{M}\subset\mathfrak{M}'\subset(\ker A^*\cap\mathcal{D}(D)^{[\perp]})$, we get that $A_{D',\mathfrak{M}'}$ is a dissipative extension of $A_{D,\mathfrak{M}}$.

For the other direction, let us assume that $A_{D,\mathfrak{M}}$ is not maximally dissipative. It is clear that the operator $A_{D,\widehat{\mathfrak{M}}}$, where $\widehat{\mathfrak{M}}=\ker A^*\cap\mathcal{D}(D)^{[\perp]}$, is a dissipative extension of $A_{D,\mathfrak{M}}$ and from now on, we will therefore only consider this case. By Proposition 2.4.5, we know that there exists a proper maximally dissipative extension $\widehat{A}$ of the dual pair $(A_{D,\widehat{\mathfrak{M}}},\widetilde{A})$ and by what we have shown above, there exists an operator D' that satisfies (8.1) and a subspace $\mathfrak{M}'\subset(\ker A^*\cap\mathcal{D}(D')^{[\perp]})$ such that $\widehat{A}=A_{D',\mathfrak{M}'}$, where $D\subset D'$ and $(\ker A^*\cap\mathcal{D}(D)^{[\perp]})=\widehat{\mathfrak{M}}\subset\mathfrak{M}'$. However, $\mathcal{D}(D)\subset\mathcal{D}(D')$ implies that $\mathcal{D}(D)^{[\perp]}\supset\mathcal{D}(D')^{[\perp]}$, from which it immediately follows that $(\ker A^*\cap\mathcal{D}(D)^{[\perp]})\supset(\ker A^*\cap\mathcal{D}(D')^{[\perp]})$. This implies that $(\ker A^*\cap\mathcal{D}(D)^{[\perp]})=(\ker A^*\cap\mathcal{D}(D')^{[\perp]})$, which shows that D was not maximal in the sense as stated in the theorem. □

Remark 8.5. Note that the correspondence between the pairs $(D,\mathfrak{M})$ and the proper maximally dissipative extensions $A_{D,\mathfrak{M}}$ of $(A,\widetilde{A})$ is not one-to-one. This follows from the fact that for any $\widetilde{k}\in\mathcal{D}(D)$ and any $k\in(\ker A^*\cap\mathcal{D}(D)^{[\perp]})$, we can write

$$\mathrm{span}\{A_F^{-1}D\widetilde{k}+\widetilde{k}\}\dot{+}\mathrm{span}\{A_F^{-1}k\}=\mathrm{span}\{A_F^{-1}(D\widetilde{k}+k)+\widetilde{k}\}\dot{+}\mathrm{span}\{A_F^{-1}k\}\,,$$

i.e. if we have an auxiliary operator D' with $\mathcal{D}(D') = \mathcal{D}(D)$ and $D'\widetilde{k} - D\widetilde{k} \in (\ker A^* \cap \mathcal{D}(D)^{[\perp]})$ for all $\widetilde{k} \in (\ker A^* \cap \mathcal{D}(D)^{[\perp]})$ we would get that $A_{D',\mathfrak{M}} = A_{D,\mathfrak{M}}$. However, we could for example restrict our considerations to auxiliary operators that satisfy $D\widetilde{k} \perp k$ for all $\widetilde{k} \in \mathcal{D}(D)$ and all $k \in (\ker A^* \cap \mathcal{D}(D)^{[\perp]})$. With this additional requirement, the correspondence between $(D, \mathfrak{M})$ and proper dissipative extensions $A_{D,\mathfrak{M}}$ of $(A, \widetilde{A})$ becomes one-to-one.

Remark 8.6. For the case that $\mathcal{D}(D)$ is finite-dimensional the maximality condition on D is automatically satisfied. In this case, $A_{D,\mathfrak{M}}$ is therefore maximally dissipative if and only if $\mathfrak{M} = (\ker A^* \cap \mathcal{D}(D)^{[\perp]})$.

Remark 8.7. This parametrization of all maximally dissipative extensions of the dual pair $(A, \widetilde{A})$, where A is sectorial can be generalized to truly dissipative dual pairs in some situations. The proof of Theorem 8.4 still carries through as long as one assumes the existence of a proper maximally dissipative extension $\widehat{A}$ such that $0 \in \rho(\widehat{A})$ and an analog of (8.3) is still valid, i.e. we would need that for all $v \in \mathcal{D}(\widehat{A})//\mathcal{D}(A)$ it holds that

$$\operatorname{Im}\langle v, \widehat{A}v\rangle = \|V_K^{1/2} v\|^2 \, . \tag{8.5}$$

Theorem 8.4 could then be reformulated with $\widehat{A}$ taking the role of the Friedrichs extension A_F. For the case of finite-dimensional defect indices, this means that it is necessary that there exists an isometry from $\mathcal{W}_+$ into $\mathcal{W}_-$, i.e. it is necessary that $\dim \mathcal{W}_+ \leq \dim \mathcal{W}_-$, where $\mathcal{W}_+$ and $\mathcal{W}_-$ have been introduced in Theorem 5.2.13. See Example 5.4.7 for a dissipative operator for which $\dim \mathcal{W}_+ = 1$ and $\dim \mathcal{W}_- = 0$, which means that no proper maximally dissipative extension $\widehat{A}$ of $(A, \widetilde{A})$ can satisfy (8.5) in this case. This means that the parametrization of Theorem 8.4 could not be used in this case.

Remark 8.8. Observe that this theorem reduces to the result of Theorem 7.2.2 in the case of (maximally) dissipative extensions of a dual pair of strictly positive symmetric operators (S, S), where we have $V = (2i)^{-1}(S - S) = 0$, since the condition $\operatorname{Im}[\widetilde{k}, D\widetilde{k}] \geq \|V_K^{1/2}\widetilde{k}\|^2$ is equivalent to the condition $\operatorname{Im}\langle \widetilde{k}, D\widetilde{k}\rangle \geq 0$ for all $\widetilde{k} \in \mathcal{D}(D)$ in this case. This is of course the same as requiring that D be a dissipative operator from

$\mathcal{D}(D) \subset \ker S^*$ into $\ker S^*$. The condition that S_D is maximally dissipative if and only if D is a maximally dissipative operator in $\overline{\mathcal{D}(D)}$ follows also from this theorem since $(\ker S^* \cap \mathcal{D}(D)^\perp) = (\ker S^* \cap \mathcal{D}(D')^\perp)$ is equivalent to $\overline{\mathcal{D}(D)} = \overline{\mathcal{D}(D')}$.

CONVENTION 8.9. *If $A_{D,\mathfrak{M}}$ is a maximally dissipative extension of the dual pair $(A, \widetilde{A})$ as defined in Theorem 8.4, we know that $\mathfrak{M}$ is determined by the choice of D: $\mathfrak{M} = \ker A^* \cap \mathcal{D}(D)^{[\perp]}$. Thus, if $A_{D,\mathfrak{M}}$ is maximally dissipative let us just write A_D instead of $A_{D,\mathfrak{M}}$.*

EXAMPLE 8.10. As in Section 5.4.4, let $\gamma > 0$ and $\mathcal{H} = L^2(0,1)$ and consider the dual pair $A_0 = -i\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{\gamma}{x^2}$ and $\widetilde{A}_0 = i\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{\gamma}{x^2}$ with domain $\mathcal{C}_c^\infty(0,1)$ and let $A, \widetilde{A}$ denote their respective closures. As imaginary part, we may choose $V = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ with domain $\mathcal{C}_c^\infty(0,1)$. Recall that the domain of $V_K^{1/2}$ is given by $H^1(0,1)$ and since $\ker V^* = \operatorname{span}\{1, x\}$, we have that

$$\|V_K^{1/2} f\|^2 = \|V_F^{1/2}(f(x) - (1-x)f(0) - xf(1))\|^2 = \|f'\|^2 - |f(1) - f(0)|^2 .$$

Define the numbers $\omega_\pm := \frac{1 \pm \sqrt{1+4i\gamma}}{2}$. As already shown in Section 5.4.4, we have that:

- For $0 < \gamma < \sqrt{3}$: $\ker \widetilde{A}^* = \operatorname{span}\{x^{\omega_+}, x^{\omega_-}\}$ and $\ker A^* = \operatorname{span}\{x^{\overline{\omega_+}}, x^{\overline{\omega_-}}\}$. However, we have $x^{\omega_-} \notin H^1(0,1) = \mathcal{D}(V_K^{1/2})$, since $\operatorname{Re}(\omega_-) < 1/2$. This means that $\ker \widetilde{A}^* \cap \mathcal{D}(V_K^{1/2}) = \operatorname{span}\{x^{\omega_+}\}$. Hence, the only two choices for $\mathcal{D}(D)$ are either $\mathcal{D}(D) = \{0\}$ or $\mathcal{D}(D) = \operatorname{span}\{x^{\omega_+}\}$. As $\mathcal{D}(D) = \{0\}$ just corresponds to the Friedrichs extension of A, let us now focus on the case $\mathcal{D}(D) = \operatorname{span}\{x^{\omega_+}\}$. Rather than determining $\mathcal{D}(D)^{[\perp]} \cap \ker A^* = \operatorname{span}\{x^{\omega_+}\}^{[\perp]} \cap \operatorname{span}\{x^{\overline{\omega_+}}, x^{\overline{\omega_-}}\}$, which we could obtain by finding the solution space of

$$\langle x^{\omega_+}, \lambda x^{\overline{\omega_+}} + \mu x^{\overline{\omega_-}}\rangle - 2i\langle V_K^{1/2} x^{\omega_+}, V_K^{1/2} A_F^{-1}(\lambda x^{\overline{\omega_+}} + \mu x^{\overline{\omega_-}})\rangle = 0 ,$$

we use that, in Section 5.4.4, we have already shown that any proper dissipative extension of $(A, \widetilde{A})$ was of the form

$$\mathcal{D}(A_\rho) = \mathcal{D}(A) \dot{+} \operatorname{span}\{\xi_\rho\} \dot{+} \operatorname{span}\{\chi\} . \tag{8.6}$$

Here, $\{\xi_\rho\}_\rho$ is a family of functions that is characterized by one complex parameter ρ that has to satisfy (5.4.16), while the function χ, which is given by

$$\chi(x) := (\omega_+ - \overline{\omega_-} - 2)(x^{\omega_+} - x^{\overline{\omega_+}+2}) - (\omega_+ - \overline{\omega_+} - 2)(x^{\omega_+} - x^{\overline{\omega_-}+2})$$

has to lie in the domain of *any* proper maximally dissipative extension of $(A, \widetilde{A})$. Comparing the structure of (8.6) to the structure of the proper maximally dissipative extensions as described in Theorem 8.4, we see that χ is a natural candidate for $\chi = A_F^{-1}k$, where $k \in (\mathcal{D}(D)^{[\perp]} \cap \ker A^*)$. Indeed, a lengthy but not difficult calculation shows that $[x^{\omega_+}, k] = [x^{\omega_+}, A_F\chi] = \langle x^{\omega_+}, A_F\chi\rangle - 2i\langle V_K^{1/2}x^{\omega_+}, V_K^{1/2}\chi\rangle = 0$.

- For $\gamma \geq \sqrt{3}$: $\ker \widetilde{A}^* = \operatorname{span}\{x^{\omega_+}\}$ and $\ker A^* = \operatorname{span}\{x^{\overline{\omega_+}}\}$. Moreover, $x^{\omega_+} \in H^1(0,1) = \mathcal{D}(V_K^{1/2})$ from which we get that $(\ker A^* \cap \mathcal{D}(V_K^{1/2})) = \operatorname{span}\{x^{\omega_+}\}$.

For $\gamma \geq \sqrt{3}$, any map from $\operatorname{span}\{x^{\omega_+}\}$ into $\operatorname{span}\{x^{\overline{\omega_+}}\}$ has to be of the form

$$Dx^{\omega_+} = dx^{\overline{\omega_+}}\,, \tag{8.7}$$

where $d \in \mathbb{C}$. For the case $0 < \gamma < \sqrt{3}$, let us argue that it is also sufficient to only consider maps D of the form (8.7). This follows from what has been said in Remark 8.5. To see this, assume that the map D is of the form

$$Dx^{\omega_+} = d_+x^{\overline{\omega_+}} + d_-x^{\overline{\omega_-}}\,.$$

Then, since $(A_F^{-1}x^{\overline{\omega_\pm}}) \propto (x^{\overline{\omega_\pm}+2} - x^{\omega_+})$ we can find numbers $\lambda, \mu \in \mathbb{C}$ such that

$$(A_F^{-1}Dx^{\omega_+} + \lambda\chi(x)) = \mu(x^{\overline{\omega_+}+2} - x^{\omega_+}) \propto A_F^{-1}x^{\overline{\omega_+}}\,,$$

which means that there exists another number $\nu \in \mathbb{C}$ such that

$$(A_F^{-1}Dx^{\omega_+} + \lambda\chi(x)) = A_F^{-1}(Dx^{\omega_+} + \lambda A_F\chi(x)) = \nu A_F^{-1}x^{\overline{\omega_+}}\,. \tag{8.8}$$

Thus, the operator D' given by

$$D'x^{\omega_+} = Dx^{\omega_+} + \lambda A_F\chi(x)$$

maps $\operatorname{span}\{x^{\omega_+}\}$ into $\operatorname{span}\{x^{\overline{\omega_+}}\}$, which follows from (8.8) and the fact that A_F^{-1} is injective. Moreover, since $A_F\chi \in (\ker A^* \cap \mathcal{D}(D)^{[\perp]})$, we have that $A_D = A_{D'}$. Hence, for any $\gamma > 0$, we only need to consider auxiliary operators of the form (8.7). A calculation shows that

$$[x^{\omega_+}, Dx^{\omega_+}] = d[x^{\omega_+}, x^{\overline{\omega_+}}]$$
$$= \frac{-d}{i(\overline{\omega_+}+2)(\overline{\omega_+}+1)-\gamma}\left[\frac{\overline{\omega_+}(\overline{\omega_+}+2)}{2\overline{\omega_+}+1} - \frac{|\omega_+|^2}{\omega_+ + \overline{\omega_+} - 1}\right] = d \cdot \sigma(\omega_+)\ ,$$

where we have defined

$$\sigma(\omega_+) := \frac{-1}{i(\overline{\omega_+}+2)(\overline{\omega_+}+1)-\gamma}\left[\frac{\overline{\omega_+}(\overline{\omega_+}+2)}{2\overline{\omega_+}+1} - \frac{|\omega_+|^2}{\omega_+ + \overline{\omega_+} - 1}\right]\ .$$

From another calculation, we get

$$\|V_K^{1/2} x^{\omega_+}\|^2 = \|(x^{\omega_+})'\|^2 - 1 = \frac{|\omega_+ - 1|^2}{\omega_+ + \overline{\omega_+} - 1} =: \tau(\omega_+)\ .$$

Thus, A_D is dissipative if and only if the condition

$$\text{(8.9)} \qquad \mathrm{Im}(d\sigma(\omega_+)) = \mathrm{Re}(d)\mathrm{Im}(\sigma(\omega_+)) + \mathrm{Im}(d)\mathrm{Re}(\sigma(\omega_+)) \geq \tau(\omega_+)$$

is satisfied, which means that d has to lie in a half-plane of the complex plane.

Next, we want to investigate what can be said about the quadratic form associated to the imaginary part of a maximally dissipative extension of a dual pair $(A, \widetilde{A})$ satisfying the assumptions of Theorem 8.4. Hence, let us define

Definition 8.11. *Let $(A, \widetilde{A})$ be a dual pair satisfying the assumptions of Theorem 8.4. For any proper maximally dissipative extension A_D of $(A, \widetilde{A})$ let us define the associated non-negative quadratic form $\mathfrak{im}_{D,0}$:*

$$\mathfrak{im}_{D,0}: \qquad \mathcal{D}(\mathfrak{im}_{D,0}) = \mathcal{D}(A_D)$$
$$\mathfrak{im}_{D,0}(\psi) = Im\langle\psi, A_D\psi\rangle\ .$$

Moreover, if $\mathfrak{im}_{D,0}$ is closable let us denote its closure by $\mathfrak{im}_D := \overline{\mathfrak{im}_{D,0}}$. Recall that $\mathfrak{im}_D$ is given by:

$\mathfrak{im}_D$:

$$\mathcal{D}(\mathfrak{im}_D) = \{f \in \mathcal{H} : \exists \{f_n\}_n \subset \mathcal{D}(\mathfrak{im}_{D,0}) \ s.t.\ \|f_n - f\| \overset{n\to\infty}{\longrightarrow} 0 \ and\ \|f_n - f_m\|_{\mathfrak{im}_D} \overset{n,m\to\infty}{\longrightarrow} 0\}$$

$$\mathfrak{im}_D(f) := \lim_{n\to\infty} \mathfrak{im}_{D,0}(f_n)\ ,$$

where $\|\cdot\|_{\mathfrak{im}_D}$ *denotes the norm induced by* $\mathfrak{im}_{D,0}$*:*

$$\|f\|^2_{\mathfrak{im}_D} := \|f\|^2 + \mathfrak{im}_{D,0}(f) = \|f\|^2 + Im\langle f, A_D f\rangle \quad \textit{for all } f \in \mathcal{D}(A_D)\,.$$

Moreover, let us denote the non-negative selfadjoint operator associated to $\mathfrak{im}_D$ *by* V_D*.*

By [**26**, Thm. VI, 1.27] each non-negative selfadjoint operator S induces a closable quadratic form. However, it is not always the case that the form $\mathfrak{im}_{D,0}$ is closable. Let us now give a necessary and sufficient condition for $\mathfrak{im}_{D,0}$ to be closable.

THEOREM 8.12. *Let* A_D *be defined as in Theorem 8.4 and assume that* $V \geq \varepsilon > 0$ *as well as* $\dim \mathcal{D}(D) < \infty$*. Then,* $\mathfrak{im}_{D,0}$ *is closable if and only if we have that*

$$q(A_F^{-1}D\widetilde{k} + \widetilde{k}) = Im[\widetilde{k}, D\widetilde{k}] - \|V_K^{1/2}\widetilde{k}\|^2 = 0 \quad \textit{for all} \quad \widetilde{k} \in \mathcal{D}(D) \cap \mathcal{D}(V_F^{1/2})\,. \tag{8.10}$$

PROOF. Firstly, let us show that (8.10) is necessary for $\mathfrak{im}_{D,0}$ to be closable. Thus, assume that there exists a $\widetilde{k} \in \mathcal{D}(D) \cap \mathcal{D}(V_F^{1/2})$ such that $q(A_F^{-1}D\widetilde{k} + \widetilde{k}) \neq 0$. Since by Theorem 7.3.9, we have that $A_F^{-1}D\widetilde{k} + \widetilde{k} \in \mathcal{D}(V_F^{1/2})$, there exists a sequence $\{f_n\}_n \subset \mathcal{D}(V)$ that is Cauchy with respect to $\|\cdot\|^2 + \langle\cdot, V\cdot\rangle$ such that

$$\|f_n + A_F^{-1}D\widetilde{k} + \widetilde{k}\|^2 + \|V_F^{1/2}(f_n + A_F^{-1}D\widetilde{k} + \widetilde{k})\|^2 \stackrel{n\to\infty}{\longrightarrow} 0\,.$$

This means in particular that the sequence $g_n := f_n + A_F^{-1}D\widetilde{k} + \widetilde{k}$ converges to 0:

$$\lim_{n\to\infty} \|g_n\| = 0\,.$$

Also, since $\{f_n\}_n \subset \mathcal{D}(V) \subset \mathcal{D}(V_F^{1/2})$, we can show that $\{g_n\}_n$ is Cauchy with respect to $\|\cdot\|_{\mathfrak{im}_D}$:

$$\|g_n - g_m\|^2_{\mathfrak{im}_D} = \|f_n - f_m\|^2_{\mathfrak{im}_D} = \|f_n - f_m\|^2 + \|V_F^{1/2}(f_n - f_m)\|^2 \stackrel{n,m\to\infty}{\longrightarrow} 0\,.$$

However, by Lemma 5.5.1 we have that

$$\begin{aligned}\mathrm{Im}\langle g_n, A_D g_n\rangle &= \mathrm{Im}\langle f_n + A_F^{-1}D\widetilde{k} + \widetilde{k}, A_D(f_n + A_F^{-1}D\widetilde{k} + \widetilde{k})\rangle \\ &= \|V_K^{1/2}(f_n + A_F^{-1}D\widetilde{k} + \widetilde{k})\|^2 + q(A_F^{-1}D\widetilde{k} + \widetilde{k})\,.\end{aligned}$$

Moreover, since $\|V_K^{1/2}(f_n + A_F^{-1}D\widetilde{k} + \widetilde{k})\| = \|V_F^{1/2}(f_n + A_F^{-1}D\widetilde{k} + \widetilde{k})\|$, we get

$$\begin{aligned}\|g_n\|^2_{\mathfrak{im}_D} = \|f_n + A_F^{-1}D\widetilde{k} + \widetilde{k}\|^2 &+ \|V_F^{1/2}(f_n + A_F^{-1}D\widetilde{k} + \widetilde{k})\|^2 \\ &+ q(A_F^{-1}D\widetilde{k} + \widetilde{k}) \stackrel{n\to\infty}{\longrightarrow} q(A_F^{-1}D\widetilde{k} + \widetilde{k}) \neq 0\,,\end{aligned}$$

which shows that $\mathfrak{im}_{D,0}$ is not closable.

Now, let us show that (8.10) being satisfied implies that $\mathfrak{im}_{D,0}$ is closable. To this end, let us firstly show that A_D being dissipative and (8.10) imply that

$$q(A_F^{-1}D(\widetilde{k}_1+\widetilde{k}_2)+\widetilde{k}_1+\widetilde{k}_2)=q(A_F^{-1}D\widetilde{k}_1+\widetilde{k}_1) \tag{8.11}$$

for all $\widetilde{k}_2\in\mathcal{D}(D)\cap\mathcal{D}(V_F^{1/2})$. Since $q(A_F^{-1}D\widetilde{k}_2+\widetilde{k}_2)=0$, for any $\lambda\in\mathbb{C}$, we get

$$q(A_F^{-1}D\widetilde{k}_1+\widetilde{k}_1+\lambda(A_F^{-1}D\widetilde{k}_2+\widetilde{k}_2))=q(A_F^{-1}D\widetilde{k}_1+\widetilde{k}_1)+2\mathrm{Re}[\lambda q(A_F^{-1}D\widetilde{k}_1+\widetilde{k}_1,A_F^{-1}D\widetilde{k}_2+\widetilde{k}_2)], \tag{8.12}$$

where $q(\cdot,\cdot)$ denotes the sesquilinear form associated to q. This implies that $q(A_F^{-1}D\widetilde{k}_1+\widetilde{k}_1,A_F^{-1}D\widetilde{k}_2+\widetilde{k}_2)=0$, since otherwise, we could choose $\lambda\in\mathbb{C}$ such that the right hand side of (8.12) is negative. This, however, would contradict the dissipativity of A_D, from which we have $q(A_F^{-1}D\widetilde{k}_1+\widetilde{k}_1+\lambda(A_F^{-1}D\widetilde{k}_2+\widetilde{k}_2))\geq 0$. Next, let us define the operator $\mathcal{P}$ to be the projection onto $\ker V^*$ along $\mathcal{D}(V_F^{1/2})$ according to the decomposition $\mathcal{D}(V_K^{1/2})=\mathcal{D}(V_F^{1/2})\dot{+}\ker V^*$:

$$\begin{aligned}\mathcal{P}:\qquad \mathcal{D}(\mathcal{P})=\mathcal{D}(V_K^{1/2})&=\mathcal{D}(V_F^{1/2})\dot{+}\ker V^*\\ \mathcal{P}(v_F+v^*)&=v^*\,,\end{aligned} \tag{8.13}$$

where $v_F\in\mathcal{D}(V_F^{1/2})$ and $v^*\in\ker V^*$. Moreover, let us define $\mathcal{D}_2:=\mathcal{D}(D)\cap\mathcal{D}(V_F^{1/2})$ and decompose

$$\mathcal{D}(D)=\mathcal{D}_2\dot{+}\mathcal{D}(D)//\mathcal{D}_2\,. \tag{8.14}$$

Now, let $\{f_n\}_n\subset\mathcal{D}(A_D)$ be a sequence that converges to 0 and that is Cauchy with respect to $\|\cdot\|_{\mathfrak{im}_D}$. In general form, it can be written as

$$f_n:=f_{0,n}+A_F^{-1}k_n+A_F^{-1}D\widetilde{k}_n^{(1)}+\widetilde{k}_n^{(1)}+A_F^{-1}D\widetilde{k}_n^{(2)}+\widetilde{k}_n^{(2)}\,,$$

where $\{f_{0,n}\}_n\subset\mathcal{D}(V)$, $\{k_n\}_n\subset(\ker A^*\cap\mathcal{D}(D)^{[\perp]})$, $\left\{\widetilde{k}_n^{(1)}\right\}_n\subset(\mathcal{D}(D)//\mathcal{D}_2)$ and $\left\{\widetilde{k}_n^{(2)}\right\}_n\subset\mathcal{D}_2$. At this point it becomes clear that it does not matter which specific decomposition we have chosen in (8.14) since any component $(\mathbb{1}-\mathcal{P})\widetilde{k}^{(1)}$ could be absorbed into $\widetilde{k}^{(2)}$. For convenience, let us define

$$v_{F,n}:=(\mathbb{1}-\mathcal{P})f_n=f_{0,n}+A_F^{-1}k_n+A_F^{-1}D\widetilde{k}_n^{(1)}+(\mathbb{1}-\mathcal{P})\widetilde{k}_n^{(1)}+A_F^{-1}D\widetilde{k}_n^{(2)}+\widetilde{k}_n^{(2)}\,,$$

from which we get $f_n = v_{F,n} + \mathcal{P}\widetilde{k}_n^{(1)}$, where $\{v_{F,n}\}_n \subset \mathcal{D}(V_F^{1/2})$ and $\{\mathcal{P}\widetilde{k}_n^{(1)}\}_n \subset \ker V^*$. Then, we have

$$\lim_{n\to\infty} \|f_n\| = \lim_{n\to\infty} \|v_{F,n} + \mathcal{P}\widetilde{k}_n^{(1)}\| = 0 \tag{8.15}$$

as well as

$$\begin{aligned}
&\|f_n - f_m\|_{\mathfrak{im}_D}^2 = \|f_n - f_m\|^2 + \|V_F^{1/2}(v_{F,n} - v_{F,m})\|^2 \\
&\quad + q(A_F^{-1}D(\widetilde{k}_n^{(1)} - \widetilde{k}_m^{(1)}) + (\widetilde{k}_n^{(1)} - \widetilde{k}_m^{(1)}) + A_F^{-1}D(\widetilde{k}_n^{(2)} - \widetilde{k}_m^{(2)}) + (\widetilde{k}_n^{(2)} - \widetilde{k}_m^{(2)})) \overset{n,m\to\infty}{\longrightarrow} 0\,,
\end{aligned}$$

which — using (8.10) and (8.11) — simplifies to

$$\begin{aligned}
&\|f_n - f_m\|_{\mathfrak{im}_D}^2 \\
=&\|f_n - f_m\|^2 + \|V_F^{1/2}(v_{F,n} - v_{F,m})\|^2 + q(A_F^{-1}D(\widetilde{k}_n^{(1)} - \widetilde{k}_m^{(1)}) + (\widetilde{k}_n^{(1)} - \widetilde{k}_m^{(1)})) \overset{n,m\to\infty}{\longrightarrow} 0\,.
\end{aligned}$$

Now, since

$$\varepsilon\|v_{F,n} - v_{F,m}\| \le \|V_F^{1/2}(v_{F,n} - v_{F,m})\| \overset{n,m\to\infty}{\longrightarrow} 0$$

we have that $\{v_{F,n}\}_n$ converges to an element $v_F \in \mathcal{D}(V_F^{1/2})$. Since $f_n = v_{F,n} + \mathcal{P}\widetilde{k}_n^{(1)} \overset{n\to\infty}{\longrightarrow} 0$, we have that $\{\mathcal{P}\widetilde{k}_n^{(1)}\}_n$ converges to $-v_F$. However, since $\mathcal{P}\mathcal{D}(D)$ is finite-dimensional, $\{\mathcal{P}\widetilde{k}_n^{(1)}\}_n$ converges to an element of $\mathcal{P}\mathcal{D}(D) \subset \ker V^*$, from which we get $v_F \in \ker V^*$. But since $\mathcal{D}(V_F^{1/2}) \cap \ker V^* = \{0\}$, we get that $v_F = \lim_{n\to\infty} v_{F,n} = -\lim_{n\to\infty} \mathcal{P}\widetilde{k}_n^{(1)} = 0$. Moreover, the projection $\mathcal{P}$ is injective on $(\mathcal{D}(D)//\mathcal{D}_2)$, which is finite-dimensional. Thus, there exists a number $\varepsilon' > 0$ such that

$$\varepsilon'\|\widetilde{k}_n^{(1)}\| \le \|\mathcal{P}\widetilde{k}_n^{(1)}\| \overset{n\to\infty}{\longrightarrow} 0\,,$$

which implies that

$$\|\widetilde{k}_n^{(1)}\| \overset{n\to\infty}{\longrightarrow} 0\,. \tag{8.16}$$

Now, since $\dim \mathcal{D}(D) < \infty$, there exists a constant $M < \infty$ such that

$$q(A_F^{-1}D\widetilde{k}_n^{(1)} + \widetilde{k}_n^{(1)}) \le M\|\widetilde{k}_n^{(1)}\|^2 \overset{n\to\infty}{\longrightarrow} 0 \quad \text{by (8.16)}\,. \tag{8.17}$$

Altogether, this shows that $\lim_{n\to\infty} \|f_n\|_{\mathfrak{im}_D} = 0$:

$$\begin{aligned}
\|f_n\|_{\mathfrak{im}_D}^2 &= \|f_n\|^2 + \operatorname{Im}\langle f_n, A_D f_n\rangle \\
&= \|v_{F,n} + \mathcal{P}\widetilde{k}_n^{(1)}\|^2 + \|V_F^{1/2} v_{F,n}\|^2 + q(A_F^{-1}D\widetilde{k}_n^{(1)} + \widetilde{k}_n^{(1)}) \overset{n\to\infty}{\longrightarrow} 0\,,
\end{aligned}$$

where we have used (8.15) and (8.17) as well as the fact $\{v_{F,n}\}_n$ is a sequence of elements in $\mathcal{D}(V_F^{1/2})$ that converges to 0 and that is Cauchy with respect to $\|V_F^{1/2}\cdot\|$, which implies that $V_F^{1/2}v_{F,n}\overset{n\to\infty}{\longrightarrow} 0$ as well. This shows that $\mathfrak{im}_{D,0}$ is closable. This finishes the proof. □

EXAMPLE 8.13. Let us give an example of a dual pair $(A,\widetilde{A})$ satisfying the assumptions of Theorem 8.4 for which there exists a proper maximally dissipative extension A_D for which $\mathfrak{im}_{D,0}$ is not closable. Let $\mathcal{H}=L^2(0,1)$ and consider the dual pair of operators

$$\begin{aligned}
A_0:&\qquad \mathcal{D}(A_0)=\mathcal{C}_c^\infty(0,1),\quad (A_0f)(x)=-f''(x)+\frac{i\gamma}{x^2}f(x)\\
\widetilde{A}_0:&\qquad \mathcal{D}(\widetilde{A}_0)=\mathcal{C}_c^\infty(0,1),\quad (\widetilde{A}_0f)(x)=-f''(x)-\frac{i\gamma}{x^2}f(x)\,,
\end{aligned}$$

where for simplicity, we choose $\gamma\geq\sqrt{3}$ in order to ensure that $\dim\ker A^*=\dim\ker\widetilde{A}^*=1$. In this example, the imaginary part V is just given by the multiplication by the function γx^{-2}, where we may choose $\mathcal{D}=\mathcal{C}_c^\infty(0,1)$. Since V is essentially selfadjoint, we get that $V_F=V_K=\overline{V}$, which is the maximal multiplication operator by the function γx^{-2}. In particular, since $V\geq\gamma>0$, we have that $\ker V^*=\ker\overline{V}=\{0\}$. By Lemma 5.5.1, we know that for any proper maximally dissipative extension A_D of the dual pair $(A,\widetilde{A})$, we have that

$$\mathrm{Im}\langle f+v,A_D(f+v)\rangle=\|V_K^{1/2}(f+v)\|^2+q(v)\,, \tag{8.18}$$

where $f\in\mathcal{D}(A)$ and $v\in\mathcal{V}=\mathcal{D}(A_D)//\mathcal{D}(A)$. Moreover, using Equation (5.4.8) for $n=2$, an integration by parts yields that the form $q(v)=\mathrm{Im}\langle v,\widetilde{A}^*v\rangle-\|V_K^{1/2}v\|^2$ is equal to

$$q(v)=-\mathrm{Im}(\overline{v(1)}v'(1))\,. \tag{8.19}$$

In Example 7.3.11, we have parametrized all maximally dissipative extensions of $(A,\widetilde{A})$ by the family of operators $\{A_\rho\}_{\mathrm{Im}\rho\geq 0}\cup\{A_F\}$, where A_ρ was given by (7.3.19) for $\gamma<\sqrt{3}$ and by (7.3.22) for $\gamma\geq\sqrt{3}$. This means that $(\mathcal{C}_c^\infty(0,1)\dot{+}\mathrm{span}\{\xi_\rho\})\subset\mathcal{D}(A_\rho)$. By (8.18) and (8.19) we get for $\rho \neq \infty$ that the form $\mathfrak{im}_{D(\rho),0}$ is given by

$$\begin{aligned}\mathfrak{im}_{D(\rho),0}: \qquad \mathcal{D}(\mathfrak{im}_{D(\rho),0}) &= \mathcal{D}(A)\dot{+}\mathrm{span}\{\xi_\rho\} \\ \mathfrak{im}_{D(\rho),0}(f+\lambda\xi_\rho) &= \|V_K^{1/2}(f+\lambda\xi_\rho)\|^2 + |\lambda|^2\mathrm{Im}(\rho)\,,\end{aligned}$$

where $f \in \mathcal{D}(A)$ and $\lambda \in \mathbb{C}$. For $\rho = \infty$, we just get that

$$\begin{aligned}\mathfrak{im}_{D(\infty),0}: \qquad \mathcal{D}(\mathfrak{im}_{D(\infty),0}) &= \mathcal{D}(A)\dot{+}\mathrm{span}\{\xi_\infty\} \\ \mathfrak{im}_{D(\infty),0}(f+\lambda\xi_\infty) &= \|V_K^{1/2}(f+\lambda\xi_\infty)\|^2\,.\end{aligned}$$

Here, the notation $D(\rho)$ indicates that for any ρ such that $\mathrm{Im}(\rho) \geq 0$, there exists an auxiliary operator $D(\rho)$ from $\ker \widetilde{A}^*$ into $\ker A^*$ such that $A_\rho = A_{D(\rho)}$. Since $\xi_\rho \in \mathcal{D}(V_K^{1/2}) = \mathcal{D}(V_F^{1/2})$, we have by Theorem 8.12 that $\mathfrak{im}_{D(\rho),0}$ is closable if and only if $q(\xi_\rho) = \mathrm{Im}(\rho) = 0$ or $\rho = \infty$, since $\mathcal{D}(D) = \{0\}$ in this case. Indeed, assume that $\rho \neq \infty$ and that $\mathrm{Im}\rho > 0$. Since $\mathcal{C}_c^\infty(0,1)$ is a core for $V_F^{1/2}$, we can pick a sequence $\{f_n\}_n \subset \mathcal{C}_c^\infty(0,1)$ such that

$$\|f_n - \xi_\rho\|^2 + \|V_F^{1/2}(f_n-\xi_\rho)\|^2 \overset{n\to\infty}{\longrightarrow} 0\,,$$

which means that the sequence $g_n := f_n - \xi_\rho$ converges to 0 with respect to the graph norm of $V_F^{1/2}$. Moreover, $\{g_n\}_n \subset \mathcal{D}(A_{D(\rho)})$ is Cauchy with respect to $\|\cdot\|_{\mathfrak{im}_{D(\rho),0}}$:

$$\|g_n - g_m\|^2_{\mathfrak{im}_{D(\rho),0}} = \|f_n - f_m\|^2 + \|V_F^{1/2}(f_n - f_m)\|^2 \overset{n,m\to\infty}{\longrightarrow} 0\,.$$

However $\mathfrak{im}_{D(\rho),0}$ is not closable, which follows from

$$\|g_n\|^2_{\mathfrak{im}_{D(\rho),0}} = \|f_n-\xi_\rho\|^2 + \|V_F^{1/2}(f_n-\xi_\rho)\| + \mathrm{Im}(\rho) \overset{n\to\infty}{\longrightarrow} \mathrm{Im}(\rho) \neq 0\,.$$

REMARK 8.14. If $(-iA_D)$ is sectorial in the sense of Kato, recall that by [**26**, Chapter VI, Thm. 1.27], the form $\mathfrak{im}_{D,0}$ is always closable.

If $\mathfrak{im}_{D,0}$ is closable, there exists a selfadjoint operator V_D associated to the closure $\mathfrak{im}_D$. Let us now show that this operator is an extension of V:

LEMMA 8.15. *Let $(A,\widetilde{A})$ be a dual pair that satisfies the assumptions of Theorem 8.4. Also, let $\mathcal{D}$ be a common core for $(A,\widetilde{A})$ and define $V := \frac{1}{2i}(A-\widetilde{A})\restriction_{\mathcal{D}}$. Moreover, assume that $\mathfrak{im}_{D,0}$ is closable. Then, $V \subset V_D$.*

PROOF. Let $f \in \mathcal{D}(\mathfrak{im}_D)$ and $g \in \mathcal{D}(V)$. Moreover, let $\mathfrak{im}_D(\cdot,\cdot)$ be the sesquilinear form associated to $\mathfrak{im}_D$. Since $\mathcal{D}(\mathfrak{im}_D)$ is the closure of $\mathcal{D}(A_D)$ with respect to $\|\cdot\|_{\mathfrak{im}_D}$, this means that there exists a sequence $\{f_n\}_n \subset \mathcal{D}(A_D)$ such that

$$\mathfrak{im}_D(f,g) = \lim_{n\to\infty} \mathfrak{im}_D(f_n, g)\,. \tag{8.20}$$

Now, for any $f_n \in \mathcal{D}(A_D)$ and $g \in \mathcal{D}(V)$ observe that $\mathfrak{im}_D(f_n,g)$ can be written as

$$\begin{aligned}\mathfrak{im}_D(f_n,g) &= \frac{1}{2i}(\langle f_n, A_D g\rangle - \langle A_D f_n, g\rangle) = \frac{1}{2i}(\langle f_n, \widetilde{A}^* g\rangle - \langle \widetilde{A}^* f_n, g\rangle)\\ &= \langle f_n, \frac{1}{2i}(A-\widetilde{A})g\rangle = \langle f_n, Vg\rangle\,,\end{aligned} \tag{8.21}$$

from which we get

$$\mathfrak{im}_D(f,g) \overset{(8.20)}{=} \lim_{n\to\infty} \mathfrak{im}_D(f_n,g) \overset{(8.21)}{=} \lim_{n\to\infty}\langle f_n, Vg\rangle = \langle f, Vg\rangle$$

for all $f \in \mathcal{D}(\mathfrak{im}_D)$ and all $g \in \mathcal{D}(V)$, which implies that $V \subset V_D$ and thus, in addition, the lemma. $\square$

Next, let us determine the form domain of V_D. Using that V_D is a non-negative selfadjoint extension of V, we know by the Birman–Kreĭn–Vishik theory of non-negative selfadjoint extensions that $V_K \leq V_D \leq V_F$. This implies that $\mathcal{D}(V_F^{1/2}) \subset \mathcal{D}(V_D^{1/2})$ and that there exists a subspace $\mathcal{M} \subset \ker V^*$ such that $\mathcal{D}(V_D^{1/2}) = \mathcal{D}(V_F^{1/2}) \dot{+} \mathcal{M}$. In the case that $\mathcal{P}\mathcal{D}(D)$ is finite-dimensional, we will show that $\mathcal{M} = \mathcal{P}\mathcal{D}(D)$, i.e. the part of $\mathcal{D}(D)$ that can be projected onto $\ker V^*$.

LEMMA 8.16. *Let V be strictly positive, i.e. $V \geq \varepsilon > 0$ and assume that $\mathfrak{im}_{D,0}$ is closable. Then, the domain of $V_D^{1/2}$ is given by*

$$\mathcal{D}(V_D^{1/2}) = \overline{\mathcal{D}(V_F^{1/2}) \dot{+} \mathcal{P}\mathcal{D}(D)}^{\|\cdot\|_{\mathfrak{im}_D}}\,.$$

In particular, if $\dim(\mathcal{P}\mathcal{D}(D)) < \infty$, *we get*

$$\mathcal{D}(V_D^{1/2}) = \mathcal{D}(V_F^{1/2}) \dot{+} \mathcal{P}\mathcal{D}(D)\,.$$

PROOF. By Theorem 7.3.9, we have that $\mathcal{D}(A_F) \subset \mathcal{D}(V_F^{1/2})$, which implies that any element of

$$\mathcal{D}(A_D) = \mathcal{D}(A) \dot{+} \{A_F^{-1} D\widetilde{k} + \widetilde{k} : \widetilde{k} \in \mathcal{D}(D)\} \dot{+} \{A_F^{-1} k : k \in \ker A^* \cap \mathcal{D}(D)^{[\perp]}\}$$

can be written as

$$f + A_F^{-1}D\widetilde{k} + \widetilde{k} + A_F^{-1}k = \underbrace{\left(f + A_F^{-1}D\widetilde{k} + (\mathbb{1} - \mathcal{P})\widetilde{k} + A_F^{-1}k\right)}_{\in \mathcal{D}(V_F^{1/2})} + \underbrace{\mathcal{P}\widetilde{k}}_{\in \mathcal{P}\mathcal{D}(D)} ,$$

which implies that $\mathcal{D}(A_D) \subset \mathcal{D}(V_F^{1/2}) \dot{+} \mathcal{P}\mathcal{D}(D)$. On the other hand, we have by Lemma 8.15 that V_D is a positive selfadjoint extension of V, from which we get by [**2**] that $\mathcal{D}(V_F^{1/2}) \subset \mathcal{D}(V_D^{1/2})$. As any $\mathcal{P}\widetilde{k} \in \mathcal{P}\mathcal{D}(D)$ can be written as

$$\mathcal{P}\widetilde{k} = \underbrace{\left(A_F^{-1}D\widetilde{k} + \widetilde{k}\right)}_{\in \mathcal{D}(A_D)} - \underbrace{\left(A_F^{-1}D\widetilde{k} + (\mathbb{1} - \mathcal{P})\widetilde{k}\right)}_{\in \mathcal{D}(V_F^{1/2})}$$

and since $\mathcal{D}(A_D) \subset \mathcal{D}(V_D^{1/2})$ and $\mathcal{D}(V_F^{1/2}) \subset \mathcal{D}(V_D^{1/2})$, this implies that $\mathcal{P}\widetilde{k} \in \mathcal{D}(V_D^{1/2})$, and thus $\mathcal{P}\mathcal{D}(D) \subset \mathcal{D}(V_D^{1/2})$. Consequently, we have $\mathcal{D}(A_D) \subset \mathcal{D}(V_F^{1/2}) \dot{+} \mathcal{P}\mathcal{D}(D) \subset \mathcal{D}(V_D^{1/2})$ and since $\overline{\mathcal{D}(A_D)}^{\|\cdot\|_{\mathfrak{im}_D}} = \mathcal{D}(V_D^{1/2})$, we get

$$\mathcal{D}(V_D^{1/2}) = \overline{\mathcal{D}(A_D)}^{\|\cdot\|_{\mathfrak{im}_D}} \subset \overline{\mathcal{D}(V_F^{1/2}) \dot{+} \mathcal{P}\mathcal{D}(D)}^{\|\cdot\|_{\mathfrak{im}_D}} \subset \mathcal{D}(V_D^{1/2}) ,$$

which proves the first assertion of the lemma. Next, let us show that $\mathcal{D}(V_F^{1/2})$ is a closed subspace of $\mathcal{D}(V_D^{1/2})$ with respect to $\|\cdot\|_{\mathfrak{im}_D}$. This follows from the fact that for any $f \in \mathcal{D}(V)$ we get that

$$\|f\|_{\mathfrak{im}_D}^2 = \|f\|^2 + \mathrm{Im}\langle f, A_D f\rangle = \|f\|^2 + \langle f, Vf\rangle ,$$

which means that $\overline{\mathcal{D}(V)}^{\|\cdot\|_{\mathfrak{im}_D}} = \mathcal{D}(V_F^{1/2})$. Since $\dim(\mathcal{P}\mathcal{D}(D)) < \infty$, we have by [**24**, Problem 13] that $\mathcal{D}(V_F^{1/2}) \dot{+} \mathcal{P}\mathcal{D}(D)$ is a closed subspace of $\mathcal{D}(V_D^{1/2})$ with respect to the $\|\cdot\|_{\mathfrak{im}_D}$-norm and by what we have shown before, this yields

$$\mathcal{D}(V_D^{1/2}) = \overline{\mathcal{D}(V_F^{1/2}) \dot{+} \mathcal{P}\mathcal{D}(D)}^{\|\cdot\|_{\mathfrak{im}_D}} = \mathcal{D}(V_F^{1/2}) \dot{+} \mathcal{P}\mathcal{D}(D) \subset \mathcal{D}(V_D^{1/2}) ,$$

which is the desired result. □

Finally, let us determine the action of $\mathfrak{im}_D$.

THEOREM 8.17. *Let V be as in Lemma 8.16 and moreover, assume that*

$$\dim \mathcal{P}\mathcal{D}(D) < \infty .$$

Then, there exists a non-negative selfadjoint operator B with $\mathcal{D}(B) = \mathcal{PD}(D)$ such that for any $v_F \in \mathcal{D}(V_F^{1/2})$ and any $\eta \in \mathcal{PD}(D)$, we have

$$\mathfrak{im}_D(v_F + \eta) = \|V_D^{1/2}(v_F + \eta)\|^2 = \|V_F^{1/2} v_F\|^2 + q_B(\eta)\,,$$

where q_B denotes the quadratic form associated to B. It is given by

$$q_B(\eta) = Im[\mathcal{P}^{-1}\eta, D\mathcal{P}^{-1}\eta] - \|V_K^{1/2}\mathcal{P}^{-1}\eta\|^2\,.$$

Here, $\mathcal{P}^{-1}$ denotes the inverse of $\mathcal{P}$ restricted to a subspace of $\mathcal{D}(D)$ that is complementary to $\mathcal{D}(D) \cap \mathcal{D}(V_F^{1/2})$. The form q_B does not depend on the specific choice of this subspace. Moreover, if we choose $\{\eta_i\}_{i=1}^n$ to be an orthonormal basis of $\mathcal{PD}(D)$, the elements of the matrix representation of B with respect to $\{\eta_i\}_{i=1}^n$ are given by

$$\begin{aligned} B &= (b_{ij})_{i,j=1}^n \quad \textit{where} \\ b_{ij} &= \frac{1}{2i}\left([\mathcal{P}^{-1}\eta_i, D\mathcal{P}^{-1}\eta_j] - [D\mathcal{P}^{-1}\eta_i, \mathcal{P}^{-1}\eta_j]\right) - \langle V_K^{1/2}\mathcal{P}^{-1}\eta_i, V_K^{1/2}\mathcal{P}^{-1}\eta_j\rangle\,. \end{aligned}$$

PROOF. For any $\eta \in \mathcal{PD}(D)$, there exists a $\widetilde{k} \in \mathcal{D}(D)$ such that $\eta = \mathcal{P}\widetilde{k}$. In the case that $\mathcal{D}(V_F^{1/2}) \cap \mathcal{D}(D)$ is non-trivial, which means that $\ker(\mathcal{P}) \cap \mathcal{D}(D)$ is non-trivial, this choice of $\widetilde{k}$ is not unique as for any $\chi \in \mathcal{D}(D) \cap \mathcal{D}(V_F^{1/2})$ we would still have that $\mathcal{P}(\widetilde{k} + \chi) = \eta$. However, if we choose a subspace of $\mathcal{S} \subset \mathcal{D}(D)$ that is complementary to $\mathcal{D}(V_F^{1/2}) \cap \mathcal{D}(D)$ in $\mathcal{D}(D)$, then we can define the inverse of $\mathcal{P}$ on $\mathcal{S}$:

$$\begin{aligned} \mathcal{P}^{-1} : \mathcal{D}(\mathcal{P}^{-1}) &= \mathcal{PS} \\ &\quad \mathcal{P}\widetilde{k} \mapsto \widetilde{k}, \quad \widetilde{k} \in \mathcal{S}\,. \end{aligned}$$

Now, for any $v_F \in \mathcal{D}(V_F^{1/2})$ and $\widetilde{k} \in \mathcal{S}$, let us pick a sequence $\{f_n\}_n \subset \mathcal{D}(V)$ such that

$$\|\cdot\|_{\mathfrak{im}_D} - \lim_{n\to\infty} f_n = \left[v_F - A_F^{-1}D\widetilde{k} - (\mathbb{1} - \mathcal{P})\widetilde{k}\right] \in \mathcal{D}(V_F^{1/2})\,,$$

where "$\|\cdot\|_{\mathfrak{im}_D} - \lim$" denotes the limit with respect to the $\|\cdot\|_{\mathfrak{im}_D}$-norm. This implies that the sequence $g_n := f_n + A_F^{-1}D\widetilde{k} + \widetilde{k}$ converges to $v_F + \mathcal{P}\widetilde{k}$ in the usual norm $\|\cdot\|$. Next, let us show that $\{g_n\}_n$ is Cauchy with respect to $\|\cdot\|_{\mathfrak{im}_D}$:

$$\|g_n - g_m\|^2_{\mathfrak{im}_D} = \|f_n - f_m\|^2 + \|V_F^{1/2}(f_n - f_m)\|^2 = \|f_n - f_m\|^2_{\mathfrak{im}_D} \stackrel{n,m\to\infty}{\longrightarrow} 0\,,$$

since $\{f_n\}_n$ has a limit with respect to $\|\cdot\|_{\mathfrak{im}_D}$. Thus, we get

$$
\begin{aligned}
\|v_F + \mathcal{P}\widetilde{k}\|^2_{\mathfrak{im}_D} &= \lim_{n\to\infty} \|g_n\|^2_{\mathfrak{im}_D} \\
&= \lim_{n\to\infty} (\|g_n\|^2 + \|V_F^{1/2}(f_n + A_F^{-1}D\widetilde{k} + (\mathbb{1} - \mathcal{P})\widetilde{k})\|^2 + q(A_F^{-1}D\widetilde{k} + \widetilde{k})) \\
&= \|v_F + \mathcal{P}\widetilde{k}\|^2 + \|V_F^{1/2}v_F\|^2 + q(A_F^{-1}D\widetilde{k} + \widetilde{k}) \\
&= \|v_F + \mathcal{P}\widetilde{k}\|^2 + \|V_F^{1/2}v_F\|^2 + \mathrm{Im}[\widetilde{k}, D\widetilde{k}] - \|V_K^{1/2}\widetilde{k}\|^2
\end{aligned}
$$

and since for any $\phi \in \mathcal{D}(\mathfrak{im}_D)$ we have $\|\phi\|^2_{\mathfrak{im}_D} = \|\phi\|^2 + \mathfrak{im}_D(\phi)$, this allows us to read off

$$
\mathfrak{im}_D(v_F + \mathcal{P}\widetilde{k}) = \|V_F^{1/2}v_F\|^2 + \mathrm{Im}[\widetilde{k}, D\widetilde{k}] - \|V_K^{1/2}\widetilde{k}\|^2 \,.
$$

However, if $\mathcal{D}(V_F^{1/2}) \cap \mathcal{D}(D)$ is non-trivial, we could have added a $\chi \in (\mathcal{D}(V_F^{1/2}) \cap \mathcal{D}(D))$ such that $\mathcal{P}(\widetilde{k} + \chi) = \mathcal{P}\widetilde{k} = \eta$. But since $\mathfrak{im}_{D,0}$ was assumed to be closable, we have by Theorem 8.12 that $q(A_F^{-1}D\chi + \chi) = 0$ and by (8.11), we have in addition that

$$
q(A_F^{-1}D\widetilde{k} + \widetilde{k} + A_F^{-1}D\chi + \chi) = q(A_F^{-1}D\widetilde{k} + \widetilde{k}) \,. \tag{8.22}
$$

Thus, the specific choice of $\mathcal{S} \subset \mathcal{D}(D)$ — as long as it is complementary to $(\mathcal{D}(D) \cap \mathcal{D}(V_F^{1/2}))$ in $\mathcal{D}(D)$ — does not affect the value of $\mathfrak{im}_D(v_F + \eta)$, where $v_F \in \mathcal{D}(V_F^{1/2})$ and $\eta \in \mathcal{P}\mathcal{D}(D)$, where $\eta = \mathcal{P}\widetilde{k}$ for a unique $\widetilde{k} \in \mathcal{S}$. Hence, by Equation (8.22), we get

$$
\mathfrak{im}_D(v_F + \eta) = \mathfrak{im}_D(v_F + \mathcal{P}\widetilde{k}_\eta) = \|V_F^{1/2}v_F\|^2 + \mathrm{Im}[\widetilde{k}_\eta, D\widetilde{k}_\eta] - \|V_K^{1/2}\widetilde{k}_\eta\|^2 \,, \tag{8.23}
$$

where $\widetilde{k}_\eta$ is the unique element of $\mathcal{S}$ such that $\mathcal{P}\widetilde{k}_\eta = \eta$, or in other words, we get $\widetilde{k}_\eta = \mathcal{P}^{-1}\eta$. Plugged into (8.23), this yields

$$
\mathfrak{im}_D(v_F + \eta) = \|V_F^{1/2}v_F\|^2 + \mathrm{Im}[\mathcal{P}^{-1}\eta, D\mathcal{P}^{-1}\eta] - \|V_K^{1/2}\mathcal{P}^{-1}\eta\|^2 \,. \tag{8.24}
$$

Now, since we have shown in Lemma 8.15 that V_D is a non-negative selfadjoint extension of V, we know by [**2**] that there exists a subspace $\mathcal{D}(B) \subset \ker V^*$ and a non-negative auxiliary operator B from $\mathcal{D}(B)$ into $\mathcal{D}(B)$ such that

$$
\mathfrak{im}_D(v_F + \eta) = \|V_D^{1/2}(v_F + \eta)\|^2 = \|V_F^{1/2}v_F\|^2 + q_B(\eta) \,, \tag{8.25}
$$

where $v_F \in \mathcal{D}(V_F^{1/2})$ and $\eta \in \mathcal{D}(B)$. The form q_B is given by $q_B(\eta) = \langle \eta, B\eta \rangle$ for all $\eta \in \mathcal{D}(B)$.[8.1] Comparing Equations (8.25) and (8.24), we can read off that

$$q_B(\eta) = \mathrm{Im}[\mathcal{P}^{-1}\eta, D\mathcal{P}^{-1}\eta] - \|V_K^{1/2}\mathcal{P}^{-1}\eta\|^2 \,, \tag{8.26}$$

which is the desired result. To determine the entries of the non-negative matrix $(b_{ij})_{ij}$, we use that the sesquilinear form $q_B(\cdot,\cdot)$ associated to q_B is given by

$$q_B(\eta_i, \eta_j) = \frac{1}{2i}([\mathcal{P}^{-1}\eta_i, D\mathcal{P}^{-1}\eta_j] - [D\mathcal{P}^{-1}\eta_i, \mathcal{P}^{-1}\eta_j]) - \langle V_K^{1/2}\mathcal{P}^{-1}\eta_i, V_K^{1/2}\mathcal{P}^{-1}\eta_j \rangle \,.$$

This immediately follows from the fact that $\mathfrak{im}_D(\eta,\eta) = \mathfrak{im}_D(\eta)$, which can be seen by direct inspection. Now, since $b_{ij} = \langle \eta_i, B\eta_j \rangle = q_B(\eta_i, \eta_j)$, this finishes the proof. □

The previous result allows us to deduce a way of comparing the imaginary parts V_{D_1} and V_{D_2} of two different extensions A_{D_1} and A_{D_2}:

Corollary 8.18. *Let D_1 and D_2 parametrize two different proper maximally dissipative extensions of $(A, \widetilde{A})$ and let B_1 and B_2 be the two associated non-negative auxiliary operators whose quadratic form is given in* (8.26). *Then $B_1 \geq B_2$ if and only if $\mathcal{P}\mathcal{D}(D_1) \subset \mathcal{P}\mathcal{D}(D_2)$ and*

$$Im[\widetilde{k}, D_1\widetilde{k}] \geq Im[\widetilde{k}, D_2\widetilde{k}] \tag{8.27}$$

for all $\widetilde{k} \in \mathcal{D}(D_1)$.

Proof. By definition, $B_1 \geq B_2$ as operators on a finite-dimensional space if and only if $\mathcal{D}(B_1) \subset \mathcal{D}(B_2)$ and $q_{B_1}(\eta) \geq q_{B_2}(\eta)$ for all $\eta \in \mathcal{D}(B_1)$. Since $\mathcal{D}(B_{1,2}) = \mathcal{P}\mathcal{D}(D_{1,2})$, this shows the first condition of the corollary. Now, for any $\eta \in \mathcal{D}(B_1) = \mathcal{P}\mathcal{D}(D_1)$ we have by (8.26)

$$\begin{aligned} q_{B_1}(\eta) - q_{B_2}(\eta) & \\ &= \mathrm{Im}[\mathcal{P}^{-1}\eta, D_1\mathcal{P}^{-1}\eta] - \|V_K^{1/2}\mathcal{P}^{-1}\eta\|^2 - (\mathrm{Im}[\mathcal{P}^{-1}\eta, D_2\mathcal{P}^{-1}\eta] - \|V_K^{1/2}\mathcal{P}^{-1}\eta\|^2) \\ &= \mathrm{Im}[\mathcal{P}^{-1}\eta, D_1\mathcal{P}^{-1}\eta] - \mathrm{Im}[\mathcal{P}^{-1}\eta, D_2\mathcal{P}^{-1}\eta] \,, \qquad (8.28) \end{aligned}$$

which is non-negative for all $\eta \in \mathcal{D}(B_1)$ if and only if

$$\mathrm{Im}[\mathcal{P}^{-1}\eta, D_1\mathcal{P}^{-1}\eta] \geq \mathrm{Im}[\mathcal{P}^{-1}\eta, D_2\mathcal{P}^{-1}\eta]$$

[8.1] Note that we are only considering the finite-dimensional case, which means that we do not have to worry about closures and domains.

for all $\eta \in \mathcal{D}(B_1) = \mathcal{P}\mathcal{D}(D_1)$. Thus, Condition (8.27) being satisfied is sufficient for $B_1 \geq B_2$. Let us now show that it is also necessary. Assume that there exists a $\widetilde{k} \in \mathcal{D}(D_1)$ such that

$$\mathrm{Im}[\widetilde{k}, D_1\widetilde{k}] < \mathrm{Im}[\widetilde{k}, D_2\widetilde{k}] \;\;\Leftrightarrow\;\; \mathrm{Im}[\widetilde{k}, D_1\widetilde{k}] - \|V_K^{1/2}\widetilde{k}\|^2 < \mathrm{Im}[\widetilde{k}, D_2\widetilde{k}] - \|V_K^{1/2}\widetilde{k}\|^2 \,.$$

Observe that by Theorem 8.12, this means that $\widetilde{k} \in (\mathcal{D}(V_F^{1/2}) \cap \mathcal{D}(D_1))$ is not possible in this case, since this would imply that $\mathrm{Im}[\widetilde{k}, D_2\widetilde{k}] - \|V_K^{1/2}\widetilde{k}\|^2 = 0$, but by dissipativity of A_{D_1} we have by virtue of Theorem 8.4 that $\mathrm{Im}[\widetilde{k}, D_1\widetilde{k}] - \|V_K^{1/2}\widetilde{k}\|^2 \geq 0$. Hence, $0 \neq \mathcal{P}\widetilde{k} =: \eta \in \mathcal{P}\mathcal{D}(D_1)$ or $\mathcal{P}^{-1}\eta = \widetilde{k}$. Therefore we get

$$q_{B_1}(\eta) - q_{B_2}(\eta) = \mathrm{Im}[\widetilde{k}, D_1\widetilde{k}] - \mathrm{Im}[\widetilde{k}, D_2\widetilde{k}] < 0 \,,$$

which shows that $B_1 \ngeq B_2$ if Condition (8.27) is not satisfied, which therefore is necessary for $B_1 \geq B_2$ to be true. This shows the corollary. $\square$

Remark 8.19. These results allow us to give first estimates of the lower bound of the imaginary part. From [**2**, Thm. 2.13], it follows that

$$\frac{\alpha\delta}{1+\delta} \leq \inf_{0 \neq \psi \in \mathcal{D}(A_D)} \frac{\mathrm{Im}\langle \psi, A_D\psi\rangle}{\|\psi\|^2} \leq \alpha\delta \,, \tag{8.29}$$

where α is the lower bound of the imaginary part of A and δ is the lower bound of the quadratic form q_B:

$$\alpha := \inf\left\{\frac{\mathrm{Im}\langle \psi, A\psi\rangle}{\|\psi\|^2} : \psi \in \mathcal{D}(A), \psi \neq 0\right\} \;\text{ and }\; \delta := \inf\left\{\frac{q_B(\eta)}{\|\eta\|^2} : \eta \in \mathcal{P}\mathcal{D}(D), \eta \neq 0\right\} .$$

As mentioned in [**2**, Thm. 2.13], this means in particular that

$$\inf\left\{\frac{\mathrm{Im}\langle \psi, A_D\psi\rangle}{\|\psi\|^2} : \psi \in \mathcal{D}(A_D), \psi \neq 0\right\} = 0$$

if and only if $\delta = 0$. (Recall that we have assumed that $\alpha \geq \varepsilon > 0$.)

Example 8.20 (Continuation of Example 8.10). Consider the dual pair $(A, \widetilde{A})$ as defined in Example 8.10. Now, since $\mathcal{D}(D) = \mathrm{span}\{x^{\omega_+}\}$ and

$$x^{\omega_+} = \underbrace{(x^{\omega_+} - x)}_{\in \mathcal{D}(V_F^{1/2})} + \underbrace{x}_{\in \ker V^*} \,,$$

we get $\mathcal{P}\mathcal{D}(D) = \mathrm{span}\{x\}$. In particular, we have that $\mathcal{D}(D) \cap \mathcal{D}(V_F^{1/2}) = \{0\}$ from which we see by virtue of Theorem 8.12 that $\mathfrak{im}_{D,0}$ is closable. Moreover, the operator

$\mathcal{P} \restriction_{\mathcal{D}(D)}$ is injective and thus, we define $\mathcal{P}^{-1}x = x^{\omega_+}$. By Lemma 8.16, the associated operator V_D has form domain

$$\mathcal{D}(V_D^{1/2}) = \mathcal{D}(V_F^{1/2}) \dot{+} \mathcal{P}\mathrm{span}\{x^{\omega_+}\} = \mathcal{D}(V_F^{1/2}) \dot{+} \mathrm{span}\{x\} = H_0^1(0,1) \dot{+} \mathrm{span}\{x\}$$

and the quadratic form acts like

$$\begin{aligned} \|{V_D}^{1/2}(f + \lambda x)\|^2 &= \|V_F^{1/2} f\|^2 + |\lambda|^2 \left(\mathrm{Im}[\mathcal{P}^{-1}x, D\mathcal{P}^{-1}x] - \|V_K^{1/2}\mathcal{P}^{-1}x\|^2 \right) \\ &= \|V_F^{1/2} f\|^2 + |\lambda|^2 \left(\mathrm{Im}(d\sigma(\omega_+)) - \tau(\omega_+) \right) , \end{aligned}$$

where $f \in \mathcal{D}(V_F^{1/2})$. The operator B_D associated to the quadratic form is a map from $\mathrm{span}\{x\}$ to $\mathrm{span}\{x\}$ and is therefore of the form $B_D x = bx$, where $b \in \mathbb{C}$. By Theorem 8.17, we have that b is given by

$$b = \mathrm{Im}[\mathcal{P}^{-1}(\sqrt{3}x), D\mathcal{P}^{-1}(\sqrt{3}x)] - \|V_K^{1/2}\mathcal{P}^{-1}\sqrt{3}x\|^2 = 3(\mathrm{Im}(d\sigma(\omega_+)) - \tau(\omega_+)) ,$$

where the factor $\sqrt{3}$ comes from normalizing the function x. Now, for two different maximally dissipative extensions A_{D_1} and A_{D_2}, we have that if $\mathrm{Im}(d_1\sigma) = \mathrm{Im}[\widetilde{k}, D_1\widetilde{k}] \geq \mathrm{Im}(d_2\sigma) = \mathrm{Im}[\widetilde{k}, D_2\widetilde{k}]$, this implies that $B_{D_1} \geq B_{D_2}$.

Finally, let us construct the selfadjoint operators V_D using the Birman–Kreĭn–Vishik theory for positive symmetric operators. For $\mathcal{D}(D) = \{0\}$ we get the Friedrichs extension. The other possibility is that $\mathcal{D}(D) = \mathrm{span}\{x^{\omega_+}\}$ with $\mathcal{P}\mathrm{span}\{x^{\omega_+}\} = \mathrm{span}\{x\}$. We then get

$$\begin{aligned} V_D :\ \mathcal{D}(V_D) &= \mathcal{D}(V) \dot{+} \mathrm{span}\{V_F^{-1}B_D x + x\} \dot{+} \mathrm{span}\{V_F^{-1}(2 - 3x)\} \\ &= \mathcal{D}(V) \dot{+} \mathrm{span}\left\{3[\mathrm{Im}(d\sigma(\omega_+)) - \tau(\omega_+)]V_F^{-1}x + x\right\} \dot{+} \mathrm{span}\{V_F^{-1}(2 - 3x)\} \\ f &\mapsto -f'' , \end{aligned}$$

where the last span comes from the fact that $(2 - 3x) \perp x$. Also, note that it is not difficult to compute $V_F^{-1}1$ and $V_F^{-1}x$:

$$V_F^{-1}1 = \frac{x^2 - x}{2} \quad \text{and} \quad V_F^{-1}x = \frac{x^3 - x}{6} .$$

CHAPTER 9

More general dissipative extensions

In this chapter, we are going to discuss ideas on how to construct non-proper extensions of a dissipative operator A. As a starting point, we will consider dual pairs $(A, \widetilde{A})$ that satisfy the common core condition and try to construct dissipative extensions of A whose domain is contained in $\mathcal{D}(\widetilde{A}^*)$. We will apply our results to symmetric operators with bounded dissipative perturbations and obtain a full description of their dissipative extensions. After this, we consider dissipative operators A for which the quadratic form $\psi \mapsto \mathrm{Im}\langle \psi, A\psi \rangle$ is closable and strictly positive and give necessary and sufficient conditions for an arbitrary extension $A \subset B$ to be dissipative.

9.1. Construction of non-proper extensions using dual pairs

Let A be dissipative and $\widetilde{A}$ be antidissipative and assume that $(A, \widetilde{A})$ is a dual pair satisfying the common core condition. In this section, we will construct all dissipative extensions of A that have domain contained in $\mathcal{D}(\widetilde{A}^*)$. We will need the following two lemmas:

Lemma 9.1.1. *Let V be a non-negative symmetric operator. Then, ran$(V_F^{1/2} \upharpoonright_{\mathcal{D}(V)})$ is dense in $\overline{\mathrm{ran}(V_F^{1/2})}$.*

Proof. By construction of the Friedrichs extension, we know that for any $\psi \in \mathcal{D}(V_F^{1/2})$, there exists a sequence $\{\psi_n\}_n \subset \mathcal{D}(V)$, such that

$$\lim_{n\to\infty} (\|\psi - \psi_n\|^2 + \|V_F^{1/2}(\psi - \psi_n)\|^2) = 0\,,$$

which implies in particular that $\lim_{n\to\infty} V_F^{1/2}\psi_n = V_F^{1/2}\psi$, i.e. $\mathrm{ran}(V_F^{1/2}) \subset \overline{\mathrm{ran}(V_F^{1/2} \upharpoonright_{\mathcal{D}(V)})}$. On the other hand, since $\mathrm{ran}(V_F^{1/2} \upharpoonright_{\mathcal{D}(V)}) \subset \mathrm{ran}(V_F^{1/2})$, the assertion follows from taking closures. $\square$

Lemma 9.1.2. *Let V be a non-negative symmetric operator and let V_F and V_K denote its Friedrichs, resp. its Kreĭn extension. Then there exists a partial isometry $\mathcal{U}$ on $\mathcal{H}$*

such that

$$V_K^{1/2}h = \mathcal{U}V_F^{1/2}h \tag{9.1.1}$$

for all $h \in \mathcal{D}(V_F^{1/2})$. The map $\mathcal{U}$ is an isometry on $\overline{\mathrm{ran}(V_F^{1/2})}$ and its range ran($\mathcal{U}$) is contained in $\overline{\mathrm{ran}(V_K^{1/2})}$.

Proof. Since we have that $V_K \leq V_F$, it is clear that $\mathcal{D}(V_F^{1/2}) \subset \mathcal{D}(V_K^{1/2})$. Moreover, by Proposition 5.2.3, for any $h \in \mathcal{D}(V_F^{1/2}) \subset \mathcal{D}(V_K^{1/2})$, we have that

$$\begin{aligned}\|V_K^{1/2}h\|^2 &= \sup_{f\in\mathcal{D}(V):Vf\neq 0} \frac{|\langle h, Vf\rangle|^2}{\langle f, Vf\rangle} = \sup_{f\in\mathcal{D}(V):Vf\neq 0} \frac{|\langle h, V_F^{1/2}V_F^{1/2}f\rangle|^2}{\langle f, V_F^{1/2}V_F^{1/2}f\rangle} \\ &= \sup_{f\in\mathcal{D}(V):Vf\neq 0} \frac{|\langle V_F^{1/2}h, V_F^{1/2}f\rangle|^2}{\|V_F^{1/2}f\|^2} = \|V_F^{1/2}h\|^2 ,\end{aligned}$$

where we have used that $\mathrm{ran}(V_F^{1/2} \restriction_{\mathcal{D}(V)})$ is dense in $\overline{\mathrm{ran}(V_F^{1/2})}$ by Lemma 9.1.1. This implies that the linear map

$$\begin{aligned}\mathcal{U}_0 : \quad \mathrm{ran}(V_F^{1/2}) &\to \mathrm{ran}\left(V_K^{1/2} \restriction_{\mathcal{D}(V_F^{1/2})}\right) \\ V_F^{1/2}h &\mapsto V_K^{1/2}h\end{aligned}$$

is isometric. Since, trivially, $\mathrm{ran}(V_F^{1/2})$ is dense in $\overline{\mathrm{ran}(V_F^{1/2})}$, there exists a unique isometric extension $\mathcal{U}_0 \subset \mathcal{U}$ on $\overline{\mathrm{ran}(V_F^{1/2})}$. Setting $\mathcal{U}k = 0$ for all $k \in \ker(V_F^{1/2}) = \mathrm{ran}(V_F^{1/2})^\perp$ defines $\mathcal{U}$ as a partial isometry on the whole Hilbert space $\mathcal{H}$. Moreover, since

$$\mathrm{ran}(\mathcal{U}_0) = \mathrm{ran}\left(V_K^{1/2} \restriction_{\mathcal{D}(V_F^{1/2})}\right) \subset \mathrm{ran}(V_K^{1/2}) ,$$

this implies that ran($\mathcal{U}$) is contained in $\overline{\mathrm{ran}(V_K^{1/2})}$ and thus the lemma. □

It will be convenient to introduce the following notation:

Definition 9.1.3. *Let $(A, \widetilde{A})$ be a dual pair, where A is dissipative and $\widetilde{A}$ is antidissipative. Let $\mathcal{V} \subset \mathcal{D}(\widetilde{A}^*)//\mathcal{D}(A)$ and let $\mathcal{L}$ be a linear operator from $\mathcal{V}$ into $\mathcal{H}$. Then, the operator $A_{\mathcal{V},\mathcal{L}}$ is given by*

$$\begin{aligned}A_{\mathcal{V},\mathcal{L}} : \qquad \mathcal{D}(A_{\mathcal{V},\mathcal{L}}) &= \mathcal{D}(A)\dot{+}\mathcal{V} \\ (f+v) &\mapsto \widetilde{A}^*(f+v) + \mathcal{L}v ,\end{aligned}$$

where $f \in \mathcal{D}(A)$ and $v \in \mathcal{V}$. Clearly, if we choose $\mathcal{L}$ to be the zero-operator, i.e. $\mathcal{L} = 0$, we get the previous description of a proper extension: $A_{\mathcal{V},0} = A_{\mathcal{V}}$. (Cf. Definition 5.2.7)

We now want to find conditions on $\mathcal{V}$ and $\mathcal{L}$ for $A_{\mathcal{V},\mathcal{L}}$ to be dissipative. To this end, we will look at $\mathrm{Im}\langle f+v, A_{\mathcal{V},\mathcal{L}}(f+v)\rangle$ for any $f \in \mathcal{D}$ and any $v \in \mathcal{V}$, where $\mathcal{D} \subset \mathcal{D}(A) \cap \mathcal{D}(\widetilde{A})$ is a common core for $(A, \widetilde{A})$. We therefore will get

$$\text{(9.1.2)}\quad \mathrm{Im}\langle(f+v, A_{\mathcal{V},\mathcal{L}}(f+v)\rangle = \langle f, Vf\rangle + \mathrm{Im}\langle v, 2iVf\rangle + \mathrm{Im}\langle v, (\widetilde{A}^* + \mathcal{L})v\rangle - \mathrm{Im}\langle \mathcal{L}v, f\rangle .$$

For the case of proper extensions ($\mathcal{L} = 0$), we have seen that it is necessary that $\mathcal{V} \subset \mathcal{D}(V_K^{1/2})$ for $A_{\mathcal{V}}$ to be dissipative. The idea was that for any $v \notin \mathcal{D}(V_K^{1/2})$, there exists a normalized sequence $\{V_F^{1/2} f_n\}_n \subset \mathrm{ran}(V_F^{1/2} \restriction_{\mathcal{D}(V)})$ such that

$$\lim_{n\to\infty} \mathrm{Im}\langle v, 2iV_F^{1/2} V_F^{1/2} f_n\rangle = -\infty ,$$

which means that

$$\mathrm{Im}\langle(f_n + v, A_{\mathcal{V}}(f_n + v)\rangle = 1 + \mathrm{Im}\langle v, 2iV_F^{1/2} V_F^{1/2} f_n\rangle + \mathrm{Im}\langle v, \widetilde{A}^* v\rangle \overset{n\to\infty}{\longrightarrow} -\infty .$$

Now, for $\mathcal{L} \neq 0$ it could happen that the last term in (9.1.2) does not stay bounded either and instead "competes" against the part of (9.1.2) that would go to $-\infty$. Since — at least formally — $\langle \mathcal{L}v, f\rangle = \langle \mathcal{L}v, V_F^{-1/2} V_F^{1/2} f\rangle$, this might be the case if $\mathcal{L}v \notin \mathcal{D}(V_F^{-1/2}) = \mathrm{ran}(V_F^{1/2})$. Thus, in the situation $v \notin \mathcal{D}(V_K^{1/2})$ and $\mathcal{L}v \notin \mathrm{ran}(V_F^{1/2})$ it is not clear whether it is in general possible that $A_{\mathcal{V},\mathcal{L}}$ is dissipative. Moreover, since it is difficult to compute $V_F^{1/2}$, $V_F^{-1/2}$ and $V_K^{1/2}$ explicitly, we were not able to construct such an example. (The elementary case of V being a multiplication operator or — more generally — an essentially selfadjoint operator will be discussed in Lemma 9.1.6.)

However, if one of the two cases $\mathcal{V} \subset \mathcal{D}(V_K^{1/2})$ or $\mathrm{ran}(\mathcal{L}) \subset \mathcal{D}(V_F^{1/2})$ are given, we can show that the other one must be true as well for $A_{\mathcal{V},\mathcal{L}}$ to have a chance to be dissipative.

Lemma 9.1.4. *Let $(A, \widetilde{A})$ be a dual pair satisfying the common core condition, where A is dissipative.*

i) If ran$(\mathcal{L}) \subset$ ran$(V_F^{1/2})$, then it is necessary that $\mathcal{V} \subset \mathcal{D}(V_K^{1/2})$ for $A_{\mathcal{V},\mathcal{L}}$ to be dissipative.

ii) If $\mathcal{V} \subset \mathcal{D}(V_K^{1/2})$, then it is necessary that ran$(\mathcal{L}) \subset$ ran$(V_F^{1/2})$ for $A_{\mathcal{V},\mathcal{L}}$ to be dissipative.

Proof. i) If $\operatorname{ran}(\mathcal{L}) \subset \operatorname{ran}(V_F^{1/2})$, this means that for any $v \in \mathcal{V}$ there exists a ϕ_v such that $\mathcal{L}v = V_F^{1/2}\phi_v$. Thus, we can rewrite (9.1.2) as

(9.1.3)
$$\operatorname{Im}\langle (f+v, A_{\mathcal{V},\mathcal{L}}(f+v)\rangle = \|V_F^{1/2}f\|^2 + \operatorname{Im}\langle v, 2iV_F^{1/2}V_F^{1/2}f\rangle + \operatorname{Im}\langle v, (\widetilde{A}^* + \mathcal{L})v\rangle - \operatorname{Im}\langle \phi_v, V_F^{1/2}f\rangle.$$

Now, assume that there exists a $v \in \mathcal{V}$ such that $v \notin \mathcal{D}(V_K^{1/2})$. By Corollary 5.2.5, this means that there exists a normalized sequence $\{V_F^{1/2}f_n\}_n \subset \operatorname{ran}(V_F^{1/2} \restriction_{\mathcal{D}(V)})$ such that

$$\lim_{n\to\infty} \operatorname{Im}\langle v, 2iV_F^{1/2}V_F^{1/2}f_n\rangle = -\infty\,.$$

Since all other terms in (9.1.3) stay bounded, this shows that $A_{\mathcal{V},\mathcal{L}}$ cannot be dissipative in this case.

ii) We start by showing that in this case, it is necessary that $\mathcal{L}v \perp \ker V_F^{1/2}$. Assume this is not the case, i.e. that there exists a $v \in \mathcal{V}$ and a $k \in \ker(V_F^{1/2})$ such that $\langle \mathcal{L}v, k\rangle \neq 0$. Without loss of generality we may assume that $\operatorname{Im}\langle \mathcal{L}v, k\rangle = 1$. Now, since $\mathcal{D}(V)$ is a core for $V_F^{1/2}$, we can pick a sequence $\{f_n\}_n \subset \mathcal{D}(V)$ such that $f_n \to \lambda k$ and $V_F^{1/2}f_n \to \lambda V_F^{1/2}k = 0$, where $\lambda \in \mathbb{C}$ is an arbitrary complex number. We then get

$$\begin{aligned}
&\lim_{n\to\infty} \operatorname{Im}\langle (f_n + v, A_{\mathcal{V},\mathcal{L}}(f_n + v)\rangle \\
&= \lim_{n\to\infty}\Big(\|V_F^{1/2}f_n\|^2 + \operatorname{Im}\langle v, 2iV_K^{1/2}V_K^{1/2}f_n\rangle + \operatorname{Im}\langle v, (\widetilde{A}^* + \mathcal{L})v\rangle - \operatorname{Im}\langle \mathcal{L}v, f_n\rangle\Big) \\
&\overset{(9.1.1)}{=} \lim_{n\to\infty}\Big(\|V_F^{1/2}f_n\|^2 + \operatorname{Im}\langle V_K^{1/2}v, 2i\mathcal{U}V_F^{1/2}f_n\rangle + \operatorname{Im}\langle v, (\widetilde{A}^* + \mathcal{L})v\rangle - \operatorname{Im}\langle \mathcal{L}v, f_n\rangle\Big) \\
&= \lim_{n\to\infty}\Big(\|V_F^{1/2}f_n\|^2 + \operatorname{Im}\langle \mathcal{U}^*V_K^{1/2}v, 2iV_F^{1/2}f_n\rangle + \operatorname{Im}\langle v, (\widetilde{A}^* + \mathcal{L})v\rangle - \operatorname{Im}\langle \mathcal{L}v, f_n\rangle\Big) \\
&= \operatorname{Im}\langle v, (\widetilde{A}^* + \mathcal{L})v\rangle - \operatorname{Im}\lambda\,,
\end{aligned}$$

which is negative if we choose $\operatorname{Im}\lambda$ large enough. This contradicts the dissipativity of $A_{\mathcal{V},\mathcal{L}}$. Hence $\operatorname{ran}(\mathcal{L}) \subset (\ker V_F^{1/2})^\perp = \overline{\operatorname{ran}(V_F^{1/2})}$. Now, since $\ker V_F^{1/2}$ is a reducing subspace for $V_F^{1/2}$, we have that the operator $V_F^{-1/2}$ given by

$$\begin{aligned}
V_F^{-1/2}: \qquad \mathcal{D}(V_F^{-1/2}) = \operatorname{ran}V_F^{1/2} &\to \mathcal{D}(V_F^{1/2}) \cap \overline{\operatorname{ran}(V_F^{1/2})} \\
V_F^{1/2}f &\mapsto f
\end{aligned}$$

is a well-defined non-negative selfadjoint operator on the Hilbert space $\overline{\operatorname{ran}(V_F^{1/2})}$, which reduces $V_F^{1/2}$. Now, assume that there is a $v \in \mathcal{V}$, such that $\mathcal{L}v \notin \operatorname{ran}(V_F^{1/2}) = \mathcal{D}(V_F^{-1/2})$.

Since $V_F^{1/2} = \overline{V_F^{1/2} \restriction_{\mathcal{D}(V)}}$, we have $\overline{\mathrm{ran}(V_F^{1/2})} = \overline{\mathrm{ran}(V_F^{1/2} \restriction_{\mathcal{D}(V)})}$. This means that we can pick a sequence $\{V_F^{1/2} f_n\}_n \subset \mathrm{ran}\,(V_F^{1/2} \restriction_{\mathcal{D}(V)})$, where $\|V_F^{1/2} f_n\| = 1$ for all n, such that

$$\lim_{n\to\infty} \mathrm{Im}\langle \mathcal{L}v, V_F^{-1/2} V_F^{1/2} f_n\rangle = +\infty\ ,$$

since otherwise the map $g \mapsto \langle \mathcal{L}v, V_F^{-1/2} g\rangle$ would be a bounded linear functional on $\mathrm{ran}(V_F^{1/2} \restriction_{\mathcal{D}(V)})$, which is dense in $\overline{\mathrm{ran}(V_F^{1/2})}$ — a contradiction to $\mathcal{L}v \notin \mathcal{D}(V_F^{-1/2})$. Thus, we get

$$\begin{aligned}
&\mathrm{Im}\langle (f_n + v, A_{\mathcal{V},\mathcal{L}}(f_n + v)\rangle \\
=&\|V_F^{1/2} f_n\|^2 + \mathrm{Im}\langle \mathcal{U}^* V_K^{1/2} v, 2iV_F^{1/2} f_n\rangle + \mathrm{Im}\langle v, (\widetilde{A}^* + \mathcal{L})v\rangle - \mathrm{Im}\langle \mathcal{L}v, f_n\rangle \\
\leq& 1 + 2\|\mathcal{U}^* V_K^{1/2} v\| + \mathrm{Im}\langle v, (\widetilde{A}^* + \mathcal{L})v\rangle - \mathrm{Im}\langle \mathcal{L}v, V_F^{-1/2} V_F^{1/2} f_n\rangle \stackrel{n\to\infty}{\longrightarrow} -\infty\ ,
\end{aligned}$$

which means that $A_{\mathcal{V},\mathcal{L}}$ cannot be dissipative in this case either. This shows the lemma.

$\square$

REMARK 9.1.5. If V is strictly positive, i.e. if there exists an $\varepsilon > 0$ such that

$$\mathrm{Im}\langle f, Af\rangle \geq \varepsilon \|f\|^2$$

for all $f \in \mathcal{D}$, we have that $V_F^{-1/2}$ is a bounded operator on $\mathcal{H}$. In this case, the condition $\mathrm{ran}(\mathcal{L}) \subset \mathrm{ran}(V_F^{1/2}) = \mathcal{H}$ is always satisfied. Hence, in this case it is necessary that $\mathcal{V} \subset \mathcal{D}(V_K^{1/2})$ for $A_{\mathcal{V},\mathcal{L}}$ to be dissipative.

For the special case that V is essentially selfadjoint, we will prove that both conditions, $\mathcal{V} \subset \mathcal{D}(\overline{V}^{1/2})$ and $\mathrm{ran}(\mathcal{L}) \subset \mathrm{ran}(\overline{V}^{1/2})$ are independently necessary for $A_{\mathcal{V},\mathcal{L}}$ to be dissipative.

LEMMA 9.1.6. *Let $(A, \widetilde{A})$ be as in Lemma 9.1.4 and assume in addition that the imaginary part V is essentially selfadjoint. Then, for $A_{\mathcal{V},\mathcal{L}}$ to be dissipative it is necessary that $\mathcal{V} \subset \mathcal{D}(\overline{V}^{1/2})$ and ran$(\mathcal{L}) \subset$ ran$(\overline{V}^{1/2})$.*

PROOF. Since $\overline{V}^{1/2} = V_F^{1/2} = V_K^{1/2}$, we only need to show that $\mathcal{V} \subset \mathcal{D}(\overline{V}^{1/2})$ is necessary for $A_{\mathcal{V},\mathcal{L}}$ to be dissipative. The condition $\mathrm{ran}(\mathcal{L}) \subset \mathrm{ran}(\overline{V}^{1/2})$ will then just follow from Lemma 9.1.4, ii). Thus, assume that there exists a $v \in \mathcal{V}$ such that

$v \notin \mathcal{D}(\overline{V}^{1/2})$. In this case, we will show that

$$\inf_{f \in \mathcal{D}(\overline{V})} (\langle f, \overline{V} f\rangle + \mathrm{Im}(\langle v, 2i\overline{V} f\rangle - \langle \mathcal{L}v, f\rangle)) = -\infty \,.$$

Since $V_F^{1/2} = V_K^{1/2} = \overline{V}^{1/2}$, we get that $\mathrm{ran}(V_F^{1/2} \restriction_{\mathcal{D}(V)}) = \mathrm{ran}(\overline{V}^{1/2} \restriction_{\mathcal{D}(V)})$. Hence, if $v \notin \mathcal{D}(\overline{V}^{1/2}) = \mathcal{D}(V_K^{1/2})$, we have by Corollary 5.2.5 that there exists a sequence $\{\overline{V}^{1/2} f_n\}_n \subset \mathrm{ran}(\overline{V}^{1/2} \restriction_{\mathcal{D}(V)}) \subset \mathrm{ran}(\overline{V}^{1/2} \restriction_{\mathcal{D}(\overline{V})})$ such that $\|\overline{V}^{1/2} f_n\| = 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} |\langle v, 2i\overline{V}^{1/2}\overline{V}^{1/2} f_n\rangle| = +\infty$. Now, let P denote the projection-valued measure corresponding to $\overline{V}$ and define $P_1 := P([0,1))$ and $P_2 := P([1,\infty))$ as well as $\mathcal{H}_{1,2} := P_{1,2}\mathcal{H}$. Since $\overline{V} \geq 0$, we have $P_1 + P_2 = \mathbb{1}$, resp. $\mathcal{H}_1 \oplus \mathcal{H}_2 = \mathcal{H}$. For any $n \in \mathbb{N}$, define $\widetilde{f}_n := P_2 f_n$, which is an element of $\mathcal{D}(\overline{V})$ since $\mathcal{H}_2$ reduces $\overline{V}$. We now claim that the sequence $\{\overline{V}^{1/2}\widetilde{f}_n\}_n$ satisfies

$$\|\overline{V}^{1/2}\widetilde{f}_n\| \leq 1 \quad \text{and} \quad \lim_{n\to\infty} |\langle v, 2i\overline{V}^{1/2}\overline{V}^{1/2}\widetilde{f}_n\rangle| = +\infty \,.$$

The first statement follows from

$$\|\overline{V}^{1/2}\widetilde{f}_n\| = \|\overline{V}^{1/2} P_2 f_n\| = \|P_2 \overline{V}^{1/2} f_n\| \leq \|\overline{V}^{1/2} f_n\| = 1 \,. \tag{9.1.4}$$

To see the second statement consider

$$\begin{aligned}
&|\langle v, 2i\overline{V}^{1/2}\overline{V}^{1/2} f_n\rangle| = |\langle v, 2i\overline{V}^{1/2}\overline{V}^{1/2}(P_1+P_2) f_n\rangle| \\
\leq& |\langle v, 2i\overline{V}^{1/2}\overline{V}^{1/2} P_1 f_n\rangle| + |\langle v, 2i\overline{V}^{1/2}\overline{V}^{1/2} P_2 f_n\rangle| \\
\leq& 2\|v\| \|\overline{V}^{1/2} P_1 \overline{V}^{1/2} f_n\| + |\langle v, 2i\overline{V}^{1/2}\overline{V}^{1/2}\widetilde{f}_n\rangle| \\
=& 2\|v\| \sqrt{\int_{[0,1)} \lambda \,\mathrm{d}\|P(\lambda)(\overline{V}^{1/2} f_n)\|^2} + |\langle v, 2i\overline{V}^{1/2}\overline{V}^{1/2}\widetilde{f}_n\rangle| \\
\leq& 2\|v\| \sqrt{\int_{[0,1)} \mathrm{d}\|P(\lambda)(\overline{V}^{1/2} f_n)\|^2} + |\langle v, 2i\overline{V}^{1/2}\overline{V}^{1/2}\widetilde{f}_n\rangle| \\
=& 2\|v\| \|P_1 \overline{V}^{1/2} f_n\| + |\langle v, 2i\overline{V}^{1/2}\overline{V}^{1/2}\widetilde{f}_n\rangle| \\
\leq& 2\|v\| \underbrace{\|\overline{V}^{1/2} f_n\|}_{=1} + |\langle v, 2i\overline{V}^{1/2}\overline{V}^{1/2}\widetilde{f}_n\rangle| = 2\|v\| + |\langle v, 2i\overline{V}^{1/2}\overline{V}^{1/2}\widetilde{f}_n\rangle| \,.
\end{aligned}$$

Now, since $|\langle v, 2i\overline{V}^{1/2}\overline{V}^{1/2} f_n\rangle| \overset{n\to\infty}{\longrightarrow} \infty$, this implies that $|\langle v, 2i\overline{V}^{1/2}\overline{V}^{1/2} \widetilde{f}_n\rangle| \overset{n\to\infty}{\longrightarrow} \infty$. Next, observe that

$$\|\widetilde{f}_n\|^2 = \int_{[1,\infty)} \mathrm{d}\|P(\lambda)\widetilde{f}_n\|^2 \le \int_{[1,\infty)} \lambda \mathrm{d}\|P(\lambda)\widetilde{f}_n\|^2 = \|\overline{V}^{1/2}\widetilde{f}_n\|^2 \overset{(9.1.4)}{\le} 1\,. \tag{9.1.5}$$

For any $n \in \mathbb{N}$ choose $\phi_n \in [0, 2\pi)$ such that

$$\operatorname{Im}\langle v, 2i\overline{V}^{1/2}\overline{V}^{1/2} e^{i\phi_n}\widetilde{f}_n\rangle = -|\langle v, 2i\overline{V}^{1/2}\overline{V}^{1/2}\widetilde{f}_n\rangle|\,.$$

Altogether, we get

$$\begin{aligned}
&\|\overline{V}^{1/2} e^{i\phi_n}\widetilde{f}_n\|^2 + \operatorname{Im}\langle v, 2i\overline{V}^{1/2}\overline{V}^{1/2} e^{i\phi_n}\widetilde{f}_n\rangle - \operatorname{Im}\langle \mathcal{L}v, e^{i\phi_n}\rangle \\
&\qquad \le 1 - |\langle v, 2i\overline{V}^{1/2}\overline{V}^{1/2}\widetilde{f}_n\rangle| + \|\mathcal{L}v\|\|\widetilde{f}_n\| \\
&\qquad \overset{(9.1.5)}{\le} 1 - |\langle v, 2i\overline{V}^{1/2}\overline{V}^{1/2}\widetilde{f}_n\rangle| + \|\mathcal{L}v\| \overset{n\to\infty}{\longrightarrow} -\infty\,.
\end{aligned} \tag{9.1.6}$$

Let us now show that

$$\begin{aligned}
&\inf_{f\in\mathcal{D}(V)} \operatorname{Im}\langle f+v, A_{\mathcal{V},\mathcal{L}}(f+v)\rangle \\
&\quad = \operatorname{Im}\langle v, (\widetilde{A}^* + \mathcal{L})v\rangle + \inf_{f\in\mathcal{D}(V)} \left(\langle f, Vf\rangle + \operatorname{Im}\langle v, 2iVf\rangle - \operatorname{Im}\langle \mathcal{L}v, f\rangle\right) = -\infty\,.
\end{aligned}$$

Assume that this is not true, i.e. that there exists a $K > -\infty$ such that

$$\operatorname{Im}\langle f+v, A_{\mathcal{V},\mathcal{L}}(f+v)\rangle = \operatorname{Im}\langle v, (\widetilde{A}^* + \mathcal{L})v\rangle + \langle f, Vf\rangle + \operatorname{Im}\langle v, 2iVf\rangle - \operatorname{Im}\langle \mathcal{L}v, f\rangle \ge K \tag{9.1.7}$$

for all $f \in \mathcal{D}(V)$. Now, by (9.1.6), we can choose an $N \in \mathbb{N}$ big enough such that

$$\operatorname{Im}\langle v, (\widetilde{A}^* + \mathcal{L})v\rangle + \|\overline{V}^{1/2} e^{i\phi_N}\widetilde{f}_N\|^2 + \operatorname{Im}\langle v, 2i\overline{V}^{1/2}\overline{V}^{1/2} e^{i\phi_N}\widetilde{f}_N\rangle - \operatorname{Im}\langle \mathcal{L}v, e^{i\phi_N}\widetilde{f}_N\rangle \le K - 1\,.$$

Since $\mathcal{D}(V)$ is a core for $\overline{V}$, we get that for $\widetilde{f}_N \in \mathcal{D}(\overline{V})$, there exists a sequence $\{\widetilde{f}_{N,m}\}_m \subset \mathcal{D}(V)$ such that $\widetilde{f}_{N,m} \overset{m\to\infty}{\longrightarrow} \widetilde{f}_N$ and $V\widetilde{f}_{N,m} \overset{m\to\infty}{\longrightarrow} \overline{V}\widetilde{f}_N$, which clearly implies that

$$\begin{aligned}
K - 1 \ge &\operatorname{Im}\langle v, (\widetilde{A}^* + \mathcal{L})v\rangle + \langle \widetilde{f}_N \overline{V}\widetilde{f}_N\rangle + \operatorname{Im}\langle v, 2i\overline{V} e^{i\phi_N}\widetilde{f}_N\rangle - \operatorname{Im}\langle \mathcal{L}v, e^{i\phi_N}\widetilde{f}_N\rangle \\
= &\operatorname{Im}\langle v, (\widetilde{A}^* + \mathcal{L})v\rangle + \lim_{m\to\infty}\big(\langle \widetilde{f}_{N,m}\overline{V}\widetilde{f}_{N,m}\rangle \\
&+ \operatorname{Im}\langle v, 2i\overline{V} e^{i\phi_N}\widetilde{f}_{N,m}\rangle - \operatorname{Im}\langle \mathcal{L}v, e^{i\phi_N}\widetilde{f}_{N,m}\rangle\big)\,.
\end{aligned}$$

Hence, for m big enough we get that

$$\operatorname{Im}\langle e^{i\phi_N}\widetilde{f}_{N,m} + v, A_{\mathcal{V},\mathcal{L}}(e^{i\phi_N}\widetilde{f}_{N,m} + v)\rangle < K$$

in contradiction to (9.1.7). This shows the lemma. $\square$

REMARK 9.1.7. This result applies in particular to the case of V being bounded.

Under the assumption that the conditions $\mathcal{V} \subset \mathcal{D}(V_K^{1/2})$ and $\operatorname{ran}(\mathcal{L}) \subset \operatorname{ran}(V_F^{1/2})$ are satisfied, let us now show a necessary and sufficient condition for $A_{\mathcal{V},\mathcal{L}}$ to be dissipative. Before we do this, we will need the following lemma:

LEMMA 9.1.8. *Let V be a symmetric and non-negative operator and let V_K denote its selfadjoint Kreĭn–von Neumann extension. Then, ran$(V_K^{1/2} \restriction_{\mathcal{D}(V)})$ is dense in $\overline{\text{ran}(V_K^{1/2})}$.*

PROOF. Any element of $\operatorname{ran}(V_K^{1/2})$ is of the form $V_K^{1/2}h$, where $h \in \mathcal{D}(V_K^{1/2})$. By Proposition 5.2.3, we have that

$$\|V_K^{1/2}h\|^2 = \sup_{f\in\mathcal{D}(V):Vf\neq 0} \frac{|\langle h, Vf\rangle|^2}{\langle f, Vf\rangle}\,,$$

where $\frac{|\langle h,Vf\rangle|^2}{\langle f,Vf\rangle}$ can be rewritten as

$$\frac{|\langle h, Vf\rangle|^2}{\langle f, Vf\rangle} = \left|\left\langle V_K^{1/2}h, \frac{V_K^{1/2}f}{\|V_K^{1/2}f\|}\right\rangle\right|^2 .$$

This allows us to rewrite

$$\begin{aligned}\|V_K^{1/2}h\|^2 &= \sup\left\{\frac{|\langle h, Vf\rangle|^2}{\langle f, Vf\rangle}, f \in \mathcal{D}(V) : Vf \neq 0\right\}\\ &= \sup\left\{|\langle V_K^{1/2}h, g\rangle|^2, g \in \operatorname{ran}(V_K^{1/2} \restriction_{\mathcal{D}(V)}) : \|g\| = 1\right\} .\end{aligned} \tag{9.1.8}$$

But this implies that $\operatorname{ran}(V_K^{1/2} \restriction_{\mathcal{D}(V)})$ is dense in $\overline{\operatorname{ran}(V_K^{1/2})}$. To see why, assume that there exists a $\varphi \in \overline{\operatorname{ran}(V_K^{1/2})}$ such that $\|\varphi\| = 1$ and $\langle\varphi, g\rangle = 0$ for all $g \in \operatorname{ran}(V_K^{1/2} \restriction_{\mathcal{D}(V)})$. Take a $V_K^{1/2}h \in \operatorname{ran}(V_K^{1/2})$, with $\|V_K^{1/2}h\| = 1$ such that $\|V_K^{1/2}h - \varphi\|^2 < \varepsilon$ for some $0 < \varepsilon < 1$ small enough. Then, for any $g \in \operatorname{ran}(V_K^{1/2} \restriction_{\mathcal{D}(V)})$, we get

$$|\langle V_K^{1/2}h, g\rangle|^2 = |\langle V_K^{1/2}h - \varphi, g\rangle|^2 \leq \|V_K^{1/2}h - \varphi\|^2\|g\|^2 \leq \varepsilon\|g\|^2 .$$

Taking the supremum over all $g \in \operatorname{ran}(V_K^{1/2} \restriction_{\mathcal{D}(V)})$ with $\|g\| = 1$, we arrive at a contradiction, since the supremum of the left hand side is 1 whereas the supremum of the right hand side is $\varepsilon < 1$. This shows the lemma. $\square$

We are now prepared to show the main theorem:

THEOREM 9.1.9. *Let $(A,\widetilde{A})$ be a dual pair that has the common core property, where A is dissipative. Moreover, for any $\mathcal{V}\in\mathcal{D}(\widetilde{A}^*)//\mathcal{D}(A)$ and $\mathcal{L}$ being a linear map from $\mathcal{V}$ into $\mathcal{H}$, let the operator $A_{\mathcal{V},\mathcal{L}}$ be defined as in Definition 9.1.3. Moreover, assume that*

$$v\in\mathcal{D}(V_K^{1/2})\qquad \textit{and}\qquad \mathcal{L}v\in\operatorname{ran}(V_F^{1/2})=\mathcal{D}(V_F^{-1/2})$$

for all $v\in\mathcal{V}$. Then, $A_{\mathcal{V},\mathcal{L}}$ is dissipative if and only if for all $v\in\mathcal{V}$ we have

$$Im\langle v,(\widetilde{A}^*+\mathcal{L})v\rangle\geq\frac{1}{4}\|\mathcal{U}V_F^{-1/2}\mathcal{L}v+2iV_K^{1/2}v\|^2\,. \tag{9.1.9}$$

Here, $V_F^{-1/2}$ denotes the inverse of $V_F^{1/2}$ as an operator in $\overline{\operatorname{ran}(V_F^{1/2})}$ and $\mathcal{U}$ is the partial isometry as defined in Lemma 9.1.2.

PROOF. Let us start be showing that the above conditions are sufficient. As usual, let $\mathcal{D}$ denote a common core for A and $\widetilde{A}$. For any $f\in\mathcal{D}$ and any $v\in\mathcal{V}$, we then get

$$\begin{aligned}
\operatorname{Im}\langle f+v,A_{\mathcal{V},\mathcal{L}}(f+v)\rangle&=\operatorname{Im}\langle f+v,\widetilde{A}^*(f+v)\rangle+\operatorname{Im}\langle f+v,\mathcal{L}v\rangle\\
&=\langle f,Vf\rangle+\operatorname{Im}\langle v,2iVf\rangle+\operatorname{Im}\langle v,(\widetilde{A}^*+\mathcal{L})v\rangle+\operatorname{Im}\langle f,\mathcal{L}v\rangle\\
&=\|V_K^{1/2}f\|^2+\operatorname{Im}\langle v,2iV_K^{1/2}V_K^{1/2}f\rangle+\operatorname{Im}\langle v,(\widetilde{A}^*+\mathcal{L})v\rangle+\operatorname{Im}\langle V_F^{-1/2}V_F^{1/2}f,\mathcal{L}v\rangle\\
&=\|V_K^{1/2}f\|^2+\operatorname{Im}\langle V_K^{1/2}v,2iV_K^{1/2}f\rangle+\operatorname{Im}\langle v,(\widetilde{A}^*+\mathcal{L})v\rangle+\operatorname{Im}\langle\mathcal{U}V_F^{1/2}f,\mathcal{U}V_F^{-1/2}\mathcal{L}v\rangle\\
&=\|V_K^{1/2}f\|^2+\operatorname{Im}\langle V_K^{1/2}v,2iV_K^{1/2}f\rangle+\operatorname{Im}\langle v,(\widetilde{A}^*+\mathcal{L})v\rangle+\operatorname{Im}\langle V_K^{1/2}f,\mathcal{U}V_F^{-1/2}\mathcal{L}v\rangle\\
&=\|V_K^{1/2}f\|^2+\operatorname{Im}\langle v,(\widetilde{A}^*+\mathcal{L})v\rangle+\operatorname{Im}\langle V_K^{1/2}f,(\mathcal{U}V_F^{-1/2}\mathcal{L}+2iV_K^{1/2})v\rangle\\
&\geq\|V_K^{1/2}f\|^2+\frac{1}{4}\|\mathcal{U}V_F^{-1/2}\mathcal{L}v+2iV_K^{1/2}v\|^2+\operatorname{Im}\langle V_K^{1/2}f,(\mathcal{U}V_F^{-1/2}\mathcal{L}+2iV_K^{1/2})v\rangle\\
&\geq\|V_K^{1/2}f\|^2+\frac{1}{4}\|\mathcal{U}V_F^{-1/2}\mathcal{L}v+2iV_K^{1/2}v\|^2-\|V_K^{1/2}f\|\|(\mathcal{U}V_F^{-1/2}\mathcal{L}+2iV_K^{1/2})v\|\\
&=\left(\|V_K^{1/2}f\|-\frac{1}{2}\|\mathcal{U}V_F^{-1/2}\mathcal{L}v+2iV_K^{1/2}v\|\right)^2\geq 0\,.
\end{aligned}$$

Let us now show that Condition (9.1.9) is also necessary. Assume that it is not satisfied, i.e. that there exists a $v\in\mathcal{V}$ such that

$$\operatorname{Im}\langle v,(\widetilde{A}^*+\mathcal{L})v\rangle-\frac{1}{4}\|\mathcal{U}V_F^{-1/2}\mathcal{L}v+2iV_K^{1/2}v\|^2\leq-\varepsilon \tag{9.1.10}$$

for some $\varepsilon > 0$. By Lemma 9.1.2, we have that $(\mathcal{U}V_F^{-1/2}\mathcal{L}v + 2iV_K^{1/2}v) \in \overline{\mathrm{ran}(V_K^{1/2})}$. By Lemma 9.1.8, there exists a sequence $\{V_K^{1/2}f_n\}_n \subset \mathrm{ran}(V_K^{1/2} \upharpoonright_{\mathcal{D}(V)})$ such that

$$V_K^{1/2}f_n \overset{n\to\infty}{\longrightarrow} \frac{-i}{2}(\mathcal{U}V_F^{-1/2}\mathcal{L}v + 2iV_K^{1/2}v)\ ,$$

which means by (9.1.10) that

$$\begin{aligned}&\mathrm{Im}\langle f_n + v, A_{\mathcal{V},\mathcal{L}}(f_n+v)\rangle \\ &= \|V_K^{1/2}f_n\|^2 + \mathrm{Im}\langle v, (\widetilde{A}^* + \mathcal{L})v\rangle + \mathrm{Im}\langle V_K^{1/2}f_n, \mathcal{U}V_F^{-1/2}\mathcal{L}v + 2iV_K^{1/2}v\rangle \overset{n\to\infty}{\longrightarrow} -\varepsilon < 0\ .\end{aligned}$$

This shows the theorem. $\square$

Since one would need further knowledge of the explicit form of the operators $V_F^{1/2}$ and $V_K^{1/2}$ in order to compute $\mathcal{U}$, this result seems not to be very useful for practical applications. However, in the following, we will describe three special situations where significant simplifications occur:
Firstly let us consider the case when the imaginary part V is strictly positive, i.e. when there exists a positive number $\varepsilon > 0$ such that $\langle f, Vf\rangle \geq \varepsilon\|f\|^2$ for all $f \in \mathcal{D}(V)$.

Corollary 9.1.10. *Let $(A, \widetilde{A})$ be a dual pair satisfying the common core property, where A is dissipative. Moreover, let the imaginary part V be strictly positive. Then, $A_{\mathcal{V},\mathcal{L}}$ is dissipative if and only if $\mathcal{V} \subset \mathcal{D}(V_K^{1/2})$ and for all $v \in \mathcal{V}$ we have that*

$$Im\langle v, \widetilde{A}^* v\rangle + Im\langle \mathcal{P}v, \mathcal{L}v\rangle \geq \frac{1}{4}\|V_F^{-1/2}\mathcal{L}v\|^2 + \|V_K^{1/2}v\|^2\ . \tag{9.1.11}$$

Here, $\mathcal{P}$ denotes the projection onto $\ker V^*$ *along $\mathcal{D}(V_F^{1/2})$, according to the decomposition $\mathcal{D}(V_K^{1/2}) = \mathcal{D}(V_F^{1/2}) \dot{+} \ker V^*$ as defined in* (8.13).

Proof. Since $V \geq \varepsilon > 0$, we have that $\mathrm{ran}(V_F) = \mathrm{ran}(V_F^{1/2}) = \mathcal{H}$, which means that the condition $\mathrm{ran}(\mathcal{L}) \subset \mathrm{ran}(V_F^{1/2})$ is always satisfied. Thus, by Lemma 9.1.4, it is necessary that $\mathcal{V} \subset \mathcal{D}(V_K^{1/2})$ for $A_{\mathcal{V},\mathcal{L}}$ to be dissipative.
Since $V \geq \varepsilon > 0$, we have that $\mathcal{D}(V_K) = \mathcal{D}(V) \dot{+} \ker V^*$ with $V_K = V^* \upharpoonright_{\mathcal{D}(V_K)}$. This implies that $\ker V^* = \ker V_K$ and since V_K is non-negative, we also get that $\ker V_K^{1/2} =$

$\ker V^*$. By (5.2.1), it is known that $\mathcal{D}(V_K^{1/2}) = \mathcal{D}(V_F^{1/2}) \dot{+} \ker V^*$. Thus, we can rewrite

$$\begin{aligned}
&\frac{1}{4}\|\mathcal{U}V_F^{-1/2}\mathcal{L}v + 2iV_K^{1/2}v\|^2 = \frac{1}{4}\|\mathcal{U}V_F^{-1/2}\mathcal{L}v + 2iV_K^{1/2}(\mathbb{1}-\mathcal{P})v\|^2 \\
\overset{(9.1.1)}{=}&\frac{1}{4}\|\mathcal{U}(V_F^{-1/2}\mathcal{L}v + 2iV_F^{1/2}(\mathbb{1}-\mathcal{P}))v\|^2 = \frac{1}{4}\|V_F^{-1/2}\mathcal{L}v + 2iV_F^{1/2}(\mathbb{1}-\mathcal{P})v\|^2 \\
=&\frac{1}{4}\|V_F^{-1/2}\mathcal{L}v\|^2 + \|V_F^{1/2}(\mathbb{1}-\mathcal{P})v\|^2 + \mathrm{Im}\langle V_F^{1/2}(\mathbb{1}-\mathcal{P})v, V_F^{-1/2}\mathcal{L}v\rangle \\
=&\frac{1}{4}\|V_F^{-1/2}\mathcal{L}v\|^2 + \|V_K^{1/2}v\|^2 + \mathrm{Im}\langle(\mathbb{1}-\mathcal{P})v, \mathcal{L}v\rangle\,.
\end{aligned}$$

With this, Condition (9.1.9) from Theorem 9.1.9 can be rewritten as

$$\mathrm{Im}\langle v, \widetilde{A}^*v\rangle + \mathrm{Im}\langle\mathcal{P}v, \mathcal{L}v\rangle \geq \frac{1}{4}\|V_F^{-1/2}\mathcal{L}v\|^2 + \|V_K^{1/2}v\|^2\,,$$

which is the desired result. □

Remark 9.1.11. For explicit computations, it seems useful to use the fact that $\mathrm{ran}(V_F) = \mathcal{H}$. This means that any $\mathcal{L}v$ can be written as $\mathcal{L}v = V_F\phi_v$ for some $\phi_v \in \mathcal{D}(V_F)$. Then, we can rewrite (9.1.11) as follows

$$\mathrm{Im}\langle v, \widetilde{A}^*v\rangle + \mathrm{Im}\langle\mathcal{P}v, V_F\phi_v\rangle \geq \frac{1}{4}\|V_F^{1/2}\phi_v\|^2 + \|V_K^{1/2}v\|^2\,, \tag{9.1.12}$$

which is more accessible to explicit computations. A similar idea for generic non-negative imaginary parts V will be discussed in Corollary 9.1.13.

Example 9.1.12. As in Section 5.4.4, consider the dual pair $(A_0, \widetilde{A}_0)$, given by

$$\begin{aligned}
A_0: &\qquad \mathcal{D}(A_0) = \mathcal{C}_c^\infty(0,1), \quad (A_0f)(x) = -if''(x) - \gamma\frac{f(x)}{x^2}, \\
\widetilde{A}_0: &\qquad \mathcal{D}(\widetilde{A}_0) = \mathcal{C}_c^\infty(0,1), \quad \left(\widetilde{A}_0f\right)(x) = if''(x) - \gamma\frac{f(x)}{x^2}\,.
\end{aligned}$$

Define the dual pair $(A, \widetilde{A})$, where $A := \overline{A_0}$ and $\widetilde{A} := \overline{\widetilde{A}_0}$. By construction, $(A, \widetilde{A})$ has the common core property, where we choose $\mathcal{C}_c^\infty(0,1) =: \mathcal{D}$ to be the common core. The "imaginary part" V is given by

$$\begin{aligned}
V: \qquad &\mathcal{D}(V) = \mathcal{C}_c^\infty(0,1) \\
&f \mapsto -f''\,,
\end{aligned}$$

which is a strictly positive operator, since in (5.4.11), we have already argued that

$$\langle f, Vf\rangle \geq \pi^2\|f\|^2 \text{ for all } f \in \mathcal{C}_c^\infty(0,1)\,.$$

For simplicity, assume that $\gamma \geq \sqrt{3}$, which ensures that $\dim\ker \widetilde{A}^* = \dim\ker A^* = 1$. Recall that $\mathcal{D}(\widetilde{A}^*)$ can be written as

$$\mathcal{D}(\widetilde{A}^*) = \mathcal{D}(A) \dot{+} \mathrm{span}\{x^{\omega_+}, x^{\overline{\omega_+}+2}\} ,$$

where we have defined $\omega_+ := (1+\sqrt{1+4i\gamma})/2$. We therefore choose $\mathcal{D}(\widetilde{A}^*)//\mathcal{D}(A) = \mathrm{span}\{x^{\omega_+}, x^{\overline{\omega_+}+2}\}$. Recall that in (5.4.15), we have parametrized all proper one-dimensional extensions of $(A, \widetilde{A})$, with the family of operators $\{A_\rho\}_{\rho\in\mathbb{C}\cup\{\infty\}}$ given by

$$A_\rho : \qquad \mathcal{D}(A_\rho) = \mathcal{D}(A) \dot{+} \mathrm{span}\{\xi_\rho\}, \quad A_\rho = \widetilde{A}^* \restriction_{\mathcal{D}(A_\rho)} ,$$

where

$$\mathrm{span}\{x^{\omega_+}, x^{\overline{\omega_+}+2}\} \ni \xi_\rho(x) := \begin{cases} \rho\left(\frac{(2+\overline{\omega_+})x^{\omega_+} - \omega_+ x^{\overline{\omega_+}+2}}{2+\overline{\omega_+}-\omega_+}\right) - \frac{x^{\omega_+} - x^{\overline{\omega_+}+2}}{2+\overline{\omega_+}-\omega_+} & \text{for } \rho \in \mathbb{C} \\ \frac{(2+\overline{\omega_+})x^{\omega_+} - \omega_+ x^{\overline{\omega_+}+2}}{2+\overline{\omega_+}-\omega_+} & \text{for } \rho = \infty \end{cases}$$

satisfies the boundary conditions

$$\xi_\rho(0) = \xi_\rho'(0) = 0 \quad \text{for} \quad \rho \in \mathbb{C}\cup\{\infty\}$$
$$\xi_\rho(1) = \rho,\ \ \xi_\rho'(1) = 1 \quad \text{for} \quad \rho \in \mathbb{C} \quad \text{and} \quad \xi_\rho(1) = 1,\ \ \xi_\rho'(1) = 0 \quad \text{for} \quad \rho = \infty .$$

Next, observe that for $\rho \in \mathbb{C}$, we get $\mathcal{P}\xi_\rho(x) = \rho x$, whereas for $\rho = \infty$, we get $\mathcal{P}\xi_\infty(x) = x$. This follows from the fact that $\mathcal{D}(V_F^{1/2}) = H_0^1(0,1)$ and for any $\rho \in \mathbb{C}$, we have $\xi_\rho(0) = \xi_\infty(0) = 0$ as well as $\xi_\rho(1) = \rho$ and $\xi_\infty(1) = 1$. Now, since V is strictly positive, we know that its Friedrichs extension V_F is bijective, which means that any function $\mathcal{L}\xi_\rho \in L^2(0,1)$ can be written as $\mathcal{L}\xi_\rho = V_F\phi = -\phi''$ for some unique $\phi \in \mathcal{D}(V_F) = \{\phi \in H^2(0,1), \phi(0) = \phi(1) = 0\}$. Hence, let us use the parameter $\rho \in \mathbb{C}\cup\{\infty\}$ and the functions $\phi \in \mathcal{D}(V_F)$ to label all one-dimensional extensions of $\mathcal{D}(A)$ that have domain contained in $\mathcal{D}(\widetilde{A}^*)$. They are given by

$$A_{\rho,\phi} : \qquad \mathcal{D}(A_{\rho,\phi}) = \mathcal{D}(A) \dot{+} \mathrm{span}\{\xi_\rho\}$$
$$[A_{\rho,\phi}(f+\lambda\xi_\rho)](x) = (-if''(x) - \lambda i\xi_\rho''(x)) - \gamma\frac{f(x)+\lambda\xi_\rho(x)}{x^2} - \lambda\phi''(x) ,$$

where $f \in \mathcal{D}(A)$ and $\lambda \in \mathbb{C}$. By (9.1.12), we have that $A_{\rho,\phi}$ is dissipative if and only if

$$\mathrm{Im}\langle\xi_\rho, \widetilde{A}^*\xi_\rho\rangle - \|V_K^{1/2}\xi_\rho\|^2 \geq \frac{1}{4}\|V_F^{1/2}\phi\|^2 - \mathrm{Im}\langle\mathcal{P}\xi_\rho, V_F\phi\rangle .$$

is satisfied. In (5.4.14), we have shown that for any $v \in \operatorname{span}\{x^{\omega_+}, x^{\overline{\omega_+}+2}\}$, we have

$$\operatorname{Im}\langle v, \widetilde{A}^* v\rangle - \|V_K^{1/2} v\|^2 = -\operatorname{Re}\left(\overline{v(1)} v'(1)\right) + |v(1)|^2 ,$$

which means that

$$\operatorname{Im}\langle \xi_\rho, \widetilde{A}^* \xi_\rho\rangle - \|V_K^{1/2} \xi_\rho\|^2 = \begin{cases} |\rho|^2 - \operatorname{Re}(\rho) & \text{if} \quad \rho \in \mathbb{C} \\ 1 & \text{if} \quad \rho = \infty . \end{cases}$$

Moreover, since $\|V_F^{1/2}\phi\| = \|\phi'\|$ and

$$\operatorname{Im}\left(\int_0^1 x\phi''(x)\mathrm{d}x\right) = \operatorname{Im}(\phi'(1)) \tag{9.1.13}$$

for any $\phi \in \mathcal{D}(V_F)$, the above yields the conditions for $A_{\rho,\phi}$ to be dissipative:

$$\begin{aligned} &\frac{1}{4}\|\phi'\|^2 + \operatorname{Im}(\overline{\rho}\phi'(1)) \le |\rho|^2 - \operatorname{Re}\rho \quad && \text{for} \quad \rho \in \mathbb{C} \\ &\frac{1}{4}\|\phi'\|^2 + \operatorname{Im}(\phi'(1)) \le 1 && \text{for} \quad \rho = \infty . \end{aligned}$$

For the case of proper extensions $A_{\rho,\phi=0}$ we had the condition that either $\rho = \infty$ or $|\rho|^2 - \operatorname{Re}\rho \ge 0$ for $A_{\rho,\phi=0}$ to be dissipative. In the non-proper case, for a suitable choice of ϕ, it is no longer necessary that ρ satisfies this condition. For instance, let $\phi(x) := x^2 - x \in \mathcal{D}(V_F)$. We then get the condition

$$\frac{1}{4}\|\phi'\|^2 + \operatorname{Im}(\overline{\rho}\phi'(1)) = \frac{1}{12} - \operatorname{Im}(\rho) \le |\rho|^2 - \operatorname{Re}\rho$$

for $A_{\rho,(x^2-x)}$ to be dissipative. This condition is for example satisfied by $\rho = \frac{1}{2} + \frac{3}{8}i$, i.e. $A_{\left(\frac{1}{2}+\frac{3}{8}i\right),(x^2-x)}$ is dissipative, while $A_{\left(\frac{1}{2}+\frac{3}{8}i\right),\phi=0}$ is not. In Corollary 9.1.15, we will show that the phenomenon that we have a dissipative non-proper extension, defined on a domain on which the corresponding proper extension would not be dissipative, can only occur if V is not essentially selfadjoint.

Next, let us consider the special case that $\operatorname{ran}(\mathcal{L}) \subset \operatorname{ran}(V_F)$.

Corollary 9.1.13. *Let $(A, \widetilde{A})$ be dual pair satisfying the common core property, where A is dissipative. Moreover, assume that $\operatorname{ran}(\mathcal{L}) \subset \operatorname{ran}(V_F)$. In this case, we write $\mathcal{L}v = V_F\phi_v$, where $\phi_v \in \mathcal{D}(V_F)$. Then, $A_{\mathcal{V},\mathcal{L}}$ is dissipative if and only if $\mathcal{V} \subset \mathcal{D}(V_K^{1/2})$ and for all $v \in \mathcal{V}$, we have that*

$$Im\langle v, \widetilde{A}^* v\rangle + Im\langle v, V_F\phi_v\rangle \ge \frac{1}{4}\|V_K^{1/2}(\phi_v + 2iv)\|^2 . \tag{9.1.14}$$

Proof. Since $\operatorname{ran}(\mathcal{L}) \subset \operatorname{ran}(V_F) \subset \operatorname{ran}(V_F^{1/2})$ by assumption, it follows from Lemma 9.1.6 that it is necessary that $\mathcal{V} \subset \mathcal{D}(V_K^{1/2})$ for $A_{\mathcal{V},\mathcal{L}}$ to be dissipative. Again, condition (9.1.14) follows from (9.1.9), where we substitute $\mathcal{L}v = V_F\phi_v$ to get

$$\operatorname{Im}\langle v, \widetilde{A}^* v\rangle + \operatorname{Im}\langle v, V_F\phi_v\rangle \geq \frac{1}{4}\|\mathcal{U}V_F^{-1/2}V_F\phi_v + 2iV_K^{1/2}v\|^2 = \frac{1}{4}\|V_K^{1/2}(\phi_v + 2iv)\|^2 ,$$

which is the desired result. $\square$

Example 9.1.14. Let $\mathcal{H} = L^2(0,\infty)$ and consider the dual pair of closed operators $(A, \widetilde{A})$ given by

$$\begin{aligned} A: &\qquad \mathcal{D}(A) = H_0^2(0,\infty), \quad f \mapsto -if'' \\ \widetilde{A}: &\qquad \mathcal{D}(\widetilde{A}) = H_0^2(0,\infty), \quad f \mapsto if'' , \end{aligned}$$

which has the common core property since $\mathcal{D}(A) = \mathcal{D}(\widetilde{A})$ and $(A, \widetilde{A})$ are closed. Their adjoints are given by

$$\begin{aligned} \widetilde{A}^*: &\qquad \mathcal{D}(\widetilde{A}^*) = H^2(0,\infty), \quad f \mapsto -if'' \\ A^*: &\qquad \mathcal{D}(A^*) = H^2(0,\infty), \quad f \mapsto if'' . \end{aligned}$$

Moreover, the "imaginary part" V and its adjoint V^* are given by

$$\begin{aligned} V: &\qquad \mathcal{D}(V) = H_0^2(0,\infty), \quad f \mapsto -f'' \\ V^*: &\qquad \mathcal{D}(V^*) = H^2(0,\infty), \quad f \mapsto -f'' . \end{aligned}$$

As $A = iV$ and $\widetilde{A} = -iV$, we get that $\widetilde{A}^* = iV^*$ and $A^* = -iV^*$. Since

$$\ker(V^* \pm i) = \operatorname{span}\left\{\exp\left(-\frac{1 \pm i}{\sqrt{2}}x\right)\right\} ,$$

and

$$\mathcal{D}(\widetilde{A}^*) = \mathcal{D}(V^*) = \mathcal{D}(V) \dotplus \ker(V^* + i) \dotplus \ker(V^* - i) = \mathcal{D}(A) \dotplus \ker(V^* + i) \dotplus \ker(V^* - i) ,$$

we may choose

$$\begin{aligned} \mathcal{D}(V^*)//\mathcal{D}(V) &= \mathcal{D}(\widetilde{A}^*)//\mathcal{D}(A) \\ &= \operatorname{span}\left\{\exp\left(-\frac{1+i}{\sqrt{2}}x\right), \exp\left(-\frac{1-i}{\sqrt{2}}x\right)\right\} = \operatorname{span}\{\sigma, \tau\} . \end{aligned}$$

The functions σ and τ are suitable linear combinations of the elements of $\mathcal{D}(\widetilde{A}^*)//\mathcal{D}(A)$ such that $\sigma(0) = \tau'(0) = 1$ and $\sigma'(0) = \tau(0) = 0$. For $\rho \in \mathbb{C}$, define the function $\zeta_\rho(x) := \sigma(x) + \rho\tau(x)$ and let $\zeta_\infty(x) := \tau(x)$. In order to be able to use Corollary 9.1.13, we will only consider $\mathcal{L}\zeta_\rho \in \operatorname{ran}(V_F)$, i.e. we can write $\mathcal{L}\zeta_\rho = V_F\phi$ for some $\phi \in \mathcal{D}(V_F) = \{f \in H^2(0,\infty), f(0) = 0\}$. Thus, as in Example 9.1.12, let us use the parameter $\rho \in \mathbb{C} \cup \{\infty\}$ and the function $\phi \in \mathcal{D}(V_F)$ to describe all extensions $A_{\rho,\phi}$ of the form

$$A_{\rho,\phi}: \qquad \begin{aligned} &\mathcal{D}(A_{\rho,\phi}) = \mathcal{D}(A)\dot{+}\operatorname{span}\{\zeta_\rho\} \\ &f + \lambda\zeta_\rho \mapsto -i(f'' + \lambda\zeta_\rho'') - \lambda\phi''\,, \end{aligned}$$

where $f \in \mathcal{D}(A)$ and $\lambda \in \mathbb{C}$. Next, let us use Corollary 9.1.13 to find the conditions on ρ and ϕ for $A_{\rho,\phi}$ to be dissipative. Firstly, observe that V_K is the Neumann-Laplacian on the half-line. This can be seen from

$$\langle f, V^*f\rangle = \overline{f(0)}f'(0) + \int_0^\infty |f'(x)|^2 \mathrm{d}x \tag{9.1.15}$$

for all $f \in \mathcal{D}(V^*)$. In order to find the selfadjoint restrictions of V^*, observe that any additional selfadjoint boundary condition has to be of the form $f'(0) = rf(0)$, where $r \in \mathbb{R}$. The choice $r = \infty$ corresponds to a Dirichlet condition at 0, i.e. $f(0) = 0$ and describes the Friedrichs extension of V. For any $r < 0$, we get that $\langle f, V^*f\rangle$ can be made negative, which therefore does not describe a non-negative selfadjoint extension of V. For $r \geq 0$, it is obvious that $r = 0$ describes the smallest non-negative extension of V. Hence, the Kreĭn–von Neumann extension is given by the Neumann-Laplacian with domain $\mathcal{D}(V_K) = \{f \in H^2(0,\infty), f'(0) = 0\}$. It is also not hard to see that if we close $\mathcal{D}(V_K)$ with respect to the norm induced by (9.1.15), we get $\mathcal{D}(V_K^{1/2}) = H^1(0,\infty)$. Now, since $\mathcal{D}(\widetilde{A}^*)//\mathcal{D}(A) \subset H^1(0,\infty) = \mathcal{D}(V_K^{1/2})$, we get that the first necessary condition for $A_{\rho,\phi}$ to be dissipative is satisfied for any $\rho \in \mathbb{C} \cup \{\infty\}$. Next, let us determine for which $\rho \in \mathbb{C} \cup \{\infty\}$ and $\phi \in \mathcal{D}(V_F)$ Condition (9.1.14) is satisfied. For $\rho \in \mathbb{C}$, it reads as

$$
\begin{aligned}
&\mathrm{Im}\langle \zeta_\rho, -i\zeta_\rho''\rangle + \mathrm{Im}\langle \zeta_\rho, -\phi''\rangle \geq \frac{1}{4}\|\phi' + 2i\zeta_\rho'\|^2 = \frac{1}{4}\|\phi'\|^2 + \|\zeta_\rho'\|^2 + \mathrm{Re}\langle \phi', i\zeta_\rho'\rangle \\
\Leftrightarrow &\mathrm{Im}(\overline{\zeta_\rho(0)} i\zeta_\rho'(0)) + \|\zeta_\rho'\|^2 + \mathrm{Im}\langle \phi'', \zeta_\rho\rangle \geq \frac{1}{4}\|\phi'\|^2 + \|\zeta_\rho'\|^2 + \mathrm{Im}\langle \phi'', \zeta_\rho\rangle + \mathrm{Im}(\overline{\phi'(0)}\zeta_\rho(0)) \\
\Leftrightarrow &\mathrm{Re}\rho \geq \frac{1}{4}\|\phi'\|^2 - \mathrm{Im}(\phi'(0))\,.
\end{aligned}
$$

For $\rho = \infty$, we get the condition that

$$0 \geq \frac{1}{4}\|\phi'\|^2\,,$$

which means that the only allowed choice is $\phi(x) \equiv 0$ in this case.

Finally, let us consider the case that V is essentially selfadjoint.

Corollary 9.1.15. *Let $(A, \widetilde{A})$ be dual pair satisfying the common core property, where A is dissipative. Moreover, let the imaginary part V be essentially selfadjoint. Then, $A_{\mathcal{V},\mathcal{L}}$ is dissipative if and only if $\mathcal{V} \subset \mathcal{D}(\overline{V}^{1/2})$, ran$(\mathcal{L}) \subset$ ran$(\overline{V}^{1/2})$ and for all $v \in \mathcal{V}$ we have that*

$$Im\langle v, \widetilde{A}^* v\rangle \geq \frac{1}{4}\|\overline{V}^{-1/2}\mathcal{L}v\|^2 + \|\overline{V}^{1/2}v\|^2\,. \tag{9.1.16}$$

In particular, this implies that for $A_{\mathcal{V},\mathcal{L}}$ to be dissipative, it is necessary that $A_{\mathcal{V}}$ is dissipative.

Proof. The conditions that $\mathcal{V} \subset \mathcal{D}(\overline{V}^{1/2})$ and $\mathrm{ran}(\mathcal{L}) \subset \mathrm{ran}(\overline{V}^{1/2})$ for $A_{\mathcal{V},\mathcal{L}}$ to be dissipative follow from Lemma 9.1.6. Condition (9.1.16) follows from (9.1.9) using that $V_K = V_F = \overline{V}$, which implies that $\mathcal{U}$ acts like the identity on $\overline{\mathrm{ran}(\overline{V}^{1/2})}$. Moreover, by Theorem 5.2.8, $A_{\mathcal{V}}$ is dissipative if and only if $\mathcal{V} \subset \mathcal{D}(\overline{V}^{1/2})$ and $\mathrm{Im}\langle v, \widetilde{A}^* v\rangle \geq \|\overline{V}^{1/2}v\|^2$ for all $v \in \mathcal{V}$. Thus, if $A_{\mathcal{V}}$ is not dissipative then it is either true that $\mathcal{V} \not\subset \mathcal{D}(\overline{V}^{1/2})$ or we have $\mathcal{V} \subset \mathcal{D}(\overline{V}^{1/2})$ but there exists a $v \in \mathcal{V}$ such that

$$\mathrm{Im}\langle v, \widetilde{A}^* v\rangle - \|\overline{V}^{1/2}v\|^2 < 0\,.$$

In the first case, $A_{\mathcal{V},\mathcal{L}}$ would not be dissipative, since by what we have just shown, it is necessary that $\mathcal{V} \subset \mathcal{D}(\overline{V}^{1/2})$ for $A_{\mathcal{V},\mathcal{L}}$ to be dissipative. In the second case, Condition (9.1.16) would read as

$$0 > \mathrm{Im}\langle v, \widetilde{A}^* v\rangle - \|\overline{V}^{1/2}v\|^2 \geq \frac{1}{4}\|\overline{V}^{-1/2}\mathcal{L}v\|^2\,,$$

which is impossible. This shows the corollary. $\square$

EXAMPLE 9.1.16. Let $0 < \gamma < \frac{1}{2}$ and let $(A, \widetilde{A})$ be the dual pair as discussed in Example 2.4.4, where we have chosen $\mathcal{D} = \mathcal{C}_c^\infty(0,1)$ to be the common core. Recall that we have $\mathcal{D}(\widetilde{A}^*)//\mathcal{D}(A) = \operatorname{span}\{x^{-\gamma}, x^{\gamma+1}\}$ and that the imaginary part V is the multiplication operator by the function $\frac{\gamma}{x}$ which has closure to the selfadjoint maximal multiplication operator by $\frac{\gamma}{x}$. As argued before, $x^{-\gamma} \notin \mathcal{D}(\overline{V}^{1/2})$, which means that the only choice for $\mathcal{V} \subset \operatorname{span}\{x^{-\gamma}, x^{\gamma+1}\}$ in order to have a chance for $A_{\mathcal{V},\mathcal{L}}$ to be dissipative is $\mathcal{V} := \operatorname{span}\{x^{\gamma+1}\}$. Let us define $v(x) := x^{\gamma+1}$ and $\mathcal{L}v =: \ell \in \mathcal{H}$ and let us use the functions v and ℓ to label $A_{\mathcal{V},\mathcal{L}} := A_{v,\ell}$. Since $\langle f, \overline{V} f\rangle \geq \|f\|^2$ for all $f \in \mathcal{D}(\overline{V})$, we get that $\overline{V}$ and $\overline{V}^{1/2}$ are both boundedly invertible, in particular that $\operatorname{ran}(\overline{V}^{1/2}) = \mathcal{H}$. Thus, by Corollary 9.1.15, it only remains to check whether Condition (9.1.16) is satisfied, which reads as

$$\operatorname{Im}\langle v, \widetilde{A}^* v\rangle - \|\overline{V}^{1/2} v\|^2 \geq \frac{1}{4}\|\overline{V}^{-1/2}\ell\|^2 \,.$$

In (5.4.9), we have already shown that

$$\operatorname{Im}\langle v, \widetilde{A}^* v\rangle - \|\overline{V}^{1/2} v\|^2 \overset{(5.4.9)}{=} \frac{1}{2}\left(|v(1)|^2 - |v(0)|^2\right) = \frac{1}{2}\,.$$

Hence, $A_{v,\ell}$ is dissipative if and only if

$$\|\overline{V}^{\,-1/2}\ell\|^2 = \frac{1}{\gamma}\int_0^1 x|\ell(x)|^2 \mathrm{d}x \leq 2\,.$$

This means that all dissipative extension of A that have domain contained in $\mathcal{D}(\widetilde{A}^*)$ are given by

$$A_{v,\ell}: \qquad \mathcal{D}(A_{v,\ell}) = \mathcal{D}(A)\dot{+}\operatorname{span}\{v\}$$

$$(A_{v,\ell}(f+\lambda v))(x) = if'(x) + i\lambda v'(x) + i\gamma\frac{f(x)+\lambda v(x)}{x} + \lambda\ell(x)\,, \tag{9.1.17}$$

where $f \in \mathcal{D}(A)$ and $\lambda \in \mathbb{C}$. The function $\ell \in L^2(0,1)$ has to satisfy

$$\int_0^1 x|\ell(x)|^2 \mathrm{d}x \leq 2\gamma\,. \tag{9.1.18}$$

Moreover, by Lemma 2.3.8, we have that $A_{v,\ell}$ is maximally dissipative since it is a one-dimensional extension of A.

9.2. Application: Operators with bounded imaginary part

Let us apply the result of Corollary 9.1.15 in order to construct all dissipative extensions of a dissipative operator with bounded imaginary part. To start with, let us show that it is sufficient to only consider operators of the form $S + iV$, where S is symmetric and $V \geq 0$ is bounded:

Lemma 9.2.1. *Let A be a dissipative operator and assume that the quadratic form q given by*

$$q: \quad \mathcal{D}(q) = \mathcal{D}(A), \quad f \mapsto Im\langle f, Af\rangle$$

is bounded. Then there exists a unique symmetric operator S with $\mathcal{D}(S) = \mathcal{D}(A)$ and a unique selfadjoint bounded operator $V \geq 0$ such that $A = S + iV$.

Proof. Firstly, observe that the quadratic form q is closable and let V be the selfadjoint operator associated to the closure of q. Define the operator $S = A - iV$ with $\mathcal{D}(S) = \mathcal{D}(A)$, which is symmetric:

$$\mathrm{Im}\langle f, Sf\rangle = \mathrm{Im}\langle f, (A - iV)f\rangle = \mathrm{Im}\langle f, Af\rangle - \langle f, Vf\rangle = 0\,.$$

Thus, $A = S + iV$, where $\mathcal{D}(S) = \mathcal{D}(A)$.
To show that this choice is unique, assume that there exists another symmetric S' with $\mathcal{D}(S') = \mathcal{D}(A)$ and a bounded $V' \geq 0$, such that $A = S' + iV'$. Then, for all $f \in \mathcal{D}(A)$, we get

$$\langle f, [(S - S') + i(V - V')]\, f\rangle = 0\,,$$

which means in particular that for all $f \in \mathcal{D}(A)$, we have:

$$\mathrm{Re}\langle f, ((S - S') + i(V - V'))f\rangle = \langle f, (S - S')f\rangle = 0$$

and

$$\mathrm{Im}\langle f, ((S - S') + i(V - V'))f\rangle = \langle f, (V - V')f\rangle = 0\,.$$

By Lemma 9.3.8, we get that $S = S'$ and $V \restriction_{\mathcal{D}(A)} = V' \restriction_{\mathcal{D}(A)}$ and as V and V' coincide on a dense set, so do their unique continuous extensions to $\mathcal{H}$. This shows the lemma. $\square$

Next, let us show that for *any* dissipative extension of $S + iV$, it is necessary that its domain is contained in $\mathcal{D}(S^*)$:

LEMMA 9.2.2. *Let $A := S + iV$, where S is symmetric and $V \geq 0$ is bounded. Then, for an extension $A \subset B$ to be dissipative, it is necessary that $\mathcal{D}(B) \subset \mathcal{D}(S^*)$.*

PROOF. Assume that $\mathcal{D}(B) \not\subset \mathcal{D}(S^*)$, i.e. that there exists a $v \in \mathcal{D}(B)$ such that $v \notin \mathcal{D}(S^*)$. For any $f \in \mathcal{D}(A) = \mathcal{D}(S)$, consider

$$\begin{aligned} \mathrm{Im}\langle f+v, B(f+v)\rangle &= \mathrm{Im}\langle f, (S+iV)f\rangle + \mathrm{Im}\langle v, (S+iV)f\rangle + \mathrm{Im}\langle f+v, Bv\rangle \\ &= \langle f, Vf\rangle + \mathrm{Im}\langle v, Sf\rangle + \mathrm{Im}\langle v, iVf\rangle + \mathrm{Im}\langle f+v, Bv\rangle \\ &\leq \|V\|\|f\|^2 + \mathrm{Im}\langle v, Sf\rangle + \|V\|\|v\|\|f\| + \|f\|\|Bv\| + \|v\|\|Bv\| \,. \end{aligned} \tag{9.2.1}$$

Since $v \notin \mathcal{D}(S^*)$, there exists a normalized sequence $\{f_n\}_n \subset \mathcal{D}(S)$ such that

$$\lim_{n\to\infty} \mathrm{Im}\langle v, Sf\rangle = -\infty \,.$$

Using (9.2.1), we therefore get

$$\mathrm{Im}\langle f_n+v, B(f_n+v)\rangle \leq \|V\| + \|V\|\|v\| + \|Bv\| + \|v\|\|Bv\| + \mathrm{Im}\langle v, Sf_n\rangle \overset{n\to\infty}{\longrightarrow} -\infty \,,$$

which shows that B cannot be dissipative in this case. This finishes the proof. $\square$

We are now able to describe all dissipative extensions of $A = S + iV$:

THEOREM 9.2.3. *Let $A = S + iV$ be a dissipative operator with bounded imaginary part. Moreover, let $\mathcal{V} \subset \mathcal{D}(S^*)//\mathcal{D}(S)$ and let $\mathcal{L}$ be a linear map from $\mathcal{V}$ into $\mathcal{H}$. Define the operator $S_{\mathcal{V},\mathcal{L}}$ via*

$$\begin{aligned} S_{\mathcal{V},\mathcal{L}}: \quad \mathcal{D}(S_{\mathcal{V},\mathcal{L}}) &= \mathcal{D}(S)\dot{+}\mathcal{V} \\ S_{\mathcal{V},\mathcal{L}}(f+v) &= S^*(f+v) + \mathcal{L}v \,, \end{aligned} \tag{9.2.2}$$

where $f \in \mathcal{D}(S)$ and $v \in \mathcal{V}$. Then $S_{\mathcal{V},\mathcal{L}} + iV$ is a dissipative extension of $S + iV$ if and only if for all $v \in \mathcal{V}$ we have that $\mathcal{L}v \in ran(V^{1/2})$ and the condition

$$Im\langle v, S^*v\rangle \geq \frac{1}{4}\|V^{-1/2}\mathcal{L}v\|^2 \tag{9.2.3}$$

is satisfied. As before, $V^{-1/2}$ denotes the inverse of $V^{1/2}$ on the reducing subspace $\overline{\mathrm{ran}(V^{1/2})}$. *Moreover,* all *dissipative extensions of $S + iV$ are of this form.*

Proof. Since V is bounded, $S_{\mathcal{V},\mathcal{L}}$ is an extension of S if and only if $A_{\mathcal{V},\mathcal{L}} = S_{\mathcal{V},\mathcal{L}}+iV$ is an extension of $A := S + iV$. Clearly, for $A := S + iV$ and $\widetilde{A} := S - iV$, we have that $(A, \widetilde{A})$ is a dual pair and we get that $\mathcal{D}(A) = \mathcal{D}(\widetilde{A}) = \mathcal{D}(S)$, which means that it has the common core property. Moreover, by boundedness of V, we get that $\widetilde{A}^* = S^* + iV$, where $\mathcal{D}(\widetilde{A}^*) = \mathcal{D}(S^*)$. Also, observe that $V \restriction_{\mathcal{D}(S)}$ is essentially selfadjoint, which means that we can apply Corollary 9.1.15. Since V is bounded, we have that $\mathcal{D}(V^{1/2}) = \mathcal{D}(V) = \mathcal{H}$, which means that the Condition that $\mathcal{V} \subset \mathcal{D}(V^{1/2})$ is always satisfied. Thus, by Corollary 9.1.15, it is necessary that $\operatorname{ran}(\mathcal{L}) \subset \operatorname{ran}(V^{1/2})$ for $A_{\mathcal{V},\mathcal{L}}$ to be dissipative. Condition (9.1.16) reads as

$$\operatorname{Im}\langle v, (S^* + iV)v\rangle \geq \|V^{1/2}v\|^2 + \frac{1}{4}\|V^{-1/2}\mathcal{L}v\|^2 \Leftrightarrow \operatorname{Im}\langle v, S^*v\rangle \geq \frac{1}{4}\|V^{-1/2}\mathcal{L}v\|^2 ,$$

which is the desired result. Let us finish by showing that *all* dissipative extensions of $S + iV$ are parametrized by the operators $S_{\mathcal{V},\mathcal{L}} + iV$. By Lemma 9.2.2, we know that all dissipative extensions have domain contained in $\mathcal{D}(S^*) = \mathcal{D}(\widetilde{A}^*)$. On the other hand, since $\mathcal{V}$ is an arbitrary subspace of $\mathcal{D}(S^*)//\mathcal{D}(S)$, the extensions $S_{\mathcal{V},\mathcal{L}}$ as defined in (9.2.2) describe all possible extensions of S that have domain contained in $\mathcal{D}(S^*)$. As they are dissipative if and only if $\mathcal{V}$ and $\mathcal{L}$ satisfy the assumptions of this Theorem, we have found all dissipative extensions of $(S + iV)$. □

Remark 9.2.4. This is not a new result, it was first shown by Crandall and Phillips [**14**, Theorem 1].

From Condition (9.2.3), we see that if $\operatorname{Im}\langle v, S^*v\rangle \equiv 0$ on $\mathcal{V}$, it is necessary that $\mathcal{L}v = 0$ for all $v \in \mathcal{V}$:

Corollary 9.2.5. *Let* $\mathcal{V} \subset \mathcal{D}(S^*)//\mathcal{D}(S)$.

a) If $S_{\mathcal{V}}$ *is symmetric, then* $(S_{\mathcal{V}} + iV)$ *is the only dissipative extension of* $(S + iV)$ *with domain equal to* $\mathcal{D}(S_{\mathcal{V}})$*. Moreover, the imaginary part of any other extension of the form* $(S_{\mathcal{V},\mathcal{L}} + iV)$ *is not bounded from below, i.e. for* $\mathcal{L} \neq 0$*, there exists no* $\gamma \in \mathbb{R}^+$ *such that*

$$\inf_{\psi\in\mathcal{D}(S_{\mathcal{V},\mathcal{L}}):\|\psi\|=1} Im\langle\psi, (S_{\mathcal{V},\mathcal{L}} + iV)\psi\rangle \geq -\gamma\|\psi\|^2 . \tag{9.2.4}$$

b) On the other hand, if there exists an $\varepsilon > 0$ such that

$$Im\langle v, S_{\mathcal{V}} v\rangle \geq \varepsilon \|v\|^2$$

for all $v \in \mathcal{V}$ and if the operator $\mathcal{L}$ is bounded, we get that

$$Im\langle \psi, S_{\mathcal{V},\mathcal{L}} \psi\rangle \geq -\frac{\|\mathcal{L}\|^2}{4\varepsilon}\|\psi\|^2 \tag{9.2.5}$$

for all $\psi \in \mathcal{D}(S_{\mathcal{V},\mathcal{L}})$. This implies in particular that for any bounded $V \geq \frac{\|\mathcal{L}\|^2}{4\varepsilon}$, we get

$$Im\langle \psi, (S_{\mathcal{V},\mathcal{L}} + iV)\psi\rangle \geq 0$$

for all $\psi \in \mathcal{D}(S_{\mathcal{V},\mathcal{L}})$.

Proof. a) By Theorem 9.2.3, Condition (9.2.3), it is necessary that

$$\mathrm{Im}\langle v, S^* v\rangle \geq \frac{1}{4}\|V^{-1/2}\mathcal{L}v\|^2$$

for all $v \in \mathcal{V}$. But since $S_{\mathcal{V}} = S^* \restriction_{\mathcal{D}(S_{\mathcal{V}})}$ is symmetric, we get $\mathrm{Im}\langle v, S^* v\rangle = 0$ for all $v \in \mathcal{V}$, which makes it necessary that $\mathcal{L}v = 0$ for all $v \in \mathcal{V}$ for $(S_{\mathcal{V},\mathcal{L}} + iV)$ to be dissipative. In other words, only for $\mathcal{L} \equiv 0$ do we have that $A_{\mathcal{V},\mathcal{L}=0} = (S_{\mathcal{V},\mathcal{L}=0} + iV)$ is dissipative. For the second part of a), assume that the imaginary part of $A_{\mathcal{V},\mathcal{L}}$ is semibounded with semibound $-\gamma$ (cf. (9.2.4)). This would mean that the operator $S_{\mathcal{V},\mathcal{L}} + i(V+\gamma)$ is dissipative, which by Condition (9.2.3) means that for all $v \in \mathcal{V}$, it is necessary that

$$0 = \mathrm{Im}\langle v, S^* v\rangle \geq \frac{1}{4}\|(V+\gamma)^{-1/2}\mathcal{L}v\|^2 \,,$$

which is impossible if $\mathcal{L} \neq 0$.

b) Assume now that there exists an $\varepsilon > 0$ such that $\mathrm{Im}\langle v, S_{\mathcal{V}} v\rangle = \mathrm{Im}\langle v, S^* v\rangle \geq \varepsilon\|v\|^2$ for all $v \in \mathcal{V}$. If $\mathcal{L} = 0$, (9.2.5) clearly holds with $\|\mathcal{L}\| = 0$. Now, let $\mathcal{L} \neq 0$. Again, by Condition (9.2.3) of Theorem 9.2.3, the operator $S_{\mathcal{V},\mathcal{L}} + i\frac{\|\mathcal{L}\|^2}{4\varepsilon}$ is dissipative if and only if

$$\mathrm{Im}\langle v, S^* v\rangle \geq \frac{1}{4}\left\|\left(\frac{\|\mathcal{L}\|^2}{4\varepsilon}\right)^{-1/2}\mathcal{L}v\right\|^2 \tag{9.2.6}$$

for all $v \in \mathcal{V}$. Since for all $v \in \mathcal{V}$ we may estimate

$$\frac{1}{4}\left\|\left(\frac{\|\mathcal{L}\|^2}{4\varepsilon}\right)^{-1/2}\mathcal{L}v\right\|^2 \leq \frac{4\varepsilon}{4\|\mathcal{L}\|^2}\|\mathcal{L}v\|^2 \leq \varepsilon\|v\|^2 \leq \mathrm{Im}\langle v, S^* v\rangle \,,$$

this proves that (9.2.6) is satisfied. Hence the operator $S_{\mathcal{V},\mathcal{L}}+i\frac{\|\mathcal{L}\|^2}{4\varepsilon}$ is dissipative, which is equivalent to

$$\mathrm{Im}\langle\psi, S_{\mathcal{V},\mathcal{L}}\psi\rangle \geq -\frac{\|\mathcal{L}\|^2}{4\varepsilon}\|\psi\|^2$$

for all $\psi \in \mathcal{D}(S_{\mathcal{V},\mathcal{L}})$. This finishes the proof. $\square$

EXAMPLE 9.2.6 (Schrödinger operator on the half-line). Let $\mathcal{H} = L^2(\mathbb{R}^+)$ and consider the closed symmetric operator S given by:

$$S : \qquad \mathcal{D}(S) = \{f \in H^2(\mathbb{R}^+) : f(0) = f'(0) = 0\}, \quad f \mapsto -f''\,.$$

Its adjoint is given by

$$S^* : \qquad \mathcal{D}(S^*) = H^2(\mathbb{R}^+), \quad f \mapsto -f''\,,$$

where in both cases, f'' denotes the second weak derivative of f. Since for any $f \in \mathcal{D}(S^*)$ we have

$$\mathrm{Im}\langle f, S^* f\rangle = -\mathrm{Im}\left(\int_0^\infty \overline{f(x)}f''(x)dx\right) = \mathrm{Im}\left(\overline{f(0)}f'(0)\right)\,,$$

this shows that all maximally dissipative extensions of S are parametrized by the boundary condition

$$\begin{aligned} S_h : \qquad \mathcal{D}(S_h) &= \{f \in H^2(\mathbb{R}^+) : f'(0) = hf(0)\} \\ f &\mapsto -f''\,, \end{aligned}$$

where $\mathrm{Im}(h) \geq 0$. Since S is symmetric, we may choose

$$\mathcal{D}(S^*)//\mathcal{D}(S) = \ker(S^* + i)\dot{+}\ker(S^* - i)\,.$$

Now pick $\eta_h \in \mathcal{D}(S^*)//\mathcal{D}(S)$ such that $\eta_h'(0) = h$ and $\eta_h(0) = 1$, which means that $\mathcal{D}(S_h) = \mathcal{D}(S)\dot{+}\mathrm{span}\{\eta_h\}$ with the understanding that $h = \infty$ corresponds to Dirichlet boundary conditions at the origin. This implies that

$$\mathrm{Im}\langle\eta_h, S^*\eta_h\rangle = \mathrm{Im}h\,, \tag{9.2.7}$$

where we introduce the convention that $\mathrm{Im}(\infty) = 0$ since S_∞ is selfadjoint. By Theorem 9.2.3, Condition (9.2.3), we get that for $h = \infty$ the only linear map $\mathcal{L}$ that describes a dissipative extension $S_{\mathcal{V}_\infty,\mathcal{L}}$ is given by $\mathcal{L} \equiv 0$, which corresponds to a proper dissipative extension. Here, $\mathcal{V}_\infty := \mathrm{span}\{\eta_\infty\}$. Hence, we will not treat this case anymore from now on. Now, for $h \neq \infty$, the map $\mathcal{L}$ from $\mathcal{V} = \mathrm{span}\{\eta_h\}$ has to be of the form $\mathcal{L}\eta_h = k$ for some $k \in \mathcal{H}$. Thus, any $f \in \mathcal{D}(S_h)$ can be written as $f = (f - f(0)\eta_h) + f(0)\eta_h$, where $(f - f(0)\eta_h) \in \mathcal{D}(S)$. This means that the operator $S_{\mathcal{V},\mathcal{L}}$ is given by

$$
\begin{aligned}
S_{\mathcal{V},\mathcal{L}}: \qquad & \mathcal{D}(S_{\mathcal{V},\mathcal{L}}) = \mathcal{D}(S_h) \\
& S_{\mathcal{V},\mathcal{L}} f = -f'' + f(0)k \,.
\end{aligned}
$$

Since $S_{\mathcal{V},\mathcal{L}}$ only depends on our choice of $h \in \mathbb{C}$ and $k \in \mathcal{H}$, let us use these two parameters to label $S_{\mathcal{V},\mathcal{L}} = S_{h,k}$. Let us now consider a few bounded dissipative perturbations:

- Let us start with a rank-one perturbation of the form $V = \alpha|\varphi\rangle\langle\varphi|$, where $\alpha > 0$ and $\|\varphi\| = 1$. Since $\mathrm{ran}\, V = \mathrm{ran}\, V^{1/2} = \mathrm{span}\{\varphi\}$, the first condition of Theorem 9.2.3 yields that $k \in \mathrm{span}\{\varphi\}$. Moreover, on $\mathrm{span}\{\varphi\}$, the operator $V^{-1/2}$ is given by $\varphi \mapsto \alpha^{-1/2}\varphi$. Thus, the second condition of Theorem 9.2.3 reads as

$$
\frac{1}{4}\|\alpha^{-1/2}\lambda\varphi\|^2 \leq \mathrm{Im}h \;\;\Leftrightarrow\;\; |\lambda|^2 \leq 4\alpha\mathrm{Im}h \,, \tag{9.2.8}
$$

  where we have parametrized $k = \lambda\varphi$. Thus, all (maximally) dissipative extensions of the operator

$$
\begin{aligned}
A: \quad & \mathcal{D}(A) = \{f \in H^2(\mathbb{R}^+) : f(0) = f'(0) = 0\} \\
& f \mapsto -f'' + i\alpha\langle\varphi, f\rangle\varphi
\end{aligned}
$$

  are given by the family of operators $A_{h,\lambda}$, where $|\lambda|^2 \leq 4\alpha\mathrm{Im}h$:

$$
\begin{aligned}
A_{h,\lambda}: \qquad & \mathcal{D}(A_{h,\lambda}) = \{f \in H^2(\mathbb{R}^+) : f'(0) = hf(0)\} \\
& f \mapsto -f'' + f(0)\lambda\varphi + i\alpha\varphi\langle\varphi, f\rangle \,.
\end{aligned}
$$

- Let us generalize the previous case to $V \geq 0$ being compact. In this case, V can be written as $V = \sum_{i=1}^{\infty}\alpha_i|\varphi_i\rangle\langle\varphi_i|$, where $\alpha_i \geq \alpha_{i+1} > 0$ for all $i \in \mathbb{N}$ and $\lim_{i\to\infty}\alpha_i = 0$. This implies that $V^{1/2} = \sum_{i=1}^{\infty}\alpha_i^{1/2}|\varphi_i\rangle\langle\varphi_i|$. Clearly, $\overline{\mathrm{ran}\, V} = \overline{\mathrm{span}\{\varphi_i\}_i}$, however, V is certainly not boundedly invertible on its range. Thus, the function $k \in \overline{\mathrm{ran}\, V}$ such that $\mathcal{L}\eta_h = k$ therefore has to satisfy the additional requirement

$$
\sum_{i=1}^{\infty}\frac{|\langle k, \varphi_i\rangle|^2}{\alpha_i} < \infty \,,
$$

which guarantees that $k \in \mathcal{D}(V^{-1/2})$. Now, the second condition of Theorem 9.2.3 reads as

$$\sum_{i=1}^{\infty} \frac{|\langle k, \varphi_i \rangle|^2}{\alpha_i} \leq 4\mathrm{Im} h \,. \tag{9.2.9}$$

We therefore have constructed the (maximally) dissipative extensions of the operator

$$A: \quad \mathcal{D}(A) = \{f \in H^2(\mathbb{R}^+) : f(0) = f'(0) = 0\}$$
$$f \mapsto -f'' + i \sum_{i=1}^{\infty} \alpha_i \varphi_i \langle \varphi_i, f \rangle \,,$$

which are given by the family of operators $A_{h,k}$:

$$A_{h,k}: \qquad \mathcal{D}(A_{h,\lambda}) = \{f \in H^2(\mathbb{R}^+) : f'(0) = hf(0)\}$$
$$f \mapsto -f'' + f(0)k + i \sum_{i=1}^{\infty} \lambda_i \varphi_i \langle \varphi_i, f \rangle \,,$$

where $k \in \overline{\operatorname{ran} V}$ satisfies (9.2.9).

- Now, let V be the multiplication operator by an a.e. non-negative function $V(x) \in L^\infty(\mathbb{R}^+)$. Clearly, $\overline{\operatorname{ran} V} = L^2(\operatorname{ess\,supp} V(x))$, where we recall that $\operatorname{ess\,supp} V(x)$ is the smallest closed subset of $\mathbb{R}^+$ such that $V(x) = 0$ a.e. in $\mathbb{R}^+ \setminus (\operatorname{ess\,supp} V(x))$. Hence, the first condition of Theorem 9.2.3 yields the requirement that $\operatorname{ess\,supp}(k(x)) \subset \operatorname{ess\,supp}(V(x))$. Next, $k \in \mathcal{D}(V^{-1/2})$ implies that k has to be such that

$$\int_{\operatorname{ess\,supp}(V(x))} \frac{|k(x)|^2}{V(x)} dx < \infty \,.$$

Lastly, the second condition of Theorem 9.2.3 reads as

$$\int_{\operatorname{ess\,supp}(V(x))} \frac{|k(x)|^2}{V(x)} dx \leq 4\mathrm{Im} h \,.$$

Thus, all (maximally) dissipative extensions of the operator

$$A: \quad \mathcal{D}(A) = \{f \in H^2(\mathbb{R}^+) : f(0) = f'(0) = 0\}$$
$$(Af)(x) = -f''(x) + iV(x)f(x)$$

are given by the family of operators $A_{h,k}$, where $k \in \mathcal{H}$ such that ess supp$(k(x)) \subset$ ess supp$(V(x))$ and

$$\int_{\text{ess supp}(V(x))} \frac{|k(x)|^2}{V(x)} dx \leq 4\mathrm{Im}h \, .$$

They are given by:

$$\begin{aligned} A_{h,k}: \qquad & \mathcal{D}(A_{h,\lambda}) = \{f \in H^2(\mathbb{R}^+) : f'(0) = hf(0)\} \\ & (A_{h,k}f)(x) = -f''(x) + f(0)k(x) + iV(x)f(x) \, . \end{aligned}$$

9.3. Dissipative extensions of operators with closable imaginary part

In this section, we are going to determine the dissipative extensions of dissipative operators A for which the form given by

$$\begin{aligned} \mathfrak{q}_A : \mathcal{D}(\mathfrak{q}_A) &= \mathcal{D}(A) \\ f &\mapsto \mathrm{Im}\langle f, Af\rangle \end{aligned} \tag{9.3.1}$$

is closable (cf. Definition 7.3.1). This is not always the case as the following counterexample illustrates:

Counterexample 9.3.1. Let $\mathcal{H} = L^2(0,1) \oplus L^2(0,\infty)$ and consider the operator

$$\begin{aligned} A: \qquad & \mathcal{D}(A) = \{(f_1, f_2) \in \mathcal{H} : f_1 \in H^1(0,1), f_1(0) = f_1(1) = 0, f_2 \in H^1(0,\infty)\} \\ & (f_1, f_2) \mapsto (-if_1', -if_2') \, . \end{aligned}$$

For any $(f_1, f_2) \in \mathcal{D}(A)$, we have — using integration by parts —

$$\mathrm{Im}\langle (f_1,f_2), A(f_1,f_2)\rangle = \mathrm{Im}\langle (f_1,f_2), (-if_1', -if_2')\rangle = \frac{|f_2(0)|^2}{2} \geq 0 \, ,$$

which means that A is dissipative. However, since we have added the first Hilbert space $L^2(0,1)$, the operator is *not* maximally dissipative. Now, the form given by

$$\mathfrak{q}_A((f_1,f_2)) := \mathrm{Im}\langle (f_1,f_2), A(f_1,f_2)\rangle = \frac{|f_2(0)|^2}{2}$$

is not closable. To see this, pick e.g. the sequence $g_n = (0, g_n^{(2)})$, where

$$g_n^{(2)}(x) = \begin{cases} 1 - nx & \text{for} \quad x \in [0, 1/n] \\ 0 & \text{for} \quad x > 1/n \, . \end{cases}$$

A calculation shows that $\|g_n\|^2 = \frac{1}{3n} \to 0$ and $\mathfrak{q}_A(g_n - g_m) = 0$ for all n, m. However, for any n we have that $\mathfrak{q}_A(g_n) = 1 \not\to 0$, which means that $\mathfrak{q}_A$ is not closable.

Remark 9.3.2. If A is dissipative and there exists an antidissipative $\widetilde{A}$ such that $(A, \widetilde{A})$ is a dual pair that has the common core property with common core $\mathcal{D}$, then the form $\mathfrak{q}_A \restriction_{\mathcal{D}}$ is closable since $\mathfrak{q}_A(f) = \langle f, \frac{1}{2i}(A - \widetilde{A}) \restriction_{\mathcal{D}} f\rangle$, where $\frac{1}{2i}(A - \widetilde{A}) \restriction_{\mathcal{D}}$ is symmetric (cf. [**26**, Thm. VI, 1.27]). Hence, in this case, we could consider extensions of the operator $A_0 := A \restriction_{\mathcal{D}}$ instead. Thus, if we have found a dissipative extension B_0 of A_0, where $B_0 \not\subset A$, we can use Lemma 5.2.1 and close B_0 to obtain a dissipative extension $B := \overline{B_0}$ of A.

Let A be dissipative and assume that it induces a strictly positive closable imaginary part. In the following, we will determine a necessary and sufficient condition for an extension $A \subset B$ to be dissipative.

Theorem 9.3.3. *Let A be dissipative and let $\mathfrak{q}_A$ be the quadratic form as defined in* (9.3.1). *Assume that $\mathfrak{q}_A$ is closable and that there exists an $\varepsilon > 0$ such that*

$$\mathfrak{q}_A(f) \geq \varepsilon \|f\|^2 \tag{9.3.2}$$

for all $f \in \mathcal{D}(A) = \mathcal{D}(\mathfrak{q}_A)$. Let V be the selfadjoint operator associated to the closure of $\mathfrak{q}_A$. Moreover, let W be the operator given by

$$\begin{aligned} W: \qquad \mathcal{D}(W) &= ran(V^{1/2} \restriction_{\mathcal{D}(A)}) \\ g &\mapsto AV^{-1/2} g\,. \end{aligned}$$

An extension $A \subset B$ is dissipative if and only if for any $v \in \mathcal{D}(B)//\mathcal{D}(A)$ we have $v \in \mathcal{D}(W^*)$ and it satisfies the inequality

$$Im\langle v, Bv\rangle \geq \frac{1}{4}\|(V^{-1/2}B - W^*)v\|^2\,. \tag{9.3.3}$$

Proof. We need to show that $\mathrm{Im}\langle f + v, B(f + v)\rangle \geq 0$ for any $f \in \mathcal{D}(A)$ and any $v \in \mathcal{D}(B)//\mathcal{D}(A)$. Using that for any $f \in \mathcal{D}(A)$, we get $\mathrm{Im}\langle f, Af\rangle = \|V^{1/2}f\|^2$, we can write

$$\begin{aligned}\operatorname{Im}\langle f+v, B(f+v)\rangle &= \operatorname{Im}\langle f, Af\rangle + \operatorname{Im}\langle v, Bv\rangle + \operatorname{Im}\langle f, Bv\rangle + \operatorname{Im}\langle v, Af\rangle \\ &= \|V^{1/2}f\|^2 + \operatorname{Im}\langle v, Bv\rangle + \operatorname{Im}\langle f, Bv\rangle + \operatorname{Im}\langle v, Af\rangle\,.\end{aligned}$$

Now, since $V \geq \varepsilon$, it follows that $V^{1/2}$ is boundedly invertible. Hence, for any $f \in \mathcal{D}(A) \subset \mathcal{D}(V^{1/2})$, there exists a unique $g \in \mathcal{H}$ such that $f = V^{-1/2}g$. Moreover, note that $\operatorname{ran}(V^{1/2} \restriction_{\mathcal{D}(A)})$ is dense in $\mathcal{H}$. This follows from the fact that $\operatorname{ran}(V^{1/2}) = \mathcal{H}$ and that for any $V^{1/2}f$, where $f \in \mathcal{D}(V^{1/2})$, there exists a sequence $\{f_n\}_n \subset \mathcal{D}(A)$ such that $V^{1/2}f_n \to V^{1/2}f$ since $\mathcal{D}(A)$ is a core for $V^{1/2}$. This means that W is a densely defined operator. Now, let us write

$$\begin{aligned}\operatorname{Im}\langle f+v, B(f+v)\rangle &= \operatorname{Im}\langle V^{-1/2}g+v, B(V^{-1/2}g+v)\rangle \\ &= \|V^{1/2}V^{-1/2}g\|^2 + \operatorname{Im}\langle v, Bv\rangle + \operatorname{Im}\langle V^{-1/2}g, Bv\rangle + \operatorname{Im}\langle v, AV^{-1/2}g\rangle \\ &= \|g\|^2 + \operatorname{Im}\langle v, Bv\rangle + \operatorname{Im}\langle g, V^{-1/2}Bv\rangle + \operatorname{Im}\langle v, Wg\rangle\,. \qquad (9.3.4)\end{aligned}$$

Assume that $v \notin \mathcal{D}(W^*)$. This means that the map $g \mapsto \langle v, Wg\rangle$ is an unbounded linear functional on $\mathcal{D}(W) = \operatorname{ran}(V^{1/2} \restriction_{\mathcal{D}(A)})$. Hence, there exists a normalized sequence $\{g_n\}_n \subset \mathcal{D}(W)$ such that $\operatorname{Im}\langle v, Wg_n\rangle \overset{n\to\infty}{\longrightarrow} -\infty$. Looking back at Equation (9.3.4), we see

$$\begin{aligned}&\|g_n\|^2 + \operatorname{Im}\langle v, Bv\rangle + \operatorname{Im}\langle g_n, V^{-1/2}Bv\rangle + \operatorname{Im}\langle v, Wg_n\rangle \\ \leq\,& 1 + \operatorname{Im}\langle v, Bv\rangle + \|V^{-1/2}Bv\| + \operatorname{Im}\langle v, Wg_n\rangle \overset{n\to\infty}{\longrightarrow} -\infty\,,\end{aligned}$$

which means that B cannot be dissipative in this case. Thus, suppose that for any $v \in \mathcal{D}(B)//\mathcal{D}(A)$, we have $v \in \mathcal{D}(W^*)$ from now on. Let us now show that if (9.3.3) is satisfied for all $v \in \mathcal{D}(B)//\mathcal{D}(A)$, we get that B is dissipative. We proceed to estimate (9.3.4):

$$\begin{aligned}&\|g\|^2 + \operatorname{Im}\langle v, Bv\rangle + \operatorname{Im}\langle g, V^{-1/2}Bv\rangle + \operatorname{Im}\langle v, Wg\rangle \\ \overset{(9.3.3)}{\geq}\,& \|g\|^2 + \frac{1}{4}\|(V^{-1/2}B - W^*)v\|^2 + \operatorname{Im}\langle W^*v, g\rangle - \operatorname{Im}\langle V^{-1/2}Bv, g\rangle \\ \geq\,& \|g\|^2 + \frac{1}{4}\|(V^{-1/2}B - W^*)v\|^2 - \|g\|\|(V^{-1/2}B - W^*)v\| \\ =\,& \left(\|g\| - \frac{1}{2}\|(V^{-1/2}B - W^*)v\|\right)^2 \geq 0\,,\end{aligned}$$

which shows that (9.3.3) is sufficient for B to be dissipative. Let us finish the proof by showing that it is also necessary. To this end, assume that there exists a $v \in \mathcal{D}(B)//\mathcal{D}(A)$ such that

$$\text{(9.3.5)} \qquad \operatorname{Im}\langle v, Bv\rangle - \frac{1}{4}\|(V^{-1/2}B - W^*)v\|^2 \leq -\varepsilon$$

for some $\varepsilon > 0$. Since $\mathcal{D}(W)$ is dense, we may pick a sequence $\{g_n\}_n \subset \mathcal{D}(W)$ such that $g_n \overset{n\to\infty}{\longrightarrow} \frac{-i}{2}(V^{-1/2}B - W^*)v$. Plugging this sequence into (9.3.4), we get

$$\begin{aligned}&\|g_n\|^2 + \operatorname{Im}\langle v, Bv\rangle + \operatorname{Im}\langle (W^* - V^{-1/2}B)v, g_n\rangle \\ \overset{n\to\infty}{\longrightarrow}& \operatorname{Im}\langle v, Bv\rangle - \frac{1}{4}\|(V^{-1/2}B - W^*)v\|^2 \overset{(9.3.5)}{\leq} -\varepsilon\,.\end{aligned}$$

This shows that B cannot be dissipative in this case either. This finishes the proof. $\square$

Note that even though by construction we have that $\mathcal{D}(A) \subset \mathcal{D}(V^{1/2})$, it is not in general true that $\mathcal{D}(A) \subset \mathcal{D}(V)$ as the following counterexample shows:

Counterexample 9.3.4. Let b be such that $\operatorname{Re}(b) \geq 0$ and $\operatorname{Im}(b) \neq 0$ and consider the maximally dissipative operator A on $\mathcal{H} = L^2(0,\infty)$ given by

$$\begin{aligned} A: \qquad \mathcal{D}(A) &= \{\psi \in H^2(0,\infty) : \psi'(0) = b\psi(0)\} \\ f &\mapsto -if'' + if\,. \end{aligned}$$

The quadratic form $\mathfrak{q}_A$ induced by the imaginary part is given by

$$\begin{aligned}\mathfrak{q}_A(\psi) = \operatorname{Im}\langle \psi, A\psi\rangle &= \operatorname{Im}\left(\int_0^\infty \overline{\psi(x)}\,(-i\psi''(x))\,\mathrm{d}x\right) + \|\psi\|^2 \\ &= \operatorname{Im}\left(\overline{\psi(0)}i\psi'(0) + i\int_0^\infty |\psi'(x)|\mathrm{d}x\right) + \|\psi\|^2 \\ &= \operatorname{Im}(ib)\cdot|\psi(0)|^2 + \|\psi'\|^2 + \|\psi\|^2 \\ &= \operatorname{Re}(b)\cdot|\psi(0)|^2 + \|\psi'\|^2 + \|\psi\|^2 \geq 0\end{aligned}$$

and since

$$\text{(9.3.6)} \quad \left||\psi(0)|^2\right| = \left|\int_0^\infty 2\operatorname{Re}(\overline{\psi(x)}\psi'(x))\mathrm{d}x\right| \leq 2\int_0^\infty |\psi(x)||\psi'(x)|\mathrm{d}x \leq \|\psi\|^2 + \|\psi'\|^2$$

we have that

$$\|\psi\|^2 + \|\psi'\|^2 \leq \mathfrak{q}_A(\psi) \overset{(9.3.6)}{\leq} (1 + \operatorname{Re}(b))\left(\|\psi\|^2 + \|\psi'\|^2\right)\,,$$

which means that the norm induced by $\mathfrak{q}_A$ is equivalent to the first Sobolev norm (it is in particular closable). Closing $\mathcal{D}(A)$ with respect to this norm just yields the first Sobolev space $H^1(0,\infty)$ and the selfadjoint operator V associated to the closed form is given by

$$V: \qquad \mathcal{D}(V_F) = \{v \in H^2(0,\infty) : \psi'(0) = \mathrm{Re}(b)\psi(0)\}$$
$$f \mapsto -f'' + f \, .$$

Hence, we have constructed an example, where $\mathcal{D}(A) \not\subset \mathcal{D}(V)$.

Even though the previous theorem provides us with a criterion for an extension to be dissipative under rather mild assumptions on the original operator, it seems rather difficult to apply it to concrete problems, since it can be quite difficult to compute the operator W. In the following, we are going to discuss two examples, where Theorem 9.3.3 can be used.

Example 9.3.5. As in Example 2.4.4, consider the dissipative operator A_0 on the Hilbert space $\mathcal{H} = L^2(0,1)$ given by

$$A_0: \quad \mathcal{D}(A_0) = \mathcal{C}_c^\infty(0,1), \quad (A_0 f)(x) = if'(x) + i\frac{\gamma}{x} f(x) \, ,$$

where $\gamma > 0$. As shown in Example 2.4.4, we have that its closure $A := \overline{A_0}$ has domain $\mathcal{D}(A) = H_0^1(0,1)$. Moreover, since $0 \in \widehat{\rho}(A)$ and $\dim \ker A^* = 1$, we have by Lemma 2.3.8 that all maximally dissipative extensions $\widehat{A}$ have to satisfy $\dim(\mathcal{D}(\widehat{A})/\mathcal{D}(A)) = 1$, i.e. there exists a $v \notin H_0^1(0,1)$ such that $\mathcal{D}(\widehat{A}) = \mathcal{D}(A) \dot{+} \mathrm{span}\{v\}$. Now, since $\mathcal{C}_c^\infty(0,1)$ is a core for A, we get from Lemma 5.2.1 that $\mathcal{C}_c^\infty(0,1) \dot{+} \mathrm{span}\{v\}$ is a core for $\widehat{A}$. Thus, we apply Theorem 9.3.3 to extensions B_0 of A_0, whose domain is of the form $\mathcal{D}(B_0) = \mathcal{D}(A_0) \dot{+} \mathrm{span}\{v\}$, where $v \notin H_0^1(0,1)$. If B_0 is dissipative, then we get that $B := \overline{B_0}$ is a maximally dissipative extension of A. We start by determining the "imaginary part" V. Since we have that $\mathfrak{q}_{A_0}$ is given by

$$\mathfrak{q}_{A_0}: \qquad \mathcal{D}(\mathfrak{q}_{A_0}) = \mathcal{C}_c^\infty(0,1)$$
$$f \mapsto \mathrm{Im}\langle f, A_0 f\rangle = \gamma \int_0^1 \frac{|f(x)|^2}{x} \mathrm{d}x \, ,$$

this means that V is given by the selfadjoint maximal multiplication operator by the function $\frac{\gamma}{x}$, which implies that $V^{-1/2}$ is the bounded selfadjoint operator of multiplication by $\sqrt{\frac{x}{\gamma}}$. In order to be able to apply Theorem 9.3.3, let us firstly determine $\mathcal{D}(W^*)$. To this end, observe that $\mathcal{D}(W) = \operatorname{ran}(V^{1/2} \restriction_{\mathcal{D}(A_0)}) = \mathcal{C}_c^\infty(0,1)$. This follows from the fact that $\mathcal{D}(A_0) = \mathcal{C}_c^\infty(0,1)$ and that $V^{1/2}$ — the operator of multiplication by $\sqrt{\frac{\gamma}{x}}$ — is a bijection from $\mathcal{C}_c^\infty(0,1)$ to $\mathcal{C}_c^\infty(0,1)$. We therefore get

$$
\begin{aligned}
W: \qquad & \mathcal{D}(W) = \mathcal{C}_c^\infty(0,1) \\
& (Wf)(x) = \left(i\frac{\mathrm{d}}{\mathrm{d}x} + i\frac{\gamma}{x}\right)\left(\sqrt{\frac{x}{\gamma}} f(x)\right) = \sqrt{\frac{x}{\gamma}}\left(if'(x) + i\frac{2\gamma+1}{2x} f(x)\right).
\end{aligned}
$$

Now, $v \in \mathcal{D}(W^*)$ means that the map

$$
f \mapsto \int_0^1 \overline{v(x)} \sqrt{\frac{x}{\gamma}}\left(if'(x) + i\frac{2\gamma+1}{2x} f(x)\right) \mathrm{d}x
$$

is a bounded linear functional on $\mathcal{C}_c^\infty(0,1)$. This implies that $v \in \mathcal{D}(W^*)$ if and only if $\left(v(x)\sqrt{\frac{x}{\gamma}}\right) \in \mathcal{D}(K^*)$, where K is the operator given by

$$
\mathcal{D}(K) = \mathcal{C}_c^\infty(0,1), \quad (Kf)(x) = if'(x) + i\frac{2\gamma+1}{2x} f(x).
$$

But K is of the same structure as the operator A_0, which means that we can consider the dual pair $(K, \widetilde{K}_0)$, where $\widetilde{K}_0$ is given by

$$
\mathcal{D}(\widetilde{K}_0) = \mathcal{C}_c^\infty(0,1), \quad (\widetilde{K}_0 f)(x) = if'(x) - i\frac{2\gamma+1}{2x} f(x).
$$

Using Proposition 2.4.3 for this dual pair for $\lambda = 0$, we get from completely analogous reasoning as in Example 2.4.4 that

$$
K^*: \quad \mathcal{D}(K^*) = \mathcal{D}(\widetilde{K}) \dot{+} \operatorname{span}\{x^{\gamma+\frac{1}{2}}\}, \quad (K^* f)(x) = if'(x) - i\frac{2\gamma+1}{2x} f(x). \tag{9.3.7}
$$

Here, $\widetilde{K}$ denotes the closure of the operator $\widetilde{K}_0$. However, note that since $\frac{2\gamma+1}{2} > \frac{1}{2}$, we have that $\dim(\mathcal{D}(K^*)/\mathcal{D}(\overline{K})) = 1$, since $x^{-\frac{2\gamma+1}{2}} \notin L^2(0,1)$ in this case. Thus we get

$$
W^*: \quad \mathcal{D}(W^*) = \left\{ v \in L^2(0,1) : (\sqrt{x} v(x)) \in \mathcal{D}(\widetilde{K}) \dot{+} \operatorname{span}\{x^{\gamma+\frac{1}{2}}\} \right\}
$$

$$
(W^* v)(x) = \left(i\frac{\mathrm{d}}{\mathrm{d}x} - i\frac{2\gamma+1}{2x}\right)\left(\sqrt{\frac{x}{\gamma}} v(x)\right) = i\left(\sqrt{\frac{x}{\gamma}} v(x)\right)' - i\frac{2\gamma+1}{2\sqrt{\gamma x}} v(x), \tag{9.3.8}
$$

where we assume in addition that $v \notin H_0^1(0,1)$ to ensure that $\mathcal{C}_c^\infty(0,1)\dot{+}\mathrm{span}\{v\}$ is a core of a maximally dissipative extension of A. Let us now denote $Bv =: \kappa$ and use $v \in \mathcal{D}(W^*)$, where $v \notin H_0^1(0,1)$ and $\kappa \in L^2(0,1)$ to parametrize all such extensions via

$$\begin{aligned} A_{0,v,\kappa}: \quad & \mathcal{D}(A_{0,v,\kappa}) = \mathcal{C}_c^\infty(0,1)\dot{+}\mathrm{span}\{v\} \\ & f + \lambda v \mapsto A_0 f + \lambda\kappa \,, \end{aligned}$$

where $f \in \mathcal{C}_c^\infty(0,1)$ and $\lambda \in \mathbb{C}$. Now, by (9.3.3), $A_{0,v,\kappa}$ is dissipative if and only if

$$\begin{aligned} & \mathrm{Im}\langle v, Bv\rangle \geq \frac{1}{4}\|(V^{-1/2}B - W^*)v\|^2 \\ & \Leftrightarrow \mathrm{Im}\langle v,\kappa\rangle \geq \frac{1}{4}\int_0^1 \left|\sqrt{\frac{x}{\gamma}}\kappa(x) - i\left(\sqrt{\frac{x}{\gamma}}v(x)\right)' + i\frac{2\gamma+1}{2\sqrt{\gamma x}}v(x)\right|^2 \mathrm{d}x \,. \end{aligned} \tag{9.3.9}$$

Hence, if $A_{0,v,\kappa}$ satisfies this condition, we can close it ($A_{v,\kappa} := \overline{A_{0,v,\kappa}}$) to obtain a maximally dissipative extension of A:

$$\begin{aligned} A_{v,\kappa}: \quad & \mathcal{D}(A_{v,\kappa}) = \mathcal{D}(A)\dot{+}\mathrm{span}\{v\} \\ & (f + \lambda v) \mapsto Af + \lambda\kappa \,, \end{aligned}$$

where $f \in \mathcal{D}(A) = H_0^1(0,1)$ and $\lambda \in \mathbb{C}$. Let us compare this result to Example 9.1.16, where we have constructed all dissipative extensions of A that have domain contained in $\mathcal{D}(\widetilde{A}^*)$. This meant that the only choice we had was adding $v(x) := x^{\gamma+1}$ to the domain of $\mathcal{D}(A)$. Firstly, observe that $v \in \mathcal{D}(W^*)$, since we already know by Example 9.1.16 that $\mathcal{D}(A)\dot{+}\mathrm{span}\{v\}$ is the domain of a dissipative extension of A and note that $v \notin H_0^1(0,1)$. From (9.1.17), we get that the $L^2(0,1)$-valued parameter ℓ characterizing the extensions $A_{v,\ell}$ is related to κ in the following way:

$$\kappa(x) = iv'(x) + \frac{i\gamma}{x}v(x) + \ell(x) = i(1+2\gamma)x^\gamma + \ell(x) \,. \tag{9.3.10}$$

Plugging (9.3.10) into Condition (9.3.9) yields the condition

$$\begin{aligned} & \mathrm{Im}\left(\int_0^1 \overline{v(x)}\left[iv'(x) + \frac{i\gamma}{x}v(x) + \ell(x)\right]\mathrm{d}x\right) \\ \geq & \frac{1}{4}\int_0^1 \left|\sqrt{\frac{x}{\gamma}}\left(iv'(x) + \frac{i\gamma}{x}v(x) + \ell(x)\right) - i\left(\sqrt{\frac{x}{\gamma}}v(x)\right)' + i\frac{2\gamma+1}{2\sqrt{\gamma x}}v(x)\right|^2 \mathrm{d}x \,, \end{aligned}$$

which — after a calculation — can be shown to be equivalent to the condition

$$\int_0^1 x|\ell(x)|^2 \mathrm{d}x \leq 2\gamma \,,$$

which is in accordance with (9.1.18).

As a second example, we apply Theorem 9.3.3 in order to determine the accretive extensions of a strictly positive symmetric operator.

Example 9.3.6. Let the dissipative operator A be of the form $A = iS$, where $S \geq \varepsilon > 0$ is a strictly positive and closed symmetric operator. Clearly, the form $\mathfrak{q}_A$ is closable, since for any $f \in \mathcal{D}(A) = \mathcal{D}(S)$, we have

$$\mathfrak{q}_A(f) = \operatorname{Im}\langle f, iSf\rangle = \langle f, Sf\rangle \, .$$

Closablility of $\mathfrak{q}_A$ follows from [**26**, Thm. VI, 1.27] and we get that $V = S_F$, where — as usual — S_F denotes the Friedrichs extension of S. Let us now determine W and W^*. The operator W is given by

$$W: \qquad \mathcal{D}(W) = \operatorname{ran}(S_F^{1/2} \upharpoonright_{\mathcal{D}(S)}) \\ f \mapsto iSS_F^{-1/2} f \, .$$

We now claim that its adjoint W^* is given by

$$W^*: \qquad \mathcal{D}(W^*) = \mathcal{D}(S_K^{1/2}) = \mathcal{D}(S_F^{1/2}) \dot{+} \ker S^* \\ f \mapsto -iS_F^{1/2}(\mathbb{1} - \mathcal{P}) f \, ,$$

where $\mathcal{P}$ is the projection onto $\ker S^*$ along $\mathcal{D}(S_F^{1/2})$ according to the decomposition $\mathcal{D}(S_K^{1/2}) = \mathcal{D}(S_F^{1/2}) \dot{+} \ker S^*$ as defined in (8.13). To see that this is true, let us start by assuming that $v \in \mathcal{D}(S_K^{1/2})$. For any $f \in \operatorname{ran}(S_F^{1/2} \upharpoonright_{\mathcal{D}(S)})$, we then get that

$$\langle v, Wf\rangle = \langle v, iSS_F^{-1/2} f\rangle = \langle v, iS_K^{1/2} S_K^{1/2} S_F^{-1/2} f\rangle = \langle -iS_K^{1/2} v, S_K^{1/2} S_F^{-1/2} f\rangle \\ = \langle -iS_K^{1/2}(\mathbb{1} - \mathcal{P})v, S_K^{1/2} S_F^{-1/2} f\rangle = \langle -iS_F^{1/2}(\mathbb{1} - \mathcal{P})v, S_F^{1/2} S_F^{-1/2} f\rangle = \langle -iS_F^{1/2}(\mathbb{1} - \mathcal{P})v, f\rangle \, ,$$

which shows that $\mathcal{D}(S_K^{1/2}) \subset \mathcal{D}(W^*)$ and that $W^* v = -iS_F^{1/2}(\mathbb{1} - \mathcal{P})v$ for $v \in \mathcal{D}(S_K^{1/2})$. Let us now show that $\mathcal{D}(W^*) \subset \mathcal{D}(S_K^{1/2})$. Assume that this is not true, i.e. that there exists a $v \in \mathcal{D}(W^*)$ such that $v \notin \mathcal{D}(S_K^{1/2})$. If $v \in \mathcal{D}(W^*)$, this means that there exists a $C < \infty$ such that for any $f \in \mathcal{D}(W) = \operatorname{ran}(S_F^{1/2} \upharpoonright_{\mathcal{D}(S)})$ we have

$$|\langle v, Wf\rangle| \leq C\|f\| \, . \tag{9.3.11}$$

Note that $|\langle v, Wf\rangle| = |\langle v, SS_F^{-1/2}f\rangle| = |\langle v, S_F^{1/2}f\rangle|$. Since $v \notin \mathcal{D}(S_K^{1/2})$, it now follows from Corollary 5.2.5, that there exists a normalized sequence $\{f_n\}_n \subset \operatorname{ran}(S_F^{1/2} \restriction_{\mathcal{D}(S)})$ such that $\lim_{n\to\infty} |\langle v, S_F^{1/2} f_n\rangle| = +\infty$. But this means that (9.3.11) cannot be satisfied in this case, which shows that $\mathcal{D}(W^*) \subset \mathcal{D}(S_K^{1/2})$. This means that for any extensions $A \subset B$ to be dissipative, it is necessary that $\mathcal{D}(B) \subset \mathcal{D}(S_K^{1/2})$. Let us now apply Condition (9.3.3) of Theorem 9.3.3 to see when such a B is dissipative. To this end, let $\mathcal{V} \subset \mathcal{D}(B)//\mathcal{D}(A)$. Since $S \geq \varepsilon$, we get that S_F is a bijection, which means that for any $v \in \mathcal{V}$ there exists a unique $\phi_v \in \mathcal{D}(S_F)$ such that $Bv = iS_F\phi_v$. We therefore parametrize all extensions of A using complementary subspaces $\mathcal{V} \subset \mathcal{D}(S_K^{1/2})$ and operators $\Phi : \mathcal{V} \to \mathcal{H}$ such that $\Phi v = \phi_v$ for all $v \in \mathcal{V}$. These extensions are given by

$$
\begin{aligned}
A_{\mathcal{V},\Phi}: \qquad & \mathcal{D}(A_{\mathcal{V},\Phi}) = \mathcal{D}(A)\dot{+}\mathcal{V} \\
& (f+v) \mapsto iSf + iS_F\phi_v \, ,
\end{aligned}
$$

where $f \in \mathcal{D}(A) = \mathcal{D}(S)$ and $v \in \mathcal{V}$. Plugged into (9.3.3), we get the condition

$$
\begin{aligned}
& \operatorname{Im}\langle v, iS_F\phi_v\rangle \geq \frac{1}{4}\|S_F^{-1/2}(iS_F)\phi_v + iS_F^{1/2}(\mathbb{1}-\mathcal{P})v\|^2 \\
\Leftrightarrow\ & \operatorname{Re}\langle v, S_F\phi_v\rangle \geq \frac{1}{4}\|S_F^{1/2}(\phi_v + (\mathbb{1}-\mathcal{P})v)\|^2 \\
\Leftrightarrow\ & \operatorname{Re}\langle v, S_F\phi_v\rangle \geq \frac{1}{4}\|S_K^{1/2}(\phi_v + v)\|^2 \, . \qquad (9.3.12)
\end{aligned}
$$

Let us apply this result to the operator $A = iS$ on $L^2(\mathbb{R}^+)$, where S is given by

$$
\begin{aligned}
S: \qquad & \mathcal{D}(S) = H_0^2(\mathbb{R}^+) \\
& f \mapsto -f'' + f \, .
\end{aligned}
$$

Clearly, S is symmetric and strictly positive: $S \geq \mathbb{1}$. It is not difficult to show that

$$
\begin{aligned}
\mathcal{D}(S_F^{1/2}) &= H_0^1(\mathbb{R}^+), \quad \|S_F^{1/2}f\|^2 = \|f\|^2 + \|f'\|^2 \\
\mathcal{D}(S_K^{1/2}) &= H^1(\mathbb{R}^+), \quad \|S_K^{1/2}f\|^2 = \|f\|^2 + \|f'\|^2 - |f(0)|^2 \, .
\end{aligned}
$$

Since $\ker S^*$ is one-dimensional, $\ker S^* = \operatorname{span}\{\exp(-x)\}$, we know by Lemma 2.3.8 that a one-dimensional dissipative extension of A will be maximally dissipative. Since $\mathcal{D}(W^*) = \mathcal{D}(S_K^{1/2}) = H^1(\mathbb{R}^+)$, we parametrize all one-dimensional extensions of A that have domain contained in $H^1(\mathbb{R}^+)$ using a $v \in H^1(\mathbb{R}^+)$ such that $v \notin H_0^2(\mathbb{R}^+)$ and a $\phi \in \mathcal{D}(S_F) = \{f \in H^2(\mathbb{R}^+) : f(0) = 0\}$ via

$$\begin{aligned} A_{v,\phi} : \qquad & \mathcal{D}(A_{v,\phi}) = H_0^2(\mathbb{R}^+) \dot{+} \mathrm{span}\{v\} \\ & (f + \lambda v) \mapsto -if'' + if + \lambda(-i\phi'' + i\phi) \, , \end{aligned}$$

where $f \in H_0^2(\mathbb{R}^+)$ and $\lambda \in \mathbb{C}$. Plugging this into (9.3.12), we see that $A_{v,\phi}$ is dissipative if and only if

$$\begin{aligned} \mathrm{Re}\langle v, -\phi'' + \phi\rangle &\geq \frac{1}{4}\|\phi' + v'\|^2 + \frac{1}{4}\|\phi + v\|^2 - \frac{|v(0)|^2}{4} \\ \Leftrightarrow \mathrm{Re}\left(\overline{v(0)}\phi'(0) + \langle v', \phi'\rangle + \langle v, \phi\rangle\right) &\geq \frac{1}{4}\|\phi' + v'\|^2 + \frac{1}{4}\|\phi + v\|^2 - \frac{|v(0)|^2}{4} \\ \Leftrightarrow \frac{|v(0)|^2}{4} + \mathrm{Re}\left(\overline{v(0)}\phi'(0)\right) &\geq \frac{1}{4}\|\phi' - v'\|^2 + \frac{1}{4}\|\phi - v\|^2 \, . \end{aligned}$$

Appendix

Lemma 9.3.7. *Let $\mathcal{M}$ and $\mathcal{N}$ be subspaces. Then it holds that $\mathcal{M}^\perp \cap \mathcal{N}^\perp = (\mathcal{M} + \mathcal{N})^\perp$. Moreover, $(\overline{\mathcal{M}} \cap \overline{\mathcal{N}})^\perp = \overline{\mathcal{M}^\perp + \mathcal{N}^\perp}$.*

Proof. Firstly, let us show that $\mathcal{M}^\perp \cap \mathcal{N}^\perp = (\mathcal{M} + \mathcal{N})^\perp$:
"$\subset$": If $\langle f, m\rangle = \langle f, n\rangle = 0$ for all $m \in \mathcal{M}$ and $n \in \mathcal{N}$, this implies in particular that $\langle f, m+n\rangle = 0$ for all $m + n \in \mathcal{M} + \mathcal{N}$.
"$\supset$": If $\langle f, m+n\rangle = 0$ for all $m \in \mathcal{M}$, $n \in \mathcal{N}$, this holds in particular for all elements of $\mathcal{M} + \mathcal{N}$, which are of the form $m + 0$ or $0 + n$.
Replacing $\mathcal{M} \Rightarrow \mathcal{M}^\perp$ and $\mathcal{N} \Rightarrow \mathcal{N}^\perp$ in the obtained result yields $\overline{\mathcal{M}} \cap \overline{\mathcal{N}} = (\mathcal{M}^\perp + \mathcal{N}^\perp)^\perp$, which — after taking orthogonal complements — gives

$$(\overline{\mathcal{M}} \cap \overline{\mathcal{N}})^\perp = \overline{\mathcal{M}^\perp + \mathcal{N}^\perp}. \tag{9.3.13}$$

□

Lemma 9.3.8. *Let Z be a densely defined operator on a complex Hilbert space $\mathcal{H}$ and assume that for all $f \in \mathcal{D}(Z)$, we have*

$$\langle f, Zf\rangle = 0 \,.$$

Then, Z is the zero operator with domain $\mathcal{D}(Z)$.

Proof. Firstly, observe that by [**43**, Thm. 4.18], Z is symmetric. Now, pick any two $f, g \in \mathcal{D}(Z)$ and a parameter $\lambda \in \mathbb{C}$ and consider

$$0 = \langle \lambda f + g, Z(\lambda f + g)\rangle = |\lambda|^2\langle f, Zf\rangle + \overline{\lambda}\langle f, Zg\rangle + \lambda\langle g, Zf\rangle + \langle g, Zg\rangle = 2\mathrm{Re}(\lambda\langle g, Zf\rangle) \,,$$

where by varying λ, we get that for all $f, g \in \mathcal{D}(Z)$, we have

$$\langle g, Zf\rangle = 0 \,.$$

Since $\mathcal{D}(Z)$ is dense, this implies that $Zf = 0$ for all $f \in \mathcal{D}(Z)$ and thus the lemma. □

REMARK 9.3.9. Note that this is not true for real Hilbert spaces. As a counterexample, consider $\mathcal{H} = \mathbb{R}^2$ and the rotation by $\frac{\pi}{2}$, given by

$$Z = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .$$

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