Properties of Null Knotted Solutions to Maxwell's Equations

Gregory Smith[a*], Paul Strange[a]
[a]School of Physical Sciences, Ingram Building, University of Kent, Canterbury, CT2 7NZ, United Kingdom

ABSTRACT

We discuss null knotted solutions to Maxwell's equations, their creation through Bateman's construction, and their relation to the Hopf-fibration. These solutions have well-known, conserved properties, related to their winding numbers. For example: energy; momentum; angular momentum; and helicity. The current research has focused on Lipkin's zilches, a set of little-known, conserved quantities within electromagnetic theory that has been explored mathematically, but over which there is still considerable debate regarding physical interpretation. The aim of this work is to contribute to the discussion of these knotted solutions of Maxwell's equations by examining the relation between the knots, the zilches, and their symmetries through Noether's theorem. We show that the zilches demonstrate either linear or more complicated relations to the p-q winding numbers of torus knots, and can be written in terms of the total energy of the electromagnetic field. As part of this work, a systematic multipole expansion of the vector potential of the knotted solutions is being carried out.

Keywords: Hopf-fibration, Knotted Solutions of Maxwell's equations, Lipkin's zilches, Multipole Expansion

1. INTRODUCTION

In 1989 Rañada discovered a particularly interesting analogy between the knotted solution to Maxwell's equations[1,2] and the Hopf-fibration. However, Rañada's method did not preserve the topology of the field lines with time. In 2008 Irvine *et al.* reformulated this solution[3,4] using Bateman's construction[5]. This method involves setting up a complex vector field where the real parts are the electric field and the imaginary parts are the magnetic field - whose topology is preserved with time (See section 2.1). Kedia *et al.* extended this solution[6] to give the whole family of torus knots and links by taking co-prime powers of the complex input functions (See section 2.2). We begin by explaining the methods used by Irvine *et al.* and Kedia *et al.* Then the current research will be discussed, where we look at the analytic form of Lipkin's zilches for the knotted electromagnetic fields.

2. NULL KNOTTED SOLUTIONS TO MAXWELL'S EQUATIONS

2.1 Initial Field Construction

To create our fields we follow Bateman's construction[5], as demonstrated clearly by Hoyos *et al.*[7] and Irvine *et al.*[3] We use rational units throughout with wave vectors set equal to unity in the free field Maxwell equations. To ensure the equations have a high degree of symmetry, the fields are redefined as follows,

$$\boldsymbol{E} \rightarrow \boldsymbol{E},$$

$$\boldsymbol{B} \rightarrow \frac{1}{c}\boldsymbol{B}.$$

The symmetrised free field Maxwell equations become,

$$\nabla \cdot \boldsymbol{E} = 0, \tag{1}$$

$$\nabla \cdot \boldsymbol{B} = 0, \tag{2}$$

$$\nabla \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t}, \tag{3}$$

$$\nabla \times \boldsymbol{B} = \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t}, \tag{4}$$

*gs357@kent.ac.uk; phone +44 7836 669 919

Complex Light and Optical Forces XI, edited by David L. Andrews, Enrique J. Galvez,
Jesper Glückstad, Proc. of SPIE Vol. 10120, 101201C · © 2017 SPIE
CCC code: 0277-786X/17/$18 · doi: 10.1117/12.2260484

where the symbols take on their usual meaning. In Bateman's construction the Riemann-Silberstein vector[8], $\boldsymbol{R}$, in dimensions $(3+1)$, is created where the electric field and magnetic field are the real and imaginary parts respectively,

$$\boldsymbol{R} = \boldsymbol{E} + i\boldsymbol{B}. \tag{5}$$

This newly constructed vector field is then expressed in terms of two complex functions, α and β, with the ansatz:

$$\boldsymbol{R} = \nabla\alpha \times \nabla\beta. \tag{6}$$

Maxwell's equations are satisfied, provided $\boldsymbol{R}$ also takes the form,

$$\boldsymbol{R} = \frac{i}{c}\left(\frac{\partial\alpha}{\partial t}\nabla\beta - \frac{\partial\beta}{\partial t}\nabla\alpha\right). \tag{7}$$

Up to this point, the complex vector field, $\boldsymbol{R}$, has been constructed in terms of the unknown functions α and β. Bateman's construction also requires that the field should have null-norm,

$$\boldsymbol{R}^2 = 0. \tag{8}$$

Physically, this forces the electric and magnetic fields to have the same magnitudes and to be orthogonal.

2.2 The Hopf-Fibration, Knots, and the Electromagnetic Field

In the literature an analogy between the knotted solutions of Maxwell's equations and the Hopf-fibration has been noted and explored[2,3,4,6,7]. The mathematical construction[9,10] of the Hopf-fibration is via the method of stereographic projection from the four dimensional spherical space, S^3, into three dimensional Euclidean space, R^3. The equations for the Hopf-fibration are[7],

$$\alpha = \frac{A - 1 + iz}{A + it}, \tag{9}$$

$$\beta = \frac{(x - iy)}{A + it}, \tag{10}$$

where,

$$A = \frac{1}{2}(x^2 + y^2 + z^2 - t^2 + 1),$$

where, x, y, z are the three spacial coordinates, and t is that of time. To create the null Hopf-knotted electromagnetic field, these equations need to be substituted into those for the newly constructed complex vector field given by equation (7). The resulting field is,

$$\boldsymbol{R} = \frac{4}{(A + 2it)^3}\begin{pmatrix} (t - x - z + i(y-1)(t + x - z - i(y+1)), \\ -i(t - y - z - i(x+1))(t + y - z + i(x-1)), \\ 2(x - iy)(t - z - i) \end{pmatrix}. \tag{11}$$

Both fields, $\boldsymbol{E}$ and $\boldsymbol{B}$, have every level of the Hopf-fibration projected into them separately, whilst being orthogonal to each other. Each field line forms a closed loop (Villarceau circle), that traverses the surface of a torus and links with every field line through the centre of the torus - that forms one level of the Hopf-fibration. The remaining levels are nested inside each other like Russian dolls filling all of space - with only two levels appearing slightly different - the projections through the north pole (the infinitely large torus) and south pole (the infinitely thin torus) of S^3. Figure 1 demonstrates a few levels of the Hopf-fibration that will be projected into each field.

Kedia *et al.* extended this work[6] to create a whole family of torus knot solutions, where the electric and magnetic fields wrap around each other according to the co-prime, $p - q$ winding numbers. The $p - q$ winding numbers refer to the direction of rotation around a torus: p is the toroidal direction (around the large circumference); q

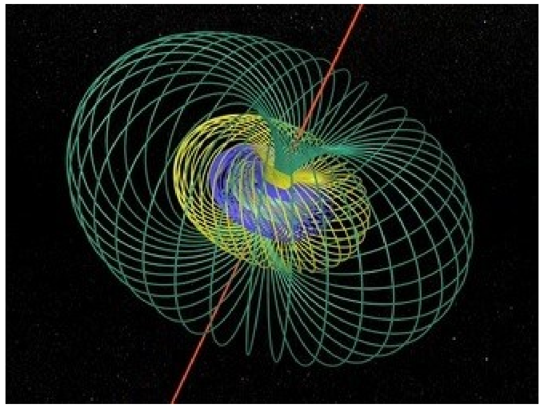

Figure 1. Five levels of S^3 from the Hopf-fibration projected into R^3 including the points at the north pole (red line) and south pole (thin, inner green ring)[11]. This is the form the electric and magnetic field lines take separately, whilst remaining orthogonal.

is the poloidal direction (around the small circumference). These winding numbers were incorporated into the construction of the complex vector field as powers of α and β,

$$\boldsymbol{R} = \nabla\alpha^p \times \nabla\beta^q. \tag{12}$$

and these powers also yield solutions of Maxwell's equations. Field line plots of the knotted and linked electromagnetic vector fields become more convoluted the more complicated the knotting becomes. For some vector fields, it is possible to have an orthogonal set of vector fields that are constructed of gradients of underlying scalar fields[6,7]. These underlying scalar fields become useful in the visualisation of our knotted solutions as they illuminate the cores of the tori twisting around each other, and clarify the different knottings (See figure 2). The equations for the scalar fields are given by,

$$\phi = \phi_B + i\phi_e = \alpha^p\beta^q, \tag{13}$$

whilst holding the condition,

$$\boldsymbol{E} \cdot \nabla\phi_E = 0; \;\; \boldsymbol{B} \cdot \nabla\phi_B = 0. \tag{14}$$

3. PROPERTIES OF KNOTTED ELECTROMAGNETIC FIELDS

3.1 Zilches of Electromagnetic Fields

In 1964 Lipkin discovered a new set of electromagnetic conserved quantities which he named zilches[12]. There are ten unique zilches which are understood mathematically, but their physical nature is still debated. Some have argued that the zilches are a measure of chirality of light fields, and have been using them to make successful predictions in experiments[13,14]. Others have suggested that the zilches could just be one of an infinite set of recurring conserved quantities that can be produced by taking curls of properties of fields[15,16]. This research aims to illuminate their true physical nature by exploring the zilches in the context of knotted electromagnetic fields and their relationship to the total energy of the field, and the $p - q$ winding numbers of the knots. The method used to understand the zilches is via a combination of algebraic and numerical computing, where each zilch is calculated using the components of the electric and magnetic fields from our overall field - equation(12).

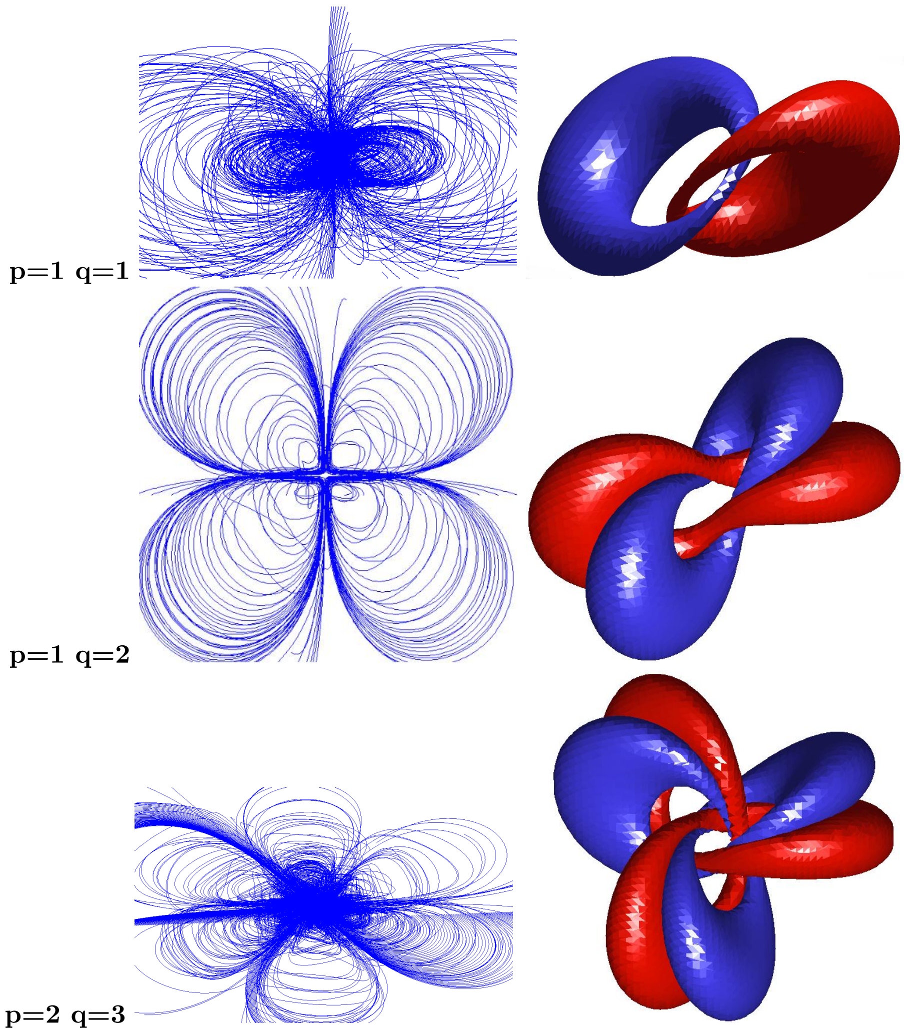

Figure 2. (Left hand column) The field lines for the electric and magnetic field are demonstrated here, becoming more twisted with the increasing $p-q$ winding numbers; (Right hand column) The underlying scalar field plots illuminate the knotting cores of our vector fields and allow for a better interpretation of each solution

The zilches take on the following mathematical form[15],

$$Z^{000} = \tfrac{1}{2}\left[\boldsymbol{E}\cdot(\boldsymbol{\nabla}\times\boldsymbol{E}) + \boldsymbol{B}\cdot(\boldsymbol{\nabla}\times\boldsymbol{B})\right],$$

$$Z^{0i0} = \tfrac{1}{2}\left[\boldsymbol{E}\times(\boldsymbol{\nabla}\times\boldsymbol{B}) - \boldsymbol{B}\times(\boldsymbol{\nabla}\times\boldsymbol{E})\right]_i,$$

$$Z^{ij0} = \tfrac{1}{2}\left(\delta_{ij}\left[\boldsymbol{E}\cdot(\boldsymbol{\nabla}\times\boldsymbol{E}) + \boldsymbol{B}\cdot(\boldsymbol{\nabla}\times\boldsymbol{B})\right] - E_i(\boldsymbol{\nabla}\times\boldsymbol{E})_j - E_j(\boldsymbol{\nabla}\times\boldsymbol{E})_i - B_i(\boldsymbol{\nabla}\times\boldsymbol{B})_j - B_j(\boldsymbol{\nabla}\times\boldsymbol{B})_i\right). \tag{15}$$

These elements and their fluxes form a third order tensor with symmetry properties described in detail by Lipkin[12]. Through Maxwell's equations, the zilches can also be expressed in terms of the time derivatives of the fields. The analytic expressions obtained for Z^{000} show that it can be related to the total energy of the field for each knot respectively and a relation can be derived in terms of the $p-q$ winding numbers.

$$Z^{000} = \frac{1}{2}\,(3p+2q)\,E_{pq} \tag{16}$$

where E_{pq} is the total energy of the electromagnetic field. First indications are that similar, but more complicated, expressions can be found for the other zilches and this work is in progress. These expressions reflect symmetries and our aim is to confirm these expressions and then to attempt to relate them to Noether's theorem[17].

3.2 Multipolar Expansion of the Knotted Electromagnetic Fields.

To further our understanding of the zilches, we are also performing a multipole expansion of the electric and magnetic fields and their corresponding potentials and may possibly extend this to the zilches themselves. Initially, we followed the multipole expansion approach utilised by Irvine[3,4]. Irvine found that the $(p, q) = (1, 1)$ field was a composed entirely of the $l = 1, m = \pm 1$ dipolar terms. The equations deployed in this analysis assumed an $\exp(-i\omega t)$ dependence of the electric field[18] which these knotted solutions do not have. However, Irvine argued that his analysis is valid at time $t = 0$. We have performed a more general analysis[19] and have confirmed Irvine's result for the $(p, q) = (1, 1)$ field. For higher values of $(p, q) = (2, 1)$ we have found that this is not a single multipole field, but has a number of multipolar components. We are currently investigating higher values of p and q to determine how the multipolar coefficients depend on the winding numbers p and q and whether this approach can yield further information about the interpretation of the zilches.

4. ACKNOWLEDGEMENTS

We would like to take the time to thank the School of Physical Sciences at the University of Kent for helping to partially fund the expenses for this conference.

5. REFERENCES

[1] Maxwell, J. C. "A Dynamical Theory of the Electromagnetic Field", Phil. Trans. R. Soc. Lond. **155**, 459-512, (1865).
[2] Rañada, A. F., "A Topological Theory of the Electromagnetic Field", Lett. Math. Phys, 1**8**, 97, (1989).
[3] Irvine, W. T. M. and Bouwmeester, D., "Linked and Knotted Beams of Light", Nat. Phys., **4**, 71620, (2008).
[4] Irvine, W. T. M., "Linked and knotted beams of light, conservation of helicity and the flow of null electromagnetic fields", J. Phys. A: Math. Theor. **43** 385203, (2010).
[5] Bateman, H. [The Mathematical Analysis of Electrical and Optical Wave-motion on the Basis of Maxwells Equations], Cambridge: Cambridge University Press, 1915.
[6] Kedia, H., Bialynicki-Birula, I., Peralta-Salas, D. and Irvine, W. T. M., "Tying Knots in Light Fields", Phys. Rev. Lett. **111** 150404, (2013).
[7] Hoyos, C. Sircar, N. and Sonnenschein, "New Knotted Solutions of Maxwell's Equations", J. Phys. A: Math. Theor. **48** 255204, (2015).
[8] Bialynicki-Birula, I. and Bialynicka-Birula, Z., "The role of the RiemannSilberstein vector in classical and quantum theories of electromagnetism", J. Phys. A: Math. Theor. **46** 053001, (2013).

[9] Lyons, D.W. "An Elementary Introduction to the Hopf Fibration", http://mas.lvc.edu/ lyons/pubs/hopf_ paper_ preprint.pdf, accessed 05/11/2014.
[10] Treisman, Z., "A Young Person's Guide to the Hopf Fibration". arXiv:0908.1205 [math.HO], (2009).
[11] Image taken from website: http://www.dimensions-math.org/Dim_CH7_E.htm, on 03/01/2017.
[12] Lipkin, D. M., "Existence of a new conservation law in electromagnetic theory", Journal of Mathematical Physics, (1964).
[13] Tang, Y. and Cohen, A. E., "Optical chirality and its interaction with matter ", Phys. Rev. Lett. **104** 163901, (2010).
[14] Yang, N. and Cohen, A. E., "Local geometry of electromagnetic fields and its role in molecular multipole transitions", J. Phys. Chem. B. **115** 5304-11, (2011).
[15] Cameron, R. P. and Barnett, "Optical helicity, optical spin and related quantities in electromagnetic theory", S. M. New J. Phys. **14** 053050, (2012).
[16] Kibble, T. W. B., "Conservation laws for free fields ", J. Maths. Phys. **6** 1022-6, (1964).
[17] Noether, E., "Invariante Variationsprobleme", Nachr. Akad. Wiss. Gottingen, Math. Phys. Kl., **2** 235-57, (1918).
[18] Jackson, J. D., [Classical Electrodynamics] New York: Wiley, 1999.
[19] Blatt J. M. and Weisskopf V. F., [Theoretical Nuclear Physics], John Wiley and Sons, new York, 1962.