BOUSFIELD LOCALISATIONS ALONG QUILLEN BIFUNCTORS AND APPLICATIONS

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ABSTRACT. We describe left and right Bousfield localisations along Quillen adjunctions of two variables. These localised model structures can be used to define Postnikov sections and homological localisations of arbitrary model categories, and to study the homotopy limit model structure on the category of sections of a left Quillen presheaf of localised model structures. We obtain explicit results in this direction in concrete examples of towers and fiber products of model categories. In particular, we prove that the category of simplicial sets is Quillen equivalent to the homotopy limit model structure of its Postnikov tower, and that the category of symmetric spectra is Quillen equivalent to the homotopy fiber product of its Bousfield arithmetic square. For spectral model categories, we show that the homotopy fiber of a stable left Bousfield localisation is a stable right Bousfield localisation.

INTRODUCTION

Quillen adjunctions between spectra or spaces and other model categories are a useful way to study homotopy structures. For example, one can gain insight into a model category $\mathcal{C}$ by studying the action of the homotopy category of simplicial sets $\mathrm{Ho}(\mathrm{sSet})$ or the stable homotopy category $\mathrm{Ho}(\mathrm{Sp})$ on the homotopy category $\mathrm{Ho}(\mathcal{C})$.

In [4] it was studied how this set-up is compatible with homological localisations of spectra, that is, left Bousfield localisation at E_*-isomorphisms for a homology theory E. For a stable model category $\mathcal{C}$, [4] constructed a corresponding Bousfield localisation $\mathcal{C}_E$ of $\mathcal{C}$, called stable E-familiarisation, with useful universal properties. Namely, $\mathcal{C}_E$ is the "closest" model category to $\mathcal{C}$ such that every left Quillen functor $\mathrm{Sp} \to \mathcal{C}_E$ factors over E-local spectra $L_E \mathrm{Sp}$. In this paper, we take this notion further by studying the compatibility of Quillen adjunctions of two variables $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ with Bousfield localisations of $\mathcal{C}$ or $\mathcal{D}$.

Our first application is describing Postnikov sections. For the category of simplicial sets sSet, the model structure $P_n \, \mathrm{sSet}$ for nth Postnikov sections is obtained via localizing sSet with respect to the map $f_n : S^{n+1} \to D^{n+2}$. Using our localisation construction and combining it with the theory of framings [20] we can now consider Postnikov sections $P_n \mathcal{C}$ in model categories $\mathcal{C}$ that are not necessarily simplicial.

Together, the model categories $P_n \mathcal{C}$ for $n \geq 0$ form a left Quillen presheaf which can be used to model Postnikov towers of objects in $\mathcal{C}$. We can then study the notion of "hypercompleteness" which encodes whether any object in $\mathcal{C}$ is the homotopy limit of its Postnikov tower. The classical result that this is the case for sSet fits

into this framework, as does the non-simplicial example $\mathrm{Ch}_b(\mathbb{Z})$ of bounded below chain complexes. Moreover, we show that the category of simplicial sets is Quillen equivalent to the homotopy limit model structure of the left Quillen presheaf for Postnikov towers; cf. [8, Section 4].

We also turn to applications from classical stable homotopy theory. It is well-known that any spectrum X can be built, using Bousfield's arithmetic square [9], as a homotopy pullback of the diagram of homological localisations

$$L_{M\mathbb{Z}_J}X \longrightarrow L_{M\mathbb{Q}}X \longleftarrow L_{M\mathbb{Z}_K}X,$$

where J and K form any partition of the set of prime numbers. Furthermore, the chromatic convergence theorem [26, Theorem 7.5.7] states that a finite p-local spectrum X is the homotopy limit of its chromatic localisations $L_{E(n)}X$. We present categorified versions of these statements. Firstly, we prove that the category Sp is Quillen equivalent to the homotopy limit model structure of the left Quillen presheaf for Bousfield arithmetic squares of spectra. Next, we consider the homotopy limit model structure on the left Quillen presheaf of chromatic towers Chrom(Sp) and show that the Quillen adjunction

$$\mathrm{const} : \mathrm{Sp} \rightleftarrows \mathrm{Chrom(Sp)} : \mathrm{lim}$$

induces a composite

$$\mathrm{Ho(Sp)}^{fin} \xrightarrow{\mathbb{L}\mathrm{const}} \mathrm{Ho(Chrom(Sp))}^F \xrightarrow{\mathrm{holim}} \mathrm{Ho(Sp)}^{fin}$$

which is isomorphic to the identity. (Here, F and fin denote suitable finiteness conditions.) This set-up is a step towards new insights into the structure of the stable homotopy category.

As a final application we focus on a correspondence between the homotopy fibre of a left Bousfield localisation $\mathcal{C} \to L_S\mathcal{C}$ and certain right Bousfield localisations. This is then used, among other examples, to understand the layers of the Postnikov towers established earlier, and to study the correspondence between stable localisations and stable colocalisations.

The paper is organised as follows. In Section 1, we recall some terminology and basic results on locally presentable categories and combinatorial model categories. In Section 2, we discuss how Quillen bifunctors are compatible with left and right Bousfield localisations. Given a Quillen adjunction of two variables $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ we describe Bousfield localisations of $\mathcal{E}$ based on localisations of $\mathcal{C}$ or $\mathcal{D}$ and their universal properties. As particular examples, we recover enriched localisations [5], enriched colocalisations and E-familiarisations [3, 4]. Section 3 reviews the case of k-types in combinatorial model categories. Finally, in Section 4, we recall the injective model structure and the homotopy limit model structure on the category of sections of a left Quillen presheaf, and we study it in the case of towers and fiber products of model categories. Finally, we provide an explicit criterion for hypercompleteness.

1. Review of combinatorial model categories

In this section, we recall some terminology on locally presentable categories and combinatorial model categories. The essentials of the theory of locally presentable categories can be found in [1], [14] or [24]. Foundations on the theory of combinatorial model categories may be found in [6], [11] and [23].

1.1. Locally presentable categories.
Let λ be a regular cardinal. A small category $\mathcal{I}$ is called λ-*filtered* if it is nonempty and satisfies the following two conditions:

(i) Given any set of objects $\{a_i \mid i \in I\}$ in $\mathcal{I}$, where $|I| < \lambda$, there is an object a and a morphism $a_i \to a$ for each $i \in I$.

(ii) Given any set of parallel morphisms $\{\alpha_j \colon a \to a' \mid j \in J\}$ in $\mathcal{I}$ between two fixed objects, where $|J| < \lambda$, there is a morphism $\gamma \colon a' \to a''$ such that $\gamma \circ \alpha_j$ is the same morphism for all $j \in J$.

An object X of a category $\mathcal{C}$ is called λ-*presentable* if the functor $\mathcal{C}(X, -)$ from $\mathcal{C}$ to sets preserves λ-filtered colimits.

A category $\mathcal{C}$ is called λ-*accessible* if λ-filtered colimits exist in $\mathcal{C}$ and there is a set of λ-presentable objects $\mathcal{A}$ such that every object of $\mathcal{C}$ is a λ-filtered colimit of objects from $\mathcal{A}$. In fact, if $\mathcal{C}$ is λ-accessible, then the collection of isomorphism classes of λ-presentable objects $\mathcal{C}_\lambda$ is a set, and for every object X, the overcategory $(\mathcal{C}_\lambda \downarrow X)$ is λ-filtered and the canonical map

$$\operatorname{colim}(\mathcal{C}_\lambda \downarrow X) \longrightarrow X$$

is an isomorphism. A category is *accessible* if it is λ-accessible for some regular cardinal λ.

A full subcategory $\mathcal{D}$ of an accessible category $\mathcal{C}$ is called is *accessibly embedded* if there is a regular cardinal λ such that $\mathcal{D}$ is closed under λ-filtered colimits in $\mathcal{C}$.

A cocomplete category $\mathcal{C}$ is *locally presentable* if it is cocomplete and accessible. Every locally λ-presentable category is equivalent to a full, reflective subcategory closed under λ-filtered colimits of the category of presheaves on some small category; see [1, Proposition 1.46].

1.2. Combinatorial model categories.
A model category $\mathcal{C}$ is cofibrantly generated if there exists a set $I_\mathcal{C}$ of *generating cofibrations* and a set $J_\mathcal{C}$ of *generating trivial cofibrations* that one can use to perform the small object argument (see [19, Definition 11.1.2] or [20, Definition 2.1.17] for a precise definition).

A *homotopy function complex* in a model category $\mathcal{C}$ is a functorial choice of a fibrant simplicial set $\operatorname{map}_\mathcal{C}(X, Y)$, for every two objects X and Y in $\mathcal{C}$, whose homotopy type is the same as the diagonal of the bisimplicial set $\mathcal{C}(\widetilde{\mathbf{X}}, \widehat{\mathbf{Y}})$, where $\widetilde{\mathbf{X}}$ is a cosimplicial resolution of X and $\widehat{\mathbf{Y}}$ is a simplicial resolution of Y; for more details, see [19, Chapter 17]. Functorial homotopy function complexes exist in every model category; see [19, Proposition 17.5.18].

Let $\mathcal{C}$ be a model category with homotopy function complex $\operatorname{map}_\mathcal{C}(-, -)$ and let $i \colon A \to B$ and $p \colon X \to Y$ be two morphisms in $\mathcal{C}$. Then the pair (i, p) is a *homotopy orthogonal pair* if the diagram

$$
\begin{array}{ccc}
\operatorname{map}_\mathcal{C}(B, X) & \longrightarrow & \operatorname{map}_\mathcal{C}(B, Y) \\
\downarrow & & \downarrow \\
\operatorname{map}_\mathcal{C}(A, X) & \longrightarrow & \operatorname{map}_\mathcal{C}(A, Y)
\end{array}
$$

is a homotopy fiber square [19, Definition 17.8.1]. In particular, the pair $(\emptyset \to W, p)$ is homotopy orthogonal if the induced map

$$p_* \colon \operatorname{map}_{\mathcal{C}}(W, X) \longrightarrow \operatorname{map}_{\mathcal{C}}(W, Y)$$

is a weak equivalence of simplicial sets.

Recall that a model category is *left proper* if pushouts of weak equivalences along cofibrations are weak equivalences, and *right proper* if pullbacks of weak equivalences along fibrations are weak equivalences. A model category is *proper* if it is left and right proper.

In a cofibrantly generated model category the set of generating cofibrations can be used to detect weak equivalences. A proof of the following result can be found in [19, Theorem 17.8.18].

Proposition 1.1. *Let $\mathcal{C}$ be a cofibrantly generated model category and let $I_{\mathcal{C}}$ be a set of generating cofibrations. Assume that $\mathcal{C}$ is left proper or that the domains of the elements of $I_{\mathcal{C}}$ are cofibrant. Then, a map f in $\mathcal{C}$ is a weak equivalence if and only if for every map i in $I_{\mathcal{C}}$ the pair (i, f) is a homotopy orthogonal pair.* $\square$

A set of *homotopy generators* for a model category $\mathcal{C}$ consists of a small full subcategory $\mathcal{G}$ such that every object of $\mathcal{C}$ is weakly equivalent to a filtered homotopy colimit of objects of $\mathcal{G}$. A set of homotopy generators also detects weak equivalences.

Proposition 1.2. *Let $\mathcal{C}$ be a model category with homotopy function complex $\operatorname{map}_{\mathcal{C}}(-, -)$ and a set of cofibrant homotopy generators $\mathcal{G}$. Then a map $f \colon X \to Y$ in $\mathcal{C}$ is a weak equivalence if and only if for every G in $\mathcal{G}$ the pair (j_G, f) is a homotopy orthogonal pair, where j_G denotes the morphism $\emptyset \to G$.*

Proof. Let j_W denote the map $\emptyset \to W$. By [19, Theorem 17.7.7] a map $f \colon X \to Y$ is a weak equivalence if and only if for every object W, where j_W denotes the map $\emptyset \to W$, that is, if and only if the induced map

$$f_* \colon \operatorname{map}_{\mathcal{C}}(W, X) \longrightarrow \operatorname{map}_{\mathcal{C}}(W, Y)$$

is a weak equivalence. Let $\widehat{f} \colon \widehat{X} \to \widehat{Y}$ a fibrant approximation. By assumption every object W is weakly equivalent to a filtered homotopy colimit hocolim G_α of objects of $\mathcal{G}$, and hence [19, Theorem 19.4.2(2)] and [19, Theorem 19.4.4] imply that

$$\operatorname{map}_{\mathcal{C}}(\operatorname{hocolim} G_\alpha, \widehat{X}) \simeq \operatorname{holim}(\operatorname{map}_{\mathcal{C}}(G_\alpha, \widehat{X}))$$

and that the map

$$\operatorname{holim}(\operatorname{map}_{\mathcal{C}}(G_\alpha, \widehat{X})) \longrightarrow \operatorname{holim}(\operatorname{map}_{\mathcal{C}}(G_\alpha, \widehat{Y}))$$

is a weak equivalence. The result now follows from the fact that homotopy function complexes are homotopy invariant; see [19, Theorem 17.7.7]. $\square$

Let λ be a regular cardinal. A model category $\mathcal{C}$ is called λ-*combinatorial* if it is cofibrantly generated and the underlying category is locally λ-presentable. A model category $\mathcal{C}$ is called *combinatorial* if it is λ-combinatorial for some regular cardinal λ.

Every combinatorial model category is Quillen equivalent to a left Bousfield localisation of a category of diagrams of simplicial sets equipped with the projective model structure [11, Theorem 1.1] and many model categories of interest are combinatorial. Examples are pointed or unpointed simplicial sets, pointed or unpointed motivic spaces, symmetric spectra over simplicial sets [21, §3.4] or over

motivic spaces, module spectra over a ring spectrum [27, Theorem 4.1], bounded or unbounded chain complexes of modules over a ring [20, § 2.3], or any locally presentable category equipped with the *discrete* model structure, where the weak equivalences are the isomorphisms and all morphisms are fibrations and cofibrations.

Dugger also proved in [11, Proposition 4.7] that every combinatorial model category has a set of homotopy generators and that, moreover, they can be chosen to be cofibrant. We denote by $\mathcal{C} \downarrow X$ the slice category of $\mathcal{C}$ over an object X.

Proposition 1.3 (Dugger)**.** *Let λ be a regular cardinal and let $\mathcal{C}$ be a λ-combinatorial model category. Let $\mathcal{C}_\lambda$ the full subcategory of the λ-presentable objects. Then every object X is a canonical filtered homotopy colimit of objects of $\mathcal{C}_\lambda$. More precisely, the canonical map*

$$\mathrm{hocolim}(\mathcal{C}_\lambda \downarrow X) \longrightarrow X$$

is a weak equivalence. Moreover, there is regular cardinal $\mu > \lambda$ such that the canonical map

$$\mathrm{hocolim}(\mathcal{C}_\mu^{\mathrm{cof}} \downarrow X) \longrightarrow X$$

is a weak equivalence, where $\mathcal{C}_\mu^{\mathrm{cof}}$ denotes the full subcategory of $\mathcal{C}_\mu$ consisting of the cofibrant objects. □

Given a combinatorial model category $\mathcal{C}$, we will denote by $\mathcal{G}_\mathcal{C}$ the set of cofibrant homotopy generators given by the previous proposition.

Corollary 1.4. *Let $\mathcal{C}$ be a combinatorial model category with a set of generating cofibrations $I_\mathcal{C}$ and a set of cofibrant homotopy generators $\mathcal{G}_\mathcal{C}$. Assume that $\mathcal{C}$ is left proper or that the domains of the elements of $I_\mathcal{C}$ are cofibrant. Then, for every map f in $\mathcal{C}$, the pair (i, f) is a homotopy orthogonal pair for all i in $I_\mathcal{C}$ if and only for every G in $\mathcal{G}_\mathcal{C}$, the pair (j_G, f) is a homotopy orthogonal pair, where j_G denotes the morphism $\emptyset \to G$.*

Proof. This is a consequence of Proposition 1.1 and Proposition 1.2. □

2. Left and right Bousfield localisations along Quillen bifunctors

In this section we are going to discuss how Quillen bifunctors are compatible with left and right Bousfield localisation.

2.1. **Quillen bifunctors.** Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be categories. An *adjunction of two variables* from $\mathcal{C} \times \mathcal{D}$ to $\mathcal{E}$ is given by functors

$$- \otimes - : \mathcal{C} \times \mathcal{D} \longrightarrow \mathcal{E},$$

$$\mathrm{Hom}_r(-, -) : \mathcal{D}^{op} \times \mathcal{E} \longrightarrow \mathcal{C},$$

$$\mathrm{Hom}_l(-, -) : \mathcal{C}^{op} \times \mathcal{E} \longrightarrow \mathcal{D},$$

and natural isomorphisms

$$\mathcal{C}(X, \mathrm{Hom}_r(Y, Z)) \cong \mathcal{E}(X \otimes Y, Z) \cong \mathcal{D}(Y, \mathrm{Hom}_l(X, Z)).$$

We will sometimes denote an adjunction of two variables from $\mathcal{C} \times \mathcal{D}$ to $\mathcal{E}$ just by the left adjoint $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$. The analog notion for model categories appears in [20, Definition 4.2.1].

Definition 2.1. Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be model categories. An adjunction of two variables from $\mathcal{C} \times \mathcal{D}$ to $\mathcal{E}$ is a *Quillen adjunction of two variables* if for every cofibration $f \colon A \to B$ in $\mathcal{C}$ and every cofibration $g \colon X \to Y$ in $\mathcal{D}$, the pushout-product

$$f \square g \colon B \otimes X \coprod_{A \otimes X} A \otimes Y \longrightarrow B \otimes Y$$

is a cofibration in $\mathcal{E}$ which is a trivial cofibration if f or g are trivial cofibrations. We will refer to the left adjoint $\otimes$ of a Quillen adjunction of two variables as a *Quillen bifunctor.*

Remark 2.2. There are equivalent formulations of the previous condition satisfied by a Quillen adjunction of two variables in terms of $\mathrm{Hom}_r^{\square}$ and $\mathrm{Hom}_l^{\square}$, where $\mathrm{Hom}_r^{\square}$ and $\mathrm{Hom}_l^{\square}$ denote the respective adjoints of the pushout-product; see [20, Lemma 4.2.2].

Remark 2.3. If $(\otimes, \mathrm{Hom}_r, \mathrm{Hom}_l)$ is a Quillen adjunction of two variables from $\mathcal{C} \times \mathcal{D}$ to $\mathcal{E}$ and $F_1 \colon \mathcal{C}' \to \mathcal{C}$, $F_2 \colon \mathcal{D}' \to \mathcal{D}$ and $F_3 \colon \mathcal{E} \to \mathcal{E}'$ are left Quillen functors (with right adjoints G_1, G_2 and G_3, respectively), then

$$(F_3(F_1(-) \otimes F_2(-)), G_1 \, \mathrm{Hom}_r(F_2(-), G_3(-)), G_2 \, \mathrm{Hom}_l(F_1(-), G_1(-)))$$

is a Quillen adjunction of two variables from $\mathcal{C}' \times \mathcal{D}'$ to $\mathcal{E}'$.

Example 2.4. Let sSet denote the category of simplicial sets with the Kan–Quillen model structure. A *simplicial model structure* on a model category $\mathcal{C}$ is the same as a Quillen bifunctor $\mathcal{C} \times \mathrm{sSet} \to \mathcal{C}$. A *topological model structure* can be defined similarly, by replacing simplicial sets with the category of compactly generated Hausdorff spaces equipped with the Quillen model structure.

Let $(\mathcal{E}, \otimes, I, \mathrm{Hom}_{\mathcal{E}})$ be a closed symmetric monoidal category. A model structure on $\mathcal{E}$ is called a *monoidal model structure* if $- \otimes - \colon \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ is a Quillen bifunctor and the unit I is cofibrant.

Let $\mathcal{E}$ be a monoidal model category. An $\mathcal{E}$-*model category* is a category $\mathcal{C}$ enriched, tensored and cotensored over $\mathcal{E}$ together with a model structure such that the tensor, enrichment and cotensor define a Quillen adjunction of two variables.

The following two lemmas are an immediate consequence of the bifunctor adjunctions and we state them without proof. We will use the terminology $f \pitchfork g$ to indicate that a morphism f has the *left lifting property* with respect to g (or that g has the *right lifting property* with respect to f), that is, $f \pitchfork g$ if for every commutative diagram of the form

$$
\begin{array}{ccc}
A & \xrightarrow{\ i\ } & X \\
{\scriptstyle f}\downarrow & {\scriptstyle h}\ \ \nearrow & \downarrow{\scriptstyle g} \\
B & \xrightarrow{\ p\ } & Y,
\end{array}
$$

there is a diagonal lifting h such that $i = hf$ and $p = gh$.

Lemma 2.5. *Let $(\otimes, \mathrm{Hom}_r, \mathrm{Hom}_l)$ be an adjunction of two variables from $\mathcal{C} \times \mathcal{D}$ to $\mathcal{E}$ and let f, g and h be morphisms in $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$, respectively. The following are equivalent:*

   (i) $(f \square g) \pitchfork h$.

   (ii) $f \pitchfork \mathrm{Hom}_r^{\square}(g, h)$.

(iii) $g \pitchfork \mathrm{Hom}_l^\square(f, h)$. $\square$

Lemma 2.6. *Let* $(\otimes, \mathrm{Hom}_r, \mathrm{Hom}_l)$ *be an adjunction of two variables from* $\mathcal{C} \times \mathcal{D}$ *to* $\mathcal{E}$ *between model categories.*

   (i) *The following are equivalent:*
      (a) *Given a cofibration* f *in* $\mathcal{C}$ *and a cofibration* g *in* $\mathcal{D}$*, the morphism* $f \square g$ *is a cofibration in* $\mathcal{E}$*.*
      (b) *Given a cofibration* g *in* $\mathcal{D}$ *and a trivial fibration* h *in* $\mathcal{E}$*, the morphism* $\mathrm{Hom}_r^\square(g, h)$ *is a trivial fibration in* $\mathcal{C}$*.*
      (c) *Given a cofibration* f *in* $\mathcal{C}$ *and a trivial fibration* h *in* $\mathcal{E}$*, the morphism* $\mathrm{Hom}_l^\square(f, h)$ *is a trivial fibration in* $\mathcal{D}$*.*
   (ii) *The following are equivalent:*
      (a) *Given a trivial cofibration* f *in* $\mathcal{C}$ *and a cofibration* g *in* $\mathcal{D}$*, the morphism* $f \square g$ *is a trivial cofibration in* $\mathcal{E}$*.*
      (b) *Given a cofibration* g *in* $\mathcal{D}$ *and a fibration* h *in* $\mathcal{E}$*, the morphism* $\mathrm{Hom}_r^\square(g, h)$ *is a fibration in* $\mathcal{C}$*.*
      (c) *Given a trivial cofibration* f *in* $\mathcal{C}$ *and a fibration* h *in* $\mathcal{E}$*, the morphism* $\mathrm{Hom}_l^\square(f, h)$ *is a trivial fibration in* $\mathcal{D}$*.* $\square$

Note that if X is cofibrant in $\mathcal{C}$, then $X \otimes -$ is a left Quillen functor with right adjoint $\mathrm{Hom}_l(X, -)$. Similarly, if Y is cofibrant in $\mathcal{D}$, then $- \otimes Y$ is a left Quillen functor with right adjoint $\mathrm{Hom}_r(-, Y)$.

Just as in the case of Quillen functors (see [19, Proposition 8.5.4]) we have the following result which will be useful to test whether an adjunction of two variables is a Quillen bifunctor. In order to prove it, we will make use the following key result, which appears as [22, Lemma 7.14].

Lemma 2.7. *A cofibration in a model category is a trivial cofibration if and only if it has the left lifting property with respect to every fibration between fibrant objects. Dually, a fibration in a model category is a trivial fibration if and only if it has the right lifting property with respect to every cofibration between cofibrant objects.* $\square$

Proposition 2.8. *Let* $(\otimes, \mathrm{Hom}_r, \mathrm{Hom}_l)$ *be an adjunction of two variables from* $\mathcal{C} \otimes \mathcal{D}$ *to* $\mathcal{E}$ *between model categories. Suppose that if* g *is a cofibration (respectively trivial cofibration) in* $\mathcal{D}$ *and* h *is a trivial fibration (respectively fibration) in* $\mathcal{E}$*, then* $\mathrm{Hom}_r^\square(g, h)$ *is a trivial fibration in* $\mathcal{C}$*. Then the following are equivalent:*

   (i) $(\otimes, \mathrm{Hom}_r, \mathrm{Hom}_l)$ *is a Quillen adjunction of two variables.*
   (ii) *Given a cofibration* g *in* $\mathcal{D}$ *and a fibration between fibrant objects* $\hat{h}$ *in* $\mathcal{E}$*, the morphism* $\mathrm{Hom}_r^\square(g, \hat{h})$ *is a fibration in* $\mathcal{C}$*.*
  (iii) *Given a cofibration between cofibrant objects* $\tilde{g}$ *in* $\mathcal{D}$ *and a fibration* h *in* $\mathcal{E}$*, the morphism* $\mathrm{Hom}_r^\square(\tilde{g}, h)$ *is a fibration in* $\mathcal{C}$*.*
  (iv) *Given a cofibration between cofibrant objects* $\tilde{g}$ *in* $\mathcal{D}$ *and a fibration between fibrant objects* $\hat{h}$ *in* $\mathcal{E}$*, the morphism* $\mathrm{Hom}_r^\square(\tilde{g}, \hat{h})$ *is a fibration in* $\mathcal{C}$*.*

Proof. It is clear that (i) implies (ii), (iii) and (iv), that (ii) implies (iv) and that (iii) implies (iv). It then suffices, for example, to prove that (ii) implies (i) and that (iv) implies (ii).

In order to prove that (ii) implies (i), let g be any cofibration in $\mathcal{D}$ and h any fibration in $\mathcal{E}$. Then $\mathrm{Hom}_r^\square(g, h)$ is a fibration in $\mathcal{C}$ if and only if for every trivial cofibration j in $\mathcal{C}$, we have that $j \pitchfork \mathrm{Hom}_r^\square(g, h)$. But by Lemma 2.5, this

is equivalent to $(j \square g) \pitchfork h$, in other words, $j \square g$ being a trivial cofibration. By Lemma 2.6(i), we know that $j \square g$ is a cofibration. Hence, by Lemma 2.7 the previous condition is equivalent to $(j \square g) \pitchfork \hat{h}$ for $\hat{h}$ being any fibration between fibrant objets in $\mathcal{E}$. Again, by Lemma 2.5 this is equivalent to $j \pitchfork \mathrm{Hom}_r^\square(g, \hat{h})$ for $\hat{h}$ any fibration between fibrant objects. Since we are assuming that $\mathrm{Hom}_r^\square(g, \hat{h})$ is a fibration in $\mathcal{C}$, the last statement is true, so we can conclude that $\mathrm{Hom}_r^\square(g, h)$ is a fibration for any cofibration g and fibration h as required, which was the missing part for $(\otimes, \mathrm{Hom}_r, \mathrm{Hom}_l)$ to be a Quillen adjunction of two variables.

That part (iv) implies (ii) is proved in a very similar way to the previous point. Let g be any cofibration in $\mathcal{D}$ and let $\hat{h}$ be a fibration between fibrant objects in $\mathcal{E}$. Then $\mathrm{Hom}_r^\square(g, \hat{h})$ is a fibration in $\mathcal{C}$ if and only if $j \pitchfork \mathrm{Hom}_r^\square(g, \hat{h})$ for every trivial cofibration j in $\mathcal{C}$. By Lemma 2.5 this is equivalent to $g \pitchfork \mathrm{Hom}_l^\square(j, \hat{h})$ for every trivial cofibration j in $\mathcal{C}$. By Lemma 2.6(ii) the morphism $\mathrm{Hom}_l^\square(j, \hat{h})$ is a fibration, and therefore, by Lemma 2.7, the previous condition is equivalent to $\tilde{g} \pitchfork \mathrm{Hom}_l^\square(j, \hat{h})$ for every cofibration $\tilde{g}$ between cofibrant objects in $\mathcal{D}$. By adjunction, this is equivalent to saying that $j \pitchfork \mathrm{Hom}_r^\square(\tilde{g}, \hat{h})$ for every trivial cofibration j in $\mathcal{C}$, every cofibration between cofibrant objects $\tilde{g}$ in $\mathcal{D}$, and every fibration between fibrant objects $\hat{h}$ in $\mathcal{E}$. But $\mathrm{Hom}_r^\square(\tilde{g}, \hat{h})$ is a fibration, by assumption, hence (iv) is equivalent to (ii), which is what we wanted to prove. $\square$

2.2. **Left and right Bousfield localisation.** We recall the notion of *left Bousfield localisation* and *right Bousfield localisation* (also called *Bousfield colocalisation*) for model categories. Let $\mathcal{C}$ be a model category with homotopy function complex $\mathrm{map}_{\mathcal{C}}(-, -)$ and let $\mathcal{S}$ be a class of morphisms of $\mathcal{C}$ and $\mathcal{K}$ a class of objects in $\mathcal{C}$. We say that an object Z in $\mathcal{C}$ is $\mathcal{S}$-*local* if it is fibrant and for every morphism $f \colon A \to B$ in $\mathcal{S}$ the induced map

$$f^* \colon \mathrm{map}_{\mathcal{C}}(B, Z) \longrightarrow \mathrm{map}_{\mathcal{C}}(A, Z)$$

is a weak equivalence of simplicial sets. We say that a map $h \colon X \to Y$ in $\mathcal{C}$ is a $\mathcal{K}$-*colocal equivalence* if for every object K in $\mathcal{K}$ the induced map

$$h_* \colon \mathrm{map}_{\mathcal{C}}(K, X) \longrightarrow \mathrm{map}_{\mathcal{C}}(K, Y)$$

is a weak equivalence of simplicial sets.

The left Bousfield localisation of $\mathcal{C}$ with respect to $\mathcal{S}$ (if it exists) is a new model structure $L_{\mathcal{S}}\mathcal{C}$ on $\mathcal{C}$ such that

(i) the cofibrations of $L_{\mathcal{S}}\mathcal{C}$ are the same as those of $\mathcal{C}$,
(ii) the weak equivalences of $L_{\mathcal{S}}\mathcal{C}$ are the $\mathcal{S}$-*local equivalences*, that is, those maps $g \colon X \to Y$ such that the induced map

$$g^* \colon \mathrm{map}_{\mathcal{C}}(Y, Z) \longrightarrow \mathrm{map}_{\mathcal{C}}(X, Z)$$

is a weak equivalence of simplicial sets for every $\mathcal{S}$-local object Z,
(iii) the fibrant objects of $L_{\mathcal{S}}\mathcal{C}$ are the $\mathcal{S}$-local objects.

The $\mathcal{S}$-local equivalences between $\mathcal{S}$-local objects are weak equivalences in $\mathcal{C}$.

The right Bousfield localisation (or Bousfield colocalisation) of $\mathcal{C}$ with respect to $\mathcal{K}$ (if it exists) is a new model structure $C_{\mathcal{K}}\mathcal{C}$ on $\mathcal{C}$ such that

(i) the fibrations of $C_{\mathcal{K}}\mathcal{C}$ are the same as those of $\mathcal{C}$,
(ii) the weak equivalences of $C_{\mathcal{K}}\mathcal{C}$ are the $\mathcal{K}$-colocal equivalences,

(iii) the cofibrant objects of $C_{\mathcal{K}}\mathcal{C}$ are the $\mathcal{K}$-colocal objects, that is, those objects W that are cofibrant in $\mathcal{C}$ and such that

$$h_* \colon \operatorname{map}_{\mathcal{C}}(W, X) \longrightarrow \operatorname{map}_{\mathcal{C}}(W, Y)$$

is a weak equivalence of simplicial sets for every $\mathcal{K}$-colocal equivalence h.

The $\mathcal{K}$-colocal equivalences between $\mathcal{K}$-colocal objects are weak equivalences in $\mathcal{C}$.

Remark 2.9. Note that the definition of the $\mathcal{S}$-local objects depends only on the homotopy function complex, which is homotopy invariant. Therefore, we can always replace the morphisms in $\mathcal{S}$ by weakly equivalent ones consisting of cofibrations between cofibrant objects, without changing the model structure $L_{\mathcal{S}}\mathcal{C}$. Hence, without loss of generality we will often assume that when we localize with respect to a class of morphisms, these morphisms are cofibrations between cofibrant objects.

Similarly, we can assume without loss of generality that when we colocalise with respect to a class of objects, they are cofibrant.

There are two main classes of model categories where localisations with respect to a *set* of morphisms and colocalisations with respect to a *set* of objects are always known to exist. These are the left/right proper cellular model categories [19, Theorem 4.1.1 and Theorem 5.1.1] and the left/right proper combinatorial model categories [5, Theorem 4.7 and Proposition 5.13]. If $\mathcal{C}$ is left proper and combinatorial (or cellular) and $\mathcal{S}$ is a set of morphisms of $\mathcal{C}$, then $L_{\mathcal{S}}\mathcal{C}$ is also left proper and combinatorial (or cellular). If $\mathcal{C}$ is right proper and combinatorial (or cellular) and $\mathcal{K}$ is a set of objects of $\mathcal{C}$, then $C_{\mathcal{K}}\mathcal{C}$ is also right proper, but it is not cofibrantly generated in general.

Definition 2.10. Let $\otimes \colon \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ be a Quillen bifunctor, where $\mathcal{D}$ is cofibrantly generated with set of generating cofibrations $I_{\mathcal{D}}$ and set of cofibrant homotopy generators $\mathcal{G}_{\mathcal{D}}$. Assume that $\mathcal{E}$ is proper and combinatorial and let $\mathcal{S}$ and $\mathcal{K}$ be sets of morphisms and objects in $\mathcal{C}$, respectively.

(i) The $\mathcal{S}$-*local model structure* on $\mathcal{E}$, denoted by $L_{\mathcal{S}}\mathcal{E}$, is the left Bousfield localisation $L_{\mathcal{S}\square I_{\mathcal{D}}}\mathcal{E}$ of $\mathcal{E}$ with respect to $\mathcal{S}\square I_{\mathcal{D}}$.
(ii) The $\mathcal{K}$-*colocal model structure* on $\mathcal{E}$, denoted by $C_{\mathcal{K}}\mathcal{E}$ is the right Bousfield localisation $C_{\mathcal{K}\otimes\mathcal{G}_{\mathcal{D}}}\mathcal{E}$ of $\mathcal{E}$ with respect to $\mathcal{K} \otimes \mathcal{G}_{\mathcal{D}}$.

Remark 2.11. If $(\otimes, \operatorname{Hom}_r, \operatorname{Hom}_l)$ is a Quillen adjunction of two variables from $\mathcal{C} \times \mathcal{D}$ to $\mathcal{E}$ and $\mathcal{S}$ is a set of morphisms in $\mathcal{D}$ (instead of in $\mathcal{C}$), then we can also define an $\mathcal{S}$-localised model structure on $\mathcal{E}$ as $L_{I_{\mathcal{C}}\square\mathcal{S}}\mathcal{E}$, where $I_{\mathcal{C}}$ is the set of generating cofibrations of $\mathcal{C}$. All the results from this section can be rephrased in terms of a set of morphisms in $\mathcal{D}$, by suitably replacing Hom_l by Hom_r and vice versa. This is due to the fact that if $(\otimes, \operatorname{Hom}_r, \operatorname{Hom}_l)$ is an adjunction of two variables from $\mathcal{C} \times \mathcal{D}$ to $\mathcal{E}$ and $\tau \colon \mathcal{D} \times \mathcal{C} \to \mathcal{C} \times \mathcal{D}$ is the functor that interchanges the components, then $(\otimes \circ \tau, \operatorname{Hom}_l, \operatorname{Hom}_r)$ is an adjunction of two variables from $\mathcal{D} \times \mathcal{C}$ to $\mathcal{E}$.

Theorem 2.12. *Let $(\otimes, \operatorname{Hom}_l, \operatorname{Hom}_r)$ be a Quillen adjunction of two variables from $\mathcal{C} \times \mathcal{D}$ to $\mathcal{E}$. Let $\mathcal{S}$ and $\mathcal{K}$ be classes of morphisms and objects in $\mathcal{C}$, respectively. Assume that $\mathcal{D}$ is combinatorial with set of generating cofibrations $I_{\mathcal{D}}$ and set of cofibrant homotopy generators $\mathcal{G}_{\mathcal{D}}$ and that it is either left proper or the domains of the elements of $I_{\mathcal{D}}$ are cofibrant.*

(i) *The following are equivalent for an object Z of $\mathcal{E}$:*
  (a) *Z is $\mathcal{S}\square I_{\mathcal{D}}$-local.*

(b) Z is $\mathcal{S} \otimes \mathcal{G}_{\mathcal{D}}$-*local.*

(c) Z *is fibrant and* $\mathrm{Hom}_r(G, Z)$ *is* $\mathcal{S}$-*local for every* G *in* $\mathcal{G}_D$.

(d) Z *is fibrant and for every* $f \colon A \to B$ *in* $\mathcal{S}$ *the induced map*

$$f^* \colon \mathrm{Hom}_l(B, Z) \longrightarrow \mathrm{Hom}_l(A, Z)$$

is a weak equivalence in $\mathcal{D}$.

(ii) *The following are equivalent for a morphism* $h \colon X \to Y$ *of* $\mathcal{E}$:

(a) h *is a* $\mathcal{K} \otimes \mathcal{G}_{\mathcal{D}}$-*colocal equivalence.*

(b) *For every* G *in* $\mathcal{G}_{\mathcal{D}}$ *the induced map*

$$\hat{h}_* \colon \mathrm{Hom}_r(G, \hat{X}) \longrightarrow \mathrm{Hom}_r(G, \hat{Y})$$

is a $\mathcal{K}$-*colocal equivalence, where* $\hat{h}$ *is a fibrant replacement of* h.

(c) *For every* K *in* $\mathcal{K}$ *the induced map*

$$\hat{h}_* \colon \mathrm{Hom}_l(K, \hat{X}) \longrightarrow \mathrm{Hom}_l(K, \hat{Y})$$

is a weak equivalence in $\mathcal{D}$, *where* $\hat{h}$ *is a fibrant replacement of* h.

Proof. Let Z be any object of $\mathcal{E}$. Then Z is $\mathcal{S} \square I_{\mathcal{D}}$-local if and only if it is fibrant and

$$\mathrm{map}_{\mathcal{E}}(B \otimes Y, Z) \longrightarrow \mathrm{map}_{\mathcal{E}}\left(A \otimes Y \coprod_{A \otimes X} B \otimes X, Z\right)$$

is a weak equivalence of simplicial sets for every map $A \to B$ in $\mathcal{S}$ and every map $X \to Y$ in $I_{\mathcal{D}}$. By adjunction and the compatibility of homotopy function complexes with Quillen pairs (see [19, Proposition 17.4.16]), the previous condition is equivalent to the diagram

$$
\begin{array}{ccc}
\mathrm{map}_{\mathcal{D}}(Y, \mathrm{Hom}_l(B, Z)) & \longrightarrow & \mathrm{map}_{\mathcal{D}}(Y, \mathrm{Hom}_l(A, Z)) \\
\downarrow & & \downarrow \\
\mathrm{map}_{\mathcal{D}}(X, \mathrm{Hom}_l(B, Z)) & \longrightarrow & \mathrm{map}_{\mathcal{D}}(X, \mathrm{Hom}_l(A, Z))
\end{array}
$$

being a homotopy fiber square. This is the same as saying that for every morphism $A \to B$ in $\mathcal{S}$ and every morphism $X \to Y$ in $I_{\mathcal{D}}$, the pair given by the morphisms $X \to Y$ and $\mathrm{Hom}_l(B, Z) \to \mathrm{Hom}_l(A, Z)$ is a homotopy orthogonal pair.

By Corollary 1.4 the previous condition amounts to saying that the pair given by $\emptyset \to G$ and $\mathrm{Hom}_l(B, Z) \to \mathrm{Hom}_l(A, Z)$ is a homotopy orthogonal pair for every G in $\mathcal{G}_{\mathcal{D}}$, that is,

$$\mathrm{map}_{\mathcal{D}}(G, \mathrm{Hom}_l(B, Z)) \longrightarrow \mathrm{map}_{\mathcal{D}}(G, \mathrm{Hom}_l(A, Z))$$

is a weak equivalence. Again by adjunction and the compatibility of homotopy function complexes with Quillen adjunctions, this is equivalent to saying that

$$\mathrm{map}_{\mathcal{E}}(B \otimes G, Z) \longrightarrow \mathrm{map}_{\mathcal{E}}(A \otimes G, Z)$$

is a weak equivalence for every G in $\mathcal{G}_{\mathcal{D}}$, and this is precisely the condition of Z being $\mathcal{S} \otimes \mathcal{G}_{\mathcal{D}}$-local. This proves that (a) and (b) are equivalent.

By adjunction (b) is equivalent to the fact that

$$\mathrm{map}_{\mathcal{C}}(B, \mathrm{Hom}_r(G, Z)) \longrightarrow \mathrm{map}_{\mathcal{C}}(A, \mathrm{Hom}_r(G, Z))$$

is a weak equivalence for every map $A \to B$ in $\mathcal{S}$. Hence (b) and (c) are equivalent.

Now, Proposition 1.2 shows that (b) is equivalent to $\mathrm{Hom}_l(B, Z) \to \mathrm{Hom}_l(A, Z)$ being a weak equivalence in $\mathcal{D}$, which concludes the proof of part (i).

To prove part (ii), first observe that a morphism $h\colon X \to Y$ is a $\mathcal{K} \otimes \mathcal{G}_\mathcal{D}$-colocal equivalence if and only if $\hat{h}\colon \hat{X} \to \hat{Y}$ is a $\mathcal{K} \otimes \mathcal{G}_\mathcal{D}$-colocal equivalence. This happens if and only if

$$\mathrm{map}_\mathcal{E}(K \otimes G, \hat{X}) \longrightarrow \mathrm{map}_\mathcal{E}(K \otimes G, \hat{Y})$$

is a weak equivalence of simplicial sets for every K in $\mathcal{K}$ and every G in $\mathcal{G}_\mathcal{D}$. As in the proof of part (i), by adjunction and the compatibility of homotopy function complexes with Quillen adjunctions, this is equivalent to saying that

$$\mathrm{map}_\mathcal{C}(K \operatorname{Hom}_r(G, \hat{X})) \longrightarrow \mathrm{map}_\mathcal{C}(K, \operatorname{Hom}_r(G, \hat{Y}))$$

is a weak equivalence for every K in $\mathcal{K}$ and every G in $\mathcal{G}_\mathcal{D}$, or that

$$\mathrm{map}_\mathcal{D}(G \operatorname{Hom}_l(K, \hat{X})) \longrightarrow \mathrm{map}_\mathcal{D}(G, \operatorname{Hom}_r(K, \hat{Y}))$$

is a weak equivalence for every K in $\mathcal{K}$ and every G in $\mathcal{G}_\mathcal{D}$. $\qquad\square$

Corollary 2.13. *Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be left proper combinatorial model categories and let $\otimes\colon \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ be a Quillen bifunctor. Let $\mathcal{S}$ be a set morphisms in $\mathcal{C}$ and let $\mathcal{G}_\mathcal{D}$ be a set of cofibrant homotopy generators of $\mathcal{D}$. Then $L_\mathcal{S}\mathcal{E} = L_{\mathcal{S} \otimes \mathcal{G}_\mathcal{D}}\mathcal{E}$.*

Proof. The result follows immediately from Theorem 2.12 $\qquad\square$

Proposition 2.14. *Let $\mathcal{C}$ be a combinatorial model category and $\mathcal{S}$ a set of morphisms. If $J_\mathcal{S}$ is a set of generating trivial cofibrations of $L_\mathcal{S}\mathcal{C}$, then $L_\mathcal{S}\mathcal{C} = L_{J_\mathcal{S}}\mathcal{C}$.*

Proof. This is [23, Proposition A.3.7.4] $\qquad\square$

Proposition 2.15. *Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be left proper combinatorial model categories and let $\otimes\colon \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ be a Quillen bifunctor. Let $\mathcal{S}$ be a set of morphisms in $\mathcal{C}$. Then $\otimes\colon L_\mathcal{S}\mathcal{C} \times \mathcal{D} \to L_\mathcal{S}\mathcal{E}$ is a Quillen bifunctor.*

Proof. By [20, Corollary 4.2.5] it is enough to prove that the pushout-product axiom holds for the sets of generating cofibrations and trivial cofibrations of $L_\mathcal{S}\mathcal{C}$ and $\mathcal{D}$. As the cofibrations in $L_\mathcal{S}\mathcal{C}$ and $\mathcal{C}$ as well as the cofibrations in $L_\mathcal{S}\mathcal{E}$ and $\mathcal{E}$ agree, it is sufficient to only consider the following case. Let $J_\mathcal{S}$ be a set of generating trivial cofibrations of $L_\mathcal{S}\mathcal{C}$ and let $I_\mathcal{D}$ be a set of generating cofibrations of $\mathcal{D}$. Since the cofibrations of $L_\mathcal{S}\mathcal{C}$ are the same as those in $\mathcal{C}$, it suffices to prove that if i is in $J_\mathcal{S}$ and j is in $I_\mathcal{D}$, then $i \square j$ is a $\mathcal{S} \square I_\mathcal{D}$-equivalence in $\mathcal{C}$. In fact, we will prove that the $J_\mathcal{S} \square I_\mathcal{D}$-equivalences coincide with the $\mathcal{S} \square I_\mathcal{D}$-equivalences.

Let $\mathcal{G}_\mathcal{D}$ be a set of cofibrant homotopy generators of $\mathcal{D}$. By Theorem 2.12(i), an object Z of $\mathcal{E}$ is $\mathcal{S} \square I_\mathcal{D}$-local if and only if $\operatorname{Hom}_r(G, Z)$ is $\mathcal{S}$-local for every G in $\mathcal{G}_D$. But by Proposition 2.14, $\mathcal{S}$-local objects coincide with $J_\mathcal{S}$-local objects. Hence $\operatorname{Hom}_r(G, Z)$ is $J_\mathcal{S}$-local for every G in $\mathcal{G}_\mathcal{D}$ and thus Z is $J_\mathcal{S} \square I_\mathcal{D}$-local. $\qquad\square$

Proposition 2.16. *Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be model categories with sets of cofibrant homotopy generators $\mathcal{G}_\mathcal{C}$, $\mathcal{G}_\mathcal{D}$ and $\mathcal{G}_\mathcal{E}$, respectively. Suppose that $\mathcal{D}$ is left proper and combinatorial. Let $(\otimes, \operatorname{Hom}_l, \operatorname{Hom}_r)$ be a Quillen adjunction of two variables from $\mathcal{C} \times \mathcal{D}$ to $\mathcal{E}$ and let $\mathcal{S}$ be a class of morphisms in $\mathcal{C}$. Let $f\colon X \to Y$ be a map in $\mathcal{E}$ and let $\hat{f}\colon \hat{X} \to \hat{Y}$ be a fibrant approximation to f in $L_{S_e}\mathcal{E}$. If the induced map*

$$\hat{f}_*\colon \operatorname{Hom}_r(G, \hat{X}) \longrightarrow \operatorname{Hom}_r(G, \hat{Y})$$

is an $\mathcal{S}$-equivalence in $\mathcal{C}$ for every G in $\mathcal{G}_\mathcal{D}$ and $\mathcal{G}_\mathcal{E} \subset \mathcal{G}_\mathcal{C} \otimes \mathcal{G}_\mathcal{D}$, then f is an $\mathcal{S}$-equivalence in $\mathcal{E}$.

Proof. By Theorem 2.12(i) the objects $\mathrm{Hom}_r(G, \hat{X})$ and $\mathrm{Hom}_r(G, \hat{Y})$ are both $\mathcal{S}$-local. Thus $\hat{f}_*$ is an $\mathcal{S}$-equivalence between $\mathcal{S}$-local objects and hence a weak equivalence in $\mathcal{C}$. This implies that

$$\mathrm{map}_{\mathcal{C}}(W, \mathrm{Hom}_r(G, \hat{X})) \longrightarrow \mathrm{map}_{\mathcal{C}}(W, \mathrm{Hom}_r(G, \hat{Y}))$$

is a weak equivalence of simplicial sets for every W in $\mathcal{G}_{\mathcal{C}}$ and every G in $\mathcal{G}_{\mathcal{D}}$. By adjunction and compatibility of homotopy function complexes with Quillen functors this is equivalent to

$$\mathrm{map}_{\mathcal{E}}(W \otimes G, \hat{X}) \longrightarrow \mathrm{map}_{\mathcal{E}}(W \otimes G, \hat{Y})$$

being a weak equivalence of simplicial sets for every W in $\mathcal{G}_{\mathcal{C}}$ and every G in $\mathcal{G}_{\mathcal{D}}$. Since by assumption $\mathcal{G}_{\mathcal{E}} \subset \mathcal{G}_{\mathcal{C}} \otimes \mathcal{G}_{\mathcal{D}}$, this implies that $\hat{f}$ is a weak equivalence in $\mathcal{E}$. Now, by the 2-out-of-3 axiom and the fact that weak equivalences in $\mathcal{E}$ are $\mathcal{S}$-equivalences, it follows that f is an $\mathcal{S}$-equivalence. $\square$
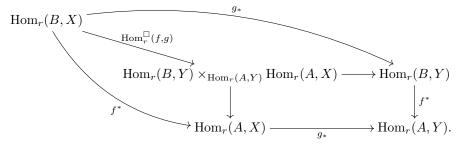
Definition 2.17. Let $\otimes\colon \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ be a Quillen bifunctor and let $\mathcal{S}$ and be a set of maps in $\mathcal{C}$. We say that $\mathcal{E}$ *is $\mathcal{S}$-familiar* if $\otimes\colon L_{\mathcal{S}}\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ is a Quillen bifunctor.

Remark 2.18. In particular, it follows from Proposition 2.15 that the $\mathcal{S}$-local model structure $L_{\mathcal{S}}\mathcal{E}$ is $\mathcal{S}$-familiar.

Proposition 2.19. *Let $(\otimes, \mathrm{Hom}_r, \mathrm{Hom}_l)$ be a Quillen adjunction of two variables from $\mathcal{C} \times \mathcal{D}$ to $\mathcal{E}$ and let $\mathcal{S}$ be a set of maps in $\mathcal{C}$. Then $\mathcal{E}$ is $\mathcal{S}$-familiar if and only if $\mathrm{Hom}_r(X, Y)$ is $\mathcal{S}$-local for every X cofibrant in $\mathcal{D}$ and Y fibrant in $\mathcal{E}$.*

Proof. The "only if" part follows from the fact that if $\mathcal{E}$ is $\mathcal{S}$-familiar and X is cofibrant in $\mathcal{D}$, then the functor $\mathrm{Hom}_r(X, -)\colon \mathcal{E} \to L_{\mathcal{S}}\mathcal{C}$ is right Quillen. Hence, for every Y fibrant in $\mathcal{E}$, we have that $\mathrm{Hom}_r(X, Y)$ is fibrant is fibrant in $L_{S_{\mathcal{C}}}\mathcal{C}$, that is, $S_{\mathcal{C}}$-local.

Conversely, we want to show that if $\mathrm{Hom}_r(X, Y)$ is $\mathcal{S}$ local for every cofibrant X and fibrant Y, then $L_{S_{\mathcal{C}}}\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ is also a Quillen bifunctor. Let f be a cofibration (respectively, a trivial cofibration) in $\mathcal{D}$ and let g be a trivial fibration (respectively, a fibration) in $\mathcal{E}$. Because $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ is assumed to be a Quillen bifunctor, the map $\mathrm{Hom}_r^{\square}(f, g)$ is a trivial fibration in $\mathcal{C}$. Therefore, by Proposition 2.8 it suffices to prove that if $f\colon A \to B$ is a cofibration between cofibrant objects in $\mathcal{D}$ and $g\colon X \to Y$ is a fibration between fibrant objects in $\mathcal{E}$, then $\mathrm{Hom}_r^{\square}(f, g)$ is a fibration in $L_{\mathcal{S}}\mathcal{C}$. Consider the pullback diagram

The right vertical map f^* is a fibration in $L_{\mathcal{S}}\mathcal{C}$, since it is a fibration in $\mathcal{C}$ between $\mathcal{S}$-local objects (see [19, Proposition 3.3.16]). Since fibrations are closed under pullbacks, the left vertical map is a also fibration in $L_{S_{\mathcal{C}}}$. But $\mathrm{Hom}_r(A, X)$ is $\mathcal{S}$-local (that is, fibrant in $L_{\mathcal{S}}\mathcal{C}$) and therefore so is $\mathrm{Hom}_r(B, Y) \times_{\mathrm{Hom}_r(A,Y)} \mathrm{Hom}_r(A, X)$.

Hence, we have proved that $\mathrm{Hom}^{\square}_r(f, g)$ is a fibration in $\mathcal{C}$ between $S_{\mathcal{C}}$-local objects. By [19, Proposition 3.3.16] this means that $\mathrm{Hom}^{\square}_r(f, g)$ is a fibration in $L_S\mathcal{C}$. □

We have seen that for a Quillen bifunctor $\otimes\colon \mathcal{C}\times\mathcal{D}\to\mathcal{E}$ and a set S of morphisms in $\mathcal{C}$, the new model structure $L_{S_{\mathcal{C}}}\mathcal{E}$ on $\mathcal{E}$ gives rise to a Quillen bifunctor

$$\otimes\colon L_S\mathcal{C}\times\mathcal{D}\longrightarrow L_S\mathcal{E}.$$

We can now state that this model structure $L_S\mathcal{E}$ is the "closest" model structure to $\mathcal{E}$ with this property in the following sense.

Proposition 2.20. *Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{E}$ be left proper combinatorial model categories and let $\otimes\colon \mathcal{C}\times\mathcal{D}\to\mathcal{E}$ be a Quillen bifunctor. Let $F\colon\mathcal{E}\to\mathcal{E}'$ be a left Quillen functor and S a set of morphisms in $\mathcal{C}$. If $\mathcal{E}'$ is S-familiar with respect to the Quillen bifunctor $F\circ\otimes\colon\mathcal{C}\times\mathcal{D}\to\mathcal{E}\to\mathcal{E}'$, then*

$$F\colon L_S\mathcal{E}\longrightarrow\mathcal{E}'$$

is also a left Quillen functor, that is, F factors over the S-localisation of $\mathcal{E}$.

Proof. By Corollary 2.13 we have that $L_S\mathcal{E}=L_{S\otimes\mathcal{G}_{\mathcal{D}}}\mathcal{E}$, where $\mathcal{G}_{\mathcal{D}}$ is a set of cofibrant homotopy generators of $\mathcal{D}$. Thus, by [19, Proposition 3.3.18] it is enough to show that $F(f\otimes G)$ is a weak equivalence in $\mathcal{E}'$ for every f in S and G in $\mathcal{G}_{\mathcal{D}}$. But, by assumption, $F\circ\otimes\colon L_S\times\mathcal{D}\to\mathcal{E}'$, is a Quillen bifunctor. Hence $F(f\otimes G)$ is a weak equivalence in $\mathcal{E}'$ since f is a weak equivalence in $L_S\mathcal{C}$ between cofibrant objects and G is cofibrant in $\mathcal{D}$. □

2.3. **Examples.**

2.3.1. *Enriched localisations and colocalisations.* Let $\mathcal{V}$ be a monoidal model category and let $\mathcal{C}$ be a $\mathcal{V}$-enriched model category. Then there is a Quillen adjunction of two variables $\mathcal{C}\times\mathcal{V}\to\mathcal{C}$. If $\mathcal{V}$ is combinatorial, $\mathcal{C}$ is left proper combinatorial and S is a set of maps in $\mathcal{C}$, then the S-localised model structure (see Remark 2.11) is the $\mathcal{V}$-*enriched left Bousfield localisation* of $\mathcal{C}$ with respect to S, as in [5, Definition 4.42]. Similarly if $\mathcal{K}$ is a set of objects in $\mathcal{C}$, then the $\mathcal{K}$-colocalised model structure of $\mathcal{C}$ along the Quillen bifunctor is the *enriched right Bousfield localisation* of $\mathcal{C}$ with respect to $\mathcal{K}$.

If $\mathcal{V}=\mathrm{sSet}$, the category of simplicial sets, then we recover left and right Bousfield localisations of simplicial model categories.

2.3.2. *Familiarisations.* Let $\mathcal{C}$ be a spectral category. Then there is a Quillen adjunction of two variables $\mathcal{C}\times\mathrm{Sp}\to\mathcal{C}$, where Sp denotes the model category of symmetric spectra. Let E be any spectrum and let S_E be the set of generating trivial cofibrations of the E-local model structure $L_E\,\mathrm{Sp}$. Then the S_E-localised model structure on $\mathcal{C}$ is the E-*familiarisation* of $\mathcal{C}$ in the sense of [4, Section 5].

If S is a set of morphisms in Sp, then we call the S-localised model structure on $\mathcal{C}$ the *stable S-familiarisation*.

3. k-TYPES

3.1. **The classical case: spaces.** We are going to recall some results for Postnikov towers and k-types in simplicial sets. For details, see [19, Section 1.5]. Note that in [19] this is formulated for topological spaces rather than simplicial sets, but due

to the compatibility of localisation with the geometric realisation and total singular complex functors this will not be an issue; see [19, Section 1.6].

Let $f_k \colon S^{k+1} \longrightarrow D^{k+2}$ denote the morphism in sSet from the $(k+1)$-sphere to the $(k+2)$-disk. We form the left Bousfield localisation of sSet with respect to this map, obtaining the model structure L_{f_k} sSet. This is called the *category of k-types* of simplicial sets. In fact, a simplicial set X is f_k-local if and only if it is a Kan complex and its homotopy groups vanish in degrees $k+1$ and higher, for every choice of basepoint in X. The localisation map

$$l_k \colon X \longrightarrow L_{f_k}X,$$

which is defined as the fibrant replacement of X in L_{f_k} sSet, is a π_i-isomorphism for $i \leq k$ and every choice of a basepoint in X.

Remark 3.1. The model category L_{f_k} sSet is cofibrantly generated/cellular, since it is a left Bousfield localisation of a cofibrantly generated/cellular model category; see for example [19, Theorem 4.1.1].

Proposition 3.2. *If a map of fibrant simplicial sets $X \to Y$ is a π_i-isomorphism for $i \leq k$ and every choice of a basepoint in X, then it is an f_k-equivalence, that is, a weak equivalence in L_{f_k} sSet.*

Proof. This is [19, Propositions 1.5.2 and 1.5.4]. $\qquad\qquad\qquad\qquad\square$

As a consequence of the above, we see that the localisation map l_k of a simplicial set X to its f_k-localisation is nothing but the projection of X onto its k-th Postnikov section P_kX. For details on Postnikov sections, see for instance [15, VI.3] or [18, Section 4.3].

If $i \geq j$, then $L_{f_j}X$ is fibrant in L_{f_i} sSet, that is, $L_{f_j}X$ is f_i-local. Hence, there is a commutative triangle

$$\begin{array}{ccc} X & \xrightarrow{\ l_j\ } & P_jX \\ {\scriptstyle l_i}\downarrow & \nearrow & \\ P_iX, & & \end{array}$$

since, by definition, l_i is a trivial cofibration in L_{f_i} sSet.

Furthermore, let $X \to Y$ be a weak equivalence in P_k sSet. Consider the commutative square

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ P_kX & \longrightarrow & P_kY. \end{array}$$

We know that the vertical maps are π_i-isomorphisms for $i \leq k$ by definition. As the top horizontal and the two vertical maps are P_k-equivalences, then so is the map $P_kX \to P_kY$. But of course P_kX and P_kY are P_k-local, so the bottom map is in fact a π_i-isomorphism for all i. Thus, any weak equivalence in P_k sSet is a π_i-isomorphism for $i \leq k$. Together with Proposition 3.2 we can conclude that $X \to Y$ is a weak equivalence in P_k sSet if and only if it is a π_i-isomorphism for $i \leq k$.

3.2. **The general case.** Let $\mathcal{C}$ be now a simplicial, left proper, combinatorial model category. Again, by f_k we denote the map $S^{k+1} \to D^{k+2}$ in simplicial sets, and denote $W_k = I_{\mathcal{C}} \square f_k$, where $I_{\mathcal{C}}$ denotes the set of generating cofibrations in $\mathcal{C}$. We then form the Bousfield localisation $P_k\mathcal{C} = L_{W_k}\mathcal{C}$ which we will call the *model structure for k-types* in $\mathcal{C}$.

When $\mathcal{C}$ is a model category that is not necessarily simplicial, we can still define the model structure for k-types in $\mathcal{C}$. In this case we use the technique of *framings*; see [20, Section 5] or [2, Section 3] for details. Framings provide any model category $\mathcal{C}$ with bifunctors

$$- \otimes - \colon \mathcal{C} \times \mathrm{sSet} \longrightarrow \mathcal{C},$$
$$(-)^{(-)} \colon \mathrm{sSet}^{op} \times \mathcal{C} \longrightarrow \mathcal{C},$$
$$\mathrm{map}_l(-,-) \colon \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathrm{sSet},$$
$$\mathrm{map}_r(-,-) \colon \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathrm{sSet},$$

and adjunctions

$$\mathcal{C}(X \otimes K, Y) \cong \mathrm{sSet}(K, \mathrm{map}_l(X, Y)) \quad \text{and} \quad \mathcal{C}^{op}(Y^K, X) \cong \mathrm{sSet}(K, \mathrm{map}_r(X, Y)).$$

The homotopy function complex $\mathrm{map}_{\mathcal{C}}(-,-)$ agrees with the derived functors $R\,\mathrm{map}_l(-,-)$ and $R\,\mathrm{map}_r(-,-)$. Moreover, if X is a cofibrant object in $\mathcal{C}$ and Y is a fibrant object in $\mathcal{C}$, then

$$X \otimes - \colon \mathrm{sSet} \rightleftarrows \mathcal{C} \colon \mathrm{map}_l(X, -) \quad \text{and} \quad Y^{(-)} \colon \mathrm{sSet} \rightleftarrows \mathcal{C}^{op} \colon \mathrm{map}_r(-, Y).$$

are Quillen pairs; see [20, Corollary 5.4.4].

Note that a framing does not provide $\mathcal{C}$ with a simplicial model structure though, as map_l and map_r only agree up to a zig-zag of weak equivalences [20, Proposition 5.4.7]. However, it does mean that $\mathrm{Ho}(\mathcal{C})$ is a closed $\mathrm{Ho}(\mathrm{sSet})$-module category. If $\mathcal{C}$ is already a simplicial model category, the action from the simplicial structure agrees with the $\mathrm{Ho}(\mathrm{sSet})$-action coming from framings. In our previous notation, for a simplicial model category $\mathcal{C}$, the simplicial enrichment $\mathrm{Map}(-,-) = \mathrm{Hom}_l(-,-)$ coincides with $\mathrm{map}_l(-,-)$ and $\mathrm{map}_r(-,-)$, and the cotensor is $\mathrm{Hom}_r(-,-)$.

Thus, if our model category $\mathcal{C}$ is not simplicial we can define $W_k = I_{\mathcal{C}} \square f_k$ just as before, where the pushout-product is constructed using the functor $\otimes$ coming from the framing.

Remark 3.3. If $\mathcal{C}$ is a *pointed* model category, then it is equipped with a *pointed framing* [20, Section 5.7], where the category of simplicial sets is replaced by *pointed* simplicial sets sSet_*.

Definition 3.4. Let $\mathcal{C}$ be a left proper combinatorial model category. We call $P_k\mathcal{C} = L_{W_k}\mathcal{C}$ the *model category of k-types in $\mathcal{C}$*. An object of $\mathcal{C}$ is called a *k-type* if it is W_k-local, that is, fibrant in $P_k\mathcal{C}$.

Before we look further into the properties of this localisation, we need an analogue of Theorem 2.12(ii) using framings.

Proposition 3.5. *Let $\mathcal{C}$ be a combinatorial, left proper model category with generating cofibrations $I_{\mathcal{C}}$ and set of cofibrant homotopy generators $\mathcal{G}_{\mathcal{C}}$. Furthermore, let S be a class of maps in sSet. Then the following are equivalent for an object Z of $\mathcal{C}$:*

(i) *Z is $I_{\mathcal{C}} \square S$-local*

(ii) Z is $\mathcal{G}_{\mathcal{C}} \otimes S$-local

(iii) Z is fibrant and $\mathrm{map}_{\mathcal{C}}(G, Z)$ is $S_{\mathcal{C}}$-local for every G in $\mathcal{G}_{\mathcal{C}}$.

(iv) Z is fibrant and for every $g \colon X \to Y$ in S the induced map

$$g^* \colon Z^Y \longrightarrow Z^X$$

is a weak equivalence in $\mathcal{C}$.

Proof. The proof follows exactly the same pattern as Theorem 2.12(ii), so we are not spelling it out here. The occurring functors $\otimes$, Hom_r and Hom_l have been replaced by the functors $\otimes$, $(-)^{(-)}$, map_l and map_r coming from framings. The only properties needed are that when X is cofibrant and Y is fibrant in $\mathcal{C}$, the adjunctions $(X \otimes -, \mathrm{map}_l(X, -))$ and $(Y^{(-)}, \mathrm{map}_r(-, Y))$ are Quillen pairs, and that $\mathrm{map}_l(X, Y)$ is weakly equivalent to $\mathrm{map}_r(X, Y)$; see [20, Proposition 5.4.7]. As the homotopy mapping objects are also derived from framings, these are all compatible and the necessary adjunctions hold just as before. $\square$

Proposition 3.6. *Let $\mathcal{C}$ be a left proper combinatorial model category with set of cofibrant homotopy generators $\mathcal{G}_{\mathcal{C}}$. A fibrant object Z of $\mathcal{C}$ is a k-type if and only if $\pi_i(\mathrm{map}_{\mathcal{C}}(X, Z)) = 0$ for all X in $\mathcal{C}$ and $i > k$, or equivalently, $\pi_i(\mathrm{map}_{\mathcal{C}}(G, Z)) = 0$ for all G in $\mathcal{G}_{\mathcal{C}}$ and $i > k$.*

Proof. By Proposition 3.5 we have that Z is W_k-local if and only if Z is fibrant in $\mathcal{C}$ and $\mathrm{map}_{\mathcal{C}}(G, Z)$ is a k-type in sSet for every G in $\mathcal{G}_{\mathcal{C}}$. Since every object in $\mathcal{C}$ is weakly equivalent to a homotopy colimit of objects of $\mathcal{G}_{\mathcal{C}}$ and those commute with homotopy function complexes, the result follows. $\square$

In combination with Proposition 3.5 we also have the following.

Corollary 3.7. *Let $\mathcal{C}$ be a left proper combinatorial model category with set of cofibrant homotopy generators $\mathcal{G}_{\mathcal{C}}$, and let $f_k \colon S^{k+1} \to D^{k+2}$ in sSet. Then the model category of k-types $P_k \mathcal{C}$ coincides with $L_{\mathcal{G}_{\mathcal{C}} \otimes f_k} \mathcal{C}$.* $\square$

Remark 3.8. When $\mathcal{C}$ is a simplicial model category, then model structure $P_k \mathcal{C}$ agrees with the model structure for k-types defined by Barwick in [5, Proposition 5.28].

In the context of *familiarisation* as defined by [4], one would define $P_k \mathcal{C}$ to be $L_{I_{\mathcal{C}} \square J_{f_k}} \mathcal{C}$ where J_{f_k} denotes the generating acyclic cofibrations of L_{f_k} sSet. However, those two model structures agree since L_{f_k} sSet $= L_{J_{f_k}}$ sSet by Proposition 2.14. The reason one works with the acyclic cofibrations in [4] is to actually cut down the localised weak equivalences of some L_S sSet to a generating set if S is not a set. However, in our case we only localize simplicial sets at one morphism, making this technicality unnecessary.

Proposition 3.9. *Let $\mathcal{C}$ be a left proper combinatorial model category. The model category of k-types $P_k \mathcal{C}$ has the following properties:*

(i) *Every Quillen adjunction $\mathrm{sSet} \rightleftarrows \mathcal{C}$ gives rise to a Quillen adjunction $L_{f_k} \mathrm{sSet} \rightleftarrows P_k \mathcal{C}$, and $P_k \mathcal{C}$ is the closest model structure to $\mathcal{C}$ with this property. This means that if $\mathcal{C} \rightleftarrows \mathcal{D}$ is a Quillen adjunction such that the composite $\mathrm{sSet} \rightleftarrows \mathcal{D}$ factors over $L_{f_k} \mathrm{sSet}$, then $P_k \mathcal{C} \rightleftarrows \mathcal{D}$ is also a Quillen adjunction.*

(ii) *If $\mathcal{C}$ is a simplicial model category, then $P_k \mathcal{C}$ is a $L_{f_k} \mathrm{sSet}$-model category.*

(iii) *For every $k \geq 0$ the model structures $P_k P_{k+1} \mathcal{C}$ and $P_k \mathcal{C}$ coincide.*

Proof. Let $F\colon \mathrm{sSet} \rightleftarrows \mathcal{C}\colon U$ be a Quillen adjunction. By [19, Proposition 3.3.18], in order for this to be a Quillen adjunction between $L_{f_k}\mathrm{sSet}$ and $P_k\mathcal{C}$, we need to show that $F(f_k)$ is a weak equivalence in $P_k\mathcal{C}$. By [20, Chapter 5], all Quillen adjunctions arise from framings, that is, they are of the form $F = A \otimes -$ for some $A \in \mathcal{C}$. (Every adjunction between sSet and $\mathcal{C}$ is of the form $(A^\bullet \otimes -, \mathrm{Hom}(A^\bullet, -))$ for some cosimplicial object $A^\bullet \in \mathcal{C}^\Delta$, and every Quillen adjunction is given by a framing on $A^\bullet[0] = A$; see [20, Proposition 3.1.5 and Section 5.2] and [2, Section 3].) So we have to show that $A \otimes f_k$ is a weak equivalence in $P_k\mathcal{C}$. By Proposition 3.5, all maps of the form $G \otimes f_k$ are weak equivalences for all generators $G \in \mathcal{G}$. But as every A is a filtered colimit of such generators, and $- \otimes f_k$ commutes with colimits, $A \otimes f_k$ is a weak equivalence as well.

Now let $F'\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon U'$ be another Quillen adjunction such that $F'(F(f_k))$ is a weak equivalence in $\mathcal{D}$ for any left Quillen functor F as before. This means that $F'(A \otimes f_k)$ is a weak equivalence in $\mathcal{D}$ for any $A \in \mathcal{C}$. So in particular, F' sends all morphisms $G \otimes f_k$ to weak equivalences, where $G \in \mathcal{G}$. As $P_k\mathcal{C} = L_{\mathcal{G} \otimes f_k}\mathcal{C}$, this means that F' sends all the weak equivalences in $P_k\mathcal{C}$ to weak equivalences in $\mathcal{D}$, which is what we wanted to prove.

Part (ii) follows from Proposition 2.15(ii), and part (iii) follows from the fact that both model structures have the same cofibrations and the same fibrant objects. This last point can be easily checked using the characterisation of local objects given in Proposition 3.5. $\qquad\square$

Before we move on to the next result, let us note the following. The fact that a model category is λ-presentable only depends on the underlying category, not on its model structure. Also, the left Bousfield localisation of a cofibrantly generated model category is again cofibrantly generated. Thus, if a model category is combinatorial, so is any left Bousfield localisation of it. Also, as Bousfield localisation does not change cofibrations and preserves weak equivalences, if $\mathcal{G}_\mathcal{C}$ is a set of homotopy generators for a combinatorial model category $\mathcal{C}$, then $\mathcal{G}_\mathcal{C}$ will also be a set of homotopy generators for any left Bousfield localisation of $\mathcal{C}$.

We can now characterise the weak equivalences of $P_k\mathcal{C}$.

Proposition 3.10. *Let* $f\colon X \to Y$ *be a morphism in* $\mathcal{C}$. *If its fibrant replacement* $\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}$ *in* $P_k\mathcal{C}$ *induces a weak equivalence*

$$\widetilde{f}_*\colon \mathrm{map}_\mathcal{C}(G, \widetilde{X}) \longrightarrow \mathrm{map}_\mathcal{C}(G, \widetilde{Y})$$

in $L_{f_k}\mathrm{sSet}$ *for all homotopy generators* G *in* $\mathcal{G}_\mathcal{C}$, *then the morphism* f *is a weak equivalence in* $P_k\mathcal{C}$.

Proof. We have that $\mathcal{G}_\mathcal{C} \subset \mathcal{G}_\mathcal{C} \otimes \mathcal{G}_{\mathrm{sSet}}$ as we can, without loss of generality, add the single point to $\mathcal{G}_{\mathrm{sSet}}$. Thus, the statement follows from Proposition 2.16. Note that if $\mathcal{C}$ is not simplicial, then we have to replace the mapping objects in that proof by the mapping objects given by framings. $\qquad\square$

Corollary 3.11. *If* $f\colon X \to Y$ *is a morphism in* $\mathcal{C}$ *such that its fibrant replacement* $\widetilde{f}\colon \widetilde{X} \to \widetilde{Y}$ *in* $P_k\mathcal{C}$ *induces an isomorphism*

$$\pi_i(\widetilde{f}_*)\colon \pi_i(\mathrm{map}_\mathcal{C}(G, \tilde{X})) \longrightarrow \pi_i(\mathrm{map}_\mathcal{C}(G, \tilde{Y}))$$

for all $i \leq k$ *and homotopy generators* G *in* $\mathcal{G}_\mathcal{C}$, *then* f *is a weak equivalence in* $P_k\mathcal{C}$. $\qquad\square$

3.3. **Example: $\mathbb{S}$-local simplicial sets.** Let us consider the example of left Bousfield localisations of pointed simplicial sets, $\mathcal{C} = L_{\mathbb{S}} \, \mathrm{sSet}_*$. We can easily describe Postnikov sections in this model category. By definition, $P_k L_{\mathbb{S}} \, \mathrm{sSet}_* = L_{W_k} L_{\mathbb{S}} \, \mathrm{sSet}_*$ where $W_k = I_{L_{\mathbb{S}} \, \mathrm{sSet}_*} \square f_k$ and $f_k : S^{k+1} \to D^{k+2}$. As the generating cofibrations $I_{L_{\mathbb{S}} \, \mathrm{sSet}_*}$ of $L_{\mathbb{S}} \, \mathrm{sSet}_*$ are the same as the generating cofibrations of sSet_* we can conclude that

$$P_k L_{\mathbb{S}} \, \mathrm{sSet}_* = L_{f_k} L_{\mathbb{S}} \, \mathrm{sSet}_* \, .$$

Thus, X is fibrant in $P_k L_{\mathbb{S}} \, \mathrm{sSet}_*$ if and only if it is a Kan complex, $\mathbb{S}$-local and $\pi_i X = \pi_i L_{\mathbb{S}} X = 0$ for $i > k$.

3.4. **Example: k-types in chain complexes.** We are going to apply the results from the previous section to the category of bounded chain complexes of R-modules, $\mathrm{Ch}_b(R)$, where R is a commutative ring with unit. This is a particularly interesting example as it concerns a model category that is not simplicial. We are going to describe the k-types in $\mathrm{Ch}_b(R)$ as well as describe some of the weak equivalences. The results are just what one would expect and fit very neatly with our general setup.

Let $\mathrm{Ch}_b(R)$ denote the category of bounded chain complexes of R-modules with the standard projective model structure; see [13, Section 7]. The weak equivalences are given by quasi-isomorphisms, fibrations are morphisms which are surjective in positive degrees, and cofibrations are monomorphisms with projective cokernel in every degree. Consider the model category of k-types of chain complexes, $P_k \, \mathrm{Ch}_b(R)$. According to Definition 3.4, this is the left Bousfield localisation with respect to the set

$$W_k = I_{\mathrm{Ch}_b(R)} \square \{ f_k \colon S^{k+1} \longrightarrow D^{k+2} \}.$$

Now the generating cofibrations in the standard projective model structure are given by the inclusions

$$I_{\mathrm{Ch}_b(R)} = \{ \mathbb{S}^{n-1} \longrightarrow \mathbb{D}^n \mid n \geq 1 \},$$

where $\mathbb{S}^{n-1}$ denotes the chain complex which is R in degree $n-1$ and zero everywhere else, and $\mathbb{D}^n$ denotes the chain complex with R in degrees $n-1$ and n with the identity differential between them, and zero everywhere else. To avoid notational confusion with the sphere and disk in spaces, we will use bold face for these.

Recall that the suspension functor Σ in a pointed model category $\mathcal{C}$ can be defined using pointed framings; see [20, Definition 6.1.1]. If X is a cofibrant object then $\Sigma X = X \otimes S^1$, that is, ΣX is the pushout of the diagram

$$
\begin{array}{ccc}
X \otimes \partial \Delta[1] & \longrightarrow & X \otimes \Delta[1] \\
\downarrow & & \\
* & &
\end{array}
$$

In the category $\mathrm{Ch}_b(R)$, the suspension is given by shifting. Hence, putting this into the above definition, we obtain

$$W_k = \{ \mathbb{S}^{n+k+1} \longrightarrow \mathbb{D}^{n+k+2} \mid n \geq 0 \},$$

so $P_k \, \mathrm{Ch}_b(R)$ is just localizing $\mathrm{Ch}_b(R)$ at the map $g_k \colon \mathbb{S}^{k+1} \to \mathbb{D}^{k+2}$. Note that local equivalences are closed under (positive) suspensions, and hence localizing with respect to g_k is the same as localizing with respect to $\{ \Sigma^n g_k \mid n \geq 0 \} = W_k$.

Proposition 3.12. *A fibrant chain complex M in $\mathrm{Ch}_b(R)$ is a k-type if and only if $H_i(M) = 0$ for all $i > k$.*

Proof. The chain complex M is g_k-local if and only if

$$\pi_i(\mathrm{map}_{\mathrm{Ch}_b(R)}(\mathbb{D}^{k+2}, M)) \longrightarrow \pi_i(\mathrm{map}_{\mathrm{Ch}_b(R)}(\mathbb{S}^{k+1}, M))$$

is an isomorphism for all $i \geq 0$. By adjunction, this is equivalent to

$$[\mathbb{D}^{i+k+2}, M] \longrightarrow [\mathbb{S}^{i+k+1}, M]$$

being an isomorphism for all $i \geq 0$, where the square brackets denote morphisms in the derived category $D_b(R)$. But as the chain complex $\mathbb{D}^{i+k+2}$ is acyclic and the right hand side equals the homology $H_{i+k+1}(M)$ of M, the above is equivalent to $H_i(M) = 0$ for all $i > k$. $\qquad\square$

We can now say something about the weak equivalences in $P_k \mathrm{Ch}_b(R)$. Recall that if M is a chain complex in $\mathrm{Ch}_b(R)$, we denote by $M[n]$ the n-fold suspension of M.

Proposition 3.13. *Let $f\colon M \to N$ be a morphism of chain complexes such that $H_i(f)$ is an isomorphism for $0 \leq i \leq k$. Then f is a weak equivalence in $P_k \mathrm{Ch}_b(R)$.*

Proof. This is very similar to [19, Proposition 1.5.2]. Without loss of generality, let $f\colon M \to N$ be a cofibration of chain complexes, that is, a degreewise monomorphism with projective cokernel.

We know that f is a weak equivalence in $P_k \mathrm{Ch}_b(R)$ if and only if

$$\mathrm{map}_{\mathrm{Ch}_b(R)}(N, Z) \longrightarrow \mathrm{map}_{\mathrm{Ch}_b(R)}(M, Z)$$

is an acyclic fibration in simplicial sets for all g_k-local Z; see [19, Section 1.3.1]. This is equivalent to having a lift in the diagram

$$
\begin{array}{ccc}
\partial\Delta[n] & \longrightarrow & \mathrm{map}_{\mathrm{Ch}_b(R)}(N, Z) \\
\downarrow & \nearrow & \downarrow \\
\Delta[n] & \longrightarrow & \mathrm{map}_{\mathrm{Ch}_b(R)}(M, Z)
\end{array}
$$

for all $n \geq 0$. By adjunction, this is equivalent to having a lift in the diagram

$$
\begin{array}{ccc}
M \otimes \Delta[n] \coprod_{M \otimes \partial\Delta[n]} N \otimes \partial\Delta[n] & \longrightarrow & Z \\
\downarrow & \nearrow & \downarrow \\
N \otimes \Delta[n] & \longrightarrow & 0
\end{array}
$$

for all $n \geq 0$.

We know by Proposition 3.12 that $H_j(Z) = 0$ for $j \geq k + 1$. Moreover, the pushout in the top left corner of the diagram is a shift of the mapping cone of f (that is, $M[n+1] \oplus N[n]$), whereas the bottom left corner is a shift of the cone of Y (that is, $N[n+1] \oplus N[n]$). Thus, the left vertical map is also a cofibration that is a homology isomorphism in degrees 0 to $k+1$ (rather than just k). This means that we have a square in $\mathrm{Ch}_b(R)$ where the left vertical map is a cofibration and the right vertical map a fibration. In order to have the desired lift, one of those maps would have to be a homology isomorphism.

As the left vertical map is a homology isomorphism in degrees 0 to $k + 1$, we can use methods analogous to [13, Section 7.7] to construct a lift in those degrees. Then we can use the same method as in [13, Section 7.5] to inductively construct the lift from degrees $k + 2$ onwards, which uses that $H_j(N) = 0$ for $j \geq k + 1$.

So we have constructed a lift in the above square, which means that $f \colon M \to N$ is a weak equivalence in $P_k \operatorname{Ch}_b(R)$. $\square$

As a consequence of Proposition 3.12 and Proposition 3.13 we get the following.

Corollary 3.14. *If M is a chain complex in $\operatorname{Ch}_b(R)$, then the W_k-localisation is given by the k-truncation $\tau_{\geq k}M$ of X, defined by*

$$(\tau_{\geq k}M)_n = \begin{cases} M_n & \text{if } n < k, \\ M_k/B_k & \text{if } n = k, \\ 0 & \text{if } n > k, \end{cases}$$

where $B_k = \operatorname{im}(d_k)$ denotes the group of k-boundaries. $\square$

4. Towers and fiber products of model categories

In this section we recall the injective model structure on the category of sections of diagrams of model categories. We will state the existence of this model structure in general, although we will be mainly interested in the cases of sections of towers and fiber products of model categories. Details about these model structures can be found in [5, Section 2, Application II], [7], [8], [16, Section 3] and [28, Section 4].

Let $\mathcal{I}$ be a small category. A *left Quillen presheaf on $\mathcal{I}$* is a presheaf of categories $F \colon \mathcal{I}^{\mathrm{op}} \to \mathrm{CAT}$ such that for every i in $\mathcal{I}$ the category $F(i)$ has a model structure, and for every map $f \colon i \to j$ in $\mathcal{I}$ the induced functor $f^* \colon F(j) \to F(i)$ has a right adjoint and they form a Quillen pair.

Definition 4.1. A *section* of a left Quillen presheaf $F \colon \mathcal{I}^{\mathrm{op}} \to \mathrm{CAT}$ consists of a tuple $X = (X_i)_{i \in \mathcal{I}}$, where each X_i is in $F(i)$, and, for every morphism $f \colon i \to j$ in $\mathcal{I}$ a morphism $\varphi_f \colon f^*X_j \to X_i$ in $F(i)$ such that the diagram

$$
\begin{array}{ccc}
(f \circ g)^*X_k & \xrightarrow{\varphi_{f \circ g}} & X_i \\
{\scriptstyle f^*\varphi_g} \downarrow & \nearrow {\scriptstyle \varphi_f} & \\
f^*X_j & &
\end{array}
$$

commutes for every pair of composable morphisms $f \colon i \to j$ and $g \colon j \to k$.

A *morphism of sections* $\phi \colon (X, \varphi) \to (Y, \varphi')$ is given by morphisms $\phi_i \colon X_i \to Y_i$ in $F(i)$ such that the diagram

$$
\begin{array}{ccc}
f^*X_j & \xrightarrow{f^*\phi_j} & f^*Y_j \\
{\scriptstyle \varphi_f} \downarrow & & \downarrow {\scriptstyle \varphi'_f} \\
X_i & \xrightarrow{\phi_i} & Y_i
\end{array}
$$

commutes for every morphism $f \colon i \to j$ in $\mathcal{I}$.

A section (X, φ) is called *homotopy cartesian* if for every $f \colon i \to j$, the morphism $\varphi_f \colon f^*X_j \to X_i$ is a weak equivalence in $F(i)$.

As proved in [5, Theorem 2.30], the category of sections admits an *injective* model structure.

Theorem 4.2. *Let $F\colon \mathfrak{I}^{\mathrm{op}} \to \mathrm{CAT}$ be a left Quillen presheaf such that $F(i)$ is combinatorial for every i in $\mathfrak{I}$. Then there exists a combinatorial model structure on the category of sections of F, denoted by $\mathrm{Sect}(\mathfrak{I}, F)$ and called the* injective *model structure, such that a morphism of sections ϕ is a weak equivalence or a cofibration if ϕ_i is a weak equivalence or a cofibration in $F(i)$ for every i in $\mathfrak{I}$, respectively. Moreover, if $F(i)$ is left or right proper for every $i \in \mathfrak{I}$, then so is the model structure on $\mathrm{Sect}(\mathfrak{I}, F)$.* $\square$

Now, in order to model the homotopy limit of a diagram of left Quillen presheaf, we would like to construct a model structure on the category of sections whose cofibrant objects are precisely the levelwise cofibrant homotopy cartesian sections. This will be done by taking a right Bousfield localisation of $\mathrm{Sect}(\mathfrak{I}, F)$. The resulting model structure will be called the *homotopy limit model structure*.

The existence of the homotopy limit model structure as a *right* model structure is proved in [5, Theorem 5.25]. It follows directly from that result that if $F(i)$ is right proper for every i in $\mathfrak{I}$, then we get a *full* model structure. For the reader's convenience we spell this out in a little more detail.

Theorem 4.3. *Let $F\colon \mathfrak{I}^{\mathrm{op}} \to \mathrm{CAT}$ be a left Quillen presheaf such that $F(i)$ is right proper and combinatorial for every i in $\mathfrak{I}$. Then there exists a combinatorial model structure on the category of sections of F, called the* homotopy limit model *structure, with the same fibrations as $\mathrm{Sect}(\mathfrak{I}, F)$ and whose cofibrant objects are the sections that are cofibrant in $\mathrm{Sect}(\mathfrak{I}, F)$ and homotopy cartesian.* $\square$

Proof. Let $\mathcal{D}$ be the full subcategory of $\mathrm{Sect}(\mathfrak{I}, F)$ consisting of the homotopy cartesian sections. Consider the functor

$$\Phi\colon \mathrm{Sect}(\mathfrak{I}, F) \longrightarrow \prod_{f\colon i \to j} \mathrm{Arr}(F(i))$$

defined as $\Phi((X_i)_{i \in \mathfrak{I}}) = \prod_{f\colon i \to j} \varphi_f$, where f runs over all morphisms of $\mathfrak{I}$ and $\mathrm{Arr}(-)$ denotes the category of arrows.

The categories $\mathrm{Sect}(\mathfrak{I}, F)$ and $\prod_{f\colon i \to j} \mathrm{Arr}(F(i))$ are accessible (in fact, they are locally presentable; see [1, Corollary 1.54]) and the functor Φ is an accessible functor since it preserves all colimits (they are computed levelwise). Hence Φ is an accessible functor between accessible categories.

Each $F(i)$ is combinatorial for every i in $\mathfrak{I}$, and hence by [23, Corollary A.2.6.6] the subcategory of weak equivalences $\mathrm{weq}(F(i))$ is an accessible and accessibly embedded subcategory of $\mathrm{Arr}(F(i))$. Therefore, $\prod_{f\colon i \to j} \mathrm{weq}(F(i))$ is an accessible and accessibly embedded subcategory of $\prod_{f\colon i \to j} \mathrm{Arr}(F(i))$. By [1, Remark 2.50], the preimage $\Phi^{-1}(\prod_{f\colon i \to j} \mathrm{weq}(F(i)))$ is an accessible and accessibly embedded subcategory of $\mathrm{Sect}(\mathfrak{I}, F)$. But this preimage is precisely $\mathcal{D}$.

Now, since $\mathcal{D}$ is accessible there exists a set $\mathcal{K}$ and a regular cardinal λ such that every object of $\mathcal{D}$ is a λ-filtered colimit (and hence a homotopy colimit if we choose λ big enough; see [11, Proposition 7.3]) of objects in $\mathcal{K}$. Moreover, since $\mathcal{D}$ is accessibly embedded this homotopy colimit lies in $\mathcal{D}$.

The homotopy limit model structure is then the right Bousfield localisation $R_{\mathcal{K}} \mathrm{Sect}(\mathfrak{I}, F)$. (We can perform this right Bousfield localisation because every $F(i)$, and hence $\mathrm{Sect}(\mathfrak{I}, F)$ are right proper.) The fact that the cofibrant objects of

this new model structure are precisely the levelwise cofibrant homotopy cartesian
sections follows from [19, Theorem 5.1.5].   □

4.1. **Towers of model categories.** Let $\mathbb{N}$ be the category $0 \to 1 \to 2 \to \cdots$.
A *tower of model categories* is a left Quillen presheaf $F\colon \mathbb{N}^{\mathrm{op}} \to \mathrm{CAT}$. The objects
of the category of sections are then sequences $X_0, X_1, \ldots, X_n, \ldots$, where each X_i
is an object of $F(i)$, together with morphisms $\varphi_i\colon f^*X_{i+1} \to X_i$ in $F(i)$ for every
$i \geq 0$, where $f\colon i \to i+1$ is the unique morphism from i to $i+1$ in $\mathbb{N}$. A morphism
between two sections $\phi_\bullet\colon X_\bullet \to Y_\bullet$ consist of morphisms $\phi_i\colon X_i \to Y_i$ in $F(i)$ such
that the diagram

$$
\begin{array}{ccc}
f^*X_{i+1} & \longrightarrow & X_i \\
{\scriptstyle f^*\phi_{i+1}}\big\downarrow & & \big\downarrow{\scriptstyle \phi_i} \\
f^*Y_{i+1} & \longrightarrow & Y_i
\end{array}
$$

commutes for every $i \geq 0$.

Proposition 4.4. *Let $F\colon \mathbb{N}^{\mathrm{op}} \to \mathrm{CAT}$ be a tower of model categories, where each
$F(i)$ is a combinatorial model category for every $i \geq 0$. There exist a combinato-
rial model structure on the category of sections, denoted by $\mathrm{Sect}(\mathbb{N}^{\mathrm{op}}, F)$, where a
map $\phi_\bullet$ is a weak equivalence or a cofibration if for every $i \geq 0$ the map ϕ_n is a
weak equivalence or a cofibration in $F(i)$, respectively. The fibrations are the maps
$\phi_\bullet\colon X_\bullet \to Y_\bullet$ such that ϕ_0 is a fibration in $F(0)$ and*

$$
X_{i+1} \longrightarrow Y_{i+1} \times_{f_*Y_i} f_*X_i
$$

*is a fibration in $F(i+1)$ for every $i \geq 0$, where f_* denotes the right adjoint to f^*.
The fibrant objects are those sections $X_\bullet$ such that X_i is fibrant in $F(i)$ and the
morphism*

$$
X_{i+1} \longrightarrow f_*X_i
$$

is a fibration in $F(i+1)$ for every $i \geq 0$.

Proof. The existence of the required model structure follows from Theorem 4.2.
The description of the fibrations follows from [16, Theorem 3.1].   □

Proposition 4.5. *Let $F\colon \mathbb{N}^{\mathrm{op}} \to \mathrm{CAT}$ be a tower of model categories, where each
$F(i)$ is combinatorial and right proper for every $i \geq 0$. Then there is a model
structure $\mathrm{Tow}(F)$ on the category of sections of F with the following properties:*

   (i) *A morphism $\phi_\bullet$ is a fibration in $\mathrm{Tow}(F)$ if $\phi_\bullet$ is a fibration in $\mathrm{Sect}(\mathbb{N}^{\mathrm{op}}, F)$.*
  (ii) *A section $X_\bullet$ is cofibrant in $\mathrm{Tow}(F)$ if X_i is cofibrant in $F(i)$ and the
       morphism $f^*X_{i+1} \to X_i$ is a weak equivalence in $F(i)$ for every $i \geq 0$.*
 (iii) *A morphism $\phi_\bullet$ between cofibrant sections is a weak equivalence in $\mathrm{Tow}(F)$
       if and only if ϕ_i is weak equivalence in $F(i)$ for every $i \geq 0$.*

Proof. The existence of the model structure $\mathrm{Tow}(F)$ follows from Theorem 4.3
applied to the left Quillen presheaf F. The characterisation of the weak equivalences
between cofibrant objects follows since $\mathrm{Tow}(F)$ is a right Bousfield localisation of
$\mathrm{Sect}(\mathbb{N}^{\mathrm{op}}, F)$.   □

4.2. **Postnikov towers.** Let $\mathcal{C}$ be a left proper combinatorial model category and, for every $n \geq 0$, consider the model structure $P_n\mathcal{C}$ of n-types in $\mathcal{C}$ as described in Section 3.2. For every $n < m$ the identity is a left Quillen functor $P_m\mathcal{C} \to P_n\mathcal{C}$. Thus we have a tower of model categories $P_\bullet\mathcal{C} \colon \mathbb{N}^{\mathrm{op}} \to \mathrm{CAT}$. The objects $X_\bullet$ of the category of sections are sequences

$$\cdots \longrightarrow X_n \longrightarrow \cdots \longrightarrow X_2 \longrightarrow X_1 \longrightarrow X_0$$

of morphisms in $\mathcal{C}$, and its morphisms $f_\bullet \colon X_\bullet \to Y_\bullet$ are given by commutative ladders

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & X_n & \longrightarrow & \cdots & \longrightarrow & X_2 & \longrightarrow & X_1 & \longrightarrow & X_0 \\
& & \downarrow{\scriptstyle f_n} & & & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} \\
\cdots & \longrightarrow & Y_n & \longrightarrow & \cdots & \longrightarrow & Y_2 & \longrightarrow & Y_1 & \longrightarrow & Y_0.
\end{array}
$$

By Proposition 4.4, if $\mathcal{C}$ is a left proper combinatorial model category, then there exist a left proper combinatorial model structure on the category of sections $\mathrm{Sect}(\mathbb{N}^{\mathrm{op}}, P_\bullet\mathcal{C})$, where a map $f_\bullet$ is a weak equivalence or a cofibration if for every $n \geq 0$ the map f_n is a weak equivalence or a cofibration in $P_n\mathcal{C}$, respectively. The fibrations are the maps $f_\bullet \colon X_\bullet \to Y_\bullet$ such that f_0 is a fibration in $P_0\mathcal{C}$ and

$$X_n \longrightarrow Y_n \times_{Y_{n-1}} X_{n-1}$$

is a fibration in $P_n\mathcal{C}$ for every $n \geq 1$. The fibrant objects can be characterised as follows:

Lemma 4.6. *Let $X_\bullet$ be a section of $P_\bullet\mathcal{C}$. The following are equivalent:*

(i) *$X_\bullet$ is fibrant in $\mathrm{Sect}(\mathbb{N}^{\mathrm{op}}, P_\bullet\mathcal{C})$.*
(ii) *X_0 is fibrant in $P_0\mathcal{C}$ and $X_{n+1} \to X_n$ is a fibration in $P_{n+1}\mathcal{C}$ for all $n \geq 0$.*
(iii) *X_n is fibrant in $P_n\mathcal{C}$ and $X_{n+1} \to X_n$ is a fibration in $\mathcal{C}$ for all $n \geq 0$.*

Proof. This follows because a fibration in $P_n\mathcal{C}$ is also a fibration in $P_{n+1}\mathcal{C}$ as well as a fibration in $\mathcal{C}$. $\qquad\square$

If the model structures for n-types $P_n\mathcal{C}$ are right proper for every $n \geq 0$, then by Proposition 4.5, the model structure $\mathrm{Tow}(P_\bullet\mathcal{C})$ exists and will be denoted by $\mathrm{Post}(\mathcal{C})$. It has the following properties:

(i) A morphism $f_\bullet$ is a fibration in $\mathrm{Post}(\mathcal{C})$ if $f_\bullet$ is a fibration in $\mathrm{Sect}(\mathbb{N}^{\mathrm{op}}, P_\bullet\mathcal{C})$.
(ii) A section $X_\bullet$ is cofibrant if X_n is cofibrant in $\mathcal{C}$ and $X_{n+1} \to X_n$ is a weak equivalence in $P_n\mathcal{C}$ for every $n \geq 0$.
(iii) A morphism $f_\bullet$ between cofibrant sections is a weak equivalence if and only if f_n is a weak equivalence in $P_n\mathcal{C}$ for every $n \geq 0$.

For every $n \geq 0$ the identity functors give a Quillen pair $\mathrm{id} \colon \mathcal{C} \rightleftarrows P_n\mathcal{C} \colon \mathrm{id}$, since $P_n\mathcal{C}$ is a left Bousfield localisation of $\mathcal{C}$. This extends to a Quillen pair

$$\mathrm{id} \colon \mathcal{C}^{\mathbb{N}^{\mathrm{op}}}_{\mathrm{inj}} \rightleftarrows \mathrm{Sect}(\mathbb{N}^{\mathrm{op}}, P_\bullet\mathcal{C}) \colon \mathrm{id},$$

where $\mathcal{C}^{\mathbb{N}^{\mathrm{op}}}_{\mathrm{inj}}$ denotes the category of $\mathbb{N}^{\mathrm{op}}$-indexed diagrams with the injective model structure. Indeed weak equivalences and cofibrations in $\mathcal{C}^{\mathbb{N}^{\mathrm{op}}}_{\mathrm{inj}}$ are defined levelwise and every weak equivalence in $\mathcal{C}$ is a weak equivalence in $P_n\mathcal{C}$ for all $n \geq 0$. Hence, there is a Quillen pair

$$\mathcal{C} \underset{\mathrm{lim}}{\overset{\mathrm{const}}{\rightleftarrows}} \mathcal{C}^{\mathbb{N}^{\mathrm{op}}}_{\mathrm{inj}} \underset{\mathrm{id}}{\overset{\mathrm{id}}{\rightleftarrows}} \mathrm{Sect}(\mathbb{N}^{\mathrm{op}}, P_\bullet\mathcal{C}),$$

where const denotes the constant diagram functor.

Lemma 4.7. *The adjunction* $\mathrm{const}\colon \mathcal{C} \rightleftarrows \mathrm{Post}(\mathcal{C})\colon \lim$ *is a Quillen pair.*

Proof. It is enough to check that the left adjoint preserves trivial cofibrations and cofibrations between cofibrant objects. If f is a trivial cofibration in $\mathcal{C}$ then $\mathrm{const}(f)$ is a trivial cofibration in $\mathrm{Sect}(\mathbb{N}^{\mathrm{op}}, P_\bullet\mathcal{C})$. But since $\mathrm{Post}(\mathcal{C})$ is a right Bousfield localisation of $\mathrm{Sect}(\mathbb{N}^{\mathrm{op}}, P_\bullet\mathcal{C})$ it has the same trivial cofibrations. Hence $\mathrm{const}(f)$ is a trivial cofibration in $\mathrm{Post}(\mathcal{C})$.

Let $f\colon X \to Y$ be a cofibration between cofibrant objets in $\mathcal{C}$. Then $\mathrm{const}(f)$ is a cofibration between cofibrant objects in $\mathrm{Sect}(\mathbb{N}^{\mathrm{op}}, P_\bullet\mathcal{C})$. But $\mathrm{const}(X)$ and $\mathrm{const}(Y)$ are both cofibrant in $\mathrm{Post}(\mathcal{C})$ by Proposition 4.5. Hence $\mathrm{const}(f)$ is a cofibration in $\mathrm{Post}(\mathcal{C})$ if and only if it is a cofibration in $\mathrm{Sect}(\mathbb{N}^{\mathrm{op}}, P_\bullet\mathcal{C})$ (see [19, Proposition 3.3.16(ii)]). $\square$

If $\mathcal{C} = \mathrm{sSet}$, then the model structure $\mathrm{Post}(\mathrm{sSet})$ exists, since $P_n(\mathrm{sSet})$ is right proper for every $n \geq 0$; see [10, Theorem 9.9].

Theorem 4.8. *Let $\mathcal{C} = \mathrm{sSet}$ be the category of simplicial sets. Then, the Quillen pair* $\mathrm{const}\colon \mathrm{sSet} \rightleftarrows \mathrm{Post}(\mathrm{sSet})\colon \lim$ *is a Quillen equivalence.*

Proof. By [20, Proposition 1.3.13] it suffices to check that the derived unit and counit are weak equivalences. Let X be a fibrant simplicial set. Then $\mathrm{const}(X)$ is cofibrant in $\mathrm{Post}(\mathrm{sSet})$, since const is a left Quillen functor. Let

$$\cdots \longrightarrow X_n \longrightarrow \cdots \longrightarrow X_2 \longrightarrow X_1 \longrightarrow X_0$$

be a fibrant replacement of $\mathrm{const}(X)$ in $\mathrm{Post}(\mathrm{sSet})$. Hence we have that X_n is fibrant in $P_n \mathrm{sSet}$ and $X_{n+1} \to X_n$ is a fibration in sSet and a weak equivalence in $P_n \mathrm{sSet}$ for all $n \geq 0$. Now, by [15, Ch.VI, Theorem 3.5], the map $X \to \lim X_\bullet$ is a weak equivalence.

Now, let $X_\bullet$ be any fibrant and cofibrant object in $\mathrm{Post}(\mathrm{sSet})$. We have to see that the map $\mathrm{const}(\lim X_\bullet) \to X_\bullet$ is a weak equivalence in $\mathrm{Post}(\mathrm{sSet})$. This is equivalent to seeing that the map $\lim X_\bullet \to X_n$ is a weak equivalence in $P_n \mathrm{sSet}$ for every $n \geq 0$. First note that since the category $\mathbb{N}^{\mathrm{op}}_{>n} = \cdots \to n{+}3 \to n{+}2 \to n{+}1$ is homotopy left cofinal in $\mathbb{N}^{\mathrm{op}}$ we have that $\lim X_\bullet$ is weakly equivalent to $\lim_{\mathbb{N}^{\mathrm{op}}_{>n}} X_\bullet$ for every n (see [19, Theorem 19.6.13]). Hence it is enough to check that the map $\lim_{\mathbb{N}^{\mathrm{op}}_{>n}} X_\bullet \to X_n$ is a weak equivalence in $P_n \mathrm{sSet}$ for all $n \geq 0$. For every $n \geq 0$ we have a map of towers

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & X_m & \longrightarrow & \cdots & \longrightarrow & X_2 & \longrightarrow & X_{n+2} & \longrightarrow & X_{n+1} \\
 & & \downarrow & & & & \downarrow & & \downarrow & & \| \\
\cdots & = & X_{n+1} & = & \cdots & = & X_{n+1} & = & X_{n+1} & = & X_{n+1},
\end{array}$$

where each vertical map is a weak equivalence in $P_{n+1} \mathrm{sSet}$. Applying Milnor exact sequence (see [15, Ch.VI, Proposition 2.15]) we get a morphism of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \lim^1_{\mathbb{N}^{\mathrm{op}}_{>n}} \pi_{i+1} X_\bullet & \longrightarrow & \pi_i(\lim_{\mathbb{N}^{\mathrm{op}}_{>n}} X_\bullet) & \longrightarrow & \lim_{\mathbb{N}^{\mathrm{op}}_{>n}} \pi_i X_\bullet & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \lim^1_{\mathbb{N}^{\mathrm{op}}_{>n}} \pi_{i+1} X_{n+1} & \longrightarrow & \pi_i(\lim_{\mathbb{N}^{\mathrm{op}}_{>n}} X_{n+1}) & \longrightarrow & \lim_{\mathbb{N}^{\mathrm{op}}_{>n}} \pi_i X_{n+1} & \longrightarrow & 0.
\end{array}$$

For $0 \leq i < n$ the left and right morphisms are isomorphisms, hence the map $\lim_{\mathbb{N}^{op}_{>n}} X_\bullet \to X_{n+1}$ is a weak equivalence in P_n sSet. Therefore the map

$$\lim_{\mathbb{N}^{op}_{>n}} X_\bullet \longrightarrow X_{n+1} \longrightarrow X_n$$

is a weak equivalence in P_n sSet for $n \geq 0$. □

4.3. **Chromatic towers of localisations.** We can also use the homotopy limit model structure on towers of categories to obtain a categorified version of yet another classical result. The chromatic convergence theorem states that for a finite p-local spectrum X,

$$X \simeq \operatorname{holim}_n L_n X$$

where L_n denotes left localisation at the chromatic homology theory $E(n)$, see [26, Theorem 7.5.7]. We will see that the Quillen adjunction between spectra and the left Quillen presheaf of chromatic localisations of spectra induces an adjunction between the homotopy category of finite spectra and the homotopy category of chromatic towers subject to a suitable finiteness condition. The chromatic convergence theorem then shows that the derived unit of this adjunction is a weak equivalence. (Note that by spectra in this section we always mean *p-local spectra*.)

Let $\mathcal{C}$ be a proper and combinatorial stable model category. Define $L_n\mathcal{C}$ to be the localisation of $\mathcal{C}$ with respect to $E(n)$-equivalences. This defines a left Quillen presheaf

$$L_\bullet\mathcal{C} \colon \mathbb{N}^{op} \longrightarrow \mathrm{CAT} \,.$$

By Proposition 4.4 we get the following.

Proposition 4.9. *There is a left proper, combinatorial and stable model structure on the category of sections* $\mathrm{Sect}(\mathbb{N}^{op}, L_\bullet\mathcal{C})$, *such that a map is a weak equivalence (respectively, a cofibration) if and only if each*

$$f_n \colon X_n \longrightarrow Y_n$$

is a weak equivalence (respectively, a cofibration) in $L_n\mathcal{C}$. *A map* $f_n \colon X_n \to Y_n$ *is a fibration if and only if* f_0 *is a fibration in* $L_0\mathcal{C}$ *and*

$$X_{n+1} \longrightarrow Y_{n+1} \times_{Y_n} X_n$$

is a fibration in $L_{n+1}\mathcal{C}$ *for all* $n \geq 1$. □

Note that the resulting model structure is stable as each $L_n\mathcal{C}$ is stable. We then perform a right Bousfield localisation to obtain the homotopy limit model structure. Note that this again results in a stable model category as this right localisation is stable in the sense of [4, Definition 5.3]. As left localisation with respect to $E(n)$ is also stable in the sense of [4, Definition 4.2], $L_n\mathcal{C}$ is both left and right proper if $\mathcal{C}$ is; see [3, Proposition 4.7]. Hence, Proposition 4.5 implies

Proposition 4.10. *Let* $\mathcal{C}$ *be a proper, combinatorial and stable model category. There is a model structure* $\mathrm{Chrom}(\mathcal{C})$ *in* $\mathrm{Sect}(\mathbb{N}^{op}, L_\bullet\mathcal{C})$ *with the following properties.*

(i) *A morphism is a fibration in* $\mathrm{Chrom}(\mathcal{C})$ *if and only if it is a fibration in* $\mathrm{Sect}(\mathbb{N}^{op}, L_\bullet\mathcal{C})$.

(ii) *An object* $X_\bullet$ *is cofibrant in* $\mathrm{Chrom}(\mathcal{C})$ *if all the* X_n *are cofibrant in* $\mathcal{C}$ *and* $X_{n+1} \to X_n$ *is an* $E(n)$-*equivalence for each* n.

$\square$

The following is useful to justify the name "homotopy limit model structure".

Lemma 4.11. *Let* $f\colon X_\bullet \to Y_\bullet$ *be a weak equivalence in* $\mathrm{Chrom}(\mathrm{Sp})$. *Then*

$$\mathrm{holim}\, X_\bullet \longrightarrow \mathrm{holim}\, Y_\bullet$$

is a weak equivalence of spectra.

Proof. Let $f\colon X_\bullet \to Y_\bullet$ be a weak equivalence in $\mathrm{Chrom}(\mathcal{C})$. This implies that

$$\mathrm{Ho}(\mathrm{Chrom}(\mathcal{C}))(\mathrm{const}(A), X_\bullet) \longrightarrow \mathrm{Ho}(\mathrm{Chrom}(\mathcal{C}))(\mathrm{const}(A), Y_\bullet)$$

is an isomorphism for all cofibrant $A \in \mathcal{C}$. By Lemma 4.7, $(\mathrm{const}, \lim)$ is a Quillen pair, so the above is equivalent to

$$[A, \mathrm{holim}\, X_\bullet] \longrightarrow [A, \mathrm{holim}\, Y_\bullet]$$

is an isomorphism for all cofibrant $A \in \mathcal{C}$, where the square brackets denote morphisms in the stable homotopy category. But as the class of all cofibrant spectra detects isomorphisms in the stable homotopy category, this is equivalent to

$$\mathrm{holim}\, X_\bullet \longrightarrow \mathrm{holim}\, Y_\bullet$$

being a weak equivalence of spectra as desired. $\square$

It is important to note that we do not know if the converse is true. Looking at the proof of this lemma, we see that the following are equivalent:

(i) There is a set of constant generators $\mathrm{const}(G)$ for $\mathrm{Chrom}(\mathcal{C})$.
(ii) The weak equivalences in $\mathrm{Chrom}(\mathcal{C})$ are precisely the holim-isomorphisms.

Unfortunately, it is not known from the definition of the homotopy limit model structure whether any of those equivalent conditions hold.

We can now turn to the main result of this subsection. For this, we need to specify our finiteness conditions. Recall that a (p-local) spectrum is called finite if it is in the full subcategory of the stable homotopy category $\mathrm{Ho}(\mathrm{Sp})$ which contains the sphere spectrum and is closed under exact triangles and retracts. We denote this full subcategory by $\mathrm{Ho}(\mathrm{Sp})^{fin}$.

Definition 4.12. We call a diagram $X_\bullet$ in $\mathrm{Chrom}(\mathrm{Sp})$ *finitary* if $\mathrm{holim}\, X_\bullet$ is a finite spectrum. By $\mathrm{Ho}(\mathrm{Chrom}(\mathrm{Sp}))^F$ we denote the full subcategory of the finitary diagrams in the homotopy category of $\mathrm{Chrom}(\mathrm{Sp})$.

Theorem 4.13. *The Quillen adjunction* $\mathrm{const}\colon \mathrm{Sp} \rightleftarrows \mathrm{Chrom}(\mathrm{Sp})\colon \lim$ *induces an adjunction*

$$\mathrm{Ho}(\mathrm{Sp})^{fin} \rightleftarrows \mathrm{Ho}(\mathrm{Chrom}(\mathrm{Sp}))^F.$$

The composite

$$\mathrm{Ho}(\mathrm{Sp})^{fin} \xrightarrow{\mathbb{L}\mathrm{const}} \mathrm{Ho}(\mathrm{Chrom}(\mathrm{Sp}))^F \xrightarrow{\mathrm{holim}} \mathrm{Ho}(\mathrm{Sp})^{fin}$$

is isomorphic to the identity.

Proof. Firstly, we notice that the derived adjunction

$$\mathbb{L}\mathrm{const}\colon \mathrm{Ho}(\mathrm{Sp}) \rightleftarrows \mathrm{Ho}(\mathrm{Chrom}(\mathrm{Sp}))\colon \mathbb{R}\lim = \mathrm{holim}$$

restricts to an adjunction

$$\mathbb{L}\mathrm{const}\colon \mathrm{Ho}(\mathrm{Sp})^{fin} \rightleftarrows \mathrm{Ho}(\mathrm{Chrom}(\mathrm{Sp}))^F\colon \mathbb{R}\lim = \mathrm{holim}.$$

By definition, the homotopy limit of each finitary diagram is assumed to be a finite spectrum. On the other side,

$$\text{holim}(\mathbb{L}\text{const}(X)) \simeq X$$

is exactly the chromatic convergence theorem for finite spectra. The derived unit of the above adjunction is a weak equivalence. For a cofibrant spectrum

$$X \longrightarrow (\text{holim}(\text{const}(X)) = \text{holim}_n L_n X)$$

is again the chromatic convergence theorem. $\qquad \square$

We would really like to show that the above adjunction is an equivalence of categories, that is, that the counit is a weak equivalence, meaning that

$$\text{const}(\text{holim } Y_\bullet) \longrightarrow Y_\bullet$$

is a weak equivalence for $Y_\bullet$ a fibrant and cofibrant finitary diagram in $\text{Chrom}(\text{Sp})$). However, to show this we would need to know that the weak equivalences in $\text{Chrom}(\text{Sp})$ are exactly the holim-isomorphisms; see earlier remark. Furthermore, we would not just have to know that $\text{Chrom}(\text{Sp})$ has a constant set of generators but also that those generators are finitary, that is, have finite homotopy limit.

4.4. Homotopy pullbacks of model categories. Let $\mathcal{I}$ be the small category

$$1 \xleftarrow{\alpha} 0 \xrightarrow{\beta} 2.$$

A *pullback diagram of model categories* is a left Quillen presheaf $F\colon \mathcal{I}^{\text{op}} \to \text{CAT}$. The objects $X_\bullet$ of the category of sections are given by three objects X_0, X_1 and X_2 in $F(0), F(1)$ and $F(2)$, respectively, together with morphisms

$$\alpha^* X_1 \longrightarrow X_0 \longleftarrow \beta^* X_2$$

in $F(0)$. A morphism $\phi_\bullet\colon X_\bullet \to Y_\bullet$ consists of morphisms $\phi_i\colon X_i \to Y_i$ in $F(i)$ for $i = 0, 1, 2$, such that the diagram

$$
\begin{array}{ccccc}
\alpha^* X_1 & \longrightarrow & X_0 & \longleftarrow & \beta^* X_2 \\
\downarrow{\scriptstyle \alpha^* \phi_1} & & \downarrow{\scriptstyle \phi_0} & & \downarrow{\scriptstyle \beta^* \phi_2} \\
\alpha^* Y_1 & \longrightarrow & Y_0 & \longleftarrow & \beta^* Y_2
\end{array}
$$

commutes.

Proposition 4.14. *Let $F\colon \mathcal{I}^{\text{op}} \to \text{CAT}$ be a pullback diagram of model categories such that each $F(i)$ combinatorial model category for every i in $\mathcal{I}$. Then there exist a combinatorial model structure on the category of sections $\text{Sect}(\mathcal{I}^{\text{op}}, F)$, where a map $\phi_\bullet$ is a weak equivalence or a cofibration if ϕ_i is a weak equivalence or cofibration in $F(i)$ for every i in $\mathcal{I}$. The fibrations are the maps $\phi_\bullet\colon X_\bullet \to Y_\bullet$ such that f_0 is a fibration in $F(0)$ and*

$$X_1 \longrightarrow Y_1 \times_{\alpha_* Y_0} \alpha_* X_0 \quad \text{and} \quad X_2 \longrightarrow Y_2 \times_{\beta_* Y_0} \beta_* X_0$$

are fibrations in $F(1)$ and $F(2)$, respectively. In particular, $X_\bullet$ is fibrant if X_i is fibrant in $F(i)$ and

$$X_1 \longrightarrow \alpha_* X_0 \quad \text{and} \quad X_2 \longrightarrow \beta_* X_0$$

are fibrations in $F(1)$ and $F(2)$, respectively

Proof. The existence of the required model structure follows from Theorem 4.2. The description of the fibrations follows from [16, Theorem 3.1]. $\qquad \square$

Proposition 4.15. *Let $F\colon \mathcal{I}^{\mathrm{op}} \to \mathrm{CAT}$ be a pullback diagram of model categories such that each $F(i)$ is combinatorial and right proper for every i in $\mathcal{I}$. Then there is a model structure $\mathrm{Pull}(\mathcal{C})$ on the category of sections of F, called the homotopy pullback model structure, with the following properties:*

(i) *A morphism $\phi_\bullet$ is a fibration in $\mathrm{Pull}(F)$ if $\phi_\bullet$ is a fibration in $\mathrm{Sect}(\mathcal{I}^{\mathrm{op}}, F)$.*

(ii) *A section $X_\bullet$ is cofibrant in $\mathrm{Pull}(F)$ if X_i is cofibrant in $F(i)$ for every i in $\mathcal{I}$ and the morphisms $\alpha^* X_1 \to X_0$ and $\beta^* X_2 \to X_0$ are weak equivalences in $F(0)$.*

(iii) *A morphism $\phi_\bullet$ between cofibrant sections is a weak equivalence if and only if ϕ_i is a weak equivalence in $F(i)$ for every i in $\mathcal{I}$.*

Proof. The existence of the model structure $\mathrm{Pull}(F)$ follows from Theorem 4.3 applied to the left Quillen presheaf F. The characterisation of the weak equivalences between cofibrant objects follows since $\mathrm{Pull}(F)$ is a right Bousfield localisation of $\mathrm{Sect}(\mathcal{I}^{\mathrm{op}}, F)$. $\square$

4.5. **Bousfield arithmetic squares of homological localisations.** Let $\mathcal{C}$ be a left proper spectral combinatorial model category and, let J and K be a partition of the set of primes numbers. By $\mathbb{Z}_J$ we denote the J-local integers, and by MG the Moore spectrum of the group G. Consider the model structures $L_{M\mathbb{Z}_J}\mathcal{C}$, $L_{M\mathbb{Z}_K}\mathcal{C}$ and $L_{M\mathbb{Q}}\mathcal{C}$, as described in Section 2.3.2. Since for every set of primes P, every $M\mathbb{Z}_P$-equivalence is an $M\mathbb{Q}$-equivalence, the identities $L_{M\mathbb{Z}_J}\mathcal{C} \to L_{M\mathbb{Q}}\mathcal{C}$ and $L_{M\mathbb{Z}_K}\mathcal{C} \to L_{M\mathbb{Q}}\mathcal{C}$ are left Quillen functors.

Thus we have a pullback diagram of model categories $L_\bullet\mathcal{C}\colon \mathcal{I}^{\mathrm{op}} \to \mathrm{CAT}$, where $\mathcal{I} = 1 \leftarrow 0 \to 2$ and $L_0\mathcal{C} = L_{M\mathbb{Q}}\mathcal{C}$, $L_1\mathcal{C} = L_{M\mathbb{Z}_J}\mathcal{C}$ and $L_2\mathcal{C} = L_{M\mathbb{Z}_K}\mathcal{C}$.

If $\mathcal{C}$ is a left proper stable combinatorial model category, then by Proposition 4.14, the model structure $\mathrm{Sect}(\mathcal{I}^{\mathrm{op}}, L_\bullet\mathcal{C})$ exists, and it is also a stable model structure because each of the involved model categories is stable.

Moreover, if in addition the model structures $L_{M\mathbb{Z}_J}\mathcal{C}$, $L_{M\mathbb{Z}_K}\mathcal{C}$ and $L_{M\mathbb{Q}}\mathcal{C}$ are right proper, then by Proposition 4.15 the model structure $\mathrm{Pull}(L_\bullet\mathcal{C})$, which we denote by $\mathrm{Bou}(\mathcal{C})$ also exists. The model structure $\mathrm{Bou}(\mathcal{C})$ is also stable, since it is a right Bousfield localisation with respect to a set of stable objects; see [3, Proposition 5.6].

Lemma 4.16. *The adjunction $\mathrm{const}\colon \mathcal{C} \rightleftarrows \mathrm{Bou}(\mathcal{C})\colon \lim$ is a Quillen pair.*

Proof. The proof is the same as the one for Lemma 4.7. $\square$

Now let Sp be a suitable model structure for the category of spectra, e.g., symmetric spectra. Note that for any spectrum E, the model structure $L_E\mathrm{Sp}$ is right proper [3, Proposition 4.7], hence the model structure $\mathrm{Bou}(\mathrm{Sp})$ exists.

Theorem 4.17. *The Quillen pair $\mathrm{const}\colon \mathrm{Sp} \rightleftarrows \mathrm{Bou}(\mathrm{Sp})\colon \lim$ is a Quillen equivalence.*

Proof. By [20, Proposition 1.3.13] it suffices to check that the derived unit and counit are weak equivalences.

Let X be a fibrant and cofibrant spectrum. We need to show that

$$X \longrightarrow \lim(\mathrm{const}(X)^{fib})$$

is a weak equivalence in Sp, where $(-)^{fib}$ denotes the fibrant replacement in $\mathrm{Bou}(\mathrm{Sp})$. The constant diagram $\mathrm{const}(X)$ is cofibrant in $\mathrm{Bou}(\mathrm{Sp})$ since const is

a left Quillen functor. Let

$$L_{M\mathbb{Z}_J} X \longrightarrow L_{M\mathbb{Q}} X \longleftarrow L_{M\mathbb{Z}_K} X$$

be a fibrant replacement of $\mathrm{const}(X)$ in $\mathrm{Bou}(\mathrm{Sp})$. We have that $L_{M\mathbb{Z}_K} X$, $L_{M\mathbb{Z}_J} X$ and $L_{M\mathbb{Q}} X$ are fibrant in $L_{M\mathbb{Z}_K} \mathrm{Sp}$, $L_{M\mathbb{Z}_J} \mathrm{Sp}$ and $L_{M\mathbb{Q}} \mathrm{Sp}$, respectively, and the two maps are fibrations in Sp and weak equivalences in $L_{M\mathbb{Q}} \mathrm{Sp}$. Now, by [9, Proposition 2.10], the map

$$X \longrightarrow \lim(L_{M\mathbb{Z}_K} X \longrightarrow L_{M\mathbb{Q}} X \longleftarrow L_{M\mathbb{Z}_J} X)$$
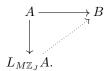
is a weak equivalence.

Now, let $X_\bullet$ be any fibrant and cofibrant object in $\mathrm{Bou}(\mathrm{Sp})$. We have to see that the map

$$\mathrm{const}(\lim X_\bullet) \longrightarrow X_\bullet$$

is a weak equivalence in $\mathrm{Bou}(\mathrm{Sp})$. This is equivalent to saying that the map $\lim X_\bullet \to X_1$ is a weak equivalence in $L_{M\mathbb{Z}_J} \mathrm{Sp}$, $\lim X_\bullet \to X_2$ is a weak equivalence in $L_{M\mathbb{Z}_K} \mathrm{Sp}$ and $\lim X_\bullet \to X_{12}$ is a weak equivalence in $L_{M\mathbb{Q}} \mathrm{Sp}$.

Note that if $A \to B$ is a map such that is a weak equivalence in $L_{M\mathbb{Q}} \mathrm{Sp}$, A is fibrant in $L_{M\mathbb{Z}_K} \mathrm{Sp}$ and B is fibrant in $L_{M\mathbb{Q}} \mathrm{Sp}$, then $A \to B$ is a weak equivalence in $L_{M\mathbb{Z}_J} \mathrm{Sp}$. To see this, let $A \to L_{M\mathbb{Z}_J} A$ be a fibrant replacement of A in $L_{M\mathbb{Z}_J} \mathrm{Sp}$. Since B is fibrant $L_{M\mathbb{Q}} \mathrm{Sp}$, it is in $L_{M\mathbb{Z}_J} \mathrm{Sp}$. Thus, there is a lifting

$$
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & \nearrow & \\
L_{M\mathbb{Z}_J} A. & &
\end{array}
$$

The left arrow is a weak equivalence in $L_{M\mathbb{Z}_J} \mathrm{Sp}$ and hence a weak equivalence in $L_{M\mathbb{Q}} \mathrm{Sp}$. Therefore the dotted arrow is a weak equivalence in $L_{M\mathbb{Q}} \mathrm{Sp}$ between fibrant objects in $L_{M\mathbb{Q}} \mathrm{Sp}$. (Observe that $L_{M\mathbb{Z}_J} A$ is fibrant in $L_{M\mathbb{Z}_J} \mathrm{Sp}$ and $L_{M\mathbb{Z}_K} \mathrm{Sp}$, and hence in $L_{M\mathbb{Q}} \mathrm{Sp}$). Thus, it is a weak equivalence in Sp. This completes the proof of the claim since weak equivalences in Sp are weak equivalences in $L_{M\mathbb{Z}_J} \mathrm{Sp}$.

Since $X_\bullet$ is fibrant and cofibrant, we have that in the pullback diagram

$$
\begin{array}{ccc}
\lim X_\bullet & \xrightarrow{\ f_2\ } & X_2 \\
{\scriptstyle f_1}\downarrow & & \downarrow \\
X_1 & \longrightarrow & X_{12}
\end{array}
$$

X_1, X_2 and X_{12} are fibrant in $L_{M\mathbb{Z}_J} \mathrm{Sp}$, $L_{M\mathbb{Z}_K} \mathrm{Sp}$ and $L_{M\mathbb{Q}} \mathrm{Sp}$, respectively, and the right and bottom arrows weak equivalences in $L_{M\mathbb{Q}} \mathrm{Sp}$ and fibrations in $L_{M\mathbb{Z}_K} \mathrm{Sp}$ and $L_{M\mathbb{Z}_J} \mathrm{Sp}$, respectively. By the previous observation and right properness of the model structures involved, the map $f_1 : \lim X_\bullet \to X_1$ is a weak equivalence in $L_{M\mathbb{Z}_J}$, and $f_2 : \lim X_\bullet \to X_2$ is a weak equivalence in $L_{M\mathbb{Z}_K} \mathrm{Sp}$, respectively. Thus, the map $\lim X_\bullet \to X_{12}$ is also a weak equivalence in $M\mathbb{Q}$, which means that $\mathrm{const}(\lim X_\bullet) \longrightarrow X_\bullet$ is a vertexwise weak equivalence, and thus a weak equivalence in $\mathrm{Bou}(\mathrm{Sp})$ as claimed. $\qquad\square$

Remark 4.18. There is a higher chromatic version of the objectwise statement. There is a homotopy fibre square

$$
\begin{array}{ccc}
L_n X & \longrightarrow & L_{K(n)} X \\
\downarrow & & \downarrow \\
L_{n-1} X & \longrightarrow & L_{n-1} L_{K(n)} X,
\end{array}
$$

see [12, Section 3.9]. However, we cannot apply the methods of this section to get a result analogously to Theorem 4.17. This is due to the fact that $L_{K(n)} L_{n-1} \operatorname{Sp}$ is trivial as a model category. (By [25, Theorem 2.1], a spectrum is $E(n-1)$-local if and only if it is $K(i)$-local for $1 \leq i \leq n-1$. But the $K(n)$-localisation of a $K(m)$-local spectrum is trivial for $n \neq m$.) Consider the homotopy limit model structure on

$$
L_{n-1} \operatorname{Sp} \longrightarrow L_{n-1} L_{K(n)} \operatorname{Sp} \longleftarrow L_{K(n)} \operatorname{Sp} .
$$

A fibrant and cofibrant diagram

$$
X_1 \xrightarrow{f_1} X_0 \xleftarrow{f_2} X_2
$$

would have to satisfy that X_1 is $E(n-1)$-local and f_1 is a $L_{n-1} L_{K(n)}$ localisation. By the universal property of localisations, this means that f_1 factors over $L_{n-1} L_{K(n)} X_1 \longrightarrow X_0$. However, as X_1 is $E(n-1)$-local and thus $K(n)$-acyclic, this map (and thus f_1) is trivial. Thus we cannot reconstruct a pullback square like the above from this model structure.

4.6. **Homotopy fibers of localised model categories.** We will use the homotopy pullback model structure to describe the homotopy fibre of Bousfield localisations. We can then use this to describe the layers of a Postnikov tower, among other examples.

Let $\mathcal{C}$ be a left proper pointed combinatorial model category and let $\mathcal{S}$ be a set of morphisms in $\mathcal{C}$. The identity $\mathcal{C} \to L_{\mathcal{S}} \mathcal{C}$ is a left Quillen functor and thus we have a pullback diagram of model categories $L_{\bullet}^{\mathcal{S}} \mathcal{C} \colon \mathcal{J}^{\mathrm{op}} \to \mathrm{CAT}$, where $\mathcal{J} = 1 \leftarrow 0 \to 2$, and $L_0^{\mathcal{S}} \mathcal{C} = L_{\mathcal{S}} \mathcal{C}$, $L_1^{\mathcal{S}} \mathcal{C} = *$ and $L_2^{\mathcal{S}} \mathcal{C} = \mathcal{C}$. (Here $*$ denotes the category with one object and one identity morphism with the trivial model structure.)

A section of $L_{\bullet}^{\mathcal{S}} \mathcal{C}$ is a diagram $* \to Y \leftarrow X$ in $\mathcal{C}$ where $*$ denotes the zero object. There is an adjunction

$$
\mathrm{const} \colon \mathcal{C} \rightleftarrows \mathrm{Sect}(\mathcal{J}^{\mathrm{op}}, L_{\bullet}^{\mathcal{S}} \mathcal{C}) \colon ev_2,
$$

where $\mathrm{const}(X) = (* \to X \xleftarrow{1} X)$ and $ev_2(* \to Y \leftarrow X) = X$. We will denote $\mathrm{Pull}(L_{\bullet}^{\mathcal{S}})$ by $\mathrm{Fib}(L_{\bullet}^{\mathcal{S}})$ and we will call it the *homotopy fiber* of the Quillen pair $\mathcal{C} \rightleftarrows L_{\mathcal{S}} \mathcal{C}$.

Definition 4.19. Let $\mathcal{C}$ be a proper pointed combinatorial model category and let $\mathcal{K}$ be a set of objects and $\mathcal{S}$ be a set of morphisms in $\mathcal{C}$. We say that the colocalised model structure $C_{\mathcal{K}} \mathcal{C}$ and the localised model structure $L_{\mathcal{S}} \mathcal{C}$ are *compatible* when for every object X in $\mathcal{C}$, X is $\mathcal{K}$-colocal if and only if X is cofibrant in $\mathcal{C}$ and the map $* \to X$ is an $\mathcal{S}$-local equivalence;

The stable case is discussed in detail in [4, Section 10] where such model structures are called "orthogonal", see also Section 4.6.3.

Remark 4.20. Note that if $C_{\mathcal{K}}\mathcal{C}$ and $L_{\mathcal{S}}\mathcal{C}$ are compatible, then it follows from the definitions $* \to Y \leftarrow X$ is cofibrant in $\mathrm{Fib}(L^{\mathcal{S}}_{\bullet}\mathcal{C})$, if and only both X and Y are $\mathcal{K}$-colocal and cofibrant in $\mathcal{C}$. If $* \to Y \leftarrow X$ is moreover fibrant in $\mathrm{Fib}(L^{\mathcal{S}}_{\bullet}\mathcal{C})$, then Y is weakly contractible since Y is $\mathcal{S}$-local and $* \to Y$ is an $\mathcal{S}$-equivalence and $X \to Y$ is a fibration in $\mathcal{C}$.

Theorem 4.21. *Let $\mathcal{C}$ be a proper pointed combinatorial model category and let $\mathcal{K}$ be a set of objects and $\mathcal{S}$ be a set of morphisms in $\mathcal{C}$. If $C_{\mathcal{K}}\mathcal{C}$ and $L_{\mathcal{S}}\mathcal{C}$ are compatible, then the adjunction*

$$\mathrm{const} \colon C_{\mathcal{K}}\mathcal{C} \rightleftarrows \mathrm{Fib}(L^{\mathcal{S}}_{\bullet}\mathcal{C}) \colon ev_2,$$

is a Quillen equivalence.

Proof. We will first show that the adjunction is a Quillen pair. For this, it is enough to check that the left adjoint preserves trivial cofibrations and sends cofibrations between cofibrant objects to cofibrations.

Let f be a trivial cofibration in $C_{\mathcal{K}}\mathcal{C}$. Then f is a trivial cofibration in $\mathcal{C}$ and therefore $\mathrm{const}(f)$ is a trivial cofibration in $\mathrm{Sect}(\mathcal{J}^{\mathrm{op}}, L^{\mathcal{S}}_{\bullet}\mathcal{C})$ and thus a trivial cofibration in $\mathrm{Fib}(L^{\mathcal{S}}_{\bullet}\mathcal{C})$.

Now let $f\colon X \to Y$ be a cofibration between cofibrant objects in $C_{\mathcal{K}}\mathcal{C}$. Then f is a cofibration between cofibrant objects in $\mathcal{C}$ and hence $\mathrm{const}(f)$ is also a cofibration between cofibrant objects in $\mathrm{Sect}(\mathcal{J}^{\mathrm{op}}, L^{\mathcal{S}}_{\bullet}\mathcal{C})$. But $\mathrm{const}(X)$ and $\mathrm{const}(Y)$ are cofibrant in $\mathrm{Fib}(L^{\mathcal{S}}_{\bullet}\mathcal{C})$, since $C_{\mathcal{K}}\mathcal{C}$ and $L_{\mathcal{S}}\mathcal{C}$ are compatible and therefore the maps $* \to X$ and $* \to Y$ are $\mathcal{S}$-local equivalences. Hence $\mathrm{const}(f)$ is a cofibration in $\mathrm{Fib}(L^{\mathcal{S}}_{\bullet}\mathcal{C})$.

To prove that it is a Quillen equivalence, it suffices to show that the derived unit and counit are weak equivalences; see [20, Proposition 1.3.13]. Let X be a cofibrant object in $C_{\mathcal{K}}\mathcal{C}$. Then we can construct a fibrant replacement for $\mathrm{const}(X)$ in $\mathrm{Fib}(L^{\mathcal{S}}_{\bullet}\mathcal{C})$ as follows

where the map $X \to L_{\mathcal{S}}X$ is a trivial cofibration in $L_{\mathcal{S}}\mathcal{C}$ and $X \to X' \to L_{\mathcal{S}}X$ is a factorisation in $\mathcal{C}$ of the previous map as a trivial cofibration followed by a fibration. Indeed, the map between the two sections is a trivial cofibration in $\mathrm{Fib}(L^{\mathcal{S}}_{\bullet}\mathcal{C})$ since it is a levelwise trivial cofibration, and $* \to L_{\mathcal{S}}X \leftarrow X'$ is fibrant in $\mathrm{Fib}(L^{\mathcal{S}}_{\bullet}\mathcal{C})$ since $L_{\mathcal{S}}X$ is fibrant in $L_{\mathcal{S}}\mathcal{C}$, X' is fibrant in $\mathcal{C}$ and $X' \to L_{\mathcal{S}}X$ is a fibration in $\mathcal{C}$.

Therefore the map $X \to ev_2(\mathrm{const}(X)) \to ev_2(R(\mathrm{const}(X)))$, where R denotes fibrant replacement in $\mathrm{Fib}(L^{\mathcal{S}}_{\bullet}\mathcal{C})$, is precisely the map $X \to X'$, which is a weak equivalence in $C_{\mathcal{K}}\mathcal{C}$ since it was already a weak equivalence in $\mathcal{C}$.

Finally, let $* \to Y \leftarrow X$ be a fibrant and cofibrant section in $\mathrm{Fib}(L^{\mathcal{S}}_{\bullet}\mathcal{C})$. We need check that the composite

$$\mathrm{const}(Q(ev_2(* \to Y \leftarrow X))) \longrightarrow \mathrm{const}(ev_2(* \to Y \leftarrow X)) \longrightarrow (* \to Y \leftarrow X)$$

is a weak equivalence in $\mathrm{Fib}(L_\bullet^{\mathcal{S}}\mathcal{C})$. But $ev_2(* \to Y \leftarrow X) = X$ is already cofibrant in $C_\mathcal{K}\mathcal{C}$, by Remark 4.20. Therefore, we need to show that the map of sections

$$
\begin{array}{ccc}
* \longrightarrow X & == & X \\
\parallel \quad\quad \downarrow & & \parallel \\
* \longrightarrow Y & \longleftarrow & X
\end{array}
$$

is a weak equivalence in $\mathrm{Fib}(L_\bullet^{\mathcal{S}}\mathcal{C})$. Since both sections are cofibrant, it is enough to see that the map in the middle is a weak equivalence in $L_\mathcal{S}\mathcal{C}$, which follows again from Remark 4.20. $\square$

4.6.1. *Postnikov sections and connective covers of simplicial sets.* We can use this setup to describe the "layers" of Postnikov towers. Let sSet_* denote the category of pointed simplicial sets. Consider the model structure $P_k\,\mathrm{sSet}_* = L_\mathcal{S}\,\mathrm{sSet}_*$ for k-types, as in Section 3, where $\mathcal{S} = \{S^{k+1} \to D^{k+2}\}$. If $\mathcal{K} = \{S^{k+1}\}$, then $P_k\,\mathrm{sSet}_*$ and $C_k\,\mathrm{sSet}_* := C_{S^{k+1}}\,\mathrm{sSet}_*$ are compatible, since for every X there is a fiber sequence

$$C_k X \longrightarrow X \longrightarrow P_k X,$$

where $C_k X$ is the kth connective cover of X. By Theorem 4.21 the model categories $C_k\,\mathrm{sSet}_*$ and $\mathrm{Fib}(L_\bullet^{\mathcal{S}}\,\mathrm{sSet}_*)$ are Quillen equivalent.

Note that in the general case the localisations $L_\mathcal{S}\mathcal{C}$ and $C_\mathcal{K}\mathcal{C}$ for

$$\mathcal{S} = \{G \otimes S^{k+1} \longrightarrow G \otimes D^{k+2} \mid G \in \mathcal{G}\}$$

and $\mathcal{K} = \{G \otimes S^{k+1} \mid G \in \mathcal{G}\}$ are not necessarily compatible, so this construction cannot be performed with general $\mathcal{C}$. However, examples of $\mathcal{C}$ where compatibility holds include chain complexes $Ch_b(R)$ and stable localisations; see Section 4.6.3.

We can also consider $\mathrm{Fib}(L_\bullet^{\mathcal{S}}P_{k+1}\,\mathrm{sSet}_*)$. Since for every X we have a fibration

$$K(\pi_{k+1}X, k+1) \longrightarrow P_{k+1}X \longrightarrow P_k X,$$

the model structures $C_k P_{k+1}\,\mathrm{sSet}_*$ and $P_k P_{k+1}\,\mathrm{sSet}_* = P_k\,\mathrm{sSet}_*$ are compatible. Hence Theorem 4.21 implies that $C_k P_{k+1}\,\mathrm{sSet}_*$ and $\mathrm{Fib}(L_\bullet^{\mathcal{S}}P_{k+1}\,\mathrm{sSet}_*)$ are Quillen equivalent. This means that we can view $C_k P_{k+1}\,\mathrm{sSet}_*$ as the kth layer of the Postnikov tower model structure. Note that $\mathrm{Ho}(C_k P_{k+1}\,\mathrm{sSet}_*)$ is equivalent to the category of abelian groups.

4.6.2. *Nullifications and cellularisations of spectra.* Let Sp be a suitable model structure for the category of spectra, for instance, symmetric spectra and let $\mathcal{S}$ be a set of maps. If $\mathcal{S} = \{E \to *\}$ then $L_\mathcal{S}\,\mathrm{Sp} = P_E\,\mathrm{Sp}$ is called the E-*nullification* of Sp and $C_E\,\mathrm{Sp}$ is called E-cellularisation of Sp. As follows from [17, Theorem 3.6] we have the following compatibility between localised and colocalised model structures:

(i) If the induced map $\mathrm{Ho}(\mathrm{Sp})(\Sigma^{-1}E, C_E X) \to \mathrm{Ho}(\mathrm{Sp})(\Sigma^{-1}E, X)$ is injective for every X, then $C_E\,\mathrm{Sp}$ and $P_E\,\mathrm{Sp}$ are compatible.
(ii) If the induced map $\mathrm{Ho}(\mathrm{Sp})(E, X) \to \mathrm{Ho}(\mathrm{Sp})(E, P_{\Sigma E}X)$ is the zero map for every X, then $C_E\,\mathrm{Sp}$ and $P_{\Sigma E}\,\mathrm{Sp}$ are compatible.

4.6.3. *Stable localisations and colocalisations.* Let $\mathcal{C}$ be a proper spectral combinatorial model category. Let $\mathcal{S}$ be a stable set of morphisms in $\mathcal{C}$ and let $\mathcal{K} = \mathrm{cof}(\mathcal{S})$ be the set of cofibers of the elements of $\mathcal{S}$. Since $\mathcal{C}$ is spectral, we have a Quillen bifunctor $\otimes \colon \mathcal{C} \times \mathrm{Sp} \to \mathcal{C}$, and $\mathrm{cof}(\mathcal{S} \otimes \mathcal{G}_{\mathrm{Sp}}) = \mathrm{cof}(\mathcal{S}) \otimes \mathcal{G}_{\mathrm{Sp}} = \mathcal{K} \otimes \mathcal{G}_{\mathrm{Sp}}$. Hence, by [3, Proposition 10.3] it follows that $L_{\mathcal{S} \otimes \mathcal{G}_{\mathrm{Sp}}} \mathcal{C}$ and $C_{\mathcal{K} \otimes \mathcal{G}_{\mathrm{Sp}}} \mathcal{C}$ are compatible. Therefore, Theorem 4.21 implies that the model categories $C_{\mathcal{K} \otimes \mathcal{G}_{\mathrm{Sp}}} \mathcal{C}$ and $\mathrm{Fib}(L_{\bullet}^{\mathcal{S} \otimes \mathcal{G}_{\mathrm{Sp}}} \mathcal{C})$ are Quillen equivalent.

5. Convergence of towers

In this section we are going to take a closer look at what it means for a tower in $\mathrm{Post}(\mathcal{C})$ to converge. Recall that we have a Quillen adjunction

$$\mathrm{const} : \mathcal{C} \rightleftarrows \mathrm{Post}(\mathcal{C}) : \lim.$$

The following terminology appears in [5, Definition 5.35].

Definition 5.1. The model category $\mathcal{C}$ is *hypercomplete* if the composite

$$\mathrm{Ho}(\mathcal{C}) \xrightarrow{\mathbb{L}\mathrm{const}} \mathrm{Ho}(\mathrm{Post}(\mathcal{C})) \xrightarrow{\mathrm{holim}} \mathrm{Ho}(\mathcal{C})$$

is isomorphic to the identity, that is, for every cofibrant X in $\mathcal{C}$, the natura map

$$X \longrightarrow \mathrm{holim}(\mathrm{const}X)$$

is a weak equivalence.

We have seen in Section 4.2 that this is true for $\mathcal{C} = \mathrm{sSet}$. We have also seen in Theorem 4.13 that, under a finiteness assumption, the chromatic tower of spectra $\mathrm{Chrom}(\mathrm{Sp})$ is hypercomplete in this sense. We can also consider the case of $\mathcal{C} = L_{\mathcal{S}} \mathrm{sSet}_*$ as in Section 3.3. In general, this model category will not be hypercomplete. Let X be fibrant in $L_{\mathcal{S}} \mathrm{sSet}_*$, that is, fibrant as a simplicial set and $\mathcal{S}$-local. If we take the fibrant replacement of the constant tower $\mathrm{const}(Y)$ in $\mathrm{Post}(L_{\mathcal{S}} \mathrm{sSet}_*)$, we obtain a tower

$$(\mathrm{const}(Y))^{fib} = (\cdots \longrightarrow Y_n \longrightarrow Y_{n-1} \longrightarrow \cdots \longrightarrow Y_0)$$

such that all the Y_i are $\mathcal{S}$-local, Y_i is P_i-local for all i and $Y_n \to Y_{n-1}$ is a weak equivalence in $P_{n-1} L_{\mathcal{S}} \mathrm{sSet}_*$. However, this is not a fibrant replacement of $\mathrm{const}(Y)$ in $\mathrm{Post}(\mathrm{sSet}_*)$, unless $L_{\mathcal{S}}$ commutes with all the localisations P_n. In this case, a Postnikov tower in $L_{\mathcal{S}} \mathrm{sSet}_*$ is also a Postnikov tower in sSet_*, and hypercompleteness holds. This would be the case for $L_{\mathcal{S}} = L_{MR}$ for R a subring of the rational numbers $\mathbb{Q}$, but it cannot be expected in general.

Let us recapture the classical case to get a more general insight into hypercompleteness. For X in sSet we know that $X \to \lim_n P_n X$ is a weak equivalence. This equivalent to saying that for all i,

$$\pi_i(X) \longrightarrow \pi_i(\lim_n P_n X)$$

is an isomorphism of groups. But we have also seen that

$$\pi_i(\lim_n P_n X) = \lim_n \pi_i(P_n X)$$

as well as

$$\pi_i(P_n X) = (\pi_i(X)) \langle n \rangle = \begin{cases} \pi_i(X) & \text{if } i \leq n, \\ 0 & \text{if } i > n. \end{cases}$$

Putting this together we get that indeed, $\pi_i(\lim_n P_n X) \cong \pi_i(X)$ for all i. This is a special case of the following. Let $\mathcal{C}$ be a proper combinatorial model category with a set of generators $\mathcal{G}$. Then for a cofibrant X, the map $X \to \lim_n P_n X$ is a weak equivalence in $\mathcal{C}$ if and only if

$$\operatorname{map}_{\mathcal{C}}(G, X) \longrightarrow \operatorname{map}_{\mathcal{C}}(G, \operatorname{holim}_n P_n X) = \lim_n \operatorname{map}_{\mathcal{C}}(G, P_n X)$$

is a weak equivalence in sSet for all $G \in \mathcal{G}$.

So from this we can see that if we had $\operatorname{map}_{\mathcal{C}}(G, P_n X) \cong P_n \operatorname{map}_{\mathcal{C}}(G, X)$ for all G in $\mathcal{G}$, then we would get the desired weak equivalence because again

$$\pi_i \operatorname{map}_{\mathcal{C}}(G, P_n X) = \pi_i(P_n \operatorname{map}_{\mathcal{C}}(G, X)) = \pi_i(\operatorname{map}_{\mathcal{C}}(G, X)) \langle n \rangle.$$

We could also reformulate this statement by not using the full set of generators $\mathcal{G}$, since we are only making use of the fact that they detect weak equivalences.

Proposition 5.2. *Let $h\mathcal{G}$ be a set in $\mathcal{C}$ that detects weak equivalences. If*

$$\operatorname{map}_{\mathcal{C}}(G, P_n X) \cong P_n \operatorname{map}_{\mathcal{C}}(G, X)$$

for every G in $h\mathcal{G}$, then $\mathcal{C}$ is hypercomplete. $\qquad\qquad\square$

We can follow this through with our non-simplicial example, bounded chain complexes of $\mathbb{Z}$-modules $\operatorname{Ch}_b(\mathbb{Z})$. Let $\operatorname{Hom}(M, N)$ denote the mapping chain complex for M, N in $\operatorname{Ch}_b(\mathbb{Z})$, that is,

$$\operatorname{Hom}(M, N)_k = \prod_i \operatorname{Hom}_{\mathbb{Z}}(M_i, N_{i+k})$$

with differential $(df)(x) = d(f(x)) + (-1)^{k+1} f(d(x))$; see for example [20, Chapter 4.2]. We note that

$$\pi_i(\operatorname{map}_{\operatorname{Ch}_b(\mathbb{Z})}(M, N)) = H_i(\operatorname{Hom}(M, N))$$

because

$$\pi_i(\operatorname{map}_{\operatorname{Ch}_b(\mathbb{Z})}(M, N)) = [S^i, \operatorname{map}_{\operatorname{Ch}_b(\mathbb{Z})}(M, N)]_{\text{sSet}} = [M \otimes^L S^i, N]_{\operatorname{Ch}_b(\mathbb{Z})}$$
$$= [M[i], N]_{\operatorname{Ch}_b(\mathbb{Z})} = [M \otimes \mathbb{Z}[i], N]_{\operatorname{Ch}_b(\mathbb{Z})} = [\mathbb{Z}[i], \operatorname{Hom}(M, N)]_{\operatorname{Ch}_b(\mathbb{Z})}$$
$$= H_i(\operatorname{Hom}(M, N)).$$

So $\operatorname{Ch}_b(\mathbb{Z})$ is hypercomplete if $\operatorname{Hom}(G, P_n N)$ is quasi-isomorphic to $P_n \operatorname{Hom}(G, N)$ for all G in $h\mathcal{G}$. For bounded below chain complexes, a set that detects weak equivalences can be taken to be

$$h\mathcal{G} = \{\mathbb{S}^i = \mathbb{Z}[i] \mid i \geq 0\}.$$

Note that in general it is not true that $\operatorname{Hom}(M, P_n N) \simeq P_n \operatorname{Hom}(M, N)$. We have the following diagram of short exact sequences.

$$\begin{array}{ccccc}
\operatorname{Ext}_{\mathbb{Z}}(H_i(M), H_{i+1}(N)) & \longrightarrow & H_i(\operatorname{Hom}(M, N)) & \longrightarrow & \operatorname{Hom}_{\mathbb{Z}}(H_i(M), H_i(N)) \\
\downarrow & & \downarrow & & \downarrow \\
\operatorname{Ext}_{\mathbb{Z}}(H_i(M), H_{i+1}(P_n N)) & \longrightarrow & H_i(\operatorname{Hom}(M, P_n N)) & \longrightarrow & \operatorname{Hom}_{\mathbb{Z}}(H_i(M), H_i(P_n N)).
\end{array}$$

Using the 5-lemma we can read off that $H_i(\operatorname{Hom}(M, P_n N)) = 0$ for $i > n$ as desired and that

$$H_i(\operatorname{Hom}(M, P_n N)) = H_i(\operatorname{Hom}(M, N))$$

for $i \leq n-1$, but unless $\mathrm{Ext}_{\mathbb{Z}}(H_n(M), H_{n+1}(N)) = 0$ we do *not* get that

$$H_n(\mathrm{Hom}(M, P_nN)) = H_n(\mathrm{Hom}(M, N)).$$

However, as we only require the case $M = \mathbb{S}^i$, we have that

$$\mathrm{Hom}(\mathbb{S}^i, N) = N[n],$$

where $N[n]$ is the n-fold suspension of N. Thus,

$$\mathrm{Hom}(G, P_nN) = P_n\mathrm{Hom}(G, N)$$

for all G in $h\mathcal{G}$, so $\mathrm{Ch}_b(\mathbb{Z})$ is hypercomplete as expected.

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