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ON THE PRIVATE PROVISION OF PUBLIC GOODS ON NETWORKS

NIZAR ALLOUCH

Queen Mary, University of London

School of Economics and Finance

Mile End Rd, London, E1 4NS

United Kingdom

ABSTRACT. This paper analyzes the private provision of public goods where consumers interact within a fixed network structure and may benefit only from their direct neighbors' provisions. We present a proof of the existence and uniqueness of a Nash equilibrium for general networks and best-reply functions. In addition, we investigate the neutrality result of Warr [38] and Bergstrom, Blume, and Varian [6] whereby consumers are able to undo the impact of income redistribution as well as public provision financed by lump-sum taxes. To this effect, we show that the neutrality result has a limited scope of application beyond a special network architecture in the neighborhood of the set of contributors.

E-mail address: n.allouch@qmul.ac.uk.

1. Introduction

The private provision of public goods is a subject of ongoing interest in several strands of the economics literature ranging from taxation to political economy. Private contributions to public goods are important phenomena for many reasons. Voluntary contributions by members of a community are vital for the provision of essential social infrastructure, while at the aggregate level charitable giving accounts for a significant proportion of GDP in many countries. The seminal contribution of Bergstrom, Blume, and Varian [6], built on an earlier striking result by Warr [38], provides a rigorous investigation of the standard model of private provision of pure public goods.[1] Their main results, with sharp testable implications, are the invariance of individual private and public goods consumption, the so-called neutrality result, to income redistribution among contributors that leaves unchanged the composition of the set of contributors, and the related complete crowding-out of public provision financed by lump-sum taxes.

The findings of the private provision model rest on the assumption that each consumer benefits from the public goods provisions of all other consumers. Often, for various public goods such as information gathering, new products experimentation, and local amenities, a consumer may benefit from provisions accessible only through his social interactions or geographical position. For instance, there is strong empirical evidence that farmers perceive the experimentation of a new technology as a public good and adjust their experimentation level in the opposite direction to their neighbors' provision (see, for example, Foster and Rosenzweig [22]). Moreover, consumers often first seek information from friends, colleagues, or even their various online communities before sampling the products themselves.

In this paper, we investigate the private provision of public goods where consumers interact within a fixed network structure and benefit only from their direct neighbors' provisions. Recently, the economics of networks has gained prominence as a new approach to understanding varied economic interactions (see Goyal [27] and Jackson [32]). The main insights on formation and stability of networks are powerful predictive tools for both positive and normative analysis in many fields, including development economics and labor economics. Public goods provision within networks was first studied in

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[1]There is a special issue in the *Journal of Public Economics* celebrating the 20th anniversary of Bergstrom, Blume, and Varian [6].

the key paper by Bramoullé and Kranton [10]. Their analysis, under complete information, distinguishes between specialized and hybrid contribution equilibria and shows that specialized contribution equilibria correspond to the maximal independent sets of the network. Galeotti, Goyal, Jackson, Vega-Redondo, and Yariv [25] show that the possibility that consumers hold partial information about the network can shrink considerably the potentially large set of equilibria that arise under complete information. Galeotti and Goyal [25] study a more general model of information sharing where consumers simultaneously decide on their information acquisition, that is, their contribution to the public good, and connections. Bramoullé, Kranton, and D'Amours [11] investigate games of strategic substitutes[2] on networks with linear best-reply functions and unearth a network measure, related to the lowest eigenvalue of the adjacency matrix of the network,[3] as a key to equilibrium analysis. Elliott and Golub [19] develop an innovative normative approach to public goods provision on networks that, among other things, decentralizes a Lindahl-like allocation. Acemoglu, García-Jimeno, and Robinson [1] propose a network based approach to explore the link between investments in local state capacity, including public goods provision, and economic development.

This paper presents a proof of the existence and uniqueness of a Nash equilibrium in the private provision of public goods on networks. The existence of a Nash equilibrium is guaranteed by Brouwer's fixed point theorem. Our key assumption to establish the uniqueness of a Nash equilibrium, called network normality, stipulates that each consumer's marginal propensity to consume the public good is strictly bounded: it must be less than one, and greater than one plus the inverse of the lowest eigenvalue. Together, these conditions correspond to a standard normality of the private good and a strong normality of the public good. Our existence and uniqueness results simultaneously extend similar results on the private provision of public goods in Bergstrom, Blume, and Varian [6] to a network setting and results on games of strategic substitutes in Bramoullé, Kranton, and D'Amours [11] to nonlinear best-reply functions.

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[2]The private provision of public goods falls into this category since a consumer has incentives to adjust his public goods provision in the opposite direction to his neighbors' provisions.

[3]As far as we know, such a measure has not been used previously in any of the fields related to networks, including social networks, biology, and physics. Moreover, Bramoullé, Kranton, and D'Amours [11] provide an excellent discussion on the structural properties of the network that may affect the magnitude of the lowest eigenvalue.

The closely related literature on clubs/local public goods also investigates the strategic interactions underlying the formation of clubs and communities. If one thinks of a network as a collection of clubs formed by either the vertices or the edges then the public goods network literature and the club/local public goods literature are essentially equivalent. However, such an equivalence is not very useful since a network is then a collection of overlapping clubs and, so far, only a few papers have explored the Nash equilibrium with an overlapping clubs structure. Bloch and Zenginobuz [8] present a model of local public goods allowing spillovers between communities, and hence violating one of Tiebout's assumptions, which may be interpreted as a weighted network. Eshel, Samuelson, and Shaked [21] and Corazzini and Gianazza [14] adapt Ellison's [20] local interaction model to public good games played on a spatial structure, which in a network setting correspond to a circulant network.

It is worth noting that the range of the network normality assumption may be quite small since, depending on the structure and size of the network, the magnitude of the lowest eigenvalue may be quite large. This constitutes the main limitation of our uniqueness analysis. Nonetheless, this also leads to another important contribution in Bramoullé, Kranton, and D'Amours [11]: in addition to being key to the uniqueness of a Nash equilibrium, the lowest eigenvalue is a measure of how a small change in consumers' provisions is dampened or amplified through the network. More precisely, the larger the magnitude of the lowest eigenvalue, the more a small change in players' actions reverberates in the network. A key insight that emerges from their analysis, when multiple equilibria arise and in the presence of players with identical payoffs and symmetric network positions, is that symmetric contribution Nash equilibria not only may be unstable, but also may coexist with asymmetric contribution Nash equilibria that are stable. As a consequence, when there are multiple equilibria, the focus on symmetric and interior Nash equilibria may seem inappropriate when selecting among them.

Of the policy questions that arise in connection with the private provision of public goods, the impact of income redistribution is of central importance. In motivation, this has similar implications for policy design to the Second Welfare Theorem although, unlike the competitive equilibrium, the Nash equilibrium of private provision will typically be inefficient. For pure public goods, which correspond to a complete network of interactions, the question has, to a large extent, been settled by the neutrality result mentioned above.

However, it appears that there has been no attempt in the literature on the economics of networks to explore whether the neutrality result holds beyond pure public goods. To this effect, we provide an innovative approach to explore the impact of income redistribution in networks. In particular, for preferences of general form, our results relate the neutrality of a transfer to the network structure of interactions in a simple and intuitive way. As a consequence, we are able to show that the neutrality result will not in general hold beyond a special network architecture in the neighborhood of the set of contributors.

Furthermore, we show that the impact of income redistribution on the aggregate provision of public goods is related to the Bonacich centrality, due to Bonacich [9]. This was first introduced to economics in the seminal paper of Ballester, Calvó-Armengol, and Zenou [5] as being proportional to Nash equilibrium actions in a linear best-reply game. Bonacich centrality, which usually measures prestige and influence in social networks, is shown here to summarize information on each consumer's impact on the aggregate provision after income redistribution. In particular, such information is useful in outlining conditions on the network structure for the invariance of aggregate provision to hold. In addition, partly due to our spectral approach to Bonacich centrality, sharper predictions on the patterns of change in aggregate provision can be obtained for the particular case of Gorman polar form preferences.

The paper is organized as follows. In Section 2, we present the model of private provision of public goods on networks. In Section 3, we establish the existence and uniqueness of a Nash equilibrium. In Section 4, we investigate the stability of the Nash equilibrium. Section 5 investigates the validity of the neutrality result in general networks. Section 6, for the particular case of Gorman polar form preferences, explores the composition of the set of contributors and further investigates the impact of income redistribution. Section 7 concludes the paper.

2. The model

There are n consumers embedded in a connected fixed network $\mathbf{g}$. Let $\mathbf{G} = [g_{ij}]$ denote the adjacency matrix of the network $\mathbf{g}$, where $g_{ij} = 1$ indicates that consumer i and consumer j are neighbors in the network $\mathbf{g}$ and $g_{ij} = 0$ otherwise. In particular, we assume that $g_{ii} = 0$ for each consumer i. We denote by $N = \{1, \ldots, n\}$ the set of consumers and by $\mathcal{N}_i = \{j \in N \mid g_{ij} = 1\}$ the set of consumer i's neighbors. Given a

subset of consumers S, let $\mathbf{g}_S$ denote the subnetwork induced by S, that is, the network obtained by removing consumers not belonging to S as well as all the links emanating from them and $\mathbf{G}_S$ denote the adjacency matrix of $\mathbf{g}_S$.

The adjacency matrix of the network, $\mathbf{G}$, is symmetric with nonnegative entries and therefore has a complete set of real eigenvalues (not necessarily distinct), denoted by $\lambda_{\min}(\mathbf{G}) = \lambda_n \leq \ldots \leq \lambda_1 = \lambda_{\max}(\mathbf{G})$, where $\lambda_{\min}(\mathbf{G})$ is the lowest eigenvalue and $\lambda_{\max}(\mathbf{G})$ is the largest eigenvalue of $\mathbf{G}$. By the Perron–Frobenius Theorem, it holds that

$$0 < -\lambda_{\min}(\mathbf{G}) \leq \lambda_{\max}(\mathbf{G}). \tag{2.1}$$

Moreover, it holds that $\mathbf{G} = \mathbf{VDV}^T$, where $\mathbf{D} = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ is a diagonal matrix whose diagonal entries are the eigenvalues of $\mathbf{G}$ and $\mathbf{V}$ is a matrix whose columns, $\mathbf{v}_1, \ldots, \mathbf{v}_n$, are the corresponding eigenvectors of $\mathbf{G}$ that form an orthonormal basis of $\mathbb{R}^n$. As usual, let $\mathbf{I}$ denote the identity matrix.

The preferences of each consumer $i = 1, \ldots, n$ are represented by the utility function $u_i(x_i, q_i + Q_{-i})$, where x_i is consumer i's private good consumption, q_i is consumer i's public good provision, and $Q_{-i} = \sum_{j \in \mathcal{N}_i} q_j$ is the sum of public good provisions of consumer i's neighbors. For simplicity, we assume the public good can be produced from the private good with a unit-linear production technology. The utility function u_i is continuous, strictly increasing in both arguments, and strictly quasi-concave. Consumer i faces the following maximization problem:

$$\max_{x_i, q_i} u_i(x_i, q_i + Q_{-i})$$

$$\text{s.t.} \quad x_i + q_i = w_i \text{ and } q_i \geq 0,$$

where $w_i > 0$ is his income (exogenously fixed). It follows from the strict quasi-concavity that consumer i's public good provision is determined by a (single-valued) best-reply function f_i.

At a Nash equilibrium $\mathbf{q}^* = (q_1^*, \ldots, q_n^*)^T$, every consumer's choice is a best-reply to the sum of his neighbors' public good provisions, that is, $q_i^* = f_i(Q_{-i}^*)$ for each consumer $i = 1, \ldots, n$. Let $C = \{i \in N \mid q_i^* > 0\}$ denote the set of contributors. A subset of contributors S that induces a component of $\mathbf{g}_C$[4] will be called a "component of contributors".

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[4]As usual, a component of a network is a maximal connected subnetwork.

Following a standard modification in the public economics literature, the utility maximization problem can be rewritten with consumer i choosing his (local) public good consumption, Q_i, rather than his public good provision, q_i, that is,

$$\max_{x_i, Q_i} u_i(x_i, Q_i)$$

$$\text{s.t.} \quad x_i + Q_i = w_i + Q_{-i} \text{ and } Q_i \geq Q_{-i}.$$

If we ignore the last constraint $Q_i \geq Q_{-i}$ in the above maximization problem, we obtain a standard utility maximization problem of consumer demand theory. Hence a standard demand function for consumer i's public good consumption can be expressed by $Q_i = \gamma_i(w_i + Q_{-i})$, where $w_i + Q_{-i}$ may be interpreted as consumer i's "social income" and γ_i is consumer i's Engel curve. In view of this, acknowledging the constraint $Q_i \geq Q_{-i}$ again leads to $Q_i = \max\{\gamma_i(w_i + Q_{-i}), Q_{-i}\}$, which in turn implies

$$q_i = Q_i - Q_{-i} = \max\{\gamma_i(w_i + Q_{-i}) - Q_{-i}, 0\} = f_i(Q_{-i}). \tag{2.2}$$

Hence, consumers can only contribute a positive amount of the public good determined by their autarkic demand for the public good, their income, and also their neighbors' public good provisions.

3. Existence and uniqueness of a Nash equilibrium

In this section, we shall prove the existence and uniqueness of a Nash equilibrium for general networks and best-reply functions. In the case of a pure public good, which corresponds to a complete network structure of interactions, Bergstrom, Blume, and Varian [6] rely on the assumption of normality of private and public goods to establish the existence and uniqueness of the Nash equilibrium. We introduce the following network-specific normality assumption:

Network normality. For each consumer $i = 1, \ldots, n$, the Engel curve γ_i is differentiable and it holds that $1 + \frac{1}{\lambda_{\min}(\mathbf{G})} < \gamma_i'(\cdot) < 1$.

The network normality assumption allows for a nonlinear relationship between the demand for both private and public goods, and the income of each consumer, but places bounds on the marginal propensity to consume these goods. Indeed, the left-hand-side inequality stipulates a strong normality of the public good, which depends on the lowest eigenvalue of the network, while the right-hand-side inequality is the standard normality

of the private good. Nonlinear Engel curves arise in many areas of economics, including public economics (see, for example, Moffitt [34]), since it is plausible that consumers may not have a constant marginal propensity to spend on some given goods.

Theorem 1. *Assume network normality. Then there exists a unique Nash equilibrium in the private provision of public goods on networks.*

Proof. The proof of Theorem 1, together with all of our other proofs, appears in Appendix A.□

We have the following two corollaries:

First, note that for a complete network it holds that $\lambda_{\min}(\mathbf{G}) = -1$.[5] Thus, in the case of a pure public good, the normality of both private and public goods implies network normality.

Corollary 1. *(Bergstrom, Blume, and Varian [6]) Assume that the public good is pure and that both private and public goods are normal goods. Then there exists a unique Nash equilibrium.*

Second, a linear strategic substitute game coincides with a suitably constructed private provision game, which, provided that network normality holds, has a unique Nash equilibrium.

Corollary 2. *(Bramoullé, Kranton, and D'Amours [11]) Consider a game such that for each consumer $i = 1, \ldots, n$, it holds that $q_i = \max\{1 - \alpha_i \sum_{j \in \mathcal{N}_i} q_j, 0\}$ with $\alpha_i \in]0, -\frac{1}{\lambda_{\min}(\mathbf{G})}[$. This game coincides with the private provision game, where $\gamma_i'(\cdot) = 1 - \alpha_i$ and $w_i = \frac{1}{1-\alpha_i}$. Thus, there exists a unique Nash equilibrium.*

Discussion of related literature. At the heart of the equilibrium analysis of the pure public good model of Bergstrom, Blume, and Varian [6] lies the elegant proof of the uniqueness of a Nash equilibrium. The proof is divided into various steps, in which consumers' best-reply functions are transformed in order to take advantage of the normality of both private and public goods. Yet, the many subtleties of their uniqueness proof technique may not have fully revealed the intuition behind the proof or shown what a

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[5]The adjacency matrix of the complete network $\mathbf{G} = \mathbf{F} - \mathbf{I}$, where $\mathbf{F}$ is the all-ones matrix. Since $\mathbf{F}$ has eigenvalues n and 0 with multiplicities 1 and $n-1$, respectively, we see that the complete network has eigenvalues $n-1$ and -1 with multiplicities 1 and $n-1$.

familiar uniqueness argument is at work (see, for example, Bergstrom, Blume, and Varian [7], Fraser [23], and Cornes and Hartley [15] for discussions and alternative proofs). Theorem 1 extends the uniqueness result of Bergstrom, Blume, and Varian [6] to the more general setting of local public goods shaped by local interactions. As a consequence, an inherent advantage of the proof technique of Theorem 1 is the insights it provides on what is driving the uniqueness of the Nash equilibrium in the private provision of public goods.

Bramoullé, Kranton, and D'Amours [11] show that the local interaction model with linear best-reply functions lends itself nicely to the theory of potential games of Monderer and Shapley [35]. In a potential game, consumers' optimal strategies concur in a common maximization problem of a potential function whose strict concavity provides the uniqueness of a Nash equilibrium.[6] Games of strategic substitutes with nonlinear best-reply functions can also be investigated, as suggested in Bramoullé, Kranton, and D'Amours [11], using local approximation techniques, although, as they acknowledge, Nash equilibrium uniqueness in a game with nonlinear best-reply functions cannot be inferred from uniqueness in the locally approximating games with linear best-reply functions. Unlike local approximation techniques, the proof of Theorem 1 overcomes the lack of linear structure by appealing to the network normality assumption, which stipulates global bounds on the slopes of the nonlinear best-reply functions.

Finally, key to the uniqueness of the Nash equilibrium in Theorem 1 is our assumption of network normality, whose range is the inverse of the magnitude of the lowest eigenvalue. Given that the magnitude of the lowest eigenvalue, which depends on both the structure and the size of the network, may be quite large, the range of network normality may be quite small.[7] This is the main limitation of our uniqueness analysis. Yet, for some network structures the lowest eigenvalue is bounded from below, regardless of the size of the network, and as a result the network normality assumption can be checked easily for a fixed range of marginal propensities. An example of such a network structure is the complete network since $\lambda_{\min}(\mathbf{G}) = -1$, as mentioned above. The following example provides another network structure:

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[6]The uniqueness result of Bramoullé, Kranton, and D'Amours [11] also can be viewed as a network application of the well-known uniqueness result of Rosen [37], which we discuss in Appendix B.

[7]Even though, in view of (2.1), it encompasses the range of strategic interactions that yields uniqueness in Ballester, Calvó-Armengol, and Zenou [5].

Example 1. The "line network" of a network $\mathbf{g}$, denoted by $L(\mathbf{g})$, is obtained by interchanging vertices and edges, that is, the vertices of $L(\mathbf{g})$ are the edges of $\mathbf{g}$ and two vertices of $L(\mathbf{g})$ are adjacent whenever the corresponding edges of $\mathbf{g}$ have a vertex in common. Below we have an example of a network $\mathbf{g}$ and its line network $L(\mathbf{g})$:
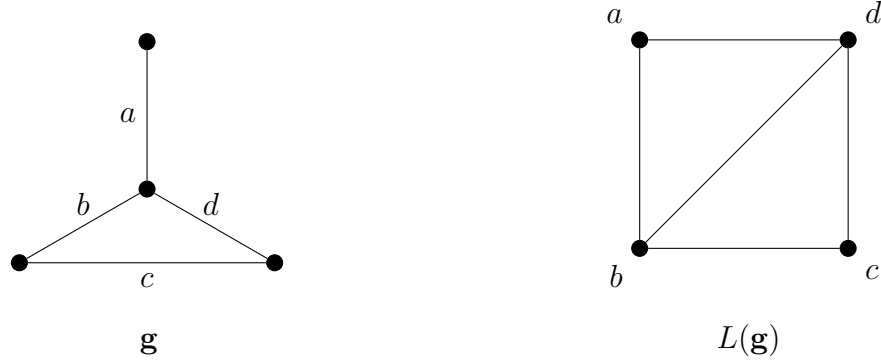

Figure 1: The line network.

In interpretation, in the network $L(\mathbf{g})$ consumers are the edges of $\mathbf{g}$, whose end vertices may be thought of as (overlapping) clubs in which a member's provision benefits all other members. İlkiliç [31] shows that line networks are useful to characterize equilibrium flows in a network of multiple commons and users. Here, quite differently, we show that a spectral property of line networks ensures a fixed range for which the network normality assumption holds. Indeed, let $L(\mathbf{G})$ denote the adjacency matrix of the line network $L(\mathbf{g})$; then it holds that $-2 \leq \lambda_{\min}(L(\mathbf{G}))$.[8] Hence, given that $1 + \frac{1}{\lambda_{\min}(L(\mathbf{G}))} \leq \frac{1}{2}$, whenever $\frac{1}{2} < \gamma_i'(\cdot) < 1$ for each consumer $i = 1, \ldots, n$, the network normality assumption holds in the network $L(\mathbf{g})$.

4. Stability of the Nash equilibrium

We shall now investigate the issue of stability of the Nash equilibrium in the private provision of public goods on networks. Stability is of paramount importance to the study of comparative statics. If, following a small perturbation of parameters, the new equilibrium can be reached by a dynamic adjustment process, then the analysis of comparative statics is strengthened. In this respect we closely follow Dixit [18] and consider a myopic

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[8]This follows immediately from the fact that the matrix $L(\mathbf{G}) + 2\mathbf{I} = \mathbf{N}^T\mathbf{N}$, where $\mathbf{N}$ is the incidence matrix of $\mathbf{g}$, is obviously positive semidefinite.

adjustment process defined, for each consumer $i = 1, \ldots, n$, by

$$\dot{q}_i = \frac{\mathrm{d}q_i}{\mathrm{d}t} = \sigma_i(f_i(Q_{-i}) - q_i),$$

where $\sigma_1, \ldots, \sigma_n > 0$ are the adjustment speeds.

In interpretation, starting at a public good provision level, each consumer increases his provision if he anticipates a higher utility level from doing so. The Nash equilibrium $\mathbf{q}^*$ is "locally asymptotically stable" if there exists a neighborhood of $\mathbf{q}^*$ such that if the above system starts at any point inside this neighborhood, it converges to $\mathbf{q}^*$. For simplicity, for the remainder of this paper, we assume that the set of knife-edge non-contributors, characterized by $\gamma_i(w_i + Q^*_{-i}) = Q^*_{-i}$, is empty, which is likely to be the case.[9]

The following result, as far as we know, is the first to investigate stability for general networks and best-reply functions:

Theorem 2. *Assume network normality. Then the unique Nash equilibrium of the private provision of public goods on networks is locally asymptotically stable.*

Theorem 2 shows that under the network normality assumption, whose range depends on the magnitude of the lowest eigenvalue, stability and uniqueness of a Nash equilibrium are closely related. More generally, Bramoullé, Kranton, and D'Amours [11] show that not only does the magnitude of the lowest eigenvalue reveal interesting details about the structural properties of the network,[10] but it is also a key measure of how the network absorbs or reverberates a small change in consumers' provisions, with linear best-reply functions. Following their insightful approach for nonlinear best-reply functions, the network normality assumption can be restated as:

For each consumer $i = 1, \ldots, n$, it holds that $\gamma_i'(\cdot) < 1$ and $|\lambda_{\min}(\mathbf{G})| < 1/(1 - \gamma_i'(\cdot))$. This has a two fold implication. First, for fixed marginal propensities to consume the public good, the larger the magnitude of the lowest eigenvalue, the more a small change in consumers' provisions reverberates in the network. For instance, a small change generates

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[9]Bramoullé, Kranton, and D'Amours [11] show that, in their setting, the set of knife-edge non-contributors is empty for any network and for any degree of substitutability between own and neighbors' actions except may be for a finite number of values.

[10]In particular, it emerges that, among all possible network structures, the magnitude of the lowest eigenvalue is maximized when the network has a similar structure to a complete bipartite network with almost equal sides. A complete bipartite graph consists of vertices divided into two sets with the edges connecting each vertex from one set to all the vertices in the other set.

larger adjustment effects in a complete bipartite network than in a complete network. Second, given a fixed network structure of interactions, the higher the marginal propensities to consume the public good, the less a small change in consumers' provisions reverberates in the network. In interpretation, the higher the marginal propensities to consume the public good, the less consumers substitute and adjust to their neighbors' provisions.

5. Income redistribution in networks

In this section, we shall explore the impact of income redistribution on private provision of public goods on networks, as a tool of economic policy. Similar to the Second Welfare Theorem, the policy instrument employed is a lump-sum redistribution, even though, unlike the competitive equilibrium, the Nash equilibrium of private provision of a public good will typically be inefficient.

Let t_i denote the income transfer made to consumer i, which may be either a tax ($t_i < 0$) or a subsidy ($t_i \geq 0$), and let "transfer" denote $\mathbf{t} = (t_1, \ldots, t_n)^T \in \mathbb{R}^n$, which lists all income transfers made to consumers. Every transfer is budget balanced; hence $\sum_{i=1}^n t_i = 0$. Similar to Warr [38] and Bergstrom, Blume, and Varian [6], we will focus on transfers that are relatively small in magnitude so that the income redistribution does not change the composition of the set of contributors, and we will call such transfers "relatively small". Note that transfers between non-contributors and contributors are relatively small as long as the composition of the set of contributors is unchanged.

Let $\mathbf{q}_C^*$ (resp. $\mathbf{q}_C^{\mathbf{t}}$) denote the provisions of contributors before income redistribution (resp. after income redistribution). The following result describes the impact of relatively small income redistributions on provisions in general networks:

Proposition 1. *Assume network normality. Then, for any relatively small transfer* $\mathbf{t}$ *it holds that*

$$\mathbf{q}_C^{\mathbf{t}} - \mathbf{q}_C^* = (\mathbf{I} + \mathbf{A}_C \mathbf{G}_C)^{-1} (\mathbf{I} - \mathbf{A}_C) \mathbf{t}_C,$$

where $\mathbf{A}_C = \mathrm{diag}(1 - \gamma_i'(\beta_i))_{i \in C}$ *for some* β_i *and* $\mathbf{t}_C = (t_i)_{i \in C}$.

Proposition 1, which will be key to the rest of our analysis of the impact of income redistribution, relates the change in consumers' public good provisions to the network structure of interactions, the marginal propensities of consumers, and the transfer.

5.1. **Neutrality of income redistribution in networks.** In the case of a pure public good, which corresponds to a complete network of interactions, the invariance result of Warr [38] and Bergstrom, Blume, and Varian [6], the so-called neutrality result, shows that income redistribution among contributors that does not change the composition of the set of contributors, yields a new equilibrium such that each consumer has precisely the same individual consumption of the private and the public good as he had before. What is remarkable about the neutrality result is that it holds regardless of the form of the preferences. The intuition behind it may be explained as follows: assume that after income redistribution each consumer adjusts his public good provision by exactly the amount of the income transfer made to him and leaves unchanged his private good consumption. Since the transfer, being budget balanced, leaves unchanged the aggregate public good provision, it follows that such allocation, with unchanged private and public goods consumption for each consumer, is not only individually optimal but also a Nash equilibrium. In order to investigate the neutrality of income redistribution in general networks, we first shed some light on how the neutrality of a particular transfer is related to the network structure of interactions. Formally, a transfer $\mathbf{t}$ is "neutral" if it leaves unchanged the individual consumption of the private and public goods for each consumer i, that is,

$$(x_i^{\mathbf{t}}, Q_i^{\mathbf{t}}) = (x_i^*, Q_i^*).$$

Now let us introduce a local balance condition for the transfer determined by the network. We say a transfer $\mathbf{t}$ is "neighborhood balanced" if it is balanced in each consumer's neighborhood, that is, for each consumer i in his neighborhood, $i \cup \mathcal{N}_i$, formed by himself and his neighbors, it holds that

$$t_i + \sum_{j \in \mathcal{N}_i} t_j = 0.$$

Neighborhood balance seems compelling since it requires that each consumer's neighborhood is, in aggregate, neither taxed nor subsidized. The following proposition shows that a further relationship holds: neighborhood balance together with another basic condition on the transfer is equivalent to transfer neutrality:

Proposition 2. *Assume network normality. Then a relatively small transfer is neutral if and only if it is neighborhood balanced and confined to contributors.*

Proposition 2 characterizes, in a simple and intuitive way, relatively small neutral transfers. Quite similar to what has already been emphasized by Warr [38] and Bergstrom, Blume, and Varian [6] in the pure public good case, the condition that the transfer is confined to contributors is essential for neutrality to hold.

Now suppose that for some political, institutional, or even practical constraints, transfers are confined to a subset of contributors rather than all contributors. A question that may arise naturally is "Under what conditions on the network structure does neutrality hold for all such relatively small transfers?" To proceed further with our analysis, we introduce a condition on the network architecture in the neighborhood of the subset of contributors. We say that a subset of consumers S is "neighborhood-homogenous" if all consumers in S have identical neighborhoods in the network, that is, for any i, j in S it holds that

$$i \cup \mathcal{N}_i = j \cup \mathcal{N}_j.$$

An example of a neighborhood-homogenous subset of consumers is the core of a fully connected core-periphery network.[11] Actually, (an alternative definition of) neighborhood homogeneity of a subset of consumers amounts to removing all edges not emanating from it, inducing a fully connected core-periphery component, with the subset being the core. Note that the network architecture is arbitrary beyond the component and hence the network may have multiple, and possibly non-overlapping, neighborhood homogenous subsets of consumers. An example of this is the complete network, where any subset of consumers is neighborhood homogenous.

Intuitively, as far as income redistribution is concerned, if a subset of consumers is neighborhood homogenous, then its members are not only fully connected but also indistinguishable in terms of their network position from the other consumers, which is, as shown below, an ideal network structure for neutrality to hold.

Theorem 3. *Assume network normality. Given a subset of contributors S, neutrality holds for all relatively small S-confined transfers if and only if S is neighborhood homogenous. In particular, neutrality holds for all relatively small transfers among contributors if and only if the set of contributors is neighborhood-homogenous.*

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[11]A fully connected core-periphery network structure is obtained as follows: start with a star network and simply add duplicates of the center to the network, and connect them to each other and to the periphery vertices.

Theorem 3 shows that neighborhood homogeneity of a subset of contributors ensures the neutrality of all relatively small transfers confined to it. The following example illustrates some neighborhood homogenous subsets of contributors:

Example 2. In the networks $\mathbf{g}_1$ and $\mathbf{g}_2$ below, let vertices in black represent contributors and vertices in white represent non-contributors. The smallest neighborhood-homogenous subset of contributors is composed of two contributors, and $\mathbf{g}_1$ has three of them: $\{c_4, c_5\}$, $\{c_8, c_9\}$, and $\{c_1, c_{12}\}$. Understandably, a larger neighborhood-homogenous subset of contributors, such as $\{c_1, c_2, c_3\}$ in $\mathbf{g}_2$, imposes greater restrictions on the network structure.
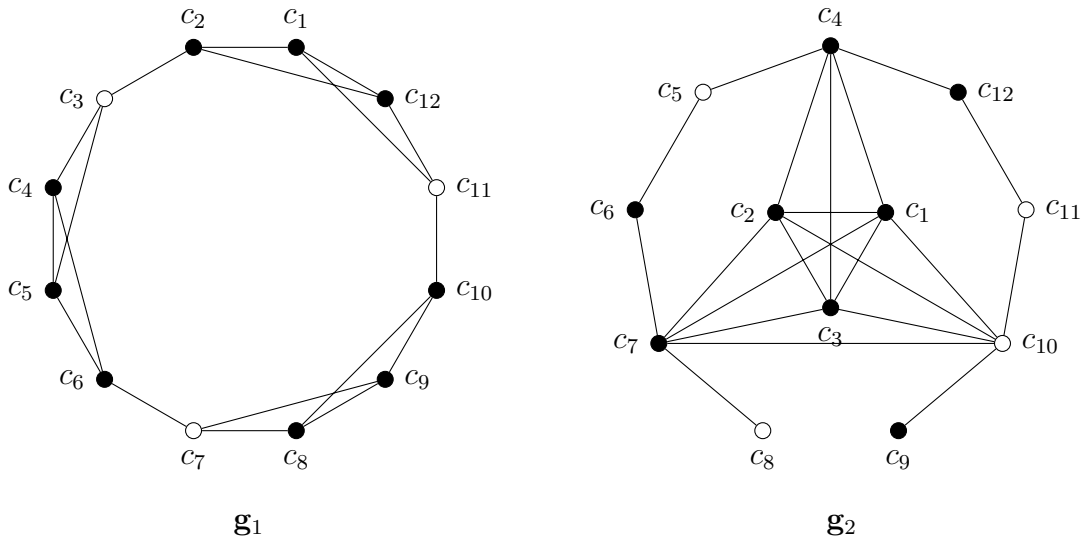

Figure 2: Neighborhood-homogenous subsets of contributors.

Note that in Example 2 the network position of contributors/non-contributors beyond a neighborhood-homogenous subset of contributors, including immediate neighbors of the subset, is arbitrary. Nevertheless, if this is further restricted by assuming transfers are confined to the entire set of contributors, then Theorem 3 shows that neighborhood homogeneity of the set of contributors characterizes the neutrality of relatively small transfers among contributors.

In particular, given that the set of contributors in a complete network is always neighborhood homogenous, it follows that:

Corollary 3. *(Bergstrom, Blume, and Varian [6]) Suppose the public good is pure, that is, the network is complete, and assume both public and private goods are normal goods. Then relatively small transfers among contributors are neutral.*

Corollary 3 establishes the standard neutrality result of Warr [38] and Bergstrom, Blume, and Varian [6] by following an alternative approach based on analysis of the private provision of public goods on networks. More generally, the light shed by Theorem 3 on the neutrality of income redistribution in networks is insightful. In interpretation, although one might not expect the neutrality result from the usual pure public good setting to extend to other settings with local interaction patterns accounted for, it is still important to point out that the neutrality result has some serious limitations as it fails in all networks where the set of contributors is not neighborhood homogenous.

5.2. **Invariance of aggregate provision in networks.** A fundamental question in public economics, especially in the charitable giving literature, is how private provision of public goods responds to public provision. In this regard, the standard model of private provision of pure public goods identifies the testable prediction that public provision financed by lump-sum taxation completely crowds-out "*dollar-for-dollar*" private provision. Intuitively, the reason that crowding-out is complete is that, since public provision is financed by lump-sum taxation, in view of budget balance it may be simply interpreted as a form of income redistribution, which, being neutral in the case of a pure public good, leaves aggregate provision unchanged. Yet, while the aggregate public good provision coincides with the individual public good consumption enjoyed by each consumer in the case of a pure public good, this may no longer be true when the public good is locally enjoyed. As a consequence, the invariance of aggregate provision is characterized differently from neutrality (invariance of individual private and public good consumption of each consumer) in general networks.

In the following, we show that the impact of income redistribution on the aggregate public good provision in general networks is closely related to the Bonacich centrality. Bonacich centrality, due to Bonacich [9], has been widely employed in the theoretical and empirical economics of networks literature since it was first shown by Ballester, Calvó-Armengol, and Zenou [5] to be proportional to the Nash equilibrium actions of a linear

best-reply game. Bonacich centrality is defined, for a ψ that is small in magnitude, by

$$\mathbf{b}(\mathbf{G}, \psi) = (\mathbf{I} - \psi\mathbf{G})^{-1}\mathbf{1} = \sum_{k=0}^{+\infty} \psi^k \mathbf{G}^k \mathbf{1}, \qquad (5.1)$$

where $\mathbf{1}$ is the all-ones vector. Since the ij^{th} entry of the matrix $\mathbf{G}^k$ denotes the number of walks of length k in $\mathbf{g}$ emanating from i and terminating at j, it follows that the ith coordinate $b_i(\mathbf{G}, \psi)$ of Bonacich centrality is the sum of all walks in $\mathbf{g}$ emanating from i weighted by ψ to the power of their length. In that sense, Bonacich centrality is interpreted as a measure of prestige, power, and network influence.

Let $Q^* = \sum_{i=1}^n q_i^*$ (resp. $Q^{\mathbf{t}} = \sum_{i=1}^n q_i^{\mathbf{t}}$) denote the aggregate public good provision before income redistribution (resp. after income redistribution). The next result follows directly from Proposition 1.

Proposition 3. *Assume network normality. Then for any relatively small transfer* $\mathbf{t}$ *it holds that*

$$Q^{\mathbf{t}} - Q^* = \mathbf{b}^{dw}(\mathbf{G}_C, -\mathbf{A}_C)^T (\mathbf{I} - \mathbf{A}_C)\mathbf{t}_C, \quad where \quad \mathbf{b}^{dw}(\mathbf{G}_C, -\mathbf{A}_C)^T = \mathbf{1}^T (\mathbf{I} + \mathbf{A}_C\mathbf{G}_C)^{-1}.$$

In the case of nonlinear best-reply functions, the above proposition shows that the impact of income redistribution on aggregate provision is determined by $\mathbf{b}^{dw}(\mathbf{G}_C, -\mathbf{A}_C)$, which may be thought of as a "diagonally weighted" Bonacich centrality since consumers carry different diagonal weights. Recent contributions of Golub and Lever [26] and Candogan, Bimpikis, and Ozdaglar [13] have proposed other useful generalizations of Bonacich centrality, which, provided that they are well defined, characterize equilibria outcomes in some classes of games. In this paper, quite differently, the diagonally weighted Bonacich centrality summarizes information concerning each consumer's impact on aggregate provision. Understandably, such information is useful in outlining conditions on the network structure for the invariance of aggregate provision to hold.

Theorem 4. *Assume network normality. Then aggregate public good provision is invariant to relatively small transfers among contributors if and only if the set of contributors is a clique.*[12]

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[12]A clique is a fully connected subset of vertices.

Theorem 4 shows that the invariance of aggregate provision to relatively small transfers among contributors is equivalent to the set of contributors being a clique, which obviously holds if the set of contributors is neighborhood homogenous as stipulated by neutrality.

To conclude, although the invariance of aggregate provision is a weaker requirement than neutrality it still imposes the considerable restriction that contributors must be closely tied as a clique. Notwithstanding, as far as testable implications are concerned, this may sound quite promising in view of the ample empirical and experimental evidence that public provision crowds-out private provision at a rate significantly less than complete "*dollar-for-dollar*" (see, for example, Andreoni [4] for a discussion of the various results in the literature).

6. Gorman polar form preferences

In order to obtain further insights into the impact of income redistribution, as well as to gain a basic understanding of the composition of the set of contributors, we confine our attention to particular preferences of consumers, the so-called Gorman polar form, of which Cobb–Douglas preferences are a special case.[13]

Gorman polar form preferences. Preferences of consumers yield Engel curves with identical constant slopes, that is, $\gamma_i'(\cdot) = 1 - a$ for each consumer $i = 1, \ldots, n$.

For Gorman polar form preferences, consumers have constant, as well as identical, marginal propensities to consume the public good. More precisely, it holds that the autarkic public good provision of consumer i is

$$\hat{q}_i = (1 - a)w_i + q_i^s, \tag{6.1}$$

where q_i^s is the intercept of the Engel curve. In interpretation, although consumers may allocate their income differently between private and public goods, each consumer must allocate an additional unit of income in the same proportion.

6.1. **The set of contributors.** So far we have carried out our analysis of the private provision of public goods on networks without discussing the important issue of who contributes to the public good. In the standard model of private provision of a pure

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[13]In the literature, the theoretical and empirical attraction of preferences of the Gorman polar form is that one can treat a society of utility-maximizing consumers as a single consumer. Such a concept, albeit different, bears a great methodological similarity to the concept of potential games of Monderer and Shapley [35].

public good, the distribution of income and preferences of consumers, together, determine the composition of the set of contributors. For instance, Bergstrom, Blume, and Varian [6] show that if consumers' preferences are identical, then only the upper tail of the income distribution contributes to the public good. In the private provision of public goods on networks, matters become more complicated since the network structure is also a determinant of the composition of the set of contributors. As highlighted in Bramoullé, Kranton, and D'Amours [11], if consumers have identical linear best-reply functions and substitute little of their neighbors' provisions so that initially they are all contributors, then gradually increasing the degree of substitutability between own and neighbors' provisions induces more (Bonacich) central consumers to cease contributing first to the public good.

Now, for the particular case of Gorman polar form preferences, we explore how income distribution, consumers' preferences, and network structure interact to yield Nash equilibrium. Indeed, in view of (2.2) and (6.1), the public good provision of each consumer i can be expressed as:

$$q_i = \max\{\hat{q}_i - aQ_{-i}, 0\}.$$

Let $\hat{\mathbf{q}} = (\hat{q}_1, \ldots, \hat{q}_n)^T$ denote the autarkic provisions of consumers and $\mathbf{G}_{N \setminus C, C}$ denote edges from non-contributors to contributors. The following proposition simply extends to arbitrary autarkic provisions the conditions for Nash equilibrium of proposition 1 in Bramoullé, Kranton, and D'Amours [11] and, as shown there, it can provide a procedure to identify the set of contributors in Nash equilibrium.

Proposition 4. *Assume network normality holds and preferences are of the Gorman polar form. Then* $\mathbf{q}^*$ *is the unique Nash equilibrium if and only if*

(1) $(\mathbf{I} + a\mathbf{G}_C)\mathbf{q}_C^* = \hat{\mathbf{q}}_C,$

(2) $a\mathbf{G}_{N \setminus C, C}\mathbf{q}_C^* \geq \hat{\mathbf{q}}_{N \setminus C}.$

From Proposition 4 it follows that $\hat{\mathbf{q}}$ yields a Nash equilibrium where all consumers are contributors whenever condition (1) holds for the entire set of consumers, that is, $(\mathbf{I} + a\mathbf{G})\mathbf{q}^* = \hat{\mathbf{q}}$, which is equivalent to $\hat{\mathbf{q}}$ being spanned by the columns of the matrix $\mathbf{I} + a\mathbf{G}$. In particular, a $\hat{\mathbf{q}}$ proportional to the principal eigenvector[14] $\mathbf{v}_1$ always yields a Nash equilibrium where all consumers are contributors since $\mathbf{G}\hat{\mathbf{q}} = \lambda_{\max}(\mathbf{G})\hat{\mathbf{q}}$ implies

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[14]Also known as the eigenvector centrality.

that $(\mathbf{I} + a\mathbf{G})\frac{\hat{\mathbf{q}}}{1+a\lambda_{\max}(\mathbf{G})} = \hat{\mathbf{q}}$. In this case for each consumer i it holds that

$$\hat{q}_i = \frac{1}{\lambda_{\max}(\mathbf{G})} \sum_{j \in \mathcal{N}_i} \hat{q}_j,$$

which has a recursive nature as it stipulates that the autarkic provision of a consumer is proportional to the sum of his neighbors' autarkic provisions. Intuitively, all other things being constant, a consumer whose neighbors have high autarkic provision must have a proportionally high autarkic provision himself in order to contribute to the public good.

Note that in a regular network almost equal autarkic provisions, being almost proportional to the principal eigenvector $\mathbf{v}_1 = \mathbf{1}$, will always lead to a Nash equilibrium where all consumers are contributors. The following example illustrates in a star network the set of contributors for different income distributions:

Example 3. There are four consumers with identical Cobb–Douglas preferences $u_i(x_i, q_i + Q_{-i}) = x_i^{\frac{1}{2}}(q_i + Q_{-i})^{\frac{1}{2}}$ for $i = 1, 2, 3, 4$, located on a star network $\mathbf{g}$ with consumer 1 being the center. It is worth recalling that Cobb–Douglas preferences fall within the family of Gorman polar form preferences, and in particular it holds that $\hat{\mathbf{q}} = (1 - a)\mathbf{w}$. Since $\lambda_{\max}(\mathbf{G}) = -\lambda_{\min}(\mathbf{G}) = \sqrt{3}$,[15] network normality holds and guarantees a unique Nash equilibrium. Note that $\mathbf{w}_1$ is proportional to $\mathbf{v}_1 = (\frac{3\sqrt{2}}{6}, \frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{6}, \frac{\sqrt{6}}{6})$; hence all consumers are contributors. This also holds for income distributions almost proportional to $\mathbf{v}_1$ but, obviously, not $\mathbf{w}_2$ or $\mathbf{w}_3$. As usual, vertices in black represent contributors while vertices in white represent non-contributors.
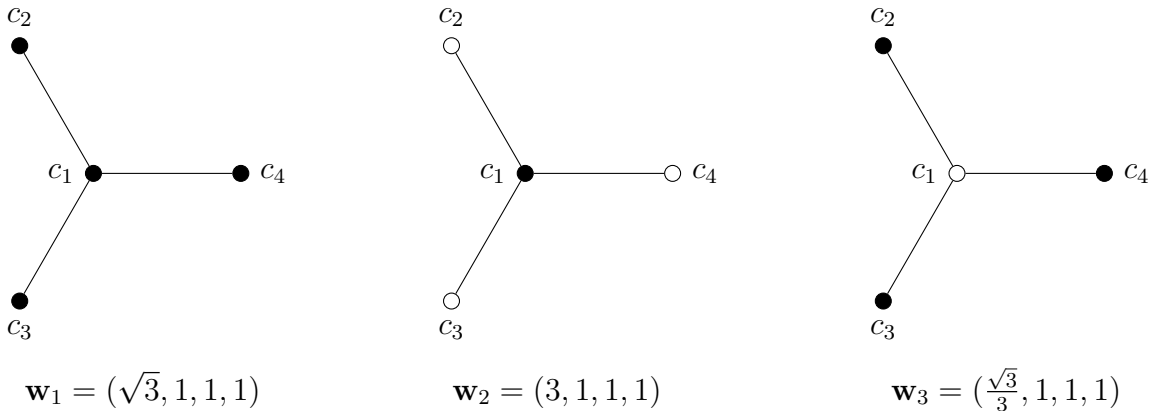

$$\mathbf{w}_1 = (\sqrt{3}, 1, 1, 1) \qquad \mathbf{w}_2 = (3, 1, 1, 1) \qquad \mathbf{w}_3 = (\tfrac{\sqrt{3}}{3}, 1, 1, 1)$$

Figure 3: The set of contributors in a star network.

[15]More generally, provided that $\mathbf{G}$ is connected, the equality $-\lambda_{\min}(\mathbf{G}) = \lambda_{\max}(\mathbf{G})$ holds if and only if $\mathbf{G}$ is a bipartite network (see, for example, Cvetković, Rowlinson, and Simić ([17], p.15)).

6.2. **Aggregate provision and Bonacich centrality.** Now we further investigate the impact of income redistribution on the aggregate provision of public goods in the particular case of Gorman polar form preferences. Recall that, in view of Proposition 3, the impact of income redistribution on the aggregate provision is determined by a diagonally weighted Bonacich centrality, which obviously coincides with the original Bonacich centrality for Gorman polar form preferences. Hence the next result.

Proposition 5. *Assume network normality holds and preferences are of the Gorman polar form. Given a component of contributors S, for any relatively small S-confined transfer* $\mathbf{t}$ *it holds that*

$$Q^{\mathbf{t}} - Q^* = (1 - a)\mathbf{b}(\mathbf{G}_S, -a) \cdot \mathbf{t}_S = (1 - a)\sum_{i \in S} b_i(\mathbf{G}_S, -a) \cdot t_i.$$

Proposition 5 shows that, within each component of contributors, a transfer from a low Bonacich centrality contributor to a high Bonacich centrality contributor will always raise the aggregate provision, while a transfer between contributors with identical Bonacich centrality has no effect on the aggregate provision.

Given that Bonacich centrality is key to the impact of income redistribution on the aggregate public good provision in the particular case of Gorman polar form preferences, it may be useful to explore other possible expressions of Bonacich centrality. For this purpose, we turn to spectral graph theory and introduce the concept of main eigenvalue, due to Cvetković [16]. An eigenvalue μ_i of the adjacency matrix $\mathbf{G}$ is called a main eigenvalue if it has a (unit) eigenvector $\mathbf{u}_i$ not orthogonal to $\mathbf{1}$, that is, $\mathbf{1} \cdot \mathbf{u}_i \neq 0$. Since for eigenvalues with multiplicity greater than one we can choose the corresponding eigenvectors in such a way that, at most, one of them is not orthogonal to $\mathbf{1}$, without loss of generality, we may also assume that $\mathbf{u}_i \in \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, the orthonormal basis of $\mathbb{R}^n$ formed by the eigenvectors of $\mathbf{G}$. In addition, it also holds that the main eigenvalues of $\mathbf{G}$ are distinct and may, consequently, be ordered $\mu_s < \ldots < \mu_2 < \mu_1$. Recall that by the Perron–Frobenius Theorem, the principal eigenvector $\mathbf{v}_1$ has positive entries and, hence, $\mu_1 = \lambda_{\max}(\mathbf{G})$. The set of main eigenvalues $\mathcal{M} = \{\mu_1, \mu_2, \ldots, \mu_s\}$ is called the "main part of the spectrum" and is shown by Harary and Schwenk [29] to be the minimum set of eigenvalues the span of whose eigenvectors includes $\mathbf{1}$.

Proposition 6. *Assume $a \in]0, -\frac{1}{\lambda_{\min}(\mathbf{G})}[$. Then it holds that*

$$\mathbf{b}(\mathbf{G}, -a) = \sum_{i=1}^{s} \frac{\mathbf{1} \cdot \mathbf{u}_i}{1 + a\mu_i} \mathbf{u}_i = \sum_{i=1}^{s} \frac{1}{1 + a\mu_i} \prod_{k=1, k \neq i}^{s} \left(\frac{\mathbf{G} - \mu_k \mathbf{I}}{\mu_i - \mu_k} \right) \mathbf{1}.$$

As far as we know, Proposition 6 is the first to express Bonacich centrality in terms of the main part of the spectrum. Note that the non-main eigenvectors, being orthogonal to $\mathbf{1}$, do not contribute to Bonacich centrality; hence the reduced expression of Bonacich centrality. Proposition 6 also further expands on the links between Bonacich centrality and the main part of the spectrum to express Bonacich centrality as a sum of the number of walks of length k, rather than unbounded as in (5.1), strictly bounded by s, the cardinality of $\mathcal{M}$. That is, Bonacich centrality satisfies $\mathbf{b}(\mathbf{G}, -a) = \sum_{k=0}^{s-1} \phi_k \mathbf{G}^k \mathbf{1}$, where ϕ_k are constants determined by a and $\mathcal{M}$.

Understandably, as shown below, our alternative expressions of Bonacich centrality can be useful in computing Bonacich centrality for networks with few main eigenvalues. For instance:

Proposition 7. *Assume $a \in]0, -\frac{1}{\lambda_{\min}(\mathbf{G})}[$. Then the following are equivalent:*

  (i) *consumers have identical Bonacich centrality,*

 (ii) *the network is regular,*

(iii) *Bonacich centrality and degree centrality are proportional,*

(iv) *the network has exactly one main eigenvalue.*

Therefore, in view of Proposition 5, for Gorman polar form preferences, the invariance of aggregate provision to relatively small transfers confined to a component of contributors is equivalent to the component being regular.

Corollary 4. *Assume network normality holds and preferences are of the Gorman polar form. Given a component of contributors S, aggregate public good provision is invariant to relatively small S-confined transfers if and only if $\mathbf{g}_S$ is regular.*

Given that regular components of contributors are characterized by having exactly one main eigenvalue, it may be informative to learn about the patterns of change in aggregate provision in components of contributors with exactly two main eigenvalues, which are the first instance in which the invariance of aggregate provision fails to hold in the case of

Gorman polar form preferences.[16] Although, in view of Proposition 7, Bonacich centrality and degree centrality are not proportional for components of contributors with exactly two main eigenvalues, as shown in the following result, their impacts on transfers are (negatively) proportional.

Proposition 8. *Assume network normality holds and preferences are of the Gorman polar form. Given a component of contributors S with exactly two main eigenvalues, for any relatively small S-confined transfer* $\mathbf{t}$ *it holds that*

$$Q^{\mathbf{t}} - Q^* = \frac{-a(1-a)}{(1+a\mu_1)(1+a\mu_2)}\mathbf{d}_S \cdot \mathbf{t}_S,$$

where $\mathbf{d}_S$ *denotes degree centrality in* $\mathbf{g}_S$.

Hence, for components of contributors with, at most, two main eigenvalues ($s = 1, 2$), our results generate precise and clear predictions about the impact of income redistribution on the aggregate public good provision. Indeed, it turns out that the change in aggregate public good provision is determined by degree centrality rather than the more sophisticated Bonacich centrality. In this case, our results are in line with similar observations in the literature on the economics of networks. Galeotti, Goyal, Jackson, Vega-Redondo, and Yariv [25] emphasize the importance of degree centrality as a measure of immediate influence and local knowledge of the network and König, Tessone, and Zenou [33] present a model of dynamic network formation where the degree and Bonacich centrality rankings coincide.

7. Conclusion

In this paper, we have established that beyond a special network architecture in the neighborhood of the set of contributors, consumers are no longer able to undo the impact of income redistribution by changes in their public good provision. Our result restores, to some extent, the role of income redistribution as a main channel for policy intervention in the private provision of public goods.

In the literature, various lines of research have been proposed to counter the neutrality of income redistribution in the private provision of public goods given that, among

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[16]The simplest examples of networks with exactly two main eigenvalues are the complete bipartite networks with two unequal sides such as star networks and fully connected core-periphery networks. See Hagos [28] and subsequent literature for a treatment of networks with exactly two main eigenvalues.

other things, its closely related prediction of complete crowding-out of public provision is sharply contradicted by empirical and experimental evidence. Often, the reason neutrality breaks down appears to hinge on the imperfect substitution among the various consumers' private provisions. For instance, consumers may have access to different technologies to produce the public good or may care differently in their preferences about their own and other consumers' public good provisions (see Andreoni [3]). Our result suggests that neutrality fails for similar reasons in the private provision of public goods on networks. However, unlike the various technological and behavioral explanations in the literature, in the absence of a neighborhood homogeneous set of contributors, the lack of perfect substitution seems to be brought about by a heterogeneity in the network structure. Therefore, in view of the large body of empirical and experimental research on the impact of income redistribution, including the rate of crowding-out of public provision, this paper may help identify some testable predictions, whenever local interaction patterns are accounted for.

Finally, it is worth noting that since most of our findings regarding the private provision of public goods on networks, including existence, uniqueness, and stability of a Nash equilibrium, are based on properties of nonlinear best-reply functions, they may be accommodated within the general class of games of strategic substitutes on networks, which is a cornerstone of the study of many areas of economics (see, for example, Bulow, Geanakoplos, and Klemperer [12] and Bramoullé, Kranton, and D'Amours [11]).

Acknowledgments

Appendices

In Appendix A we provide all our proofs. In Appendix B we provide a further discussion of the result on the uniqueness of a Nash equilibrium.

APPENDIX A

Proof of Theorem 1. The existence of a Nash equilibrium is guaranteed by Brouwer's fixed point theorem. Suppose there are two Nash equilibria

$$\mathbf{q}^1 = (q_1^1, \ldots, q_n^1)^T \neq (q_1^2, \ldots, q_n^2)^T = \mathbf{q}^2;$$

then for each consumer $i = 1, \ldots, n$, it holds that

$$q_i^1 = \max\{\gamma_i(w_i + Q_{-i}^1) - Q_{-i}^1, 0\} \quad \text{and} \quad q_i^2 = \max\{\gamma_i(w_i + Q_{-i}^2) - Q_{-i}^2, 0\}.$$

Since $\mathbf{q}^1 \neq \mathbf{q}^2$, it follows that the set $\mathcal{D} = \{i \in N \mid Q_{-i}^1 \neq Q_{-i}^2\} \neq \emptyset$. Moreover, from the mean value theorem, for each $i \in \mathcal{D}$ there exists a real number β_i such that

$$\gamma_i(w_i + Q_{-i}^1) - \gamma_i(w_i + Q_{-i}^2) = \gamma_i'(\beta_i)(Q_{-i}^1 - Q_{-i}^2). \tag{A.1}$$

Note that if one sets $\beta_i = Q_{-i}^1 = Q_{-i}^2$ for each $i \notin D$, then (A.1) holds for each $i \in N$. Let $a = \max_{i \in \mathcal{N}}\{1 - \gamma_i'(\beta_i)\}$; then from the network normality assumption, for each consumer $i = 1, \ldots, n$, it holds that

$$0 < 1 - \gamma_i'(\beta_i) \leq a < -\frac{1}{\lambda_{\min}(\mathbf{G})}. \tag{A.2}$$

For each consumer $i = 1, \ldots, n$, define s_i as follows:

$$s_i = \begin{cases} 1 & \text{if } Q_{-i}^1 \leq Q_{-i}^2, \\ -1 & \text{otherwise.} \end{cases}$$

Hence for each consumer $i = 1, \ldots, n$, it holds[17]

$$
\begin{aligned}
0 \leq s_i(q_i^1 - q_i^2) &= |q_i^1 - q_i^2| \\
&= |\max\{\gamma_i(w_i + Q_{-i}^1) - Q_{-i}^1, 0\} - \max\{\gamma_i(w_i + Q_{-i}^2) - Q_{-i}^2, 0\}| \\
&\leq |(\gamma_i(w_i + Q_{-i}^1) - Q_{-i}^1) - (\gamma_i(w_i + Q_{-i}^2) - Q_{-i}^2)| \\
&= |(1 - \gamma_i'(\beta_i))(Q_{-i}^2 - Q_{-i}^1)| \leq a|Q_{-i}^2 - Q_{-i}^1| = s_i a(Q_{-i}^2 - Q_{-i}^1).
\end{aligned}
$$

---

[17]Note that for $z \in \mathbb{R}$ it holds that $\max\{z, 0\} = \frac{|z|+z}{2}$. Hence, it holds that $|\max\{z^1, 0\} - \max\{z^2, 0\}| = |\frac{|z^1|+z^1}{2} - \frac{|z^2|+z^2}{2}| \leq \frac{1}{2}(||z^1| - |z^2|| + |z^1 - z^2|) \leq |z^1 - z^2|$.

Let $\mathbf{S} = \mathrm{diag}(s_1, \ldots, s_n)$ be the diagonal matrix whose diagonal entries are s_i; then it follows from the above inequalities that[18]

$$
\begin{aligned}
0 \leq \mathbf{S}(\mathbf{q}^1 - \mathbf{q}^2) &= (s_1(q_1^1 - q_1^2), \ldots, s_n(q_n^1 - q_n^2))^T \\
&\leq a(s_1(Q_{-1}^2 - Q_{-1}^1), \ldots, s_n(Q_{-n}^2 - Q_{-n}^1))^T \\
&= a\mathbf{S}(\sum_{j \in N_1}(q_j^2 - q_j^1), \ldots, \sum_{j \in N_n}(q_j^2 - q_j^1))^T = a\mathbf{S}\mathbf{G}(\mathbf{q}^2 - \mathbf{q}^1).
\end{aligned}
$$

Rearranging terms, it follows that $0 \leq \mathbf{S}(\mathbf{q}^1 - \mathbf{q}^2)$ and $\mathbf{S}(\mathbf{I} + a\mathbf{G})(\mathbf{q}^1 - \mathbf{q}^2) \leq 0$, which together imply

$$
(\mathbf{q}^1 - \mathbf{q}^2)^T(\mathbf{I} + a\mathbf{G})(\mathbf{q}^1 - \mathbf{q}^2) = (\mathbf{S}(\mathbf{q}^1 - \mathbf{q}^2))^T\mathbf{S}(\mathbf{I} + a\mathbf{G})(\mathbf{q}^1 - \mathbf{q}^2) \leq 0.
$$

Given that $\mathbf{q}^1 - \mathbf{q}^2 \neq 0$ it follows that $(\mathbf{I} + a\mathbf{G})$ is not positive definite, which is a contradiction since (A.2). Therefore, there exists a unique Nash equilibrium.□

Proof of Theorem 2. We investigate stability by linearizing around the Nash equilibrium and then examining the location of the eigenvalues of the Jacobian matrix

$$
\mathbf{J}_C = \mathbf{U}(\frac{\partial\{f_i(Q_{-i}) - q_i\}}{\partial q_j} \mid \mathbf{q}_C^*)_{i,j \in C} = -\mathbf{U}(\mathbf{I} + \mathbf{B}\mathbf{G}_C),
$$

where $\mathbf{B} = \mathrm{diag}(1 - \gamma_i'(w_i + Q_{-i}^*))_{i \in C}$ and $\mathbf{U} = \mathrm{diag}(\sigma_i)_{i \in C}$. The unique Nash equilibrium $\mathbf{q}^*$ is locally asymptotically stable if all eigenvalues of the Jacobian matrix $\mathbf{J}_C$ have negative real parts, which we obtain from the lemma below.

Lemma 1. *Assume network normality. Then the eigenvalues of $\mathbf{U}(\mathbf{I} + \mathbf{B}\mathbf{G}_C)$ are positive real numbers.*

Proof of Lemma 1. From the sharp bounds provided by Ostrowski [36], it holds that the eigenvalues of the symmetric matrix $\mathbf{B}^{\frac{1}{2}}\mathbf{G}_C\mathbf{B}^{\frac{1}{2}}$ are given by $\theta_i\lambda_i$, where λ_i is an eigenvalue of $\mathbf{G}_C$ and θ_i lies between the smallest and the largest eigenvalues of $\mathbf{B}$. From the network normality assumption, it follows that for each $i \in C$

$$
0 < \min_{i \in C}\{1 - \gamma_i'(w_i + Q_{-i}^*)\} \leq \theta_i \leq \max_{i \in C}\{1 - \gamma_i'(w_i + Q_{-i}^*)\} < -\frac{1}{\lambda_{\min}(\mathbf{G})}.
$$

---

[18]Consider $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$; then $\mathbf{x} \geq 0$ if $x_i \geq 0$ for each $i = 1, \ldots, n$ and $\mathbf{x} > 0$ if $\mathbf{x} \geq 0$ and $x_i > 0$ for some i.

From the interlacing eigenvalue theorem,[19] it follows that $\lambda_{\min}(\mathbf{G}) \leq \lambda_{\min}(\mathbf{G}_C)$. Hence, the eigenvalues of $\mathbf{I} + \mathbf{B}^{\frac{1}{2}}\mathbf{G}_C\mathbf{B}^{\frac{1}{2}}$, given by $1 + \theta_i\lambda_i$, are positive since for each $i \in C$, it holds that $0 = 1 - 1 < 1 + \theta_i\lambda_{\min}(\mathbf{G}) \leq 1 + \theta_i\lambda_{\min}(\mathbf{G}_C) \leq 1 + \theta_i\lambda_i$. From Ostrowski [36] again, it follows that the eigenvalues of the symmetric matrix $\mathbf{U}^{\frac{1}{2}}(\mathbf{I} + \mathbf{B}^{\frac{1}{2}}\mathbf{G}_C\mathbf{B}^{\frac{1}{2}})\mathbf{U}^{\frac{1}{2}}$ are positive. Hence, the matrix $\mathbf{U}(\mathbf{I} + \mathbf{B}\mathbf{G}_C)$ also has positive eigenvalues, being similar to $\mathbf{U}^{\frac{1}{2}}(\mathbf{I} + \mathbf{B}^{\frac{1}{2}}\mathbf{G}_C\mathbf{B}^{\frac{1}{2}})\mathbf{U}^{\frac{1}{2}}$ since

$$\mathbf{U}(\mathbf{I} + \mathbf{B}\mathbf{G}_C) = (\mathbf{U}\mathbf{B})^{\frac{1}{2}}[\mathbf{U}^{\frac{1}{2}}(\mathbf{I} + \mathbf{B}^{\frac{1}{2}}\mathbf{G}_C\mathbf{B}^{\frac{1}{2}})\mathbf{U}^{\frac{1}{2}}](\mathbf{U}\mathbf{B})^{-\frac{1}{2}}.\square$$

Proof of Proposition 1. From (2.2), it follows that for each consumer $i \in C$

$$q_i^{\mathbf{t}} - q_i^* = (\gamma_i(w_i + t_i + Q_{-i}^{\mathbf{t}}) - Q_{-i}^{\mathbf{t}}) - (\gamma_i(w_i + Q_{-i}^*) - Q_{-i}^*).$$

From the mean value theorem, it follows that for each $i \in C$ such that $t_i + Q_{-i}^{\mathbf{t}} \neq Q_{-i}^*$, there exists a real number β_i such that $q_i^{\mathbf{t}} - q_i^* = \gamma_i'(\beta_i)(t_i + Q_{-i}^{\mathbf{t}} - Q_{-i}^*) - (Q_{-i}^{\mathbf{t}} - Q_{-i}^*)$. Let $\beta_i = Q_{-i}^*$ if $t_i + Q_{-i}^{\mathbf{t}} = Q_{-i}^*$; then for each consumer $i \in C$ it holds that

$$q_i^{\mathbf{t}} - q_i^* + (1 - \gamma_i'(\beta_i)) \sum_{j \in N_i \cap C} (q_j^{\mathbf{t}} - q_j^*) = \gamma_i'(\beta_i)t_i.$$

Consequently, it holds that

$$(\mathbf{I} + \mathbf{A}_C\mathbf{G}_C)(\mathbf{q}_C^{\mathbf{t}} - \mathbf{q}_C^*) = (\mathbf{I} - \mathbf{A}_C)\mathbf{t}_C,$$

where $\mathbf{A}_C = \mathrm{diag}(1 - \gamma_i'(\beta_i))_{i \in C}$. Applying Lemma 1 for $\mathbf{B} = \mathbf{A}_C$ and $\mathbf{U} = \mathbf{I}$, it follows that $\mathbf{I} + \mathbf{A}_C\mathbf{G}_C$ is invertible since it has positive eigenvalues. Hence,

$$\mathbf{q}_C^{\mathbf{t}} - \mathbf{q}_C^* = (\mathbf{I} + \mathbf{A}_C\mathbf{G}_C)^{-1}(\mathbf{I} - \mathbf{A}_C)\mathbf{t}_C.\square$$

Proof of Proposition 2. First, note that subtracting the budget constraints before and after a relatively small transfer $\mathbf{t}$, for a consumer $i \in N$, it holds that

$$q_i^{\mathbf{t}} - q_i^* = (w_i + t_i - x_i^{\mathbf{t}}) - (w_i - x_i^*) = t_i - (x_i^{\mathbf{t}} - x_i^*). \tag{A.3}$$

Moreover, if $q_i^{\mathbf{t}} - q_i^* = t_i$, for each consumer $i \in N$, then it holds that

$$Q_i^{\mathbf{t}} - Q_i^* = q_i^{\mathbf{t}} + \sum_{j \in \mathcal{N}_i} q_j^{\mathbf{t}} - q_i^* - \sum_{j \in \mathcal{N}_i} q_j^* = t_i + \sum_{j \in \mathcal{N}_i} t_j. \tag{A.4}$$

---

[19]See, for example, Horn and Johnson ([30], p.185).

Now, suppose that a relatively small transfer $\mathbf{t}$ is neutral, that is, for each consumer $i \in N$, it holds that $(x_i^{\mathbf{t}}, Q_i^{\mathbf{t}}) = (x_i^*, Q_i^*)$. Then, in view of (A.3), for each consumer $i \in N$, since $x_i^{\mathbf{t}} = x_i^*$ it follows that $q_i^{\mathbf{t}} - q_i^* = t_i$. In particular, for each consumer $i \notin C$, since $q_i^{\mathbf{t}} = q_i^* = 0$ it holds that $t_i = 0$. Thus the transfer $\mathbf{t}$ is C-confined. Moreover, since $q_i^{\mathbf{t}} - q_i^* = t_i$ and $Q_i^{\mathbf{t}} = Q_i^*$, for each consumer $i \in N$, in view of (A.4), it follows that $t_i + \sum_{j \in \mathcal{N}_i} t_j = 0$ for each consumer $i \in N$. Hence the transfer $\mathbf{t}$ is neighborhood balanced.

Conversely, if a relatively small transfer $\mathbf{t}$ is C-confined and neighborhood balanced, then for each consumer $i \in N$ it holds that

$$t_i + \sum_{j \in \mathcal{N}_i} t_j = t_i + \sum_{j \in \mathcal{N}_i \cap C} t_j = 0. \tag{A.5}$$

From Proposition 1,

$$\begin{aligned}
\mathbf{q}_C^{\mathbf{t}} - \mathbf{q}_C^* &= (\mathbf{I} + \mathbf{A}_C \mathbf{G}_C)^{-1}(\mathbf{I} - \mathbf{A}_C)\mathbf{t}_C = (\mathbf{I} + \mathbf{A}_C \mathbf{G}_C)^{-1}(\mathbf{I} + \mathbf{A}_C \mathbf{G}_C - \mathbf{A}_C - \mathbf{A}_C \mathbf{G}_C)\mathbf{t}_C \\
&= (\mathbf{I} + \mathbf{A}_C \mathbf{G}_C)^{-1}(\mathbf{I} + \mathbf{A}_C \mathbf{G}_C)\mathbf{t}_C - (\mathbf{I} + \mathbf{A}_C \mathbf{G}_C)^{-1}\mathbf{A}_C(\mathbf{I} + \mathbf{G}_C)\mathbf{t}_C = \mathbf{t}_C,
\end{aligned}$$

which, together with $q_i^{\mathbf{t}} - q_i^* = 0 = t_i$ for each consumer $i \notin C$, implies that $q_i^{\mathbf{t}} - q_i^* = t_i$ for each consumer $i \in N$. Therefore, for each consumer $i \in N$, it holds that $x_i^{\mathbf{t}} = x_i^*$ since (A.3) and $Q_i^{\mathbf{t}} = Q_i^*$ since (A.4) and (A.5). $\square$

Proof of Theorem 3. First, suppose that all relatively small S-confined transfers are neutral and S is not neighborhood homogeneous. Hence there exists i, j in S such that for a consumer $k \in N$ it holds that $k \in i \cup \mathcal{N}_i$ but $k \notin j \cup \mathcal{N}_j$. Let $\mathbf{t}^{i,j} \neq 0$ denote a relatively small bilateral transfer, that is, $t_l^{i,j} = 0$ for all $l \notin \{i, j\}$. Obviously, the transfer $\mathbf{t}^{i,j}$ is S-confined but not neighborhood balanced since in the neighborhood of consumer k it holds that $t_k^{i,j} + \sum_{l \in \mathcal{N}_k} t_l^{i,j} = t_i^{i,j} \neq 0$. From Proposition 2, it follows that $\mathbf{t}^{i,j}$ is not neutral, which is a contradiction.

Conversely, suppose that S is neighborhood homogeneous and that there exists a relatively small S-confined transfer $\mathbf{t}$ that is not neutral. From Proposition 2, the transfer $\mathbf{t}$ is not neighborhood balanced and therefore for a consumer $k \in N$ it holds that $t_k + \sum_{l \in \mathcal{N}_k} t_l \neq 0$. Since $\mathbf{t}$ is S-confined it follows that $\{k \cup \mathcal{N}_k\} \cap S \neq S$ or $\emptyset$ (otherwise, $t_k + \sum_{l \in \mathcal{N}_k} t_l = 0$). Hence there exists i, j in S such that $i \in k \cup \mathcal{N}_k$ but $j \notin k \cup \mathcal{N}_k$. Therefore, it holds that $k \in i \cup \mathcal{N}_i$ but $k \notin j \cup \mathcal{N}_j$, which is a contradiction. $\square$

Proof of Theorem 4. From Proposition 3, it holds that

$$Q^{\mathbf{t}} - Q^* = \mathbf{1}^T(\mathbf{I} + \mathbf{A}_C\mathbf{G}_C)^{-1}(\mathbf{I} - \mathbf{A}_C)\mathbf{t}_C.$$

Hence, the aggregate public good provision is invariant to income redistribution if and only if the vectors $\mathbf{1}^T(\mathbf{I} - \mathbf{A}_C)^{-1}(\mathbf{I} + \mathbf{A}_C\mathbf{G}_C)$ and $\mathbf{1}^T$ are proportional, that is, if and only if there exists a nonnull real number λ such that

$$\begin{aligned}
\mathbf{1}^T &= \lambda\mathbf{1}^T(\mathbf{I} - \mathbf{A}_C)^{-1}(\mathbf{I} + \mathbf{A}_C\mathbf{G}_C) = \lambda\mathbf{1}^T(\mathbf{I} - \mathbf{A}_C)^{-1}(\mathbf{I} - \mathbf{A}_C + \mathbf{A}_C + \mathbf{A}_C\mathbf{G}_C) \\
&= \lambda\mathbf{1}^T + \lambda\mathbf{1}^T(\mathbf{I} - \mathbf{A}_C)^{-1}\mathbf{A}_C(\mathbf{I} + \mathbf{G}_C),
\end{aligned}$$

which is equivalent to

$$\mathbf{1}^T(\mathbf{I} - \mathbf{A}_C)^{-1}\mathbf{A}_C(\mathbf{I} + \mathbf{G}_C) = \frac{1 - \lambda}{\lambda}\mathbf{1}^T.$$

Recall that $\mathbf{A}_C = \mathrm{diag}(1 - \gamma_i'(\beta_i))_{i \in C}$; hence $\mathbf{1}^T(\mathbf{I} - \mathbf{A}_C)^{-1}\mathbf{A}_C = (\frac{1 - \gamma_i'(\beta_i)}{\gamma_i'(\beta_i)})_{i \in C}^T$. Therefore, the aggregate public good provision is invariant to income redistribution if and only if, for each $i, j \in C$,

$$\frac{1 - \gamma_i'(\beta_i)}{\gamma_i'(\beta_i)} + \sum_{l \in \mathcal{N}_i \cap C} \frac{1 - \gamma_l'(\beta_l)}{\gamma_l'(\beta_l)} = \frac{1 - \gamma_j'(\beta_j)}{\gamma_j'(\beta_j)} + \sum_{k \in \mathcal{N}_j \cap C} \frac{1 - \gamma_k'(\beta_k)}{\gamma_k'(\beta_k)}. \tag{A.6}$$

Observe that, if C is a clique of the network, then (A.6) holds. Conversely, suppose that (A.6) holds and for some $i, j \in C$ it holds that $j \notin \mathcal{N}_i$. Then, given that (A.6) holds for any arbitrary marginal propensities provided that network normality holds, keeping constant the marginal propensity of consumer i while choosing the marginal propensities of all other consumers in C very close to 1 will result in the left-hand side of (A.6) getting close to $\frac{1 - \gamma_i'(\beta_i)}{\gamma_i'(\beta_i)}$ while the right-hand side of (A.6) gets close to 0, which is a contradiction. Hence C is a clique of the network.□

Proof of Proposition 4. See Bramoullé, Kranton, and D'Amours [11].□

Proof of Proposition 6. Recall that $\mathbf{G} = \mathbf{V}\mathbf{D}\mathbf{V}^T$, where $\mathbf{D} = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ is a diagonal matrix whose diagonal entries are the eigenvalues of $\mathbf{G}$ and $\mathbf{V}$ is a matrix whose columns, $\mathbf{v}_1, \ldots, \mathbf{v}_n$, are the corresponding eigenvectors of $\mathbf{G}$ that form an orthonormal basis of $\mathbb{R}^n$. Since $\{\mathbf{u}_1, \ldots, \mathbf{u}_s\} \subset \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, it follows that

$$\mathbf{b}(\mathbf{G}, -a) = (\mathbf{I} + a\mathbf{G})^{-1}\mathbf{1} = \mathbf{V}(\mathbf{I} + a\mathbf{D})^{-1}\mathbf{V}^T\mathbf{1} = \sum_{i=1}^n \frac{\mathbf{1} \cdot \mathbf{v}_i}{1 + a\lambda_i}\mathbf{v}_i = \sum_{i=1}^s \frac{\mathbf{1} \cdot \mathbf{u}_i}{1 + a\mu_i}\mathbf{u}_i. \tag{A.7}$$

Moreover, since $\mathbf{1} = \sum_{l=1}^{s}(\mathbf{1} \cdot \mathbf{u}_l)\mathbf{u}_l$, it follows that

$$\prod_{k=1,k\neq i}^{s}\left(\frac{\mathbf{G} - \mu_k\mathbf{I}}{\mu_i - \mu_k}\right)\mathbf{1} = \prod_{k=1,k\neq i}^{s}\left(\frac{\mathbf{G} - \mu_k\mathbf{I}}{\mu_i - \mu_k}\right)\sum_{l=1}^{s}(\mathbf{1} \cdot \mathbf{u}_l)\mathbf{u}_l = \sum_{l=1}^{s}(\mathbf{1} \cdot \mathbf{u}_l)\prod_{k=1,k\neq i}^{s}\left(\frac{\mathbf{G} - \mu_k\mathbf{I}}{\mu_i - \mu_k}\right)\mathbf{u}_l$$

$$= \sum_{l=1,l\neq i}^{s}(\mathbf{1} \cdot \mathbf{u}_l)\prod_{k=1,k\notin\{i,l\}}^{s}\left(\frac{\mathbf{G} - \mu_k\mathbf{I}}{\mu_i - \mu_k}\right)\left(\frac{\mathbf{G} - \mu_l\mathbf{I}}{\mu_i - \mu_l}\right)\mathbf{u}_l$$

$$+ (\mathbf{1} \cdot \mathbf{u}_i)\prod_{k=1,k\neq i}^{s}\left(\frac{\mathbf{G} - \mu_k\mathbf{I}}{\mu_i - \mu_k}\right)\mathbf{u}_i$$

$$= (\mathbf{1} \cdot \mathbf{u}_i)\prod_{k=1,k\neq i}^{s}\left(\frac{\mu_i - \mu_k}{\mu_i - \mu_k}\right)\mathbf{u}_i = (\mathbf{1} \cdot \mathbf{u}_i)\mathbf{u}_i.$$

Hence it follows from (A.7) that

$$\mathbf{b}(\mathbf{G}, -a) = \sum_{i=1}^{s}\frac{1}{1 + a\mu_i}(\mathbf{1} \cdot \mathbf{u}_i)\mathbf{u}_i = \sum_{i=1}^{s}\frac{1}{1 + a\mu_i}\prod_{k=1,k\neq i}^{s}\left(\frac{\mathbf{G} - \mu_k\mathbf{I}}{\mu_i - \mu_k}\right)\mathbf{1}.\square$$

Proof of Proposition 7.

$(i) \Rightarrow (ii)$ If $\mathbf{b}(\mathbf{G}, -a)$ and $\mathbf{1}^T$ are proportional, then there exists a (nonnull) real number λ such that $(\mathbf{I} + a\mathbf{G})^{-1}\mathbf{1} = \lambda\mathbf{1}$, or equivalently, $\mathbf{1} = \lambda(\mathbf{I} + a\mathbf{G})\mathbf{1}$. This implies that $\mathbf{d} = \mathbf{G}\mathbf{1} = \frac{1-\lambda}{a\lambda}\mathbf{1}$. Hence $\mathbf{g}$ is a regular network.

$(ii) \Rightarrow (iii)$ If $\mathbf{g}$ is a regular network of degree r, then it holds that $\mathbf{d} = \mathbf{G}\mathbf{1} = r\mathbf{1}$. Hence $\mathbf{b}(\mathbf{G}, -a) = (\mathbf{I} + a\mathbf{G})^{-1}\mathbf{1} = \frac{1}{1+ar}\mathbf{1} = \frac{1}{r(1+ar)}\mathbf{d}$.

$(iii) \Rightarrow (iv)$ If $\mathbf{b}(\mathbf{G}, -a)$ and $\mathbf{d}$ are proportional, then there exists a (nonnull) real number κ such that $(\mathbf{I} + a\mathbf{G})^{-1}\mathbf{1} = \kappa\mathbf{d} = \kappa\mathbf{G}\mathbf{1}$. Since $\mathbf{1} = \sum_{i=1}^{s}(\mathbf{1} \cdot \mathbf{u}_i)\mathbf{u}_i$, it holds that

$$\sum_{i=1}^{s}\frac{\mathbf{1} \cdot \mathbf{u}_i}{1 + a\mu_i}\mathbf{u}_i = \kappa\sum_{i=1}^{s}\mu_i(\mathbf{1} \cdot \mathbf{u}_i)\mathbf{u}_i.$$

Recall that the main eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_s$ are linearly independent and thus it holds

$$\kappa = \frac{1}{\mu_1(1 + a\mu_1)} = \frac{1}{\mu_2(1 + a\mu_2)} = \ldots = \frac{1}{\mu_s(1 + a\mu_s)}. \tag{A.8}$$

Moreover, since $a \in]0, -\frac{1}{\lambda_{\min}(\mathbf{G})}[$, it follows that $1 + a\mu_1 > 1 + a\mu_2 > \ldots > 1 + a\mu_s > 0$. Now, suppose that $s > 1$; then since $\mu_1 > 0$ and $\mu_1 > \mu_s$, which in view of $a \in]0, -\frac{1}{\lambda_{\min}(\mathbf{G})}[$ implies that $1 + a\mu_1 > 1 + a\mu_s > 0$, it holds that

$$\mu_1(1 + a\mu_1) > \mu_1(1 + a\mu_s) > \mu_s(1 + a\mu_s),$$

which is a contradiction to (A.8).

$(iv) \Rightarrow (i)$ It follows from Proposition 6 that $\mathbf{b}(\mathbf{G}, -a) = \frac{1}{1+a\mu_1}\mathbf{1}.\square$

Proof of Proposition 8. It follows from Propositions 5 and 6 that

$$
\begin{aligned}
Q^{\mathbf{t}} - Q^* &= (1-a)[\frac{(\mathbf{G}_S - \mu_2\mathbf{I})\mathbf{1}}{(1+a\mu_1)(\mu_1 - \mu_2)} + \frac{(\mathbf{G}_S - \mu_1\mathbf{I})\mathbf{1}}{(1+a\mu_2)(\mu_2 - \mu_1)}] \cdot \mathbf{t}_S \\
&= (1-a)\frac{\mathbf{G}_S\mathbf{1} \cdot \mathbf{t}_S}{(1+a\mu_1)(\mu_1 - \mu_2)} + \frac{\mathbf{G}_S\mathbf{1} \cdot \mathbf{t}_S}{(1+a\mu_2)(\mu_2 - \mu_1)} \\
&= (1-a)\frac{[(1+a\mu_2) - (1+a\mu_1)]}{(1+a\mu_1)(1+a\mu_2)(\mu_1 - \mu_2)}\mathbf{d}_S \cdot \mathbf{t}_S \\
&= \frac{-a(1-a)}{(1+a\mu_1)(1+a\mu_2)}\mathbf{d}_S \cdot \mathbf{t}_S.\square
\end{aligned}
$$

Appendix B

Now we will discuss the well-known uniqueness result of Rosen [37]. First, it is worth noting that the uniqueness result of Rosen [37] does not apply to the private provision of public goods on networks. Indeed, key to the equilibrium analysis of Rosen [37] is the assumption that the each player's payoff function is concave in his own strategy, which is obviously more restrictive than the assumption made in the private provision model that each consumer has a quasi-concave utility function. However, we can get around this difficulty by constructing an auxiliary game with concave-in-own-strategy payoff functions that yield best-reply functions identical to the private provision game. Yet, in the following, we provide an example where the network normality assumption may not imply Rosen's condition of "diagonal strict concavity" in the auxiliary game.

Let us consider the game $\mathcal{G} = (N, (S_i)_{i\in N}, (\Phi_i)_{i\in N})$, where player i's strategy space is $S_i = [0, w_i]$ and payoff function is

$$
\Phi_i(\mathbf{q}) = \Phi_i(q_i, Q_{-i}) = -\frac{1}{2}(q_i + Q_{-i} - \gamma_i(w_i + Q_{-i}))^2.
$$

Note that the payoff function of each player is concave in his own strategy. Moreover, player i's best-reply function in the game $\mathcal{G}$ is identical to the one in the private provision game, that is,

$$
q_i = f_i(Q_i) = \max\{\gamma_i(w_i + Q_{-i}) - Q_{-i}, 0\}.
$$

Rosen [37]'s uniqueness of a Nash equilibrium result requires that the game $\mathcal{G}$ is "diagonally strictly concave", that is, for a fixed $\mathbf{r} = (r_1, \ldots, r_n)^T \in \mathbf{R}_{++}^n$ it holds that for any

$$\mathbf{q}^1 \neq \mathbf{q}^2$$

$$\mathbf{h}(\mathbf{q}^1, \mathbf{r})(\mathbf{q}^2 - \mathbf{q}^1) + \mathbf{h}(\mathbf{q}^2, \mathbf{r})(\mathbf{q}^1 - \mathbf{q}^2) > 0,$$

where $\mathbf{h}(\mathbf{q}, \mathbf{r})^T = \left(r_1 \frac{\partial \Phi_1(\mathbf{q})}{\partial q_1}, \ldots, r_n \frac{\partial \Phi_n(\mathbf{q})}{\partial q_n} \right)$. Intuitively, diagonal strict concavity guarantees that a player has more control over his payoff than the other players. A sufficient condition for the game to be diagonally strictly concave is that the symmetric matrix

$$\left(\mathbf{H}(\mathbf{q}, \mathbf{r}) + \mathbf{H}(\mathbf{q}, \mathbf{r})^T \right)$$

is negative definite for all $\mathbf{q} \in \Pi_{i \in N} S_i$, where $\mathbf{H}(\mathbf{q}, \mathbf{r})$ is the Jacobian of $\mathbf{h}(\mathbf{q}, \mathbf{r})$ with respect to $\mathbf{q}$.

Now, let us consider a symmetric private provision game of a pure public good with six identical consumers (in preferences and incomes). Recall that for a pure public good, the network normality stipulates that $0 < \gamma_i'(\cdot) < 1$. Thus, for some $\bar{\mathbf{q}}$ it can hold that

$$(\gamma_1'(w_1 + \bar{Q}_{-1}) - 1, \ldots, \gamma_6'(w_6 + \bar{Q}_{-6}) - 1) = -(0.99, 0.01, 0.01, 0.01, 0.01, 0.01).$$

Since $\mathbf{G}$ is the complete network, it follows that

$$\mathbf{H}(\bar{\mathbf{q}}, \mathbf{1}) = - \begin{pmatrix} 1 & 0.99 & 0.99 & 0.99 & 0.99 & 0.99 \\ 0.01 & 1 & 0.01 & 0.01 & 0.01 & 0.01 \\ 0.01 & 0.01 & 1 & 0.01 & 0.01 & 0.01 \\ 0.01 & 0.01 & 0.01 & 1 & 0.01 & 0.01 \\ 0.01 & 0.01 & 0.01 & 0.01 & 1 & 0.01 \\ 0.01 & 0.01 & 0.01 & 0.01 & 0.01 & 1 \end{pmatrix}.$$

The matrix $\mathbf{H}(\bar{\mathbf{q}}, \mathbf{1})$ was introduced in Al-Nowaihi and Levine [2] to investigate the stability of the Cournot oligopoly model and was shown there not to be negative definite since for $\boldsymbol{\zeta} = (2, -1, -1, -1, -1, -1)^T$ it holds that

$$\boldsymbol{\zeta}^T \left(\frac{\mathbf{H}(\bar{\mathbf{q}}, \mathbf{1}) + \mathbf{H}(\bar{\mathbf{q}}, \mathbf{1})^T}{2} \right) \boldsymbol{\zeta} = \boldsymbol{\zeta}^T \mathbf{H}(\bar{\mathbf{q}}, \mathbf{1}) \boldsymbol{\zeta} = 0.8. \tag{B.1}$$

Hence diagonal strict concavity does not hold for the vector of weights $\mathbf{r} = \mathbf{1}$, in spite of the symmetry of the private provision game.

Note that from Lemma 1, the matrix $\mathbf{H}(\bar{\mathbf{q}}, \mathbf{1}) = -(\mathbf{I} + \bar{\mathbf{B}}\mathbf{G})$, where $\bar{\mathbf{B}} = \mathrm{diag}(1 - \gamma_i'(w_i + \bar{Q}_{-i}))_{i=1}^6$, has negative eigenvalues, which together with (B.1) implies that

$$0 < -\lambda_{\min}(\bar{\mathbf{B}}\mathbf{G}) < 1 < -\lambda_{\min}\left(\frac{\bar{\mathbf{B}}\mathbf{G} + (\bar{\mathbf{B}}\mathbf{G})^T}{2} \right).$$

Finally, we conclude by observing that if we assume that $\frac{4}{5} = 1 - \frac{1}{\lambda_{\max}(\mathbf{G})} < \gamma_i'(\cdot) < 1$, for each consumer $i = 1, \ldots, n$, rather than network normality, then for $\mathbf{q} \in \Pi_{i \in N} S_i$ it holds that $\max_{i \in N}\{1 - \gamma_i'(Q_{-i})\} = \Omega_{\mathbf{q}} < \frac{1}{\lambda_{\max}(\mathbf{G})}$. By the monotonicity of $\lambda_{\max}(\cdot)$, with respect to the entries of a nonnegative matrix, and the Perron–Frobenius Theorem, it holds that

$$0 < -\lambda_{\min}\Big(\frac{\mathbf{BG} + (\mathbf{BG})^T}{2}\Big) \leq \lambda_{\max}\Big(\frac{\mathbf{BG} + (\mathbf{BG})^T}{2}\Big) \leq \Omega_{\mathbf{q}}\,\lambda_{\max}(\mathbf{G}) < 1,$$

where $\mathbf{B} = \mathrm{diag}(1 - \gamma_i'(w_i + Q_{-i}))_{i \in N}$. Thus

$$\mathbf{H}(\mathbf{q}, \mathbf{1}) + \mathbf{H}(\mathbf{q}, \mathbf{1})^T = -\Big(\mathbf{I} + \frac{\mathbf{BG} + (\mathbf{BG})^T}{2}\Big)$$

is negative definite for all $\mathbf{q} \in \Pi_{i \in N} S_i$ and, as a consequence, Rosen [37] diagonal strict concavity holds for the vector of weights $\mathbf{r} = \mathbf{1}$.$\square$

References

[1] Acemoglu, D., C. García-Jimeno, and J.A. Robinson (2014) "State capacity and economic development: a network approach," Preprint, available at http://economics.mit.edu/files/9437

[2] Al-Nowaihi, A. and P. L. Levine (1985) "The stability of the Cournot oligopoly model: a reassessment," *Journal of Economic Theory* 35, 307-321.

[3] Andreoni, J. (1990) "Impure altruism and donations to public goods: a theory of warm glow giving," *Economic Journal* 100, 464-477.

[4] Andreoni, J. (2006) "Philanthropy," in S.-C. Kolm and J. M. Ythier (eds), *Handbook of the Economics of Giving, Altruism and Reciprocity*, Elsevier/North-Holland, Amsterdam.

[5] Ballester, C., A. Calvó-Armengol, and Y. Zenou (2006) "Who's who in networks. Wanted: the key player," *Econometrica* 74, 1403-1417.

[6] Bergstrom, T., L. Blume, and H. Varian (1986) "On the private provision of public goods," *Journal of Public Economics* 29, 25-49.

[7] Bergstrom, T., L. Blume, and H. Varian (1992) "Uniqueness of Nash equilibrium in private provision of public goods: an improved proof," *Journal of Public Economics* 49, 391-392.

[8] Bloch, F. and Ü. Zenginobuz (2007) "The effect of spillovers on the provision of local public goods," *Review of Economic Design* 11, 199-216.

[9] Bonacich, P. (1987) "Power and centrality: a family of measures," *American Journal of Sociology* 92, 1170-1182.

[10] Bramoullé, Y. and R. Kranton (2007) "Public goods in networks," *Journal of Economic Theory* 135, 478-494.

[11] Bramoullé, Y., R. Kranton, and M. D'Amours (2014) "Strategic interaction and networks," *American Economic Review* 104, 898-930.

[12] Bulow, J., J. Geanakoplos, and P. Klemperer (1985) "Multimarket oligopoly: strategic substitutes and complements," *Journal of Political Economy* 93, 488-511.

[13] Candogan, O., K. Bimpikis, and A. Ozdaglar (2012) "Optimal pricing in the presence of local network effects," *Operations Research,* 60, 883-905.

[14] Corazzini, L. and U. Gianazza (2008) "Unequal contributions from symmetric agents in a local interaction model," *Journal of Public Economic Theory* 10, 351-370.

[15] Cornes, R. and R. Hartley (2007) "Aggregative public good games," *Journal of Public Economic Theory* 9, 201-219.

[16] Cvetković, D. M. (1970) "The generating function for variations with restrictions and paths of the graph and self-complementary graphs," *Univ. Beograd, Publ. Elektrotehn. Fak. Ser. Mat. Fiz.* 320-328, 27-34.

[17] Cvetković, D. M., P. Rowlinson, and S. Simić (1997) *Eigenspaces of Graphs,* Cambridge University Press, Cambridge.

[18] Dixit, A. (1986) "Comparative statics for oligopoly," *International Economic Review* 27, 107-122.

[19] Elliott, M. and B. Golub (2014) "A network approach to public goods," Preprint, available at http://www.people.fas.harvard.edu/∼bgolub/papers/jmp.pdf.

[20] Ellison, G. (1993) "Learning, local interaction, and coordination," *Econometrica* 61, 1047-1071.

[21] Eshel, I., L. Samuelson, and A. Shaked (1998) "Altruists, egoists and hooligans in a local interaction model," *American Economic Review* 88, 157-179.

[22] Foster, A. and M. Rosenzweig (1995) "Learning by doing and learning from others: human capital and technical change," *Journal of Political Economy* 103, 1176-1209.

[23] Fraser, C. D. (1992) "The uniqueness of Nash equilibrium in the private provision of public goods: an alternative proof," *Journal of Public Economics* 49, 389-390.

[24] Galeotti, A. and S. Goyal (2010) "The Law of the Few," *American Economic Review* 100, 1468-1492.

[25] Galeotti, A., S. Goyal, M. Jackson, F. Vega-Redondo, and L. Yariv (2010) "Network games," *Review of Economic Studies* 77, 218-244.

[26] Golub, B. and C. Lever (2010) "The leverage of weak ties: how linking groups affects inequality," Preprint, available at http://www.people.fas.harvard.edu/∼bgolub/papers/intergroup.pdf.

[27] Goyal, S. (2007) *Connections: An Introduction to the Economics of Networks,* Princeton University Press, Princeton.

[28] Hagos, E. M. (2002) "Some results on graph spectra," *Linear Algebra and its Applications* 356, 103-111.

[29] Harary, F. and A. J. Schwenk (1979) "The spectral approach to determining the number of walks in a graph," *Pacific Journal of Mathematics* 80, 443-449.

[30] Horn, R. and C. Johnson (1985) *Matrix Analysis,* Cambridge University Press, Cambridge.

[31] İlkiliç, R. (2011) "Networks of common property resources," *Economic Theory* 47, 105-134.

[32] Jackson, M. O. (2008) *Social and Economic Networks,* Princeton University Press, Princeton.

[33] König, M. D., J. C. Tessone, and Y. Zenou (2014) "Nestedness in networks: a theoretical model and some applications," *Theoretical Economics* 3, 695-752.

[34] Moffitt, R. (1989) "Estimating the value of an in-kind transfer: the case of food stamps," *Econometrica* 57, 385-409.

[35] Monderer, D. and L. S. Shapley (1996) "Potential games," *Games and Economic Behavior* 14, 124-143.

[36] Ostrowski, A. M. (1959) "A quantitative formulation of Sylvester's law of inertia," *Proceedings of the National Academy of Sciences* 45, 740-744.

[37] Rosen, J. B. (1965) "Existence and uniqueness of equilibrium points for concave n-person games," *Econometrica* 33, 520-534.

[38] Warr, P. G. (1983) "The private provision of a public good is independent of the distribution of income," *Economics Letters* 13, 207-211.