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An order theoretic characterization of spin factors

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Abstract

The famous Koecher-Vinberg theorem characterizes the Euclidean Jordan algebras among the finite dimensional order unit spaces as the ones that have a symmetric cone. Recently Walsh gave an alternative characterization of the Euclidean Jordan algebras. He showed that the Euclidean Jordan algebras correspond to the finite dimensional order unit spaces (V, C, u) for which there exists a bijective map $g\colon C^\circ \to C^\circ$ with the property that g is antihomogeneous, i.e., $g(\lambda x) = \lambda^{-1} g(x)$ for all $\lambda > 0$ and $x \in C^\circ$, and g is an order-antimorphism, i.e., $x \leq_C y$ if and only if $g(y) \leq_C g(x)$. In this paper we make a first step towards extending this order theoretic characterization to infinite dimensional JB-algebras. We show that if (V, C, u) is a complete order unit space with a strictly convex cone and $\dim V \geq 3$, then there exists a bijective antihomogeneous order-antimorphism $g\colon C^\circ \to C^\circ$ if and only if (V, C, u) is a spin factor.

Keywords: Spin factors, order-antimorphisms, order unit spaces, JB-algebras, symmetric Banach-Finsler manifolds

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1 Introduction

Let C be a cone in a real vector space V, so C is convex, $\lambda C \subseteq C$ for all $\lambda \geq 0$ and $C \cap -C = \{0\}$. Then C induces a partial ordering $\leq_C$ on V by $x \leq_C y$ if $y - x \in C$. Recall that C is *Archimedean* if for each $x \in V$ and $y \in C$ with $nx \leq_C y$ for all $n = 1, 2, \ldots$, we have that $x \leq_C 0$. Moreover, $u \in C$ is said to be an *order unit* if for each $x \in V$ there exists $\lambda \geq 0$ such that $x \leq_C \lambda u$. The triple (V, C, u) is called an *order unit space* if C is an Archimedean cone with order unit u. An order unit space can be equipped with the so called *order unit norm*,

$$\|x\|_u := \inf\{\lambda > 0 \colon -\lambda u \leq_C x \leq_C \lambda u\}.$$

With respect to the order unit norm topology the cone C is closed and has nonempty interior, denoted by C°. In the paper we will study order unit spaces that are complete with respect to $\|\cdot\|_u$

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and which have a strictly convex cone. Recall that a cone C is *strictly convex* if for each linearly independent $x, y \in \partial C$, the segment $\{(1-\lambda)x + \lambda y \colon 0 < \lambda < 1\}$ is contained in C°.

An important class of complete order unit spaces are JB-algebras (with unit). A *Jordan algebra over* $\mathbb{R}$ is a real vector space A equipped with a commutative bilinear product $\circ$ that satisfies

$$a^2 \circ (a \circ b) = a \circ (a^2 \circ b) \quad \text{for all } a, b \in A.$$

A *(unital) JB-algebra* A is a normed, complete Jordan algebra over $\mathbb{R}$ with unit e satisfying

$$\|a \circ b\| \leq \|a\|\|b\|, \quad \|a^2\| = \|a\|^2, \quad \text{and } \|a^2\| \leq \|a^2 + b^2\| \quad \text{for all } a, b \in A.$$

A JB-algebra A gives rise to a complete order unit space, where the cone A_+ is the set of squares $\{a^2 \colon a \in A\}$, the unit e is an order unit, and $\|\cdot\|_e$ coincides with norm of A, see [2, Theorem 1.11]. A special class of JB-algebras are spin factors. A *spin factor* M is a real vector space with $\dim M \geq 3$ such that $M = H \oplus \mathbb{R}e$ (vector space direct sum) with $(H, (\cdot \mid \cdot))$ a Hilbert space and $\mathbb{R}e$ the linear span of e, where M is given the Jordan product

$$(a + \alpha e) \circ (b + \beta e) = \beta a + \alpha b + ((a \mid b) + \alpha\beta)e \tag{1}$$

and norm $\|a + \lambda e\| := \|a\|_2 + |\lambda|$, with $\|\cdot\|_2$ the norm of H.

The famous Koecher-Vinberg theorem ([6] and [12]) says that the finite dimensional JB-algebras are in one-to-one correspondence with symmetric cones, i.e., self-dual cones in a Euclidean space V for which $\mathrm{Aut}(C) := \{T \in \mathrm{GL}(V) \colon T(C) = C\}$ acts transitively on C°. As JB-algebras are merely Banach spaces instead of Hilbert spaces, no such characterization exists in infinite dimensions. It is, however, interesting to ask if one could characterize the JB-algebras among the complete order unit spaces in order theoretic terms. One such characterization was obtained by Kai [4] who characterized the symmetric cones among the homogeneous cones. More recently Walsh [14] gave an order theoretic characterization of finite dimensional JB-algebras using order-antimorphisms. A map $g \colon C^\circ \to C^\circ$ is an *order-antimorphism* if for each $x, y \in C^\circ$ we have that $x \leq_C y$ if and only if $g(y) \leq_C g(x)$. It is said to be *antihomogeneous* if $g(\lambda x) = \frac{1}{\lambda} g(x)$ for all $\lambda > 0$ and $x \in C^\circ$.

Walsh [14, Theorem 1.1] showed that if (V, C, u) is a finite dimensional order unit space, then there exists an antihomogeneous order-antimorphism $g \colon C^\circ \to C^\circ$ if and only if (V, C, u) is a JB-algebra. At present it is unknown if this characterization can be extended to infinite dimensional JB-algebras. In this paper we make the following contribution to this problem.

Theorem 1.1. *If (V, C, u) is a complete order unit space with a strictly convex cone and $\dim V \geq 3$, then there exists a bijective antihomogeneous order-antimorphism $g \colon C^\circ \to C^\circ$ if and only if (V, C, u) is a spin factor.*

As our general approach is similar to Walsh's [14], we briefly discuss the main similarities and differences. To prove that the cone is homogeneous [14, Lemma 3.5] Walsh uses the fact that a bijective antihomogeneous order-antimorphism is a locally Lipschitz map, and hence almost everywhere Fréchet differentiable by Rademacher's Theorem. There is, however, no infinite dimensional version of Rademacher's Theorem. To overcome this difficulty, we show that a bijective antihomogeneous order-antimorphism is Gateaux differentiable at each point in a strictly convex cone, and work with the Gateaux derivative, see Proposition 2.4. Like Walsh we will also use ideas from metric geometry such as Hilbert's and Thompson's metrics. In particular, Walsh applies his characterization of the Hilbert's metric horofunctions [13], which, at present, is not known for infinite dimensional spaces.

Instead we shall show that if there exists a bijective antihomogeneous order-antimorphism on a strictly convex cone, then the cone is smooth, see Theorem 3.2. This will allow us to avoid the use of horofunctions completely, but implicitly some of Walsh's horofunction method is still present in the proof of Proposition 4.2.

2 Order-antimorphisms and symmetries

For $x, y \in V$ linearly independent we write $V(x,y) := \operatorname{span}(x,y)$, $C(x,y) := V(x,y) \cap C$, and $C^\circ(x,y) := V(x,y) \cap C^\circ$. Note that as C is Archimedean, $C(x,y)$ is a closed 2-dimensional cone in $V(x,y)$, if $x \in C^\circ$.

Useful tools in the analysis are Hilbert's and Thompson's metrics on C°. They are defined in terms of the following function. For $x \in C$ and $y \in C^\circ$ let

$$M(x/y) := \inf\{\beta > 0 \colon x \leq_C \beta y\}.$$

Note that $0 \leq M(x/y) < \infty$ for all $x \in C$ and $y \in C^\circ$, if (V, C, u) is an order unit space. Moreover, $M(\sigma x/\mu y) = \frac{\sigma}{\mu} M(x/y)$ for all $\sigma, \mu > 0$ and $x \in C$ and $y \in C^\circ$.

Now *Hilbert's metric* on C° is defined by

$$d_H(x,y) := \log M(x/y) + \log M(y/x),$$

and *Thompson's metric* on C° is given by

$$d_T(x,y) := \max\{\log M(x/y), \log M(y/x)\}$$

for $x, y \in C^\circ$. Note that $d_H(\sigma x, \mu y) = d_H(x,y)$ for all $x, y \in C^\circ$ and $\sigma, \mu > 0$. So, d_H is not a metric on C°. However, for cones in an order unit space it is known [7, Chapter 2] that d_H is a metric between pairs of rays in C°, as $d_H(x,y) = 0$ if and only if $x = \lambda y$ for some $\lambda > 0$ in that case. Thompson's metric is a metric on C° in an order unit space. Moreover, its topology coincides with the order unit norm topology on C°, see [7, Chapter 2].

The following basic lemma is well known, see e.g., [10], and implies that each antihomogeneous order-antimorphism is an isometry under d_H and d_T. For the reader's convenience we include the simple proof.

Lemma 2.1. *Let (V, C, u) be an order unit space. Then $g \colon C^\circ \to C^\circ$ is an antihomogeneous order-antimorphism if and only if $M(x/y) = M(g(y)/g(x))$ for all $x, y \in C^\circ$. In particular, a bijective antihomogeneous order-antimorphism $g \colon C^\circ \to C^\circ$ is an isometry under d_H and d_T, and the inverse $g^{-1} \colon C^\circ \to C^\circ$ is an antihomogeneous order-antimorphism.*

Proof. Clearly, if $g \colon C^\circ \to C^\circ$ is antihomogeneous order-antimorphism and $x \leq_C \beta y$, then $g(\beta y) \leq_C g(x)$, so that $g(y) \leq_C \beta g(x)$. This implies that $M(g(y)/g(x)) \leq M(x/y)$. On the other hand, $g(y) \leq_C \beta g(x)$ implies $g(\beta y) \leq_C g(x)$, so that $x \leq_C \beta y$ from which we conclude that $M(x/y) \leq M(g(y)/g(x))$. This shows that $M(x/y) = M(g(y)/g(x))$ for all $x, y \in C^\circ$.

Now suppose that $M(x/y) = M(g(y)/g(x))$ for all $x, y \in C^\circ$. If $x \leq_C y$, then $M(g(y)/g(x)) = M(x/y) \leq 1$, so that $g(y) \leq_C g(x)$. Likewise $g(y) \leq_C g(x)$ implies $M(x/y) = M(g(y)/g(x)) \leq 1$, so that $x \leq_C y$, which shows that g is an order-antimorphism. To see that g is antihomogeneous note that if $x \in C^\circ$ and $\lambda > 0$, then $y := \lambda x$ satisfies $M(g(y)/g(x)) = M(x/y) = 1/\lambda$ and $M(g(x)/g(y)) = M(y/x) = \lambda$. This implies that $\lambda g(y) \leq_C g(x) \leq_C \lambda g(y)$ from which we conclude that $g(\lambda x) = g(y) = \frac{1}{\lambda} g(x)$. □

Every JB-algebra A has a bijective antihomogeneous order-antimorphism namely, the map $\iota\colon A_+^\circ \to A_+^\circ$ given by $\iota(a) = a^{-1}$. As shown in [9, Section 2.4], we have that $M(\iota(a)/\iota(b)) = M(b/a)$ for all $a, b \in A_+^\circ$, and hence ι is a bijective antihomogeneous order-antimorphism by Lemma 2.1.

A linear functional $\varphi\colon V \to \mathbb{R}$ is said to be *positive* if $\varphi(C) \subseteq [0, \infty)$, and it is called *strictly positive* if $\varphi(C \setminus \{0\}) \subseteq (0, \infty)$. A positive functional φ is called a *state* of (V, C, u) if $\varphi(u) = 1$. The set $S(V) := \{\varphi \in V^*\colon \varphi \text{ is a state}\}$ is called the *state space*, which is a w*-closed convex subset of the unit ball in V^*, and hence $S(V)$ is w*-compact by the Banach-Alaoglu Theorem. Moreover, as $x \leq_C \beta y$ is equivalent to $\varphi(x) \leq \beta\varphi(y)$ for all $\varphi \in S(V)$, we get that

$$M(x/y) = \max_{\varphi \in S(V)} \frac{\varphi(x)}{\varphi(y)} \quad \text{for all } x, y \in C^\circ. \tag{2}$$

If (V, C, u) is an order unit space with a strictly convex cone, then there exists a strictly positive state on V as the following lemma shows.

Lemma 2.2. *If (V, C, u) is an order unit space with a strictly convex cone, then there exists a strictly positive state $\rho \in S(V)$.*

Proof. Let $r \in \partial C \setminus \{0\}$. Then $C(r, u)$ is a 2-dimensional closed cone in V. By [7, A.5.1] there exists an $s \in \partial C \setminus \{0\}$ such that $C(r, u) = \{\alpha r + \beta s\colon \alpha, \beta \geq 0\}$. Let φ and ψ be linear functionals on $V(r, u)$ such that $\varphi(r) = 0 = \psi(s)$, $\varphi(s), \psi(r) > 0$, and $\varphi(u) = 1 = \psi(u)$. By the Hahn-Banach theorem we can extend φ and ψ to linear functional on V such that $\|\varphi\| = \varphi(u) = 1$ and $\|\psi\| = \psi(u) = 1$. It follows from [1, 1.16 Lemma] that $\varphi, \psi \in S(V)$.

Now let $\rho := \frac{1}{2}(\varphi + \psi) \in S(V)$. Note that $\varphi(x) = 0$ for $x \in C$ if and only if $x = \lambda r$ for some $\lambda \geq 0$, as C is strictly convex. Likewise, $\psi(x) = 0$ for $x \in C$ if and only if $x = \lambda s$ for some $\lambda \geq 0$. This implies that $\rho(x) > 0$ for all $x \in C \setminus \{0\}$. □

Next we shall show that antihomogeneous order-antimorphisms on strictly convex cones map 2-dimensional subcones to 2-dimensional subcones. To prove this we use unique geodesics. Recall that given a metric space (X, d_X) a *geodesic path* $\gamma\colon I \to X$, where $I \subseteq \mathbb{R}$ is a possibly unbounded interval, is a map such that

$$d_X(\gamma(s), \gamma(t)) = |s - t| \quad \text{for all } s, t \in I.$$

The image $\gamma(I)$ is simply called a *geodesic*, and $\gamma(\mathbb{R})$ is said to be a *geodesic line* in (X, d_X). A geodesic line γ is called *unique* if for each x and y on γ we have that γ is the only geodesic line through x and y in (X, d_X).

If (V, C, u) is an order unit space with a strictly positive functional $\rho \in S(V)$, then d_H is a metric on

$$\Sigma_\rho := \{x \in C^\circ\colon \rho(x) = 1\}.$$

Straight line segments are geodesic in the Hilbert's metic space (Σ_ρ, d_H). Moreover, if the cone is strictly convex, then it is well known, see for example [3, Section 18], that each geodesic in the Hilbert's metic space (Σ_ρ, d_H) is a straight line segment.

Lemma 2.3. *Let (V, C, u) be an order unit space with a strictly convex cone, and $g\colon C^\circ \to C^\circ$ be a bijective antihomogeneous order-antimorphism. If $x, y \in C^\circ$ are linearly independent, then $g(x)$ and $g(y)$ are linearly independent and g maps $C^\circ(x, y)$ onto $C^\circ(g(x), g(y))$.*

Proof. Let $\rho \in S(V)$ be a strictly positive state, which we know exists by Lemma 2.2. Now define $f\colon \Sigma_\rho \to \Sigma_\rho$ by

$$f(x) := \frac{g(x)}{\rho(g(x))} \quad \text{for all } x \in \Sigma_\rho.$$

Then f is an isometry on (Σ_ρ, d_H) by Lemma 2.1. If $x, y \in C^\circ$ are linearly independent, then the straight line ℓ through $x/\rho(x)$ and $y/\rho(y)$ intersected with Σ_ρ is a geodesic line in (Σ_ρ, d_H). Thus, $f(\ell \cap \Sigma_\rho)$ is also a geodesic line, and hence a straight line segment, as C is strictly convex. In fact, its image is the intersection of the straight line through $g(x)/\rho(g(x))$ and $g(y)/\rho(g(y))$ and Σ_ρ. It follows that $g(x)/\rho(g(x))$ and $g(y)/\rho(g(y))$ are linearly independent and that g maps $C^\circ(x, y)$ onto $C^\circ(g(x), g(y))$, as g is antihomogeneous. □

We note that the proof of Lemma 2.3 goes through if one only assumes that (Σ_ρ, d_H) is uniquely geodesic.

Using this lemma we can now prove the following proposition.

Proposition 2.4. *Let (V, C, u) be an order unit space with a strictly convex cone. If $g\colon C^\circ \to C^\circ$ is a bijective antihomogeneous order-antimorphism, then the following assertions hold.*

(1) For each linearly independent $x, y \in C^\circ$ the restriction g_{xy} of g to $C^\circ(x, y)$ is a Fréchet differentiable map, and its Fréchet derivative $Dg_{xy}(z)$ at $z \in C^\circ(x, y)$ is an invertible linear map from $V(x, y)$ onto $V(g(x), g(y))$.

(2) For each $x \in C^\circ$ and $z \in V$ we have that

$$\Delta_x^z g(x) := \lim_{t \to 0} \frac{g(x + tz) - g(x)}{t}$$

exists, and $-\Delta_x^z g(x) \in C$ for all $z \in C$.

(3) For each $x \in C^\circ$ we have $\Delta_x^{\lambda x} g(x) = -\lambda g(x)$ for all $\lambda \in \mathbb{R}$.

Proof. Let $x, y \in C^\circ$ be linearly independent and $g\colon C^\circ \to C^\circ$ be an antihomogeneous order-antimorphism. By Lemma 2.3 the restriction g_{xy} of g maps $C^\circ(x, y)$ onto $C^\circ(g(x), g(y))$. The 2-dimensional closed cones $C(x, y)$ and $C(g(x), g(y))$ are order-isomorphic to $\mathbb{R}^2_+ := \{(x_1, x_2) \in \mathbb{R}^2\colon x_1, x_2 \geq 0\}$, i.e., there exist linear maps $A\colon V(x, y) \to \mathbb{R}^2$ and $B\colon V(g(x), g(y)) \to \mathbb{R}^2$ such that $A(C(x, y)) = \mathbb{R}^2_+$ and $B(C(g(x), g(y))) = \mathbb{R}^2_+$. Thus, the map $h\colon (\mathbb{R}^2_+)^\circ \to (\mathbb{R}^2_+)^\circ$ given by $h(z) = B(g_{xy}(A^{-1}(z)))$ is a bijective antihomogeneous order-antimorphism on $(\mathbb{R}^2_+)^\circ$, and hence h is a d_T-isometry on $(\mathbb{R}^2_+)^\circ$. We know from [9, Theorem 3.2] that h is of the form:

$$h((z_1, z_2)) = (a_1/z_{\sigma(1)}, a_2/z_{\sigma(2)}) \quad \text{for } (z_1, z_2) \in (\mathbb{R}^2_+)^\circ,$$

where σ is a permutation on $\{1, 2\}$ and $a_1, a_2 > 0$ are fixed. Clearly the map h is Fréchet differentiable on $(\mathbb{R}^2_+)^\circ$, and hence g_{xy} is Fréchet differentiable on $C^\circ(x, y)$. Moreover, the Fréchet derivative $Dh(z)$ is an invertible linear map on $\mathbb{R}^2$ at each $z \in (\mathbb{R}^2_+)^\circ$, so that $Dg_{xy}(z)$ an invertible linear map from $V(x, y)$ onto $V(g(x), g(y))$ for all $z \in C^\circ(x, y)$.

To prove the second statement note that if z is linearly independent of x, then there exists a $y \in C^\circ$ such that $z \in V(x, y)$. From (1) we get that $\Delta_x^z g(x) = Dg_{xy}(x)(z)$, as g_{xy} is Fréchet differentiable on $C^\circ(x, y)$. Also, if $z = \lambda x$ for some $\lambda \neq 0$, then

$$\Delta_x^{\lambda x} g(x) = \lim_{t \to 0} \frac{g(x + t\lambda x) - g(x)}{t} = \lim_{t \to 0} \frac{-\lambda t}{t(1 + \lambda t)} g(x) = -\lambda g(x),$$

and $\Delta_x^0 g(x) = 0$. Furthermore, if $z \in C$, then

$$\Delta_x^z g(x) = \lim_{t \to 0} \frac{g(x+tz) - g(x)}{t} \in -C,$$

as g is an order-antimorphism. This completes the proofs of (2) and (3). □

Given a bijective antihomogeneous order-antimorphism $g\colon C^\circ \to C^\circ$ on a strictly convex cone C in an order unit space, and $x \in C^\circ$ we define $G_x = G_{g,x}\colon V \to V$ by

$$G_x(z) := -\Delta_x^z g(x) \quad \text{for all } z \in V.$$

Lemma 2.5. *If $x \in C^\circ$ and $G_x(x) = x$, then $g(x) = x$.*

Proof. Simply note that $x = G_x(x) = -\Delta_x^x g(x) = g(x)$ by Proposition 2.4(3). □

The map G_x has the following property.

Proposition 2.6. *The map $G_x\colon V \to V$ is a bijective homogeneous order-isomorphism with inverse $G_{g^{-1},g(x)}\colon V \to V$.*

Proof. Let $z \in V(x,y)$, $x, y \in C^\circ$ linearly independent, and $\lambda \neq 0$. Then

$$G_x(\lambda z) = -\lim_{t \to 0} \frac{g(x+t\lambda z) - g(x)}{t} = -\lambda \lim_{t \to 0} \frac{g(x+t\lambda z) - g(x)}{\lambda t} = \lambda G_x(z).$$

Also if $w \leq_C z$, then

$$G_x(w) = -\lim_{t \to 0} \frac{g(x+tw) - g(x)}{t} \leq_C -\lim_{t \to 0} \frac{g(x+tz) - g(x)}{t} = G_x(z),$$

as $x + tw \leq_C x + tz$ for all $t > 0$ and g is an order-antimorphism.

To show that G_x is a surjective map on V let $h := g_{xy} \circ g^{-1}_{g(x)g(y)}$. So, $h\colon C^\circ(g(x), g(y)) \to C^\circ(g(x), g(y))$ and $h(z) = z$ for all $z \in C^\circ(g(x), g(y))$. For each $w \in V(g(x), g(y))$ we have by the chain rule that

$$w = Dh(g_{xy}(x))(w) = Dg_{xy}(x) Dg^{-1}_{g(x)g(y)}(g_{xy}(x))w = G_x(G_{g^{-1},g(x)}(w)).$$

Interchanging the roles of g and g^{-1} we also have that $G_{g^{-1},g(x)}(G_x(v)) = v$ for all $v \in V(x,y)$, and hence $G_{g^{-1},g(x)}$ is the inverse of G_x on V. □

Combining Proposition 2.6 and [10, Theorem B] we conclude that $G_x \in \mathrm{Aut}(C) := \{T \in \mathrm{GL}(V)\colon T(C) = C\}$ and G_x is continuous with respect to $\|\cdot\|_u$ on V, as $\|G_x\|_u = \|G_x(u)\|_u$.

Now for $x \in C^\circ$ define the *symmetry at x* by

$$S_x := G_x^{-1} \circ g. \tag{3}$$

So, $S_x\colon C^\circ \to C^\circ$ is a bijective antihomogeneous order-antimorphism, with inverse $S_x^{-1} = g^{-1} \circ G_x$. We derive some further properties of the symmetries. Let us begin by making the following useful observation.

Lemma 2.7. *Let $x \in C^\circ$ and $y \in V$ be linearly independent of x. Then for each $w \in V(x,y)$ we have that $D(S_x)_{xy}(x)(w) = -w$.*

Proof. Note that

$$\begin{aligned} D(S_x)_{xy}(x)(w) &= \lim_{t\to 0} \frac{S_x(x+tw) - S_x(x)}{t} \\ &= \lim_{t\to 0} \frac{G_x^{-1}(g(x+tw)) - G_x^{-1}(g(x))}{t} \\ &= G_x^{-1}\left(\lim_{t\to 0} \frac{g(x+tw) - g(x)}{t}\right) \\ &= G_x^{-1}(-G_x(w)) \\ &= -w, \end{aligned}$$

as $G_x^{-1} = G_{g^{-1},g(x)}$ is a bounded linear map on $(V, \|\cdot\|_u)$ by Proposition 2.6. □

Theorem 2.8. *For each $x \in C^\circ$ we have that*

(1) $S_x(x) = x$.

(2) $S_x \circ S_x = \mathrm{Id}$ on C°.

Proof. To prove (1) note that for $x \in C^\circ$ we have by Propositions 2.4(3) and 2.6 that

$$S_x(x) = G_x^{-1}(g(x)) = G_{g^{-1},g(x)}(g(x)) = g^{-1}(g(x)) = x.$$

To show (2) let $x, y \in C^\circ$ be linearly independent. For simplicity we write $T := (S_x)_{S_x(x)S_x(y)}$ and $S := (S_x)_{xy}$, so $(S_x^2)_{xy} = T \circ S$ and S, T are Fréchet differentiable on $C^\circ(x,y)$ and $C^\circ(S_x(x), S_x(y))$ respectively. Then using the chain rule and Lemma 2.7 we find that

$$\Delta_x^y S_x^2(x) = \lim_{t\to 0} \frac{T(S(x+ty)) - T(S(x))}{t} = DT(S(x))(DS(x))(y) = -DS(x)(y) = y.$$

Note that S_x^2 is a homogeneous order-isomorphism on C°, and hence by [10, Theorem B] we know that it is linear. So, it follows from the previous equality that $S_x^2 = \mathrm{Id}$ on C°. □

To proceed it is useful to recall a few facts about unique geodesics for Thompson's metric from [8, Section 2]. If $x \in (C^\circ, d_T)$, then there are two special types of geodesic lines through x. There are the so-called *type I geodesic lines* γ which are the images of the geodesic paths,

$$\gamma(t) := e^t r + e^{-t} s \quad \text{for } t \in \mathbb{R}, \tag{4}$$

with $r, s \in \partial C$ and $r + s = x$. The *type II geodesic line* μ through x is the image of the geodesic path $\mu(t); = e^t x$ with $t \in \mathbb{R}$. The type I geodesics γ have the property that $M(u/v) = M(v/u)$ for all u and v on γ, and the type II geodesics have the property that $M(u/v) = M(v/u)^{-1}$ for all u and v on μ.

Each unique geodesic line in (C°, d_T) is either of type I or type II, see [8, Section 2]. Moreover, the type II geodesic is always unique [8, Proposition 4.1], but the type I geodesics may not be unique. However, if C is strictly convex, then all type I geodesic lines are unique, see [8, Theorem 4.3].

Lemma 2.9. *Let (V, C, u) be an order unit space with a strictly convex cone. If $\gamma\colon \mathbb{R} \to (C^\circ, d_T)$ is a geodesic path with $\gamma(0) = x$, and $\gamma(\mathbb{R})$ is a type I geodesic line, then $S_x(\gamma(t)) = \gamma(-t)$ for all $t \in \mathbb{R}$.*

Proof. If $\gamma\colon \mathbb{R} \to (C^\circ, d_T)$ is a geodesic path with $\gamma(0) = x$, and $\gamma(\mathbb{R})$ is a type I geodesic line, then there exist $r, s \in \partial C$ with $r + s = x$ and $\gamma(t) = e^t r + e^{-t} s$ for all $t \in \mathbb{R}$ by [8, Lemma 3.7]. As C is strictly convex, we know from [8, Theorem 4.3] that $\gamma\colon \mathbb{R} \to (C^\circ, d_T)$ is a unique geodesic path. This implies that $\hat{\gamma}(t) := S_x(\gamma(t))$, $t \in \mathbb{R}$, is also a unique geodesic path in (C°, d_T), as S_x is an isometry under d_T. Moreover, as $M(S_x(y)/S_x(z)) = M(z/y)$ for all $y, z \in C^\circ$, we know that

$$M(S_x(\gamma(t_1))/S_x(\gamma(t_2))) = M(\gamma(t_2)/\gamma(t_1)) = M(\gamma(t_1)/\gamma(t_2)) = M(S_x(\gamma(t_2))/S_x(\gamma(t_1))),$$

so that $\hat{\gamma}(\mathbb{R})$ is a type I geodesic line though x.

It now follows again from [8, Lemma 3.7] that there exists $u, v \in \partial C$ such that $u + v = x$ and $\hat{\gamma}(t) = e^t u + e^{-t} v$ for all $t \in \mathbb{R}$. Recall from Proposition 2.4 that the restriction $(S_x)_{rx}$ of S_x to $C^\circ(r, x)$ is Fréchet differentiable, and hence

$$\hat{\gamma}'(0) = D(S_x)_{rx}(\gamma(0))(\gamma'(0)) = D(S_x)_{rx}(x)(r - s) = -r + s$$

by Lemma 2.7. But also $\hat{\gamma}'(0) = u - v$. Combining this with the equalities $r + s = x = u + v$, we find that $u = s$ and $v = r$. Thus, $S_x(\gamma(t)) = \hat{\gamma}(t) = e^t s + e^{-t} r = \gamma(-t)$ for all $t \in \mathbb{R}$. □

Proposition 2.10. *Let (V, C, u) be an order unit space with a strictly convex cone. For each $x \in C^\circ$ we have that S_x has x as a unique fixed point.*

Proof. Suppose by way of contradiction that $y \in C^\circ$ is a fixed point of S_x and $y \neq x$. Then y is linearly independent of x, as S_x is antihomogeneous and $S_x(x) = x$. Define $\mu := M(x/y)^{1/2} M(y/x)^{-1/2}$ and $z := \mu y \in C^\circ$. Then $M(x/z) = M(z/x)$ and hence there exists a type I geodesic path $\gamma\colon \mathbb{R} \to (C^\circ, d_T)$ through x and z, with $\gamma(0) = x$. From Lemma 2.9 it follows that $S_x(\gamma(\mathbb{R})) = \gamma(\mathbb{R})$, As z is the unique point of intersection of $\gamma(\mathbb{R})$ with the invariant ray $R_y := \{\lambda y\colon \lambda > 0\}$, we conclude that $S_x(z) = z$. This, however, contradicts Lemma 2.9, as $z \neq x$. □

Remark 2.11. The metric space (C°, d_T) is a natural example of a Banach-Finsler manifold, see [11]. So, the results in this section show that if there exists a bijective antihomogeneous order-antimorphism on C° in a complete order unit space with strictly convex cone, then (C°, d_T) is a *globally symmetric* Banach-Finsler manifold, in the sense that for each $x \in C^\circ$ there exists an isometry $\sigma_x\colon C^\circ \to C^\circ$ such that $\sigma_x^2 = \mathrm{Id}$ and x is an isolated fixed point of σ_x. Indeed, we can take $\sigma_x = S_x$. It is interesting to understand which complete order unit spaces (C°, d_T) are globally symmetric Banach-Finsler manifolds. It might well be true that these are precisely the JB-algebras.

3 Smoothness of the cone

Throughout this section we will assume that $\dim V \geq 3$.

We will show that if (V, C, u) is a complete order unit space with a strictly convex cone and there exists an antihomogeneous order-antimorphism $g\colon C^\circ \to C^\circ$, then C is a *smooth* cone, that is to say, for each $\eta \in \partial C$ with $\eta \neq 0$ there exists a unique $\varphi \in S(V)$ such that $\varphi(\eta) = 0$. Before we prove this we make the following elementary observation.

Lemma 3.1. *If (V, C, u) is an order unit space and $\eta \in \partial C$ with $\eta \neq 0$, then for each $x \in C^\circ$ and $y := (1 - s)\eta + sx$, with $0 < s \leq 1$, we have that*

$$M(x/y) = \frac{\varphi(x)}{\varphi(y)} = \frac{1}{s}$$

for each $\varphi \in S(V)$ with $\varphi(\eta) = 0$.

Proof. By [7, Section 2.1] we know that

$$M(x/y) = \frac{\|\eta - x\|_u}{\|\eta - y\|_u} = \frac{1}{s}.$$

But also $1/s = \varphi(x)/\varphi(y)$ for all states $\varphi \in S(V)$ with $\varphi(\eta) = 0$. □

Theorem 3.2. *If (V, C, u) is an order unit space with a strictly convex cone and there exists a bijective antihomogeneous order-antimorphism $g\colon C^\circ \to C^\circ$, then C is a smooth cone.*

Proof. Let $\rho \in S(V)$ be a strictly positive state, which exists by Lemma 2.2. Suppose by way of contradiction that there exist $\eta \in \partial C$ with $\rho(\eta) = 1$ and states $\varphi \neq \psi$ such that $\varphi(\eta) = 0 = \psi(\eta)$. As $\varphi \neq \psi$, there exists $x \in V$ such that $\varphi(x) \neq \psi(x)$. Note that if $\alpha x + \beta\eta + \gamma u = 0$ for some $\alpha, \beta, \gamma \in \mathbb{R}$, then $\alpha\varphi(x) + \gamma = \alpha\psi(x) + \gamma = 0$, which yields $\alpha = 0$ and $\gamma = 0$. This shows that x, η and u are linearly independent.

Let $W := \operatorname{span}(x, \eta, u)$ and $K := W \cap C$. As $\dim V \geq 3$ and $u \in C^\circ$, K is a 3-dimensional, strictly convex, closed cone in W containing u in its interior. Let $S(W)$ be the state space of the order unit space (W, K, u). Note that the restrictions of φ, ψ, ρ to W, denoted $\bar{\varphi}$, $\bar{\psi}$, and $\bar{\rho}$ respectively, are in $S(W)$. Moreover $\bar{\rho}(w) > 0$ for all $w \in K \setminus \{0\}$, and hence

$$\Omega := \{w \in K\colon \bar{\rho}(w) = 1\}$$

is a 2-dimensional, strictly convex, compact set, with η in its (relative) boundary. We also know that $S(W)$ is a compact, convex subset of W^*.

Let $F := \{\zeta \in S(W)\colon \zeta(\eta) = 0\}$, which is a closed face of $S(W)$. As F contains $\bar{\varphi}$ and $\bar{\psi}$ which are not equal, F is a straight line segment, say $[\tau, \nu]$ with $\tau \neq \nu$. Let $x, y \in \partial\Omega$ be such that u is between the straight line segments $[\eta, x]$ and $[\eta, y]$, as in Figure 1.

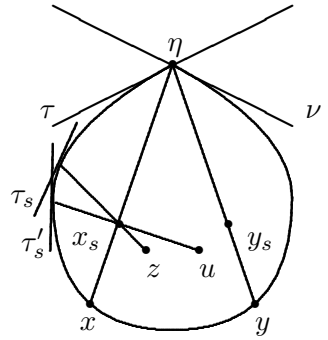


Figure 1: Point of non-smoothness

Now let $z \in \Omega \cap C^\circ$ also be between the segments $[\eta, x]$ and $[\eta, y]$ such that $\operatorname{span}(z, \eta, u) = W$. For $0 < s < 1$, let $x_s := (1 - s)\eta + sx$ and $y_s := (1 - s)\eta + sy$. By Lemma 3.1 there exists $\tau_s, \tau'_s \in S(W)$ such that

$$M(z/x_s) = \frac{\tau_s(z)}{\tau_s(x_s)} \quad \text{and} \quad M(u/x_s) = \frac{\tau'_s(u)}{\tau'_s(x_s)}$$

for $0 < s < 1$.

Then

$$\tau_s'(z) = \frac{\tau_s'(z)}{\tau_s'(x_s)}\frac{\tau_s'(x_s)}{\tau_s'(u)} \leq M(z/x_s)M(u/x_s)^{-1} \leq \frac{\tau_s(z)}{\tau_s(x_s)}\frac{\tau_s(x_s)}{\tau_s(u)} \leq \tau_s(z)$$

for all $0 < s < 1$. As $\tau_s(z) \to \tau(z)$ and $\tau_s'(z) \to \tau(z)$ as $s \to 0$, we conclude that

$$\lim_{s\to 0} M(z/x_s)M(u/x_s)^{-1} = \tau(z).$$

In the same way it can be shown that

$$\lim_{s\to 0} M(z/y_s)M(u/y_s)^{-1} = \nu(z).$$

We will now show that $\tau(z) = \nu(z)$, which implies that $\tau = \nu$, as $\tau(\eta) = \nu(\eta) = 0$, $\tau(u) = \nu(u) = 1$ and $\operatorname{span}(z, \eta, u) = W$. This gives the desired contradiction. To prove the equality we use the symmetry $S_u \colon C^\circ \to C^\circ$ at u. Let $f \colon \Sigma_\rho \to \Sigma_\rho$ be given by

$$f(v) = \frac{S_u(u)}{\rho(S_u(v))} \quad \text{for all } v \in \Sigma_\rho = \{w \in C^\circ \colon \rho(w) = 1\}.$$

Thus, f is an isometry on (Σ_ρ, d_H). As C is strictly convex, the segments (x, η) and (y, η) are unique geodesic lines in (Σ_ρ, d_H). So, $f((x, \eta))$ and $f((y, \eta))$ are unique geodesic lines, and hence there exist $x', y', \zeta_1, \zeta_2 \in \partial\Sigma_\rho$ so that $f((x, \eta)) = (x', \zeta_1)$ with $\lim_{s\to 0} f(x_s) = \zeta_1$, and $f((y, \eta)) = (y', \zeta_2)$ with $\lim_{s\to 0} f(y_s) = \zeta_2$.

We claim that $\zeta_1 = \zeta_2$. Suppose by way of contradiction that $\zeta_1 \neq \zeta_2$. Then using [5, Theorem 5.2] we know that there exists a constant $C_0 < \infty$ such that

$$\limsup_{s\to 0} d_H(f(x_s), u) + d_H(f(y_s), u) - d_H(f(x_s), f(y_s)) \leq C_0, \tag{5}$$

as Σ_ρ is strictly convex.

However, we know (see [7, Section 2.1]) that

$$d_H(x_s, y_s) = \log \frac{\|y_s - w_s'\|}{\|x_s - w_s'\|}\frac{\|x_s - v_s'\|}{\|y_s - v_s'\|}$$

for all $0 < s < 1$, where $w_s', v_s' \in \partial\Omega$. Let w_s, v_s be on the lines ℓ_1 and ℓ_2 as in Figure 2, where ℓ_1 and ℓ_2 are fixed. For $s > 0$ sufficiently small

$$\frac{\|y_s - w_s'\|}{\|x_s - w_s'\|}\frac{\|x_s - v_s'\|}{\|y_s - v_s'\|} \leq \frac{\|y_s - w_s\|}{\|x_s - w_s\|}\frac{\|x_s - v_s\|}{\|y_s - v_s\|}.$$

By projective invariance of the cross-ratio we know there exists $C_1 < \infty$ such that

$$\frac{\|y_s - w_s\|}{\|x_s - w_s\|}\frac{\|x_s - v_s\|}{\|y_s - v_s\|} = C_1 \quad \text{for all } s > 0 \text{ sufficiently small.}$$

Thus, $\limsup_{s\to 0} d_H(x_s, y_s) \leq \log C_1$.

As f is an isometry under d_H with $f(u) = u$, we deduce that

$$d_H(f(x_s), u) + d_H(f(y_s), u) - d_H(f(x_s), f(y_s)) = d_H(x_s, u) + d_H(y_s, u) - d_H(x_s, y_s) \to \infty,$$

as $s \to 0$. This contradicts (5), and hence $\zeta_1 = \zeta_2$.

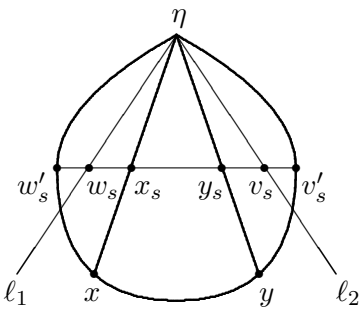


Figure 2: cross-ratios

Now note that

$$\begin{aligned}\tau(z) &= \lim_{s\to 0} M(z/x_s)M(u/x_s)^{-1}\\ &= \lim_{s\to 0} M(S_u(x_s)/S_u(z))M(S_u(x_s)/u)^{-1}\\ &= \lim_{s\to 0} M(f(x_s)/S_u(z))M(f(x_s)/u)^{-1}\\ &= M(\zeta_1/S_u(z))M(\zeta_1/u)^{-1}.\end{aligned}$$

Likewise $\nu(z) = M(\zeta_2/S_u(z))M(\zeta_2/u)^{-1}$, which shows that $\tau(z) = \nu(z)$, as $\zeta_1 = \zeta_2$. This completes the proof. □

Lemma 3.3. *Let (V, C, u) be an order unit space with a smooth cone, $\eta \in \partial C \setminus \{0\}$, and $\varphi \in S(V)$ be such that $\varphi(\eta) = 0$. Suppose that $z \in C$ with $\varphi(z) > 0$, and for $0 < s \leq 1$ let $y_s := (1-s)\eta + su$ and $z_s := (1-s)z + su$ in C°. If $\varphi_s \in S(V)$ is such that $M(z_s/y_s) = \varphi_s(z_s)/\varphi_s(y_s)$ for $0 < s \leq 1$, then $\varphi_s(\eta) \to 0$, as $s \to 0$, and (φ_s) w*-converges to φ.*

Proof. Note that $M(z_s/y_s) = \varphi_s(z_s)/\varphi_s(y_s) \geq \varphi(z_s)/\varphi(y_s) = \frac{1-s}{s}\varphi(z) + 1 \to \infty$, as $s \to 0$. As $|\varphi_s(z_s)| \leq \|z_s\|_u \leq (1-s)\|z\|_u + s\|u\|_u \leq \|z\|_u + 1$, we deduce that $\varphi_s(y_s) \to 0$ as $s \to 0$. So,

$$|\varphi_s(\eta)| \leq |\varphi_s(\eta) - \varphi_s(y_s)| + |\varphi_s(y_s)| \leq \|\eta - y_s\|_u + |\varphi_s(y_s)| \to 0 \quad \text{as } s \to 0.$$

Now consider any subnet $(\varphi_{s'})$ of (φ_s) in $S(V)$. It has a w*-convergent subnet with limit say ψ, as $S(V)$ is w*-compact. By the first part of the lemma we know that $\psi(\eta) = 0$, and hence $\psi = \varphi$, since C is smooth. This shows that (φ_s) w*-converges to φ. □

Proposition 3.4. *Let (V, C, u) be an order unit space with a smooth cone, $\eta \in \partial C \setminus \{0\}$, and $\varphi \in S(V)$ be such that $\varphi(\eta) = 0$. Suppose that $z \in C$ with $\varphi(z) > 0$ and for $0 < s \leq 1$ let $y_s := (1-s)\eta + su$ and $z_s := (1-s)z + su$ in C°. Then*

$$\lim_{s\to 0} M(z_s/y_s)M(u/y_s)^{-1} = \varphi(z).$$

Proof. For $0 < s \leq 1$ let $\varphi_s \in S(V)$ be such that $M(z_s/y_s) = \varphi_s(z_s)/\varphi_s(y_s)$. So, (φ_s) w*-converges

to φ by Lemma 3.3. Note that

$$\begin{aligned} M(z_s/y_s)M(u/y_s)^{-1} &\leq \frac{\varphi_s(z_s)}{\varphi_s(y_s)}\left(\frac{\varphi(u)}{\varphi(y_s)}\right)^{-1} \\ &= \frac{\varphi_s(z_s)}{\hat{\varphi}(u)}\frac{\varphi(y_s)}{\varphi_s(y_s)} \\ &= \varphi_s(z_s)\frac{\varphi((1-s)\eta+su)}{\varphi_s((1-s)\eta+su)} \\ &\leq \varphi_s(z_s) \end{aligned}$$

as $\varphi(\eta) = 0$ and $\varphi_s(\eta) \geq 0$ for all $0 < s \leq 1$. The right-hand side of the inequality converges to $\varphi(z)$ as $s \to 0$, since (φ_s) w*-converges to φ.

On the other hand, if we let $\psi_s \in S(V)$ be such that $M(u/y_s) = \psi_s(u)/\psi_s(y_s)$, then (ψ_s) w*-converges to φ by taking $z = u$ in Lemma 3.3. Moreover,

$$\begin{aligned} M(z_s/y_s)M(u/y_s)^{-1} &\geq \frac{\varphi(z_s)}{\varphi(y_s)}\left(\frac{\psi_s(u)}{\psi_s(y_s)}\right)^{-1} \\ &= \frac{\varphi(z_s)}{\psi_s(u)}\frac{\psi_s(y_s)}{\varphi(y_s)} \\ &\geq \varphi(z_s), \end{aligned}$$

as $\psi_s(\eta) \geq 0$. The right-hand side converges to $\varphi(z)$ as $s \to 0$, which completes the proof. □

4 Proof of Theorem 1.1

Define

$$\mathcal{P} := \{p \in \partial C \colon M(p/u) = \|p\|_u = 1\}.$$

Lemma 4.1. *If (V, C, u) is an order unit space, then for each $p \in \mathcal{P}$ there exists a unique $p' \in \mathcal{P}$ with $p + p' = u$.*

Proof. Note that $p \leq_C M(p/u)u = u$, so that $w := u - p \in (\partial C \setminus \{0\}) \cap V(p, u)$. So,

$$M(w/u) := \inf\{\beta > 0 \colon u - p \leq_C \beta u\} = \inf\{\beta > 0 \colon 0 \leq_C (\beta - 1)u + p\} = 1,$$

as otherwise $p - \delta u \in C$ for some $\delta > 0$. This would imply that $p = \delta u + (p - \delta u) \in C^\circ$, as $\delta u \in C^\circ$, which is impossible. Thus, if we let $p' := w$, then clearly p' is unique, $p' \in \mathcal{P}$ and $p + p' = u$. □

Note that $V = \operatorname{span}(\mathcal{P})$. Indeed, if $v \in V$ is linearly independent of u, then $V(u, v)$ is a 2-dimensional subspace with a 2-dimensional closed cone $C(u, v)$. By [7, A.5.1] there exists $r, s \in \partial C$ such that $C(u, v) = \{\lambda r + \mu s \colon \lambda, \mu \geq 0\}$ and $\operatorname{span}(r, s) = V(u, v)$. So, if we let $p := M(r/u)^{-1}r$ and $q := M(s/u)^{-1}s$, then $p, q \in \mathcal{P}$ and $v \in \operatorname{span}(p, q)$. On the other hand, if $v = \lambda u$ with $\lambda \in \mathbb{R}$, then $v = \lambda(p + p')$ for some $p \in \mathcal{P}$ by Lemma 4.1.

Now let (V, C, u) be an order unit space with a strictly convex cone and $\dim V \geq 3$. Suppose there exists a bijective antihomogeneous order-antimorphism $g \colon C^\circ \to C^\circ$. Then C is a smooth cone by Theorem 3.2. Denote by $\varphi_p \in S(V)$ the unique supporting functional at $p \in \mathcal{P}$, so $\varphi_p(p) = 0$ and $\varphi_p(p') = \varphi_p(u) = 1$. For $p \in \mathcal{P}$ define the linear form $B(p, \cdot)$ on V by

$$B(p, v) := \varphi_{p'}(v) \quad \text{for all } v \in V.$$

Proposition 4.2. *If $p, q \in \mathcal{P}$, then $B(p,q) = B(q,p)$.*

Proof. Let $p, q \in \mathcal{P}$ and for $0 < s \leq 1$ define

$$\begin{aligned} p_s &:= (1-s)p + su, & p'_s &:= (1-s)p' + su, \\ q_s &:= (1-s)q + su, & q'_s &:= (1-s)q' + su. \end{aligned}$$

We wish to show that $S_u(p_s) = \frac{1}{s}p'_s$ and $S_u(q_s) = \frac{1}{s}q'_s$. By interchanging the roles of p_s and q_s it suffices to prove the first equality.

Note that if $\beta > 0$ is such that $u \leq_C \beta p_s$, then $(1-\beta s)u \leq_C \beta(1-s)p$, so that $\beta s \geq 1$, as $p \in \partial C$ and $u \in C^\circ$. Thus, $M(u/p_s) = 1/s$. The same argument shows that $M(u/p'_s) = 1/s$. Furthermore, it is easy to check that $M(p_s/u) = 1 = M(p'_s/u)$, and hence $d_T(u, p_s) = -\log s = d_T(u, p'_s)$ for all $0 < s \leq 1$.

Let $\delta_s := M(u/p_s)^{1/2} M(p_s/u)^{-1/2} = 1/\sqrt{s}$ and put $x_s := \delta_s p_s$ and $y_s := \delta_s p'_s$. Then $M(x_s/u) = M(u/x_s) = 1/\sqrt{s} = M(y_s/u) = M(u/y_s)$. Thus, x_s and y_s are on the unique type I geodesic line γ through u in $C^\circ(p, p')$. Let $\gamma\colon \mathbb{R} \to (C^\circ, d_T)$ be the geodesic path with $\gamma = \gamma(\mathbb{R})$ and $\gamma(0) = u$. As S_u is a d_T-isometry and $S_u(u) = u$, we find that $d_T(u, x_s) = d_T(u, S_u(x_s)) = -\log\sqrt{s} = d_T(u, y_s)$. Using Lemma 2.9 and the fact that $x_s \neq y_s$, we conclude that $S_u(x_s) = y_s$. Thus, $S_u(\delta_s p_s) = \delta_s p'_s$, which shows that $S_u(p_s) = \frac{1}{s}p'_s$.

Now let $p, q \in \mathcal{P}$ and suppose that $q \neq p'$. Then by Proposition 3.4 we have that

$$\begin{aligned} B(p,q) &= \varphi_{p'}(q) \\ &= \lim_{s\to 0} M(q_s/p'_s) M(u/p'_s)^{-1} \\ &= \lim_{s\to o} M(q_s/S_u(p_s)) M(u/S_u(p_s))^{-1} \\ &= \lim_{s\to 0} M(p_s/S_u(q_s)) M(p_s/u)^{-1} \\ &= \lim_{s\to 0} M(p_s/S_u(q_s)), \end{aligned}$$

where we used the identity $S_u(p_s) = \frac{1}{s}p'_s$ and the fact that $S_u^2 = \mathrm{Id}$ (Theorem 2.8) in the third equality.

Likewise,

$$B(q,p) = \lim_{s\to 0} M(q_s/S_u(p_s)).$$

Now using the fact that $M(p_s/S_u(q_s)) = M(q_s/S_u(p_s))$ for all $0 < s \leq 1$, we deduce that $B(p,q) = B(q,p)$ if $q \neq p'$. On the other hand, if $q = p'$, then $B(p,q) = 0$ and $B(q,p) = 0$. □

We now extend B linearly to V by letting

$$B\left(\sum_{i=1}^n \alpha_i p_i, v\right) := \sum_{i=1}^n \alpha_i B(p_i, v) \quad \text{for all } v \in V.$$

To see that B is a well-defined bilinear form suppose that $w = \sum_i \alpha_i p_i = \sum_j \beta_j q_j$ for some $\alpha_i, \beta_j \in \mathbb{R}$

and $p_i, q_j \in \mathcal{P}$. Write $v = \sum_k \gamma_k r_k$ with $r_k \in \mathcal{P}$. Then by Proposition 4.2 we get that

$$\begin{aligned}\sum_i \alpha_i B(p_i, v) &= \sum_{i,k} \alpha_i \gamma_k B(p_i, r_k) \\ &= \sum_{i,k} \gamma_k \alpha_i B(r_k, p_i) \\ &= \sum_k \gamma_k B(r_k, w).\end{aligned}$$

Likewise $\sum_j \beta_j B(q_j, v) = \sum_k \gamma_k B(r_k, w)$, which shows that B is a well defined symmetric bilinear form on $V \times V$.

Let $H := \operatorname{span}\{p - p' \colon p \in \mathcal{P}\}$ and $\mathbb{R}u := \operatorname{span}(u)$.

Lemma 4.3. *We have that $V = H \oplus \mathbb{R}u$ (vector space direct sum), and H is a closed subspace of $(V, \|\cdot\|_u)$.*

Proof. Note that for each $v \in V$ there exists $p \in \mathcal{P}$ and $\alpha, \beta \in \mathbb{R}$ such that $v = \alpha p + \beta p'$. So,

$$v = \frac{1}{2}(\alpha - \beta)(p - p') + \frac{1}{2}(\alpha + \beta)u, \tag{6}$$

by Lemma 4.1. This shows that $V = H + \mathbb{R}u$. Now let $\psi_u \colon V \to \mathbb{R}$ be given by $\psi_u(v) := B(v, u)$ for all $v \in V$. Note that if $v = p - p'$, then

$$\psi_u(v) = B(p, u) - B(p', u) = \varphi_{p'}(u) - \varphi_p(u) = 1 - 1 = 0,$$

and hence $H \subseteq \ker(\psi_u)$. Moreover, $B(u, u) = B(p, u) + B(p', u) = 2$. Also for $v = \alpha s + \beta u$ with $s = p - p' \in H$ we have that $\psi_u(v) = 2\beta = 0$ if and only if $\beta = 0$. Thus, $H = \ker(\psi_u)$, which shows that $V = H \oplus \mathbb{R}u$.

To see that H is closed it suffices to show that ψ_u is bounded with respect to $\|\cdot\|_u$. Let $v = \alpha p + \beta p' \in V$. Then

$$\|v\|_u = \inf\{\lambda > 0 \colon -\lambda u \leq_C \alpha p + \beta p' \leq_C \lambda u\} = \max\{|\alpha|, |\beta|\}. \tag{7}$$

It follows that

$$|\psi_u(v)| \leq |\alpha|\psi_u(p) + |\beta|\psi_u(p') = |\alpha| + |\beta| \leq 2\|v\|_u,$$

and hence ψ_u is bounded. □

Define a bilinear form $(x \mid y)$ on H by

$$(x \mid y) := \frac{1}{2}B(x, y) \quad \text{for all } x, y \in H.$$

Proof of Theorem 1.1. We will first show that $(H, (\cdot \mid \cdot))$ is a Hilbert space. Note that if $x \in H$, then there exists $p \in \mathcal{P}$ and $\alpha \in \mathbb{R}$ such that $x = \alpha(p - p')$ by (6). Clearly

$$\|x\|_2^2 = (x \mid x) = \frac{1}{2}\left(\alpha^2 B(p, p - p') - \alpha^2 B(p', p - p')\right) = \frac{\alpha^2}{2}(1 + 1) = \alpha^2 = \|x\|_u^2, \tag{8}$$

by (7). It follows that $(x \mid x) \geq 0$ for all $x \in H$, $(x \mid x) = 0$ if and only if $x = 0$, and $(H, (\cdot \mid \cdot))$ is a Hilbert space, as H is closed in $(V, \|\cdot\|_u)$.

We already know from Lemma 4.3 that $V = H \oplus \mathbb{R}u$, where $(H, (\cdot \mid \cdot))$ is a Hilbert space. Note that if $x = \alpha(p - p') \in H$, then $\|x + \beta u\|_u = \max\{|\alpha + \beta|, |\alpha - \beta|\} = |\alpha| + |\beta| = \|x\|_u + |\beta|$ by (7). So, we deduce from equality (8) that

$$\|x + \beta u\|_u = \|x\|_2 + |\beta| \quad \text{for } x \in H \text{ and } \beta \in \mathbb{R}.$$

It remains to show that $\{a^2 \colon a \in V\} = C$, where the Jordan product is given by (1). Note that if $a = x + \sigma u$ where $x = \delta(p - p') \in H$ and $\sigma, \delta \in \mathbb{R}$, then

$$\begin{aligned} a^2 &= 2\sigma x + ((x \mid x) + \sigma^2)u \\ &= 2\sigma\delta(p - p') + \left(\frac{\delta^2}{2} B(p - p', p - p') + \sigma^2\right) u \\ &= 2\sigma\delta(p - p') + (\delta^2 + \sigma^2)(p + p') \\ &= (\sigma + \delta)^2 p + (\sigma - \delta)^2 p' \in C. \end{aligned}$$

Conversely, if $v \in C$, then $v = \lambda p + \mu p'$ for some $\lambda, \mu \geq 0$ and $p, p' \in \mathcal{P}$. Let

$$w := \sqrt{\lambda} p + \sqrt{\mu} p' = \frac{1}{2}\left((\sqrt{\lambda} - \sqrt{\mu})(p - p') + (\sqrt{\lambda} + \sqrt{\mu})(p + p')\right).$$

So,

$$w^2 = \frac{1}{4}\left(2(\sqrt{\lambda} - \sqrt{\mu})(\sqrt{\lambda} + \sqrt{\mu})(p - p') + ((\sqrt{\lambda} - \sqrt{\mu})^2 + (\sqrt{\lambda} + \sqrt{\mu})^2)(p + p')\right) = \lambda p + \mu p' = v,$$

which shows that $v \in \{a^2 \colon a \in V\}$. □

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