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Bousfield Localisations along Quillen Bifunctors

Javier J. Gutiérrez[1,3] ⓘ · Constanze Roitzheim[2]

Abstract Consider a Quillen adjunction of two variables between combinatorial model categories from $\mathcal{C} \times \mathcal{D}$ to $\mathcal{E}$, a set $\mathcal{S}$ of morphisms in $\mathcal{C}$ and a set $\mathcal{K}$ of objects in $\mathcal{C}$. We prove that there is a localised model structure $L_\mathcal{S}\mathcal{E}$ on $\mathcal{E}$, where the local objects are the $\mathcal{S}$-local objects in $\mathcal{E}$ described via the right adjoint. Dually, we show that there is a colocalised model structure $C_\mathcal{K}\mathcal{E}$ on $\mathcal{E}$, where the colocal equivalences are the $\mathcal{K}$-colocal equivalences in $\mathcal{E}$ described via the right adjoint. These localised and colocalised model structures generalise left and right Bousfield localisations of simplicial model categories, Barnes and Roitzheim's familiar model structures, and Barwick's enriched left and right Bousfield localisations.

✉ Javier J. Gutiérrez
javier.gutierrez@ub.edu
http://www.ub.edu/topologia/gutierrez

Constanze Roitzheim
C.Roitzheim@kent.ac.uk
http://www.kent.ac.uk/smsas/personal/csrr

[1] Institute for Mathematics, Astrophysics, and Particle Physics, Radboud Universiteit Nijmegen, Heyendaalseweg 135, 6525, AJ, Nijmegen, The Netherlands

[2] School of Mathematics, Statistics and Actuarial Science, University of Kent, Canterbury, Kent, CT2 7NF, UK

[3] Present address: Departament de Matemàtiques i Informàtica, Facultat de Matemàtiques i Informàtica, Universitat de Barcelona, Gran Via de les Corts Catalanes 585, 08007, Barcelona, Spain

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1 Introduction

Quillen adjunctions between spectra or spaces and other model categories are a useful way to study homotopy structures. For example, one can gain insight into a model category $\mathcal{C}$ by studying the canonical action of the homotopy category of simplicial sets Ho(sSet) or of the stable homotopy category Ho(Sp) (provided that $\mathcal{C}$ is a stable model category) on the homotopy category Ho($\mathcal{C}$).

In [4] it was studied how this set-up is compatible with homological localisations of spectra, that is, left Bousfield localisation at E_*-isomorphisms for a homology theory E. For a stable model category $\mathcal{C}$, Barnes and Roitzheim constructed in [4] a corresponding Bousfield localisation $\mathcal{C}_E$ of $\mathcal{C}$ called *stable E-familiarisation* with appropriate universal properties. One of them implies that $\mathcal{C}_E$ is the "closest" model category to $\mathcal{C}$ such that every left Quillen functor from the model category of symmetric spectra Sp to $\mathcal{C}_E$ factors over E-local spectra L_E Sp.

In this paper, we take this notion further by studying the compatibility of Quillen adjunctions of two variables from $\mathcal{C} \times \mathcal{D}$ to $\mathcal{E}$ with Bousfield localisations and colocalisations of $\mathcal{C}$ or $\mathcal{D}$. Given a Quillen adjunction of two variables between combinatorial model structures

$$- \otimes -\colon \mathcal{C} \times \mathcal{D} \longrightarrow \mathcal{E},$$

$$\mathrm{Hom}_r(-, -)\colon \mathcal{D}^{op} \times \mathcal{E} \longrightarrow \mathcal{C},$$

$$\mathrm{Hom}_l(-, -)\colon \mathcal{C}^{op} \times \mathcal{E} \longrightarrow \mathcal{D},$$

where $\mathcal{E}$ is left proper, and a set $\mathcal{S}$ of morphisms in $\mathcal{C}$, we define an $\mathcal{S}$-localised model structure $L_{\mathcal{S}}\mathcal{E}$ on $\mathcal{E}$. The cofibrations of $L_{\mathcal{S}}\mathcal{E}$ are the same as the ones in $\mathcal{E}$, and the fibrant objects are the objects Z that are fibrant in $\mathcal{E}$ and such that for every morphism $f\colon A \to B$ in $\mathcal{S}$ the induced map

$$f^*\colon \mathrm{Hom}_l(B, Z) \longrightarrow \mathrm{Hom}_l(A, Z)$$

is a weak equivalence in $\mathcal{D}$. We show that the fibrant objects of $L_{\mathcal{S}}\mathcal{E}$ can be equivalently characterised in terms of a set of homotopy generators $\mathcal{G}_{\mathcal{D}}$ of $\mathcal{D}$. Namely, Z is fibrant in $L_{\mathcal{S}}\mathcal{E}$ if it is fibrant in $\mathcal{E}$ and $\mathrm{Hom}_r(G, Z)$ is $\mathcal{S}$-local in $\mathcal{C}$ for every G in $\mathcal{G}_{\mathcal{D}}$.

In fact, if $I_{\mathcal{D}}$ denotes the set of generating cofibrations of $\mathcal{D}$ and $\mathcal{G}_{\mathcal{D}}$ a set of homotopy generators of $\mathcal{D}$, then the model structure $L_{\mathcal{S}}\mathcal{E}$ can be obtained as the left Bousfield localisation of $\mathcal{E}$ with respect to the set of morphisms $\mathcal{S}\square I_{\mathcal{D}}$, where $\square$ denotes the pushout-product, or equivalently as the left Bousfield localisation of $\mathcal{E}$ with respect to $\mathcal{S} \otimes \mathcal{G}_{\mathcal{D}}$. Moreover, we show that the left Quillen bifunctor $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ induces a left Quillen bifunctor between the localised model structures $L_{\mathcal{S}}\mathcal{C} \times \mathcal{D} \to L_{\mathcal{S}}\mathcal{E}$, and that the model structure $L_{\mathcal{S}}\mathcal{E}$ is the "closest" model structure to $\mathcal{E}$ with this property.

Dually, if we start with a set of objects $\mathcal{K}$ in $\mathcal{C}$, we can obtain similar statements for the corresponding $\mathcal{K}$-colocalised model structure $C_{\mathcal{K}}\mathcal{E}$.

The $\mathcal{S}$-localised model structure and the $\mathcal{K}$-colocalised model structure are useful as they now generalise two known constructions: the enriched left and right Bousfield localisations of enriched model categories [5] and the E-familiarisation of spectral model categories [4, Section 5].

One application of our results is the description of Postnikov sections of arbitrary left proper combinatorial model categories. For the category sSet of simplicial sets, the model structure P_k sSet for kth Postnikov sections is obtained via localising sSet with respect to

the map $f_k : S^{k+1} \to D^{k+2}$. Using our localisation construction and combining it with the theory of framings [15] we can now consider Postnikov sections $P_k\mathcal{C}$ in model categories $\mathcal{C}$ that are not necessarily simplicial, for instance, when $\mathcal{C} = \mathrm{Ch}_b(R)$ is the category of non-negatively graded chain complexes of R-modules endowed with the standard projective model structure.

The $\mathcal{S}$-localised and $\mathcal{K}$-colocalised model structures play also an important role in the study of towers and homotopy pullbacks of model categories, such as Postnikov towers, Bousfield arithmetic squares, and homotopy fibers [12].

The paper is organised as follows. In Section 1, we recall some terminology and basic results on locally presentabl categories and combinatorial model categories. In Section 2, we discuss how Quillen adjunctions of two variables are compatible with left and right Bousfield localisations. Given a Quillen adjunction of two variables $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ we describe Bousfield localisations of $\mathcal{E}$ based on localisations of $\mathcal{C}$ or $\mathcal{D}$ and their universal properties. As particular examples, we recover enriched localisations [5], enriched colocalisations and E-familiarisations [3, 4]. Finally, in Section 3 we study the special case of Postnikov k-types in combinatorial model categories.

2 Review of Combinatorial Model Categories

In this section, we recall some terminology on locally presentable categories and combinatorial model categories. The essentials of the theory of locally presentable categories can be found in [1, 10] or [20]. Foundations on the theory of combinatorial model categories may be found in [6, 8] and [19]. As in [14] and [15] we will assume that all our model categories are equipped with functorial factorisations.

2.1 Locally Presentable Categories

Let λ be a regular cardinal. A small category $\mathcal{I}$ is called λ-*filtered* if it is nonempty and satisfies the following two conditions:

(i) Given any set of objects $\{a_i \mid i \in I\}$ in $\mathcal{I}$, where $|I| < \lambda$, there is an object a and a morphism $a_i \to a$ for each $i \in I$.

(ii) Given any set of parallel morphisms $\{\alpha_j : a \to a' \mid j \in J\}$ in $\mathcal{I}$ between two fixed objects, where $|J| < \lambda$, there is a morphism $\gamma : a' \to a''$ such that $\gamma \circ \alpha_j = \gamma \circ \alpha_{j'}$ for all $j, j' \in J$.

An object X of a category $\mathcal{C}$ is called λ-*presentable* if the functor $\mathcal{C}(X, -)$ from $\mathcal{C}$ to sets preserves λ-filtered colimits.

A cocomplete category $\mathcal{C}$ is *locally λ-presentable* if there is a set of λ-presentable objects $\mathcal{A}$ such that every object of $\mathcal{C}$ is a λ-filtered colimit of objects from $\mathcal{A}$. In fact, if $\mathcal{C}$ is locally λ-presentable, then the collection of all λ-presentable objects has a set of representatives with respect to isomorphism, and we will denote by $\mathcal{C}_\lambda$ the full subcategory determined by any such set. The overcategory $\mathcal{C}_\lambda \downarrow X$ is λ-filtered and if we denote by $\mathrm{colim}(\mathcal{C}_\lambda \downarrow X)$ the colimit of the diagram $\mathcal{C}_\lambda \downarrow X \to \mathcal{C}_\lambda \hookrightarrow \mathcal{C}$, where the first functor is the projection, then the canonical map

$$\mathrm{colim}(\mathcal{C}_\lambda \downarrow X) \longrightarrow X$$

is an isomorphism. A category is *locally presentable* if it is locally λ-presentable for some regular cardinal λ.

Every locally λ-presentable category is equivalent to a full, reflective subcategory closed under λ-filtered colimits of the category of presheaves on some small category; see [1, Proposition 1.46].

2.2 Combinatorial Model Categories

A model category $\mathcal{C}$ is *cofibrantly generated* if there exists a set $I_{\mathcal{C}}$ of *generating cofibrations* and a set $J_{\mathcal{C}}$ of *generating trivial cofibrations* that one can use to perform the small object argument (see [14, Definition 11.1.2] or [15, Definition 2.1.17] for a precise definition).

A *homotopy function complex* in a model category $\mathcal{C}$ is a functorial choice of a fibrant simplicial set $\mathrm{map}_{\mathcal{C}}(X, Y)$, for every two objects X and Y in $\mathcal{C}$, whose homotopy type is the same as the diagonal of the bisimplicial set $\mathcal{C}(\widetilde{\mathbf{X}}, \widehat{\mathbf{Y}})$, where $\widetilde{\mathbf{X}}$ is a cosimplicial resolution of X and $\widehat{\mathbf{Y}}$ is a simplicial resolution of Y; for more details, see [14, Chapter 17]. Functorial homotopy function complexes exist in every model category and they are unique up to homotopy; see [14, Proposition 17.5.18 and Theorem 17.5.21].

Let $\mathcal{C}$ be a model category with homotopy function complex $\mathrm{map}_{\mathcal{C}}(-, -)$ and let $i \colon A \to B$ and $p \colon X \to Y$ be two morphisms in $\mathcal{C}$. Then the pair (i, p) is a *homotopy orthogonal pair* if the diagram

$$
\begin{array}{ccc}
\mathrm{map}_{\mathcal{C}}(B, X) & \longrightarrow & \mathrm{map}_{\mathcal{C}}(B, Y) \\
\downarrow & & \downarrow \\
\mathrm{map}_{\mathcal{C}}(A, X) & \longrightarrow & \mathrm{map}_{\mathcal{C}}(A, Y)
\end{array}
$$

is a homotopy fiber square [14, Definition 17.8.1]. The notion of homotopy orthogonal pair does not depend on the choice of homotopy function complexes, as shown in [14, Proposition 17.8.2]. In particular, we have that the pair $(\emptyset \to W, p)$ is homotopy orthogonal if the induced map

$$
p_* \colon \mathrm{map}_{\mathcal{C}}(W, X) \longrightarrow \mathrm{map}_{\mathcal{C}}(W, Y)
$$

is a weak equivalence of simplicial sets.

Recall that a model category is *left proper* if pushouts of weak equivalences along cofibrations are weak equivalences, and *right proper* if pullbacks of weak equivalences along fibrations are weak equivalences. A model category is *proper* if it is left and right proper.

In a cofibrantly generated model category the set of generating cofibrations can be used to detect weak equivalences. A proof of the following result can be found in [14, Theorem 17.8.18].

Proposition 2.1 *Let $\mathcal{C}$ be a cofibrantly generated model category and let $I_{\mathcal{C}}$ be a set of generating cofibrations. Assume that $\mathcal{C}$ is left proper or that the domains of the elements of $I_{\mathcal{C}}$ are cofibrant. Then, a map f in $\mathcal{C}$ is a weak equivalence if and only if for every map i in $I_{\mathcal{C}}$ the pair (i, f) is a homotopy orthogonal pair.*

A set of *homotopy generators* for a model category $\mathcal{C}$ consists of a small full subcategory $\mathcal{G}$ such that every object of $\mathcal{C}$ is weakly equivalent to a filtered homotopy colimit of objects of $\mathcal{G}$. A set of homotopy generators also detects weak equivalences.

Proposition 2.2 *Let $\mathcal{C}$ be a model category with homotopy function complex $\mathrm{map}_{\mathcal{C}}(-, -)$ and a set of cofibrant homotopy generators $\mathcal{G}$. Then a map $f \colon X \to Y$ in $\mathcal{C}$ is a weak equivalence if and only if for every G in $\mathcal{G}$ the pair (j_G, f) is a homotopy orthogonal pair, where j_G denotes the morphism $\emptyset \to G$.*

Proof Let j_W denote the map $\emptyset \to W$. By [14, Theorem 17.7.7] a map $f : X \to Y$ is a weak equivalence if and only if the pair (j_W, f) is a homotopy orthogonal pair for every object W, that is, if and only if the induced map

$$f_* : \mathrm{map}_{\mathcal{C}}(W, X) \longrightarrow \mathrm{map}_{\mathcal{C}}(W, Y)$$

is a weak equivalence. Let $\hat{f} : \hat{X} \to \hat{Y}$ be a fibrant approximation of f. By assumption every object W is weakly equivalent to a filtered homotopy colimit hocolim G_α of objects of $\mathcal{G}$, and hence [14, Theorem 19.4.2(2)] and [14, Theorem 19.4.4] imply that

$$\mathrm{map}_{\mathcal{C}}(\mathrm{hocolim}\, G_\alpha, \hat{X}) \simeq \mathrm{holim}(\mathrm{map}_{\mathcal{C}}(G_\alpha, \hat{X}))$$

and that the map

$$\mathrm{holim}(\mathrm{map}_{\mathcal{C}}(G_\alpha, \hat{X})) \longrightarrow \mathrm{holim}(\mathrm{map}_{\mathcal{C}}(G_\alpha, \hat{Y}))$$

is a weak equivalence. The result now follows from the fact that homotopy function complexes are homotopy invariant; see [14, Theorem 17.7.7]. $\square$

Let λ be a regular cardinal. A model category $\mathcal{C}$ is called λ-*combinatorial* if it is cofibrantly generated and the underlying category is locally λ-presentable. A model category $\mathcal{C}$ is called *combinatorial* if it is λ-combinatorial for some regular cardinal λ.

Every combinatorial model category is Quillen equivalent to a left Bousfield localisation of a category of diagrams of simplicial sets equipped with the projective model structure [8, Theorem 1.1] and many model categories of interest are combinatorial. Examples are pointed or unpointed simplicial sets, pointed or unpointed motivic spaces [7, 21], symmetric spectra over simplicial sets [16, § 3.4] or over motivic spaces [17], module spectra over a ring spectrum [22, Theorem 4.1], bounded or unbounded chain complexes of modules over a ring [15, § 2.3], or any locally presentable category equipped with the *discrete* model structure, where the weak equivalences are the isomorphisms and all morphisms are fibrations and cofibrations.

Dugger also proved in [8, Proposition 4.7] that every combinatorial model category has a set of homotopy generators (in the sense that every object is a homotopy colimit of those) and that, moreover, they can be chosen to be cofibrant. Recall that we denote by $\mathcal{C} \downarrow X$ the slice category of $\mathcal{C}$ over an object X, and that by hocolim$(\mathcal{C}_\lambda \downarrow X)$ we mean the homotopy colimit of the projection followed by inclusion functor $\mathcal{C}_\lambda \downarrow X \to \mathcal{C}_\lambda \hookrightarrow \mathcal{C}$.

Proposition 2.3 (Dugger) *Let λ be a regular cardinal and let $\mathcal{C}$ be a λ-combinatorial model category. Let $\mathcal{C}_\lambda \subseteq \mathcal{C}$ denote the full subcategory of the λ-presentable objects. Then every object X is a canonical filtered homotopy colimit of objects of $\mathcal{C}_\lambda$. More precisely, the canonical map*

$$\mathrm{hocolim}(\mathcal{C}_\lambda \downarrow X) \longrightarrow X$$

is a weak equivalence. Moreover, there is a regular cardinal $\mu > \lambda$ such that the canonical map

$$\mathrm{hocolim}(\mathcal{C}_\mu^{cof} \downarrow X) \longrightarrow X$$

is a weak equivalence, where $\mathcal{C}_\mu^{cof}$ denotes the full subcategory of $\mathcal{C}_\mu$ consisting of the cofibrant objects.

Given a combinatorial model category $\mathcal{C}$, we will denote by $\mathcal{G}_{\mathcal{C}}$ a set of cofibrant homotopy generators, whose existence is guaranteed by the previous proposition.

Corollary 2.4 *Let $\mathcal{C}$ be a combinatorial model category with a set of generating cofibrations $I_{\mathcal{C}}$ and a set of cofibrant homotopy generators $\mathcal{G}_{\mathcal{C}}$. Assume that $\mathcal{C}$ is left proper or that the domains of the elements of $I_{\mathcal{C}}$ are cofibrant. Then, for every map f in $\mathcal{C}$, the pair (i, f) is a homotopy orthogonal pair for all i in $I_{\mathcal{C}}$ if and only if for every G in $\mathcal{G}_{\mathcal{C}}$, the pair (j_G, f) is a homotopy orthogonal pair, where j_G denotes the morphism $\emptyset \to G$.*

Proof This is a consequence of Proposition 1.1 and Proposition 1.2. □

3 Left and Right Bousfield Localisations along Quillen Bifunctors

In this section we are going to discuss how Quillen adjunctions of two variables are compatible with left and right Bousfield localisation.

3.1 Quillen Bifunctors

Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be categories. An *adjunction of two variables* from $\mathcal{C} \times \mathcal{D}$ to $\mathcal{E}$ is given by functors

$$- \otimes -\colon \mathcal{C} \times \mathcal{D} \longrightarrow \mathcal{E},$$
$$\operatorname{Hom}_r(-, -)\colon \mathcal{D}^{op} \times \mathcal{E} \longrightarrow \mathcal{C},$$
$$\operatorname{Hom}_l(-, -)\colon \mathcal{C}^{op} \times \mathcal{E} \longrightarrow \mathcal{D},$$

and natural isomorphisms

$$\mathcal{C}(X, \operatorname{Hom}_r(Y, Z)) \cong \mathcal{E}(X \otimes Y, Z) \cong \mathcal{D}(Y, \operatorname{Hom}_l(X, Z)).$$

We will sometimes denote an adjunction of two variables from $\mathcal{C} \times \mathcal{D}$ to $\mathcal{E}$ just by the left adjoint $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$. The following analogous notion for model categories appears in [15, Definition 4.2.1].

Definition 3.1 Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be model categories. An adjunction of two variables from $\mathcal{C} \times \mathcal{D}$ to $\mathcal{E}$ is a *Quillen adjunction of two variables* if for every cofibration $f\colon A \to B$ in $\mathcal{C}$ and every cofibration $g\colon X \to Y$ in $\mathcal{D}$, the pushout-product

$$f \square g\colon B \otimes X \coprod_{A \otimes X} A \otimes Y \longrightarrow B \otimes Y$$

is a cofibration in $\mathcal{E}$ which is a trivial cofibration if either f or g is a trivial cofibration. We will refer to the left adjoint $\otimes$ of a Quillen adjunction of two variables as a *left Quillen bifunctor*.

Every Quillen adjunction of two variables induces a derived adjunction of two variables on the corresponding homotopy categories.

Remark 3.2 There are equivalent formulations of the previous condition satisfied by a Quillen adjunction of two variables in term of $\operatorname{Hom}_r^{\square}$ and $\operatorname{Hom}_l^{\square}$, where $\operatorname{Hom}_r^{\square}$ and $\operatorname{Hom}_l^{\square}$ denote the respective adjoints of the pushout-product; see [15, Lemma 4.2.2]. Explicitly, an adjunction of two variables $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ is a Quillen adjunction of two variables if and only if any of the following equivalent conditions hold:

(i) For every cofibration $f\colon A \to B$ in $\mathcal{C}$ and every fibration $h\colon U \to V$ in $\mathcal{E}$, the map

$$\operatorname{Hom}_l^{\square}(f, h)\colon \operatorname{Hom}_l(B, U) \longrightarrow \operatorname{Hom}_l(B, V) \times_{\operatorname{Hom}_l(A, V)} \operatorname{Hom}_l(A, U)$$

is a fibration in $\mathcal{D}$ which is a trivial fibration if either f is a trivial cofibration or h is a trivial fibration.

(ii)   For every cofibration $g\colon X \to Y$ in $\mathcal{D}$ and every fibration $h\colon U \to V$ in $\mathcal{E}$, the map

$$\operatorname{Hom}_r^{\square}(g, h)\colon \operatorname{Hom}_r(Y, U) \longrightarrow \operatorname{Hom}_r(Y, V) \times_{\operatorname{Hom}_r(X, V)} \operatorname{Hom}_r(X, U)$$

is a fibration in $\mathcal{C}$ which is a trivial fibration if either g is a trivial cofibration or h is a trivial fibration.

Remark 3.3 If $(\otimes, \operatorname{Hom}_r, \operatorname{Hom}_l)$ is a Quillen adjunction of two variables from $\mathcal{C} \times \mathcal{D}$ to $\mathcal{E}$ and $F_1\colon \mathcal{C}' \to \mathcal{C}$, $F_2\colon \mathcal{D}' \to \mathcal{D}$ and $F_3\colon \mathcal{E} \to \mathcal{E}'$ are left Quillen functors (with right adjoints G_1, G_2 and G_3, respectively), then

$$(F_3(F_1(-) \otimes F_2(-)), G_1 \operatorname{Hom}_r(F_2(-), G_3(-)), G_2 \operatorname{Hom}_l(F_1(-), G_3(-)))$$

is a Quillen adjunction of two variables from $\mathcal{C}' \times \mathcal{D}'$ to $\mathcal{E}'$.

Example 3.4 Let sSet denote the category of simplicial sets with the Kan–Quillen model structure. A *simplicial model structure* on a model category $\mathcal{C}$ is the same as a left Quillen bifunctor $\mathcal{C} \times \text{sSet} \to \mathcal{C}$. A *topological model structure* can be defined similarly, by replacing simplicial sets with the category of compactly generated Hausdorff spaces equipped with the Quillen model structure.

Let $(\mathcal{E}, \otimes, I, \operatorname{Hom}_{\mathcal{E}})$ be a closed symmetric monoidal category. A model structure on $\mathcal{E}$ is called a *monoidal model structure* if $- \otimes -\colon \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ is a left Quillen bifunctor and the unit I is cofibrant.

Let $\mathcal{E}$ be a monoidal model category. An $\mathcal{E}$-*model category* is a category $\mathcal{C}$ enriched, tensored and cotensored over $\mathcal{E}$ together with a model structure such that the tensor, enrichment and cotensor define a Quillen adjunction of two variables.

The following two lemmas are an immediate consequence of the bifunctor adjunctions and we state them without proof. We will use the terminology $f \pitchfork g$ to indicate that a morphism f has the *left lifting property* with respect to g (or that g has the *right lifting property* with respect to f), that is, $f \pitchfork g$ if for every commutative diagram of the form

there is a diagonal lifting h such that $i = hf$ and $p = gh$.

Lemma 3.5 *Let* $(\otimes, \operatorname{Hom}_r, \operatorname{Hom}_l)$ *be an adjunction of two variables from* $\mathcal{C} \times \mathcal{D}$ *to* $\mathcal{E}$ *and let* f, g *and* h *be morphisms in* $\mathcal{C}$, $\mathcal{D}$ *and* $\mathcal{E}$, *respectively. The following are equivalent:*

(i) $(f \square g) \pitchfork h$.
(ii) $f \pitchfork \operatorname{Hom}_r^{\square}(g, h)$.
(iii) $g \pitchfork \operatorname{Hom}_l^{\square}(f, h)$.

Lemma 3.6 *Let* $(\otimes, \operatorname{Hom}_r, \operatorname{Hom}_l)$ *be an adjunction of two variables from* $\mathcal{C} \times \mathcal{D}$ *to* $\mathcal{E}$ *between model categories.*

(i)   The following are equivalent:

    (a)   For every cofibration f in $\mathcal{C}$ and a every cofibration g in $\mathcal{D}$, the morphism $f \square g$ is a cofibration in $\mathcal{E}$.

    (b)   For every a cofibration g in $\mathcal{D}$ and for every trivial fibration h in $\mathcal{E}$, the morphism $\operatorname{Hom}_r^{\square}(g, h)$ is a trivial fibration in $\mathcal{C}$.

    (c)   For every cofibration f in $\mathcal{C}$ and every trivial fibration h in $\mathcal{E}$, the morphism $\operatorname{Hom}_l^{\square}(f, h)$ is a trivial fibration in $\mathcal{D}$.

(ii)   The following are equivalent:

    (a)   For every cofibration f in $\mathcal{C}$ and every trivial cofibration g in $\mathcal{D}$, the morphism $f \square g$ is a trivial cofibration in $\mathcal{E}$.

    (b)   For every trivial cofibration g in $\mathcal{D}$ and every fibration h in $\mathcal{E}$, the morphism $\operatorname{Hom}_r^{\square}(g, h)$ is a trivial fibration in $\mathcal{C}$.

    (c)   For every cofibration f in $\mathcal{C}$ and every fibration h in $\mathcal{E}$, the morphism $\operatorname{Hom}_l^{\square}(f, h)$ is a fibration in $\mathcal{D}$.

Note that in every Quillen adjunction of two variables, if X is cofibrant in $\mathcal{C}$, then $X \otimes -$ is a left Quillen functor with right adjoint $\operatorname{Hom}_l(X, -)$. Similarly, if Y is cofibrant in $\mathcal{D}$, then $- \otimes Y$ is a left Quillen functor with right adjoint $\operatorname{Hom}_r(Y, -)$.

Just as in the case of Quillen functors (see [14, Proposition 8.5.4]) we have the following proposition which will be useful to test whether an adjunction of two variables is a Quillen adjunction of two variables. In order to prove it, we will make use of the following key result, which appears as [18, Lemma 7.14].

Lemma 3.7 *A cofibration in a model category is a trivial cofibration if and only if it has the left lifting property with respect to every fibration between fibrant objects. Dually, a fibration in a model category is a trivial fibration if and only if it has the right lifting property with respect to every cofibration between cofibrant objects.*

Proposition 3.8 *Let $(\otimes, \operatorname{Hom}_r, \operatorname{Hom}_l)$ be an adjunction of two variables from $\mathcal{C} \times \mathcal{D}$ to $\mathcal{E}$ between model categories. Suppose that if g is a cofibration (respectively trivial cofibration) in $\mathcal{D}$ and h is a trivial fibration (respectively fibration) in $\mathcal{E}$, then $\operatorname{Hom}_r^{\square}(g, h)$ is a trivial fibration in $\mathcal{C}$. Then the following are equivalent:*

(i)   *$(\otimes, \operatorname{Hom}_r, \operatorname{Hom}_l)$ is a Quillen adjunction of two variables.*

(ii)   *Given a cofibration g in $\mathcal{D}$ and a fibration between fibrant objects $\hat{h}$ in $\mathcal{E}$, the morphism $\operatorname{Hom}_r^{\square}(g, \hat{h})$ is a fibration in $\mathcal{C}$.*

(iii)   *Given a cofibration between cofibrant objects $\tilde{g}$ in $\mathcal{D}$ and a fibration h in $\mathcal{E}$, the morphism $\operatorname{Hom}_r^{\square}(\tilde{g}, h)$ is a fibration in $\mathcal{C}$.*

(iv)   *Given a cofibration between cofibrant objects $\tilde{g}$ in $\mathcal{D}$ and a fibration between fibrant objects $\hat{h}$ in $\mathcal{E}$, the morphism $\operatorname{Hom}_r^{\square}(\tilde{g}, \hat{h})$ is a fibration in $\mathcal{C}$.*

Proof It is clear that (i) implies (ii), (iii) and (iv), that (ii) implies (iv) and that (iii) implies (iv). It then suffices, for example, to prove that (ii) implies (i) and that (iv) implies (ii).

In order to prove that (ii) implies (i), let g be any cofibration in $\mathcal{D}$ and h any fibration in $\mathcal{E}$. Then $\operatorname{Hom}_r^{\square}(g, h)$ is a fibration in $\mathcal{C}$ if and only if for every trivial cofibration j in $\mathcal{C}$, we have that $j \pitchfork \operatorname{Hom}_r^{\square}(g, h)$. But by Lemma 2.5, this is equivalent to $(j \square g) \pitchfork h$, in other words, $j \square g$ being a trivial cofibration. Since by assumption condition (i)(b) of Lemma 2.6

holds, Lemma 2.6(i) implies that $j \square g$ is a cofibration. Hence, by Lemma 2.7 the previous condition is equivalent to $(j \square g) \pitchfork \hat{h}$ for $\hat{h}$ being any fibration between fibrant objets in $\mathcal{E}$. Again, by Lemma 2.5 this is equivalent to $j \pitchfork \operatorname{Hom}_r^{\square}(g, \hat{h})$ for $\hat{h}$ any fibration between fibrant objects. Since we are assuming that $\operatorname{Hom}_r^{\square}(g, \hat{h})$ is a fibration in $\mathcal{C}$, the last statement is true, so we can conclude that $\operatorname{Hom}_r^{\square}(g, h)$ is a fibration for any cofibration g and fibration h as required, which was the missing part for $(\otimes, \operatorname{Hom}_r, \operatorname{Hom}_l)$ to be a Quillen adjunction of two variables.

That part (iv) implies (ii) is proved in a very similar way to the previous point. Let g be any cofibration in $\mathcal{D}$ and let $\hat{h}$ be a fibration between fibrant objects in $\mathcal{E}$. Then $\operatorname{Hom}_r^{\square}(g, \hat{h})$ is a fibration in $\mathcal{C}$ if and only if $j \pitchfork \operatorname{Hom}_r^{\square}(g, \hat{h})$ for every trivial cofibration j in $\mathcal{C}$. By Lemma 2.5 this is equivalent to $g \pitchfork \operatorname{Hom}_l^{\square}(j, \hat{h})$ for every trivial cofibration j in $\mathcal{C}$. By the assumption of the proposition and Lemma 2.6(ii) the morphism $\operatorname{Hom}_l^{\square}(j, \hat{h})$ is a fibration, and therefore, by Lemma 2.7, it is a trivial fibration if and only if $\tilde{g} \pitchfork \operatorname{Hom}_l^{\square}(j, \hat{h})$ for every cofibration $\tilde{g}$ between cofibrant objects in $\mathcal{D}$. By adjunction, this is equivalent to saying that $j \pitchfork \operatorname{Hom}_r^{\square}(\tilde{g}, \hat{h})$ for every trivial cofibration j in $\mathcal{C}$, every cofibration between cofibrant objects $\tilde{g}$ in $\mathcal{D}$, and every fibration between fibrant objects $\hat{h}$ in $\mathcal{E}$. But $\operatorname{Hom}_r^{\square}(\tilde{g}, \hat{h})$ is a fibration, by assumption, hence (iv) is equivalent to (ii), which is what we wanted to prove. $\square$

Remark 3.9 Note that if $(\otimes, \operatorname{Hom}_r, \operatorname{Hom}_l)$ is an adjunction of two variables from $\mathcal{C} \times \mathcal{D}$ to $\mathcal{E}$ and $\tau \colon \mathcal{D} \times \mathcal{C} \to \mathcal{C} \times \mathcal{D}$ is the functor that interchanges the components, then $(\otimes \circ \tau, \operatorname{Hom}_l, \operatorname{Hom}_r)$ is an adjunction of two variables from $\mathcal{D} \times \mathcal{C}$ to $\mathcal{E}$.

3.2 Left and Right Bousfield Localisation

We recall the notion of *left Bousfield localisation* and *right Bousfield localisation* (also called *Bousfield colocalisation*) for model categories; see [14, Chapters 3–5].

Let $\mathcal{C}$ be a model category with homotopy function complex $\operatorname{map}_{\mathcal{C}}(-, -)$ and let $\mathcal{S}$ be a class of morphisms of $\mathcal{C}$ and $\mathcal{K}$ a class of objects in $\mathcal{C}$. We say that an object Z in $\mathcal{C}$ is $\mathcal{S}$-*local* if it is fibrant and for every morphism $f \colon A \to B$ in $\mathcal{S}$ the induced map

$$f^* \colon \operatorname{map}_{\mathcal{C}}(B, Z) \longrightarrow \operatorname{map}_{\mathcal{C}}(A, Z)$$

is a weak equivalence of simplicial sets. We say that a map $g \colon X \to Y$ is an $\mathcal{S}$-*local equivalence* if the induced map

$$g^* \colon \operatorname{map}_{\mathcal{C}}(Y, Z) \longrightarrow \operatorname{map}_{\mathcal{C}}(X, Z)$$

is a weak equivalence of simplicial sets for every $\mathcal{S}$-local object Z.

We say that a map $h \colon X \to Y$ in $\mathcal{C}$ is a $\mathcal{K}$-*colocal equivalence* if for every object K in $\mathcal{K}$ the induced map

$$h_* \colon \operatorname{map}_{\mathcal{C}}(K, X) \longrightarrow \operatorname{map}_{\mathcal{C}}(K, Y)$$

is a weak equivalence of simplicial sets. We say that an object W in $\mathcal{C}$ is $\mathcal{K}$-*colocal* if it is cofibrant and for every $\mathcal{K}$-colocal equivalence h the induced map

$$h_* \colon \operatorname{map}_{\mathcal{C}}(W, X) \longrightarrow \operatorname{map}_{\mathcal{C}}(W, Y)$$

is a weak equivalence of simplicial sets.

The *left Bousfield localisation* of $\mathcal{C}$ with respect to $\mathcal{S}$ (if it exists) is a new model structure $L_\mathcal{S}\mathcal{C}$ on $\mathcal{C}$ such that

(i)   the cofibrations of $L_\mathcal{S}\mathcal{C}$ are the same as those of $\mathcal{C}$,
(ii)   the weak equivalences of $L_\mathcal{S}\mathcal{C}$ are the $\mathcal{S}$-local equivalences,
(iii) the fibrant objects of $L_\mathcal{S}\mathcal{C}$ are the $\mathcal{S}$-local objects.

The $\mathcal{S}$-local equivalences between $\mathcal{S}$-local objects are weak equivalences in $\mathcal{C}$.

The *right Bousfield localisation* (or Bousfield colocalisation) of $\mathcal{C}$ with respect to $\mathcal{K}$ (if it exists) is a new model structure $C_\mathcal{K}\mathcal{C}$ on $\mathcal{C}$ such that

(i)   the fibrations of $C_\mathcal{K}\mathcal{C}$ are the same as those of $\mathcal{C}$,
(ii)   the weak equivalences of $C_\mathcal{K}\mathcal{C}$ are the $\mathcal{K}$-colocal equivalences,
(iii) the cofibrant objects of $C_\mathcal{K}\mathcal{C}$ are the $\mathcal{K}$-*colocal objects*.

The $\mathcal{K}$-colocal equivalences between $\mathcal{K}$-colocal objects are weak equivalences in $\mathcal{C}$.

Remark 3.10 In [14, Chapter 3], both left and right Bousfield localisations are defined with respect to arbitrary classes of morphisms. Here we have chosen to define right Bousfield localisations with respect to objects, since the main existence result works under the assumption that we localise at the class of $\mathcal{K}$-colocal equivalences for a set of objects $\mathcal{K}$.

Remark 3.11 Note that the definition of the $\mathcal{S}$-local objects does not depend on the chosen homotopy function complexes, since they are unique up to homotopy and also homotopy invariant. Therefore, we can always replace the morphisms in $\mathcal{S}$ by weakly equivalent ones consisting of cofibrations between cofibrant objects without changing the model structure $L_\mathcal{S}\mathcal{C}$. Hence, without loss of generality we will often assume that when we localise with respect to a class of morphisms, these morphisms are cofibrations between cofibrant objects.

Similarly, we can assume without loss of generality that when we colocalise with respect to a class of objects, they are cofibrant.

There are two main classes of model categories where localisations with respect to a *set* of morphisms and colocalisations with respect to a *set* of objects are always known to exist. These are the cellular model categories and the combinatorial model categories. For both classes the assumption of left properness is needed for the existence of left Bousfield localisations (see [14, Theorem 4.1.1] and [5, Theorem 4.7]) and right properness in needed for the existence of right Bousfield localisation (see [14, Theorem 5.1.1] and [5, Proposition 5.13]). If $\mathcal{C}$ is left proper and combinatorial (or cellular) and $\mathcal{S}$ is a set of morphisms of $\mathcal{C}$, then $L_\mathcal{S}\mathcal{C}$ is also left proper and combinatorial (or cellular). If $\mathcal{C}$ is right proper and combinatorial (or cellular) and $\mathcal{K}$ is a set of objects of $\mathcal{C}$, then $C_\mathcal{K}\mathcal{C}$ is also right proper, but it is not cofibrantly generated in general.

Definition 3.12 Let $\otimes\colon \mathcal{C}\times\mathcal{D}\to\mathcal{E}$ be a left Quillen bifunctor, where $\mathcal{D}$ is cofibrantly generated with set of generating cofibrations $I_\mathcal{D}$ and set of cofibrant homotopy generators $\mathcal{G}_\mathcal{D}$. Assume that $\mathcal{E}$ is proper and combinatorial and let $\mathcal{S}$ and $\mathcal{K}$ be sets of morphisms and objects in $\mathcal{C}$, respectively.

(i)   The $\mathcal{S}$-*local model structure* on $\mathcal{E}$, denoted by $L_\mathcal{S}\mathcal{E}$, is the left Bousfield localisation $L_{\mathcal{S}\square I_\mathcal{D}}\mathcal{E}$ of $\mathcal{E}$ with respect to $\mathcal{S}\square I_\mathcal{D}$.
(ii)   The $\mathcal{K}$-*colocal model structure* on $\mathcal{E}$, denoted by $C_\mathcal{K}\mathcal{E}$ is the right Bousfield localisation $C_{\mathcal{K}\otimes\mathcal{G}_\mathcal{D}}\mathcal{E}$ of $\mathcal{E}$ with respect to $\mathcal{K}\otimes\mathcal{G}_\mathcal{D}$.

Remark 3.13 If ($\otimes$, Hom_r, Hom_l) is a Quillen adjunction of two variables from $\mathcal{C} \times \mathcal{D}$ to $\mathcal{E}$, with $\mathcal{C}$ cofibrantly generated, $\mathcal{E}$ left proper and combinatorial, and $\mathcal{S}$ is a set of morphisms in $\mathcal{D}$ (instead of in $\mathcal{C}$), then we can also define an $\mathcal{S}$-localised model structure on $\mathcal{E}$ as $L_{I_{\mathcal{C}}\square\mathcal{S}}\mathcal{E}$, where $I_{\mathcal{C}}$ is the set of generating cofibrations of $\mathcal{C}$. All the results from this section can be rephrased in terms of a set of morphisms in $\mathcal{D}$, by suitably replacing Hom_l by Hom_r and vice versa; see Remark 2.9.

Theorem 3.14 *Let ($\otimes$, Hom_r, Hom_l) be a Quillen adjunction of two variables from $\mathcal{C} \times \mathcal{D}$ to $\mathcal{E}$. Let $\mathcal{S}$ and $\mathcal{K}$ be classes of morphisms and objects in $\mathcal{C}$, respectively. Assume that $\mathcal{D}$ is combinatorial with set of generating cofibrations $I_{\mathcal{D}}$ and set of cofibrant homotopy generators $\mathcal{G}_{\mathcal{D}}$ and that it is either left proper or the domains of the elements of $I_{\mathcal{D}}$ are cofibrant.*

(i)   *The following are equivalent for an object Z of $\mathcal{E}$:*

    (a)   *Z is $\mathcal{S}\square I_{\mathcal{D}}$-local.*
    (b)   *Z is $\mathcal{S} \otimes \mathcal{G}_{\mathcal{D}}$-local.*
    (c)   *Z is fibrant and $\mathrm{Hom}_r(G, Z)$ is $\mathcal{S}$-local for every G in $\mathcal{G}_{\mathcal{D}}$.*
    (d)   *Z is fibrant and for every $f: A \to B$ in $\mathcal{S}$ the induced map*

$$f^*: \mathrm{Hom}_l(B, Z) \longrightarrow \mathrm{Hom}_l(A, Z)$$

    is a weak equivalence in $\mathcal{D}$.

(ii)   *The following are equivalent for a morphism $h: X \to Y$ of $\mathcal{E}$:*

    (a)   *h is a $\mathcal{K} \otimes \mathcal{G}_{\mathcal{D}}$-colocal equivalence.*
    (b)   *For every G in $\mathcal{G}_{\mathcal{D}}$ the induced map*

$$\hat{h}_*: \mathrm{Hom}_r(G, \hat{X}) \longrightarrow \mathrm{Hom}_r(G, \hat{Y})$$

    is a $\mathcal{K}$-colocal equivalence, where $\hat{h}$ is a fibrant approximation of h.
    (c)   *For every K in $\mathcal{K}$ the induced map*

$$\hat{h}_*: \mathrm{Hom}_l(K, \hat{X}) \longrightarrow \mathrm{Hom}_l(K, \hat{Y})$$

    is a weak equivalence in $\mathcal{D}$, where $\hat{h}$ is a fibrant approximation of h.

Proof We will prove part (i) first. Let Z be any object of $\mathcal{E}$. Then Z is $\mathcal{S}\square I_{\mathcal{D}}$-local if and only if it is fibrant and

$$\mathrm{map}_{\mathcal{E}}(B \otimes Y, Z) \longrightarrow \mathrm{map}_{\mathcal{E}}(A \otimes Y \coprod_{A \otimes X} B \otimes X, Z)$$

is a weak equivalence of simplicial sets for every map $A \to B$ in $\mathcal{S}$ and every map $X \to Y$ in $I_{\mathcal{D}}$. By adjunction and the compatibility of homotopy function complexes with Quillen pairs (see [14, Proposition 17.4.16]), the previous condition is equivalent to the diagram

$$
\begin{array}{ccc}
\mathrm{map}_{\mathcal{D}}(Y, \mathrm{Hom}_l(B, Z)) & \longrightarrow & \mathrm{map}_{\mathcal{D}}(Y, \mathrm{Hom}_l(A, Z)) \\
\downarrow & & \downarrow \\
\mathrm{map}_{\mathcal{D}}(X, \mathrm{Hom}_l(B, Z)) & \longrightarrow & \mathrm{map}_{\mathcal{D}}(X, \mathrm{Hom}_l(A, Z))
\end{array}
$$

being a homotopy fiber square. This is the same as saying that for every morphism $A \to B$ in $\mathcal{S}$ and every morphism $X \to Y$ in $I_{\mathcal{D}}$, the pair given by the morphisms $X \to Y$ and $\mathrm{Hom}_l(B, Z) \to \mathrm{Hom}_l(A, Z)$ is a homotopy orthogonal pair.

By Corollary 1.4 the previous condition amounts to saying that the pair given by $\emptyset \to G$ and $\mathrm{Hom}_l(B, Z) \to \mathrm{Hom}_l(A, Z)$ is a homotopy orthogonal pair for every G in $\mathcal{G}_\mathcal{D}$, that is,

$$\mathrm{map}_\mathcal{D}(G, \mathrm{Hom}_l(B, Z)) \longrightarrow \mathrm{map}_\mathcal{D}(G, \mathrm{Hom}_l(A, Z))$$

is a weak equivalence. Again by adjunction and the compatibility of homotopy function complexes with Quillen adjunctions, this is equivalent to saying that

$$\mathrm{map}_\mathcal{E}(B \otimes G, Z) \longrightarrow \mathrm{map}_\mathcal{E}(A \otimes G, Z)$$

is a weak equivalence for every G in $\mathcal{G}_\mathcal{D}$, and this is precisely the condition of Z being $\mathcal{S} \otimes \mathcal{G}_\mathcal{D}$-local. This proves that (a) and (b) are equivalent.

By adjunction (b) is equivalent to the fibrancy of Z and the fact that

$$\mathrm{map}_\mathcal{C}(B, \mathrm{Hom}_r(G, Z)) \longrightarrow \mathrm{map}_\mathcal{C}(A, \mathrm{Hom}_r(G, Z))$$

is a weak equivalence for every map $A \to B$ in $\mathcal{S}$. Hence (b) and (c) are equivalent.

Now, Proposition 1.2 shows that (b) is equivalent to $\mathrm{Hom}_l(B, Z) \to \mathrm{Hom}_l(A, Z)$ being a weak equivalence in $\mathcal{D}$, which concludes the proof of part (i).

To prove part (ii), first observe that a morphism $h \colon X \to Y$ is a $\mathcal{K} \otimes \mathcal{G}_\mathcal{D}$-colocal equivalence if and only if $\hat{h} \colon \hat{X} \to \hat{Y}$ is a $\mathcal{K} \otimes \mathcal{G}_\mathcal{D}$-colocal equivalence. By definition, this means that

$$\mathrm{map}_\mathcal{E}(K \otimes G, \hat{X}) \longrightarrow \mathrm{map}_\mathcal{E}(K \otimes G, \hat{Y})$$

is a weak equivalence of simplicial sets for every K in $\mathcal{K}$ and every G in $\mathcal{G}_\mathcal{D}$. As in the proof of part (i), by adjunction and the compatibility of homotopy function complexes with Quillen adjunctions, this is equivalent to saying that

$$\mathrm{map}_\mathcal{C}(K, \mathrm{Hom}_r(G, \hat{X})) \longrightarrow \mathrm{map}_\mathcal{C}(K, \mathrm{Hom}_r(G, \hat{Y}))$$

is a weak equivalence for every K in $\mathcal{K}$ and every G in $\mathcal{G}_\mathcal{D}$, or that

$$\mathrm{map}_\mathcal{D}(G, \mathrm{Hom}_l(K, \hat{X})) \longrightarrow \mathrm{map}_\mathcal{D}(G, \mathrm{Hom}_l(K, \hat{Y}))$$

is a weak equivalence for every K in $\mathcal{K}$ and every G in $\mathcal{G}_\mathcal{D}$, which follows from Proposition 1.2. □

Corollary 3.15 *Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be left proper combinatorial model categories and let $\otimes \colon \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ be a left Quillen bifunctor. Let $\mathcal{S}$ be a set of morphisms in $\mathcal{C}$ and let $\mathcal{G}_\mathcal{D}$ be a set of cofibrant homotopy generators of $\mathcal{D}$. Then $L_\mathcal{S}\mathcal{E} = L_{\mathcal{S} \otimes \mathcal{G}_\mathcal{D}}\mathcal{E}$, where as before $L_\mathcal{S}\mathcal{E}$ means $L_{\mathcal{S} \square I_\mathcal{D}}\mathcal{E}$.*

Proof The result follows immediately from Theorem 2.14. □

Recall that the left Bousfield localisation of a left proper combinatorial model category at a set of morphisms is cofibrantly generated.

Proposition 3.16 *Let $\mathcal{C}$ be a left proper combinatorial model category and $\mathcal{S}$ a set of morphisms in $\mathcal{C}$. If $J_\mathcal{S}$ is a set of generating trivial cofibrations of $L_\mathcal{S}\mathcal{C}$, then $L_\mathcal{S}\mathcal{C} = L_{J_\mathcal{S}}\mathcal{C}$.*

Proof The argument is the same as in [19, Proposition A.3.7.4], where it is proved for left proper combinatorial simplicial model categories. In our case we have to replace the simplicial enrichment by the homotopy function complex $\mathrm{map}_\mathcal{C}(-, -)$.

Both model structures $L_\mathcal{S}\mathcal{C}$ and $L_{J_\mathcal{S}}\mathcal{C}$ have the same cofibrations, so it is enough to check that they have the same trivial cofibrations. The elements in $J_\mathcal{S}$ are trivial cofibrations in $L_{J_\mathcal{S}}\mathcal{C}$. Since the set $J_\mathcal{S}$ determines the trivial cofibrations of $L_\mathcal{S}\mathcal{C}$ (these are in fact the

morphisms with the left lifting property with respect to the morphisms with the right lifting property with respect to $J_{\mathcal{S}}$) it follows that every trivial cofibration of $L_{\mathcal{S}}\mathcal{C}$ is a trivial cofibration of $L_{J_{\mathcal{S}}}\mathcal{C}$.

Conversely, let $f\colon X \to Y$ be a trivial cofibration in $L_{J_{\mathcal{S}}}\mathcal{C}$. It is in particular a cofibration in $L_{\mathcal{S}}\mathcal{C}$ so it suffices to see that it is an $\mathcal{S}$-local equivalence. By assumption

$$f^*\colon \operatorname{map}_{\mathcal{C}}(Y, Z) \longrightarrow \operatorname{map}_{\mathcal{C}}(X, Z)$$

is a weak equivalence of simplicial sets for every Z that is $J_{\mathcal{S}}$-local. But every $\mathcal{S}$-local is $J_{\mathcal{S}}$-local, since $J_{\mathcal{S}}$ consists of trivial cofibrations of $L_{\mathcal{S}}\mathcal{C}$, so f^* is a weak equivalence for every Z that is $\mathcal{S}$-local. □

Proposition 3.17 *Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be left proper combinatorial model categories and let $\otimes\colon \mathcal{C}\times\mathcal{D} \to \mathcal{E}$ be a left Quillen bifunctor. Let $\mathcal{S}$ be a set of morphisms in $\mathcal{C}$. Then $\otimes\colon L_{\mathcal{S}}\mathcal{C}\times \mathcal{D} \to L_{\mathcal{S}}\mathcal{E}$ is a left Quillen bifunctor.*

Proof By [15, Corollary 4.2.5] it is enough to prove that the pushout-product axiom holds for the sets of generating cofibrations and trivial cofibrations of $L_{\mathcal{S}}\mathcal{C}$ and $\mathcal{D}$. As the cofibrations in $L_{\mathcal{S}}\mathcal{C}$ and $\mathcal{C}$ as well as the cofibrations in $L_{\mathcal{S}}\mathcal{E}$ and $\mathcal{E}$ agree, it is sufficient to only consider the following case. Let $J_{\mathcal{S}}$ be a set of generating trivial cofibrations of $L_{\mathcal{S}}\mathcal{C}$ and let $I_{\mathcal{D}}$ be a set of generating cofibrations of $\mathcal{D}$. Since the cofibrations of $L_{\mathcal{S}}\mathcal{C}$ are the same as those in $\mathcal{C}$, it suffices to prove that if i is in $J_{\mathcal{S}}$ and j is in $I_{\mathcal{D}}$, then $i\square j$ is a $\mathcal{S}\square I_{\mathcal{D}}$-local equivalence in $\mathcal{E}$. In fact, we will prove that the $J_{\mathcal{S}}\square I_{\mathcal{D}}$-local equivalences coincide with the $\mathcal{S}\square I_{\mathcal{D}}$-local equivalences.

Let $\mathcal{G}_{\mathcal{D}}$ be a set of cofibrant homotopy generators of $\mathcal{D}$. By Theorem 2.14(i), an object Z of $\mathcal{E}$ is $\mathcal{S}\square I_{\mathcal{D}}$-local if and only if $\operatorname{Hom}_r(G, Z)$ is $\mathcal{S}$-local for every G in $\mathcal{G}_D$. But by Proposition 2.16, $\mathcal{S}$-local objects coincide with $J_{\mathcal{S}}$-local objects. Hence $\operatorname{Hom}_r(G, Z)$ is $J_{\mathcal{S}}$-local for every G in $\mathcal{G}_{\mathcal{D}}$ and thus Z is $J_{\mathcal{S}}\square I_{\mathcal{D}}$-local. □

Proposition 3.18 *Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be model categories with sets of cofibrant homotopy generators $\mathcal{G}_{\mathcal{C}}$, $\mathcal{G}_{\mathcal{D}}$ and $\mathcal{G}_{\mathcal{E}}$, respectively. Suppose that $\mathcal{D}$ is left proper and combinatorial. Let $(\otimes, \operatorname{Hom}_r, \operatorname{Hom}_l)$ be a Quillen adjunction of two variables from $\mathcal{C} \times \mathcal{D}$ to $\mathcal{E}$ and let $\mathcal{S}$ be a class of morphisms in $\mathcal{C}$. Let $f\colon X \to Y$ be a map in $\mathcal{E}$ and let $\hat{f}\colon \hat{X} \to \hat{Y}$ be a fibrant approximation to f in $L_{\mathcal{S}}\mathcal{E}$. If the induced map*

$$\hat{f}_*\colon \operatorname{Hom}_r(G, \hat{X}) \longrightarrow \operatorname{Hom}_r(G, \hat{Y})$$

is an $\mathcal{S}$-local equivalence in $\mathcal{C}$ for every G in $\mathcal{G}_{\mathcal{D}}$ and $\mathcal{G}_{\mathcal{E}} \subset \mathcal{G}_{\mathcal{C}} \otimes \mathcal{G}_{\mathcal{D}}$, then f is an $\mathcal{S} \otimes \mathcal{G}_{\mathcal{E}}$-local equivalence in $\mathcal{E}$.

Proof By Theorem 2.14(i) the objects $\operatorname{Hom}_r(G, \hat{X})$ and $\operatorname{Hom}_r(G, \hat{Y})$ are both $\mathcal{S}$-local. Thus $\hat{f}_*$ is an $\mathcal{S}$-local equivalence between $\mathcal{S}$-local objects and hence a weak equivalence in $\mathcal{C}$. This implies that

$$\operatorname{map}_{\mathcal{C}}(W, \operatorname{Hom}_r(G, \hat{X})) \longrightarrow \operatorname{map}_{\mathcal{C}}(W, \operatorname{Hom}_r(G, \hat{Y}))$$

is a weak equivalence of simplicial sets for every W in $\mathcal{G}_{\mathcal{C}}$ and every G in $\mathcal{G}_{\mathcal{D}}$. By adjunction and compatibility of homotopy function complexes with Quillen functors this is equivalent to

$$\operatorname{map}_{\mathcal{E}}(W \otimes G, \hat{X}) \longrightarrow \operatorname{map}_{\mathcal{E}}(W \otimes G, \hat{Y})$$

being a weak equivalence of simplicial sets for every W in $\mathcal{G}_{\mathcal{C}}$ and every G in $\mathcal{G}_{\mathcal{D}}$. Since by assumption $\mathcal{G}_{\mathcal{E}} \subset \mathcal{G}_{\mathcal{C}} \otimes \mathcal{G}_{\mathcal{D}}$, this implies that $\hat{f}$ is a weak equivalence in $\mathcal{E}$. Now, by the

2-out-of-3 axiom and the fact that weak equivalences in $\mathcal{E}$ are $\mathcal{S} \otimes \mathcal{G}_{\mathcal{E}}$-local equivalences, it follows that f is an $\mathcal{S} \otimes \mathcal{G}_{\mathcal{E}}$-local equivalence. $\qquad\square$

The following definition is motivated by the notion of *E-familiar model structure* described in [2, Section 4].

Definition 3.19 Let $\mathcal{C}$ be a left proper combinatorial model category and let $\mathcal{S}$ be a set of morphisms in $\mathcal{C}$. Let $\otimes \colon \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ be a left Quillen bifunctor. We say that $\mathcal{E}$ *is $\mathcal{S}$-familiar* if $\otimes \colon L_{\mathcal{S}}\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ is a left Quillen bifunctor.

Remark 3.20 In particular, it follows from Proposition 2.17 that the $\mathcal{S}$-local model structure $L_{\mathcal{S}}\mathcal{E}$ is $\mathcal{S}$-familiar.

Proposition 3.21 *Let $(\otimes, \mathrm{Hom}_r, \mathrm{Hom}_l)$ be a Quillen adjunction of two variables from $\mathcal{C} \times \mathcal{D}$ to $\mathcal{E}$, where $\mathcal{C}$ is left proper and combinatorial, and let $\mathcal{S}$ be a set of morphisms in $\mathcal{C}$. Then $\mathcal{E}$ is $\mathcal{S}$-familiar if and only if $\mathrm{Hom}_r(X, Y)$ is $\mathcal{S}$-local for every X cofibrant in $\mathcal{D}$ and Y fibrant in $\mathcal{E}$.*

Proof The "only if" part follows from the fact that if $\mathcal{E}$ is $\mathcal{S}$-familiar and X is cofibrant in $\mathcal{D}$, then the functor $\mathrm{Hom}_r(X, -) \colon \mathcal{E} \to L_{\mathcal{S}}\mathcal{C}$ is right Quillen. Hence, for every Y fibrant in $\mathcal{E}$, we have that $\mathrm{Hom}_r(X, Y)$ is fibrant in $L_{\mathcal{S}}\mathcal{C}$, that is, $\mathcal{S}$-local.

Conversely, we want to show that if $\mathrm{Hom}_r(X, Y)$ is $\mathcal{S}$-local for every cofibrant X and fibrant Y, then $L_{\mathcal{S}}\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ is also a left Quillen bifunctor. Let f be a cofibration (respectively, a trivial cofibration) in $\mathcal{D}$ and let g be a trivial fibration (respectively, a fibration) in $\mathcal{E}$. Because $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ is assumed to be a left Quillen bifunctor, the map $\mathrm{Hom}_r^{\square}(f, g)$ is a trivial fibration in $\mathcal{C}$ and hence a trivial fibration in $L_{\mathcal{S}}\mathcal{C}$ (since $\mathcal{C}$ and $L_{\mathcal{S}}\mathcal{C}$ have the same cofibrations). Therefore, by Proposition 2.8 it suffices to prove that if $f \colon A \to B$ is a cofibration between cofibrant objects in $\mathcal{D}$ and $g \colon X \to Y$ is a fibration between fibrant objects in $\mathcal{E}$, then $\mathrm{Hom}_r^{\square}(f, g)$ is a fibration in $L_{\mathcal{S}}\mathcal{C}$. Since it is already a fibration in $\mathcal{C}$, it is enough to check that the source and target are $\mathcal{S}$-local. Consider the pullback diagram
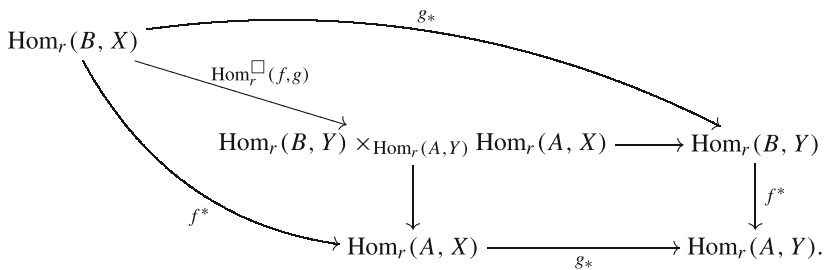
The right vertical map f^* is a fibration in $L_{\mathcal{S}}\mathcal{C}$, since it is a fibration in $\mathcal{C}$ between $\mathcal{S}$-local objects (see [14, Proposition 3.3.16]). Since fibrations are closed under pullbacks, the left vertical map is also a fibration in $L_{\mathcal{S}}\mathcal{C}$. But $\mathrm{Hom}_r(A, X)$ is $\mathcal{S}$-local (that is, fibrant in $L_{\mathcal{S}}\mathcal{C}$) and therefore so is

$$\mathrm{Hom}_r(B, Y) \times_{\mathrm{Hom}_r(A, Y)} \mathrm{Hom}_r(A, X).$$

Hence, we have proved that $\mathrm{Hom}_r^{\square}(f, g)$ is a fibration in $\mathcal{C}$ between $\mathcal{S}$-local objects. By [14, Proposition 3.3.16] this means that $\mathrm{Hom}_r^{\square}(f, g)$ is a fibration in $L_{\mathcal{S}}\mathcal{C}$. $\qquad\square$

We have seen that for a left Quillen bifunctor $\otimes\colon \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ and a set $\mathcal{S}$ of morphisms in $\mathcal{C}$, the new model structure $L_{\mathcal{S}}\mathcal{E}$ on $\mathcal{E}$ gives rise to a left Quillen bifunctor

$$\otimes\colon L_{\mathcal{S}}\mathcal{C} \times \mathcal{D} \longrightarrow L_{\mathcal{S}}\mathcal{E}.$$

We can now state that this model structure $L_{\mathcal{S}}\mathcal{E}$ is the "closest" model structure to $\mathcal{E}$ with this property in the following sense.

Proposition 3.22 *Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be left proper combinatorial model categories and let $\otimes\colon \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ be a left Quillen bifunctor. Let $F\colon \mathcal{E} \to \mathcal{E}'$ be a left Quillen functor and $\mathcal{S}$ a set of morphisms in $\mathcal{C}$. If $\mathcal{E}'$ is $\mathcal{S}$-familiar with respect to the Quillen bifunctor $F \circ \otimes\colon \mathcal{C} \times \mathcal{D} \to \mathcal{E} \to \mathcal{E}'$, then*

$$F\colon L_{\mathcal{S}}\mathcal{E} \longrightarrow \mathcal{E}'$$

is also a left Quillen functor, that is, F factors over the $\mathcal{S}$-localisation of $\mathcal{E}$.

Proof By Corollary 2.15 we have that $L_{\mathcal{S}}\mathcal{E} = L_{\mathcal{S} \otimes \mathcal{G}_{\mathcal{D}}}\mathcal{E}$, where $\mathcal{G}_{\mathcal{D}}$ is a set of cofibrant homotopy generators of $\mathcal{D}$. Thus, by [14, Proposition 3.3.18] it is enough to show that $F(f \otimes G)$ is a weak equivalence in $\mathcal{E}'$ for every f in $\mathcal{S}$ and G in $\mathcal{G}_{\mathcal{D}}$. But, by assumption, $F \circ \otimes\colon L_{\mathcal{S}} \times \mathcal{D} \to \mathcal{E}'$, is a left Quillen bifunctor. Hence $F(f \otimes G)$ is a weak equivalence in $\mathcal{E}'$ since f is a weak equivalence in $L_{\mathcal{S}}\mathcal{C}$ between cofibrant objects and G is cofibrant in $\mathcal{D}$ (recall that, by Remark 2.11, we can assume without loss of generality that the morphisms in $\mathcal{S}$ are cofibrations between cofibrant objects). $\square$

3.3 Examples

3.3.1 Enriched Localisations and Colocalisations

Let $\mathcal{V}$ be a monoidal model category and let $\mathcal{C}$ be a $\mathcal{V}$-enriched model category. Then there is a Quillen adjunction of two variables $\mathcal{C} \times \mathcal{V} \to \mathcal{C}$. If $\mathcal{V}$ is combinatorial, $\mathcal{C}$ is left proper combinatorial and $\mathcal{S}$ is a set of morphisms in $\mathcal{C}$, then the $\mathcal{S}$-localised model structure (see Remark 2.11) is the $\mathcal{V}$-*enriched left Bousfield localisation* of $\mathcal{C}$ with respect to $\mathcal{S}$, as in [5, Definition 4.42]. Similarly if $\mathcal{K}$ is a set of objects in $\mathcal{C}$, then the $\mathcal{K}$-colocalised model structure of $\mathcal{C}$ along the left Quillen bifunctor is the *enriched right Bousfield localisation* of $\mathcal{C}$ with respect to $\mathcal{K}$.

If $\mathcal{V} = $ sSet, the category of simplicial sets, then we recover left and right Bousfield localisations of simplicial model categories.

3.3.2 Familiarisations

Let $\mathcal{C}$ be a spectral model category, that is, a model category which is compatibly enriched over the model category Sp of symmetric spectra. Then there is a Quillen adjunction of two variables $\mathcal{C} \times \mathrm{Sp} \to \mathcal{C}$. Let E be any spectrum and let $\mathcal{S}_E$ be the set of generating trivial cofibrations of the E-local model structure L_E Sp; see [2, Section1] or [3, Section 2]. Then the $\mathcal{S}_E$-localised model structure on $\mathcal{C}$ is the E-*familiarisation* of $\mathcal{C}$ in the sense of [4, Section 5].

If $\mathcal{S}$ is a set of morphisms in Sp, then we call the $\mathcal{S}$-localised model structure on $\mathcal{C}$ the *stable $\mathcal{S}$-familiarisation*.

4 Postnikov Sections of Model Categories

We are going to apply a construction closely related to our localisation construction to obtain Postnikov sections in combinatorial model categories. We start by reviewing the classical case of topological spaces and then explain how we can use our construction to generalise this concept to arbitrary combinatorial model categories which are not necessarily simplicial.

4.1 The Classical Case: Spaces

We are going to recall some results for Postnikov towers and k-types in simplicial sets. For details, see [14, Section 1.5]. Note that in [14] this is formulated for topological spaces rather than simplicial sets, but due to the compatibility of localisation with the geometric realisation and total singular complex functors this will not be an issue; see [14, Section 1.6].

Let $f_k \colon S^{k+1} \to D^{k+2}$ denote the boundary inclusion in sSet from the $(k+1)$-sphere to the $(k+2)$-disk. We form the left Bousfield localisation of sSet with respect to this map, obtaining the model structure L_{f_k} sSet. This is called the *model structure for k-types* of simplicial sets. In fact, a simplicial set X is f_k-local if and only if it is a Kan complex and its homotopy groups vanish in degrees $k+1$ and higher, for every choice of basepoint in X. The localisation map

$$l_k \colon X \longrightarrow L_{f_k} X,$$

which is defined as the fibrant replacement of X in L_{f_k} sSet, is a π_i-isomorphism for $i \leq k$ and every choice of a basepoint in X.

Remark 4.1 The model category L_{f_k} sSet exists and is cellular (hence cofibrantly generated), since it is a left Bousfield localisation of a left proper cellular model category; see for example [14, Theorem 4.1.1].

Proposition 4.2 *If a map of fibrant simplicial sets $X \to Y$ is a π_i-isomorphism for $i \leq k$ and every choice of a basepoint in X, then it is an f_k-local equivalence, that is, a weak equivalence in L_{f_k} sSet.*

Proof This is [14, Propositions 1.5.2 and 1.5.4]. □

As a consequence of the above, we see that the localisation map l_k of a simplicial set X to its f_k-localisation is nothing but the projection of X onto its kth Postnikov section $P_k X$; see [14, Theorem 1.5.3]. For details on Postnikov sections, see for instance [11, VI.3] or [13, Section 4.3].

If $i \geq j$, then $P_j X$ is fibrant in L_{f_i} sSet, that is, $P_j X$ is f_i-local. Hence, there is a commutative triangle

$$\begin{array}{ccc} X & \xrightarrow{l_j} & P_j X \\ {\scriptstyle l_i}\downarrow & \nearrow & \\ P_i X, & & \end{array}$$

since, by definition, l_i is a trivial cofibration in L_{f_i} sSet.

Furthermore, let $X \to Y$ be a weak equivalence in L_{f_k} sSet. Consider the commutative square

$$
\begin{array}{ccc}
X & \longrightarrow & Y \\
\downarrow & & \downarrow \\
P_k X & \longrightarrow & P_k Y.
\end{array}
$$

We know that the vertical maps are π_i-isomorphisms for $i \leq k$ by definition. As the top horizontal and the two vertical maps are f_k-local equivalences, then so is the map $P_k X \to P_k Y$. But of course $P_k X$ and $P_k Y$ are f_k-local, so the bottom map is in fact a π_i-isomorphism for all i and every choice of basepoint. Thus, any weak equivalence in L_{f_k} sSet is a π_i-isomorphism for $i \leq k$. Together with Proposition 3.2 we can conclude that a map $X \to Y$ between fibrant simplicial sets is a weak equivalence in L_{f_k} sSet if and only if it is a π_i-isomorphism for $i \leq k$ and every choice of basepoint.

4.2 The General Case

Let $\mathcal{C}$ be now a simplicial, left proper, combinatorial model category. Again, by f_k we denote the map $S^{k+1} \to D^{k+2}$ in simplicial sets, and denote $W_k = I_{\mathcal{C}} \square f_k$, where $I_{\mathcal{C}}$ denotes the set of generating cofibrations in $\mathcal{C}$ (see Remark 2.11). We then form the Bousfield localisation $P_k \mathcal{C} = L_{W_k} \mathcal{C}$ which we will call the *model structure for k-types* in $\mathcal{C}$.

When $\mathcal{C}$ is a model category that is not necessarily simplicial (but still left proper and combinatorial), we can still define the model structure for k-types in $\mathcal{C}$. In this case we use the technique of *framings*; see [15, Section 5] or [2, Section 3] for details. As explained in [15, Remark 5.2.9], framings provide any model category $\mathcal{C}$ with bifunctors

$$
\begin{aligned}
- \otimes - &: \mathcal{C} \times \text{sSet} \longrightarrow \mathcal{C}, \\
(-)^{(-)} &: \text{sSet}^{op} \times \mathcal{C} \longrightarrow \mathcal{C}, \\
\text{map}_l(-, -) &: \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \text{sSet}, \\
\text{map}_r(-, -) &: \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \text{sSet},
\end{aligned}
$$

and adjunctions

$$
\mathcal{C}(X \otimes K, Y) \cong \text{sSet}(K, \text{map}_l(X, Y)) \quad \text{and} \quad \mathcal{C}^{op}(Y^K, X) \cong \text{sSet}(K, \text{map}_r(X, Y)).
$$

The homotopy function complex $\text{map}_{\mathcal{C}}(-, -)$ agrees with the derived functors $R \, \text{map}_l(-, -)$ and $R \, \text{map}_r(-, -)$. Moreover, if X is a cofibrant object in $\mathcal{C}$ and Y is a fibrant object in $\mathcal{C}$, then

$$
X \otimes - : \text{sSet} \rightleftarrows \mathcal{C} : \text{map}_l(X, -) \quad \text{and} \quad Y^{(-)} : \text{sSet} \rightleftarrows \mathcal{C}^{op} : \text{map}_r(-, Y).
$$

are Quillen pairs; see [15, Corollary 5.4.4].

Note that a framing does not provide $\mathcal{C}$ with a simplicial model structure though, as map_l and map_r only agree up to a zig-zag of weak equivalences [15, Proposition 5.4.7]. However, it does mean that $\text{Ho}(\mathcal{C})$ is a closed $\text{Ho}(\text{sSet})$-module category. If $\mathcal{C}$ is already a simplicial model category, the action from the simplicial structure agrees with the $\text{Ho}(\text{sSet})$-action coming from framings. In our previous notation, for a simplicial model category $\mathcal{C}$, the simplicial enrichment $\text{Map}(-, -) = \text{Hom}_l(-, -)$ coincides with $\text{map}_l(-, -)$ and $\text{map}_r(-, -)$, and the cotensor is $\text{Hom}_r(-, -)$.

Thus, if our model category $\mathcal{C}$ is not simplicial we can define $W_k = I_{\mathcal{C}} \square f_k$ just as before, where the pushout-product is constructed using the functor $\otimes$ coming from the framing.

(Note that if $\mathcal{C}$ is simplicial, the simplicial action on the homotopy category agrees with the action coming from the framing, so the rest of this section, particularly Definition 3.4, does not depend on any choice of framing.)

Remark 4.3 If $\mathcal{C}$ is a *pointed* model category, then it is equipped with a *pointed framing* [15, Section 5.7], where the category of simplicial sets is replaced by *pointed* simplicial sets sSet$_*$.

Definition 4.4 Let $\mathcal{C}$ be a left proper combinatorial model category. We call $P_k\mathcal{C} = L_{W_k}\mathcal{C}$ the *model category of k-types in* $\mathcal{C}$. An object of $\mathcal{C}$ is called a *k-type* (or *k-truncated*) if it is W_k-local, that is, fibrant in $P_k\mathcal{C}$.

Before we look further into the properties of this localisation, we need an analogue of Theorem 2.14(i) using framings. Note that we are taking the class of maps $\mathcal{S}$ in sSet (see Remark 2.11).

Proposition 4.5 *Let $\mathcal{C}$ be a combinatorial, left proper model category with generating cofibrations $I_\mathcal{C}$ and set of cofibrant homotopy generators $\mathcal{G}_\mathcal{C}$. Furthermore, let $\mathcal{S}$ be a class of maps in* sSet. *Then the following are equivalent for an object Z of $\mathcal{C}$:*

(i)   *Z is $I_\mathcal{C}\square\mathcal{S}$-local*
(ii) *Z is $\mathcal{G}_\mathcal{C} \otimes \mathcal{S}$-local*
(iii) *Z is fibrant and $\operatorname{map}_\mathcal{C}(G, Z)$ is $\mathcal{S}$-local for every G in $\mathcal{G}_\mathcal{C}$.*
(iv) *Z is fibrant and for every $g\colon X \to Y$ in $\mathcal{S}$ the induced map*

$$g^*\colon Z^Y \longrightarrow Z^X$$

is a weak equivalence in $\mathcal{C}$.

Proof The proof follows exactly the same pattern as Theorem 2.14(i), so we are not spelling it out here. The occurring functors $\otimes$, Hom$_r$ and Hom$_l$ have been replaced by the functors $\otimes$, $(-)^{(-)}$, map$_l$ and map$_r$ coming from framings. The only properties needed are that when X is cofibrant and Y is fibrant in $\mathcal{C}$, the adjunctions $(X\otimes-, \operatorname{map}_l(X, -))$ and $(Y^{(-)}, \operatorname{map}_r(-, Y))$ are Quillen pairs, and that map$_l(X, Y)$ is weakly equivalent to map$_r(X, Y)$; see [15, Proposition 5.4.7]. As the homotopy mapping objects are also derived from framings, these are all compatible and the necessary adjunctions hold just as before. $\square$

Proposition 4.6 *Let $\mathcal{C}$ be a left proper combinatorial model category with set of cofibrant homotopy generators $\mathcal{G}_\mathcal{C}$. A fibrant object Z of $\mathcal{C}$ is a k-type if and only if $\pi_i(\operatorname{map}_\mathcal{C}(X, Z)) = 0$ for all X in $\mathcal{C}$, $i > k$ and every choice of a basepoint, or equivalently, $\pi_i(\operatorname{map}_\mathcal{C}(G, Z)) = 0$ for all G in $\mathcal{G}_\mathcal{C}$, $i > k$ and every choice of a basepoint.*

Proof By Proposition 3.5 we have that Z is W_k-local if and only if Z is fibrant in $\mathcal{C}$ and map$_\mathcal{C}(G, Z)$ is a k-type in sSet for every G in $\mathcal{G}_\mathcal{C}$. Since every object in $\mathcal{C}$ is weakly equivalent to a homotopy colimit of objects of $\mathcal{G}_\mathcal{C}$ and those commute with homotopy function complexes, the result follows. $\square$

In combination with Proposition 3.5 we also have the following.

Corollary 4.7 *Let $\mathcal{C}$ be a left proper combinatorial model category with set of cofibrant homotopy generators $\mathcal{G}_{\mathcal{C}}$, and let $f_k \colon S^{k+1} \to D^{k+2}$ in* sSet. *Then the model category of k-types $P_k\mathcal{C}$ coincides with $L_{\mathcal{G}_{\mathcal{C}} \otimes f_k}\mathcal{C}$.*

Remark 4.8 When $\mathcal{C}$ is a simplicial model category, then the model structure $P_k\mathcal{C}$ agrees with the model structure for k-types defined by Barwick in [5, Proposition 5.28].

In the context of *familiarisation* as defined by [4], one would define $P_k\mathcal{C}$ to be $L_{I_{\mathcal{C}} \square J_{f_k}}\mathcal{C}$ where J_{f_k} denotes the generating acyclic cofibrations of L_{f_k} sSet. However, those two model structures agree since L_{f_k} sSet $= L_{J_{f_k}}$ sSet by Proposition 2.16. The reason one works with the acyclic cofibrations in [4] is to actually cut down the localised weak equivalences of some L_S sSet to a generating set if S is not a set. However, in our case we only localise simplicial sets at one morphism, making this technicality unnecessary.

Proposition 4.9 *Let $\mathcal{C}$ be a left proper combinatorial model category. The model category of k-types $P_k\mathcal{C}$ has the following properties:*

(i) *Every Quillen adjunction* sSet $\rightleftarrows \mathcal{C}$ *gives rise to a Quillen adjunction L_{f_k} sSet $\rightleftarrows P_k\mathcal{C}$, and $P_k\mathcal{C}$ is the closest model structure to $\mathcal{C}$ with this property. This means that if $\mathcal{C} \rightleftarrows \mathcal{D}$ is a Quillen adjunction such that the composite* sSet $\rightleftarrows \mathcal{D}$ *factors over L_{f_k} sSet, then $P_k\mathcal{C} \rightleftarrows \mathcal{D}$ is also a Quillen adjunction.*

(ii) *If $\mathcal{C}$ is a simplicial model category, then $P_k\mathcal{C}$ is a L_{f_k} sSet-model category.*

(iii) *For every $k \geq 0$ the model structures $P_k P_{k+1}\mathcal{C}$ and $P_k\mathcal{C}$ coincide.*

Proof Let $F \colon$ sSet $\rightleftarrows \mathcal{C} \colon U$ be a Quillen adjunction. By [14, Proposition 3.3.18], in order for this to be a Quillen adjunction between L_{f_k} sSet and $P_k\mathcal{C}$, we need to show that $F(f_k)$ is a weak equivalence in $P_k\mathcal{C}$.

By [15, Chapter 5], all Quillen adjunctions such that the left adjoint is defined on the Quillen model structure on simplicial sets arise from framings, that is, for every left Quillen functor F there is an object $A \in \mathcal{C}$ such that the left derived functors of F and $A \otimes -$ agree. (Every adjunction between sSet and $\mathcal{C}$ is of the form $(A^{\bullet} \otimes -, \operatorname{Hom}(A^{\bullet}, -))$ for some cosimplicial object $A^{\bullet} \in \mathcal{C}^{\Delta}$, and every Quillen adjunction is given by a framing on $A^{\bullet}[0] = A$; see [15, Proposition 3.1.5 and Section 5.2] and [2, Section 3].) So we have to show that $A \otimes f_k$ is a weak equivalence in $P_k\mathcal{C}$. By Proposition 3.5, all maps of the form $G \otimes f_k$ are weak equivalences for all homotopy generators $G \in \mathcal{G}$. But as every A is a filtered homotopy colimit of such generators, and $- \otimes f_k$ commutes with such homotopy colimits, $A \otimes f_k$ is a weak equivalence as well.

Now let $F' \colon \mathcal{C} \rightleftarrows \mathcal{D} \colon U'$ be another Quillen adjunction such that $F'(F(f_k))$ is a weak equivalence in $\mathcal{D}$ for any left Quillen functor F as before. This means that $F'(A \otimes f_k)$ is a weak equivalence in $\mathcal{D}$ for any $A \in \mathcal{C}$. So in particular, F' sends all morphisms $G \otimes f_k$ to weak equivalences, where $G \in \mathcal{G}$. As $P_k\mathcal{C} = L_{\mathcal{G} \otimes f_k}\mathcal{C}$, this means that F' sends all the weak equivalences in $P_k\mathcal{C}$ to weak equivalences in $\mathcal{D}$, which is what we wanted to prove.

Part (ii) follows from Proposition 2.17(ii), and part (iii) follows from the fact that both model structures have the same cofibrations and the same fibrant objects. This last point can be easily checked using the characterisation of local objects given in Proposition 3.5. $\square$

Before we move on to the next result, let us note the following. The fact that a model category is λ-presentable only depends on the underlying category, not on its model

structure. Also, the left Bousfield localisation of a cofibrantly generated model category is again cofibrantly generated. Thus, if a model category is combinatorial, so is any left Bousfield localisation of it. Also, as Bousfield localisation does not change cofibrations and preserves weak equivalences, if $\mathcal{G}_{\mathcal{C}}$ is a set of homotopy generators for a combinatorial model category $\mathcal{C}$, then $\mathcal{G}_{\mathcal{C}}$ will also be a set of homotopy generators for any left Bousfield localisation of $\mathcal{C}$.

We can now characterise the weak equivalences of $P_k\mathcal{C}$.

Proposition 4.10 *Let $\mathcal{C}$ be a left proper combinatorial model category and let $f : X \to Y$ be a morphism in $\mathcal{C}$. If its fibrant approximation $\hat{f} : \hat{X} \to \hat{Y}$ in $P_k\mathcal{C}$ induces a weak equivalence*

$$\hat{f}_* : \operatorname{map}_{\mathcal{C}}(G, \hat{X}) \longrightarrow \operatorname{map}_{\mathcal{C}}(G, \hat{Y})$$

in $L_{f_k} \operatorname{sSet}$ for all homotopy generators G in $\mathcal{G}_{\mathcal{C}}$, then the morphism f is a weak equivalence in $P_k\mathcal{C}$.

Proof We have that $\mathcal{G}_{\mathcal{C}} \subset \mathcal{G}_{\mathcal{C}} \otimes \mathcal{G}_{\operatorname{sSet}}$ as we can, without loss of generality, add the single point to $\mathcal{G}_{\operatorname{sSet}}$. Thus, the statement follows from Proposition 2.18. Note that if $\mathcal{C}$ is not simplicial, then we have to replace the mapping objects in that proof by the mapping objects given by framings. $\qquad\square$

Corollary 4.11 *Let $\mathcal{C}$ be a left proper combinatorial model category and let $f : X \to Y$ be a morphism in $\mathcal{C}$. If its fibrant approximation $\hat{f} : \hat{X} \to \hat{Y}$ in $P_k\mathcal{C}$ induces an isomorphism of homotopy groups*

$$\pi_i(\hat{f}_*) : \pi_i(\operatorname{map}_{\mathcal{C}}(G, \hat{X})) \longrightarrow \pi_i(\operatorname{map}_{\mathcal{C}}(G, \hat{Y}))$$

with respect to all basepoints for all $i \leq k$ and homotopy generators G in $\mathcal{G}_{\mathcal{C}}$, then f is a weak equivalence in $P_k\mathcal{C}$.

4.3 Example: $\mathcal{S}$-local Simplicial sets

Let us consider the example of left Bousfield localisations of pointed simplicial sets, $\mathcal{C} = L_{\mathcal{S}} \operatorname{sSet}_*$. We can easily describe Postnikov sections in this model category. By definition, $P_k L_{\mathcal{S}} \operatorname{sSet}_* = L_{W_k} L_{\mathcal{S}} \operatorname{sSet}_*$ where $W_k = I_{L_{\mathcal{S}} \operatorname{sSet}_*} \square f_k$ and $f_k : S^{k+1} \to D^{k+2}$. As the generating cofibrations $I_{L_{\mathcal{S}} \operatorname{sSet}_*}$ of $L_{\mathcal{S}} \operatorname{sSet}_*$ are the same as the generating cofibrations of sSet_* we have that $I_{L_{\mathcal{S}} \operatorname{sSet}_*} \square f_k = I_{\operatorname{sSet}_*} \square f_k$. Then we can conclude that

$$P_k L_{\mathcal{S}} \operatorname{sSet}_* = L_{f_k} L_{\mathcal{S}} \operatorname{sSet}_* .$$

Thus, X is fibrant in $P_k L_{\mathcal{S}} \operatorname{sSet}_*$ if and only if it is a Kan complex, $\mathcal{S}$-local and $\pi_i X = \pi_i L_{\mathcal{S}} X = 0$ for $i > k$.

4.4 Example: k-types in Chain Complexes

Let $\operatorname{Ch}_b(R)$ denote the category of *non-negatively graded* chain complexes of R-modules, where R is a commutative ring with unit. We are going to apply the results from the previous section this category. This is a particularly interesting example as it concerns a model category that is not simplicial, although it is left proper and combinatorial. We are going to describe the k-types in $\operatorname{Ch}_b(R)$ as well as describe some of the weak equivalences. The results are just what one would expect and fit very neatly with our general setup.

Consider the standard projective model structure on $\mathrm{Ch}_b(R)$; see [9, Section 7]. The weak equivalences are given by quasi-isomorphisms, fibrations are morphisms which are surjective in positive degrees, and cofibrations are monomorphisms with projective cokernel in every degree. Consider the model category of k-types of chain complexes, $P_k \mathrm{Ch}_b(R)$. According to Definition 3.4, this is the left Bousfield localisation with respect to the set

$$W_k = I_{\mathrm{Ch}_b(R)} \square \{ f_k \colon S^{k+1} \longrightarrow D^{k+2} \}.$$

Now the generating cofibrations in the standard projective model structure are given by the inclusions

$$I_{\mathrm{Ch}_b(R)} = \{ \mathbb{S}^n \longrightarrow \mathbb{D}^{n+1} \mid n \geq 0 \},$$

where $\mathbb{S}^n$ denotes the chain complex which is R in degree n and zero everywhere else, and $\mathbb{D}^{n+1}$ denotes the chain complex with R in degrees n and $n+1$ with the identity differential between them, and zero everywhere else. To avoid notational confusion with the sphere and disk in spaces, we will use bold face for these.

Recall that the suspension functor Σ in a pointed model category $\mathcal{C}$ can be defined using pointed framings; see [15, Definition 6.1.1]. If X is a cofibrant object then $\Sigma X = X \otimes S^1$, that is, ΣX is the pushout of the diagram

$$X \otimes \partial \Delta[1] \longrightarrow X \otimes \Delta[1]$$
$$\downarrow$$
$$*$$

So let us look at the pushout-product

In the category $\mathrm{Ch}_b(R)$, the suspension is given by shifting. By [15, Section 6.1], $\mathbb{S}^i \otimes S^j = \mathbb{S}^{i+j}$. Furthermore, framings are compatible with fibre and cofibre sequences [15, Section 6.2], so the above diagram is the same as

The pushout of two disks and a sphere is just another sphere, hence we obtain

$$W_k = \{ \mathbb{S}^{n+k+1} \longrightarrow \mathbb{D}^{n+k+2} \mid n \geq 0 \},$$

so $P_k \mathrm{Ch}_b(R)$ is just localising $\mathrm{Ch}_b(R)$ at the map $g_k \colon \mathbb{S}^{k+1} \to \mathbb{D}^{k+2}$. Note that local equivalences are closed under (positive) suspensions, and hence localising with respect to g_k is the same as localising with respect to $\{ \Sigma^n g_k \mid n \geq 0 \} = W_k$.

Recall that we denote by $\mathrm{map}_{\mathrm{Ch}_b(R)}(-, -)$ a homotopy function complex for the model category $\mathrm{Ch}_b(R)$.

Proposition 4.12 *A fibrant chain complex M in $\mathrm{Ch}_b(R)$ is a k-type if and only if $H_i(M) = 0$ for all $i > k$.*

Proof The chain complex M is g_k-local if and only if

$$\pi_i(\mathrm{map}_{\mathrm{Ch}_b(R)}(\mathbb{D}^{k+2}, M)) \longrightarrow \pi_i(\mathrm{map}_{\mathrm{Ch}_b(R)}(\mathbb{S}^{k+1}, M))$$

is an isomorphism for all $i \geq 0$. By adjunction, this is equivalent to

$$[\mathbb{D}^{i+k+2}, M] \longrightarrow [\mathbb{S}^{i+k+1}, M]$$

being an isomorphism for all $i \geq 0$, where the square brackets denote morphisms in the derived category $D_b(R)$. But as the chain complex $\mathbb{D}^{i+k+2}$ is acyclic and the right hand side equals the homology $H_{i+k+1}(M)$ of M, the above is equivalent to $H_i(M) = 0$ for all $i > k$. □

We can now say something about the weak equivalences in $P_k \mathrm{Ch}_b(R)$. Recall that if M is a chain complex in $\mathrm{Ch}_b(R)$, we denote by $M[n]$ the n-fold suspension of M.

Proposition 4.13 *Let $f : M \to N$ be a morphism of chain complexes such that $H_i(f)$ is an isomorphism for $0 \leq i \leq k + 1$. Then f is a weak equivalence in $P_k \mathrm{Ch}_b(R)$.*

Proof This is very similar to [14, Proposition 1.5.2]. Without loss of generality, let $f : M \to N$ be a cofibration of chain complexes, that is, a monomorphism with degreewise projective cokernel. We know that f is a weak equivalence in $P_k \mathrm{Ch}_b(R)$ if and only if

$$\mathrm{map}_{\mathrm{Ch}_b(R)}(N, Z) \longrightarrow \mathrm{map}_{\mathrm{Ch}_b(R)}(M, Z)$$

is an acyclic fibration in simplicial sets for all g_k-local Z; see [14, Section 1.3.1]. This is equivalent to having a lift in the diagram

for all $n \geq 0$. By adjunction, this is equivalent to having a lift in the diagram

for all $n \geq 0$.

Note that for a chain complex A, the complex $A \otimes \Delta[n]$ is the nth suspension of the cone of A, while $A \otimes \partial\Delta[n]$ is $A[n] \oplus A[n + 1]$ (direct sum of chain complexes). Thus, the top left corner of this square is the nth suspension of the mapping cone of f, and the left vertical map is given by $f \oplus \mathrm{id}$. As f was assumed to be a homology isomorphism in degrees 0 to $k+1$, this mapping cone is acyclic in degrees 0 to $k+1$ and the left vertical map

is a homology isomorphism in those degrees (as the cone of a chain complex is obviously acyclic).

We know by Proposition 3.12 that $H_j(Z) = 0$ for $j \geq k + 1$. This means that we have a square in $\mathrm{Ch}_b(R)$ where the left vertical map is a cofibration and the right vertical map a fibration. In order to have the desired lift, one of those maps would have to be a homology isomorphism.

As the left vertical map is a homology isomorphism in degrees 0 to $k + 1$, we can use the methods in [9, Section 7.7, proof of MC4(i)] to construct a lift in those degrees. Then we can use the same method as in [9, Section 7.5, proof of MC4(ii)] to inductively construct the lift from degrees $k + 2$ onwards, which uses that $H_j(N) = 0$ for $j \geq k + 1$.

So we have constructed a lift in the above square, which means that $f \colon M \to N$ is a weak equivalence in $P_k \mathrm{Ch}_b(R)$. $\qquad\square$

As a consequence of Proposition 3.12 and Proposition 3.13 we get the following.

Corollary 4.14 *If M is a chain complex in $\mathrm{Ch}_b(R)$, then the W_k-localisation is given by the k-truncation $\tau_{\geq k} M$ of X, defined by*

$$(\tau_{\geq k} M)_n = \begin{cases} M_n & \text{if } n < k, \\ M_k / B_k & \text{if } n = k, \\ 0 & \text{if } n > k, \end{cases}$$

where $B_k = im(d_k)$ denotes the group of k-boundaries.

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