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A Priori Error Estimate of Stochastic Galerkin Method for Optimal Control Problem Governed by Random Parabolic PDE with Constrained Control

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A stochastic Galerkin approximation scheme is proposed for an optimal control problem governed by a parabolic PDE with random perturbation in its coefficients. The objective functional is to minimize the expectation of a cost functional, and the deterministic control is of the obstacle constrained type. We obtain the necessary and sufficient optimality conditions and establish a scheme to approximate the optimality system through the discretization with respect to both the spatial space and the probability space by Galerkin method and with respect to time by the backward Euler scheme. *A priori* error estimates are derived for the state, the co-state and the control variables. Numerical examples are presented to illustrate our theoretical results.

Keywords: Priori error estimates; stochastic Galerkin method; optimal control problem; random parabolic PDE; obstacle control constraint.

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1. Introduction

Deterministic optimal control problems constrained by PDEs have been well developed and investigated for several decades. There have been extensive studies on this aspect. Some of progress in this area has been summarized in Adams [1975], Barbu [1984], Glowinski and Lions [1996], Grisvard [1986], Tiba [1995], and Fursikov [2000], and the references cited therein. Finite element approximation of optimal control problems plays a very important role in numerical methods for these problems. The authors also had some works on this field [Liu and Tiba (2001); Sun *et al.* (2013); Li *et al.* (2015); Liu (2008); Liu and Yan (2001); Liu and Yan (2008); Liu *et al.* (2010); Sun (2010)]. Nevertheless, because of the existence of uncertainty, such as uncertain parameters, arises in many complex real-world problems of physical and engineering applications, the variability of soil permeability in subsurface aquifers, heterogeneity of materials with microstructure, wall roughness in a fluid dynamics study, etc., it is natural to consider optimal control problems governed by random PDEs. Based on the works about the numerical methods for PDEs and random PDEs [Babuska and Chatzipantelidis (2002); Xiu and Karniadakis (2002); Babuska *et al.* (2003); Babuska *et al.* (2004); Chen *et al.* (2011); Cohen *et al.* (2010); Deb *et al.* (2001); Nobile and Tempone (2009); Todor and Schwab (2007); Wiener (1938)], recently, there exist some works about optimal control problem governed by PDEs with random perturbation in its coefficients [Gunzburger *et al.* (2011); Shen *et al.* (2015)].

The work [Gunzburger *et al.* (2011)] dealt with the optimal control problems for stochastic partial differential equations with Neumann boundary conditions, the existence of an optimal solution and of a Lagrange multiplier were also demonstrated for the deterministic control. The optimal control problems governed by partial differential equations with uncertainties and with uncertain controls are addressed in Rosseel and Wells [2012], and a one-shot method is combined with stochastic finite element discretizations to get the optimal solutions. In Hou *et al.* [2011] and Lee and Lee [2013], stochastic optimal control problems constrained by stochastic elliptic PDEs with deterministic distributed control function are introduced. The authors prove the existence of the optimal solution, establish the validity of the Lagrange multiplier rule and obtain stochastic optimality system. Then, they use the Wiener–Itô (WI) chaos or the Karhunen–Loève (KL) expansion as a main tool to convert stochastic optimality system to deterministic optimality system. Finally, *a priori* error estimates for Galerkin approximation of the optimality system in both physical space and stochastic space are provided. In Sun *et al.* [2015], an optimal control problem with the deterministic control is of the obstacle constrained type governed by an elliptic PDE with random perturbation in its coefficients is introduced. The authors obtain the necessary and sufficient optimality conditions by applying the well-known Lions' Lemma and *a priori* error estimate for the state, the co-state and the control variables. A stochastic finite element approximation scheme and *a priori* error estimate for the state, the co-state and the control variables are

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developed for an optimal control problem governed by an elliptic integro-differential equation with random coefficients in Shen *et al.* [2015]. However, to our best knowledge, there has been a lack of *a prior* error estimates for stochastic finite element approximation of any optimal control problem governed by random parabolic PDE, which is immensely important and yet far more complicated to be analyzed than an random elliptic control problem.

In this paper, we establish a scheme to approximate the optimality system through the discretization with respect to both the spatial space and the probability space by Galerkin method and with respect to time by the backward Euler scheme. We give *a priori* error estimate for the state, the co-state and the control variables for an optimal control problem governed by a parabolic PDE with random perturbation in its coefficients. The plan of the paper is as follows: In Sec. 2, we introduce some function spaces and the stochastic optimal control problem. In Sec. 3, we represent the stochastic parabolic PDE in term of the (KL) expansion and obtain the finite-dimensional optimal control problem. We use the well-known Lions' Lemma to the reduced optimal problem and obtain the necessary and sufficient optimality conditions. After constructing finite element spaces and theirs approximation properties with respect to both the spatial space and the probability space, we use the backward Euler method to discretize time and get the fully discrete approximation scheme in Sec. 4. Section 5 considers *a priori* error estimates for the state, the co-state and the control variables. Numerical examples are presented to illustrate our theoretical results in Sec. 6.

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2. Notations and Model Control Problem

2.1. *Function spaces and notations*

Let D be a convex bounded polygonal spatial domain in $\mathbb{R}^d (1 \leq d \leq 3)$ with boundary ∂D and $B(D)$ be the Borel σ-algebra generated by the open subset of D. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, here Ω is a set of outcomes, $\mathcal{F}$ is a σ-algebra of events and $P : \mathcal{F} \to [0,1]$ is a probability measure. Let Y be an $\mathbb{R}^N$-valued random variable in $(\Omega, \mathcal{F}, P)$, and for $q \in [1, \infty)$, let $(L_P^q(\Omega))^N$ be the set comprising those random variables Y with $\sum_{i=1}^N \int_\Omega |Y_i(\omega)|^q dP(\omega) < \infty$. If $Y \in L_P^1(\Omega)$ we denote its expected value by

$$\mathbb{E}[Y] = \int_\Omega Y(\omega) dP(\omega) = \int_{\mathbb{R}^N} y d\mu_Y(y),$$

where μ_Y is the distribution measure for Y, defined for the Borel sets $\tilde{b} \in B(\mathbb{R}^N)$, by $\mu_Y(\tilde{b}) = P(Y^{-1}(\tilde{b}))$. If μ_Y is absolutely continuous with respect to the Lebesgue measure, then there exists a density function $\rho : \mathbb{R} \to [0, +\infty)$, such that

$$\mathbb{E}[Y] = \int_{\mathbb{R}^N} y\rho(y) dy.$$

Analogously, whenever $Y \in (L_P^2(\Omega))^N$, the positive semi-definite covariance matrix of Y, $\mathrm{Cov}[Y] \in \mathbb{R}^{N \times N}$, is defined by $\mathrm{Cov}[Y](i,j) = \mathrm{Cov}(Y_i, Y_j) = \mathbb{E}[(Y_i - \mathbb{E}(Y_i))(Y_j -$

$\mathbb{E}(Y_j))]$, for $i, j = 1, 2, \ldots, N$. Similarly, for a stochastic function $Y = Y(x, \omega)$ with $x \in \bar{D}$ and $\omega \in \Omega$, we denote its covariance function by $\mathrm{Cov}[Y](x, x') = \mathrm{Cov}(Y(x, \cdot), Y(x', \cdot))$ for $x, x' \in \bar{D}$.

Throughout this paper, we use standard notations for Sobolev spaces on D as in Adams [1975]. For examples, $L^2(D)$ and $H^1(D)$ are Hilbert spaces with norms $\|\cdot\|_{L^2(D)}$ and $\|\cdot\|_{H^1(D)}$, respectively; $H_0^1(D)$ is the subspace of $H^1(D)$ whose function value is zero on ∂D. With these standard Sobolev spaces, we adopt the definition of stochastic Sobolev spaces [Babuska *et al.* (2004); Hou *et al.* (2011); Lee and Lee (2013)]. For nonnegative integers s and $1 \leq p, q, r < +\infty$, the space $L^p(\Omega; W^{s,q}(D))$ contains all the stochastic functions $v: D \times \Omega \to \mathbb{R}$, that are measurable with respect to the product σ-algebra $B(D) \otimes \mathcal{F}$ and equipped with the averaged norms

$$\begin{aligned}
\|v\|_{L^p(\Omega; W^{s,q}(D))} &= (\mathbb{E}[\|v\|^p_{W^{s,q}(D)}])^{1/p} \\
&= \left(\mathbb{E}\left(\sum_{|\alpha| \leq s} \int_D |\partial^\alpha v|^q dx\right)^{p/q}\right)^{1/p},
\end{aligned}$$

and

$$\|v\|_{L^\infty(\Omega; W^{s,q}(D))} = \operatorname*{ess\,sup}_{\Omega} \|v\|_{W^{s,q}(D)}.$$

The space $L^r(0, T; L^p(\Omega; W^{s,q}(D)))$ contains all the stochastic functions $v: [0, T] \times D \times \Omega \to \mathbb{R}$, which are measurable with respect to the product σ-algebra $B([0, T]) \otimes B(D) \otimes \mathcal{F}$ and equipped with the norms

$$\begin{aligned}
\|v\|_{L^r(0,T; L^p(\Omega; W^{s,q}(D)))} &= \left(\int_0^T \|v\|^r_{L^p(\Omega; W^{s,q}(D))} dt\right)^{1/r} \\
&= \left(\int_0^T \left(\mathbb{E}\left[\sum_{|\alpha| \leq s} \int_D |\partial^\alpha v|^q dx\right]^{p/q}\right)^{r/p} dt\right)^{1/r}.
\end{aligned}$$

For a nonnegative integer s and $1 \leq p, q < +\infty$, let $L^p(0, T; W^{s,q}(D))$ contain all functions $v: [0, T] \times D \to \mathbb{R}$, which are measurable with respect to the σ-algebra $B([0, T]) \otimes B(D)$ and equipped with the norms

$$\begin{aligned}
\|v\|_{L^p(0,T; W^{s,q}(D))} &= \left(\int_0^T \|v\|^p_{W^{s,q}(D)} dt\right)^{1/p} \\
&= \left(\int_0^T \left(\sum_{|\alpha| \leq s} \int_D |\partial^\alpha v|^q dx\right)^{p/q} dt\right)^{1/p},
\end{aligned}$$

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and

$$\|v\|_{L^\infty(0,T;W^{s,q}(D))} = \operatorname*{ess\,sup}_{[0,T]} \|v\|_{W^{s,q}(D)}.$$

When $q=2$, we can similarly define other spaces $L^p_P(\Omega; H^s(D))$, $L^r(0,T; L^p_P(\Omega; H^s(D)))$ and $L^p(0,T; H^s(D))$.

2.2. *Stochastic optimal control problem*

We will consider the following control problem governed by random parabolic equations with constrained control:

$$\min_{u\in K} \mathcal{J}(u) = \min_{u\in K} \frac{1}{2}\int_0^T \mathbb{E}(\|y-y_d\|^2_{L^2(D)})dt + \frac{\alpha}{2}\int_0^T \|u\|^2_{L^2(D)}dt \tag{1}$$

subject to

$$\begin{cases} \partial_t y(t,x,\omega) - \nabla\cdot[a(x,\omega)\nabla y(t,x,\omega)] = u(t,x), & (t,x,\omega)\in[0,T]\times D\times\Omega, \\ y(t,x,\omega)=0, & (t,x,\omega)\in[0,T]\times\partial D\times\Omega, \\ y(0,x,\omega)=0, & (x,\omega)\in D\times\Omega. \end{cases} \tag{2}$$

The operator ∇ means derivatives with respect to the spatial variable $x\in D$ only. Where $\mathcal{J}$ is a cost functional, $y:[0,T]\times\bar{D}\times\Omega\to\mathbb{R}$ is the state variable, $y_d:[0,T]\times\bar{D}\to\mathbb{R}$ is a given target solution, $a: D\times\Omega\to\mathbb{R}$ is a random function with continuous and bounded covariance function, $u:[0,T]\times D\to\mathbb{R}$ is a deterministic control, a and u are assumed measurable with respect to the σ-algebras $(B(D)\otimes\mathcal{F})$ and $B([0,T])\otimes B(D)$, respectively. α is a positive constant measuring the importance between two terms in $\mathcal{J}$. The convex admissible set K is given by

$$K = \{u\in L^2(0,T;L^2(D)) : u(t,x)\geq 0, \forall\,(t,x)\in[0,T]\times D\}, \tag{3}$$

or

$$K = \left\{u\in L^2(0,T;L^2(D)) : \int_D u(t,x)dx \geq 0, \forall\, t\in[0,T]\right\}. \tag{4}$$

Although the objective functional $\mathcal{J}$ in (1) contains stochastic function y subject to (2), its outcome is deterministic by using the expectation $\mathbb{E}$. Besides, in order to guarantee the existence and uniqueness for the solution of (2), we assume that the diffusion coefficient a is bounded and uniformly coercive, i.e., there exist positive constants $a_{\min}$ and $a_{\max}$ such that

$$a_{\min}\leq a(x,\omega)\leq a_{\max}, \quad \text{a.e. } (x,\omega)\in D\times\Omega. \tag{5}$$

Then, with the two assumptions (3) and (5), the existence and uniqueness of a solution y for (2) can be proved [Nobile and Tempone (2009)]. Further, to ensure regularity of the solution y with respect to x we assume also that a is globally Lipschitz in $D\times\Omega$.

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In the following, we will take the state space $Y = L^2(0,T;L^2(\Omega;H_0^1(D)))$ and the control space $U = L^2(0,T;L^2(D))$, let $Z = L^2(0,T;L^2(\Omega;H_0^1(D))) \cap H^1(0,T;L^2(\Omega;L^2(D)))$. Let

$$A[y,v] = \mathbb{E}\int_D a\nabla y\cdot\nabla v dx, \quad \forall\, y,\ v\in Y, \tag{6}$$

$$[u,v] = \mathbb{E}\int_D uv dx, \quad \forall\, u\in U,\ v\in Y, \tag{7}$$

and

$$[\partial_t y,v] = \mathbb{E}\int_D \partial_t y v dx, \quad \forall\, y\in Z,\ v\in Y. \tag{8}$$

Then, a weak formulation for the state equation (2) reads: find $y\in Z$, such that

$$\begin{cases} [\partial_t y,v] + A[y,v] = [u,v], & \forall\, v\in Y,\ t\in(0,T],\\ y(0,x,\omega) = 0, & \forall\,(x,\omega)\in D\times\Omega. \end{cases} \tag{9}$$

Therefore, the optimal control problem (1)–(2) can be restated as:

$$\min_{u\in K}\mathcal{J}(u) = \min_{u\in K}\frac{1}{2}\int_0^T \mathbb{E}(\|y-y_d\|^2_{L^2(D)})dt + \frac{\alpha}{2}\int_0^T \|u\|^2_{L^2(D)}dt \tag{10}$$

subject to

$$\begin{cases} [\partial_t y,v] + A[y,v] = [u,v], & \forall\, v\in Y,\ t\in(0,T],\\ y(0,x,\omega) = 0, & \forall\,(x,\omega)\in D\times\Omega. \end{cases} \tag{11}$$

By the theory of optimal control problem [Lions (1971)], the existence of an optimal solution for (10)–(11) can be proved.

2.3. *Stochastic optimality system*

Let

$$\mathcal{J}'(u)(w) = \lim_{s\to 0^+}\frac{\mathcal{J}(u+sw)-\mathcal{J}(u)}{s} \tag{12}$$

denote the directional derivative of functional $\mathcal{J}$ at $u\in K$ along the direction $w\in K$. According to the Lions' theorem Lions (1971), there exists a unique minimizer $u\in K$, which satisfies the following variational inequality

$$\mathcal{J}'(u)(w-u)\geq 0, \quad \forall\, w\in K. \tag{13}$$

Theorem 2.1. *It follows from Lions* [1971] *and Fursikov* [2000] *that the optimal control problem* (10) *and* (11) *has a unique solution* $(y,u)\in Z\times K$. *Furthermore, a pair* (y,u) *is the solution of* (10)–(11) *if and only if there is a co-state variable*

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$p \in Z$, such that the triplet (y, p, u) satisfies the following optimality system:

$$\begin{cases} [\partial_t y, v] + A[y, v] = [u, v], \quad \forall\, v \in Y, \;\; t \in (0, T], \\ -[\partial_t p, q] + A[q, p] = [y - y_d, q], \quad \forall\, q \in Y, \;\; t \in (0, T], \\ \displaystyle\int_0^T [p + \alpha u, w - u] dt \geq 0, \quad \forall\, w \in K, \\ y|_{t=0} = 0; \;\; p|_{t=T} = 0. \end{cases} \tag{14}$$

Proof. Let $\mathcal{J}(u) = g(y(u)) + j(u)$, where $y(u)$ is the solution of (2) and

$$g(y(u)) = \frac{1}{2} \int_0^T \mathbb{E}(\|y - y_d\|^2_{L^2(D)}) dt, \quad j(u) = \frac{\alpha}{2} \int_0^T \|u\|^2_{L^2(D)} dt.$$

Then, the optimal condition (13) is

$$j'(u)(w - u) + g(y(u))'(w - u) \geq 0, \quad \forall\, w \in K.$$

We have

$$\begin{aligned} j'(u)(w - u) &= \lim_{s \to 0^+} \frac{1}{s}(j(u + s(w - u)) - j(u)) \\ &= \lim_{s \to 0^+} \frac{1}{s} \left(\frac{\alpha}{2} \int_0^T \int_D (|u + s(w - u)|^2 - |u|^2) dx dt \right) \\ &= \int_0^T [\alpha u, w - u] dt, \end{aligned} \tag{15}$$

$$\begin{aligned} g(y(u))'(w - u) &= \lim_{s \to 0^+} \frac{1}{s}(g(y(u + s(w - u))) - g(y(u))) \\ &= \lim_{s \to 0^+} \frac{1}{s} \mathbb{E} \left(\frac{1}{2} \int_0^T \int_D (|y(u + s(w - u)) - y_d|^2 - |y(u) - y_d|^2) dx dt \right) \\ &= \int_0^T [y'(u)(w - u), y - y_d] dt. \end{aligned} \tag{16}$$

Now, we compute $y'(u)(w - u)$. From the state equation (11), we have

$$\frac{1}{s} \left(\int_0^T [(\partial_t y(u + sw) - \partial_t y(u)), v] dt + \int_0^T A[y(u + sw) - y(u), v] dt \right) = \int_0^T [w, v] dt.$$

Letting $s \to 0$, we have

$$\int_0^T [(y'(u)(w))_t, v] + \int_0^T A[(y'(u)(w)), v] = \int_0^T [w, v] dt. \tag{17}$$

Define the co-state p satisfying

$$\begin{cases} -[\partial_t p, q] + A[q, p] = [y - y_d, q], \quad \forall\, q \in Y, \;\; t \in (0, T], \\ p\; |_{t=T} = 0. \end{cases} \tag{18}$$

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Letting $v = p$ in Eq. (17), we have

$$\begin{aligned}\int_0^T [w-u,p]dt &= \int_0^T [-y'(u)(w-u), \partial_t p]dt + \int_0^T A[y'(u)(w-u),p]dt \\ &= \int_0^T [y-y_d, y'(u)(w-u)]dt,\end{aligned} \tag{19}$$

then,

$$g(y(u))'(w-u) = \int_0^T [y-y_d, y'(u)(w-u)]dt = \int_0^T [w-u,p]dt. \tag{20}$$

Therefore, the optimality condition is

$$\mathcal{J}'(u)(w-u) = \int_0^T [p+\alpha u, w-u]dt \geq 0, \quad \forall\, w \in K. \tag{21}$$

It is known that the inequality (21) is just the necessary and sufficient optimality condition. □

3. Finite-Dimensional Representation of Model Control Problem

3.1. *KL expansion of stochastic fields*

Consider a stochastic function $a(x,\omega)$ with continuous covariance function $\mathrm{Cov}[a]$: $D \times D \to \mathbb{R}$. Let $\{(\lambda_n, b_n)\}_{n=1}^{\infty}$ denote the sequence of eigenpairs associated with the compact self-adjoint operator that maps $g \in L^2(D) \mapsto \int_D \mathrm{Cov}[a](x,\cdot)g(x)dx \in C^0(\bar{D})$. Its nonnegative eigenvalues satisfy $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$ and $\sum_{n=1}^{+\infty} \lambda_n = \int_D \mathrm{Var}[a](x)dx$. The corresponding eigenfunctions are orthonormal, i.e., $\int_D b_i(x)b_j(x)dx = \delta_{ij}$. The truncated KL expansion [Ghanem and Spanos (1991)] of the random function a is

$$a_N(x,\omega) = \mathbb{E}[a](x) + \sum_{n=1}^{N} \sqrt{\lambda_n} b_n(x)\xi_n(\omega),$$

where the real random variables, $\{\xi_n\}_{n=1}^{\infty}$, are mutually uncorrelated, have mean zero and unit variance, and are uniquely determined by $\xi_n(\omega) = \frac{1}{\sqrt{\lambda_n}} \int_D (a(x,\omega) - \mathbb{E}[a](x))b_n(x)dx$. Then, by Mercer's theorem, we have

$$\sup_{x\in D}\mathbb{E}[(a-a_N)^2](x) = \sup_{x\in D}(\mathrm{Var}[a] - \mathrm{Var}[a_N])(x) \to 0, \quad \text{as } N \to \infty.$$

Assumption 3.1 (finite-dimensional noise). In what follows, we assume that the random functions $a(x,\omega)$ depend only on an N-dimensional random vector ξ, such as, the case when we use a joint N term KL expansion to approximate the given coefficients $a(x,\omega) = a(x,\xi(\omega))$, where $\xi = \xi(\omega) = (\xi_1(\omega),\ldots,\xi_N(\omega))$ with independent components $\xi_i(\omega), i = 1,\ldots,N \in \mathbb{N}$. Let $\Gamma_i = \xi_i(\Omega) \subset \mathbb{R}$ be a bounded interval for $i = 1,\ldots,N$ and $\rho_i : \Gamma_i \to [0,1]$ be the probability density functions of the random variables $\xi_i(\omega), \omega \in \Omega$. Then we can use the joint probability density function $\rho(\xi) = \prod_{i=1}^N \rho_i(\xi_i)$ for random vector ξ with the support $\Gamma = \prod_{i=1}^N \Gamma_i \subset \mathbb{R}^N$. On Γ, we have the probability measure $\rho(\xi)d\xi$.

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After making Assumption 3.1, by Doob–Dynkin's lemma, we know that y, the solution corresponding to the random PDE (2), can be described by just a finite number of random variables, i.e., $y(t,x,\omega) = y(t,x,\xi_1(\omega),\ldots,\xi_N(\omega))$. The number N has to be large enough so that the approximation error is sufficiently small. Then, we can replace the probability space $(\Omega,\mathcal{F},P)$ with $(\Gamma, B(\Gamma), \rho(\xi)d\xi)$ involving only the image set $\Gamma \subset \mathbb{R}^N$. We can also define the space $L^r(0,T;L^p_\rho(\Gamma;W^{s,q}(D)))$, which contains all the functions $v : [0,T]\times D\times\Gamma \to \mathbb{R}$, that are measurable with respect to the product σ-algebra $B([0,T])\otimes B(D)\otimes B(\Gamma)$ and equipped with the norms

$$\begin{aligned}\|v\|_{L^r(0,T;L^p_\rho(\Gamma;W^{s,q}(D)))} &= \left(\int_0^T \left(\int_\Gamma \|v\|^p_{W^{s,q}(D)}\rho(\xi)d\xi\right)^{r/p} dt\right)^{1/r}\\ &= \left(\int_0^T \left(\int_\Gamma \left(\sum_{|\alpha|\le s}\int_D |\partial^\alpha v|^q dx\right)^{p/q} \rho(\xi)d\xi\right)^{r/p} dt\right)^{1/r}.\end{aligned}$$

Similarly, we can define the space $L^p_\rho(\Gamma;W^{s,q}(D))$, containing all functions $v : D\times\Gamma\to\mathbb{R}$, that are measurable with respect to the product σ-algebra $B(D)\otimes B(\Gamma)$ and equipped with the norms

$$\begin{aligned}&\|v\|_{L^p_\rho(\Gamma;W^{s,q}(D))}\\ &\quad = \left(\int_\Gamma \|v\|^p_{W^{s,q}(D)}\rho(\xi)d\xi\right)^{1/p} = \left(\int_\Gamma \left(\sum_{|\alpha|\le s}\int_D |\partial^\alpha v|^q dx\right)^{p/q}\rho(\xi)d\xi\right)^{1/p}.\end{aligned}$$

3.2. *Finite-dimensional representation of control problem*

With the above assumption, we can reformulate the stochastic optimal control problem (1)–(2) as a deterministic PDE-constrained optimization problem as follows:

$$\min_{u\in K}\mathcal{J}(u) = \min_{u\in K}\frac{1}{2}\int_0^T\int_\Gamma (\|y-y_d\|^2_{L^2(D)})\rho(\xi)d\xi dt + \frac{\alpha}{2}\int_0^T \|u\|^2_{L^2(D)}dt \tag{22}$$

subject to

$$\begin{cases}\partial_t y(t,x,\xi) - \nabla\cdot[a(x,\xi)\nabla y(t,x,\xi)] = u(t,x), & (t,x,\xi)\in[0,T]\times D\times\Gamma,\\ y(t,x,\xi) = 0, & (t,x,\xi)\in[0,T]\times\partial D\times\Gamma,\\ y(0,x,\xi) = 0, & (x,\xi)\in D\times\Gamma.\end{cases} \tag{23}$$

Here, it is natural to assume the aforementioned assumption (5) changed to be

$$a_{\min} \le a(x,\xi) \le a_{\max}, \quad \text{a.e. } D\times\Gamma, \tag{24}$$

and to ask the convergence of the truncated deterministic problem (23) to the original stochastic problem (2).

We will take the deterministic state space $Y_\rho = L^2(0,T;L^2_\rho(\Gamma;H^1_0(D)))$ and $Z_\rho = L^2(0,T;L^2_\rho(\Gamma;H^1_0(D)))\cap H^1(0,T;L^2_\rho(\Gamma;L^2(D)))$. Corresponding to Eqs. (2.5)–(2.7), we have:

$$A[y,v]_\rho = \int_\Gamma\int_D a\nabla y\cdot\nabla v dx\rho(\xi)d\xi, \quad \forall\, y,\ \ v\in Y_\rho, \tag{25}$$

$$[u,v]_\rho = \int_\Gamma\int_D uv dx\rho(\xi)d\xi, \quad \forall\, u\in U,\ \ v\in Y_\rho \tag{26}$$

and

$$[\partial_t y,v]_\rho = \int_\Gamma\int_D \partial_t y v dx\rho(\xi)d\xi, \quad \forall\, y\in Z_\rho,\ \ v\in Y_\rho. \tag{27}$$

Then, we can also reformulate the optimal control problem (22)–(23) by:

$$\min_{u\in K}\mathcal{J}(u) = \min_{u\in K}\frac{1}{2}\int_0^T\int_\Gamma(\|y-y_d\|^2_{L^2(D)})\rho(\xi)d\xi dt + \frac{\alpha}{2}\int_0^T\|u\|^2_{L^2(D)}dt \tag{28}$$

subject to

$$\begin{cases}[\partial_t y,v]_\rho + A[y,v]_\rho = [u,v]_\rho, & \forall\, v\in Y_\rho,\ \ t\in(0,T],\\ y(0,x,\xi) = 0, & \forall\,(x,\xi)\in D\times\Gamma.\end{cases} \tag{29}$$

With assumption (24), the existence of solutions to (28)–(29) can be proved [Lions (1971)].

Similarly to derive of (14), the optimal control problem (28)–(29) has a unique solution $(y,u)\in Z_\rho\times K$. Furthermore, a pair (y,u) is the solution of (28)–(29) if and only if there is a co-state variable $p\in Z_\rho$, such that the triplet (y,p,u) satisfies the following optimality system:

$$\begin{cases}[\partial_t y,v]_\rho + A[y,v]_\rho = [u,v]_\rho, \quad \forall\, v\in Z_\rho,\ \ t\in(0,T],\\ -[\partial_t p,q]_\rho + A[q,p]_\rho = [y-y_d,q]_\rho, \quad \forall\, q\in Z_\rho,\ \ t\in(0,T],\\ \displaystyle\int_0^T[p+\alpha u,w-u]_\rho dt\geq 0, \quad \forall\, w\in K,\\ y|_{t=0}=0; \quad p|_{t=T}=0.\end{cases} \tag{30}$$

It is known that the inequality in (30) is just the necessary and sufficient optimality condition.

The explicit solution of the variational inequality in (30) depends heavily on the choice of the joint probability density ρ. In the simple case, if the joint probability density ρ is uniform on Γ, we have the following explicit solution

$$u = \max\left\{0,-\frac{1}{\alpha}\mathbb{E}(p)\right\} \quad \text{or} \quad u = -\frac{1}{\alpha}\mathbb{E}(p) + \max\left\{0,\frac{1}{\alpha}\frac{\int_{[0,T]\times D\times\Gamma}p}{\int_{[0,T]\times D\times\Gamma}1}\right\}$$

for the case (3) or (4), respectively.

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4. Stochastic Galerkin Method

4.1. *Finite element spaces on D and* Γ

First of all, we consider finite element spaces defined on spatial domain $D \subset \mathbb{R}^d$. Let $\{\mathcal{T}_h\}_{h>0}$ be a family of regular triangulation of D such that $\bar{D} = \bigcup_{\tau \in \mathcal{T}_h} \bar{\tau}$. Let $h_s = \max_{\tau \in \mathcal{T}_h} h_\tau$, where h_τ denotes the diameter of the element τ. Consider two finite element spaces $V_{h_s} \subset H_0^1(D)$ and $W_{h_s} \subset L^2(D)$, consisting of piecewise linear continuous functions on $\{\mathcal{T}_h\}$ and piecewise constant functions on $\{\mathcal{T}_h\}$, respectively. We assume that V_{h_s} and W_{h_s} satisfy the following approximation properties [Ciarlet (2002)]:

(i) for all $\phi \in H^2(D) \cap H_0^1(D)$, there exists

$$\inf_{\phi_{h_s} \in V_{h_s}} \|\phi - \phi_{h_s}\|_{H_0^1(D)} \leq C h_s \|\phi\|_{H^2(D)}, \tag{31}$$

(ii) for all $\phi \in H_0^1(D)$, there exists

$$\inf_{\phi_{h_s} \in W_{h_s}} \|\phi - \phi_{h_s}\|_{L^2(D)} \leq C h_s \|\phi\|_{H_0^1(D)}, \tag{32}$$

where $C > 0$ is a constant independent of ϕ and h_s.

Next, we consider a finite-dimensional space defined on $\Gamma \subset \mathbb{R}^N$ [Babuska *et al.* (2004)]. Let Γ be partitioned into a finite number of disjoint boxes $B_i^N \subset \mathbb{R}^N$, that is, for a finite index set I, we have

$$\Gamma = \bigcup_{i \in I} B_i^N = \bigcup_{i \in I} \prod_{j=1}^{N} (a_i^j,\, b_i^j),$$

where $B_k^N \cap B_l^N = \varnothing$ for $k \neq l \in I$ and $(a_i^j, b_i^j) \subset \Gamma_j$. A maximum grid size parameter $0 < h_r < 1$ is denoted by

$$h_r = \max\{|b_i^j - a_i^j|/2 : 1 \leq j \leq N \text{ and } i \in I\}.$$

Let $S_{h_r} \subset L^2(\Gamma)$ be the finite element space of piecewise polynomials with degree at most p_j on each direction ξ_j, thus if $\psi_{h_r} \in S_{h_r}$, then $\psi_{h_r}|_{B_i^N} \in \text{span}\{\Pi_{j=1}^N \xi_j^{n_j} : n_j \in \mathbb{N} \text{ and } n_j \leq p_j\}$. Letting the multi-index $P = (p_1, \ldots, p_N)$, we have (see [Ciarlet (2002)]) the following property: for all $\psi \in C^{p+1}(\Gamma)$,

$$\inf_{\psi_{h_r} \in S_{h_r}} \|\psi - \psi_{h_r}\|_{L^2(\Gamma)} \leq h_r^\gamma \sum_{j=1}^{N} \frac{\|\partial_{\xi_j}^{p_j+1} \psi\|_{L^2(\Gamma)}}{(p_j+1)!}, \tag{33}$$

where $\gamma = \min_{1 \leq j \leq N}\{p_j + 1\}$.

4.2. *Tensor product finite element spaces on* $D \times \Gamma$

Combining spaces V_{h_s}, W_{h_s} and S_{h_r} together, we now define tensor product finite element space on $D \times \Gamma$. Let $\overline{Y}_h = V_{h_s} \times S_{h_r}$.

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We define the $H_0^1(D)$-projection operator $R_{h_s}: H_0^1(D) \to V_{h_s}$ by

$$(R_{h_s}\phi, \phi_{h_s})_{H_0^1(D)} = (\phi, \phi_{h_s})_{H_0^1(D)}, \quad \forall\, \phi_{h_s} \in V_{h_s},\ \forall\, \phi \in H_0^1(D), \tag{34}$$

the $L^2(D)$-projection operator $\Pi_{h_s}: L^2(D) \to W_{h_s}$ by

$$(\Pi_{h_s}\phi, \phi_{h_s})_{L^2(D)} = (\phi, \phi_{h_s})_{L^2(D)}, \quad \forall\, \phi_{h_s} \in W_{h_s},\ \forall\, \phi \in L^2(D). \tag{35}$$

Similarly, let the $L^2(\Gamma)$-projection operator $\Pi_{h_r}: L^2(\Gamma) \to S_{h_r}$ by

$$(\Pi_{h_r}\psi, \psi_{h_r})_{L^2(\Gamma)} = (\psi, \psi_{h_r})_{L^2(\Gamma)}, \quad \forall\, \psi_{h_r} \in S_{h_r},\ \forall\, \psi \in L^2(\Gamma). \tag{36}$$

It follows from (31) that for all $\phi \in H^2(D) \cap H_0^1(D)$

$$\|\phi - R_{h_s}\phi\|_{H_0^1(D)} \le C h_s \|\phi\|_{H^2(D)}, \tag{37}$$

and from (32) that for all $\phi \in H^1(D)$

$$\|\phi - \Pi_{h_s}\phi\|_{L^2(D)} \le C h_s \|\phi\|_{H^1(D)}. \tag{38}$$

Similarly, by (33) we obtain that for all $\psi \in C^{p+1}(\Gamma)$

$$\|\psi - \Pi_{h_r}\psi\|_{L^2(\Gamma)} \le h_r^{\gamma} \sum_{j=1}^{N} \frac{\|\partial_{\xi_j}^{p_j+1}\psi\|_{L^2(\Gamma)}}{(p_j+1)!}. \tag{39}$$

Using the inequalities (37) and (39), we have the following approximation property [Babuska *et al.* (2004)]: for all $\overline{y} \in C^{P+1}(\Gamma; H^2(D) \cap H_0^1(D))$

$$\inf_{\overline{y}_h \in \overline{Y}_h} \|\overline{y} - \overline{y}_h\|_{L^2(\Gamma; H_0^1(D))} \le C \left\{ h_s \|\overline{y}\|_{L^2(\Gamma; H^2(D))} + h_r^{\gamma} \sum_{j=1}^{N} \frac{\|\partial_{\xi_j}^{p_j+1}\overline{y}\|_{L^2(\Gamma; H_0^1(D))}}{(p_j+1)!} \right\}, \tag{40}$$

where positive constant C is independent of h_s, h_r, N and P.

In order to obtain the separate error estimates in D and Γ, we define a projection operator P_h which maps onto the tensor product space $W_{h_s} \times S_{h_r}$. It is defined as follows

$$P_h\varphi = \Pi_{h_s}\Pi_{h_r}\varphi = \Pi_{h_r}\Pi_{h_s}\varphi, \quad \forall\, \varphi \in L^2(\Gamma; L^2(D)). \tag{41}$$

Furthermore, we use the following decomposition

$$\varphi - P_h\varphi = (\varphi - \Pi_{h_s}\varphi) + \Pi_{h_s}(I - \Pi_{h_r})\varphi, \quad \forall\, \varphi \in L^2(\Gamma; L^2(D)). \tag{42}$$

To derive the error estimates, we need assumption and lemmas on the regularity as follows:

Assumption 4.1. Let y, p, u satisfy the following regularity condition

$$y, p \in L^2(0, T; C^{P+1}(\Gamma; H^2(D) \cap H_0^1(D))) \cap H^1(0, T; C^{P+1}(\Gamma; L^2(D)))$$

and

$$u \in L^2(0, T; H^1(D)).$$

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4.3. *Galerkin approximation scheme*

We will use $Y_h = L^2(0,T;V_{h_s}\times S_{h_r})\cap H^1(0,T;W_{h_s}\times S_{h_r})$ for the state variable y and co-state variable p, $U_h = L^2(0,T;W_{h_s})$ for the control variable u and let $K_h = L^2(0,T;W_{h_s}\cap K)$ be the finite element space of the admissible set. Then, the semi-discrete finite element approximation scheme for optimal control problem (28)–(29) is:

$$\min_{u_h\in K_h}\widetilde{\mathcal{J}}_h(u_h) = \min_{u_h\in K_h}\left(\frac{1}{2}\int_0^T\int_\Gamma(\|y_h-y_d\|^2_{L^2(D)})\rho(\xi)d\xi dt+\frac{\alpha}{2}\int_0^T\|u_h\|^2_{L^2(D)}dt\right) \tag{43}$$

subject to

$$\begin{cases}[\partial_t y_h, v_h]_\rho + A[y_h,v_h]_\rho = [u_h,v_h]_\rho, \quad \forall\, v_h\in Y_h,\ \ t\in(0,T],\\ y_h(x,0,\xi)=0.\end{cases} \tag{44}$$

Similarly, it is known Lions (1971) that the control problem (43)–(44) has a unique pair solution $(y_h,u_h)\in Y_h\times K_h$, if and only if there is a co-state variable $p_h\in Y_h$, such that $(y_h,\, p_h,\, u_h)\in Y_h\times Y_h\times K_h$ satisfies the following system

$$\begin{cases}[\partial_t y_h, v_h]_\rho + A[y_h,v_h]_\rho = [u_h,v_h]_\rho, \quad \forall\, v_h\in Y_h,\ \ t\in(0,T],\\ -[\partial_t p_h, q_h]_\rho + A[q_h,p_h]_\rho = [y_h-y_d,q_h]_\rho, \quad \forall\, q_h\in Y_h,\ \ t\in(0,T],\\ \int_0^T[p_h+\alpha u_h, w_h-u_h]_\rho dt\ge 0, \quad \forall\, w_h\in K_h\subset U_h.\end{cases} \tag{45}$$

We have the following formulations for the discrete directional derivative of functional $\mathcal{J}$:

$$\widetilde{\mathcal{J}}'_h(u_h)(w_h) = \int_0^T([p_h,w_h]_\rho+\alpha[u_h,w_h]_\rho)dt, \quad \forall\, w_h\in K_h\subset U_h, \tag{46}$$

$$\mathcal{J}'_h(u_h)(w_h-u_h)\ge 0, \quad \forall\, w_h\in K_h\subset U_h. \tag{47}$$

Let $0=t_0<t_1<t_2<\cdots<t_M=T$ be a partition of interval $[0,T], I_k=(t_{k-1},t_k), \Delta t_k = t_k-t_{k-1}$ is the step, let $\Delta t = \max_{1\le k\le M}\Delta t_k$. We consider a particular case of the space S_{h_r} with no partition of Γ, i.e., only the polynomial degree is increased. Here, we use the tensor finite element space $S_{h_r}=\bigotimes_{n=1}^N Z_n^{p_n}$, where the one-dimensional global polynomial subspaces $Z_n^{p_n}=\{v:\Gamma_n\to\mathbb{R}: v\in \mathrm{span}(1,y_n,\ldots,y_n^{p_n})\}, n=1,\ldots,N$. Let $\{\varphi_i(x)\}$ be the basis of the space V_{h_s} and $\{\psi_j(\xi)\}$ be the basis of the space S_{h_r}. Let $Y_h^k = Y_h|_{t=t_k}$ and $K_h^k = K_h|_{t=t_k}$. Then the full discretization of the control problem (43)–(44) is to find $(y_h^k,u_h^k)\in Y_h^k\times K_h^k$, such that

$$\min_{u_h^k\in K_h^k}\mathcal{J}_h(u_h) = \frac{1}{2}\sum_{k=1}^M\Delta t_k(\|y_h^k-y_d\|^2_{L^2_\rho(\Gamma;L^2(D))}+\alpha\|u_h^k\|^2_{L^2(D)}) \tag{48}$$

satisfy

$$\begin{cases} \left[\dfrac{y_h^k - y_h^{k-1}}{\Delta t_k}, v_h\right]_\rho + A[y_h^k, v_h]_\rho \\ \qquad = [u_h^k, v_h]_\rho, \quad \forall\, v_h \in Y_h^k, \;\; k = 1, 2, \ldots, M, \\ y_h^0 = 0, \end{cases} \tag{49}$$

where

$$\begin{cases} y_h^k = \sum_{i,j} y_{ijk} \varphi_i(x) \psi_j(\xi), \\ p_h^k = \sum_{i,j} p_{ijk} \varphi_i(x) \psi_j(\xi), \\ u_h^k = \sum_i u_{ik} \varphi_i(x). \end{cases} \tag{50}$$

If $(y_h^k, u_h^k) \in Y_h^k \times K_h^k$ is the unique solution of the optimal control system (48)–(49), then if and only if there is a co-state variable $p_h^{k-1} \in Y_h^{k-1}$, such that $(y_h^k, p_h^{k-1}, u_h^k) \in Y_h^k \times Y_h^{k-1} \times K_h^k$ satisfies the following system:

$$\begin{cases} \left[\dfrac{y_h^k - y_h^{k-1}}{\Delta t_k}, v_h\right]_\rho + A[y_h^k, v_h]_\rho = [u_h^k, v_h]_\rho, \quad \forall\, v_h \in Y_h^k, \;\; k = 1, 2, \ldots, M, \\ y_h^0 = 0, \\ \left[\dfrac{p_h^{k-1} - p_h^k}{\Delta t_k}, q_h\right]_\rho + A[q_h, p_h^{k-1}]_\rho \\ \qquad = [y_h^k - y_d, q_h]_\rho, \quad \forall\, q_h \in Y_h^k, \;\; k = M, M-1, \ldots, 1, \\ p_h^M = 0, \\ [p_h^{k-1} + \alpha u_h^k, w_h - u_h^k]_\rho \geq 0, \quad \forall\, w_h \in K_h^k, \;\; k = 1, 2, \ldots, M. \end{cases} \tag{51}$$

5. *A Priori* Error Estimate

In order to derive *a priori* error estimate, similarly to the continuous case, we need an auxiliary problem:

$$\begin{aligned} \mathcal{J}_h'(u)(w - u) = \sum_{k=1}^{M} \Delta t_k ([p_h^{k-1}(u), w^k - u^k]_\rho \\ + \alpha [u^k, w^k - u^k]_\rho) \geq 0, \quad \forall\, w \in K \subset U, \end{aligned} \tag{52}$$

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where $(y_h^k(u), p_h^k(u)) \in Y_h^k \times Y_h^k$ is the solution of the system:

$$\begin{cases} \left[\dfrac{y_h^k(u) - y_h^{k-1}(u)}{\Delta t_k}, v_h\right]_\rho + A[y_h^k(u), v_h]_\rho \\ \qquad = [u^k, v_h]_\rho, \quad \forall\, v_h \in Y_h^k, \ \ k = 1, 2, \ldots, M, \\ y_h^0(u) = 0, \\ \left[\dfrac{p_h^{k-1}(u) - p_h^k(u)}{\Delta t_k}, q_h\right]_\rho + A[q_h, p_h^{k-1}(u)]_\rho \\ \qquad = [y_h^k(u) - y_d, q_h]_\rho, \quad \forall\, q_h \in Y_h^k, \ \ k = M, M-1, \ldots, 1, \\ p_h^M(u) = 0. \end{cases} \tag{53}$$

For $y \in Y_h$, we can define the norm about discretization time t,

$$|||y|||_{L^p([0,T];L^2_\rho(\Gamma;H^1_0(D)))} = \left(\sum_{k=1}^{M} \Delta t_k \|y(t_k, x, \xi)\|^p_{L^2_\rho(\Gamma;H^1_0(D))}\right)^{\frac{1}{p}},$$

for $1 \le p \le \infty$, if $p = \infty$, let

$$|||y|||_{L^\infty([0,T];L^2_\rho(\Gamma;H^1_0(D)))} = \max\nolimits_{1\le k\le M} \|y(t_k, x, \xi)\|_{L^2_\rho(\Gamma;H^1_0(D))},$$

similarly, we can define $|||u|||_{L^p([0,T];L^2(D))}$ for $u \in L^p([0,T]; L^2(D))$.

Lemma 5.1. *Under the definition of* (52), *we have the following estimate*:

$$\mathcal{J}_h'(w)(w-u) - \mathcal{J}_h'(u)(w-u) \ge \alpha |||w-u|||^2_{L^2(0,T;L^2(D))}\,. \tag{54}$$

Proof. From (52), we have

$$\begin{aligned} &\mathcal{J}_h'(w)(w-u) - \mathcal{J}_h'(u)(w-u) \\ &\quad = \sum_{k=1}^{M} \Delta t_k([p_h^{k-1}(w) - p_h^{k-1}(u), w^k - u^k]_\rho + \alpha[w^k - u^k, w^k - u^k]). \end{aligned} \tag{55}$$

Noting that (53), we have

$$\begin{aligned} &\sum_{k=1}^{M} \Delta t_k [p_h^{k-1}(w) - p_h^{k-1}(u), w^k - u^k]_\rho \\ &\quad = \sum_{k=1}^{M} (\Delta t_k A[y_h^k(w) - y_h^k(u), p_h^{k-1}(w) - p_h^{k-1}(u)]_\rho \\ &\qquad + [(y_h^k(w) - y_h^k(u)) - (y_h^{k-1}(w) - y_h^{k-1}(u)), p_h^{k-1}(w) - p_h^{k-1}(u)]_\rho) \end{aligned}$$

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$$= \sum_{k=1}^{M} (\Delta t_k A[y_h^k(w) - y_h^k(u), y_h^k(w) - y_h^k(u)]_\rho$$
$$+ [p_h^k(w) - p_h^k(u), y_h^k(w) - y_h^k(u)]_\rho$$
$$- [p_h^{k-1}(w) - p_h^{k-1}(u), y_h^{k-1}(w) - y_h^{k-1}(u)]_\rho) \geq 0. \tag{56}$$

□

Lemma 5.2. *Let* (y_h^k, p_h^k) *and* $(y_h^k(u), p_h^k(u))$ *be the resolution of* (51) *and* (53), *respectively, then*

$$|||y_h - y_h(u)|||_{L^\infty([0,T];L^2(\Gamma;H_0^1(D)))} \leq C|||u - u_h|||_{L^2([0,T];L^2(D))}, \tag{57}$$

$$|||p_h - p_h(u)|||_{L^\infty([0,T];L^2(\Gamma;H_0^1(D)))} \leq C|||u - u_h|||_{L^2([0,T];L^2(D))}. \tag{58}$$

Proof. Let $\gamma^k = y_h^k - y_h^k(u)$ and $\delta^k = p_h^k - p_h^k(u)$. By (51) and (53), we can get

$$\left[\frac{1}{\Delta t_k}(\gamma^k - \gamma^{k-1}), v_h\right]_\rho + A[\gamma^k, v_h]_\rho = [u_h^k - u^k, v_h]_\rho, \tag{59}$$

$$\left[\frac{1}{\Delta t_k}(\delta^{k-1} - \delta^k), q_h\right]_\rho + A[q_h, \delta^{k-1}]_\rho = [y_h^k - y_h^k(u), q_h]_\rho. \tag{60}$$

In Eq. (59), denoting $d_t\gamma^k = \frac{1}{\Delta t_k}(\gamma^k - \gamma^{k-1})$ and letting $v_h = d_t\gamma^k$, we have

$$\Delta t_k[d_t\gamma^k, d_t\gamma^k]_\rho + A[\gamma^k, \gamma^k]_\rho = A[\gamma^k, \gamma^{k-1}]_\rho + \Delta t_k[u_h^k - u^k, d_t\gamma^k]_\rho. \tag{61}$$

Then, by

$$\Delta t_k[d_t\gamma^k, d_t\gamma^k]_\rho + A[\gamma^k, \gamma^k]_\rho \leq \frac{1}{2}(A[\gamma^k, \gamma^k]_\rho + A[\gamma^{k-1}, \gamma^{k-1}]_\rho$$
$$+ \Delta t_k[u_h^k - u^k, u_h^k - u^k]_\rho + \Delta t_k[d_t\gamma^k, d_t\gamma^k]_\rho), \tag{62}$$

we can get

$$A[\gamma^k, \gamma^k]_\rho \leq A[\gamma^{k-1}, \gamma^{k-1}]_\rho + \Delta t_k[u_h^k - u^k, u_h^k - u^k]_\rho. \tag{63}$$

Then, for $1 \leq k \leq M$, we have

$$A[\gamma^k, \gamma^k]_\rho \leq A[\gamma^0, \gamma^0]_\rho + \sum_{i=1}^{k} \Delta t_i \|u_h^i - u^i\|_{L^2(D)}^2 \leq |||u - u_h|||_{L^2([0,T];L^2(D))}^2. \tag{64}$$

Combining the last inequality with Poincare inequality, we can get (57).

In Eq. (60), letting $p_h = d_t\delta^k$, we can get the inequality (58) by the same proof of the inequality (57). □

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Lemma 5.3. *Let* (y, p, u) *be the solution of the optimal control problem* (30) *and* (y_h, p_h, u_h) *be the solution of the discretized problem* (51). *Let Assumption* 4.1 *be fulfilled. Then the following estimate holds*:

$$\begin{aligned}
&|||u - u_h|||_{L^2(0.T;L^2(D))} \\
&\quad \le C|||p - p_h(u)|||_{L^2(0,T;L^2_\rho(\Gamma;L^2(D)))} \\
&\qquad + Ch_s\{|||u|||_{L^2(0,T;L^2(H^1(D)))} + |||p|||_{L^2(0,T;L^2_\rho(\Gamma;H^1(D)))}\} \\
&\qquad + C\Delta t(|||\partial p/\partial t|||_{L^2(0,T;L^2_\rho(\Gamma;L^2(D)))}) + Ch_r^\gamma \sum_{j=1}^{N} \frac{|||\partial_{\xi_j}^{p_j+1} p|||_{L^2(L^2_\rho(\Gamma;L^2(D)))}}{(p_j+1)!},
\end{aligned} \tag{65}$$

where $\gamma = \min_{1\le j\le N}\{p_j + 1\}$.

Proof. From (47), (52) and (54), we know

$$\begin{aligned}
&C|||u - u_h|||^2_{L^2(0,T;L^2(D))} \\
&\quad \le \mathcal{J}_h'(u)(u - u_h) - \mathcal{J}_h'(u_h)(u - u_h) \\
&\quad = \sum_{k=1}^{M} \Delta t_k[\alpha u^k + p_h^{k-1}(u), u^k - u_h^k]_\rho - \sum_{k=1}^{M} \Delta t_k[\alpha u_h^k + p_h^{k-1}, u^k - u_h^k]_\rho \\
&\quad = \sum_{k=1}^{M} \Delta t_k[\alpha u^k + p^k, u^k - u_h^k]_\rho + \sum_{k=1}^{M} \Delta t_k[\alpha u_h^k + p_h^{k-1}, u_h^k - \Pi_{h_s} u^k]_\rho \\
&\qquad + \sum_{k=1}^{M} \Delta t_k[\alpha u_h^k + p_h^{k-1}, \Pi_{h_s} u^k - u^k]_\rho + \sum_{k=1}^{M} \Delta t_k[p_h^{k-1}(u) - p^k, u^k - u_h^k]_\rho \\
&\quad \le \sum_{k=1}^{M} \Delta t_k[\alpha u_h^k + p_h^{k-1}, \Pi_{h_s} u^k - u^k]_\rho + \sum_{k=1}^{M} \Delta t_k[p_h^{k-1}(u) - p^k, u^k - u_h^k]_\rho,
\end{aligned} \tag{66}$$

$$\begin{aligned}
\sum_{k=1}^{M} \Delta t_k[p_h^{k-1}, \Pi_{h_s} u^k - u^k]_\rho &= \sum_{k=1}^{M} \Delta t_k[p^{k-1}, \Pi_{h_s} u^k - u^k]_\rho \\
&\quad + \sum_{k=1}^{M} \Delta t_k[p^{k-1} - p_h^{k-1}(u), u^k - \Pi_{h_s} u^k]_\rho \\
&\quad + \sum_{k=1}^{M} \Delta t_k[p_h^{k-1}(u) - p_h^{k-1}, u^k - \Pi_{h_s} u^k]_\rho.
\end{aligned} \tag{67}$$

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Note that

$$\sum_{k=1}^{M} \Delta t_k[\alpha u_h^k, \Pi_{h_s} u^k - u^k]_\rho = 0 \tag{68}$$

and

$$\sum_{k=1}^{M} \Delta t_k[p^{k-1}, \Pi_{h_s} u^k - u^k]_\rho = \sum_{k=1}^{M} \Delta t_k[p^{k-1} - P_h(p^{k-1}), \Pi_{h_s} u^k - u^k]_\rho, \tag{69}$$

combing (58), (66)–(69) with triangle inequality and Cauchy–Schwartz inequality, we have

$$\begin{aligned} &C|||u - u_h|||^2_{L^2(0,T;L^2(D))} \\ &\quad\le C(\epsilon)\sum_{k=1}^{M} \Delta t_k\|u^k - \Pi_{h_s} u^k\|^2_{L^2(D)} + C(\epsilon)\sum_{k=1}^{M} \Delta t_k\|p^{k-1} - P_h p^{k-1}\|^2_{L^2(\Gamma;L^2(D))} \\ &\qquad + C(\epsilon)\sum_{k=1}^{M} \Delta t_k\|p^{k-1} - p_h^{k-1}(u)\|^2_{L^2(\Gamma;L^2(D))} \\ &\qquad + C(\epsilon)\sum_{k=1}^{M} \Delta t_k\|p^{k-1} - p^k\|^2_{L^2(\Gamma;L^2(D))} \\ &\qquad + C\epsilon\sum_{k=1}^{M} \Delta t_k\|p_h^{k-1}(u) - p_h^{k-1}\|^2_{L^2(\Gamma;L^2(D))} + C\epsilon\sum_{k=1}^{M} \Delta t_k\|u^k - u_h^k\|^2_{L^2(D)}. \end{aligned}$$

If ϵ is small enough, from (37)–(40), we can get (65). □

Lemma 5.4. *Let* (y, p, u) *be the solution of the optimal control problem* (30) *and* $(y_h(u), p_h(u))$ *be the solution of the auxiliary problem* (53). *Then, the following estimates hold*:

$$|||y - y_h(u)|||_{L^\infty(0,T;L^2(\Gamma;H^1(D)))} \le C(h_s + \Delta t), \tag{70}$$

and

$$|||p - p_h(u)|||_{L^\infty(0,T;L^2(\Gamma;H^1(D)))} \le C(h_s + \Delta t). \tag{71}$$

Proof. Let

$$\begin{aligned} &\overline{\gamma}^k = y^k - y_h^k(u), d_t\overline{\gamma}^k = \frac{1}{\Delta t_k}(\overline{\gamma}^k - \overline{\gamma}^{k-1}), \quad \varsigma^k = \frac{\partial y^k}{\partial t} - \frac{y^k - y^{k-1}}{\Delta t_k}, \\ &\overline{\delta}^k = p^k - p_h^k(u), \overline{d}_t\overline{\delta}^{k-1} = \frac{1}{\Delta t_k}(\overline{\delta}^{k-1} - \overline{\delta}^k), \quad \tau^{k-1} = -\frac{\partial p^{k-1}}{\partial t} - \frac{p^{k-1} - p^k}{\Delta t_k}. \end{aligned} \tag{72}$$

From (30) and (53), for $\forall\, v_h, q_h \in Y_h$, we can get

$$[d_t\overline{\gamma}^k, v_h]_\rho + A[\overline{\gamma}^k, v_h]_\rho = -[\varsigma^k, v_h]_\rho, \tag{73}$$

$$[\overline{d}_t\overline{\delta}^{k-1}, q_h]_\rho + A[q_h, \overline{\delta}^{k-1}]_\rho = [\overline{\gamma}^k, q_h]_\rho - [\tau^{k-1}, q_h]_\rho. \tag{74}$$

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For the equality (73), letting $v_h = d_t\overline{\gamma}^k$, we have

$$\begin{aligned}&\|d_t\overline{\gamma}^k\|^2_{L^2_\rho(\Gamma;L^2(D))} + A[\overline{\gamma}^k, d_t\overline{\gamma}^k]_\rho \\ &\quad = [d_t\overline{\gamma}^k, d_t(y^k - R_{h_s}y^k)]_\rho + A[\overline{\gamma}^k, d_t(y^k - R_{h_s}y^k)]_\rho \\ &\qquad - [\varsigma^k, d_t(R_{h_s}y^k - y^k_h(u))]_\rho, \end{aligned} \tag{75}$$

Then, for $1 \le r \le M$, we can get

$$\begin{aligned}&\sum_{k=1}^{r} \Delta t_k \|d_t\overline{\gamma}^k\|^2_{L^2_\rho(\Gamma;L^2(D))} + \frac{1}{2}A[\overline{\gamma}^r, \overline{\gamma}^r]_\rho \\ &\quad \le \frac{1}{2}A[\overline{\gamma}^0, \overline{\gamma}^0]_\rho + \sum_{k=1}^{r} \Delta t_k [d_t\overline{\gamma}^k, d_t(y^k - R_{h_s}y^k)]_\rho \\ &\qquad + \sum_{k=1}^{r} \Delta t_k A[\overline{\gamma}^k, d_t(y^k - R_{h_s}y^k)]_\rho - \sum_{k=1}^{r} \Delta t_k [\varsigma^k, d_t(R_{h_s}y^k - y^k_h(u))]_\rho. \end{aligned} \tag{76}$$

Using Cauchy–Schwartz inequality and (37), we have

$$\begin{aligned}&\sum_{k=1}^{r} \Delta t_k [d_t\overline{\gamma}^k, d_t(y^k - R_{h_s}y^k)]_\rho \\ &\quad \le \frac{1}{4}\sum_{k=1}^{r} \Delta t_k \|d_t\overline{\gamma}^k\|^2_{L^2_\rho(\Gamma;L^2(D))} + C\left\|\frac{\partial(y - R_{h_s}y)}{\partial t}\right\|^2_{L^2(0,T;L^2(\Gamma;L^2(D)))} \\ &\quad \le \frac{1}{4}\sum_{k=1}^{r} \Delta t_k \|d_t\overline{\gamma}^k\|^2_{L^2_\rho(\Gamma;L^2(D))} + Ch_s^2\|y\|^2_{H^1(0,T;L^2(\Gamma;H^1(D)))} \end{aligned} \tag{77}$$

and

$$\begin{aligned}&\sum_{k=1}^{r} \Delta t_k A[\overline{\gamma}^k, d_t(y^k - R_{h_s}y^k)]_\rho \\ &\quad \le \frac{1}{4}\sum_{k=1}^{r} \Delta t_k A[\overline{\gamma}^k, \overline{\gamma}^k]_\rho + C\left\|\frac{\partial(y - R_{h_s}y)}{\partial t}\right\|^2_{L^2(0,T;L^2(\Gamma;H^1(D)))} \\ &\quad \le \frac{1}{4}\sum_{k=1}^{r} \Delta t_k A[\overline{\gamma}^k, \overline{\gamma}^k]_\rho + Ch_s^2\|y\|^2_{H^1(0,T;L^2(\Gamma;H^2(D)))}. \end{aligned} \tag{78}$$

Noting that $R_{h_s}y^k - y^k_h(u) = (R_{h_s}y^k - y^k) + (y^k - y^k_h(u))$, we have

$$\begin{aligned}&\sum_{k=1}^{r} \Delta t_k [\varsigma^k, d_t(R_{h_s}y^k - y^k_h(u))]_\rho \\ &\quad \le \frac{1}{4}\sum_{k=1}^{r} \Delta t_k \|d_t\overline{\gamma}^k\|^2_{L^2_\rho(\Gamma;L^2(D))} + C(\Delta t)^2\left\|\frac{\partial^2 y}{\partial t^2}\right\|^2_{L^2(0,T;L^2(\Gamma;L^2(D)))} \\ &\qquad + Ch_s^2\|y\|^2_{H^1(0,T;L^2(\Gamma;H^1(D)))}. \end{aligned} \tag{79}$$

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Combining inequality (76)–(79) with (24), we have

$$\sum_{k=1}^{r}\Delta t_k\|d_t\overline{\gamma}^k\|^2_{L^2_\rho(\Gamma;L^2(D))}+A[\overline{\gamma}^r,\overline{\gamma}^r]_\rho$$

$$\leq A[\overline{\gamma}^0,\overline{\gamma}^0]_\rho+C(\Delta t)^2\left\|\frac{\partial^2 y}{\partial t^2}\right\|^2_{L^2(0,T;L^2(\Gamma;L^2(D)))}$$

$$+Ch_s^2(\|y\|^2_{H^1(0,T;L^2(\Gamma;H^1(D)))}+\|y\|^2_{H^1(0,T;L^2(\Gamma;H^2(D)))})$$

$$+C\sum_{k=1}^{r}\Delta t_k A[\overline{\gamma}^k,\overline{\gamma}^k]_\rho. \tag{80}$$

We can get (70) by the Gronwall Lemma.

For Eq. (74), letting $q_h=\overline{d}_t\overline{\delta}^{k-1}$, we have

$$\|\overline{d}_t\overline{\delta}^{k-1}\|^2_{L^2_\rho(\Gamma;L^2(D))}+A[\overline{d}_t\overline{\delta}^{k-1},\overline{\delta}^{k-1}]_\rho$$

$$=[\overline{d}_t\overline{\delta}^{k-1},\overline{d}_t(p^{k-1}-R_{h_s}p^{k-1})]_\rho+A[\overline{d}_t(p^{k-1}-R_{h_s}p^{k-1}),\overline{\delta}^{k-1}]_\rho$$

$$+[\overline{\gamma}^k,\overline{d}_t(R_{h_s}p^{k-1}-p_h^{k-1}(u))]_\rho-[\tau^{k-1},\overline{d}_t(R_{h_s}p^{k-1}-p_h^{k-1}(u))]_\rho, \tag{81}$$

Then, for $1\leq r\leq M-1$, we have

$$\sum_{k=r+1}^{M}\Delta t_k\|\overline{d}_t\overline{\delta}^{k-1}\|^2_{L^2_\rho(\Gamma;L^2(D))}+\frac{1}{2}A[\overline{\delta}^r,\overline{\delta}^r]_\rho$$

$$\leq\sum_{k=r+1}^{M}\Delta t_k[\overline{d}_t\overline{\delta}^{k-1},\overline{d}_t(p^{k-1}-R_{h_s}p^{k-1})]_\rho$$

$$+\sum_{k=r+1}^{M}\Delta t_k A[\overline{d}_t(p^{k-1}-R_{h_s}p^{k-1}),\overline{\delta}^{k-1}]_\rho$$

$$+\sum_{k=r+1}^{M}\Delta t_k[\overline{\gamma}^k,\overline{d}_t(R_{h_s}p^{k-1}-p_h^{k-1}(u))]_\rho$$

$$-\sum_{k=r+1}^{M}\Delta t_k[\tau^{k-1},\overline{d}_t(R_{h_s}p^{k-1}-p_h^{k-1}(u))]_\rho. \tag{82}$$

We can get (71) by the same proof of (70). Lemma 5.4 is completed. □

Combining the results of Lemmas 5.2, 5.3 with 5.4, we have the following error estimate with respect to $y-y_h$, $p-p_h$ and $u-u_h$.

Theorem 5.5. *Let* (y,p,u) *be the solution of the optimal control problem* (30) *and* (y_h,p_h,u_h) *be the solution of the discretized problem* (51), *respectively. Assume*

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that the conditions of Lemmas 5.1–5.4 *are valid. Then, the following error estimate holds*:

$$\sum_{v=y,p} |||v - v_h|||_{L^\infty(0,T;L^2(\Gamma;H^1(D)))} + |||u - u_h|||_{L^2(0,T;L^2(D))}$$

$$\leq C\left(h_s + \Delta t + h_r^\gamma \sum_{j=1}^{N} \frac{\|\partial_{\xi_j}^{p_j+1} p\|_{L^2(\Gamma;L^2(D))}}{(p_j+1)!}\right), \tag{83}$$

where $\gamma = \min_{1\leq j\leq N}\{p_j + 1\}$.

6. Numerical Experiments

In this section, numerical examples are presented to demonstrate our proposed Galekin formulation in Sec. 4 for stochastic control problem.

For simplicity in calculation, we take $T = 1$, take space domain $D = [-1, 1]$ and each stochastic domain Γ_i are $[-\sqrt{3}, \sqrt{3}]$ after finite-dimensional representing of stochastic fields. We assume each probability density function on Γ_i is uniform, i.e., $\rho_i(\xi_i) = \frac{1}{2\sqrt{3}}, i = 1, \ldots, N$. Thus, the joint probability density function $\rho(\xi)$ of random variable $\xi = (\xi_1, \xi_2, \ldots, \xi_N)$ is $\frac{1}{(2\sqrt{3})^N}$. In the following numerical example, we will do the same KL expansion as Lee and Lee [2013] for random coefficient $a(x, \omega)$, i.e.,

$$a_N(x, \omega) = \mathbb{E}a(x, \omega) + \sum_{n=1}^{N} \sqrt{\lambda_n}\phi_n(x)\xi_n(\omega),$$

where $(\lambda_n, \phi_n)_{1\leq n\leq N}$ are eigenpairs of

$$\int_D e^{-|x_1-x_2|}\phi_n(x_1)dx_1 = \lambda_n\phi_n(x_2).$$

In the following two examples, we consider the model problem:

$$\min_{u\in K} \mathcal{J}(u) = \min_{u\in K}\left(\frac{1}{2}\int_0^1\int_{-\sqrt{3}}^{\sqrt{3}} \frac{1}{(2\sqrt{3})^N}\|y - y_d\|^2_{L^2(D)}d\xi dt + \frac{\alpha}{2}\int_0^1 \|u\|^2_{L^2(D)}dt\right) \tag{84}$$

subject to

$$\begin{cases} y_t(t,x,\xi) - \nabla\cdot[\mu a(x,\xi)\nabla y(t,x,\xi)] = u(t,x), & x\in[-1,\,1],\ \ \xi\in[-\sqrt{3},\sqrt{3}]^N, \\ y(t,\pm1,\xi) = 0, & t\in[0,\,1],\ \ \xi\in[-\sqrt{3},\sqrt{3}]^N, \\ y(0,x,\xi) = 0, & x\in[-1,\,1],\ \ \xi\in[-\sqrt{3},\sqrt{3}]^N, \end{cases} \tag{85}$$

where $\alpha = 1, \mu = 0.01$, the target solution $y_d = 10(\sin(\pi x) + \sin(2\pi x))\sin(\pi t)$, the objective is to minimize the expectation of a cost functional, and the deterministic control is of the constrained type. For convenience, we take a uniform partition to

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time t and take $\Delta t = 1/64$. We can get from Figs. 1–4 and Tables 1–4 that the values of $\mathbb{E}(\int_0^1 \|y_h - y_d\|^2 dt)$ and $\mathcal{J}_h(u_h)$ are decreasing and tending to stable as the value of space step h getting smaller.

Example 1. The deterministic control is constrained by the condition $u(t,x) \geq 0, \forall\,(t,x) \in [0,1] \times [-1,1]$.

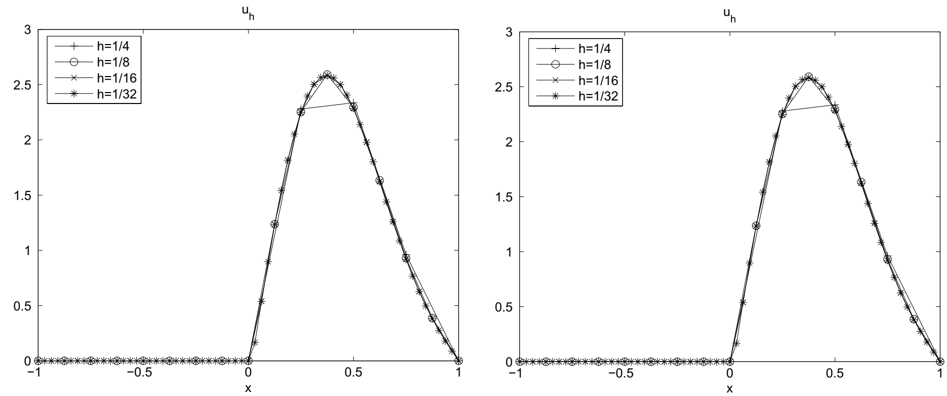

Fig. 1. $N = 2, \mathbb{E}(a) = 29,\ P = (1,1)$(left), $P = (2,2)$(right), $t = 0.25$.

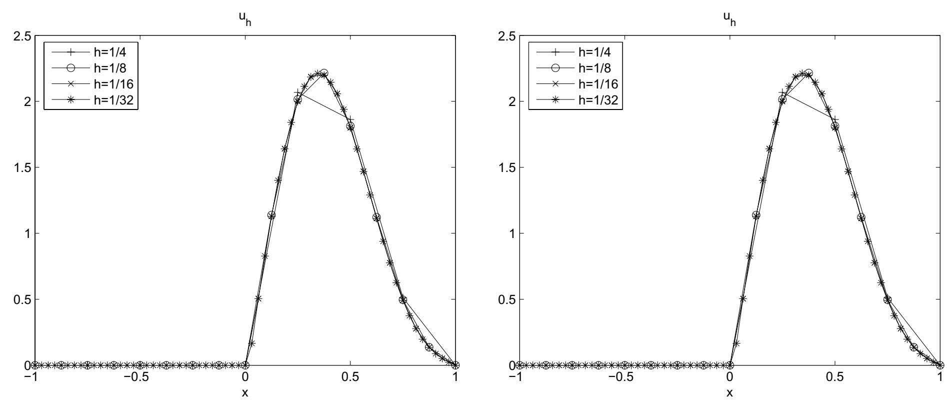

Fig. 2. $N = 2, \mathbb{E}(a) = 29,\ P = (1,1)$(left), $P = (2,2)$(right), $t = 0.5$.

Table 1. $N = 2, P = (1,1), \mathbb{E}(a) = 29$.

N	P	$\mathbb{E}(\int_0^1 \|y_h - y_d\|^2)$	$\int_0^1 \|u_h\|^2$	$\mathcal{J}_h(u_h)$	h
2	$(1,1)$	97.228177636662	1.31271080762079	49.2704442221414	1/4
2	$(1,1)$	97.0604801571598	1.3817192589109	49.2210997080353	1/8
2	$(1,1)$	97.0132898802689	1.40167803466767	49.2074839574683	1/16
2	$(1,1)$	97.0004594542209	1.4073987415786	49.2039290978997	1/32

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Table 2. $N = 2, P = (2, 2), \mathbb{E}(a) = 29$.

N	P	$\mathbb{E}(\int_0^1 \|y_h - y_d\|^2)$	$\int_0^1 \|u_h\|^2$	$\mathcal{J}_h(u_h)$	h
2	(2, 2)	97.2281773402582	1.31271094248455	49.2704441413714	1/4
2	(2, 2)	97.0604798453838	1.38171939856792	49.2210996219759	1/8
2	(2, 2)	97.0132895649192	1.40167817538437	49.2074838701518	1/16
2	(2, 2)	96.9724591357245	1.40742354166477	49.1899413386946	1/32

Example 2. The deterministic control is constrained by the condition $\int_0^1 u(t, x)dx \geq 0, \forall\, t \in [0, 1]$.

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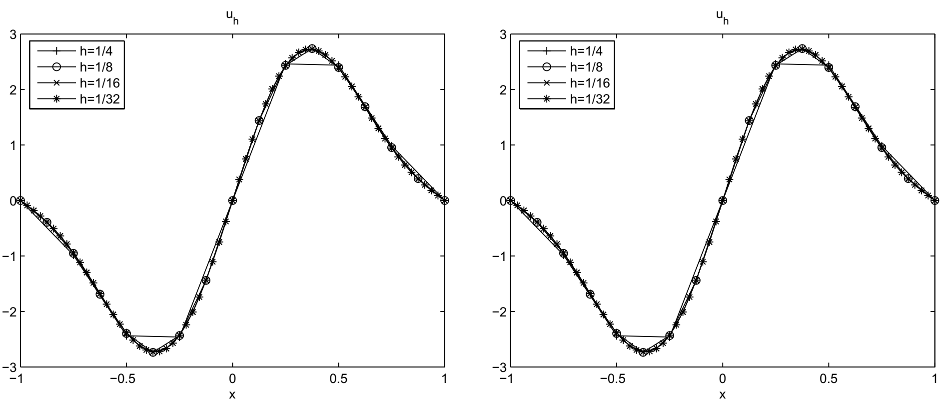

Fig. 3. $N = 2, \mathbb{E}(a) = 29,\ P = (1, 1)$(left), $P = (2, 2)$(right), $t = 0.25$.

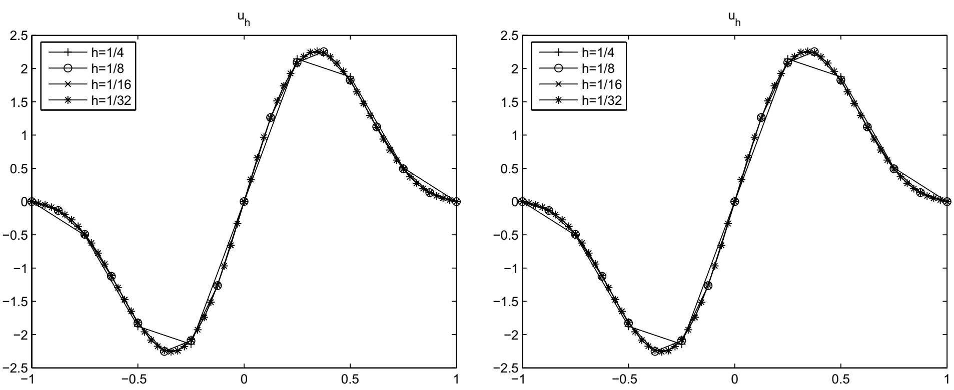

Fig. 4. $N = 2, \mathbb{E}(a) = 29,\ P = (1, 1)$(left), $P = (2, 2)$(right), $t = 0.5$.

Table 3. $N = 2, P = (1,1), \mathbb{E}(a) = 29$.

N	P	$\mathbb{E}(\int_0^1 \|y_h - y_d\|^2)$	$\int_0^1 \|u_h\|^2$	$\mathcal{J}_h(u_h)$	h
2	(1, 1)	94.0152215170601	2.93418115943041	48.4747013382452	1/4
2	(1, 1)	93.6219166191258	3.12368575649849	48.4054244495752	1/8
2	(1, 1)	93.5143904232277	3.17564276220522	48.3450165927164	1/16
2	(1, 1)	93.4869404750315	3.18893256783526	48.3379365214334	1/32

Table 4. $N = 2, P = (2,2), \mathbb{E}(a) = 29$.

N	P	$\mathbb{E}(\int_0^1 \|y_h - y_d\|^2)$	$\int_0^1 \|u_h\|^2$	$\mathcal{J}_h(u_h)$	h
2	(2, 2)	94.015220933088	2.93418142605112	48.4747011795695	1/4
2	(2, 2)	93.6199366284173	3.12368547322781	48.3718110508225	1/8
2	(2, 2)	93.5123792755425	3.17564247526523	48.3440108754038	1/16
2	(2, 2)	93.4849213627202	3.18893228002457	48.3369268213724	1/32

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