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Exponential Asymptotics for Integrals with Degenerate and Non-Isolated Critical Points

by

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ABSTRACT

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In this thesis we consider the exponentially improved asymptotic solutions to unbounded multidimensional steepest descent type integrals in the case where the phase function has sets of non-isolated critical points. These sets are connected components of the critical set of the phase function and we consider the case where these sets have both general order of degeneracy and general dimension. We consider first the case of isolated critical points of general order of degeneracy as a lead-in to the general problem.

In the isolated case, we justify the reduction of the study of the asymptotic behaviour of the general integral to the study of the asymptotic contribution to each individual critical point by appealing to the Morse lemma and results from the literature regarding the homology group of allowable integration surfaces. In the non-isolated case no such results exist, but we give a first step by appealing to the Morse-Bott lemma to suitably parameterise the integration surface. The analogous homological result does not yet exist and such a derivation is beyond the scope of this thesis, but we proceed to study individual contributions regardless, inspired by how readily the Morse-Bott lemma affords an analogous parameterisation of the integration surface in the non-isolated case.

Once this justification is established we focus on individual critical connected components of the phase function. A full hyperasymptotic expansion representing the repeatedly exponentially improved asymptotic contribution to the integral for critical points of this type is derived for the first time, with examples provided to demonstrate this new theory. The case of a general bounded integration region is briefly considered, but we demonstrate that work still needs to be done to extend the theory to this case.

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Declaration of Authorship

I, **THOMAS BENNETT**, declare that this thesis and the work presented in it are my own and has been generated by me as the result of my own original research.

Thesis Title: **Exponential Asymptotics for Integrals with Degenerate and Non-Isolated Critical Points**.

I confirm that:

1. This work was done wholly or mainly while in candidature for a research degree at this University;
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3. Where I have consulted the published work of others, this is always clearly attributed;
4. Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
5. I have acknowledged all main sources of help;
6. Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
7. Either none of this work has been published before submission, or parts of this work have been published with reference to such material made below.

Signed: ....

Date: ....

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because of her continual support and self-sacrifice that it was even possible. Dorte has done an immeasurable amount for me and I dedicate this thesis to her, for the beauty of the mathematical nature of the subject is mirrored only by that within her heart.

1 Introduction

Asymptotic analysis is the discipline of analysing limiting behaviour of mathematical objects, such as functions, integrals, and differential equations and their solutions. This analysis typically sees such objects represented by asymptotic expansions; series representations that often diverge, but not necessarily so. Knowing how to correctly derive, manipulate, and interpret these representations depending on the type of object undergoing analysis is a core and important part of the subject.

Asymptotic analysis allows us to pinpoint dominant behaviour and processes within a problem, enabling us to reduce traditionally complicated problems to discussions of these dominant ones that are often simpler. It also provides us with highly accurate numerical approximations for the object the expansion represents using very few terms - a trait that used to attract great interest from researchers in a vast variety of fields, but is somewhat lessened with the advent of computer algebra packages. Nonetheless, they do provide an additional and independent check for numerical analysis while also allowing us to efficiently uncover underlying controlling processes - a task that if carried out purely numerically may be very time consuming.

We begin this thesis by following the development of the field in §2, starting with Poincaré's formal definition of an asymptotic expansion in 1886. Due to the inability of exponentially small terms to be incorporated into this definition and the resulting theory and analysis, they caused much confusion and were often disregarded entirely as they are frequently - but not always - numerically negligible. This is significant when considering Stokes' phenomenon, whereby the form of the asymptotic solution changes due to asymptotic contributions 'switching on or off' (becoming dominant or subdominant respectively) in different regions of the complex plane. This means that these exponentially small subdominant contributions that are often discarded can grow to be the dominant contribution depending on the region of the complex plane we are looking at.

We then follow how the field developed post-Poincaré, focusing on how alternate interpretations of divergent series such as those in Dingle (1973) (hereafter referred to as just 'Dingle') gave rise to asymptotic series that did in fact include small exponentials; this discipline is in turn named exponential asymptotic analysis, or simply exponential asymptotics. Superasymptotic and hyperasymptotic expansions were then developed, producing numerical accuracy far greater than that of Poincaré expansions. Aside from the numerical benefits, hyperasymptotic expansions also allow for in-depth symbolic representation and analysis of solutions, revealing information and behavioural subtleties that otherwise would not have been noticed.

The highly important Stokes phenomenon and the related higher order Stokes phenomenon - whereby the *possibility* of a Stokes phenomenon is switched on or off - are then discussed through examples, leading into a wider literature review of more modern developments of the subject. The thesis will focus on deriving integral asymptotics based on the method of steepest descent, but the development of exponential asymptotic solutions to differential equations is also discussed, as well as the important link between differential equation and integral asymptotics.

We focus on integrals of the form

$$I(k) = \int_S d\boldsymbol{z}\, g(\boldsymbol{z})e^{-kf(\boldsymbol{z})} \tag{1.1}$$

that can be approximated using the method of steepest descent, where $f, g : \mathbb{C}^d \to \mathbb{C}$ are sufficiently holomorphic complex valued functions, $k \in \mathbb{C}$ is the asymptotic parameter, $\boldsymbol{z} = (z_1, \ldots, z_d) \in \mathbb{C}^d$, and $S \subset \mathbb{C}^d$ is a surface doubly infinite in all d complex variables that runs between two asymptotic valleys (regions where the integrand is zero as $-kf(\boldsymbol{z}) \to -\infty$). The method of steepest descent states that we can deform S into surfaces of steepest descent or ascent, making it possible in general to derive an asymptotic expansion for (1.1) in a variety of cases.

When f has finite isolated critical points (namely, no critical points at infinity), we can use concepts from Morse theory and homology to deform S into a union of steepest descent surfaces S_j that are similar to S, but which run between two asymptotic valleys and pass through exactly one critical point z_j of f (Pham, 1985). This allows us to reduce the problem of deriving an asymptotic expansion for (1.1) to deriving one for the integral

$$I^{(n)}(k) = \int_{S_n} d\boldsymbol{z}\, g(\boldsymbol{z})e^{-kf(\boldsymbol{z})}, \tag{1.2}$$

where S_n is doubly infinite in all d complex variable and runs between two asymptotic valleys, but which passes through the critical point z_n of f and no other critical points. We are then able to write (1.1) as a sum of integrals of type (1.2). A plethora of work exists regarding the derivation of asymptotic expansions for integrals of type (1.2) for both fully infinite integration surfaces and various types of bounded integration region, particularly for non-degenerate critical points (points of order two). We will focus on hyperasymptotic expansions of integral (1.2) - such as those found in Berry and Howls (1991), Howls (1992, 1997), Delabaere and Howls (2002) (among others) - and the literature we discuss will reflect that. Work involving exponentially improved expansions for degenerate critical points of f (order greater than two) exists, but is far less plentiful and not entirely explicit.

After the literature review we introduce our own original research, which generalises existing results regarding hyperasymptotic expansions of integral (1.2) by removing restrictions on the type of critical points f is allowed to have. These new results are generalisations of those found in Howls (1997) and we briefly describe those now as a starting point.

Writing

$$
\begin{aligned}
I^{(n)}(k) &= \frac{e^{-kf_n}}{k^{\frac{d}{2}}} T^{(n)}(k), \\
T^{(n)}(k) &= k^{\frac{d}{2}} \int_{S_n} d\boldsymbol{z}\, g(\boldsymbol{z}) e^{-k(f(\boldsymbol{z})-f_n)},
\end{aligned}
$$

where f has only non-degenerate isolated critical points and $f_n := f(\boldsymbol{z}_n)$, we can then express $T^{(n)}(k)$ as an formal asymptotic series in increasing negative powers of k; namely

$$
T^{(n)}(k) \sim \sum_{r=0}^{\infty} \frac{T_r^{(n)}}{k^r}, \quad T_r^{(n)} = \frac{\Gamma(r+\frac{d}{2})}{2\pi i} \oiint_{B_n} d\boldsymbol{z} \frac{g(\boldsymbol{z})}{(f(\boldsymbol{z})-f_n)^{r+\frac{d}{2}}},
$$

where B_n is a small ball around $\boldsymbol{z}_n$ and the quantities $T_r^{(n)}$ are called the asymptotic expansion coefficients. Truncating the expansion at the $(N-1)$th term allows us to write

$$
T^{(n)}(k,N) = \sum_{r=0}^{N-1} \frac{T_r^{(n)}}{k^r} + R^{(n)}(k,N),
$$

where $N \in \mathbb{N}$ and $R^{(n)}(k,N)$ is the remainder, and we are able to rewrite this remainder in terms of the asymptotic expansions around the other (non-degenerate) critical points z_{m_j} of f, with $j \in \{1,\dots,\gamma\}$. This leads to the formally exact expression

$$
T^{(n)}(k,N) = \sum_{r=0}^{N-1} \frac{T_r^{(n)}}{k^r} + \frac{1}{2\pi i} \sum_{j=1}^{\gamma} \frac{K_{nm_j}}{(kF_{nm_j})^N} \int_0^{\infty} dv\, \frac{v^{N-1}e^{-v}}{1-v/kF_{nm_j}} T^{(m_j)}\left(\frac{v}{F_{nm_j}}\right) \tag{1.3}
$$

for $T^{(n)}(k,N)$ involving $T^{(m_j)}(v/F_{nm_j})$ - the expansions around the other critical points - called a formally exact resurgence relation. In equation (1.3), v is real and positive, $F_{nm_j} := f_{m_j} - f_n$ is called the singulant, and K_{nm_j} are the Stokes multipliers defined as

$$
|K_{nm_j}| = \begin{cases} 1 & \text{if } z_{m_j} \text{ is adjacent to } z_n, \\ 0 & \text{otherwise.} \end{cases}
$$

Two critical points are said to be adjacent to one and other if there exists a steepest descent surface connecting them.

If N is the least term of the expansion, then truncating it at that term and ignoring the remainder yields exponential accuracy, or accuracy beyond all orders of the expansion parameter; this is a superasymptotic expansion. If we retain the remainder, then we can rewrite it as an asymptotic expansion by substituting the expansion $T^{(m_j)}(v/F_{nm_j})$ into (1.3) and optimally truncate the resulting expansion, providing us with a new remainder expression. This further improves the numerical accuracy of the expansion, as we obtain an even smaller exponentially small error. We can rewrite this new remainder in a similar way to get an even more accurate asymptotic expansion. By iterating this repeated substitution, we achieve increasingly accurate asymptotic expansions; expansions generated in this way are called hyperasymptotic expansions. As mentioned earlier, despite the original goal of hyperasymptotic expansions being to improve numerical accuracy, they are used nowadays to aid in the understanding of the underlying algebraic structure of the problem by giving us explicit and exact forms for both the expansion and remainder after a general number of iterations. They can be used for example to compute the Stokes multipliers, which we can use to determine the underlying Riemann surface structure of the problem in the Borel plane.

In §3 we discuss the concepts of degenerate critical points and sets of non-isolated critical points, with a view to extend the results of Howls (1997) to include the cases where f has such classes of critical points. The motivation behind studying non-isolated critical points is due to the Borel plane not distinguishing between isolated and sets of non-isolated critical points; f is constant on sets of non-isolated critical points of any dimension by definition and thus the entire set is mapped to the same point in the Borel plane. We use the term 'critical component' to describe both isolated and sets of non-isolated critical points and discuss the powers of the asymptotic parameter k we expect to see in expansions around critical components of general order and dimension. Although we do not use Borel plane techniques - favouring the geometric style derivation of Berry and Howls (1991) - the Borel plane remains our inspiration for this work. We also note the interesting possibility of sets of non-isolated critical points having *non-constant* order of degeneracy, a case which cannot occur for isolated critical points.

In §4 we look in $\mathbb{C}$ at the case where the function f in (1.2) has degenerate isolated critical points z_j of general order of degeneracy ω_j, explicitly noting that the order of each critical point can be different. This order can also be two, so the work done in this chapter also covers non-degenerate critical points. Rather than taking a uniform asymptotic approach to break the degenerate critical point up into a cluster of non-degenerate ones (such as in Berry and

Howls (1993)), we tackle the problem directly, deriving a resurgence relation similar to (1.3) that can be used to produce a full hyperasymptotic expansion. We defer derivation of this hyperasymptotic expansion until §6, as §5 covers a more general case. We restrict ourselves to integrals in $\mathbb{C}$ in this chapter in order to introduce the methods for handling degenerate critical points without the added complications associated with considering the problem in multidimensional complex space.

In §5 we study the more general case in $\mathbb{C}^d$ where the function f in (1.2) has critical components χ_j of general constant order ω_j and general dimension μ_j. When $\mu_j = 0$, χ_j is simply an isolated critical point. Again, we derive a resurgence relation that can be used to produce a full hyperasymptotic expansion and despite the added generalisations, the results bear similarities to those of §4 and - by extension - the current literature. Interestingly, we find that it is the *codimension* $q_j = d - \mu_j$ of the critical component χ_j rather than the dimension itself that influences the form of the asymptotic expansion, confirming our deduction in §3.3. The current literature on non-isolated critical point asymptotics focuses on very specific cases and only looks at non-exponentially improved expansions in real space, so the results in this chapter provide multiple generalisations to the existing results and correctly reduce to them in the appropriate cases. At the end of this chapter we discuss the practical computation of multidimensional residues; we were able to achieve such computation in a variety of cases. We decided to focus on practical computation rather than a general theoretical investigation.

Note that when f has sets of non-isolated critical points, we are no longer able to use Morse theory to deform S in integral (1.1) into a union of surfaces S_j like we did in the isolated case, as Morse theory is only valid for isolated critical points. However, Morse-Bott theory is the analogue of Morse theory in the case where f has connected components of critical points (namely, sets of non-isolated critical points), meaning that it should in fact be entirely possible to deform S in this way. It has not yet been explicitly proven that this is the case and we are not well versed enough in homology to prove it ourselves, but we believe it is a safe assumption to make like Dingle did in the isolated case. We discuss Morse theory and Morse-Bott theory in §3.2.

In §6 we use the resurgence relation derived in §5 to produce a full hyperasymptotic expansion around the critical component χ_j of general order and dimension. We derive explicit expressions for the expansion and remainder after a general number of substitutions M and write down the optimal truncation scheme. We also discuss rewriting key components of the expansion called the hyperterminants so that they are suitable for numerical evaluation, following the methods of Olde Daalhuis (1998b). This in turn enables us to numerically evaluate the full expansion.

To demonstrate the new results, we work through three examples in §7. The first is an integral in $\mathbb{C}^2$ where f has an order five and an order three critical component - both of dimension one - in which we compute level three ($M = 3$) hyperasymptotic expansions for both critical components. The second looks at the same integrand, but this time we integrate between two linear contour boundaries. Due to the way we have set up our problem in §§4 and 5, we can use the results in these chapters with $\omega = 1$ to compute boundary asymptotics in the case where f is constant on the boundary (if the boundary is a critical component, this requirement is automatically satisfied, with ω the order of the critical component). We compute level one expansions for both boundaries and use these to obtain a level one expansion for the whole problem. The third example is very different and looks at how the behaviour of the asymptotic coefficients of an expansion around a critical component change as we vary the order of degeneracy, yielding interesting results. The general 'factorial-over-power' behaviour that we come to expect from the coefficients remains, but there are differences as ω changes. For degenerate critical points, we have more than one choice of integration contour S_n and perhaps most interestingly, there can be behavioural differences in the coefficients based on which contour we choose.

In §8 we discuss the work required for us to be able to compute the asymptotic contribution of a general boundary in the case where general critical components are present, thus generalising the results of Delabaere and Howls (2002). This chapter starts by introducing miscellaneous smaller results and unfinished work, but focuses mainly on what we believe to be the most important step towards the general boundary case; namely, the development of an asymptotic framework capable of handling critical components that have non-constant order. We provide an example showing where our new results break down in this even more general case. It is worth noting that such cases are not a freak occurrences; on the contrary, such examples can be constructed with ease. A solid grasp on this case is required in order to deal with the restricted critical points that will occur when dealing with a general boundary of integration in the non-isolated case, as well as the occurrence of unrestricted critical points on the boundary, a case near unavoidable in this more general problem.

A summary of our results in presented in §9, with the main point of interest being that essentially all existing results extend quite naturally to the case of degenerate and non-isolated critical points, but such cases require substantially more consideration and careful set-up. We also briefly discuss recent work in high energy physics, noting that our work in §§5 and 6 could be applied to problems considered in Witten (2010).

2 Development of Asymptotic Analysis

In this chapter, we detail the development of asymptotic analysis, focusing on how exponential asymptotic analysis developed as a subject area. We begin with the classic and pivotal definition of an asymptotic series in 1886 due to Poincaré, along with other relevant basic definitions. A more complete historical survey of divergent series - including pre-1886 - is given in Hardy (1949). We then move on to discuss the method of steepest descent for approximating integrals and the progression of exponentially improved integral asymptotics. The regular and higher order Stokes phenomenon are discussed in detail, demonstrating their effects by way of example. The important and related field of exponentially improved solutions to differential equation is also considered in detail, before we move on to a comprehensive survey of the current work involving asymptotic contributions of non-isolated critical points to steepest descent type integrals. These non-isolated critical point results are an important first step, but we will see that there has yet to be any considerations in complex spaces or for exponential improved expansions.

2.1 Poincaré Asymptotic Expansions

We begin our discussion of the development of the subject by recalling some basic definitions. Let c be a limit point of a point set S. As $x \to c$ in S, we recall the following standard definitions from Olver et. al. (2010) (hereafter called NIST):

$$\begin{aligned} f(x) \sim \phi(x) &\iff f(x)/\phi(x) \to 1, \\ f(x) = o(\phi(x)) &\iff f(x)/\phi(x) \to 0, \\ f(x) = \mathcal{O}(\phi(x)) &\iff |f(x)/\phi(x)| \text{ is bounded.} \end{aligned}$$

Poincaré (1886) formally defined what an asymptotic expansion was in an attempt to bring rigour to the field of divergent series. Let $\sum a_r x^{-r}$ be a formal power series (that may either be convergent or divergent) and

$$f(x) = \sum_{r=0}^{N-1} \frac{a_r}{x^r} + \mathcal{O}(x^{-N}) \tag{2.1}$$

as $x \to \infty$ in an unbounded set S in $\mathbb{R}$ or $\mathbb{C}$. Then $\sum a_r x^{-r}$ is an *asymptotic expansion* of $f(x)$, written $f(x) \sim \sum a_r x^{-r}$, as $x \to \infty$ in S (NIST p.42). Truncating this series at the $(N-1)$th term produces an expression with fixed accuracy $\mathcal{O}(x^{-N})$ for all x. An equivalent condition (for all N) useful for computing terms is

$$\lim_{x\to\infty}\left\{x^N\left(f(x)-\sum_{r=0}^{N-1}\frac{a_r}{x^r}\right)\right\}=a_N. \tag{2.2}$$

These definitions provided a calculus for divergent series in that we may perform operations such as addition, subtraction, multiplication, division, and integration on the divergent series representing $f(x)$ as if we were manipulating $f(x)$ directly. However, differentiation requires additional conditions, described in NIST (p.42).

There were some problems with this definition, however; consider the function e^{-x} and begin computing its asymptotic expansion using (2.2). We see that

$$\begin{aligned}
a_0 &= \lim_{x\to\infty}\left\{e^{-x}\right\}=0,\\
a_1 &= \lim_{x\to\infty}\left\{x\left(e^{-x}-a_0\right)\right\}=0,\\
a_2 &= \lim_{x\to\infty}\left\{x^2\left(e^{-x}-a_0-\frac{a_1}{x}\right)\right\}=0,\\
&\vdots\\
a_N &= \lim_{x\to\infty}\left\{x^N\left(e^{-x}-\sum_{r=0}^{N-1}\frac{0}{x^r}\right)\right\}=0
\end{aligned}$$

and thus the asymptotic expansion of e^{-x} is

$$e^{-x}\sim\sum_{r=0}^{\infty}\frac{0}{x^r}\equiv 0.$$

Clearly this is not a helpful result and raises the additional problem of uniqueness; given two functions $f(x)$ and $f(x)+e^{-x}$, we have the result

$$f(x)\sim\sum_{r=0}^{\infty}\frac{a_r}{x^r}=\sum_{r=0}^{\infty}\frac{a_r}{x^r}+0\sim f(x)+e^{-x},$$

meaning that both functions have the same asymptotic expansion. While the series representing $f(x)$ is unique - meaning no other series can represent $f(x)$ (and thus the only asymptotic expansion of e^{-x} is 0) - the function a series represents is *not* unique. In fact, NIST (p.42) tells us that for a given set of coefficients $\{a_r\}$ and suitably restricted set S, there are *infinitely* many analytic functions that it could represent. This is due to definition (2.1) failing to represent small exponentials in any meaningful way.

Recall that by truncating this series at the $(N-1)$th term, we produce an expression with fixed accuracy $\mathcal{O}(x^{-N})$ for all x. This very general statement resulting from Poincaré's definition becomes a limitation when one is concerned with improving accuracy; expressing the '$\mathcal{O}(x^{-N})$' term in definition (2.1) as a bounded remainder term R_N, then truncating at the $(N-1)$th term produces a remainder of $\mathcal{O}(x^{-N})$.

Logically, the optimal truncation point should be when $|R_N|$ attains its least value, but the Poincaré structure gives no information at all as to which value of N this corresponds to. Furthermore, the remainder term is of course a function of x and so the optimal truncation point varies in x, whereas the Poincaré framework results in a fixed truncation point for all x. This systematic variable optimal truncation is called *superasymptotics* and produces accuracy of $\mathcal{O}(\exp(-A|x|))$, for $A > 0$ (Berry and Howls, 1991). The error is exponentially small and so we have achieved exponential accuracy; accuracy beyond all orders of the expansion variable.

Despite the flaws in Poincaré's definition, it enjoys continued use in modern day mathematics for a number of reasons. Computing the first few terms of the series is (usually) a relatively simple task and provides sufficient numerical accuracy to the function it represents after very few terms, in some cases after just one term. This is in stark contrast to the convergent Taylor series of a function, which often requires many more terms to reach the same accuracy as its divergent counterpart. Nonetheless, for some academics the aforementioned flaws loomed over the demonstrably simple and practical uses, and they thus viewed Poincaré expansions as practically useful, but mathematically unsatisfactory.

2.2 Exponential Asymptotics, Hyperasymptotics, and the Method of Steepest Descent

A different approach was taken by Dingle, who shared the viewpoint that the Poincaré definition was not mathematically satisfying and chose to focus on how divergent series were *interpreted* rather than evaluated. Consider any formal divergent series that has been generated as an alternate representation of a function, whose terms initially decrease in size, hit a minimum, and then increase indefinitely with N, one such example being a Poincaré asymptotic series. That the series *numerically* diverges is not seen as a hinderance; so long as the series has been derived through formally exact manipulations (thus ensuring no approximations have entered at any stage), we can treat it as an exact alternate representation of the function from which it is generated. These series are termed *formally exact asymptotic series*; despite numerically diverging, they are symbolically a formally exact representation of their parent function, so we can investigate their properties to gain insight that we would not otherwise see by considering the function in its original form.

Recalling our earlier example whereby - using Poincaré's framework - $e^{-x} \sim 0$ as $x \to \infty$, it was common to disregard such small terms, as from a numerical point of view they are completely subdominant and numerically insignificant in the considered region. While this is of course true, when considering formally exact series we should keep all terms to retain symbolic exactness. This practice has been named *exponential asymptotics*. Considered in this way, we are able to construct asymptotic series whose value varies smoothly as the form of the expansion changes due to Stokes' phenomenon, often producing globally valid expansions. Stokes' phenomenon is discussed in more detail later in §2.3.

Dingle studied exponential asymptotics in the context of both differential equations and integrals, with the principles being the same in each case but the exact method naturally differing. As our research involves extending current theory of integral asymptotics we focus on integrals, specifically those that can be evaluated using the method of steepest descent.

First published by Debye (1909), the method of steepest descent states that integrals of the form

$$I(k) = \int_S d\boldsymbol{z}\, g(\boldsymbol{z}) e^{-kf(\boldsymbol{z})},$$

(namely, equation (1.1)) where $f, g : \mathbb{C}^d \to \mathbb{C}$, $k \in \mathbb{C}$ is a large (asymptotic) parameter, $\boldsymbol{z} = (z_1, \ldots, z_d) \in \mathbb{C}^d$, and $S \subset \mathbb{C}^d$, can be approximated by deforming the surface of integration S into one of steepest change (descent or ascent) that hence passes through critical points of the function f. The integral can then be approximated by considering asymptotic contributions only from certain key points, such as critical points of f, end points or boundaries of integration, and other singularities of the integrand.

Dingle assumed further that each contributing point could be separated by an asymptotic valley (a fact that was later rigorously proved in Pham (1985)), which are regions of the complex plane where $\mathrm{Re}(-kf(\boldsymbol{z})) \to -\infty$, so that the integral is zero there. An asymptotic hill is then where $\mathrm{Re}(-kf(\boldsymbol{z})) \to \infty$, forcing the integral to diverge. If we start and end our integration in an asymptotic valley while passing through a critical point, the value of the integral will therefore be entirely dominated by some function of the critical point, since as we move away from the critical point into a valley, the value of the integrand rapidly approaches zero. Similarly, integrating from a contour boundary along which f is constant (an endpoint of integration in one dimension) into an asymptotic valley means that the value of this integral is entirely dominated by some function of the boundary. These are usually called the *asymptotic contributions* of the objects in question.

If we then deform our integration surface S into a union of steepest surfaces that run between asymptotic valleys, through critical points, and that starts and ends in the same place as S (either at a contour boundary or in an asymptotic valley), then our integral can be approximated extremely accurately by summing up the relevant combinations of individual asymptotic contributions. This method allowed Dingle to derive asymptotic contributions arising from each individual contributing point as a formal power series with coefficients consisting of increasingly complicated combinations of derivatives of f and g, prefactored by a constant times an exponential. It is worth noting that if a specific example dictates that we are on a Stokes line - thus preventing us from separating some contributing points by asymptotic valleys - then these contribution will be some fraction of what would normally be their 'full' asymptotic contribution.

Due to the rapidly increasing symbolic complexity of the coefficients, determining the general form of the later terms of an expansion in an alternate but exact way became desirable. Dingle found that the form of the late terms in an expansion for a contributing point were related to the asymptotic expansion coefficients of certain other contributing points, enabling us to practically calculate the late coefficients in an expansion using the first few simple coefficients from other expansions. This property was termed *resurgence* and resurgence formulae were developed to demonstrate this relationship.

In Berry and Howls (1991), the authors derived the *formally exact resurgence relation* (1.3), allowing us to iterate re-expansion of the remainder term as a function of contributions from other critical points repeatedly. This practice is named *hyperasymptotics*, giving us a *hyperasymptotic expansion* with a large amount of symbolic and algebraic depth, along with greatly increased numerical accuracy over a superasymptotic expansion. As mentioned in the introduction, we are more interested in the algebraic depth nowadays as it provides a lot of subtle detail about the problem. We briefly outline the method employed in Berry and Howls (1991) to derive an asymptotic expansion of (1.2) for the case $d = 1$ for expositionary purposes, as it forms the basis for methods used in more advanced general cases, such as Howls (1997), Delabaere and Howls (2002), and our original research discussed later.

Before we do this, we first define the *order of degeneracy* of the derivatives of a function $f : \mathbb{C} \to \mathbb{C}$ at a point $a \in \mathbb{C}$. This is the value as the value $\omega \in \mathbb{N}$ such that

$$\left.\frac{d^b f(z)}{dz^b}\right|_{z=a} = 0 \quad \text{but} \quad \left.\frac{d^\omega f(z)}{dz^\omega}\right|_{z=a} \neq 0,$$

for all $\mathbb{N} \ni b < \omega$. Essentially, if we keep taking higher order derivatives of f, the order of degeneracy is the order of the first non-zero derivative. To avoid unnecessary over-description, we refer to ω as the order of degeneracy or simply the order of the point a, with the implicit assumption that we are referring to the behaviour of f and its derivatives at that point. A non-degenerate critical point is therefore a critical point of order two, with degenerate critical points having order strictly greater than two and non-critical points corresponding to points of order one. In some literature, order two critical points are called 'quadratic' critical points, order three critical points are called 'cubic' critical points, and so on, with non-critical points being 'linear' points. We will use the term linear to describe non-critical points, but will refer to degenerate critical points by their order ω directly.

Moving on to the derivation, if z_n is a non-degenerate isolated critical point of $f(z)$, we can rewrite the one dimensional version of (1.2) as

$$\begin{aligned} I^{(n)}(k) &= \frac{e^{-kf_n}}{k^{\frac{1}{2}}} T^{(n)}(k), \\ T^{(n)}(k) &= k^{\frac{1}{2}} \int_{C_n} dz\, g(z) e^{-k(f(z)-f_n)}, \end{aligned} \tag{2.3}$$

where $f_n := f(z_n)$ and $C_n \subset \mathbb{C}$. We can then express $T^{(n)}(k)$ as a formal asymptotic series in increasing negative powers of k, namely

$$T^{(n)}(k) \sim \sum_{r=0}^{\infty} \frac{T_r^{(n)}}{k^r}.$$

The $T_r^{(n)}$ are the asymptotic expansion coefficients, which we are able to determine analytically by first making the substitution $u = k\,(f(z) - f_n)$ that transforms (2.3) into the integral

$$T^{(n)}(k) = k^{\frac{1}{2}} \left\{ e^{i\pi} \int_0^{-\infty e^{i\pi}} du\, e^{-u} \frac{g(z(u))}{kf'(z(u))} + e^{i0} \int_0^{\infty e^{i0}} du\, e^{-u} \frac{g(z(u))}{kf'(z(u))} \right\}. \tag{2.4}$$

At z_n, $u = k\,(f(z) - f_n) = 0$ and for all other $z \in C_n$, $u \geq 0$. For each value of $u \neq 0$, there are two values of z and the fact that u is double valued is the reason why there are two terms in (2.4). In the u-plane, equation (2.4) represents integrating from the z-valley $u = -\infty e^{i\pi} = \infty e^{2\pi i}$ to the critical point z_n at $u = 0$, then from the critical point into the other z-valley $u = \infty e^{i0}$, which is exactly what the contour of integration C_n does.

Since g/f' is sufficiently holomorphic, we can use Cauchy's integral formula along with algebraic simplification to rewrite (2.4) as

$$T^{(n)}(k) = \frac{1}{2\pi i k^{\frac{1}{2}}} \int_0^{\infty} du\, \frac{e^{-u}}{u^{\frac{1}{2}}} \oint_{\Gamma_n} dz\, \frac{g(z)(k\,(f(z) - f_n))^{\frac{1}{2}}}{k\,(f(z) - f_n) \left(1 - \frac{u}{k(f(z)-f_n)}\right)}, \tag{2.5}$$

where Γ_n is a loop termed a 'sausage contour' that surrounds C_n. Figures 1 and 2 - taken from Berry and Howls (1991) - illustrate the important objects of study in this problem. We now expand $x = u/k\,(f(z) - f_n)$ according to

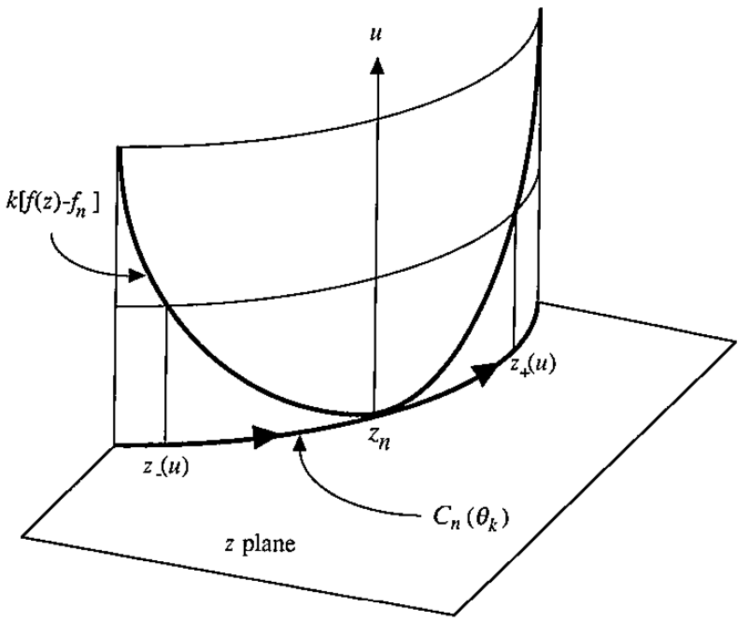


Figure 1: This is Figure 1 from Berry and Howls (1991), showing the double valued transformation $u = k\,(f(z) - f_n)$ and the steepest descent contour C_n through the isolated critical point z_n.

$$\frac{1}{1-x} = \sum_{r=0}^{N-1} x^r + \frac{x^N}{1-x}$$

to get

$$\begin{aligned} T^{(n)}(k,N) = &\sum_{r=0}^{N-1} \frac{1}{2\pi i k^r} \int_0^\infty du\, e^{-u} u^{r-\frac{1}{2}} \oint_{\Gamma_n} dz \frac{g(z)}{(f(z)-f_n)^{r+\frac{1}{2}}} \\ &+ \frac{1}{2\pi i k^N} \int_0^\infty du\, e^{-u} u^{N-\frac{1}{2}} \oint_{\Gamma_n} dz \frac{g(z)}{(f(z)-f_n)^{N+\frac{1}{2}}} \frac{1}{\left(1-\frac{u}{k(f(z)-f_n)}\right)}. \end{aligned} \tag{2.6}$$

The second term in (2.6) will become our remainder term $R^{(n)}(k,N)$.

Expanding out our contour Γ_n into a union of steepest descent contours at infinity (between two valleys and not through any critical points) and fully infinite integration contours C_{m_j} through (adjacent) critical points z_{m_j} allows us to write

$$\oint_{\Gamma_n} \equiv \sum_{j=1}^{\gamma} K_{nm_j} \int_{C_{m_j}},$$

where K_{nm_j} are the Stokes multipliers between z_n and the other non-degenerate isolated critical points z_{m_j}, for $j \in \{1,\ldots,\gamma\}$. The Stokes multipliers take an absolute value of one if z_n and z_{m_j} are adjacent (namely, can be connected by a steepest descent surface), and zero otherwise. The contours at infinity will vanish provided that the condition

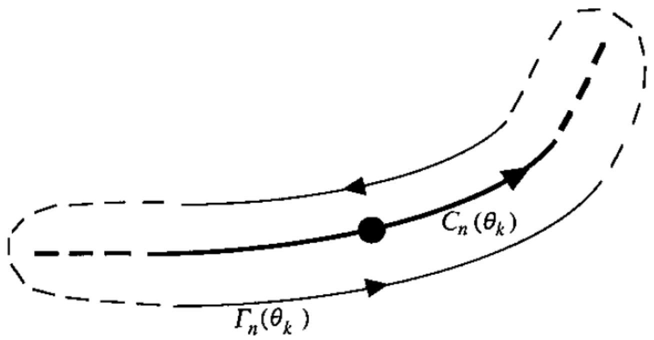


Figure 2: This is Figure 2 from Berry and Howls (1991), showing the steepest descent contour C_n and the sausage contour Γ_n that encloses it.

$$\left|\frac{g(z)}{(f(z)-f_n)^{\frac{1}{2}}}\frac{1}{\left(1-\frac{u}{k(f(z)-f_n)}\right)}\right| = o\left(\frac{1}{k(f(z)-f_n)}\right),\ |k(f(z)-f_n)| \to \infty$$

is satisfied, which can be achieved by prior restriction of the integrand. Additionally, along Γ_n we make the transformation

$$u = \frac{v(f(z)-f_n)}{F_{nm_j}},$$

where $F_{nm_j} = f_{m_j} - f_n$ is the singulant between the two critical points. Substituting these into the second term in (2.6) yields the formally exact resurgence relation

$$\begin{aligned} T^{(n)}(k,N) = & \sum_{r=0}^{N-1} \frac{\Gamma\left(r+\frac{1}{2}\right)}{2\pi i k^r} \oint_{z_n} dz \frac{g(z)}{(f(z)-f_n)^{r+\frac{1}{2}}} \\ & + \frac{1}{2\pi i}\sum_{j=1}^{\gamma} \frac{K_{nm_j}}{(kF_{nm_j})^N} \int_0^\infty dv\, \frac{v^{N-1}e^{-v}}{1-v/kF_{nm_j}}\, T^{(m_j)}\left(\frac{v}{F_{nm_j}}\right) \end{aligned} \tag{2.7}$$

that was mentioned in §1, where we have reduced the sausage contour Γ_n to a small loop around z_n as this is the only singularity of the integrand along C_n. The asymptotic coefficients are thus defined as

$$T_r^{(n)} = \frac{\Gamma\left(r+\frac{1}{2}\right)}{2\pi i}\oint_{z_n} dz \frac{g(z)}{(f(z)-f_n)^{r+\frac{1}{2}}} = \Gamma\left(r+\frac{1}{2}\right) \operatorname*{Res}_{z=z_n}\left(\frac{g(z)}{(f(z)-f_n)^{r+\frac{1}{2}}}\right) \tag{2.8}$$

and substituting this into (2.7) allows us to exactly recover (1.3) from §1. For complete details of the derivation, see Berry and Howls (1991) §§2 and 3. The ideas in this method form the basis for many other more complicated but related derivations.

Howls (1997) extends this idea into the general complex parent space $\mathbb{C}^d$, deriving the asymptotic contribution of a non-degenerate critical point to the general integral (1.2) in the case where f has only non-degenerate isolated critical points. The resulting resurgence relation is identical to (1.3), with the only difference occurring in the coefficients. These are given by

$$T_r^{(n)} = \frac{\Gamma(r+\frac{d}{2})}{2\pi i} \oiint_{B_n} d\boldsymbol{z} \frac{g(\boldsymbol{z})}{(f(\boldsymbol{z})-f_n)^{r+\frac{d}{2}}},$$

with B_n a small ball around the isolated critical point $\boldsymbol{z}_n \in \mathbb{C}^d$. Note that in this d-dimensional case, we cannot simply write the coefficients as a residue due to the complicated nature of multidimensional residues. The method used in this paper differs to the aforementioned papers involving the same author since it makes explicit use of Borel plane techniques and the Borel transform. The main ideas of the derivation remain the same despite employing a different mathematical tool kit, the newer method effectively 'modernising' previous derivations.

Beginning with the extraction of exponential dependence from (1.2), we write $I(k)$ as

$$I^{(n)}(k) = \frac{e^{-kf_n}}{k^{\frac{d}{2}}} T^{(n)}(k), \tag{2.9}$$

where

$$T^{(n)}(k) = k^{\frac{d}{2}} \int_{S_n} d\boldsymbol{z}\, g(\boldsymbol{z}) e^{-k(f(\boldsymbol{z})-f_n)} \tag{2.10}$$

and define $s = f(\boldsymbol{z}) - f_n$, where without loss of generality we set $f_n := 0$. On S_n, we define the differential form $\boldsymbol{\omega}$ by

$$d\boldsymbol{z} = ds \wedge \boldsymbol{\omega} \tag{2.11}$$

so that (by Pham (1967))

$$\boldsymbol{\omega} = \left.\frac{d\boldsymbol{z}}{ds}\right|_{\gamma_n(s)},$$

where $\gamma_n(s)$ are the real $(d-1)$-dimensional hypersurfaces $s =$ constant by which S_n is now parameterised. Symbolically, we express this parameterisation as

$$S_n = \bigcup_{s\geq 0} \gamma_n(s).$$

The hypersurfaces $\gamma_n(s)$ are called *vanishing cycles* and their span is a surface called a *Lefschetz thimble* (Pham 1967). A visualisation of these objects is given in Figure 3 (taken from Howls, 1997), where we can see that the Lefschetz thimble can be represented as a paraboloid based at the critical point $\boldsymbol{z}_n$. The transformation implicitly defined by (2.11) transforms the variables $\boldsymbol{z}$ into a new set containing s and $(d-1)$ other variables, which are locked up within $\boldsymbol{\omega}$ along with all relevant Jacobians. It is worth noting that while the individual components of $\boldsymbol{\omega}$ are not unique, their combination is and this is all that is

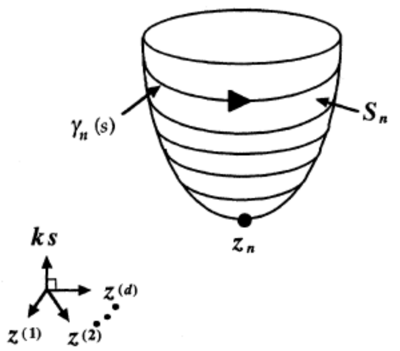


Figure 3: This is Figure 1 from Howls (1997), showing a schematic representation of vanishing cycles $\gamma_n(s)$ as the boundary of a Lefschetz thimble.

important.

Using this transformation, we can rewrite (2.10) as

$$T^{(n)}(k) = k^{\frac{d}{2}} \int_0^{\infty e^{-i\theta_k}} ds\, e^{-ks} \Delta_n G(s),$$

with

$$\Delta_n G(s) = \int_{\gamma_n(s)} \boldsymbol{\omega} g(\boldsymbol{z}).$$

Defined in this way, the s-plane is the *Borel plane* and $\Delta_n G(s)$ is the *Borel transform* of $I(k)$. Note that in this work it is more convenient to consider the ks plane, namely the Borel plane scaled by k.

Since we are assuming that $g(z)$ has no singularities in $\mathbb{C}^d$, then the function $\Delta_n G(s)$ has singularities only where $\boldsymbol{\omega}$ does. It can be shown that $\boldsymbol{\omega}$ and hence $\Delta_n G(s)$ are singular only at the critical points $\boldsymbol{z}_j$ of f. When restricted to S_n, both $\boldsymbol{\omega}$ and $\Delta_n G(s)$ are singular *only* at $\boldsymbol{z}_n$. Thus, when expanding $\Delta_n G(s)$ (in powers of s) around $\boldsymbol{z}_n$, the radius of convergence will be the distance to the nearest singularity on the same Riemann surface as $\boldsymbol{z}_n$. Integrating past this radius of convergence is the cause of the divergence of the expansion.

It is stated that in the cut plane ζ near $s = 0$, we can express $\Delta_n G(s)$ as

$$s^{1-\frac{d}{2}} \Delta_n G(s) = \frac{1}{2\pi i} \oint_{\bar{\gamma}_n(s)} d\zeta\, \frac{\Delta_n G(\zeta) \zeta^{1-\frac{d}{2}}}{\zeta - s}$$

and hence $T^{(n)}(k)$ as

$$T^{(n)}(k) = \frac{k^{\frac{d}{2}}}{2\pi i} \int_0^{\infty} ds\, e^{-ks} s^{\frac{d}{2}-1} \oint_{\Gamma_n} d\zeta\, \frac{\Delta_n G(\zeta) \zeta^{1-\frac{d}{2}}}{\zeta - s},$$

where $\bar{\gamma}_n(s)$ and Γ_n are the images in the Borel plane of $\gamma_n(s)$ and the 'sausage contour'

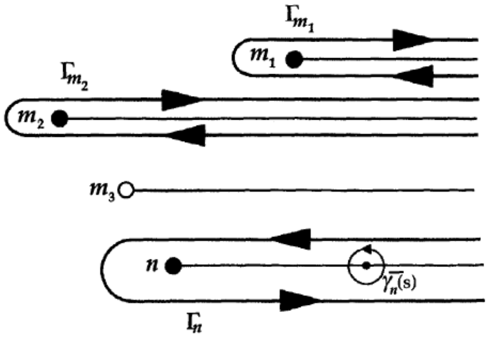


Figure 4: This is Figure 4 from Howls (1997), showing critical points z_j and their associated sausage contours Γ_j. The transformed vanishing cycle $\bar{\gamma}_n(s)$ is defined within the aforementioned text.

from Berry and Howls (1991) respectively. Figure 4 - taken from Howls (1997) - shows a visualisation of these objects. From here, the rest of the derivation is essentially an exercise in algebra that closely follows the method of Berry and Howls (1991); we arrive at

$$T_r^{(n)} = \frac{\Gamma(r+\frac{d}{2})}{2\pi i} \oint_{\zeta=0} d\zeta \, \frac{\Delta_n G(\zeta)}{\zeta^{r+\frac{d}{2}}} = \frac{\Gamma(r+\frac{d}{2})}{2\pi i} \oiint_{B_n} d\boldsymbol{z} \frac{g(\boldsymbol{z})}{(f(\boldsymbol{z})-f_n)^{r+\frac{d}{2}}}$$

as the expression for the asymptotic coefficients $T_r^{(n)}$.

As in the one-dimensional case, we deform Γ_n into a union of steepest descent contours at infinity and similar contours Γ_{m_j} that surround the other singularities of $\Delta_n G(\zeta)$ (namely, the images of the other critical points z_{m_j} in the ζ-plane), with the contours at infinity vanishing provided that

$$\left| \frac{\Delta_n G(\zeta)}{\zeta^{\frac{d}{2}}(1-s/\zeta)} \right| = o\left(\frac{1}{\zeta}\right), \quad |\zeta| \to \infty,$$

achieved by prior restriction of the integrand. Further algebraic manipulation allows us to fully recover the resurgence relation (1.3). For the complete details of the derivation, see Howls (1997). The advantage of the Borel plane technique is that it can be applied to asymptotic solutions of classes of differential equation in much the same way as detailed here and we will see this in action later on in this chapter.

We are also interested in the form of the *late terms* of the expansion (1.3); by considering the difference $T^{(n)}(k,N) - T^{(n)}(k,N+1)$, we arrive at an alternate expression for the asymptotic coefficient $T_{N_n}^{(n)}$ as

$$T_{N_n}^{(n)} = \sum_{j=1}^{\gamma} \frac{K_{nm_j}}{2\pi i} \sum_{\tau=0}^{N_{m_j}-1} \frac{\Gamma(N_n-\tau)}{F_{nm_j}^{N_n-\tau}} T_\tau^{(m_j)} + R,$$

(where R is the remainder) which is itself an asymptotic expansion. This is helpful in calculating the coefficients when r becomes large, but more importantly formally shows the 'factorial-over-power' form of the late terms. This expectation of factorial-over-power like growth is due to a theorem of Darboux (1878) (see - for example - Dingle), which tells us that series expansions (including both Taylor and asymptotic expansions) eventually become wholly dominated by the behaviour of the function in the immediate neighbourhood of the closest singularity to the expansion point. In our case, the expansion point is the critical point $\boldsymbol{z}_n$ and the other singularities are the other critical points $\boldsymbol{z}_{m_j}$, with the form of the late terms indeed revealing factorial-over-power style growth involving the expansions around the other critical points.

If we substitute in the series expansion for $T^{(m_j)}(v/F_{nm_j})$ given by

$$T^{(m_j)}\left(\frac{v}{F_{nm_j}}\right) = \sum_{s=0}^{\bar{N}-1} \frac{F_{nm_j}^s}{v^s} T_s^{(m_j)} + R, \tag{2.12}$$

where R is the remainder, into the resurgence formula (1.3), then the second term in (1.3) now splits into two parts; one that is multiplied by the finite sum to $\bar{N}-1$ - which can then be algebraically manipulated and numerically evaluated - and one that is multiplied by R to form the new and more complicated remainder $\bar{R}$. The resurgence formula (1.3) then takes the form

$$T^{(n)}(k,N) = \sum_{r=0}^{N-1} \frac{T_r^{(n)}}{k^r} + \frac{1}{2\pi i} \sum_{j=1}^{\gamma} \frac{K_{nm_j}}{(kF_{nm_j})^N} \int_0^\infty dv\, \frac{v^{N-1} e^{-v}}{1 - v/kF_{nm_j}} \sum_{s=0}^{\bar{N}-1} \frac{F_{nm_j}^s}{v^s} T_s^{(m_j)} + \bar{R}. \tag{2.13}$$

As discussed earlier, there are many ways in which we may truncate (2.13).

If we discard everything but the first sum and truncate it arbitrarily, we obtain a Poincaré expansion, with truncation at the least term yielding a superasymptotic expansion. If we now only discard $\bar{R}$ from (2.13), then we obtain what is known as an *exponentially improved* expansion. If we discard nothing and instead write down the full expression for $\bar{R}$, we can carry out a similar substitution to (2.12) to achieve an 'extra level' of exponential improvement in the accuracy of the expansion. Naturally, we can carry out this process of substitution into subsequent remainders as many times as we please, allowing us to write down an expression for the expansion after a general number of substitutions M called the *full hyperasymptotic expansion* of the integral. The number of iterations is called the *level* of the expansion and using this terminology, a superasymptotic expansion is a level zero ($M = 0$) hyperasymptotic expansion and an exponentially improved expansion is a level one ($M = 1$) hyperasymptotic expansion. The full hyperasymptotic expansion is then a level M hyperasymptotic expansion and is given by equation (6.6) in Howls (1997). We will see later that this general hyperasymptotic structure does not change as we increase the generality of the critical points involved. Writing down multiple high level hyperasymptotic expansions using this framework enables us to write down simultaneous algebraic equations for the Stokes multipliers K_{nm_j}, allowing us to compute them directly and thus discern the Riemann surface structure of the problem in the Borel plane.

To make effective use of this hyperasymptotic framework, we must have a way of knowing when to truncate the expansion at each level. Olde Daalhuis (1998a) provides us with an optimal truncation scheme - presented in a more direct way for our purposes in Howls (1997) as equation (6.3) - derived by estimating the remainder at the Mth level and using Stirling's approximation on the resulting gamma functions.

As previously mentioned, finite integration boundaries (or endpoints in one dimension) contribute to asymptotic expansions and we can derive hyperasymptotic expansions for their contributions that end up being similar to that of critical points. Howls (1992) derives the asymptotic contribution of a finite endpoint of integration using a similar method to that employed in Berry and Howls (1991). The integral of study

$$\int_{C_e} dz\, g(z) e^{-kf(z)},$$

is similar to integrals we have already considered, but this time the integration contour $C_e \subset \mathbb{C}$ is a steepest descent contour from an endpoint of integration $z_e \in \mathbb{C}$ into an asymptotic valley V.

It is found that critical points of f still cause the divergence of the integral and the cases of a linear endpoint and a non-degenerated critical endpoint are considered. A resurgence relation is obtained, with the results bearing much similarity to those we have already seen; the resurgence relations for the cases of a linear endpoint z_e and a non-degenerate critical endpoint z_n are given by

$$T^{(e)}(k) = \sum_{r=0}^{N-1} \frac{T_r^{(e)}}{k^r} + \frac{1}{2\pi i} \sum_{j=1}^{\lambda} \frac{K_{em_j}}{(kF_{em_j})^{N+\frac{1}{2}}} \int_0^\infty dv\, \frac{v^{N-\frac{1}{2}} e^{-v}}{1 - v/kF_{em_j}} T^{(m_j)}\left(\frac{v}{F_{em_j}}\right),$$
$$T^{(n)}(k) = \sum_{r=0}^{N-1} \frac{T_r^{(n)}}{k^{\frac{r}{2}}} + \frac{1}{2\pi i} \sum_{j=1}^{\gamma} \frac{K_{nm_j}}{(kF_{nm_j})^{\frac{N}{2}}} \int_0^\infty dv\, \frac{v^{\frac{N}{2}-1} e^{-v}}{1 - \sqrt{v/kF_{nm_j}}} T^{(m_j)}\left(\frac{v}{F_{nm_j}}\right)$$

respectively (recall $\{z_1, \ldots, z_\lambda\}$ is the full set of critical points of f and $\{z_1, \ldots, z_\gamma\}$ is every critical point minus z_n). The coefficients in these cases are given by

$$
\begin{aligned}
T_r^{(e)} &= \frac{\Gamma(r+1)}{2\pi i}\oint_{z_e} dz \frac{g(z)}{(f(z)-f_e)^{r+1}} = \Gamma(r+1)\operatorname*{Res}_{z=z_e}\left(\frac{g(z)}{(f(z)-f_n)^{r+1}}\right), \\
T_r^{(n)} &= \frac{\Gamma\left(\frac{r+1}{2}\right)}{2\pi i}\oint_{z_n} dz \frac{g(z)}{(f(z)-f_n)^{\frac{r+1}{2}}} = \Gamma\left(\frac{r+1}{2}\right)\operatorname*{Res}_{z=z_n}\left(\frac{g(z)}{(f(z)-f_n)^{\frac{r+1}{2}}}\right)
\end{aligned}
$$

respectively. The relevant late terms are also consistent with Darboux's theorem. More detail can be found in Howls (1992).

The multidimensional generalisation of endpoints of integration is a general boundary of integration D. Delabaere and Howls (2002) discuss this case in great algebraic depth and once the appropriate framework has been set up, they proceed - once again - in the style of Howls (1997) in order to derive a resurgence relation involving the critical points of f and the boundary D. Due to the complicated nature of the derivation, we refer the reader directly to the paper for complete details. Note that if f is constant on the boundary D, then the derivation of its hyperasymptotic expansion is substantially simpler than the fully general case. We shall see this later in §5. The important result of the paper is that the form of the expansion remains unchanged, simply augmented by the presence of a boundary.

Imposing an arbitrary boundary along which f is not constant will force new extrema to appear, so we have to distinguish between the *real* critical points of the function f and the *restricted* critical points that only exist due to the imposition of a boundary. The difference between this formulae and (1.3) is then the presence of numbers p_j called the *depths* of the critical points, which indicate how many boundaries the critical points are associated with. For example, real critical points of f that are not on the boundary have depth $p = 0$, whereas any restricted critical points by definition lie on a boundary of D in at least one dimension and so have depth $p > 0$. Once the resurgence relation has been established, expressions for late terms (that obey Darboux's theorem) and a hyperasymptotic expansion come quickly as in previous papers.

In all of the papers discussed in this chapter that showcase resurgence relations, it is shown that they all incorporate the important concept of the Stokes phenomenon. We move on now to discuss this in detail.

2.3 Stokes' Phenomenon

Throughout this section, we will use integral representations of the Airy and Pearcey functions to demonstrate key concepts related to Stokes' phenomenon.

The Airy function $\mathrm{Ai}(z)$ can be defined as

$$\mathrm{Ai}(z) = \frac{\sqrt{z}}{2\pi i} \int_{V_1(z)}^{V_2(z)} du \, \exp\left(z^{\frac{3}{2}} \left(\tfrac{1}{3}u^3 - u \right) \right), \tag{2.14}$$

where

$$V_1(z) = \infty z^{-1/2} e^{-i\pi/3} \quad \text{and} \quad V_2(z) = \infty z^{-1/2} e^{i\pi/3}.$$

However, we are only interested in the form of the integrand and so for the purposes of illustrating concepts, we instead consider the integral

$$\int_{V_1(k)}^{V_2(k)} du \, \exp\left(-k \left(u - \tfrac{1}{3}u^3 \right) \right), \tag{2.15}$$

where $z^{\frac{3}{2}} =: k \in \mathbb{C}$ is the asymptotic large parameter,

$$V_1(k) = \infty k^{-1/3} e^{-i\pi/3}, \quad \text{and} \quad V_2(k) = \infty k^{-1/3} e^{i\pi/3}.$$

The phase function (including the k in this example) is

$$f(u;k) = -k\left(u - \frac{u^3}{3} \right)$$

and has non-degenerate isolated critical points at $u = \pm 1$, which we denote by z_1 and z_2 respectively. We pick the principal branch cut for z and hence k and are interested in varying k around the unit circle. By defining $k = |k|e^{i\phi}$ and taking $|k| = 1$ (so that we are on the unit circle), we rewrite the valleys as

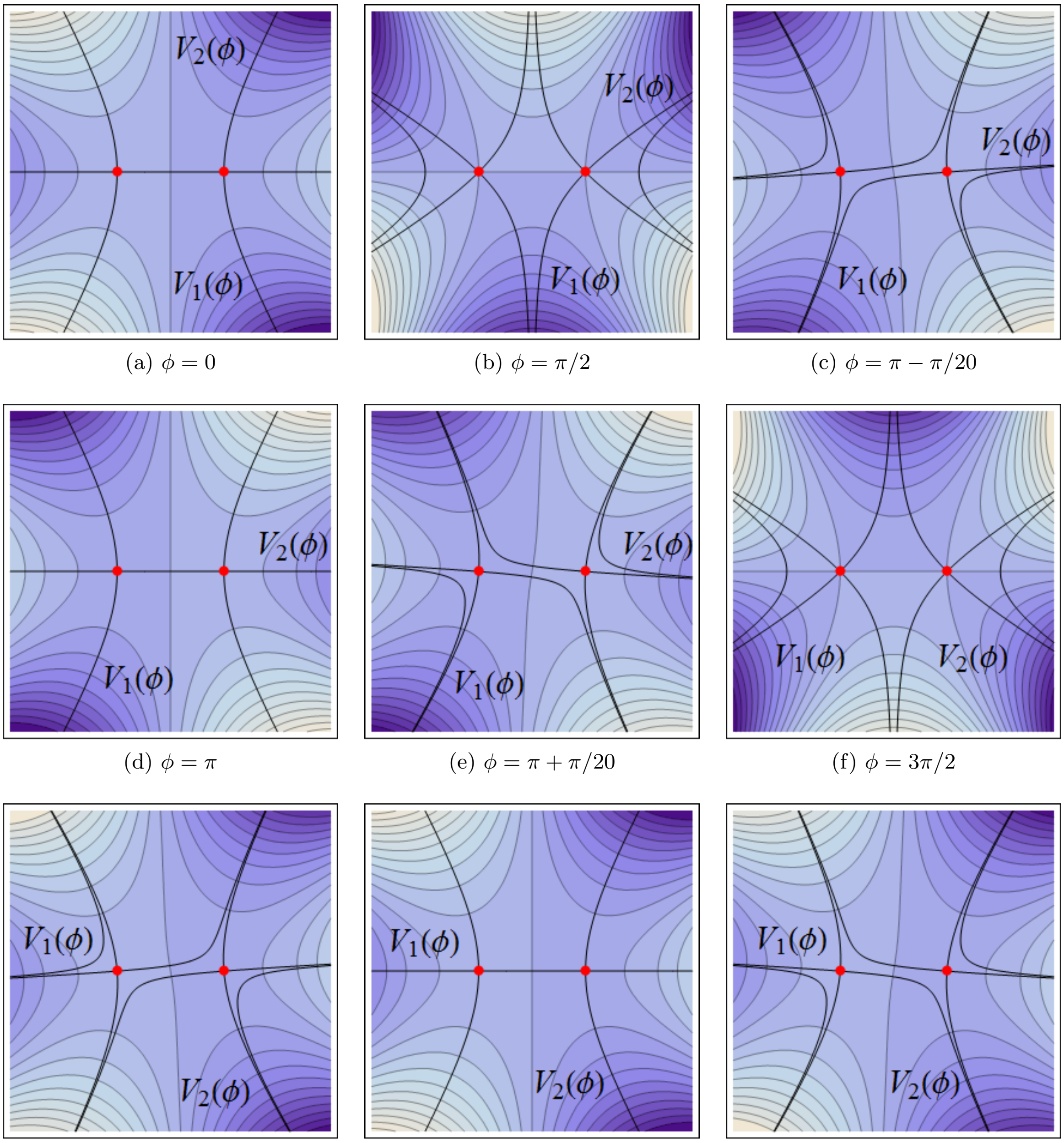
$$V_1(\phi) = \infty e^{\frac{i}{3}(-\pi-\phi)} \quad \text{and} \quad V_2(\phi) = \infty e^{\frac{i}{3}(\pi-\phi)}$$

and the phase function as

$$f(u;\phi) = -e^{i\phi}\left(u - \frac{u^3}{3} \right).$$

Note that when writing an integral between two valleys, we are implying continuous integration along steepest contours between these valleys; a similar meaning is implied when writing the 'interval' (V_a, V_b) for two valleys.

We consider the asymptotic expansion for the integral (2.15) using the method of steepest descent and by making use of contour plots of the phase function $f(u;\phi)$, we can determine which points will contribute to the asymptotics. We will study the form of the asymptotics as $k = e^{i\phi}$ varies by considering $\phi \in [0, 3\pi]$ and it will be helpful to now refer to Figure 5.



(a) $\phi = 0$ (b) $\phi = \pi/2$ (c) $\phi = \pi - \pi/20$

(d) $\phi = \pi$ (e) $\phi = \pi + \pi/20$ (f) $\phi = 3\pi/2$

(g) $\phi = 2\pi - \pi/20$ (h) $\phi = 2\pi$ (i) $\phi = 2\pi + \pi/20$

Figure 5: Contour plots of $f(u;\phi) = -e^{i\phi}(u - \frac{1}{3}u^3)$ in the complex u-plane. The red dots are the critical points $u = \pm 1$.

Note that throughout this thesis, the thicker black lines in contour plots such as Figure 5 will represent the steepest descent contours through points of interest, such as critical points of the phase function or endpoints of integration.

For $\phi = 0$, the valleys are

$$V_1(0) = \infty e^{-\frac{i\pi}{3}} \quad \text{and} \quad V_2(0) = \infty e^{\frac{i\pi}{3}}$$

and so the only asymptotic contribution comes from z_1. As we increase ϕ, we see that this remains the only contribution until $\phi = \pi$, where the contour snaps onto the other critical point z_2. This results in the contour of integration travelling up from $V_1(\pi) = \infty e^{-2\pi/3}$ and through z_2, where we pick up a second asymptotic contribution, then sharply turn onto the steepest contour through z_1 into the valley $V_2(\pi) = +\infty$. Continuing to increase ϕ, we see that immediately after leaving $\phi = \pi$, the contour snaps back off and now forces us into the third valley V_3 on our way from V_1 to V_2, meaning we retain this additional contribution. We still pick up two contributions at $\phi = 2\pi$ - where we again get a sharp turn in contour - but for $2\pi < \phi \leq 3\pi$, we only have one contribution from z_2. Further increasing ϕ will provide similar changes to the combination of contributions. Note that when we reach $\phi = 6\pi$, we are back to where we started at $\phi = 0$.

In summary, as we varied k asymptotic contributions were 'switched on and off' as the combination of contributions changed; this is known as *Stokes' phenomenon*. This phenomenon occurs across *Stokes lines*, given in this example by solutions to

$$\operatorname{Im}\left(f_1(\phi)\right) = \operatorname{Im}\left(f_2(\phi)\right) \Rightarrow \operatorname{Im}\left(f_1(\phi) - f_2(\phi)\right) = 0,$$

where $f_j(\phi) := f(z_j; \phi)$. This reduces to $\operatorname{Im}\left(e^{i\phi}\right) = 0$ that has solutions $\phi = c\pi$ for $c \in \mathbb{Z}$, matching up with the occurrence of Stokes' phenomena in our example.

For a general steepest descent integral, the Stokes lines for two critical points are curves in a complex parameter space along which the phase function is equal at both critical points. For integrals such as (2.10), the Stokes lines for z_n and z_{m_j} are given by

$$\operatorname{Im}(-k\left(f(\boldsymbol{z}) - f_n\right)) = \operatorname{Im}\left(-k\left(f(\boldsymbol{z}) - f_{m_j}\right)\right) \Rightarrow \operatorname{Im}\left(-k\left(f_n - f_{m_j}\right)\right) = 0.$$

We also have *anti-Stokes lines*, given by

$$\operatorname{Re}(-k\left(f(\boldsymbol{z}) - f_n\right)) = \operatorname{Re}\left(-k\left(f(\boldsymbol{z}) - f_{m_j}\right)\right) \Rightarrow \operatorname{Re}\left(-k\left(f_n - f_{m_j}\right)\right) = 0$$

that are important for example in the study of non-linear ordinary differential equations, but not important in our original research so we do not discuss them further here.

Stokes' phenomenon is illustrated in a simple way in the complex 'f-plane', known also as the *Borel plane*. In our specific case, we scale the Borel plane as we are considering the '$-kf(z)$-plane'. In the Borel plane, non-degenerate critical point are square root branch points, where we take cuts that are convenient. Recalling our algebraic criteria for a Stokes phenomenon as $\mathrm{Im}(-k\,(f(\boldsymbol{z})-f_n)) = \mathrm{Im}\left(-k\left(f(\boldsymbol{z})-f_{m_j}\right)\right)$, this implies that we can spot a Stokes phenomenon in the Borel plane by noticing that two contributing points line up horizontally.

In Berry (1989), it is shown that the switching on or off of a contribution is a *continuous* change. Initially, it was believed that when a Stokes line was crossed and the combination of contributions changed, the value of the asymptotic expansion changed discontinuously. Berry showed that with the appropriate description and set-up of the problem, crossing a Stokes line produces a continuous change in the value of the asymptotic expansion.

As mentioned in §2.2, by retaining exponentially small terms we can produce formally exact asymptotic expansions that have the Stokes phenomenon built in; namely, they appropriately change their form to take into account the change in the combination of contributions when a Stokes phenomenon takes place. This means we are able to generate a single asymptotic expansion for integral (1.1) that is valid for all non-zero k and whose value changes continuously with k; a globally valid asymptotic expansion. Paris (1992) rigorously extended this result to also be valid for high order differential equations.

We will now discuss what is known as the *higher order Stokes phenomenon*. This is when the *possibility* of a Stokes phenomenon is switched on and off, as the Stokes multipliers K_{nm} are no longer constant. We will again proceed by example using the Pearcey function.

The Pearcey function can be defined in a variety of ways; the definition from NIST is

$$P(X,Y) = \int_{-\infty}^{\infty} dt \, \exp\left(i\left(t^4 + Yt^2 + Xt\right)\right),$$

where X and Y are real, but as with the Airy function we will use a form which is more useful to us. By rescaling, we define

$$P(k;a) = \int_{\infty e^{-3\pi i/8}}^{\infty e^{\pi i/8}} dz \, \exp\left(ik\left(\tfrac{1}{4}z^4 + \tfrac{1}{2}z^2 + az\right)\right), \tag{2.16}$$

where $k \in \mathbb{C}$ is the positive asymptotic parameter and $a \in \mathbb{C}$ is a control parameter. The phase function for (2.16) is

$$f(z;a) = -i\left(\frac{z^4}{4} + \frac{z^2}{2} + az\right), \tag{2.17}$$

with critical points given by solutions to $z^3 + z + a = 0$. The three critical points of $f(z;a)$ are all non-degenerate and will vary with a, with

$$f_j(a) := f(z_j;a) = -\frac{iz_j}{4}(z_j + 3a), \quad j = 0, 1, 2.$$

Note that the asymptotic valleys will not vary with a, only k. We take $k \in \mathbb{R}$ positive here for ease of exposition, as the authors do in Howls, Langman, and Olde Daalhuis (2004).

A *Stokes crossing point* (SCP) is a regular point where Stokes lines cross. We consider values of a in a small circle around one such point for the integral (2.16) to demonstrate the various Stokes phenomena taking place. The contour taking us from V_1 to V_2 will always be termed C, with individual components C_n described in the caption for Figure 6 (note that Figures 6 - 9 are taken from Howls, Langman, and Olde Daalhuis, 2004). Without loss of generality, we assume that for the values of a we use here, the critical points are such that $\text{Re}(f_2) > \text{Re}(f_1) > \text{Re}(f_0)$. Figures 6 and 7 help us visualise what is happening as we change a.

Starting at a_1, we describe what happens in both the integration (z) and Borel (f) planes as we sweep around the SCP to a_9. As previously mentioned, when two or more critical point images are horizontally collinear in the Borel plane, a Stokes phenomenon is taking place.

Considering a_1, we see nothing of interest is happening in the z-plane, although the three points in the Borel plane are collinear, albeit not horizontally. We shall return to this observation later.

Moving on to a_2, we see the contour C snaps into z_1 and the corresponding Borel plane picture demonstrates the presence of a Stokes phenomenon. To save unnecessary over-description, we note now that a_5, a_7, a_8 and a_9 all share this standard behaviour, although the Stokes line through a_9 does not affect asymptotic contributions when integrating between the given valleys.

Topographically, C has the same behaviour for both a_3 and a_4 in the z-plane (hence only one plot in Figure 6). However, in the Borel plane, a_3 and a_4 give rise to different behaviour. For a_3, no points are collinear in any direction, indicating no Stokes phenomenon (hence no sketch in Figure 7); however, for a_4, the Borel plane in fact indicates that a Stokes phenomenon should be happening, despite the lack of a Stokes line in the a-plane or any indication of topological change in the z-plane. Again, we will come back to this shortly.

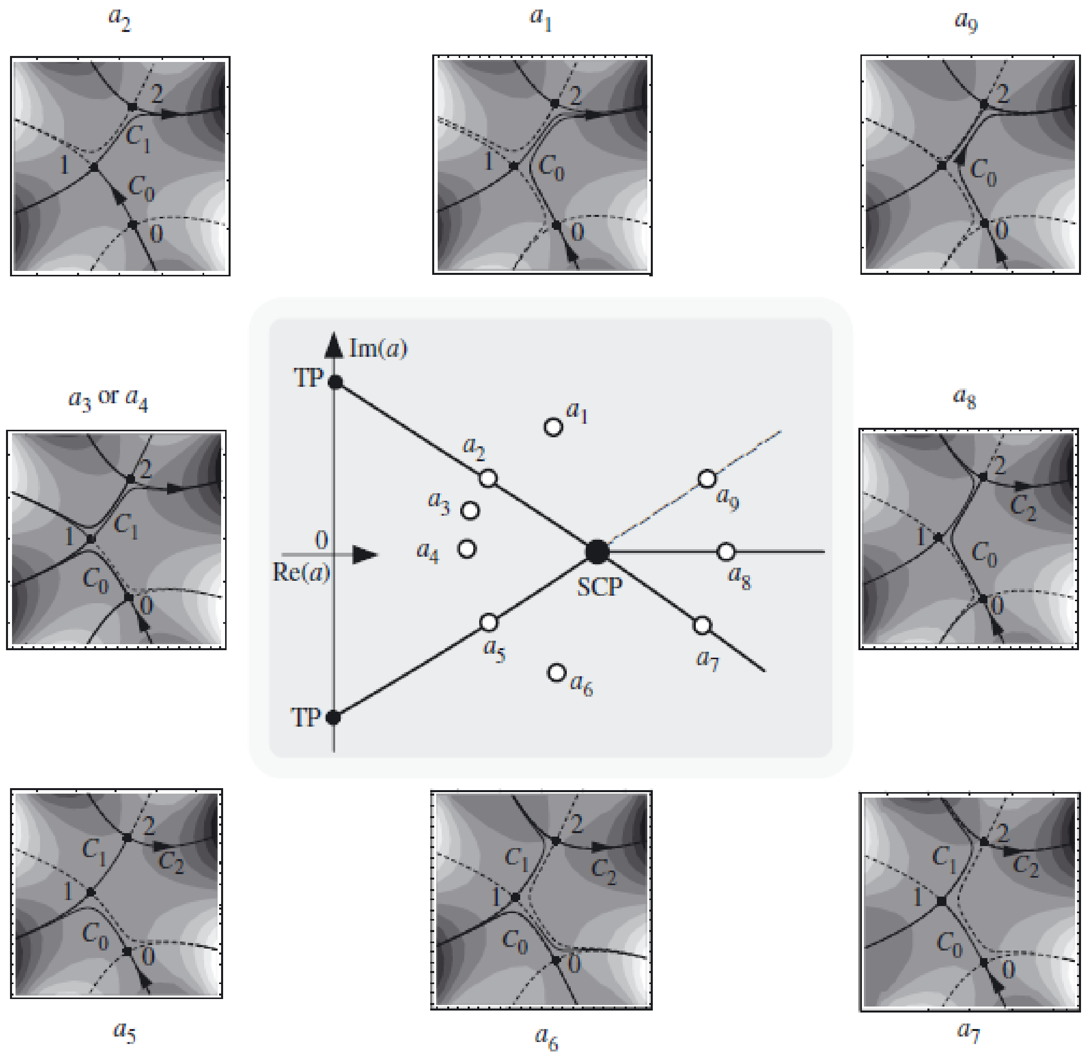


Figure 6: This is Figure 1 from Howls, Langman, and Olde Daalhuis (2004). It shows the Stokes structure in the complex a-plane, surrounded by contour plots of the function (2.17), which show steepest descent paths C_n through saddles z_n that take us from V_1 to V_2 in integral (2.16), for different values a_i. The Stokes line through a_9 is dashed because it has no effect while integrating between these valleys.

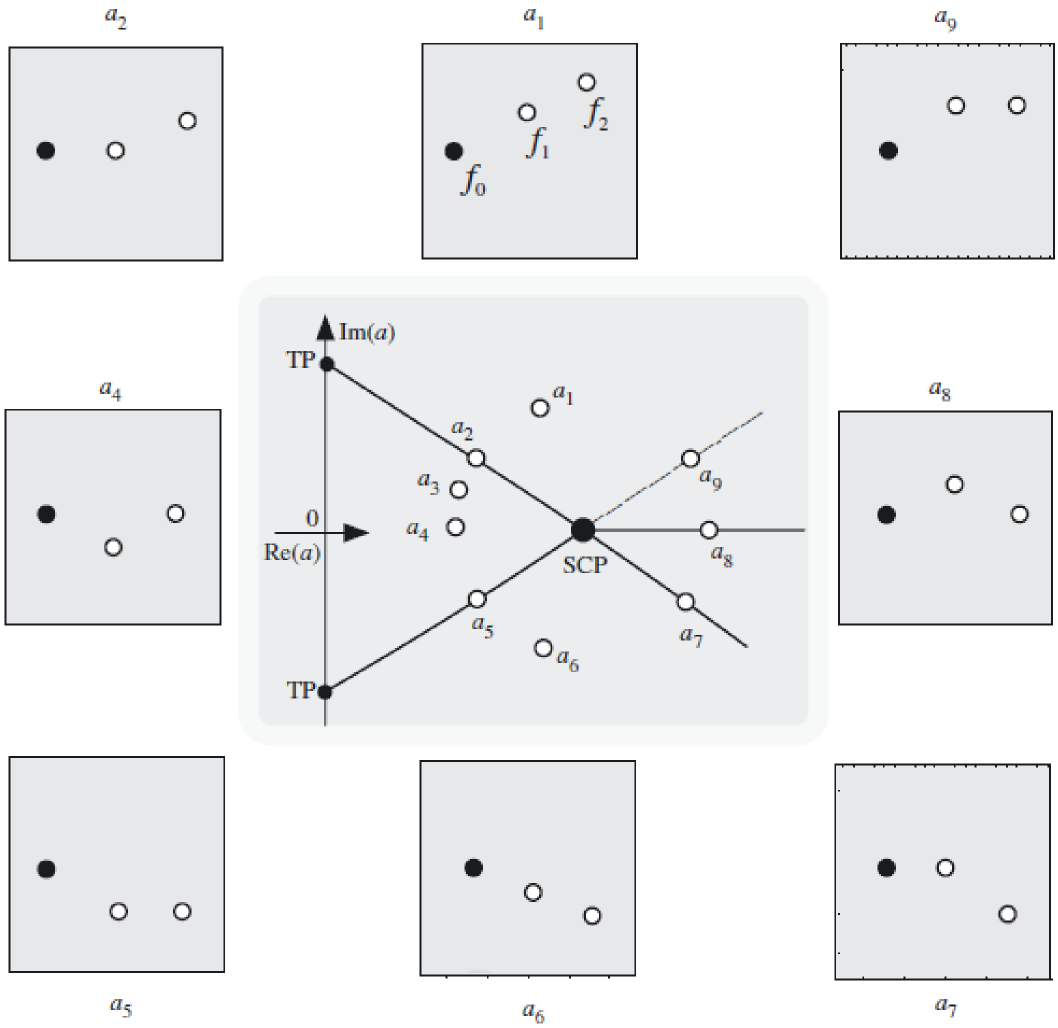


Figure 7: This is Figure 2 from Howls, Langman, and Olde Daalhuis (2004). It shows the Stokes structure in the complex a-plane, surrounded by sketches of the Borel plane for integral (2.16) at different values a_i, corresponding to those in Figure 6. The black point in the Borel plane sketches is the image $f_0(a_i)$ of the saddle point z_0, with the white points being the images $f_1(a_i)$ and $f_2(a_i)$ of z_1 and z_2 respectively.

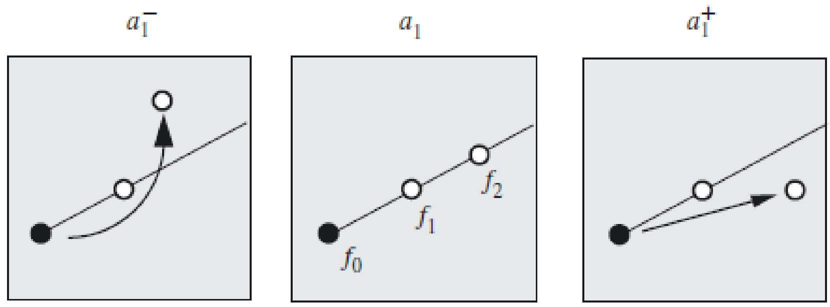


Figure 8: This is Figure 3 from Howls, Langman, and Olde Daalhuis (2004). It shows how critical point adjacency changes as a is moved past a_1.

The last remaining interesting point is a_6 that mimics the behaviour of a_1, in that nothing of interest is indicated in the plane of integration, but all three points in the Borel plane are non-horizontally collinear.

The main question raised through this analysis is with regards to a_4; why is a Stokes phenomenon occurring away from a Stokes line? To understand what is happening we introduce the concept of turning points.

If two non-degenerate isolated critical points $X_1(\alpha)$ and $X_2(\alpha)$ of some function $H(z;\alpha)$ coalesce at $\alpha = \alpha_0$, then α_0 is a *turning point* of H. This turning point (TP) is then a degenerate isolated critical point. If $X_1(\alpha)$ and $X_2(\alpha)$ are themselves degenerate, then the resulting TP will be a higher order critical degenerate point. The asymptotics of the individual critical points will break down at the TP, as it is of higher order there; we would need to describe the asymptotics at the TP separately using our new results from §4, or by using uniform asymptotics (such as Berry and Howls (1993)). At a *virtual turning point* (VTP), we have *apparent* coalescence in the Borel plane; that is, $f(X_1) = f(X_2)$ for two critical points X_1 and X_2 of f, so appear to collapse onto each other as branch points in the Borel plane. However, the branch points are on different Riemann surfaces, so are simply passing over each other at the VTP. Meanwhile, there is *nothing* of interest at all happening in the z-plane. We are in exactly this situation in this example; when $a = 0$, the critical points of (2.17) are 0 and $\pm i$ with $f(0) = 0$ and $f(\pm i) = i/4$. Thus, $a = 0$ is a VTP of (2.17). This presence of the VTP forces a change in the Riemann surface structure of the Borel plane as we vary a. Due to this, a *higher order Stokes curve* (HSC) is introduced, across which *higher order Stokes phenomena* take place (Figure (9)). In our case, the HSC would pass through a_1, a_6, the two TP's and the SCP. Referring to Figure 8, the critical point image f_2 is not on the same Riemann surface as f_0 and f_1 until all three are collinear, so no Stokes phenomenon can occur until this has happened. Past this point, f_2 is on the same Riemann surface as f_0; the critical points z_0 and z_2 have *become* adjacent, so the *possibility*

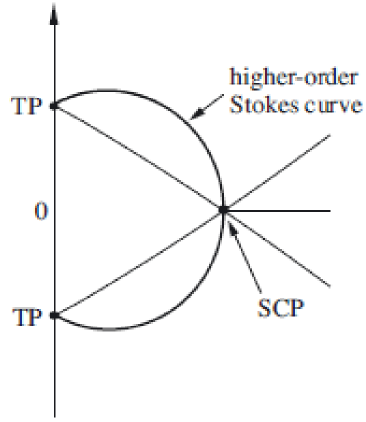


Figure 9: This is part of Figure 4 from Howls, Langman, and Olde Daalhuis (2004), illustrating the location of the higher order Stokes curve.

for a Stokes phenomenon has been switched on. This switching on and off of critical point adjacency means that the Stokes multipliers K_{nm} will be non-constant in the presence of a higher order Stokes phenomenon.

Regular Stokes lines depend on k as we saw earlier when we defined them, but Howls, Langman, and Olde Daalhuis (2004) show that the collinearity condition giving rise to the HSC is given by

$$\frac{f_{m_2}(a) - f_{m_1}(a)}{f_{m_3}(a) - f_{m_2}(a)} \in \mathbb{R},$$

meaning that the higher order Stokes curve is k independent. A consequence of this is that the SCP is k dependent, although it will always lie on the HSC. For in-depth algebraic analysis using the hyperasymptotic methods outlined earlier, the reader is referred to Howls, Langman, and Olde Daalhuis (2004).

2.4 Mellin-Barnes Integrals

The study of Mellin-Barnes integrals is a rich area, but we will focus on one specific result from Kaminski and Paris (2001) in the context of integrals whose phase function has non-isolated critical points. The method described below is for a two-dimensional integral, but the basic concepts translate into higher dimensions with full details found within the text.

Integrals can be written in Mellin-Barnes form - namely, as special combinations of gamma functions - and it is then possible to extract from this representation the asymptotic behaviour of the integral of interest. A core result from §3.3 of the text is that using the inverse Mellin transform, we may write

$$e^{-z} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds\, \Gamma(s) z^{-s}, \quad |\arg(z)| < \frac{\pi}{2}, \quad z \neq 0, \tag{2.18}$$

where the integration path is the vertical line $\mathrm{Re}(s) = c > 0$ that lies to the right of all poles of $\Gamma(s)$. If we thus consider the class of integrals

$$I(k) = \int_0^\infty \int_0^\infty dx\, dy\, g(x,y) e^{-kf(x,y)}$$

as in §7.3 of the text, where f is a polynomial of the form

$$f(x,y) = x^\mu + y^\nu + \sum_{r=1}^{p} c_r x^{m_r} y^{n_r},$$

with $\mu, \nu \in \mathbb{N}$, $m_r, n_r \in \mathbb{N}_0$, $c_r = a_r + ib_r$, $a_r \geq 0$, and $b_r \in \mathbb{R}$, we are able to use (2.18) to rewrite the entire integral in terms of gamma functions (similar but more advanced problems will work in a similar way). Note that the asymptotic parameter k is complex, but will need to be restricted in order for the integral to converge and also that for simplicity, the authors take $g \equiv 1$, but the method is applicable for more general choices of g. Thus, for the specific integral

$$I(k) = \int_0^\infty \int_0^\infty dx\, dy\, \exp\left[-k\left(x^\mu + y^\nu + \sum_{r=1}^{p} c_r x^{m_r} y^{n_r}\right)\right]$$

under consideration, its Mellin-Barnes representation is

$$I(k) = \frac{k^{-\frac{1}{\mu}-\frac{1}{\nu}}}{\mu\nu} \left[\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\right]^p d\boldsymbol{t}\, \Gamma(\boldsymbol{t})\, \Gamma\left(\frac{1 - \boldsymbol{m}\cdot\boldsymbol{t}}{\mu}\right) \Gamma\left(\frac{1 - \boldsymbol{n}\cdot\boldsymbol{t}}{\nu}\right) \boldsymbol{c}^{-\boldsymbol{t}} k^{-\boldsymbol{\delta}\cdot\boldsymbol{t}},$$

where

$$\delta_r = 1 - \frac{m_r}{\mu} - \frac{n_r}{\nu},$$

$$\boldsymbol{m} = (m_1, \ldots, m_p), \quad \boldsymbol{n} = (n_1, \ldots, n_p), \quad \boldsymbol{t} = (t_1, \ldots, t_p), \quad \boldsymbol{\delta} = (\delta_1, \ldots, \delta_p),$$

$$\Gamma(\boldsymbol{t}) = \Gamma(t_1)\cdots\Gamma(t_p), \quad \boldsymbol{c}^{-\boldsymbol{t}} = c_1^{-t_1}\cdots c_k^{-t_k}, \quad \text{and} \quad d\boldsymbol{t} = dt_1\cdots dt_p.$$

From this representation, it is possible to obtain the asymptotic behaviour of the integral and develop a hyperasymptotic expression and optimal truncation scheme. Deriving hyperasymptotic expansions using the Mellin-Barnes technique is demonstrated in §6 of the text, applying it to the confluent hypergeometric function $U(a, b, z)$.

In the Mellin-Barnes representation and in the resulting asymptotic expansion for $I(k)$, division by μ and ν is necessary and commonplace. It is stated that the presence of the terms x^μ and y^ν forces any critical point of f to be isolated and this agrees with our own experience and results later in the thesis. Hence, for non-isolated critical points, we require that at least one of the terms x^μ or y^ν is missing from the polynomial $f(x, y)$. This means that at least one of μ or ν must be zero, hence invalidating the results obtained; these results therefore can only apply when f has only isolated critical points, appearing (immediately) inextensible to cases where f has sets of non-isolated critical points. Analogous situations in higher dimensions follow a similar pattern.

As a brief addition, we mention the paper by Breen and Wood (2004) that looks at multiple integral solutions to nth order linear differential equations and rewrites them as Mellin-Barnes integrals. In doing so they indirectly handle degenerate isolated critical points and derive an exponential improved expansion involving them (among other asymptotic contributions), but this is not explicitly stated or elaborated upon.

2.5 Exponential Asymptotic Solutions to Differential Equations

The area of asymptotic solutions to differential equations is one for which a vast quantity of literature exists and we shall briefly discuss some work related to exponential asymptotic solutions to differential equations. The basic concept is much the same as in integral asymptotics; given some differential equation, we wish to construct asymptotic solutions that satisfy it and we do so by using certain properties of the differential equation under consideration. A great deal of work has been done regarding non-exponentially improved asymptotic solutions, but we do not discuss this here.

The renewed interest in exponentially improved asymptotic expansions generated by papers from Berry and Howls inspired the papers Olver (1991a, 1991b) and Olde Daalhuis (1992, 1993). The former two papers looked at exponentially improved asymptotic expansions for the generalised exponential integral and the confluent hypergeometric function $U(a, b, z)$ respectively, with the latter two looking at hyperasymptotic expansions for $U(a, b, z)$, valid away from and in the neighbourhood of Stokes lines respectively. In Olver (1993), the results of Olver (1991b) were rigorously re-derived using only differential equation theory, laying the foundations required for the derivation of hyperasymptotic solutions to differential equations. These solutions would then be able to incorporate the Stokes phenomenon, much like the hyperasymptotics for integrals discussed earlier.

Kummel's differential equation

$$z\frac{d^2w}{dz^2} + (b - z)\frac{dw}{dz} - aw = 0$$

has the two linearly independent solutions

$$
\begin{aligned}
M(a,b,z) &= \sum_{r=0}^{\infty} \frac{(a)_r}{(b)_r} \frac{z^r}{\Gamma(r+1)} = {}_1F_1\left(\begin{matrix} a \\ b \end{matrix}; z\right), \\
U(a,b,z) &= \frac{\Gamma(1-b)}{\Gamma(a-b+1)} M(a,b,z) + \frac{\Gamma(b-1)}{\Gamma(a)} z^{1-b} M(a-b+1, 2-b, z) \\
&= \frac{\Gamma(1-b)}{\Gamma(a-b+1)} {}_1F_1\left(\begin{matrix} a \\ b \end{matrix}; z\right) + \frac{\Gamma(b-1)}{\Gamma(a)} z^{1-b} {}_1F_1\left(\begin{matrix} a-b+1 \\ 2-b \end{matrix}; z\right)
\end{aligned}
$$

that are confluent hypergeometric functions, where

$$
{}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}; z\right) = \sum_{r=0}^{\infty} \frac{(a_1)_r \ldots (a_p)_r}{(b_1)_r \ldots (b_q)_r} \frac{z^r}{\Gamma(r+1)}
$$

is the generalised hypergeometric function and

$$
(a)_r = a(a+1)(a+2)\cdots(a+r-1) = \frac{\Gamma(a+r)}{\Gamma(a)}, \quad (a)_0 = 1
$$

is Pochhammer's symbol (NIST). The result that is re-derived is given by Theorem 1.1 in Olver (1993), namely that the function $U(a, a-b+1, z)$ has the exponentially improved asymptotic expansion

$$
U(a, a-b+1, z) = \frac{1}{z^a} \sum_{s=0}^{n-1} (-1)^s \frac{(a)_s (b)_s}{\Gamma(s+1) z^s} + R_n(a,b,z),
$$

where $n = |z| - \mathrm{Re}(a) - \mathrm{Re}(b) + 1 + \alpha$, with $a, b \in \mathbb{C}$ constants, $|z|$ large, and $|\alpha|$ bounded (note that in Olver (1993) equation (1.3), the Real function is missing from a and b; we infer than this is a typo based on the preceding paragraph). We then have

$$
\begin{aligned}
R_n(a,b,z) = {} & \frac{(-1)^n 2\pi z^{b-1} e^z}{\Gamma(a)\Gamma(b)} \cdots \\
& \times \sum_{s=0}^{m-1} \frac{(-1)^s (1-a)_s (1-b)_s}{\Gamma(s+1)} \frac{F_{n+a+b-s-1}(z)}{z^s} + (1-a)_m (1-b)_m R_{m,n}(a,b,z),
\end{aligned}
$$

where $m \in \mathbb{N}_0$ is constant,

$$
F_p(z) = \frac{e^{-z}}{2\pi} \int_0^\infty dt \, \frac{e^{-zt} t^{p-1}}{1+t},
$$

is a generalised exponential integral, and

$$
R_{m,n}(a,b,z) = \begin{cases} \mathcal{O}(e^{-z-|z|} z^{-m}) & \text{for } \quad |\arg(z)| \leq \pi, \\ \mathcal{O}(z^{-m}) & \text{for } \pi \leq |\arg(z)| \leq \frac{5\pi}{2} - \delta, \end{cases}
$$

with $\delta > 0$ constant. It is stated that the methods and techniques developed in Olver (1993) would serve as a valuable preliminary to a more general theory of exponentially improved solutions to differential equations.

In Olde Daalhuis and Olver (1994), the authors derived exponentially improved asymptotic solutions to the general homogeneous second order linear ordinary differential equation

$$\frac{d^2w}{dz^2} + f(z)\frac{dw}{dz} + g(z) = 0 \tag{2.19}$$

that has an irregular singularity of exponential rank one at infinity. We can express f and g as the power series

$$f(z) = \sum_{s=0}^{\infty} \frac{f_s}{z^s}, \quad g(z) = \sum_{s=0}^{\infty} \frac{g_s}{z^s},$$

which converge in unbounded open annuli that are centered at the origin. Non-exponentially improved asymptotic solutions to (2.19) are well documented, so the purpose of this paper is to present a derivation for exponentially improved asymptotic solutions to (2.19) based on the ideas developed in Olver (1993). This paper and many that follow on from it make extensive use of the Stieltjes transform in order to rewrite the remainder terms.

If we assume that the roots of the characteristic equation

$$\lambda^2 + f_0\lambda + g_0 = 0 \tag{2.20}$$

are distinct (if they are repeated, then a preliminary transformation can avoid this case), then (2.19) has the unique asymptotic solutions

$$w_1(z) \sim e^{\lambda_1 z} z^{\mu_1} \sum_{s=0}^{\infty} \frac{a_{s,1}}{z^s}, \quad w_2(z) \sim e^{\lambda_2 z} z^{\mu_2} \sum_{s=0}^{\infty} \frac{a_{s,2}}{z^s},$$

valid in the sectors

$$|\arg\{\pm(\lambda_2 - \lambda_1)z\}| \leq \frac{3\pi}{2} - \delta$$

respectively, with δ an arbitrary small positive constant. Replacing z by $z/(\lambda_2 - \lambda_1)$ forces the new characteristic values to satisfy $\lambda_2 - \lambda_1 = 1$, vastly simplifying notation while maintaining the form of (2.19). The branches of the asymptotic solutions are chosen such that solutions w_1 and w_2 are valid in the respective overlapping regions

$$|\arg(z)| \leq \frac{3\pi}{2} - \delta \quad \text{and} \quad -\frac{\pi}{2} + \delta \leq |\arg(z)| \leq \frac{5\pi}{2} - \delta,$$

with λ_j satisfying (2.20),

$$\mu_1 = f_1\lambda_1 + g_1, \quad \mu_2 = -\left(f_1\lambda_2 - g_1\right),$$

and $a_{s,j}$ given by the recurrence relations (2.5) and (2.6) in the text respectively.

It is stated that we may write what are known as the connection formulae

$$\begin{aligned} w_1(z) &= e^{2\pi i\mu_1} w_1(ze^{-2\pi i}) + C_1 w_2(z), \\ w_2(z) &= e^{-2\pi i\mu_2} w_2(ze^{2\pi i}) + C_2 w_1(z), \end{aligned}$$

with the connection coefficients C_j assumed to be known constants. The value of the C_j and hence the form of the expansions will only change when Stokes lines are crossed, namely when a Stokes phenomenon takes place. Thus, the Stokes phenomenon is automatically incorporated into these asymptotic solutions. The main results of the paper are then as follows.

Let $m \in \mathbb{N}_0$ be constant. Then, as $s \to \infty$,

$$\begin{aligned} a_{s,1} &= \frac{(-1)^s e^{(\mu_2-\mu_1)\pi i}}{2\pi i}\left(C_1 \sum_{j=0}^{m-1} (-1)^j a_{j,2}\Gamma(s+\mu_2-\mu_1-j) + \Gamma(s+\mu_2-\mu_1-m)\mathcal{O}(1)\right), \\ a_{s,2} &= -\frac{1}{2\pi i}\left(C_2 \sum_{j=0}^{m-1} (-1)^j a_{j,1}\Gamma(s+\mu_1-\mu_2-j) + \Gamma(s+\mu_1-\mu_2-m)\mathcal{O}(1)\right), \end{aligned}$$

and by defining $R_n^{(1)}(z)$ and $R_n^{(2)}(z)$ by

$$w_1(z) = e^{\lambda_1 z} z^{\mu_1} \sum_{s=0}^{n-1} \frac{a_{s,1}}{z^s} + R_n^{(1)}(z), \quad w_2(z) = e^{\lambda_2 z} z^{\mu_2} \sum_{s=0}^{n-1} \frac{a_{s,2}}{z^s} + R_n^{(2)}(z)$$

with $n = |z| + \alpha$, where $|\alpha|$ is bounded as $|z| \to \infty$, we are able to write

$$\begin{aligned} R_n^{(1)}(z) &= (-1)^{n-1} i e^{(\mu_2-\mu_1)\pi i} e^{\lambda_2 z} z^{\mu_2}\left\{C_1 \sum_{s=0}^{m-1} (-1)^s a_{s,2} \frac{F_{n+\mu_2-\mu_1-s}(z)}{z^s} + R_{m,n}^{(1)}(z)\right\}, \\ R_n^{(2)}(z) &= (-1)^{n-1} i e^{(\mu_2-\mu_1)\pi i} e^{\lambda_1 z} z^{\mu_1}\left\{C_2 \sum_{s=0}^{m-1} (-1)^s a_{s,1} \frac{F_{n+\mu_1-\mu_2-s}(ze^{-\pi i})}{z^s} + R_{m,n}^{(2)}(z)\right\}, \end{aligned}$$

where

$$R_{m,n}^{(1)}(z) = \begin{cases} \mathcal{O}(e^{-|z|-z} z^{-m}) & \text{for} \quad |\arg(z)| \le \pi, \\ \mathcal{O}(z^{-m}) & \text{for } \pi \le |\arg(z)| \le \frac{5\pi}{2} - \delta, \end{cases}$$

$$R^{(2)}_{m,n}(z) = \begin{cases} \mathcal{O}(e^{-|z|+z}z^{-m}) & \text{for } 0 \le \arg(z) \le 2\pi, \\ \mathcal{O}(z^{-m}) & \text{for } -\frac{3\pi}{2}+\delta \le \arg(z) \le 0 \text{ and } 2\pi \le \arg(z) \le \frac{7\pi}{2}-\delta, \end{cases}$$

uniformly with respect to $\arg(z)$. These sectors of validity are maximal when C_1 and C_2 are non-zero.

We can see that the remainder terms of one solution are functions of the asymptotic coefficients of the other solution, analogous to the case of exponentially improved integral asymptotics whereby remainder terms of expansions around one critical point are functions of the coefficients around the other critical points. The next natural theoretical step from here is to iterate substitution of the coefficients into remainder terms to generate hyperasymptotic solutions and this is exactly the content of Olde Daalhuis and Olver (1995) and Olde Daalhuis (1995), to which the reader is referred to for full details.

The results discussed thus far are generalised in Olde Daalhuis (1998) to nth order linear ordinary differential equations

$$\frac{d^n w}{dz^n} + f_{n-1}(z)\frac{d^{n-1}w}{dz^{n-1}} + \cdots + f_0(z)w = 0 \tag{2.21}$$

that have an irregular singularity of exponential rank one at infinity, in which the coefficients $f_m(z)$ can be expanded in power series

$$f_m(z) = \sum_{s=0}^{\infty} \frac{f_{sm}}{z^s}$$

for each m, which converge on open annuli $|z| > a$. As in the second order case, we have formal power series solutions

$$e^{\lambda_j z} z^{\mu_j} \sum_{s=0}^{\infty} a_{sj} z^{-s}$$

for $j \in \{1, \ldots, n\}$, with the λ_j, μ_j, and a_{sj} defined similarly to their second order counterparts. In §6 of Olde Daalhuis (1998) the main results of the paper are given, namely general level hyperasymptotic solutions to (2.21). The main difference in this paper is the utilisation of Borel plane techniques, similar to Howls (1997). In fact, the two papers are essentially the integral and differential equation analogue of each other, since once the problem has been recast in the Borel plane it does not matter where we came from; the techniques, methods, and even results are essentially the same. Together, Howls (1997) and Olde Daalhuis (1998) showed that the Borel plane provides an intimate and fundamental link between the asymptotics of integrals and differential equations.

Murphy and Wood (1997) extend the results of Olde Daalhuis and Olver (1995), deriving hyperasymptotic solutions for second order linear differential equations of *arbitrary* exponential rank. This is generalised further in Murphy (2001) to arbitrary order linear differential equations of arbitrary exponential rank. In both cases the set-up is more complicated, but the results are similar - albeit more complicated - and follow the same form as in the rank one case. Again, the reader is referred to these papers for full details.

When comparing differential equation and integral asymptotics, it is well known that the order n of an ordinary linear differential equation corresponds to there being n isolated critical points of the integral's phase function. Looking ahead to our work in §4, we are able to draw further parallels by looking at the arbitrary rank case. Much like the rank one case corresponds to *non-degenerate* isolated critical points of an integral's phase function, the arbitrary rank case corresponds to *degenerate* isolated critical points.

In terms of the order ω of critical points, it would seem that a linear ordinary differential equation of constant rank r corresponds to an integral whose phase function has order $\omega = r+1$ degenerate isolated critical points. Interestingly, it is stated in Murphy (2001) that the case of 'mixed exponential rank' is a far more difficult problem. This would correspond to the phase function of an integral having mixed order degenerate critical points and this is exactly the problem we solve in §4; for an integral of type (1.2) that has degenerate isolated critical points of any (mixed) order, we derive a full hyperasymptotic expansion. In §4 this is done in $\mathbb{C}$, but is extended to $\mathbb{C}^d$ in §5. We will discuss further parallels later.

In Howls and Olde Daalhuis (2003) the authors develop hyperasymptotic solutions to inhomogeneous linear ordinary differential equations with an irregular singularity of rank one at infinity. This work was motivated by Delabaere and Howls (2002), as it was noted that phenomena similar to restricted critical points occur when studying inhomogeneous differential equations. These ideas are expanded upon and the effect of the inhomogeneous terms on the hyperasymptotic structure is explored.

It is stated that without loss, the attention is restricted to the second order differential equation

$$\frac{d^2w}{dz^2} + f_1(z)\frac{dw}{dz} + f_0(z)w = e^{\lambda_3 z} z^{\mu_3} p(z), \tag{2.22}$$

where we can write

$$f_m(z) = \sum_{s=0}^{\infty} \frac{f_{sm}}{z^s}, \quad p(z) = \sum_{s=0}^{\infty} \frac{p_s}{z^s},$$

converging on open annuli $|z| > a$. The inhomogeneous differential equation (2.22) is converted into a homogeneous equation of higher order and the reason for the restriction to second order is that this naturally simpler case will demonstrate all important features of inhomogeneous equations. Interestingly, it is found in §§3 and 4 of the text that the hyperasymptotic re-expansions of the solutions of level one and above depend on λ_3 and μ_3 but are *independent* of the function $p(z)$. For full details, the reader is referred to Howls and Olde Daalhuis (2003).

Hyperasymptotic solutions to nonlinear ordinary differential equations have also been considered and were first discussed in Olde Daalhuis (2005a, b), using so called *transseries* to study the hyperasymptotic behaviour of specific nonlinear equations. A transseries is a convergent series of the form

$$\sum_n C^n v_n(z),$$

where C is a free constant and $v_n(z)$ are formal (divergent) series (Olde Daalhuis, 2005a). Despite being a more complicated problem, once transformed into a Borel setting it is essentially solved; it is stated that the Borel plane for a nonlinear ordinary differential equation will contain infinitely many singularities, of which only finitely many contribute to the asymptotics at each hyperasymptotic level. This allows us to use results from Olde Daalhuis (1998) to obtain the sought after hyperasymptotic expansion.

As a final note on differential equation asymptotics, the expositionary article Tanveer and Costin (2004) provides a good introduction to the subject as well as showcasing a wide variety of results, such as analysis on general ordinary differential equations as well as some extensions to partial differential equations.

2.6 Non-Isolated Critical Points

We move on now to a discussion of the current literature and results regarding asymptotic contributions of non-isolated critical points of the function $f(\boldsymbol{z})$ to integral (1.2), reproduced here for convenience as

$$I^{(n)}(k) = \int_{S_n} d\boldsymbol{z}\, g(\boldsymbol{z}) e^{-kf(\boldsymbol{z})}.$$

For f to have non-isolated critical points and remain holomorphic, we have to be in at least two-dimensional real or complex space (we prove this later); all of the cases considered in the literature thus far have focused on $\mathbb{R}^d$ for $d \geq 2$, are for various specific cases, and are not exponentially improved. Our aim is consider the integral in general complex space $\mathbb{C}^d$ and develop a hyperasymptotic expansion for a wide class of critical points, providing many different generalisations to the current literature. We will look at the derivations in select pieces of work in great detail in order to explore the variety of methods that have been employed thus far.

The first considerations of non-isolated critical points in this context appear to have been undertaken by Kontorovich, Karatygin, and Rozov (1970) (henceforth known as KKR), where in the parent space $\mathbb{R}^2$ they considered the integral

$$I(k) = \int_D dx\, dy\, g(x,y) e^{ikf(x,y)},$$

with $k, x, y \in \mathbb{R}$, $D \subset \mathbb{R}^2$ a closed region of integration, and $f, g : \mathbb{R}^2 \to \mathbb{R}$ real valued functions that possess sufficiently many partial derivatives. The function f has a non-degenerate critical line γ that cuts D transversely and contains the points (x_γ, y_γ), defined by $f(x_\gamma, y_\gamma) = f_0$, with f_0 constant and $f \neq f_0$ anywhere else in D. The authors assume for simplicity that $f_0 = 0$. The integral is then written in a form such that one can apply Stokes' theorem, reducing the surface integral to a contour integral along the critical line γ. A full asymptotic expansion for $I(k)$ as $k \to \infty$ is then derived, involving complicated combinations of f and g, with the expansion given explicitly to leading order as

$$I(k) = e^{\frac{i\pi}{4}} \sqrt{\frac{2\pi}{k}} \int_\gamma \frac{d\gamma\, g(x,y)}{\sqrt{f_{xx} + f_{yy}}} + \mathcal{O}\left(\frac{1}{k}\right). \tag{2.23}$$

the integral being along the part of the critical line γ within D.

Interestingly, the authors note that integrals of this type arise when f represents the distance between points on parallel curves, for example when calculating the decoupling between rectangular antennae with parallel sides or coaxial parabolic antennae. Further application appears to show itself in Servadio (1987a, b), where integrals similar to $I(k)$ - whose phase function contains a critical line - appear in the context of scattering theory. The reader is referred to these papers for full details.

In McLure and Wong (1987) (henceforth known as McW), the authors revisited the earlier work of KKR with the intent to re-derive the results by reducing the two-dimensional integral to a one-dimensional Fourier integral, in contrast to the vector analysis approach employed by KKR. The considered integral is of the form

$$I(k) = \int_D dx\, dy\, g(x,y) e^{ikf(x,y)},$$

with $k, x, y \in \mathbb{R}$, $D \subset \mathbb{R}^2$ a bounded domain of integration, and $f, g : \mathbb{R}^2 \to \mathbb{R}$. The boundary ∂D of D is piecewise smooth and f and g are assumed infinitely differentiable in $\bar{D}$, the closure of D. Again, f has a simple smooth non-degenerate critical line γ that cuts D transversely, along which f has a constant value (taken for simplicity as zero).

We parameterise γ by s - its arc length - such that $x = \xi(s)$ and $y = \eta(s)$. The part of the critical line that lies within D is said to have length L, with

$$A := (\xi(0), \eta(0)), \quad B := (\xi(L), \eta(L))$$

defined as the only two intersection points of γ and ∂D, with $A \neq B$. As in KKR we are interested in deriving an asymptotic expansion for $I(k)$, paying particular interest to the asymptotic contribution provided by γ. Since this is the contribution of interest, we are able to restrict D to the region D_δ ($\delta > 0$) containing the points of D that are a distance less than δ away from γ and instead consider the integral

$$I_0(k) = \int_{D_\delta} dxdy\, g(x,y) e^{ikf(x,y)}.$$

The boundary of D_δ is E_δ and it is stated that along this boundary, g vanishes to infinite order.

The authors show that their conditions are in fact the same as those enforced in KKR, although they allow the possibility of ∂D having corners at A and B. Instead of using Stokes' theorem to achieve the required dimensional reduction, the change of variables $m : (s,t) \to (x,y)$ such that

$$x = \xi(s) - t\eta'(s), \quad y = \eta(s) + t\xi'(s),$$

with Jacobian $|\partial(x,y)/\partial(s,t)|$, is employed. This transforms the critical line γ into the straight line segment $[0, L]$ along the s-axis at $F(s,t) = 0$ and the region D_δ into a 'more rectangular' region $R_\delta = m^{-1}(D_\delta)$ given as

$$a(t) \leq s \leq b(t), \quad -\delta \leq t \leq \delta,$$

where a and b are some functions such that $a(0) = 0$ and $b(0) = 0$ (see Figure 10, taken from McW). The value of δ is assumed small enough such that the s-boundary of R_δ is completely determined by $a(t)$ and $b(t)$.

The integral $I_0(k)$ then becomes

$$I_0(k) = \int_{-\delta}^{\delta} \int_{a(t)}^{b(t)} ds\, dt\, G(s,t) e^{ikF(s,t)},$$

with $F(s,t) = f(x,y)$ and $G(s,t) = g(x,y)|\partial(x,y)/\partial(s,t)|$. From here, we further transform

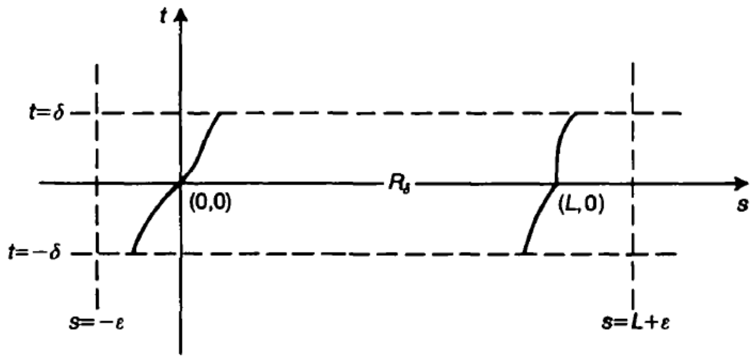


Figure 10: This is Figure 2 from McClure and Wong (1987), showing the region R_δ.

$I_0(k)$ into a one-dimensional Fourier integral using $N : (s,t) \to (w,z)$ such that

$$w = s, \quad z^2 = F(s,t) \quad \text{with} \quad \text{sgn}(z) = \text{sgn}(t).$$

The double valuedness of this transformation allows us to consider the regions either side of γ separately and then add the contributions from both subregions. Applying these changes, it can be shown that

$$I_0(k) = I_0^+(k) + I_0^-(k) = \int_0^\rho dz\, e^{ikz^2}\Phi(z) + \int_0^\rho dz\, e^{ikz^2}\Phi(-z),$$

with $\rho^2 = \sup\{F(s,t) \,|\, (s,t) \in R_\delta\}$ and

$$\Phi(z) = \int_{\alpha(c)}^{\beta(c)} dw\, G(s,t)\frac{2z}{F_t(s,t)}.$$

The variable c is the contour height $F(s,t) = c$ and the functions $\alpha(c)$ and $\beta(c)$ parameterise $a(t)$ and $b(t)$. Essentially, for a fixed contour of height c, the transformation allows us to integrate along the part of this contour that lies within the region of integration using the Φ integral, followed by integrating through all such contours - including γ at $c = 0$ - within the integration region using the full integral I. Since $F(s,t) = z^2 = c$, then γ is at $z = 0$ in the new coordinates and so the asymptotic behaviour of $I_0(k)$ is determined by that of $\Phi(z)$ near $z = 0$.

Repeated integration by parts reveals the main result of the paper; the asymptotic expansion of $I_0(k)$ as $k \to \infty$ is,

$$I_0(k) \sim \sum_{s=0}^{\infty} \frac{b_s}{k^{\frac{s+1}{2}}}, \tag{2.24}$$

$$b_s = \frac{(-1)^s e^{i(s+1)\pi}}{2}\frac{\Gamma\left(\frac{s+1}{2}\right)}{\Gamma(s+1)}\left[\Phi^{(s)}(0^+) + \Phi^{(s)}(0^-)\right], \tag{2.25}$$

with the coefficients b_s independent of k (no such 'coefficients' were derived in KKR as only an expression for the complete expansion was given). The coefficients b_0 and b_1 are calculated explicitly and it is noted that if ∂D is smooth at A and B, then $b_1 = 0$. No further coefficients are computed due to their increasingly complicated algebraic nature. The explicit computation of b_0 allows us to write the asymptotic expansion for $I_0(k)$ as

$$I_0(k) = e^{\frac{i\pi}{4}}\sqrt{\frac{2\pi}{k}}\int_\gamma \frac{ds\, g(x,y)}{\sqrt{f_{xx}+f_{yy}}} + \mathcal{O}\left(\frac{1}{k}\right), \tag{2.26}$$

with the leading order term explicitly displayed. As expected, the leading order terms of (2.23) and (2.26) are identical.

The authors believed that their method could be extended to handle a similar three-dimensional integral in $\mathbb{R}^3$ where f has a critical line in order to tackle the problem posed in Servadio (1987a, b), whereas the methods used in KKR could not. According to Kaminksi (1992), the methods of both KKR and McW generalise naturally to $\mathbb{R}^n$ in the case where f has a hypersurface of non-isolated critical points - namely, a set of codimension one - but do not generalise easily to the case of any other codimension. In order to properly consider this problem, a different approach is required.

Kaminski (1992) considered the problem of an integral in $\mathbb{R}^3$ given by

$$I(k) = \int_\Sigma d\boldsymbol{x}\, g(\boldsymbol{x})e^{ikf(\boldsymbol{x})},$$

with $k \in \mathbb{R}$, $\boldsymbol{x} \in \mathbb{R}^3$, $\Sigma \subset \mathbb{R}^3$ a three-dimensional solid over which we are integrating (that is thus bounded and closed, with boundary $\partial\Sigma$), and $f, g : \mathbb{R}^3 \to \mathbb{R}$ smooth functions in a neighbourhood of Σ. The function f will have a non-degenerate critical line γ that cuts $\partial\Sigma$ transversely and is parameterised by its arc length s. The author is interested in the asymptotic contribution of γ to $I(k)$.

Let $I \subset \mathbb{R}$ be some interval such that $(\gamma \cap \Sigma) \subset \gamma(I)$, with γ cutting $\partial\Sigma$ at $s_0, s_1 \in I$ so that

$$\gamma \cap \partial\Sigma = \{\gamma(s_0), \gamma(s_1)\}.$$

Without loss, we assume that $s_0 < s_1$, so that γ enters Σ at $\gamma(s_0)$ and leaves at $\gamma(s_1)$. Using the Frenet-Serret apparatus, a normal tubular neighbourhood of constant radius δ around $\gamma(I)$ is constructed, denoted W_δ. Figure 11 - taken from (a physical copy of) Kaminski (1992) - shows W_δ and other related relevant quantities that we will go on to describe. Any point

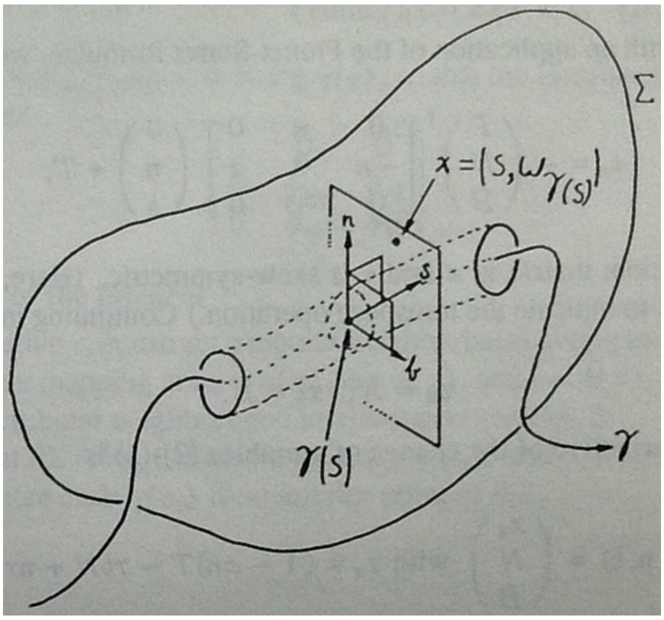


Figure 11: This is Figure 1 from Kaminski (1992), showing a tubular neighbourhood of γ along with the (s, n, b) coordinate system.

within W_δ can be fully specified by a point along $\gamma(s)$ for $s \in I$ and a point $\omega_{\gamma(s)}$ within the disk B_δ that lies in the plane normal to $\gamma(s)$, defining the map

$$\begin{aligned} W_\delta &\to I \times B_\delta \\ \boldsymbol{x} &\mapsto \left(s, \omega_{\gamma(s)}\right). \end{aligned}$$

The Frenet-Serret frame for a curve in $\mathbb{R}^3$ is the (orthogonal, linearly independent) frame that attaches a frame of reference to the curve at each point along it, consisting of the tangent vector to the curve $\mathbf{T}$, the normal vector to the curve $\mathbf{N}$, and the binormal vector to the curve $\mathbf{B} = \mathbf{T} \times \mathbf{N}$. Since the normal and binormal vectors live in the normal plane to γ at any given point, we can specify the point $\omega_{\gamma(s)}$ as a combination of these types of vectors. Formally, the transformation is

$$\begin{aligned} W_\delta &\to I \times B_\delta \\ \boldsymbol{x} &\mapsto (s, n, b), \end{aligned}$$

with

$$n = n(\boldsymbol{x}) = (\boldsymbol{x} - \gamma(s)) \cdot \mathbf{N}, \quad b = b(\boldsymbol{x}) = (\boldsymbol{x} - \gamma(s)) \cdot \mathbf{B},$$

and $s = s(\boldsymbol{x})$ being the s for which the normal plane to $\gamma(s)$ contains $\boldsymbol{x}$. Application of the Frenet-Serret formulae allows the determinant of this transformation to be calculated as

$$\det\big(\boldsymbol{x}'(s,n,b)\big) = 1-\kappa(s)n,$$

where $\kappa(s)$ is the curvature of γ at s.

The author reduces the integral of consideration to

$$I_0(k) := \int_{\Sigma\cap W_\delta} d\boldsymbol{x}\, g(\boldsymbol{x})e^{ikf(\boldsymbol{x})}$$

and the explicit calculation of the determinant allows us to write

$$\begin{aligned} I_0(k) &= \int_{\boldsymbol{x}^{-1}(\Sigma\cap W_\delta)} ds\,dn\,db\,\det\big(\boldsymbol{x}'(s,n,b)\big)\,g(\boldsymbol{x}(s,n,b))e^{ikf(\boldsymbol{x}(s,n,b))} \\ &= \int_{\boldsymbol{x}^{-1}(\Sigma\cap W_\delta)} ds\,dn\,db\,(1-\kappa(s)n)G(s,n,b)e^{ikf(\boldsymbol{x}(s,n,b))}, \end{aligned}$$

where the author sets $G = g\circ x$ for convenience. If we let $h_0(n,b)$ and $h_1(n,b)$ be the boundaries of Σ in the normal sections at s_0 and s_1 respectively, then we may rewrite $I_0(k)$ as

$$I_0(k) = \int_{n^2+b^2<\delta} dn\,db \int_{h_0(n,b)}^{h_1(n,b)} ds\,(1-\kappa(s)n)G(s,n,b)e^{ikf(\boldsymbol{x}(s,n,b))}. \tag{2.27}$$

In (2.27), the h_i not being constant restricts the order in which we may integrate. The case of constant h_i occurs when either $\partial(W_\delta\cap\Sigma)$ is made up of normal bundle sections over s from s_0 to s_1, or when $\mathrm{supp}(g)\subset W_\delta$. Both of these less general cases are handled in de Verdière (1973a, b) and Chazarain (1974), although their method allows for the handling of any codimension less than the dimension of Σ.

If we now write $f(\boldsymbol{x}(s,n,b)) = f(s,n,b)$ and - without loss - assume $f(s,0,0)=0$, then we can write f as

$$f(s,n,b) = n^2h_{11}(s,n,b) + nbh_{12}(s,n,b) + b^2h_{22}(s,n,b) + \dots$$

and carry out the transformation

$$\boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{h_{11}(s,n,b)}{h_{11}(s,0,0)}} & \frac{h_{12}(s,n,b)}{h_{11}(s,n,b)}\sqrt{\frac{h_{11}(s,n,b)}{h_{11}(s,0,0)}} \\ 0 & \sqrt{\frac{\Delta(s,n,b)}{\Delta(s,0,0)}\Big/\frac{h_{11}(s,n,b)}{h_{11}(s,0,0)}} \end{pmatrix} \begin{pmatrix} n \\ b \end{pmatrix}, \tag{2.28}$$

where

$$\Delta = \begin{vmatrix} \phi_{nn} & \phi_{nb} \\ \phi_{bn} & \phi_{bb} \end{vmatrix}.$$

The author notes that if $h_{11}(s,0,0)=0$ along γ, then we can first apply the transformation

$$\begin{pmatrix}\bar{n}\\ \bar{b}\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{vmatrix}1 & 1\\ 1 & -1\end{vmatrix}\begin{pmatrix}n\\ b\end{pmatrix}$$

before applying (2.28). Transformation (2.28) allows us to write

$$f(s,n,b)=\begin{pmatrix}y_1\\ y_2\end{pmatrix}^T\begin{vmatrix}h_{11}(s,0,0) & 0\\ 0 & \frac{\Delta(s,0,0)}{h_{11}(s,0,0)}\end{vmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix},$$

where T will now be used to represent matrix transposition, and hence write the integral as

$$I_0(k)\approx\int_{n^2+b^2<\delta}dn\,db\int_{h_0(n,b)}^{h_1(n,b)}ds\,\bar{G}(s,\boldsymbol{y})(1-n(\boldsymbol{y})\kappa(s))e^{ik\boldsymbol{y}^TQ(s)\boldsymbol{y}},$$

where the author uses the notation

$$\alpha(X)\approx\beta(X)\Rightarrow\alpha(X)=\beta(X)+\mathcal{O}(k^{-r})$$

as $k\to\infty$, $\forall r\in\mathbb{N}$. At $n=b=0$, namely $y_1=y_2=0$, it is stated that

$$\left|\left|\frac{\partial(y_1,y_2)}{\partial(n,b)}\right|\right|_{(s,0,0)}=1,$$

so that

$$I_0(k)\approx\int_{W'_\delta}ds\,d\boldsymbol{y}\,\bar{G}(s,\boldsymbol{y})(1-n(\boldsymbol{y})\kappa(s))\left|\frac{\partial(n,b)}{\partial(y_1,y_2)}\right|e^{ik\boldsymbol{y}^TQ(s)\boldsymbol{y}},$$

where W'_δ is the image of W_δ when transformed using (2.28), through which the s-axis runs. Employing the final change of variables

$$\boldsymbol{w}=\begin{pmatrix}w_1\\ w_2\end{pmatrix}=\begin{pmatrix}y_1\sqrt{|h_{11}(s,0,0)|}\\ y_2\sqrt{\left|\frac{\Delta(s,0,0)}{h_{11}(s,0,0)}\right|}\end{pmatrix}$$

with Jacobian $1/\sqrt{\Delta(s,0,0)}$, we may write

$$I_0(k)\approx\int_{W''_\delta}ds\,d\boldsymbol{w}\,\bar{G}(s,\boldsymbol{w})(1-n(\boldsymbol{w})\kappa(s))\left|\frac{\partial(n,b)}{\partial(w_1,w_2)}\right|e^{ik\boldsymbol{w}^TK\boldsymbol{w}},$$

where W''_δ is the transformed tube, through which the s-axis still runs, and

$$K = \begin{vmatrix} \mathrm{sgn}(q_{11}) & 0 \\ 0 & \mathrm{sgn}(q_{22}) \end{vmatrix}.$$

If we define

$$a(\boldsymbol{w}) = \int_{h_0(n,b)}^{h_1(n,b)} ds\,(1-\kappa(s)n(\boldsymbol{w}))\left|\frac{\partial(n,b)}{\partial(w_1,w_2)}\right|\bar{G}(s,\boldsymbol{w}),$$

and - using the authors loose terminology - 'core' W''_δ by a tube of radius $\eta > 0$, we can write the integral as

$$I_0(k) \approx \int_{||\boldsymbol{w}||<\eta} d\boldsymbol{w}\,a(\boldsymbol{w})e^{ik\boldsymbol{w}^T K\boldsymbol{w}}.$$

If the $h_i(\boldsymbol{w})$ are smooth for $\boldsymbol{w}$ in a neighbourhood of $(0,0)$, then so is $a(\boldsymbol{w})$, meaning we can Taylor expand $a(\boldsymbol{w})$ as

$$a(\boldsymbol{w}) = a_{00} + a_{10}w_1 + a_{01}w_2 + a_{20}w_1^2 + a_{11}w_1w_2 + a_{02}w_2^2 + \dots .$$

After some manipulation and reduction, the asymptotic expansion for $I_0(k)$ is found to be

$$I_0(k) \sim \pi e^{ikf(\gamma)+\frac{\nu\pi i}{4}} \sum_{l=0}^{\infty} \frac{1}{2k^{l+1}} \sum_{m=0}^{l} a_{2(l-m),2m} i^{k_{11}l} i^{(k_{22}-k_{11})m},$$

with the first term calculated explicitly as

$$I_0(k) = \frac{\pi e^{ikf(\gamma)+\frac{\nu\pi i}{4}}}{k}\left[\int_{\gamma\cap\Sigma} \frac{g(\boldsymbol{x})d\boldsymbol{x}}{\sqrt{|\det \mathrm{Hess}\bar{f}|}} + \mathcal{O}\left(\frac{1}{k}\right)\right].$$

This is similar in form to previous results, although it is not exactly the same since the codimension of γ is different. This is the first indication that the codimension of the set of non-isolated critical points is an important factor in the form of its asymptotic contribution; we shall see later in §5 that it plays a central role.

Benaissa and Rogers (2001) (henceforth known as BR01) consider an integral in $\mathbb{R}^2$ of the form

$$I(k) = \int_D dx\,dy\,g(x,y)e^{ikf(x,y)},$$

with $k, x, y \in \mathbb{R}$, $D \subset \mathbb{R}^2$ a bounded domain of integration, and $f, g : \mathbb{R}^2 \to \mathbb{R}$ with f and g assumed to be smooth in $\bar{D}$. The function f has a simple smooth critical line γ of general constant order of degeneracy ω (denoted r in the paper) along which f has a constant value. The aim of the paper is to extend the results of McW to the case where γ has general order ω. Fewer restrictions are placed on the phase function f, but otherwise we are in exactly the same set-up as in McW. The authors state that the only condition on f is that there exists an integer $\omega > 1$ such that the derivatives of order less than ω are all zero on γ and the derivatives of order ω are nowhere zero on γ.

The integral of interest is again the restricted integral

$$I_\delta(k) = \int_{D_\delta} dx\, dy\, g(x, y) e^{ikf(x,y)},$$

with D_δ retaining the same meaning as in McW. The paper proceeds in the same manner as McW, with the authors using the same transformation $m : (s, t) \to (x, y)$ defined in McW in order to bring the problem into an alternate coordinate system, where it can be reduced to a one-dimensional Fourier integral. The authors note that the Jacobian of the transformation m can be written as $1 + t\kappa_\gamma(s)$, where $\kappa_\gamma(s)$ is the curvature of γ at the point s. Since the transformation used is the same, we end up with the same transformed expression for $I_0(k)$, namely

$$I_\delta(k) = \int_{-\delta}^{\delta} \int_{a(t)}^{b(t)} ds\, dt\, G(s, t) e^{ikF(s,t)},$$

with $F(s, t) = f(x, y)$ and $G(s, t) = g(x, y)(1 + t\kappa_\gamma(s))$.

Theorem 1 in the paper states that if γ is a Jordan curve in D and the condition on the derivatives of f described above holds, then for sufficiently small δ and g zero in a neighbourhood of the boundary of D, the asymptotic expansion of $I_0(k)$ as $k \to \infty$ is given by

$$I_\delta(k) \sim \sum_{n=0}^{\infty} \frac{A_n}{k^{\frac{n+1}{\omega}}},$$

$$A_n = \frac{2e^{\frac{i(n+1)\pi}{2\omega}}}{\omega} \frac{\Gamma\left(\frac{n+1}{\omega}\right)}{\Gamma(n+1)} \int_\gamma ds\, h_s^{(n)}(0), \tag{2.29}$$

with

$$h_s(u) = g(x, y) \left(1 + t\kappa_\gamma(s)\right) \frac{dt}{du}, \quad u^\omega = F(s, t) = f(x, y).$$

The authors state that $A_n = 0$ for all odd n and give A_0 explicitly as

$$A_0 = \frac{2e^{\frac{i\pi}{2\omega}}}{\omega} \Gamma\left(\tfrac{1}{\omega}\right) \left(\Gamma(\omega + 1)\right)^{\frac{1}{\omega}} \int_\gamma ds\, g(x, y) \left[\left|\frac{\partial^\omega f}{\partial x^\omega}\right|^{\frac{2}{\omega}} + \left|\frac{\partial^\omega f}{\partial y^\omega}\right|^{\frac{2}{\omega}}\right]^{-\frac{1}{2}}.$$

When $\omega = 2$, we are in the scenario of KKR and McW and we can readily see the similarities between (2.25) and (2.29). Additionally, the first coefficient reduces to

$$A_0 = e^{\frac{i\pi}{4}} \Gamma\left(\tfrac{1}{2}\right) 2^{\frac{1}{2}} \int_\gamma ds\, g(x, y) \left[\left|\frac{\partial^2 f}{\partial x^2}\right| + \left|\frac{\partial^2 f}{\partial y^2}\right|\right]^{-\frac{1}{2}} = e^{\frac{i\pi}{4}} \sqrt{2\pi} \int_\gamma \frac{ds\, g(x, y)}{\sqrt{f_{xx} + f_{yy}}}, \tag{2.30}$$

which agrees exactly with the leading order of both (2.23) and (2.26).

The authors also briefly discuss the case of γ tangent at one point of intersection A with ∂D, but not at the other point of intersection B. Theorem 3 states that as $k \to \infty$, the expansion is of the form

$$I_\delta(k) \sim \sum_{j\geq 0} \frac{A_j}{k^{\frac{(j/p)+1}{\omega}}},$$

where $p-1>0$ is the *order of contact* between γ and ∂D at A (that is, all the derivatives of up to and including order $p-1$ are zero at A), the coefficient A_0 is given by (2.30) and $A_j = 0$ for all odd j.

In Benaissa and Rogers (2013) (henceforth known as BR13), the authors generalise their work in BR01 to the integral

$$I(k) = \int_D d\boldsymbol{x}\, g(\boldsymbol{x})e^{ikf(\boldsymbol{x})}$$

in $\mathbb{R}^d$ for the case that f has a general order critical hypersurface S (that has codimension one). Additionally, the cases that S is with and without boundary are considered. The authors use very similar methods to BR01 and decide to work in $\mathbb{R}^3$ for algebraic simplicity.

When S is unbounded, its asymptotic contribution has exactly the same form regardless of d because its codimension is constant at one; the results (3.3) - (3.5) in BR13 are near identical to the results (5) - (7) in BR01, the main differences being the additional derivative in the integrand and the power that the integrand is raised to, both due to the additional dimension. Based on equation (7) in BR01, we believe the $-1/r$ power that the integrand is raised to in (3.5) in BR13 is a typo and should be $-1/3$; generally it would be $-1/d$, but the considered case is $d = 3$.

The integral in the coefficient expression (3.4) in BR13 is decomposed into the sum of three separate integrals representing the contributions of the geometry of the critical surface S, meaning that $I_\delta(k)$ is now the sum of three separate asymptotic expansions. Specifically, if H and K are the mean and Gaussian curvatures of S respectively, then one expansion represents the contribution in the case that S is a plane ($H = K = 0$) and is hence geometrically independent, one represents the contributions of H, and the last represents the contribution of K. It is stated that the first coefficient of the H and K expansions are both zero, so that the leading term of the full expansion is independent of the geometry of S. For full details, see §5, Theorem 5.1 of BR13.

When S is bounded, it is assumed that ∂S is a simple smooth closed curve such that $\partial S = S \cap \partial D$, with ∂D smooth near ∂S (see Figure 12, taken from BR13). It is stated that corners of D in ∂S can be handled by integrating separately on both sides of S. The two

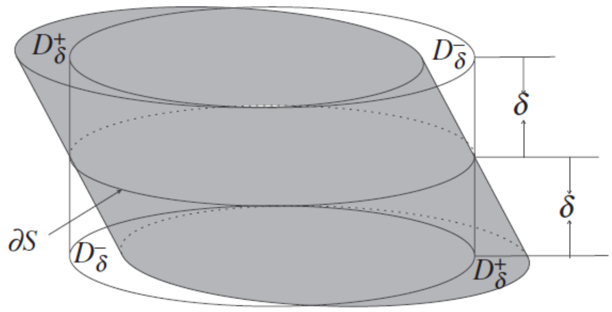


Figure 12: This is Figure 1 from Benaissa and Rogers (2013), showing the region D_δ along with various relevant related quantities.

subcases are that S and ∂D are tangent nowhere (transverse), or that S and ∂D are tangent everywhere along ∂S.

When S and ∂D are tangent nowhere, theorem 4.1 in BR13 states that

$$I_\delta(k) \sim \frac{c_0}{k^{\frac{1}{\omega}}} + \sum_{n=1}^{\infty} \frac{c_n}{k^{\frac{n+1}{\omega}}} + \sum_{n=1}^{\infty} \frac{d_n}{k^{\frac{n+1}{\omega}}}, \tag{2.31}$$

where $c_0 = b_0$ given by (3.5) in BR13, $c_n = b_n$ for n even given by (3.4) in BR13, and $c_n = d_n = 0$ for n odd. The two expansions in (2.31) represent the contribution from the non-boundary and boundary points of S respectively. Additionally, if the order of contact between every point on the boundary ∂S and the plane normal to S at each boundary point is a positive number l, then $d_n = 0$ for every $n \leq l$.

When S and ∂D are tangent everywhere along ∂S and the order of contact along ∂S is constant at $p-1$, theorem 4.2 in BR13 states that

$$I_\delta(k) \sim \frac{c_0}{k^{\frac{1}{\omega}}} + \sum_{n=1}^{\infty} \frac{c_n}{k^{\frac{n+1}{\omega}}} + \sum_{n=0}^{\infty} \frac{\tilde{d}_n}{k^{\frac{\frac{n+1}{p}+1}{\omega}}}, \tag{2.32}$$

where the c_n are identical to the c_n in the transverse case (2.31) and $\tilde{d}_n = 0$ for odd n. Once again, the two expansions in (2.31) represent the contribution from the non-boundary and boundary points of S respectively.

The authors also briefly describe how to handle the case where the critical surface S has non-constant order $\omega(\boldsymbol{x})$. Assuming that $\omega(\boldsymbol{x})$ is discontinuous only at finitely many points, it is stated that the methods in Bleistein (1967) would produce the appropriate uniform asymptotic expansion. This would break down when the discontinuities in $\omega(\boldsymbol{x})$ form their own set of non-isolated points and we look at an example of such a case in §8.

This concludes the review of literature regarding non-isolated critical points and also the discussion on the development of the subject as a whole. Many advances have been made in exponential asymptotic analysis, but it is clear that the treatment of non-isolated critical points in integral asymptotics is sporadic and somewhat cursory. Apart from BR01 and BR13 considering a general order of degeneracy, little attempt has been made to generalise the available results, with none of the literature discussed in §2.6 even discussing non-isolated critical points in complex spaces or any form of exponential improvement. With this in mind, the next chapter discusses non-isolated critical points in detail, investigating what kinds of sets of non-isolated critical points can occur.

3 Non-isolated Critical Points, Degeneracy, and Morse Theory

In this chapter we will consider properties of sets of non-isolated critical points in detail and investigate how the degeneracy of the system of equations defining critical points affects the form of the asymptotic expansion of integral (1.2). We will also discuss how to use Morse and Morse-Bott theory to split up the integral over the general surface S into an appropriate sum of integrals over surfaces S_j involving only one contributing component for isolated and non-isolated critical points respectively.

3.1 Critical Components: Definitions and Discussion

Typically, when considering points that make asymptotic contributions towards integrals of the form (1.2), the function f is assumed to only have finite non-degenerate isolated critical points (that is, none at infinity). As mentioned in §1, our aim is to extend the current results regarding non-degenerate isolated critical points to include degenerate isolated critical points and both non-degenerate and degenerate sets of non-isolated critical points. The aim of this section is to discuss these classes of critical points and make the appropriate definitions required to make these results possible, the goal being to unify all current results involving non-isolated critical points.

Considering a function

$$\begin{aligned} f : \mathbb{C}^d &\to \mathbb{C} \\ \boldsymbol{z} &\mapsto f(\boldsymbol{z}) \end{aligned}$$

in d complex variables $\boldsymbol{z} = (z_1, \ldots, z_d)$, we know that setting the derivative in every variable to zero and solving simultaneously gives us the critical points of $f(\boldsymbol{z})$. Symbolically, we write the set of all critical points of a function f as

$$C(f) = \left\{ \boldsymbol{z} \in \mathbb{C}^d \,\middle|\, \frac{\partial f}{\partial z_j} = 0, \forall j \in \{1, \ldots, d\} \right\}, \tag{3.1}$$

where the label (3.1) refers to the system of equations that define the set $C(f)$. This critical set may contain both isolated and non-isolated critical points depending on f.

We must be careful in our treatment of non-isolated critical points, as they will naturally occur in sets; an isolated critical point is just a single point in $\mathbb{C}^d$, but non-isolated points are defined by some function of $\boldsymbol{z}$. As a basic example, if $z_1 = 0$ is the only solution to (3.1) for some function $f : \mathbb{C}^2 \to \mathbb{C}$, then $C(f)$ is made up of the entire complex z_2-axis, as a 'critical complex line' of f. Moving to higher dimensions, we can have critical complex surfaces of varying dimension. In general, non-isolated critical points will be connected components (maximally connected subsets) of $C(f)$ and it is the form and behaviour of these subsets that are important, rather than the individual points that they contain.

To this end we introduce the concept of a *critical component* of f, generalising the notion of a critical point. We could define sets of non-isolated critical points as critical surfaces, but we will see that by treating isolated points simply as zero-dimensional critical components, the results in later chapters are applicable to both isolated and sets of non-isolated points. It is therefore simpler to have one all encompassing definition.

Definition 1 - Critical Component. A critical component χ is a connected component of the critical point set $C(f)$ of complex dimension $\dim_{\mathbb{C}}(\chi) = \mu$ and complex codimension $d - \mu = q$. Additionally, the set of all critical components of a function f is defined to be

$$X_f = \{\chi_1, \dots, \chi_\lambda\},$$

where $\lambda \in \mathbb{N}$, with the natural expressions

$$\bigcup_{j=1}^{\lambda} \chi_j = C(f); \quad \bigcap_{j=1}^{\lambda} \chi_j = \emptyset.$$

The set

$$X_f^{(n)} = X_f \setminus \{\chi_n\} = \{\chi_1, \dots, \chi_\gamma\}$$

contains the remaining γ critical components after removing $\{\chi_n\}$.

Formally, $M \subset S$ is a connected component of S if $M \cup \{x\}$ is disconnected for $x \in S \setminus M$. Intuitively, this simply means that M is a maximally connected subset of S; there are no more points left in S that we can add to M to produce a space that is still connected. Defined in this way, critical components are automatically mutually disjoint. Note that Definition 1 encompasses both isolated and sets of non-isolated critical points, as a connected component of $C(f)$ containing an isolated critical point $\boldsymbol{z}_c$ is simply $\{\boldsymbol{z}_c\}$. An illustrative example and diagram (Figure 14) are provided after Definition 2, where S is the critical set $C(f)$ and M is any of the individual critical components χ_1, χ_2, or χ_3; for each χ_j, including any point in $C(f)$ that is outside of χ_j in its definition would produce a clearly disconnected set and is thus disallowed under Definition 1. Each χ_j is therefore a maximally connected subset of the critical set $C(f)$.

We observe that the system (3.1) is equivalent to requiring that f has the same value at every point within a critical component χ, called the *critical value* of χ. In this thesis we dictate that each critical component has a *distinct* critical value, so that critical components will always have constant dimension. We note that it will sometimes be useful to refer specifically to critical components whose dimension is greater than zero; we will refer to these as *non-isolated critical components.*

The fact that $f(\chi)$ is constant was our inspiration to study the asymptotic contributions of non-isolated critical points to integrals of type (1.2), as in the Borel plane the entirety of any critical component χ will map to the critical value $f(\chi)$. We therefore believed that with sufficient modification, the overall concept and much of the current theory should then translate over from non-degenerate isolated critical points to more general critical components.

Recall in §2.2 we defined the order of degeneracy of the derivatives of a function $f : \mathbb{C} \to \mathbb{C}$ at a point $a \in \mathbb{C}$ as the value $\omega \in \mathbb{N}$ such that

$$\left.\frac{d^b f(z)}{dz^b}\right|_{z=a} = 0 \quad \text{but} \quad \left.\frac{d^\omega f(z)}{dz^\omega}\right|_{z=a} \neq 0,$$

for all $\mathbb{N} \ni b < \omega$. When considering functions in higher dimensions where it is possible to have sets of non-isolated critical points, we need to be more careful with our definition of order of degeneracy.

For a point to have order of degeneracy ω, we require that *all* derivatives of order up to $\omega - 1$ are zero at that point - including any mixed derivatives - and also that at least one of the derivatives of order ω is non-zero there. This is equivalent to identifying the order of the first non-constant term in the Taylor series around the point. When we have a set of non-isolated critical points, we simply consider the order of degeneracy at each point and record this information in the natural valued function $\omega(\boldsymbol{z})$. We require a function rather than a constant in this case as the order of degeneracy may in fact be different at different points within the set. We formalise this in the following definition.

Definition 2 - Order of Degeneracy. Consider a function $f : \mathbb{C}^d \to \mathbb{C}$, let $\boldsymbol{a} \in \mathbb{C}^d$, and write $\boldsymbol{z} = (z_1, \ldots, z_d)$. We make the following definitions.

(a) The *order of degeneracy of the derivatives of f at the point $\boldsymbol{a}$* is the natural number ω such that all derivatives of order $b < \omega$ are (not trivially) zero at $\boldsymbol{a}$ and at least one derivative of order ω is non-zero there. Symbolically, ω is the number such that for all

$b < \omega$ and for all solutions to $b = b_1 + \cdots + b_d$ with $b_j \in \mathbb{N}_0$, we have (non-trivially) that

$$\left.\frac{\partial^b f(\boldsymbol{z})}{\partial z_1^{b_1} \cdots \partial z_d^{b_d}}\right|_{\boldsymbol{z}=\boldsymbol{a}} = 0 \quad \text{and} \quad \left.\frac{\partial^\omega f(\boldsymbol{z})}{\partial z_1^{\omega_1} \cdots \partial z_d^{\omega_d}}\right|_{\boldsymbol{z}=\boldsymbol{a}} \neq 0$$

for some solution to $\omega = \omega_1 + \cdots + \omega_d$ with $\omega_j \in \mathbb{N}_0$. We refer to ω as the *order of degeneracy* of $\boldsymbol{a}$ or simply the *order* of $\boldsymbol{a}$.

(b) Let $\chi \subset C(f)$ be a critical component of f. Then the *order of degeneracy of the set* χ is the natural valued function $\omega(\boldsymbol{z}) : \chi \to \mathbb{N}$ that is computed by considering the order of degeneracy at each point $\boldsymbol{a} \in \chi$.

(c) A critical component χ is of *constant order of degeneracy* if $\omega(\boldsymbol{z})$ is constant on χ. Similarly, if $\omega(\boldsymbol{z})$ is non-constant on χ, then χ is of *non-constant order of degeneracy.*

(d) When $\omega(\boldsymbol{z})$ is constant on χ, we say that χ is *uniformly non-degenerate* if $\omega = 2$ and *uniformly degenerate* if $\omega > 2$. If $\omega(\boldsymbol{z})$ is not constant, then we say that χ is *non-uniformly degenerate.*

In this thesis we are mainly interested in critical components that have constant order of degeneracy so that we can invoke results from Morse theory. Uniformly non-degenerate critical components are Morse-Bott critical components and functions whose critical set comprises of only Morse-Bott critical components are Morse-Bott functions. A Morse critical point is the special case where the connected component is a non-degenerate isolated critical point and a Morse function is the special case where the critical set contains only Morse critical points. A uniformly degenerate critical component can be broken into a cluster of Morse-Bott critical components by a slight perturbation of the coefficients of f (i.e. by slight linear transformation) and can thus be included in the analysis carried out throughout this thesis.

Note that in this thesis, the terms critical point (or component) and singularity will often be used interchangeably (in particular, Morse critical points and Morse singularities). This is because during the derivation of the asymptotic expansion of integral (1.2), it is shown that the singularities of the integral are exactly the critical points of f.

Along Morse-Bott critical components, the determinant of the Hessian matrix must be non-degenerate *in the normal direction.* We must - however - be very specific with what 'degeneracy of the Hessian' actually means and implies. If the Hessian determinant is zero at a critical point $\boldsymbol{p} \in \mathbb{C}^d$, then it is sometimes said that $\boldsymbol{p}$ is 'degenerate' without further qualification.

If *all* second derivatives of a function are zero at $\boldsymbol{p}$, then the Hessian matrix and hence determinant are trivially zero there and we would say that $\boldsymbol{p}$ is degenerate in the sense of Definition 2 (that is, that the first non-constant term in the Taylor expansion around $\boldsymbol{p}$ is at least order three). On the other hand, if the Hessian matrix and determinant are zero at $\boldsymbol{p}$ but not trivially so, then $\boldsymbol{p}$ is 'degenerate' in the sense that it is actually a non-isolated critical point.

When only Morse critical points are considered this distinction is not a pressing issue as either result implies a non-Morse singularity, but for this work it is important that such distinctions are made very clear. For example, the Hessian determinant will automatically be degenerate on Morse-Bott critical components as they are made up of non-isolated critical points, but it will not be *trivially* degenerate as the critical component has constant order two; the determinant is trivially zero, but the matrix is not. A simple example is the function $f(z_1, z_2) = z_2^2$, whose derivatives are

$$\frac{\partial f}{\partial z_1} = 0, \quad \frac{\partial f}{\partial z_2} = 2z_2, \quad \text{and} \quad H_f(z_1, z_2) = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}.$$

Solving for critical points reveals that the critical set $C(f)$ contains only the (one-dimensional) complex line χ given by $z_2 = 0$. We observe that the Hessian determinant is trivially zero as χ is a non-isolated critical component, but that the Hessian *matrix* is not trivially zero as $f_{z_2 z_2} = 2$. Since $H_f(\chi)$ is clearly not the zero matrix as $f_{z_2 z_2}$ is a non-zero constant, χ is of constant order of degeneracy two. We can also see that in the normal direction to χ (namely, the z_1-direction), the Hessian determinant equals two and is hence non-degenerate, implying that χ is a Morse-Bott critical component of f.

Critical components that have non-constant order of degeneracy may sound like edge cases, but in our experience through testing many examples we found that these type of critical components occurred frequently. We shall see an example of this in §8, but it is in fact trivial to construct such an example. A simple example is the function $f(z_1, z_2) = z_1 z_2^2$, whose derivatives are

$$\frac{\partial f}{\partial z_1} = z_2^2, \quad \frac{\partial f}{\partial z_2} = 2z_1 z_2, \quad \text{and} \quad H_f(z_1, z_2) = \begin{pmatrix} 0 & 2z_2 \\ 2z_2 & 2z_1 \end{pmatrix}.$$

Solving for critical points reveals that the critical set $C(f)$ contains only the (one-dimensional) complex line χ given by $z_2 = 0$ as in the previous example, however this time the Hessian matrix and determinant behave very differently.

On χ, we have

$$H_f(\chi) = H_f(z_1, 0) = \begin{pmatrix} 0 & 0 \\ 0 & 2z_1 \end{pmatrix},$$

which has trivially degenerate determinant because χ is non-isolated. The matrix $H_f(\chi)$ is not *identically* the zero matrix because $f_{z_2z_2} = 2z_1$ is not identically zero, but we do have that all second derivatives are zero when $z_1 = 0$. This means that χ has order of degeneracy two at all of its points except $z_1 = 0$, where it is order three since $f_{z_1z_2z_2} = f_{z_2z_1z_2} = f_{z_2z_2z_2} = 2$ and all other third derivatives are zero. The order of degeneracy of χ is hence given by the function

$$\omega(z_1, z_2) = \begin{cases} 3 & \text{if } (z_1, z_2) = (0,0), \\ 2 & \text{otherwise.} \end{cases}$$

We also see that in the normal direction to χ (namely, the z_1-direction), the Hessian determinant is z_1 and is hence degenerate at $z_1 = 0$ and non-degenerate elsewhere. This implies that χ is a Morse-Bott critical component of f except at $z_1 = 0$.

Before moving on, we look at a more complicated example to provide a concrete and detailed summary of the discussion regarding critical components thus far. Consider the eighth order polynomial $f \in \mathbb{C}[z_1, z_2]$ given by

$$f(z_1, z_2) = \frac{1}{8}(z_1^2 + z_2^2)^4 - \frac{17}{6}(z_1^2 + z_2^2)^3 + 22(z_1^2 + z_2^2)^2 - 72(z_1^2 + z_2^2), \tag{3.2}$$

displayed in real space in Figure 13. We note - but do not make use of - the fact that this function also happens to be a fourth order polynomial in $\mathbb{C}[z_1^2 + z_2^2] = \mathbb{C}[r]$, implying some radial symmetry that will make itself apparent as we work through the example. The derivatives of f are

$$\frac{\partial f}{\partial z_1} = z_1(z_1^2 + z_2^2 - 4)^2(z_1^2 + z_2^2 - 9), \quad \frac{\partial f}{\partial z_2} = z_2(z_1^2 + z_2^2 - 4)^2(z_1^2 + z_2^2 - 9),$$

$$\begin{aligned}
\frac{\partial^2 f}{\partial z_1^2} &= (z_1^2 + z_2^2 - 4)(7z_1^4 + 8z_1^2z_2^2 + z_2^4 - 57z_1^2 - 13z_2^2 + 36), \\
\frac{\partial^2 f}{\partial z_2^2} &= (z_1^2 + z_2^2 - 4)(7z_2^4 + 8z_1^2z_2^2 + z_1^4 - 57z_2^2 - 13z_1^2 + 36), \\
\frac{\partial^2 f}{\partial z_1 z_2} &= 2z_1z_2(z_1^2 + z_2^2 - 4)(3z_1^2 + 3z_2^2 - 22).
\end{aligned}$$

Due to the simple factorisation that f_x and f_y admit, we can immediately spot that the critical set is given by

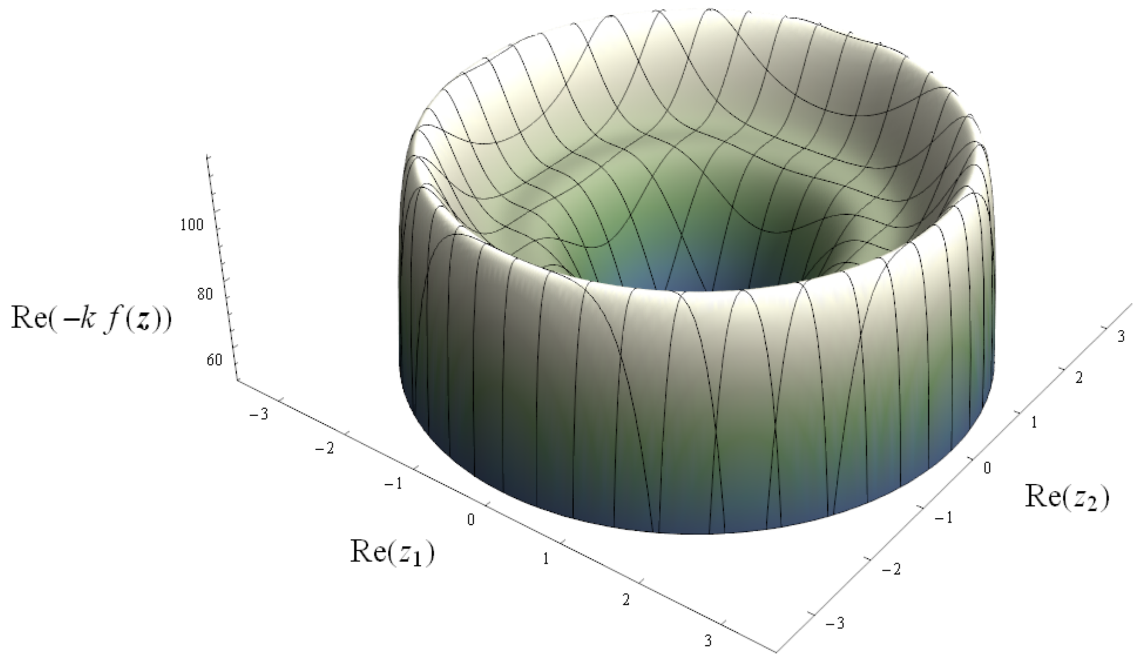


Figure 13: Three dimensional plot of $\mathrm{Re}(-kf(\boldsymbol{z}))$, for $f(\boldsymbol{z})$ given by equation (3.2). The slice shown is the subset $\{(\mathrm{Re}(z_1), \mathrm{Re}(z_2))\} \cong \mathbb{R}^2$ of $\mathbb{C}^2$. The two one-dimensional critical components χ_2 and χ_3 are clearly visible in real space as circles of radius two and three respectively, with the third order nature of the inner circle χ_2 manifesting as a 'line of inflection' in real space. The isolated critical point χ_1 at the origin is at the bottom of the dip in the center.

$$C(f) = \left\{\boldsymbol{z} \in \mathbb{C}^2 \mid (0,0) \vee \left(z_1^2 + z_2^2 = 4\right) \vee \left(z_1^2 + z_2^2 = 9\right)\right\}.$$

This set has three critical components, defined as

$$\begin{aligned} \chi_1 &= (0,0), \\ \chi_2 &= \left\{\boldsymbol{z} \in \mathbb{C}^2 \mid z_1^2 + z_2^2 = 4\right\}, \\ \chi_3 &= \left\{\boldsymbol{z} \in \mathbb{C}^2 \mid z_1^2 + z_2^2 = 9\right\}, \end{aligned}$$

and we can clearly see that

$$\chi_1 \cup \chi_2 \cup \chi_3 = C(f) \quad \text{and} \quad \chi_1 \cap \chi_2 \cap \chi_3 = \emptyset$$

as required. For this function, the critical set is made up of an isolated critical point and two complex circles of different radii, so we have $\mu_1 = 0$ and $\mu_2 = \mu_3 = 1$. Figure 14 illustrates the critical components in real space.

On χ_2, it is clear that all second derivatives are zero, since $z_1^2 + z_2^2 - 4$ is a factor of all of them and on χ_1 and χ_3, the Hessian matrix reduces to

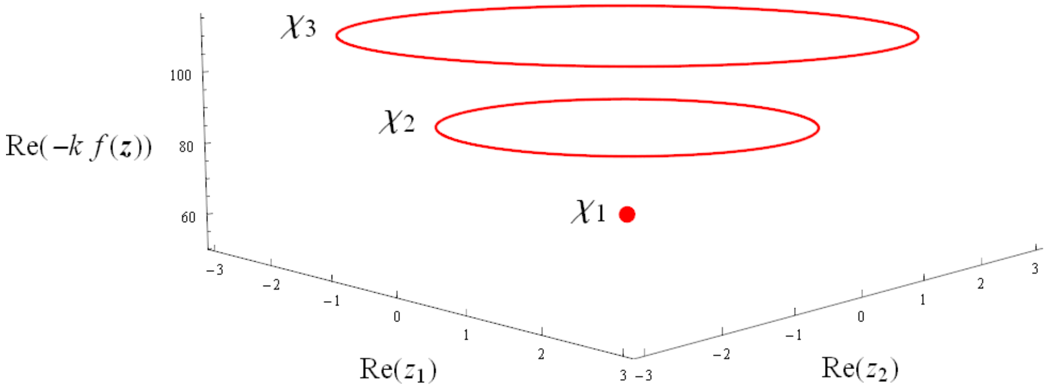


Figure 14: This is a plot of the critical components of $f(\boldsymbol{z})$, for $f(\boldsymbol{z})$ given by equation (3.2). In real space it is intuitively clear that they are mutually disjoint, but it is also algebraically clear (in general) from their definitions. If the function had more critical components, they would also have to be mutually disjoint; if any two seemingly individual critical components intersected, then under Definition 1 they would have to be considered as one component because they would be connected sets and hence form one larger connected component.

$$H_f(\chi_1) = H_f(0,0) = \begin{pmatrix} -144 & 0 \\ 0 & -144 \end{pmatrix} \quad \text{and} \quad H_f(\chi_3) = \begin{pmatrix} 50z_1^2 & 50z_1z_2 \\ 50z_1z_2 & 50z_2^2 \end{pmatrix}$$

respectively. We can see that χ_1 has constant order of degeneracy two and since $H_f(\chi_3)$ is only equal to the zero matrix at $(0,0) \notin \chi_3$, the order of χ_3 is also a constant two. By considering the eight third derivatives $\{\partial^3 f/\partial z_j \partial z_k \partial z_l\}$ for $j,k,l \in \{1,2\}$, it can be shown that the only way for all of them to be simultaneously zero on χ_2 is when the further condition $z_1 = z_2 = 0$ is satisfied. As before, since $(0,0) \notin \chi_2$, χ_2 has constant order of degeneracy three. We have therefore computed that $\omega_1 = \omega_3 = 2$ and $\omega_2 = 3$.

It is reasonable to wonder what type of critical components can exist, if they make sense in the context of steepest descent analysis or other types of analysis, and how they would contribute asymptotically. To this end, we will discuss in detail all elements of the integral (1.2), paying particular attention to the dimensionality of each element.

The parent space for the problem is $\mathbb{C}^d$ with $d \in \mathbb{N}$ and points in $\mathbb{C}^d$ are given by $\boldsymbol{z} = (z_1, \ldots, z_d)$, for $z_j \in \mathbb{C}$. We can treat $\mathbb{C}^d$ as a real vector space by taking $z_j = x_j + iy_j$, so that $\mathbb{C}^d \cong \mathbb{R}^{2d}$ is a real vector space of real dimension $2d$. Dimensionality is a key aspect in the study of critical components and it is necessary to carefully (and constantly) distinguish between real and complex dimensions; $\mathbb{C}^d$ may be a real vector space of dimension $2d$, but it is also complex space of complex dimension d. Considering the set given by $z_1 = 0$ in $\mathbb{C}^2$, we know that it is a complex line of complex dimension one, but is also a two dimensional real surface in $\mathbb{R}^4 \cong \mathbb{C}^2$. We use the function $\dim_{\mathbb{R}}(\,\cdot\,) \equiv \dim(\,\cdot\,)$ to mean the real dimension of its argument, with $\dim_{\mathbb{C}}(\,\cdot\,)$ denoting the complex dimension.

The dimension of various spaces and surfaces will play a leading part in our discussion of the elements that make up integral (1.2) and we start by looking at the steepest descent space and the integration surface. In the following we assume that exponential dependence has already been extracted from integral (1.2), so that the exponent function of the integral of interest is $-k\,(f(\boldsymbol{z}) - f_n)$, where $f(\chi_n) := f_n$.

The steepest descent space for a critical component χ_n of the function $f(\boldsymbol{z})$ is defined as the set

$$\sigma_f^{(n)} := \left\{ \boldsymbol{z} \in \mathbb{C}^d \mid \mathrm{Im}(k\,(f(\boldsymbol{z}) - f_n)) = 0 \right\}.$$

The loosely termed 'integration surface' $S_n \subset \mathbb{C}^d$ is a differentiable and hence smooth manifold within the steepest descent space $\sigma_f^{(n)}$, fully infinite in all d complex variables that runs between specified asymptotic valleys through the critical component χ_n. We recall that asymptotic valleys are given by the condition

$$\mathrm{Re}\,(k\,(f(\boldsymbol{z}) - f_n)) \to \infty.$$

The real dimension of S_n is simply d, as it is made up of small real d-dimensional patches $d\boldsymbol{z} = dz_1 \cdots dz_d$ that are formed by the product of d real one-dimensional vectors, whereas the steepest descent space is defined only by one real dimensional restriction, meaning that it has real dimension $2d - 1$.

When $d = 1$ and $\omega_n = 2$, the integration surface S_n is automatically completely specified as $\dim(S_n) = \dim(\sigma_f^{(n)}) = 1$. When $\omega_n > 2$, there are ω_n valleys connected to χ_n via steepest descent surfaces, so we are able to choose which valley we start and end integration in. However, the dimensional restriction remains and dictates that there is only one choice of integration surface S_n for each of these choices of start and end valley. This will be elaborated on in §4.

When $d > 1$, we have that $\dim(S_n) < \dim(\sigma_f^{(n)})$, introducing $d - 1$ degrees of freedom in our choice of S_n. In practice this is not an issue, since we can deform the integration surface as much as we like due to Cauchy's integral theorem, as long as it stays within the steepest descent space and the two valleys of integration remain the same. Having discussed S_n and defined $\sigma_f^{(n)}$, we turn our attention towards critical components and discuss how they interact with these sets.

When $d = 1$, the only possible critical components are zero-dimensional; that is, isolated critical points. The solutions to the critical point equation

$$\frac{df}{dz} = 0 \tag{3.3}$$

enforces some amount of real dimensional restrictions on the possible values critical points can take. If $f(z) =$ constant, then clearly its derivative is zero and so (3.3) is trivially satisfied, with f having no critical points. Any non-trivial solution will dimensionally restrict the possible set of critical points $C(f)$ and hence the dimension of any critical component χ that it includes. Defining $\dim_{\mathbb{R}}(\chi_j) := \mathcal{D}_j$, we make this explicit in the following proposition.

Proposition 1. Let $f : \mathbb{C} \to \mathbb{C}$ be a sufficiently holomorphic non-constant function with critical component set X_f. Then for all $\chi \in X_f$, $\mathcal{D} = 0$.

Proof. We prove by contradiction, showing that $\mathcal{D} \notin \{1, 2\}$.

<u>Case 1: $\mathcal{D} = 2$</u>

If $\mathcal{D} = 2$, then (3.3) introduces no dimensional restrictions on χ, which is automatically a contradiction as this is only true when f is constant. Thus, $\mathcal{D} \neq 2$.

<u>Case 2: $\mathcal{D} = 1$</u>

If $\mathcal{D} = 1$, then the equation (3.3) introduces one real dimensional restriction on χ by either restricting the real or imaginary z-axis. This means that (3.3) will be a function of $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$. Hence f cannot be holomorphic, contradicting our assumption on f. Thus, $\mathcal{D} \neq 1$. □

From this we see that to experience non-isolated critical points, we require $d \geq 2$. This introduces conceptual difficulties, as it is not possible to fully pictorially depict four or more real dimensions at once. Many of the problems that are involved in considering non-isolated critical points in $\mathbb{C}^d$ are conceptual ones, due simply to the fact that there is no analogous case in $\mathbb{C}$ to fall back on. We can consider and fully draw real critical components in $\mathbb{R}^2$ or $\mathbb{R}^3$ to help visualisation, but the work we do still focuses on the general complex case $\mathbb{C}^d$.

The system of equations determining the set $C(f)$ - where $f : \mathbb{C}^d \to \mathbb{C}$ - is given by (3.1). Each of the d separate equations in (3.1) will introduce dimensional restrictions on critical points in $C(f)$ and hence critical components χ. The real dimension of $\mathbb{C}^d$ is $2d$, so if $C(f)$ contains only isolated critical points - for which $\mathcal{D} = 0$ - then (3.1) must enforce $2d$ real dimensional restrictions on points in $C(f)$. It is worth noting however that we only end up with $2d$ restrictions because each equation in the system (3.1) gives *unique* information about the critical points of f; there are d equations, each of them contributing two real dimensional restrictions (one real and one imaginary axis restriction for each variable). What if this were not the case and two of the equations gave the same information? What if *all* of the equations in (3.1) gave the same information?

Earlier in this section we saw how simple polynomials can have non-isolated critical components; what exactly causes these non-isolated critical components to occur? Consider the function

$$f : \mathbb{C}^4 \to \mathbb{C}$$
$$(t, x, y, z) \mapsto z_1 z_2^2 + z_3 z_4^2$$

with first derivatives

$$f_{z_1} = z_2^2, f_{z_2} = 2z_1 z_2, f_{z_3} = z_4^2, f_{z_4} = 2z_3 z_4.$$

solving for critical points requires us to simultaneously solve the equations

$$\{z_2^2 = 0,\ 2z_1 z_2 = 0,\ z_4^2 = 0,\ 2z_3 z_4 = 0\}. \tag{3.4}$$

Clearly (3.4) is only satisfied when both z_2 and z_4 are zero, but what about z_1 and z_3? The fact that z_2 and z_4 are required to be zero means that equation (3.4)-2 and (3.4)-4 are trivially satisfied and so become degenerate pieces of information; from these four equations, the only thing we are able to deduce is that z_2 and z_4 must be zero, leaving z_1 and z_3 as free variables. We have lost four pieces of information (real dimensional restrictions) from our system (3.1) and thus our critical components can have a maximum real dimension of four. In this case, X_f consists of a single critical component χ of real dimension four given by the surface $(z_1, 0, z_3, 0)$.

How far can we take this idea? What is the largest dimension a critical component can attain? Equivalently, what is the *minimum* amount of information the system (3.1) can provide us with? We propose the following.

Proposition 2. Let $f : \mathbb{C}^d \to \mathbb{C}$ be a sufficiently holomorphic non-constant function with critical component set X_f. Then for all $\chi \in X_f$,

$$0 \leq \mathcal{D} \leq 2d - 2.$$

Furthermore, the real dimension of χ is always even.

Proof. The proof is very similar to that of Proposition 1. We wish to eliminate the possibilities $\mathcal{D} = 2d$ and $\mathcal{D}$ odd. We again prove by contradiction.

Case 1: $\mathcal{D} = 2d$

If $\mathcal{D} = 2d$, then (3.1) introduces no restrictions on χ, which is automatically a contradiction as this is only true when f is constant. Thus, $\mathcal{D} \neq 2d$.

Case 2: $\mathcal{D}$ is odd

Let R be the number of real dimensional restrictions the system (3.1) imposes on points in $C(f)$. If $\mathcal{D}$ is odd, R is also odd since we require that $\mathcal{D} + R = 2d$. Regardless of the actual the actual value of R, the only way this case could occur is if at least one of the equations in the system (3.1) included the terms $\mathrm{Re}(z_j)$ or $\mathrm{Im}(z_j)$, for some $j \in \{1, \ldots, d\}$. This would cause f to not be holomorphic, which is a contradiction. Thus, $\mathcal{D}$ is even.

Using the above two facts, we have the inequality

$$0 \leq \mathcal{D} \leq 2d - 2,$$

with $\mathcal{D}$ even. $\square$

Returning to the question of the maximum attainable dimension of a critical component, consider the function

$$f : \mathbb{C}^d \to \mathbb{C}$$
$$(z_1, \ldots, z_d) \mapsto \kappa(z_1 + \ldots + z_d)^X,$$

where ($\kappa \in \mathbb{C} \setminus \{0\}$ and $X \in \mathbb{N}_{\geq 2}$), whose derivative with respect to every variable z_j is $\kappa X(z_1 + \ldots + z_d)^{X-1}$. This means that all d equations in the system (3.1) give us the *same* piece of information about critical points of f, namely that critical points are given by

$$\kappa X(z_1 + \ldots + z_d)^{X-1} = 0 \Rightarrow z_1 + \ldots + z_d = 0.$$

This single equation restricts the real dimension of $C(f)$ by two and defines a single critical component of real dimension $2d-2$, showing us that $2d-2$ is indeed an attainable maximum for $\mathcal{D}$.

Proposition 2 also guarantees that critical components can never dimensionally exceed the steepest descent space $\sigma_f^{(n)}$ and will thus always remain inside it as a subset; the steepest descent space is $2d-1$ dimensional and the maximum dimension of a critical component is $2d-2$. This gives rise to the dimensional hierarchy

$$0 \leq \mathcal{D} \leq 2d-2 < \dim(\sigma_f^{(n)})(=2d-1) < 2d,$$

given more succinctly in the following proposition.

Proposition 3. Let $f : \mathbb{C}^d \to \mathbb{C}$ be a sufficiently holomorphic non-constant function with critical component set X_f. Then for all $\chi \in X_f$ with real dimension $\mathcal{D}$,

$$0 \leq \mathcal{D} < \dim(\sigma_f^{(n)}) < 2d.$$

Proof. This is implied directly by Proposition 2 and the above discussion. $\square$

As a final consideration in this area, we discuss how critical components and the integration surface interact. In the inequality chain given in Proposition 3, we cannot permanently place the quantity $\dim(S_n) = d$. The following list illustrates the differences in dimensional behaviour for various d:

$$\text{For general } d, \quad \dim(S_n) = d,\ \dim(\sigma_f^{(n)}) = 2d-1,\ \mathcal{D} \in \{0, 2, \ldots, 2d-2\};$$

$$\begin{aligned}
&\text{For } d=1, \quad \dim(S_n) = 1,\ \dim(\sigma_f^{(n)}) = 1,\ \mathcal{D} \in \{0\};\\
&\text{For } d=2, \quad \dim(S_n) = 2,\ \dim(\sigma_f^{(n)}) = 3,\ \mathcal{D} \in \{0, 2\};\\
&\text{For } d=3, \quad \dim(S_n) = 3,\ \dim(\sigma_f^{(n)}) = 5,\ \mathcal{D} \in \{0, 2, 4\};\\
&\text{For } d=4, \quad \dim(S_n) = 4,\ \dim(\sigma_f^{(n)}) = 7,\ \mathcal{D} \in \{0, 2, 4, 6\}.
\end{aligned} \tag{3.5}$$

By comparing the dimensions of S_n and χ in the above list, we can see that critical components will not always be entirely containable within the integration surface for $d > 1$; for example when $d = 4$, we have critical components of real dimensions zero and two that can be wholly contained in an integration surface S_n of real dimension four, but there can be components of real dimension four and six that will not be able to be fully covered by S_n. Initially, one might think this means that only part of the component will contribute to the integral, depending on the surface S_n. However, as noted in §1, we may deform S_n as we wish within $\sigma_f^{(n)}$ without the value of the integral changing due to Cauchy's integral theorem. This means that even though different integration surfaces may contain different slices of a critical component, the component always contributes the same amount. Thus, it is irrelevant whether or not S_n completely contains a critical component, since said component will contribute the same regardless of choice of S_n.

We define $\chi_{S_n} := S_n \cap \chi_n$ as the intersection of the critical component χ_n and the integration surface S_n. Of course, we have the natural set theoretic restriction

$$\dim(\chi_{S_n}) \leq \min\{\mathcal{D}_n, d\},$$

because the intersection of two sets cannot contain more points than each of the parent sets individually. However, we also note that $S_n \nsubseteq \chi_n$, else f would be constant along the entire integration surface and so S_n would clearly not run between asymptotic valleys. We therefore have the following:

(i) If $\mathcal{D}_n < d$, then $\chi_n \subset S_n$ and so $\dim(\chi_{S_n}) = \mathcal{D}_n$;

(ii) If $\mathcal{D}_n = d$, then $\dim(\chi_{S_n}) = d - 1$, as $\chi_n \neq S_n$;

(iii) If $\mathcal{D}_n > d$, then $\dim(\chi_{S_n}) = d - 1$, as $\chi_n \not\subset S_n$.

This can be summarised in the following proposition.

Proposition 4. The real dimension of the steepest descent integration surface S_n is given by the following:

$$\dim(\chi_{S_n}) = \min\{\mathcal{D}_n, d - 1\}.$$

Proof. This is implied by the above discussion. □

3.2 Morse Theory and Morse-Bott Theory

In this section we will discuss Morse theory in the context of decomposing integrals of type (1.1) into a finite sum of integrals of type (1.2). When f has only isolated critical points we can apply Morse theory directly, but in the presence of sets of non-isolated critical points we must appeal to Morse-Bott theory for this decomposition instead. We discuss both of these areas and use them as justification for the work done in §§4 and 5.

Let

$$I(k) = \int_S d\boldsymbol{z}\, g(\boldsymbol{z}) e^{-kf(\boldsymbol{z})}$$

(namely, integral (1.1)), where f has only finite isolated critical points of any order and S is any smooth real d-dimensional integration surface between two asymptotic valleys. Pham (1985) showed that by calculating and suitably decomposing the homology group of allowable integration surfaces S, we are able to decompose $I(k)$ into a finite sum of integrals

$$I^{(n)}(k) = \int_{S_n} d\boldsymbol{z}\, g(\boldsymbol{z}) e^{-kf(\boldsymbol{z})}$$

(namely, integral (1.2)), with S_n an unbounded integration surface similar to S but instead through the isolated critical point z_n of f and no other critical point. Therefore, the study of the asymptotic behaviour of (1.1) can be reduced to studying the contribution from each critical point of f individually, as the full asymptotic behaviour of $I(k)$ also decomposes into a finite sum of the individual expansions representing each asymptotic contribution.

This work was extended in Delabaere and Howls (2002) for the case where S is instead an arbitrary integration region. It was shown that we are able to similarly decompose S into a finite sum of (fully infinite) integrals over S_n and integrals representing the boundary contributions from restricted critical points, by again computing the homology group of allowable integration regions. In both the unbounded and bounded cases, this is in part achieved by using Morse theory. Additionally, it was shown in both cases that the resulting asymptotic expansions from each contributing point are Borel summable. We give some details on the Morse theoretical aspect of the unbounded problem below; for details on the homological aspect of the problem, the reader is referred to Pham (1985) and Delabaere and Howls (2002).

A function $f : M \to \mathbb{C}$ - where M is a complex manifold - is a *Morse function* if all of its critical points are *Morse singularities*, namely non-degenerate isolated critical points. If f has degenerate isolated critical points, we are able to break them up into clusters of non-degenerate critical points by slightly perturbing the coefficients of f, thus converting f into a Morse function and allowing us to proceed without further concern.

For our problem, we take $M = \mathbb{C}^d$. The crucial result we need to use is called the *Morse lemma*, guaranteeing the existence of special coordinate systems in the neighbourhood of Morse singularities. We present here the complex version of the Morse lemma in full.

Lemma 1 - Holomorphic Morse Lemma. Let M be a complex manifold of complex dimension m, $f : M \to \mathbb{C}$ a Morse function, and $\boldsymbol{p}$ a Morse singularity of f. Then there exists an open neighbourhood U of $\boldsymbol{p}$ and a holomorphic local chart $\phi : U \to \mathbb{C}^m$ such that:

(i) $\phi(\boldsymbol{p}) = \boldsymbol{0}$;

(ii) $(f \circ \phi^{-1})(\boldsymbol{s}) = f(\boldsymbol{p}) + s_1^2 + \cdots + s_m^2$.

This version of the lemma is stated and proved as Proposition 3.15 in Ebeling (2007) in a slightly different form, adapted from the real analogue in (for example) Morse (1934) and Milnor (1963). A slightly more precise result is given by Theorem 2.46 in Greuel, Lossen, and Shustin (2007), specifically the equivalence between points (a) and (d) in the theorem. Note that the form of the Morse lemma we provide here is the exact complex analogue of the real version given as Theorem 1 in Banyaga and Hurtubise (2004).

Following §2.3 of Delabaere and Howls (2002) (namely, the unbounded case), we start by assuming that integral (1.1) has already been decomposed into the appropriate sum of integrals of type (1.2), justified by the homological discussion earlier. Taking $M = \mathbb{C}^d$, the Morse lemma can then be applied directly to integral (1.2) in the case that f has only isolated critical points (recall that any degenerate isolated critical points of f are handled by slightly perturbing its coefficients). Thus, local to any Morse singularity $\boldsymbol{z}_n$ of f, the Morse lemma guarantees the existence of local coordinates $\boldsymbol{s} = (s_1, \ldots, s_d)$ such that

$$f(\boldsymbol{s}) - f_n = s_1^2 + \cdots + s_d^2,$$

with $f_n = f(\boldsymbol{z}_n)$. These local coordinates allow us to realise the steepest descent conditions enforced on S_n as the Lefschetz thimble

$$S_n^a = \{\boldsymbol{s} \in \mathbb{C}^d \,\big|\, \mathrm{Re}(s_1)^2 + \cdots + \mathrm{Re}(s_d)^2 \leq a,\ \mathrm{Im}(s_j) = 0\ \forall j\,\},$$

where $|f(\boldsymbol{s}) - f_n| < a$ is our local region of interest. This Lefschetz thimble has as its boundary the vanishing cycle

$$\gamma_n^a = \{\boldsymbol{s} \in \mathbb{C}^d \,\big|\, \mathrm{Re}(s_1)^2 + \cdots + \mathrm{Re}(s_d)^2 = a,\ \mathrm{Im}(s_j) = 0\ \forall j\,\},$$

namely, $\gamma_n^a = \partial S_n^a$. Thus, close to $\boldsymbol{z}_n$, S_n may be expressed as S_n^a and parameterised as

$$S_n^a = \bigcup_{j \in [f_n, f_n + a]} \gamma_n^j.$$

It is then possible to extend the Lefschetz thimble S_n^a globally into a steepest descent n-fold (simply, a smooth steepest descent integration surface) S_n by following the flow of the vector field $\nabla(\mathrm{Re}(kf))$ with the vanishing cycles as initial data. Additionally, it can be shown that the resulting asymptotics are Borel summable (Delabaere and Howls, 2002). This work provides a rigorous justification for the method used to derive our resurgence relation in §4.

In the case where f in (1.1) has non-isolated critical points, we are unable to use Morse theory and must instead turn to *Morse-Bott theory.* A function $f : M \to \mathbb{C}^d$ - where M is a complex manifold - is a *Morse-Bott function* if the critical point set $C(f)$ of f is made up of non-degenerate connected components of dimension μ. Again taking $M = \mathbb{C}^d$, Definition 1 tells us that these connected components are none other than non-degenerate critical components χ of dimension μ. As before, if f has degenerate critical components, they can be broken into clusters of non-degenerate critical components by slightly perturbing the coefficients of f, thus converting f into a Morse-Bott function.

The analogous result to the Morse lemma in this setting is the *Morse-Bott lemma*, which guarantees the existence of a special local coordinate system in a neighbourhood of every critical point within χ (namely, a hypertubular neighbourhood). It proved very difficult to find a complex version of the Morse-Bott lemma in a similar form to Lemma 1 above, so we once again present it here in full for ease of future reference.

The following is taken as Lemma 3.8 from Petro (2008) with slight notational modification. This result is itself taken as the complex analogue of the real Morse-Bott lemma from Banyaga and Hurtubise (2004). The author of the former paper states the complex version of the Morse-Bott lemma without proof, with the authors of the latter proving the real version while also stating that the Morse-Bott lemma is simply a parameterised version of the Morse lemma and its proof follows naturally from the proof of the Morse lemma. Therefore, the complex Morse-Bott lemma presented here has its proof implied by results in the literature.

Lemma 2 - Holomorphic Morse-Bott Lemma. Let M be a complex manifold of complex dimension m, $f : M \to \mathbb{C}$ a Morse-Bott function, and χ a non-degenerate critical component of f of complex dimension μ. Then for all $\boldsymbol{p} \in \chi$, there exists an open neighbourhood U of $\boldsymbol{p}$ and a holomorphic local chart $\phi : U \to \mathbb{C}^\mu \times \mathbb{C}^{m-\mu}$ such that:

(i) $\phi(\boldsymbol{p}) = \boldsymbol{0}$;

(ii) $\phi(U \cap \chi) = \{(\boldsymbol{a}, \boldsymbol{s}) \in \mathbb{C}^\mu \times \mathbb{C}^{m-\mu} \,|\, \boldsymbol{s} = \boldsymbol{0}\}$;

(iii) $(f \circ \phi^{-1})(\boldsymbol{a}, \boldsymbol{s}) = f(\chi) + s_1^2 + \cdots + s_{m-\mu}^2$.

Since we are taking $M = \mathbb{C}^d$, then $\dim(M) = d$ and so the quantities $m - \mu$ in Lemma 2 become $d - \mu = q$, the codimension of χ. In this specific case, the local chart is

$$\phi : U \to \mathbb{C}^{\mu} \times \mathbb{C}^{q}$$

and the three results of Lemma 2 are

(i) $\phi(\boldsymbol{p}) = \boldsymbol{0}$;

(ii) $\phi(U \cap \chi) = \{(\boldsymbol{a}, \boldsymbol{s}) \in \mathbb{C}^{\mu} \times \mathbb{C}^{q} \,|\, \boldsymbol{s} = \boldsymbol{0}\}$;

(iii) $(f \circ \phi^{-1})(\boldsymbol{a}, \boldsymbol{s}) = f(\chi) + s_1^2 + \cdots + s_q^2$.

While obvious, we write it out explicitly as it is the version direct applicable to our problem.

We again start by assuming that integral (1.1) has already been decomposed into the appropriate sum of integrals of type (1.2) so that we can consider the individual asymptotic contributions from each critical component separately, but this time there does not exist a rigorous homological justification for this assumption in the current literature. Nonetheless, given the existence of Morse-Bott theory, the Morse-Bott lemma, and the fact that the integration surfaces S_n in the isolated and non-isolated case are essentially the same, we believe it is a safe assumption to make and that the work just has not been done yet, so we proceed regardless. In contrast to the isolated case, the following does not follow any current literature and is original work.

With $M = \mathbb{C}^d$, the Morse-Bott lemma can be applied directly to integral (1.2) in the case that f has critical components of any dimension and order, recalling that degenerate critical components are handled by slightly perturbing the coefficients of f. Thus, local to all points within any non-degenerate critical component χ_n of f of dimension and codimension μ_n and q_n respectively, the Morse-Bott lemma guarantees the existence of local coordinates $\boldsymbol{s} = (s_1, \dots, s_{q_n})$ such that

$$f(\boldsymbol{s}) - f_n = s_1^2 + \cdots + s_{q_n}^2,$$

with $f_n = f(\chi_n)$. These local coordinates allow us to realise the steepest descent conditions enforced on S_n as the surface

$$S_n^a = \{\boldsymbol{s} \in \mathbb{C}^{q_n} \,\big|\, \mathrm{Re}(s_1)^2 + \cdots + \mathrm{Re}(s_{q_n})^2 \leq a,\ \mathrm{Im}(s_j) = 0\ \forall j\}, \tag{3.6}$$

where $|f(\boldsymbol{s}) - f_n| < a$ is our local region of interest, that has dimension μ_n as it is defined by only $q_n = d - \mu_n$ complex coordinates. This higher dimensional analogue of a Lefschetz thimble does not appear to have been named in the literature, but this type of quadratic surface (or *quadric*) is called a hyper-parabolic cylinder. Therefore, it is reasonable to term the surface (3.6) a *Lefschetz hyper-parabolic cylinder.* Under this naming convention, a Lefschetz thimble would then be a Lefschetz hyper-paraboloid. For ease of exposition, we opt not to attach the 'hyper' prefix to these terms for the remainder of the thesis.

The boundary of this Lefschetz parabolic cylinder (3.6) is the higher dimensional analogue of a vanishing cycle

$$\gamma_n^a = \{ \boldsymbol{s} \in \mathbb{C}^{q_n} \,\big|\, \mathrm{Re}(s_1)^2 + \cdots + \mathrm{Re}(s_{q_n})^2 = a,\ \mathrm{Im}(s_j) = 0\ \forall j \},$$

so that, $\gamma_n^a = \partial S_n^a$. Thus, close to χ_n, S_n may be expressed as S_n^a and parameterised as

$$S_n^a = \bigcup_{j \in [f_n, f_n + a]} \gamma_n^j,$$

just as in the isolated case above. It should then again be possible to extend the Lefschetz pencil S_n^a globally into a steepest descent n-fold S_n in exactly the same way as in the isolated case. The entirety of the preceding analysis correctly reduces to that of the isolated case when χ_n is an non-degenerate isolated critical point, since $\mu_n = 0$ means that $q_n = d$.

We do not demonstrate the Borel summability of the resulting asymptotic expansions here and there is no result in the current literature that does so. It is not a matter that needs our immediate attention, as in this thesis we deal with finitely truncated series. We again believe that this result will follow once it receives formal consideration and that it is safe to assume that the resulting asymptotics are in fact Borel summable. Once these assumptions are proved rigorously - namely that S and hence $I(k)$ decompose suitably and that the asymptotics are Borel summable - then the work here in conjunction with those proofs will provide a rigorous justification for the method used to derive our resurgence relation in §5 and hence our general hyperasymptotic expansion in §6. The proof allowing us to decompose the integration surface in this way is based on homology, and is beyond the scope of this thesis, but the reason we believe it to be true is simply due to how readily the Morse-Bott lemma generalises the analysis of the isolated case to the non-isolated case.

3.3 Degeneracy and Form of the Integral Asymptotics

Having carefully defined and discussed critical components, we now need to work out how they affect the asymptotics of integrals such as (1.2). We look at an instructive example using a class of general functions f that generates (at least) a critical component of dimension μ, allowing us to deduce the powers of k that we can expect in an asymptotic expansion around a general critical component.

Let $f : \mathbb{C}^d \to \mathbb{C}$ and write $\boldsymbol{z} = (z_1, \dots, z_\mu, z_{\mu+1}, \dots, z_d)$ for $d > \mu \in \mathbb{N}$. We look at the class of functions given by

$$f(\boldsymbol{z}) = z_{\mu+1}^{\omega} h_{\mu+1}(z_1, \dots, z_\mu) + \cdots + z_d^{\omega} h_d(z_1, \dots, z_\mu),$$

where $h_j : \mathbb{C}^\mu \to \mathbb{C}$ are non-zero functions, so that

$$\begin{aligned} f_{z_1} &= z_{\mu+1}^{\omega}(h_{\mu+1})_{z_1} + \ldots + z_d^{\omega}(h_d)_{z_1}, \\ &\vdots \\ f_{z_\mu} &= z_{\mu+1}^{\omega}(h_{\mu+1})_{z_\mu} + \ldots + z_d^{\omega}(h_d)_{z_\mu}, \\ f_{z_{\mu+1}} &= \omega z_{\mu+1}^{\omega-1} h_{\mu+1}, \\ &\vdots \\ f_{z_d} &= \omega z_d^{\omega-1} h_d. \end{aligned}$$

Critical points occur for various expressions involving h_j and also on the hypersurface

$$z_{\mu+1} = \cdots = z_d = 0.$$

Note that functions in this class are not the *only* functions that we study or that are allowed according to our work in §§3.1 and 3.2, but are simply *a* general class of functions that can be studied in order to observe their general asymptotic behaviour. Not all examples throughout this thesis will adhere to this form.

Since none of the h_j are identically zero this hypersurface is a critical component χ of complex dimension μ, arising due to $d - \mu$ complex restrictions. We also assume that the h_j are defined in such a way that χ is of constant order of degeneracy ω; if we left the h_j to be completely general, then we would have higher order degeneracies at specific points on χ, leading to ω being non-constant on χ. One such example is taking h_j as a non-zero constant function for all j, which although leads to f having only one critical component, still shows us the powers of k that appear in the asymptotic expansion.

Writing integral (1.2) in full as an indefinite integral, we have

$$I(k) = \left(\int dz_1 \cdots \int dz_\mu \int dz_{\mu+1} \cdots \int dz_d \right) g(\boldsymbol{z}) e^{-k\left(z_{\mu+1}^{\omega} h_{\mu+1}(z_1,\dots,z_\mu) + \cdots + z_d^{\omega} h_d(z_1,\dots,z_\mu)\right)}.$$

For simplicity, we set $g \equiv h_j \equiv 1$ without loss of generality in the powers of k. We can then explicitly and exactly integrate in the variables $z_{\mu+1}$ through z_d to get

$$
\begin{aligned}
I(k) &= \left(\int dz_1 \cdots \int dz_\mu\right) \left(\frac{-1}{\omega \sqrt[\omega]{k}}\right)^{d-\mu} \Gamma\left(\frac{1}{\omega}, kz_{\mu+1}^\omega\right) \cdots \Gamma\left(\frac{1}{\omega}, kz_d^\omega\right) \\
&:= \left(\frac{-1}{\omega \sqrt[\omega]{k}}\right)^q A(k),
\end{aligned}
$$

where $\Gamma(a,z)$ is the (upper) incomplete gamma function. We have an explicit power of k multiplying everything and the other factors of k are tied up in the rest of the integrand $A(k)$. Assuming that we can asymptotically expand $A(k)$ as

$$
A(k) \sim \sum_{r=0}^{\infty} \frac{A_r}{k^{\frac{r}{\omega}}},
$$

then we can write $I(k)$ asymptotically as

$$
I(k) \sim \frac{1}{k^{\frac{q}{\omega}}} \left(\frac{-1}{\omega}\right)^q \sum_{r=0}^{\infty} \frac{A_r}{k^{\frac{r}{\omega}}}.
$$

The powers of k that appear in our asymptotic expansion for $I(k)$ are thus

$$
\frac{1}{k^{\frac{r+q}{\omega}}}
$$

and these are the powers of k that we will see in our results from §§4 and 5.

Having defined and discussed critical components and seen in a basic sense how they affect integral asymptotics, we proceed towards deriving a resurgence relation similar to (1.3) for integral (1.2) in the case where the function f has any number of critical components of both general dimension and general constant order. Inspired by the discussion of Morse and Morse-Bott theory in this section, we consider the contribution from each critical component individually. To aid exposition and understanding, we break the problem into two parts by first looking at general order isolated critical points in $\mathbb{C}$, before moving on to the fully general problem of critical components of any dimension and order. The purpose of this is to introduce the techniques used to handle the general order and general dimension properties separately instead of all at once, in order to provide clearer and more focused explanations.

4 Isolated Critical Points of General Order

In this chapter, we are interested in deriving the asymptotic contribution of general order isolated critical points to the one-dimensional version of integral (1.1), namely

$$I(k) = \int_C dz\, g(z) e^{-kf(z)},$$

where $C \subset \mathbb{C}$ is a smooth contour of integration between two asymptotic valleys.

By the discussion in §3.2, we are able to decompose C and hence $I(k)$ into a finite sum of integrals

$$\begin{aligned} I^{(n)}(k) &= \int_{C_n} dz\, g(z) e^{-kf(z)} = \frac{e^{-kf_n}}{k^{\frac{1}{\omega_n}}} T^{(n)}(k), \\ T^{(n)}(k) &= k^{\frac{1}{\omega_n}} \int_{C_n} dz\, g(z) e^{-k(f(z)-f_n)}. \end{aligned}$$

We also know that due to this decomposition, the full asymptotic behaviour of $I(k)$ can be similarly decomposed into a finite sum of asymptotic expansions that represent each critical point's individual asymptotic contribution. Therefore, we only need to consider asymptotic expansions of the integrals $I^{(n)}(k)$. We thus search for a formal asymptotic expansion and respective truncated expansion of the form

$$T^{(n)}(k) \sim \sum_{r=0}^{\infty} \frac{T_r^{(n)}}{k^{\frac{r}{\omega_n}}} \quad \text{and} \quad T^{(n)}(k) = \sum_{r=0}^{N-1} \frac{T_r^{(n)}}{k^{\frac{r}{\omega_n}}} + R^{(n)}(k, N).$$

Note that for the remainder of this thesis, equations similar to those above will be combined and written as

$$T^{(n)}(k) \sim \sum_{r=0}^{\infty} \frac{T_r^{(n)}}{k^{\frac{r}{\omega_n}}} = \sum_{r=0}^{N-1} \frac{T_r^{(n)}}{k^{\frac{r}{\omega_n}}} + R^{(n)}(k, N)$$

for the sake of convenience,

In this problem, $f, g : \mathbb{C} \to \mathbb{C}$ are sufficiently holomorphic complex valued functions, $k \in \mathbb{C}$ is the asymptotic parameter, $\omega_n \in \mathbb{N}$ is the (constant) order of the isolated critical point $\chi_n = z_n$ of f, $f_n := f(z_n)$, and C_n is a fully infinite contour of integration between two asymptotic valleys and through z_n. Additionally, we make the transformation

$$u^{\omega_n} = k(f(z) - f_n) \iff u = k^{\frac{1}{\omega_n}} (f(z) - f_n)^{\frac{1}{\omega_n}}. \tag{4.1}$$

So far, this set-up is analogous to that employed in Howls (1997) and similar works, and will continue to be at many key stages. However, the method from which we are drawing the analogy only deals with second order isolated critical points, affording some simplifications that cannot be made in the general order case. Before proceeding with the derivation of the asymptotic expansion, the effects of the general order point must be carefully considered; how does each element in the integral $T^{(n)}(k)$ transform under (4.1), and how are these elements visualised in both u-space and u^{ω_n}-space?

Recall that we defined asymptotic valleys and hills as regions of $\mathbb{C}$ where

$$\mathrm{Re}(k(f(z) - f_n)) \to \pm\infty$$

respectively, with steepest descent contours given by

$$\mathrm{Im}(k(f(z) - f_n)) = \text{constant}.$$

We also have regions of ambiguous convergence, which are "halfway" (in the sense of their phase) between valleys and hills, given by

$$\mathrm{Im}(k(f(z) - f_n)) \to \pm\infty.$$

Since $u^{\omega_n} = k(f(z) - f_n)$, the above definitions can directly aid in the visualisation of the u^{ω_n}-plane. In addition, the isolated critical point of focus z_n is given by $u = 0$, with the other critical points z_{m_j} given by

$$u^{\omega_n} = k(f(z_n) - f(z_{m_j})) = kF_{nm_j}.$$

The u^{ω_n}-plane for the general case in $\mathbb{C}^d$ considered in §5 is illustrated in Figure 15, with $\chi_n = z_n$ and $\chi_{m_j} = z_{m_j}$ in this one-dimensional case. In the u^{ω_n}-plane, z_n is an ω_n order branch point, giving rise to ω_n Riemann surfaces. Therefore, there are ω_n valleys and hills, given by

$$u^{\omega_n} = \pm\infty e^{2\pi i X}$$

respectively, with $X \in \{0, 1, \ldots, \omega_n - 1\}$, $2\omega_n$ regions of ambiguous convergence given by

$$u^{\omega_n} = \pm i\infty e^{2\pi i X},$$

and - abusing notation for a real interval (a, b) - ω_n steepest descent contours through z_n given by

$$\left(-\infty e^{2\pi i X}, \infty e^{2\pi i X}\right).$$

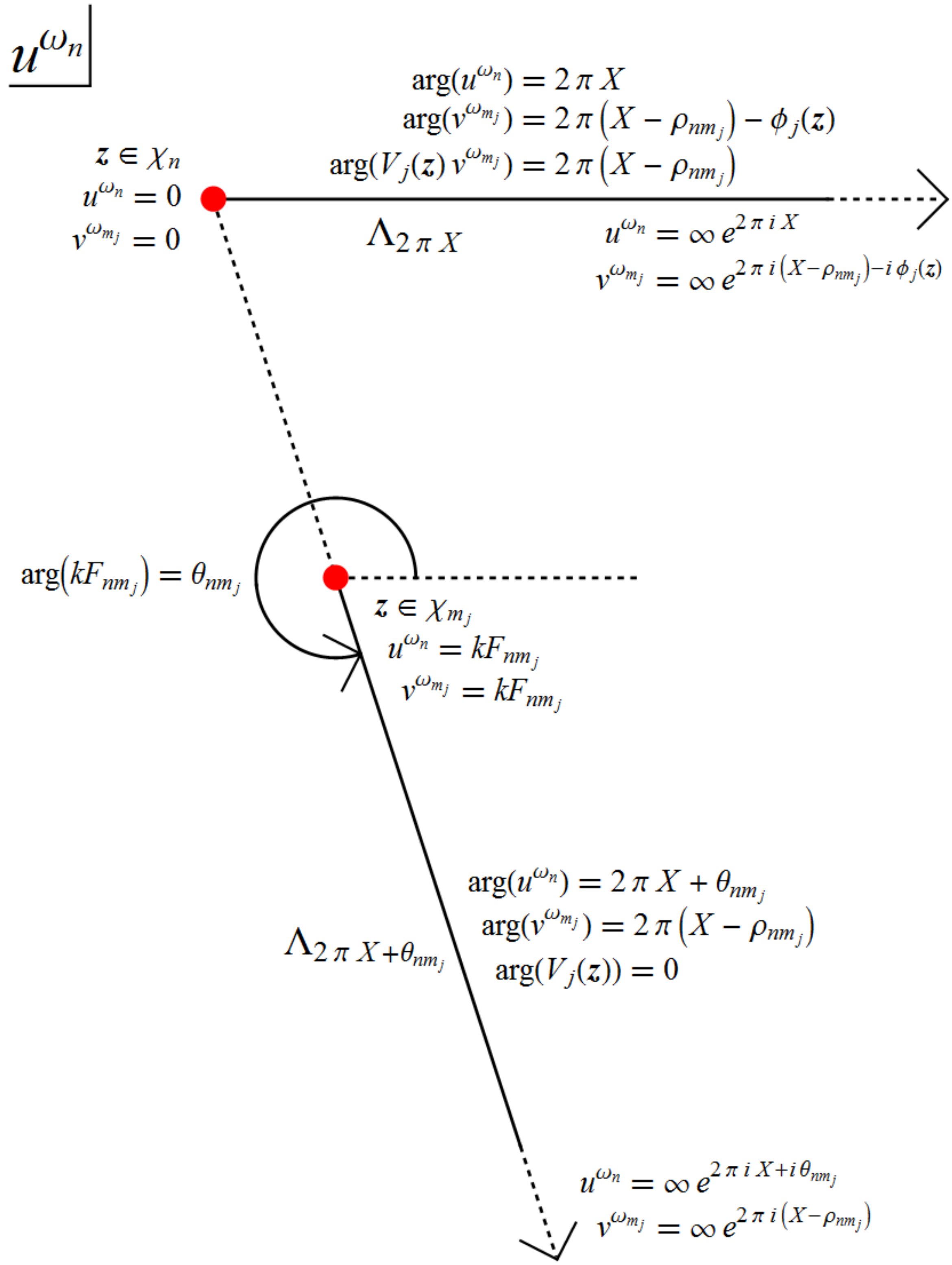


Figure 15: The complex u^{ω_n}-plane for the general case in $\mathbb{C}^d$ considered in §5, with $u^{\omega_n} = k(f(\boldsymbol{z}) - f_n)$. This illustration is still representative of the one-dimensional case considered in this chapter.

Thus, a valid integration contour will both start and end at $\infty e^{2\pi i X}$ for two different respective values of X, meaning that there are ω_n choose two, or $\frac{1}{2}\omega_n(\omega_n - 1)$, unique valid integration contours.

Defining α_n and β_n as the Riemann surface that we start and end the integration on respectively, the general integration contour in u^{ω_n}-space is

$$C_n(\alpha_n, \beta_n) = \left(\infty e^{2\pi i \alpha_n}, 0, \infty e^{2\pi i \beta_n}\right).$$

It is important to make the dependency on α_n and β_n explicit; the asymptotic framework that we develop must be able to handle all possible choices of integration contour, so intuitively it makes sense that there will be explicit dependence on α_n and β_n.

Before moving on to discuss the u-plane, we re-state the problem using our slightly updated notation. We are studying the integral

$$I^{(n)}(k; \alpha_n, \beta_n) = \int_{C_n(\alpha_n, \beta_n)} dz\, g(z) e^{-kf(z)} = \frac{e^{-kf_n}}{k^{\frac{1}{\omega_n}}} T^{(n)}(k; \alpha_n, \beta_n), \tag{4.2}$$

$$T^{(n)}(k; \alpha_n, \beta_n) = k^{\frac{1}{\omega_n}} \int_{C_n(\alpha_n, \beta_n)} dz\, g(z) e^{-k(f(z) - f_n)}, \tag{4.3}$$

and searching for a formal asymptotic expansion and respective truncated expansion of the form

$$T^{(n)}(k; \alpha_n, \beta_n) \sim \sum_{r=0}^{\infty} \frac{T_r^{(n)}(\alpha_n, \beta_n)}{k^{\frac{r}{\omega_n}}} = \sum_{r=0}^{N-1} \frac{T_r^{(n)}(\alpha_n, \beta_n)}{k^{\frac{r}{\omega_n}}} + R^{(n)}(k, N; \alpha_n, \beta_n). \tag{4.4}$$

We will come back to the u^{ω_n}-plane when discussing the remainder $R^{(n)}(k, N; \alpha_n, \beta_n)$.

The u-plane can be thought of as a condensed version of the u^{ω}-plane; all of the Riemann surface structure around z_n collapses when we take the ω-th root of the whole space and what was previously separated by a phase of $2\pi i$ - such as different Riemann surfaces - is now separated by $\frac{2\pi i}{\omega_n}$. Hence, all of the Riemann surfaces around z_n in the u^{ω_n}-plane now fit entirely in u-space, with other features following suit. Figure 16 shows the u-plane, complete with the elements we are about to discuss.

In the u-plane, z_n is given by $u = 0$, with the other critical points given by

$$u = k^{\frac{1}{\omega_n}} F_{nm_j}^{\frac{1}{\omega_n}}.$$

There are ω_n valleys and hills given by

$$u = \infty e^{\frac{2\pi i X}{\omega_n}} \quad \text{and} \quad u = \infty e^{\frac{2\pi i X + i\pi}{\omega_n}}$$

respectively, with $X \in \{0, 1, \ldots, \omega_n - 1\}$, $2\omega_n$ regions of ambiguous convergence given by

$$u = \infty e^{\frac{\pi i X + \frac{i\pi}{2}}{\omega_n}},$$

and ω_n steepest descent contours through z_n given by

$$\left(\infty e^{\frac{2\pi i X + i\pi}{\omega_n}}, \infty e^{\frac{2\pi i X}{\omega_n}}\right).$$

We note (perhaps obviously) that these quantities are simply the ω-th root of the respective quantities in u^{ω}-space when written in the form $\infty e^{i\phi}$, taking $e^{i\pi} = -1$.

The possible fully infinite integration contours are given in the u-plane by

$$C_n(\alpha_n, \beta_n) = \left(e^{\frac{2\pi i \alpha_n}{\omega_n}}, 0, e^{\frac{2\pi i \beta_n}{\omega_n}}\right),$$

where α_n, $\beta_n \in \{0, 1, \ldots, \omega_n - 1\}$ as before. In the u^{ω_n}-plane, α_n and β_n most readily described the Riemann surfaces on which we began and ended integration. In u-space, α_n and β_n most readily describe the simpler notion of which u-valley we start and end integration in (this is of course also true in u^{ω_n}-space, but denoting Riemann surfaces is more useful there). It will also be helpful to consider the semi-infinite integration contours

$$C_{n,X} = \left[0, \infty e^{\frac{2\pi i X}{\omega_n}}\right),$$

where we define $V_{n,X} = \infty e^{\frac{2\pi i X}{\omega_n}}$ to be the X-th valley in the u-plane.

Due to how steepest descent contours into valleys look in the u-plane (see Figure 16), it will be helpful to refer to $C_{n,X}$ as the "X-th leg" of z_n. The α_n-th and β_n-th legs are thus the "entry leg" and "exit leg" of the contour $C_n(\alpha_n, \beta_n)$ respectively. Using this notation, we can write

$$C_n(\alpha_n, \beta_n) = C_{n,\alpha_n} \cup C_{n,\beta_n},$$

with the implication that we still start and end integration and V_{n,α_n} and V_{n,β_n} respectively. Any integral over $C_n(\alpha_n, \beta_n)$ can thus be written

$$\int_{C_n(\alpha_n,\beta_n)} \equiv \int_{V_{n,\alpha_n}}^{z_n} + \int_{z_n}^{V_{n,\beta_n}} \equiv -\int_{z_n}^{V_{n,\alpha_n}} + \int_{z_n}^{V_{n,\beta_n}} \equiv \int_{C_{n,\beta_n}} - \int_{C_{n,\alpha_n}},$$

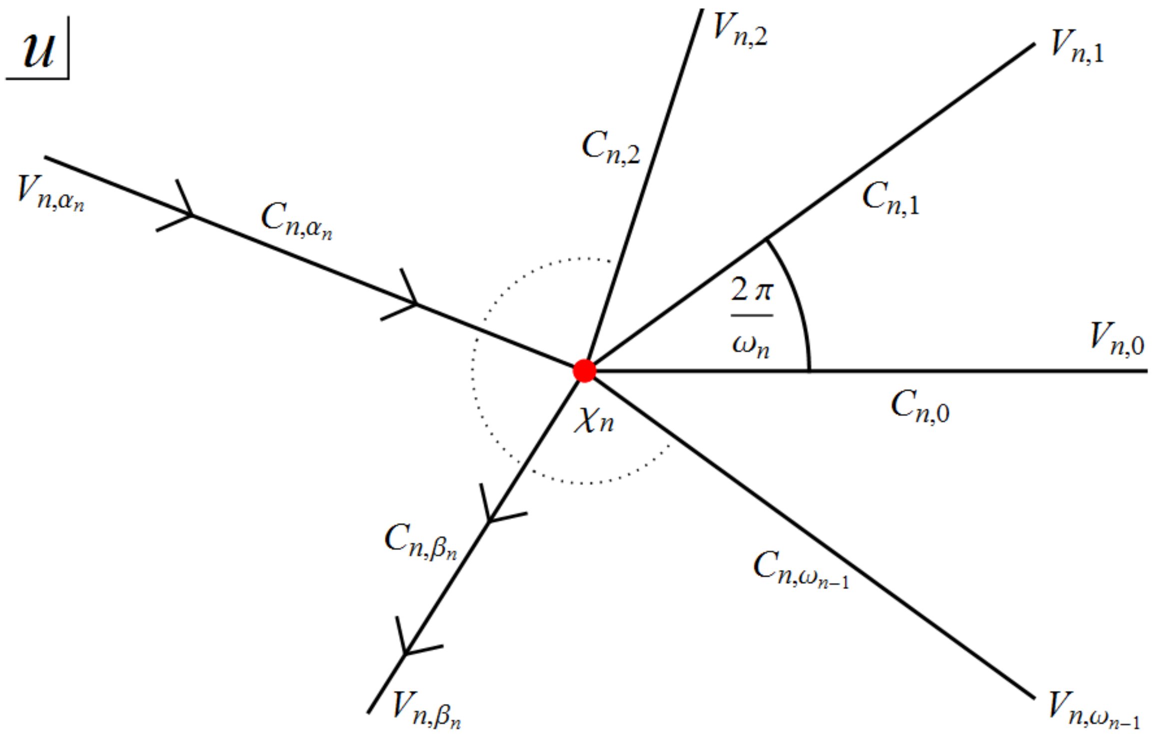


Figure 16: The complex u-plane, with $u = k^{\frac{1}{\omega_n}}(f(z) - f_n)^{\frac{1}{\omega_n}}$. The full integration contour is given by $C_n(\alpha_n, \beta_n) = C_{n,\alpha_n} \cup C_{n,\beta_n}$, with the arrows showing the direction of integration.

in turn allowing us to write

$$I^{(n)}(k; \alpha_n, \beta_n) = \left(\int_{C_{n,\beta_n}} - \int_{C_{n,\alpha_n}}\right) dz\, g(z) e^{-kf(z)},$$
$$T^{(n)}(k; \alpha_n, \beta_n) = k^{\frac{1}{\omega_n}} \left(\int_{C_{n,\beta_n}} - \int_{C_{n,\alpha_n}}\right) dz\, g(z) e^{-k(f(z)-f_n)}$$

and the respective single leg versions as

$$I^{(n)}_{(X)}(k) = \int_{C_{n,X}} dz\, g(z) e^{-kf(z)}, \tag{4.5}$$
$$T^{(n)}_{(X)}(k) = k^{\frac{1}{\omega_n}} \int_{C_{n,X}} dz\, g(z) e^{-k(f(z)-f_n)}, \tag{4.6}$$

so that

$$I^{(n)}(k; \alpha_n, \beta_n) = I^{(n)}_{(\beta_n)}(k) - I^{(n)}_{(\alpha_n)}(k),$$
$$T^{(n)}(k; \alpha_n, \beta_n) = T^{(n)}_{(\beta_n)}(k) - T^{(n)}_{(\alpha_n)}(k). \tag{4.7}$$

It will also be helpful to consider the asymptotic expansion along a single, general leg of z_n (namely, along the semi-infinite contour $C_{n,X}$) and then combine the expansions along C_{n,α_n} and C_{n,β_n} to produce the full expansion along $C_n(\alpha_n, \beta_n)$.

Since the integral along the X-th leg will have an asymptotic expansion

$$T_{(X)}^{(n)}(k) \sim \sum_{r=0}^{\infty} \frac{T_{r,X}^{(n)}}{k^{\frac{r}{\omega_n}}} = \sum_{r=0}^{N-1} \frac{T_{r,X}^{(n)}}{k^{\frac{r}{\omega_n}}} + R_{(X)}^{(n)}(k, N) \tag{4.8}$$

similar to (4.4), then using (4.7) we are able to write

$$\begin{aligned} T_{(X)}^{(n)}(k) &\sim \sum_{r=0}^{\infty} \frac{T_{r,\beta_n}^{(n)}}{k^{\frac{r}{\omega_n}}} - \sum_{r=0}^{\infty} \frac{T_{r,\alpha_n}^{(n)}}{k^{\frac{r}{\omega_n}}} = \left[\sum_{r=0}^{N-1} \frac{T_{r,\beta_n}^{(n)}}{k^{\frac{r}{\omega_n}}} + R_{(\beta_n)}^{(n)}(k, N)\right] - \left[\sum_{r=0}^{N-1} \frac{T_{r,\alpha_n}^{(n)}}{k^{\frac{r}{\omega_n}}} + R_{(\alpha_n)}^{(n)}(k, N)\right] \\ &= \sum_{r=0}^{N-1} \frac{\left[T_{r,\beta_n}^{(n)} - T_{r,\alpha_n}^{(n)}\right]}{k^{\frac{r}{\omega_n}}} + \left[R_{(\beta_n)}^{(n)}(k, N) - R_{(\alpha_n)}^{(n)}(k, N)\right], \end{aligned}$$

implying from (4.4) that

$$\begin{aligned} T_r^{(n)}(\alpha_n, \beta_n) &= T_{r,\beta_n}^{(n)} - T_{r,\alpha_n}^{(n)} \\ R^{(n)}(k, N; \alpha_n, \beta_n) &= R_{(\beta_n)}^{(n)}(k, N) - R_{(\alpha_n)}^{(n)}(k, N) \end{aligned} \tag{4.9}$$

Therefore, constructing an asymptotic expansion for $T^{(n)}(k; \alpha_n, \beta_n)$ [(4.5)] reduces to constructing one for $T_{(X)}^{(n)}(k)$ [(4.6)].

We start by looking at how each element of (4.6) transforms under

$$u = k^{\frac{1}{\omega_n}} (f(z) - f_n)^{\frac{1}{\omega_n}}.$$

We have already discussed how the integration contour $C_{n,X}$ transforms in the u-plane, so we now focus on the integrand. The Jacobian of the transformation is

$$\frac{du}{dz} = \frac{k^{\frac{1}{\omega_n}} (f(z) - f_n)^{\frac{1}{\omega_n} - 1}}{\omega_n} \frac{df}{dz} := J(z) f'(z),$$

where we have defined $J(z)$ for the sake of algebraic simplicity, so that the integral becomes

$$T_{(X)}^{(n)}(k) = k^{\frac{1}{\omega_n}} \int_0^{\infty e^{\frac{2\pi i X}{\omega_n}}} \frac{du\, e^{-u^{\omega_n}} g(z(u))}{J(z(u)) f'(z(u))}.$$

Clearly with suitable choice of $S_{n,X}$, the integral diverges only at the critical point z_n, as we are restricted to $S_{n,X}$. Analogous integrals will diverge only at critical points z_{m_j}, as we are restricted to analogous integration surfaces $S_{m_j,Y}$.

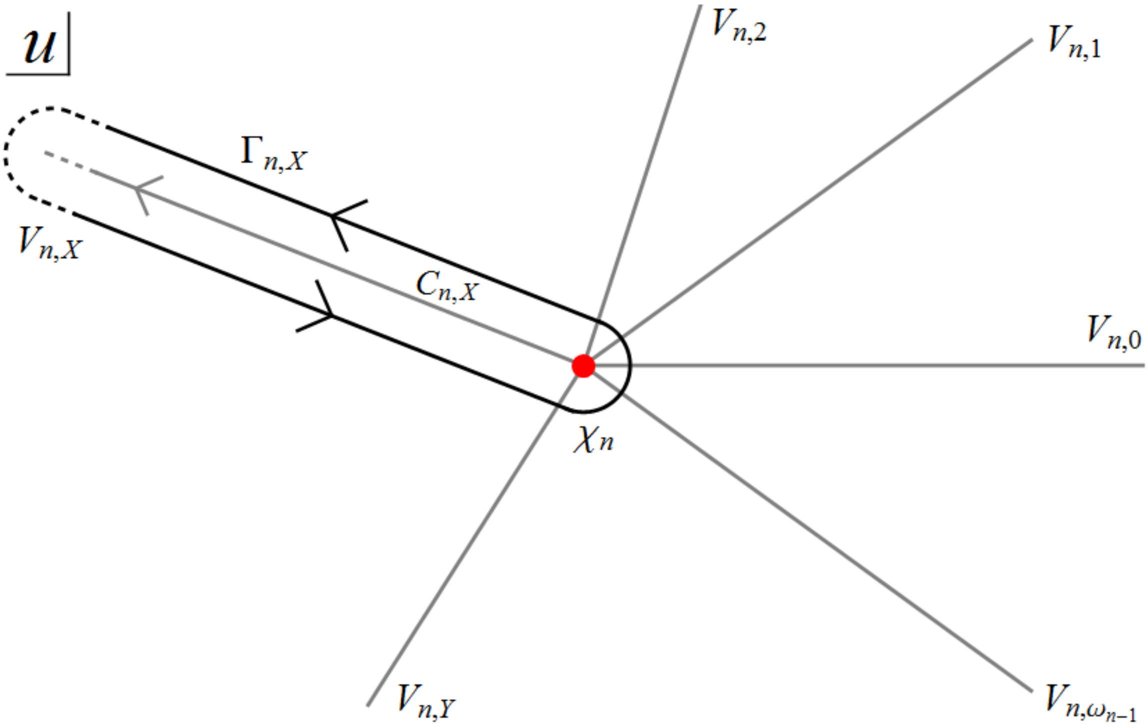


Figure 17: The complex u-plane, with $u = k^{\frac{1}{\omega_n}}(f(z)-f_n)^{\frac{1}{\omega_n}}$ and $\Gamma_{n,X}$ the sausage contour that completely encloses the semi-infinite integration contour $C_{n,X}$. The arrows show the direction of integration.

Local to z_n, we make the transformation

$$\zeta^{\omega_n} = k(f(z)-f_n) \Rightarrow \frac{d\zeta}{dz} = J(z)f'(z)$$

and since g/f' is sufficiently holomorphic, we use Cauchy's integral formula to write

$$\frac{g(z(u))}{f'(z(u))} = \frac{1}{2\pi i}\oint_{\Gamma_{n,X}} \frac{d\zeta}{\zeta-u}\frac{g(z(\zeta))}{f'(z(\zeta))},$$

where $\Gamma_{n,X}$ is a 'sausage contour' that completely encloses $C_{n,X}$, similar to that in Berry and Howls (1991) and related literature. This is illustrated in Figure 17. We can now write (4.6) as

$$\begin{aligned}
T^{(n)}_{(X)}(k) &= k^{\frac{1}{\omega_n}}\int_0^{\infty e^{\frac{2\pi i X}{\omega_n}}} \frac{du\, e^{-u^{\omega_n}}}{J(z(u))}\frac{1}{2\pi i}\oint_{\Gamma_{n,X}} \frac{d\zeta}{\zeta-u}\frac{g(z(\zeta))}{f'(z(\zeta))} \\
&= k^{\frac{1}{\omega_n}}\int_0^{\infty e^{\frac{2\pi i X}{\omega_n}}} \frac{du\, e^{-u^{\omega_n}}}{2\pi i}\oint_{\Gamma_{n,X}} \frac{dz\, g(z)}{k^{\frac{1}{\omega_n}}(f(z)-f_n)^{\frac{1}{\omega_n}}-u} \\
&= \int_0^{\infty e^{\frac{2\pi i X}{\omega_n}}} \frac{du\, e^{-u^{\omega_n}}}{2\pi i}\oint_{\Gamma_{n,X}} \frac{dz\, g(z)}{(f(z)-f_n)^{\frac{1}{\omega_n}}}\frac{1}{\left(1-\frac{u}{k^{\frac{1}{\omega_n}}(f(z)-f_n)^{\frac{1}{\omega_n}}}\right)}.
\end{aligned}$$

Making use of the expression

$$\frac{1}{1-x} = \sum_{r=0}^{N-1} x^r + \frac{x^N}{1-x}, \quad x = \frac{u}{k^{\frac{1}{\omega_n}} (f(z)-f_n)^{\frac{1}{\omega_n}}},$$

we have

$$\begin{aligned} T^{(n)}_{(X)}(k,N) = &\sum_{r=0}^{N-1} \frac{1}{2\pi i k^{\frac{r}{\omega_n}}} \int_0^{\infty e^{\frac{2\pi i X}{\omega_n}}} du\, u^r e^{-u^{\omega_n}} \oint_{\Gamma_{n,X}} \frac{dz\, g(z)}{(f(z)-f_n)^{\frac{r+1}{\omega_n}}} \\ &+ \frac{1}{2\pi i k^{\frac{N}{\omega_n}}} \int_0^{\infty e^{\frac{2\pi i X}{\omega_n}}} du\, u^N e^{-u^{\omega_n}} \oint_{\Gamma_{n,X}} \frac{dz\, g(z)}{(f(z)-f_n)^{\frac{N+1}{\omega_n}}} \frac{1}{\left(1 - \frac{u}{k^{\frac{1}{\omega_n}} (f(z)-f_n)^{\frac{1}{\omega_n}}}\right)}. \end{aligned} \tag{4.10}$$

The second term in (4.10) above is the remainder $R^{(n)}_{(X)}(k,N)$.

We want to transform u so that the upper integration limit is real infinity, so we define

$$\tilde{u} = u e^{-\frac{2\pi i X}{\omega_n}} \iff u = \tilde{u} e^{\frac{2\pi i X}{\omega_n}} \Rightarrow du = d\tilde{u}\, e^{\frac{2\pi i X}{\omega_n}}.$$

Upon substitution into (4.10) and dropping the tildes, we obtain

$$T^{(n)}_{(X)}(k,N) = \sum_{r=0}^{N-1} \frac{e^{\frac{2\pi i X(r+1)}{\omega_n}}}{2\pi i k^{\frac{r}{\omega_n}}} \int_0^{\infty} du\, u^r e^{-u^{\omega_n}} \oint_{\Gamma_{n,X}} \frac{dz\, g(z)}{(f(z)-f_n)^{\frac{r+1}{\omega_n}}} + R^{(n)}_{(X)}(k,N).$$

Note that we do not apply this transformation to the remainder expression now; we will do so later.

Since the integrand is only singular at z_n, we write

$$T^{(n)}_{(X)}(k,N) = \sum_{r=0}^{N-1} e^{\frac{2\pi i X(r+1)}{\omega_n}} \frac{\Gamma\left(\frac{r+1}{\omega_n}\right)}{\omega_n k^{\frac{r}{\omega_n}}} \operatorname*{Res}_{z=z_n} \left(\frac{g(z)}{(f(z)-f_n)^{\frac{r+1}{\omega_n}}} \right) + R^{(n)}_{(X)}(k,N), \tag{4.11}$$

so that the coefficients are given by

$$T^{(n)}_{r,X} = e^{\frac{2\pi i X(r+1)}{\omega_n}} \frac{\Gamma\left(\frac{r+1}{\omega_n}\right)}{\omega_n} \operatorname*{Res}_{z=z_n} \left(\frac{g(z)}{(f(z)-f_n)^{\frac{r+1}{\omega_n}}} \right). \tag{4.12}$$

The full expansion and coefficients are thus given by

$$\begin{aligned} T^{(n)}(k,N;\alpha_n,\beta_n) = &\sum_{r=0}^{N-1} \left(e^{\frac{2\pi i \beta_n (r+1)}{\omega_n}} - e^{\frac{2\pi i \alpha_n (r+1)}{\omega_n}} \right) \frac{\Gamma\left(\frac{r+1}{\omega_n}\right)}{\omega_n k^{\frac{r}{\omega_n}}} \operatorname*{Res}_{z=z_n} \left(\frac{g(z)}{(f(z)-f_n)^{\frac{r+1}{\omega_n}}} \right) \\ &+ R^{(n)}(k,N;\alpha_n,\beta_n), \end{aligned} \tag{4.13}$$

$$T_r^{(n)}(\alpha_n,\beta_n) = \left(e^{\frac{2\pi i\beta_n(r+1)}{\omega_n}} - e^{\frac{2\pi i\alpha_n(r+1)}{\omega_n}}\right)\frac{\Gamma\left(\frac{r+1}{\omega_n}\right)}{\omega_n}\operatorname*{Res}_{z=z_n}\left(\frac{g(z)}{(f(z)-f_n)^{\frac{r+1}{\omega_n}}}\right). \tag{4.14}$$

Looking back at the original integral (4.2), we see that the powers of k appearing in the expansion for $I^{(n)}(k;\alpha_n,\beta_n)$ are thus

$$\frac{1}{k^{\frac{r+1}{\omega_n}}}$$

as deduced in §3.3 (as $q_n = 1$).

Some of the coefficients in (4.14) will be identically zero due to the phase factors 'matching up', thereby cancelling each other out. To find these identically zero coefficients, we must identify for which values of r the phase factors are equal. This equates to solving the congruence equation

$$\frac{2\pi i\beta_n(r+1)}{\omega_n} - \frac{2\pi i\alpha_n(r+1)}{\omega_n} \equiv 2\pi i K \mod \omega_n,$$

where $K \in \mathbb{Z}$, for r. Cancelling the $2\pi i$ factors and rearranging gives us

$$(\beta_n - \alpha_n)(r+1) \equiv K\omega_n \mod \omega_n,$$

but since $K \in \mathbb{Z}$, $K\omega_n \equiv 0 \mod \omega_n$, allowing us to express this condition as the linear congruence equation

$$(\beta_n - \alpha_n)r \equiv -(\beta_n - \alpha_n) \mod \omega_n. \tag{4.15}$$

When a value of r satisfies (4.15), it means that the r-th coefficient is identically zero. Note that we do not cancel the $\beta_n - \alpha_n$ from both sides, as we lose some solutions by doing so.

We now need to manipulate the remainder

$$R_{(X)}^{(n)}(k,N) = \frac{1}{2\pi i k^{\frac{N}{\omega_n}}}\int_0^{\infty e^{\frac{2\pi i X}{\omega_n}}} du\, u^N e^{-u^{\omega_n}} \oint_{\Gamma_{n,X}} \frac{dz\, g(z)}{(f(z)-f_n)^{\frac{N+1}{\omega_n}}} \frac{1}{\left(1 - \frac{u}{k^{\frac{1}{\omega_n}}(f(z)-f_n)^{\frac{1}{\omega_n}}}\right)}$$

into a resurgence relation involving asymptotic expansions of the other critical points z_{m_j} of $f(z)$ and then write the full remainder using

$$R^{(n)}(k,N;\alpha_n,\beta_n) = R_{(\beta_n)}^{(n)}(k,N) - R_{(\alpha_n)}^{(n)}(k,N).$$

We do this by deforming $\Gamma_{n,X}$ within $\sigma_f^{(n)}$ into a union of steepest descent contours at infinity (between two valleys and not through any critical points) and fully infinite integration contours through (adjacent) critical points z_{m_j}.

The contours at infinity will vanish provided that the condition

$$\left|\frac{g(z)}{(f(z)-f_n)^{\frac{1}{\omega_n}}}\frac{1}{\left(1-\frac{u}{k^{\frac{1}{\omega_n}}(f(z)-f_n)^{\frac{1}{\omega_n}}}\right)}\right| = o\left(\frac{1}{k^{\frac{1}{\omega_n}}(f(z)-f_n)^{\frac{1}{\omega_n}}}\right), \ \left|k^{\frac{1}{\omega_n}}(f(z)-f_n)^{\frac{1}{\omega_n}}\right| \to \infty$$

is satisfied, which can be achieved by prior restriction of the integrand. Of course, the other critical points z_{m_j} will have general constant order ω_{m_j}, meaning there will be ω_{m_j} possible semi-infinite integration contours $C_{m_j,Y}$ (legs), for $Y \in \{0, 1, \dots, \omega_{m_j}-1\}$. A full contribution from one of these other critical points comes from integrating along the fully infinite contour $C_{m_j}(\alpha_{nm_j}, \beta_{nm_j})$; how do we decide the values of α_{nm_j} and β_{nm_j} for each of the other critical points?

The quantities α_{nm_j} and β_{nm_j} can be reliably determined by looking at a contour plot of the z-plane and checking which legs of z_{m_j} that z_n sits between. Regardless of the value of X, the deformed contour $\Gamma_{n,X}$ will always hit these legs first, so they will be our α_{nm_j} and β_{nm_j}. Hence, $\Gamma_{n,X}$ deforms as

$$\oint_{\Gamma_{n,X}} \equiv \sum_{j=1}^{\gamma} K_{nm_j}\left(\int_{C_{m_j,\beta_{nm_j}}} - \int_{C_{m_j,\alpha_{nm_j}}}\right),$$

along which we define

$$u^{\omega_n} = v^{\omega_{m_j}}\left(\frac{f(z)-f_n}{F_{nm_j}}\right) =: v^{\omega_{m_j}}V_j(z) \tag{4.16}$$

$$\Rightarrow \frac{du}{dv} = \frac{\omega_{m_j}}{\omega_n}\frac{v^{\frac{\omega_{m_j}}{\omega_n}-1}(f(z)-f_n)^{\frac{1}{\omega_n}}}{F_{nm_j}^{\frac{1}{\omega_n}}}.$$

To write the remainder $R^{(n)}_{(X)}(k, N)$ completely in terms of v, we must first transform the upper integration limit

$$u = \infty e^{\frac{2\pi i X}{\omega_n}}.$$

We define

$$\arg(V_j(z)) = \phi_j(z),$$
$$\arg(v^{\omega_{m_j}}) = \theta_j(z) + 2\pi\rho_{nm_j},$$
$$\arg(kF_{nm_j}) = \theta_{nm_j},$$

with $\phi_j(z)$, $\theta_{nm_j} \in [0, 2\pi)$ and $\rho_{nm_j} \in \{0, 1, \dots, \omega_n - 1\}$. When expanding the contour $\Gamma_{n,X}$, the adjacent critical points z_{m_j} may not lie on the same Riemann surface as z_n in the u^{ω_n}-plane. Geometrically, the value of ρ_{nm_j} specifics how many Riemann surfaces separate z_n and z_{m_j}; algebraically, ρ_{nm_j} specifies which ω_n-th root of v^{ω_j} we take.

The transformation (4.16) allows us to write

$$u = \infty e^{\frac{2\pi i X}{\omega_n}} \Rightarrow u^{\omega_n} = \infty e^{2\pi i X} = \infty e^{i\left(\phi_j(z)+\theta_j(z)+2\pi\rho_{nm_j}\right)},$$

implying that

$$\theta_j(z) = 2\pi(X - \rho_{nm_j}) - \phi_j(z)$$

and thus

$$v^{\omega_{m_j}} = \infty e^{2\pi i\left(X-\rho_{nm_j}\right)-i\phi_j(z)} \Rightarrow v = \infty e^{\frac{2\pi i\left(X-\rho_{nm_j}\right)-i\phi_j(z)}{\omega_{m_j}}}$$

at the upper integration limit.

To deduce the value of $\phi_j(z)$, we refer to Figure 15, which displays all information we know about various important quantities in the u^{ω_n}-plane; it shows us that we must have

$$\phi_j(z) = \theta_{nm_j}.$$

The ray $\Lambda_{2\pi X}$ is our u-plane integration contour $C_{n,X}$, along which u^{ω_n} is real and positive, with $\Lambda_{2\pi X+\theta_{nm_j}}$ being the ray along which $v^{\omega_{m_j}}$ is real and positive. It is possible to show via transformation that the value of the remainder integral will be the same along both $\Lambda_{2\pi X}$ and $\Lambda_{2\pi X+\theta_{nm_j}}$. We may also use a geometric argument to argue equality: as $\theta_{nm_j} \in [0, 2\pi)$, $\Lambda_{2\pi X}$ and $\Lambda_{2\pi X+\theta_{nm_j}}$ both lie on the same Riemann surface in u^{ω_n} space, then by Cauchy's integral theorem the remainder integral will take the same value along both.

This equality allows us to choose to integrate along $\Lambda_{2\pi X+\theta_{nm_j}}$, which affords simpler algebra than if we chose to use $\Lambda_{2\pi X} = C_{n,X}$. Our upper integration limit is thus

$$v = \infty e^{\frac{2\pi i\left(X-\rho_{nm_j}\right)}{\omega_{m_j}}},$$

enabling us to express the remainder integral entirely in terms of v as

$$R^{(n)}_{(X)}(k,N) = \sum_{j=1}^{\gamma} \frac{\omega_{m_j} K_{nm_j}}{2\pi i \omega_n k^{\frac{N}{\omega_n}} F_{nm_j}^{\frac{N+1}{\omega_n}}} \int_0^{\infty e^{\frac{2\pi i\left(X-\rho_{nm_j}\right)}{\omega_{m_j}}}} \frac{dv\, e^{-v^{\omega_{m_j}}} v^{\omega_{m_j}\left[\frac{N+1}{\omega_n}-\frac{1}{\omega_{m_j}}\right]}}{1-\left(\frac{v^{\omega_{m_j}}}{kF_{nm_j}}\right)^{\frac{1}{\omega_n}}} \cdots$$
$$\times \left(\int_{C_{m_j,\beta_{nm_j}}} - \int_{C_{m_j,\alpha_{nm_j}}}\right) dz\, g(z) e^{-\frac{v^{\omega_{m_j}}}{F_{nm_j}}\left(f(z)-f_{m_j}\right)}.$$

We make the further transformation

$$v = \tilde{v} e^{\frac{2\pi i \left(X - \rho_{nm_j}\right)}{\omega_{m_j}}} \iff \tilde{v} = v e^{-\frac{2\pi i \left(X - \rho_{nm_j}\right)}{\omega_{m_j}}}$$
$$\Rightarrow dv = d\tilde{v}\, e^{\frac{2\pi i \left(X - \rho_{nm_j}\right)}{\omega_{m_j}}},$$

so that the upper integration limit becomes real infinity, as well as noting that

$$\frac{v^{-1}}{F_{nm_j}^{-\frac{1}{\omega_{m_j}}}} T_{(Y)}^{(m_j)}\left(\frac{v^{\omega_{m_j}}}{F_{nm_j}}\right) = \int_{C_{m_j,Y}} dz\, g(z) e^{-\frac{v^{\omega_{m_j}}}{F_{nm_j}}\left(f(z) - f_{m_j}\right)},$$

so that our expression for the remainder integral - after dropping the tildes - becomes

$$R_{(X)}^{(n)}(k, N) = \sum_{j=1}^{\gamma} \frac{\omega_{m_j} K_{nm_j} e^{\frac{2\pi i \left(X - \rho_{nm_j}\right)(N+1)}{\omega_n}}}{2\pi i \omega_n k^{\frac{N}{\omega_n}} F_{nm_j}^{\frac{N+1}{\omega_n} - \frac{1}{\omega_{m_j}}}} \int_0^\infty \frac{dv\, e^{-v^{\omega_{m_j}}} v^{\omega_{m_j}\left[\frac{N+1}{\omega_n} - \frac{1}{\omega_{m_j}}\right] - 1}}{1 - \left(\frac{v^{\omega_{m_j}} e^{2\pi i \left(X - \rho_{nm_j}\right)}}{kF_{nm_j}}\right)^{\frac{1}{\omega_n}}} \cdots$$
$$\times T^{(m_j)}\left(\frac{v^{\omega_{m_j}}}{F_{nm_j}}; \alpha_{nm_j}, \beta_{nm_j}\right), \tag{4.17}$$

with the full remainder being given by

$$R^{(n)}(k, N; \alpha_n, \beta_n) = \sum_{j=1}^{\gamma} \frac{\omega_{m_j} K_{nm_j}}{2\pi i \omega_n k^{\frac{N}{\omega_n}} F_{nm_j}^{\frac{N+1}{\omega_n} - \frac{1}{\omega_{m_j}}}} \int_0^\infty dv\, e^{-v^{\omega_{m_j}}} v^{\omega_{m_j}\left[\frac{N+1}{\omega_n} - \frac{1}{\omega_{m_j}}\right] - 1} \cdots \tag{4.18}$$
$$\times \left[\frac{e^{\frac{2\pi i \left(\beta_n - \rho_{nm_j}\right)(N+1)}{\omega_n}}}{1 - \left(\frac{v^{\omega_{m_j}} e^{2\pi i \left(\beta_n - \rho_{nm_j}\right)}}{kF_{nm_j}}\right)^{\frac{1}{\omega_n}}} - \frac{e^{\frac{2\pi i \left(\alpha_n - \rho_{nm_j}\right)(N+1)}{\omega_n}}}{1 - \left(\frac{v^{\omega_{m_j}} e^{2\pi i \left(\alpha_n - \rho_{nm_j}\right)}}{kF_{nm_j}}\right)^{\frac{1}{\omega_n}}}\right] T^{(m_j)}\left(\frac{v^{\omega_{m_j}}}{F_{nm_j}}; \alpha_{nm_j}, \beta_{nm_j}\right).$$

The complete resurgence relation is thus given by

$$T^{(n)}(k, N; \alpha_n, \beta_n)$$
$$= \sum_{r=0}^{N-1} \left(e^{\frac{2\pi i \beta_n (r+1)}{\omega_n}} - e^{\frac{2\pi i \alpha_n (r+1)}{\omega_n}}\right) \frac{\Gamma\left(\frac{r+1}{\omega_n}\right)}{\omega_n k^{\frac{r}{\omega_n}}} \operatorname*{Res}_{z = z_n} \left(\frac{g(z)}{\left(f(z) - f_n\right)^{\frac{r+1}{\omega_n}}}\right)$$
$$+ \sum_{j=1}^{\gamma} \frac{\omega_{m_j} K_{nm_j}}{2\pi i \omega_n k^{\frac{N}{\omega_n}} F_{nm_j}^{\frac{N+1}{\omega_n} - \frac{1}{\omega_{m_j}}}} \int_0^\infty dv\, e^{-v^{\omega_{m_j}}} v^{\omega_{m_j}\left[\frac{N+1}{\omega_n} - \frac{1}{\omega_{m_j}}\right] - 1} \cdots \tag{4.19}$$
$$\times \left[\frac{e^{\frac{2\pi i \left(\beta_n - \rho_{nm_j}\right)(N+1)}{\omega_n}}}{1 - \left(\frac{v^{\omega_{m_j}} e^{2\pi i \left(\beta_n - \rho_{nm_j}\right)}}{kF_{nm_j}}\right)^{\frac{1}{\omega_n}}} - \frac{e^{\frac{2\pi i \left(\alpha_n - \rho_{nm_j}\right)(N+1)}{\omega_n}}}{1 - \left(\frac{v^{\omega_{m_j}} e^{2\pi i \left(\alpha_n - \rho_{nm_j}\right)}}{kF_{nm_j}}\right)^{\frac{1}{\omega_n}}}\right] T^{(m_j)}\left(\frac{v^{\omega_{m_j}}}{F_{nm_j}}; \alpha_{nm_j}, \beta_{nm_j}\right).$$

As expected, the Stokes phenomenon is incorporated in (4.19) and this can be seen once we have rationalised the denominator in the remainder. This replaces the branch points with poles that are encountered by the v-contour when Stokes phenomena occur, giving rise to the appropriate additional contributions to the integral. We will perform this rationalisation in more detail in §6. A similar situation occurs in Howls (1992), where rewriting the remainder in (21) by rationalising the denominator reveals the incorporation of the Stokes phenomenon into the resurgence relation.

Using (4.19), it is possible to derive 'late term' expressions for $T_r(\alpha_n, \beta_n)$, as well as a complete hyperasymptotic expansion for the original integral. However, we delay these derivations until §6, as we will consider a more general case in §5. In this next chapter, we consider the more general problem of deriving the asymptotic contribution of uniformly degenerate critical components of general co-dimension to integrals in the form of (1.1) in $\mathbb{C}^d$.

In the context of differential equations, the case considered in this chapter corresponds to the case of arbitrary exponential rank considered in Murphy and Wood (1997) and Murphy (2001). The main difference is that these papers do not consider mixed exponential rank; this would correspond to the critical points of f all having the same order of degeneracy, which is a simpler case than the one handled here.

5 Uniformly Degenerate Critical Components of General Dimension

In this chapter, we are interested in deriving the asymptotic contribution of critical components of general dimension and general constant order to the integral (1.1), namely

$$I(k) = \int_S dz\, g(\boldsymbol{z}) e^{-kf(\boldsymbol{z})},$$

where $S \subset \mathbb{C}^d$ is a smooth surface of integration between two asymptotic valleys. We also examine the computation of the multidimensional residues appearing in such contributions from a practical point of view.

5.1 Derivation of General Resurgence Relation

By the discussion in §3.2, our work in §4, and the successful numerical examples carried out in §7 that are based on the work done in this chapter, we make the (in our opinion safe) assumption that we are able to decompose S and hence $I(k)$ into a finite sum of integrals

$$I^{(n)}(k;\alpha_n,\beta_n) = \int_{S_n(\alpha_n,\beta_n)} dz\, g(\boldsymbol{z}) e^{-kf(\boldsymbol{z})} = \frac{e^{-kf_n}}{k^{\frac{q_n}{\omega_n}}} T^{(n)}(k;\alpha_n,\beta_n), \tag{5.1}$$

$$T^{(n)}(k;\alpha_n,\beta_n) = k^{\frac{q_n}{\omega_n}} \int_{S_n(\alpha_n,\beta_n)} d\boldsymbol{z}\, g(\boldsymbol{z}) e^{-k(f(\boldsymbol{z})-f_n)}. \tag{5.2}$$

As before, we only need to consider asymptotic expansions of the integrals $I^{(n)}(k)$, since the decomposition of $I(k)$ implies similar decomposition of its asymptotics. We thus search for a formal asymptotic expansion and respective truncated expansion of the form

$$T^{(n)}(k;\alpha_n,\beta_n) \sim \sum_{r=0}^{\infty} \frac{T_r^{(n)}(\alpha_n,\beta_n)}{k^{\frac{r}{\omega_n}}} = \sum_{r=0}^{N-1} \frac{T_r^{(n)}(\alpha_n,\beta_n)}{k^{\frac{r}{\omega_n}}} + R^{(n)}(k,N;\alpha_n,\beta_n). \tag{5.3}$$

The formal expansions (5.3) have the same structure as (4.4); the only difference will be in the expressions for the coefficients and remainders.

In this problem, $f, g : \mathbb{C}^d \to \mathbb{C}$ are sufficiently holomorphic complex valued functions, $k \in \mathbb{C}$ is the asymptotic parameter, $\omega_n \in \mathbb{N}$ is the (constant) order of the critical component χ_n of f, q_n is the co-dimension of f, and $f_n := f(\chi_n)$. The integration surface $S_n(\alpha_n,\beta_n)$ is a smooth manifold within the steepest descent space $\sigma_f^{(n)}$ that is fully infinite in all d complex variables and runs from and to the asymptotic valleys V_{α_n} and V_{β_n} respectively, as well as through χ_n.

We proceed in a similar manner to §4 - where we looked at the case $d = 1$ - making the necessary adaptations now that d is general. To look at the local behaviour around χ_n, we first make the transformation

$$\begin{aligned} F : \mathbb{C}^d &\to \mathbb{C}^d \\ \boldsymbol{z} &\mapsto \boldsymbol{w} = F(\boldsymbol{z}) \end{aligned}$$

to parameterise $S_{n,X}$ (analogous to $C_{n,X}$ and defined later), with the individual components given by $w_j = F_j(\boldsymbol{z})$, for $j \in \{1, \dots, d\}$. We then define

$$w_1 = u = k^{\frac{1}{\omega_n}} (f(\boldsymbol{z}) - f_n)^{\frac{1}{\omega_n}} \Rightarrow u^{\omega_n} = k(f(\boldsymbol{z}) - f_n)$$

and $w_j = v_j$ for the remaining coordinates, allowing us to abuse notation slightly and write $\boldsymbol{w} = (u, \boldsymbol{\nu})$. We will briefly discuss $F_j(\boldsymbol{z})$ for $j \in \{2, \dots, d\}$ later in the derivation.

The transformation now takes the form

$$\begin{aligned} u &= k^{\frac{1}{\omega_n}} (f(\boldsymbol{z}) - f_n)^{\frac{1}{\omega_n}}, \\ \nu_2 &= F_2(\boldsymbol{z}), \\ \vdots\ & \quad \vdots \\ \nu_d &= F_d(\boldsymbol{z}), \end{aligned}$$

with volume element

$$d\boldsymbol{w} = du\, d\boldsymbol{\nu} = |J_F(\boldsymbol{z})| d\boldsymbol{z},$$

where $d\boldsymbol{\nu} = d\nu_2 \cdots d\nu_d$, $d\boldsymbol{z} = dz_1 \cdots dz_d$, $d\boldsymbol{w} = dw_1 \cdots dw_d$, and

$$|J_F(\boldsymbol{z})| = \left| \frac{\partial(w_1, \dots, w_d)}{\partial(z_1, \dots, z_d)} \right| = \begin{vmatrix} u_{z_1} & u_{z_2} & \cdots & u_{z_d} \\ (\nu_2)_{z_1} & (\nu_2)_{z_2} & \cdots & (\nu_2)_{z_d} \\ \vdots & \vdots & \ddots & \vdots \\ (\nu_d)_{z_1} & (\nu_d)_{z_2} & \cdots & (\nu_d)_{z_d} \end{vmatrix}$$

is the Jacobian determinant of F. If the entries of $J_F(\boldsymbol{z})$ are $a_{i,j}$, then using a cofactor expansion

$$|J_F(\boldsymbol{z})| = \sum_{j=1}^{d} (-1)^{i+j} a_{i,j} M_{i,j} = \sum_{i=1}^{d} (-1)^{i+j} a_{i,j} M_{i,j},$$

where the $M_{i,j}$ are the Minors of $|J_F(\boldsymbol{z})|$, we can expand along the first row to obtain the expression

$$|J_F(\boldsymbol{z})| = \sum_{j=1}^{d} (-1)^{1+j} \frac{k^{\frac{1}{\omega_n}} (f(\boldsymbol{z}) - f_n)^{\frac{1}{\omega_n}-1}}{\omega_n} \frac{\partial f(\boldsymbol{z})}{\partial z_j} M_{1,j},$$

since

$$\frac{\partial u}{\partial z_j} = \frac{k^{\frac{1}{\omega_n}} (f(\boldsymbol{z}) - f_n)^{\frac{1}{\omega_n}-1}}{\omega_n} \frac{\partial f(\boldsymbol{z})}{\partial z_j}.$$

As in §4, the behaviour local to χ_n can be discerned by looking at the u^{ω_n} and u-planes (Figures 15 and 16 respectively), which behave in exactly the same way as in §4 with the definitions and notations carrying straight over. The only difference is that here we define

$$L_{n,X} := \left[0, \infty e^{\frac{2\pi i X}{\omega_n}}\right)$$

instead of $C_{n,X}$ (in the u-plane), with the term 'legs' referring to the semi-infinite integration surfaces

$$S_{n,X} = L_{n,X} \times \Upsilon_{n,X},$$

where $\Upsilon_{n,X}$ is 'whatever is left over' after splitting $L_{n,X}$ from $S_{n,X}$. The fully infinite integration surface is therefore given by

$$S_n(\alpha_n, \beta_n) = S_{n,\alpha_n} \cup S_{n,\beta_n},$$

with the implication again being that the integration starts and ends in V_{n,α_n} and V_{n,β_n} respectively, analogous to $C_n(\alpha_n, \beta_n)$ in §4.

Figure 18 shows the complex u-plane together with a schematic representation of a slice $\mathbb{R}$ of $\mathbb{C}^d$; this is essentially the higher dimensional analogue of Figure 16. Writing $S_n(\alpha_n, \beta_n)$ this way allows us to write

$$I^{(n)}(k; \alpha_n, \beta_n) = \left(\int_{S_{n,\beta_n}} - \int_{S_{n,\alpha_n}} \right) dz\, g(\boldsymbol{z}) e^{-kf(\boldsymbol{z})},$$
$$T^{(n)}(k; \alpha_n, \beta_n) = k^{\frac{q_n}{\omega_n}} \left(\int_{S_{n,\beta_n}} - \int_{S_{n,\alpha_n}} \right) dz\, g(\boldsymbol{z}) e^{-k(f(\boldsymbol{z}) - f_n)}$$

and the respective single leg versions as

$$I^{(n)}_{(X)}(k) = \int_{S_{n,X}} dz\, g(\boldsymbol{z}) e^{-kf(\boldsymbol{z})}, \tag{5.4}$$

$$T^{(n)}_{(X)}(k) = k^{\frac{q_n}{\omega_n}} \int_{S_{n,X}} dz\, g(\boldsymbol{z}) e^{-k(f(\boldsymbol{z}) - f_n)}, \tag{5.5}$$

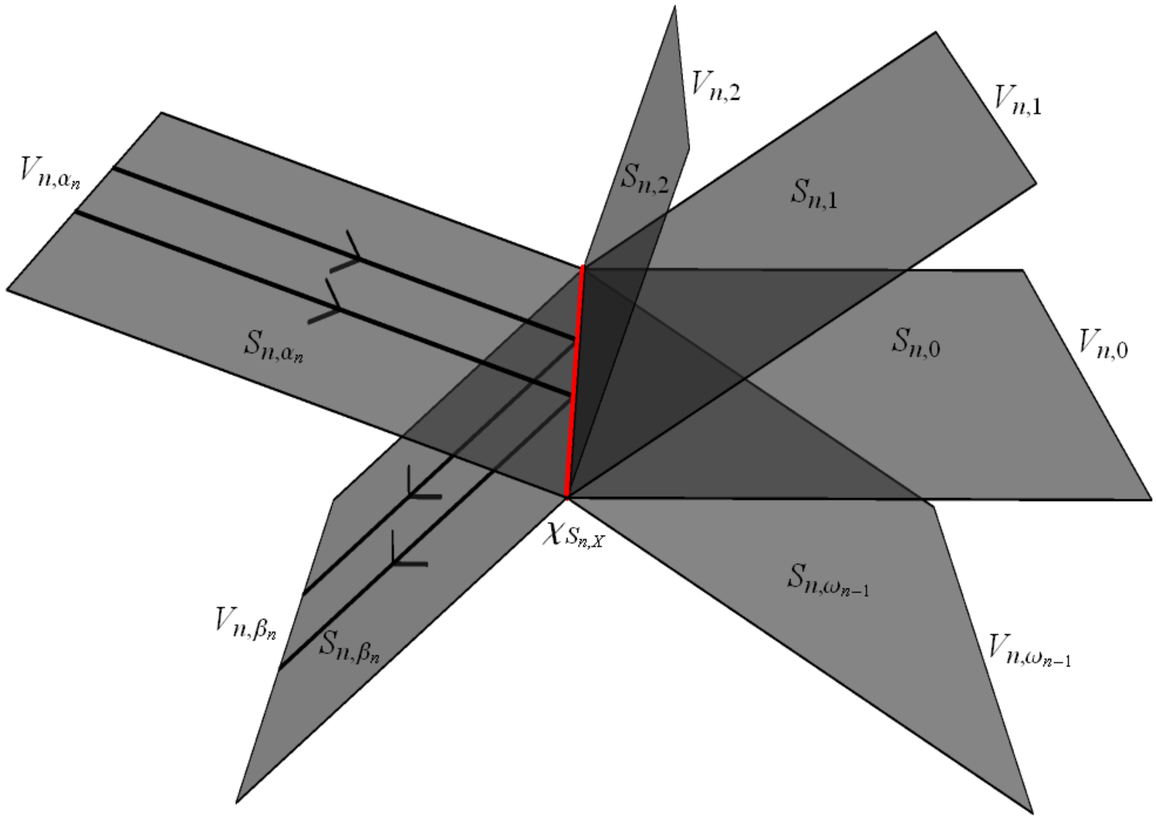


Figure 18: The complex u-plane - with $u = k^{\frac{1}{\omega_n}}(f(\boldsymbol{z}) - f_n)^{\frac{1}{\omega_n}}$ - shown on the horizontal and vertical axes together with a schematic representation of a slice $\mathbb{R} \subset \mathbb{C}^d$, shown on the axis going into the page along which $\chi_{S_{n,X}}$ is travelling. The full integration contour is given by $S_n(\alpha_n, \beta_n) = S_{n,\alpha_n} \cup S_{n,\beta_n}$, with the arrows showing the direction of integration. This figure is essentially the higher dimensional analogue of Figure 16.

analogous to (4.5) and (4.6), with these integrals also satisfying (4.7) and the coefficients and remainders satisfying (4.9). Once again, constructing an asymptotic expansion for $T^{(n)}(k; \alpha_n, \beta_n)$ [(5.2)] reduces to constructing one for $T^{(n)}_{(X)}(k)$ [(5.5)], which will have the same form as (4.8).

We look at the integral (5.5) above and check how each element of it transforms under F. Applying F and using our knowledge of how $S_{n,X}$ deconstructs into $L_{n,X}$ and $\Upsilon_{n,X}$, we can write (5.5) as

$$\begin{aligned} T^{(n)}_{(X)}(k) &= k^{\frac{q_n}{\omega_n}} \int_{L_{n,X} \times \Upsilon_{n,X}} d\boldsymbol{w}\, \frac{g(\boldsymbol{z}(\boldsymbol{w}))}{|J_F(\boldsymbol{z}(\boldsymbol{w}))|} e^{-u^{\omega_n}} = k^{\frac{q_n}{\omega_n}} \int_{L_{n,X}} du \int_{\Upsilon_{n,X}} d\boldsymbol{\nu}\, \frac{g(\boldsymbol{z}(u,\boldsymbol{\nu}))}{|J_F(\boldsymbol{z}(u,\boldsymbol{\nu}))|} e^{-u^{\omega_n}} \\ &= k^{\frac{q_n}{\omega_n}} \int_0^{\infty e^{\frac{2\pi i X}{\omega_n}}} du\, e^{-u^{\omega_n}} \int_{\Upsilon_{n,X}} d\boldsymbol{\nu}\, \frac{g(\boldsymbol{z}(u,\boldsymbol{\nu}))}{|J_F(\boldsymbol{z}(u,\boldsymbol{\nu}))|}. \end{aligned}$$

Clearly this integral diverges when the Jacobian determinant $|J_F(\boldsymbol{z})|$ is zero and if we define the transformation components $F_j(\boldsymbol{z})$ for $j \in \{2, \ldots, d\}$ suitably, the minors $M_{1,j}$ will not meaningfully impact the zeroes of $|J_F(\boldsymbol{z})|$. Thus, critical points of $f(\boldsymbol{z})$ are zeroes of $|J_F(\boldsymbol{z})|$

and when restricted to $S_{n,X}$, only points within χ_n - specifically $\chi_{S_{n,X}} = \chi_n \cap S_{n,X}$ - are zeroes, as long as $S_{n,X}$ is chosen to avoid any non-critical point zeroes of $|J_F(\boldsymbol{z})|$. On analogous integration surfaces $S_{m_j,Y}$, points within χ_{m_j} - specifically $\chi_{S_{m_j,Y}} = \chi_{m_j} \cap S_{m_j,Y}$ - are the only zeroes of $|J_F(\boldsymbol{z})|$, with F now having

$$w_1 = u = k^{\frac{1}{\omega_{m_j}}} (f(\boldsymbol{z}) - f_{m_j})^{\frac{1}{\omega_{m_j}}}$$

and suitably defined $F_j(\boldsymbol{z})$, for $j \in \{2, \dots, d\}$.

Locally around χ_n, we make the transformation

$$\begin{aligned} H : \mathbb{C}^d &\to \mathbb{C}^d \\ \boldsymbol{z} &\mapsto \boldsymbol{\eta} = H(\boldsymbol{z}) \end{aligned}$$

to locally parameterise $S_{n,X}$, with the individual components given by $\eta_j = H_j(\boldsymbol{z})$, for $j \in \{1, \dots, d\}$. We then define

$$\eta_1 = \zeta = k^{\frac{1}{\omega_n}} (f(\boldsymbol{z}) - f_n)^{\frac{1}{\omega_n}} \Rightarrow \zeta^{\omega_n} = k(f(\boldsymbol{z}) - f_n)$$

with

$$\eta_j = \xi_j \quad \text{and} \quad H_j(\boldsymbol{z}) = F_j(\boldsymbol{z})$$

for $j \in \{2, \dots, d\}$, again allowing us to abuse notation and write $\boldsymbol{\eta} = (\zeta, \boldsymbol{\xi})$. The transformation now takes the form

$$\begin{aligned} \zeta &= k^{\frac{1}{\omega_n}} (f(\boldsymbol{z}) - f_n)^{\frac{1}{\omega_n}}, \\ \xi_2 &= F_2(\boldsymbol{z}), \\ &\vdots \\ \xi_d &= F_d(\boldsymbol{z}), \end{aligned}$$

with volume element

$$d\boldsymbol{\eta} = d\zeta \, d\boldsymbol{\xi} = |J_H(\boldsymbol{z})| d\boldsymbol{z},$$

where $d\boldsymbol{\xi} = d\xi_2 \cdots d\xi_d$ and $d\boldsymbol{\eta} = d\eta_1 \cdots d\eta_d$, with local Jacobian

$$J_H(\boldsymbol{z}) = J_F(\boldsymbol{z}),$$

since $\eta_j = w_j$ locally for all j.

Since $g/|J_F(\boldsymbol{z})|$ is sufficiently holomorphic, we can use Cauchy's integral formula to write

$$\frac{g(\boldsymbol{z}(u,\boldsymbol{\nu}))}{|J_F(\boldsymbol{z}(u,\boldsymbol{\nu}))|} = \frac{1}{2\pi i}\oint_{\Gamma_{n,X}} d\zeta \frac{g(\boldsymbol{z}(\zeta,\boldsymbol{\xi}))}{|J_F(\boldsymbol{z}(\zeta,\boldsymbol{\xi}))|}\,\frac{1}{\zeta - k^{\frac{1}{\omega_n}}(f(\boldsymbol{z})-f_n)^{\frac{1}{\omega_n}}},$$

so that

$$\begin{aligned} T^{(n)}_{(X)}(k) &= \frac{k^{\frac{q_n}{\omega_n}}}{2\pi i}\int_0^{\infty e^{\frac{2\pi i X}{\omega_n}}} du\, e^{-u^{\omega_n}} \int_{\Xi_{n,X}} d\boldsymbol{\xi} \oint_{\Gamma_{n,X}} d\zeta \frac{g(\boldsymbol{z}(\zeta,\boldsymbol{\xi}))}{|J_F(\boldsymbol{z}(\zeta,\boldsymbol{\xi}))|}\,\frac{1}{\zeta-u} \\ &= \frac{k^{\frac{q_n}{\omega_n}}}{2\pi i}\int_0^{\infty e^{\frac{2\pi i X}{\omega_n}}} du\, e^{-u^{\omega_n}} \oiint_{Q_{n,X}} d\boldsymbol{\eta} \frac{g(\boldsymbol{z}(\boldsymbol{\eta}))}{|J_F(\boldsymbol{z}(\boldsymbol{\eta}))|}\,\frac{1}{\zeta-u}, \end{aligned} \tag{5.6}$$

where $\Xi_{n,X} \equiv \Upsilon_{n,X}$ locally. In equation (5.6), the parameterisation of $S_{n,X}$ is broken up into two distinct parts and then combined into one surface integral; the contour $\Gamma_{n,X}$ is a 'sausage contour' that surrounds $L_{n,X}$ and the surface $\Xi_{n,X}$ is the local parameterisation of $\Upsilon_{n,X}$. As $S_{n,X} = L_{n,X} \times \Upsilon_{n,X}$ by definition, the real d-dimensional 'sausage hypersurface' defined by

$$Q_{n,X} := \Gamma_{n,X} \times \Xi_{n,X}$$

therefore completely encloses $S_{n,X}$, similar to how $\Gamma_{n,X}$ encloses $C_{n,X}$ in §4. The surface $Q_{n,X}$ is effectively the sausage contour $\Gamma_{n,X}$ 'pushed out' over $\Xi_{n,X}$, as illustrated in Figure 19.

Transforming back to our original coordinates $\boldsymbol{z}$, we have

$$T^{(n)}_{(X)}(k) = \frac{k^{\frac{q_n-1}{\omega_n}}}{2\pi i}\int_0^{\infty e^{\frac{2\pi i X}{\omega_n}}} du\, e^{-u^{\omega_n}} \oiint_{Q_{n,X}} d\boldsymbol{z} \frac{g(\boldsymbol{z})}{(f(\boldsymbol{z})-f_n)^{\frac{1}{\omega_n}}}\,\frac{1}{\left(1-\frac{u}{k^{\frac{1}{\omega_n}}(f(\boldsymbol{z})-f_n)^{\frac{1}{\omega_n}}}\right)}$$

and by once again making use of

$$\frac{1}{1-x} = \sum_{r=0}^{N-1} x^r + \frac{x^N}{1-x}, \quad x = \frac{1}{k^{\frac{1}{\omega_n}}(f(\boldsymbol{z})-f_n)^{\frac{1}{\omega_n}}},$$

our expression for $T^{(n)}_{(X)}(k)$ becomes

$$\begin{aligned} T^{(n)}_{(X)}(k,N) = &\sum_{r=0}^{N-1} \frac{k^{\frac{q_n-1}{\omega_n}}}{2\pi i k^{\frac{r}{\omega_n}}}\int_0^{\infty e^{\frac{2\pi i X}{\omega_n}}} du\, e^{-u^{\omega_n}} u^r \oiint_{Q_{n,X}} d\boldsymbol{z} \frac{g(\boldsymbol{z})}{(f(\boldsymbol{z})-f_n)^{\frac{r+1}{\omega_n}}} \\ &+ \frac{k^{\frac{q_n-1}{\omega_n}}}{2\pi i k^{\frac{N}{\omega_n}}}\int_0^{\infty e^{\frac{2\pi i X}{\omega_n}}} du\, e^{-u^{\omega_n}} u^N \oiint_{Q_{n,X}} d\boldsymbol{z} \frac{g(\boldsymbol{z})}{(f(\boldsymbol{z})-f_n)^{\frac{N+1}{\omega_n}}}\,\frac{1}{\left(1-\frac{u}{k^{\frac{1}{\omega_n}}(f(\boldsymbol{z})-f_n)^{\frac{1}{\omega_n}}}\right)}, \end{aligned} \tag{5.7}$$

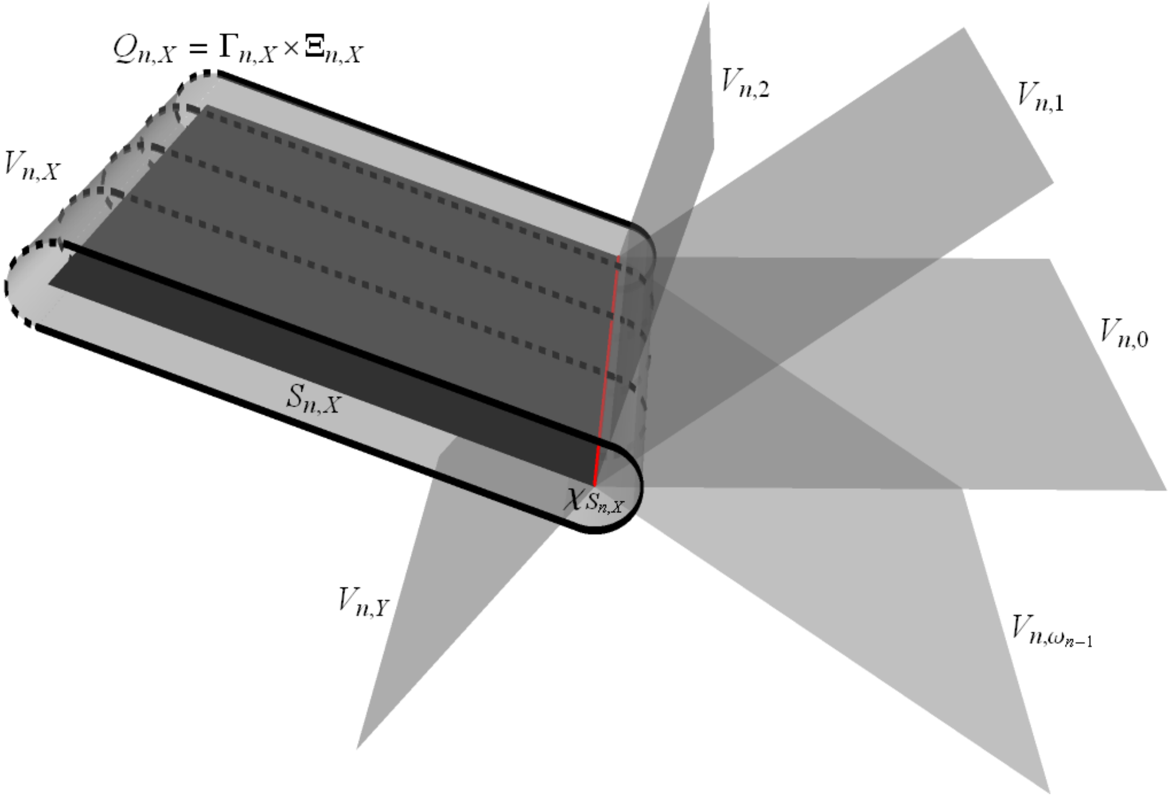


Figure 19: The complex u-plane - with $u = k^{\frac{1}{\omega_n}}(f(\boldsymbol{z}) - f_n)^{\frac{1}{\omega_n}}$ - shown on the horizontal and vertical axes together with a schematic representation of a slice $\mathbb{R} \subset \mathbb{C}^d$, shown on the axis going into the page along which $\chi_{S_{n,X}}$ is travelling. It illustrates the fact that $Q_{n,X}$ is effectively the result when $\Gamma_{n,X}$ is 'pushed out' over $\Xi_{n,X}$, resulting in the enclosure of the entire semi-infinite integration surface $S_{n,X}$. Arrows showing the direction of integration are omitted for visual clarity, but as this figure is essentially the higher dimensional analogue of Figure 17, the arrows in both figures will be analogous.

similar to (5.7) in §4.

As we are searching for an expansion of the form (5.3), it must satisfy

$$T^{(n)}_{(X)}(k, N) \in \mathbb{C}\left[k^{-\frac{r}{\omega_n}}\right],$$

namely it must be a polynomial in the variable $k^{-\frac{r}{\omega_n}}$ with complex coefficients. Multiplying (5.7) by

$$\frac{u^{q_n-1}}{u^{q_n-1}}$$

puts $T^{(n)}_{(X)}(k, N)$ in the required form, giving

$$T^{(n)}_{(X)}(k, N) = \sum_{r=0}^{N-1} \frac{1}{2\pi i k^{\frac{r}{\omega_n}}} \int_0^{\infty e^{\frac{2\pi i X}{\omega_n}}} du\, e^{-u^{\omega_n}} u^{r+q_n-1} \oiint_{Q_{n,X}} d\boldsymbol{z} \frac{g(\boldsymbol{z})}{(f(\boldsymbol{z}) - f_n)^{\frac{r+q_n}{\omega_n}}} \tag{5.8}$$
$$+ \frac{1}{2\pi i k^{\frac{N}{\omega_n}}} \int_0^{\infty e^{\frac{2\pi i X}{\omega_n}}} du\, e^{-u^{\omega_n}} u^{N+q_n-1} \oiint_{Q_{n,X}} d\boldsymbol{z} \frac{g(\boldsymbol{z})}{(f(\boldsymbol{z}) - f_n)^{\frac{N+q_n}{\omega_n}}} \frac{1}{\left(1 - \frac{u}{k^{\frac{1}{\omega_n}}(f(\boldsymbol{z})-f_n)^{\frac{1}{\omega_n}}}\right)},$$

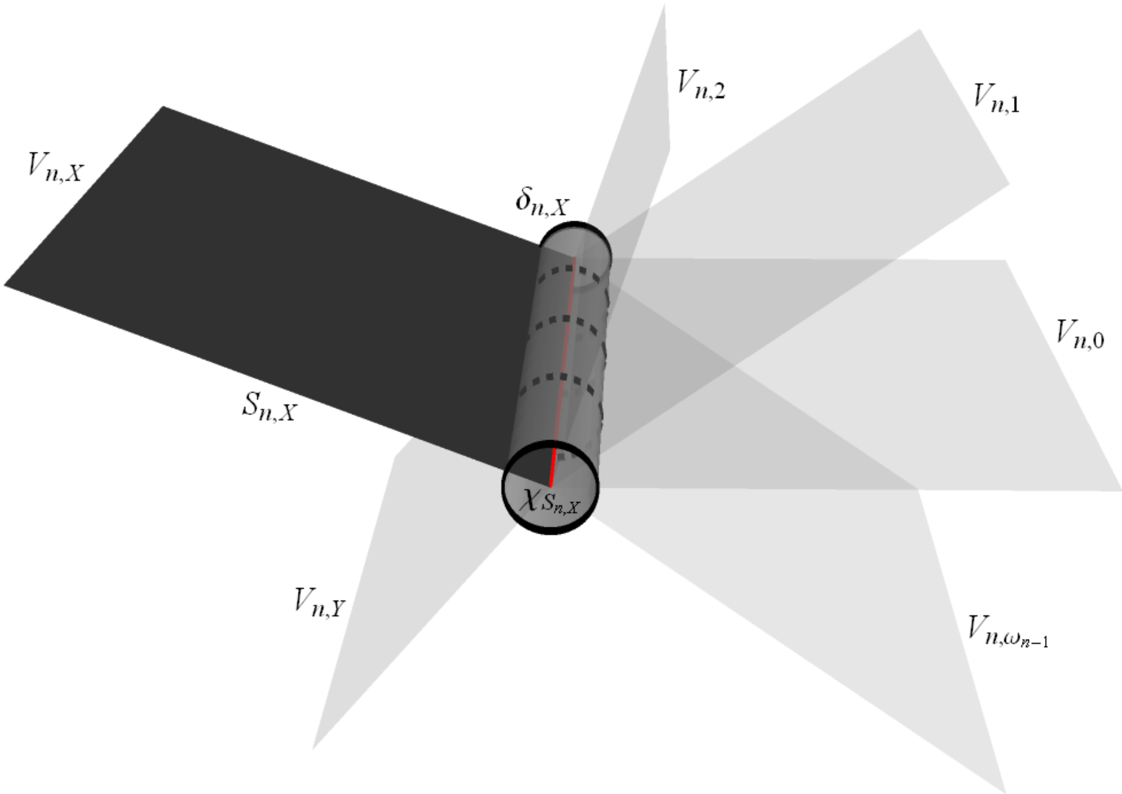


Figure 20: The complex u-plane - with $u = k^{\frac{1}{\omega_n}}(f(\boldsymbol{z}) - f_n)^{\frac{1}{\omega_n}}$ - shown on the horizontal and vertical axes together with a schematic representation of a slice $\mathbb{R} \subset \mathbb{C}^d$, shown on the axis going into the page along which $\chi_{S_{n,X}}$ is travelling. It illustrates the collapse of $Q_{n,X}$ into the hypertube $\delta_{n,X}$ that surrounds $\chi_{S_{n,X}}$, which occurs as the integrand of (5.9) is only singular on $\chi_{S_{n,X}}$.

The second term in (5.8) above is the remainder $R^{(n)}_{(X)}(k, N)$.

As in §4, we want to transform u so that the upper integration limit is real infinity, so we again define

$$\tilde{u} = ue^{-\frac{2\pi iX}{\omega_n}} \iff u = \tilde{u}e^{\frac{2\pi iX}{\omega_n}} \Rightarrow du = d\tilde{u}\, e^{\frac{2\pi iX}{\omega_n}}.$$

Upon substitution into (5.8) and dropping the tildes, we obtain

$$T^{(n)}_{(X)}(k, N) = \sum_{r=0}^{N-1} \frac{e^{\frac{2\pi iX(r+q_n)}{\omega_n}}}{2\pi i k^{\frac{r}{\omega_n}}} \int_0^\infty du\, e^{-u^{\omega_n}} u^{r+q_n-1} \oiint_{Q_{n,X}} d\boldsymbol{z} \frac{g(\boldsymbol{z})}{(f(\boldsymbol{z}) - f_n)^{\frac{r+q_n}{\omega_n}}} + R^{(n)}_{(X)}(k, N),$$

again noting that we will detail the remainder transformation later.

As we previously remarked, $T^{(n)}_{(X)}(k, N)$ is only singular on $\chi_{S_{n,X}}$, meaning we can collapse the integration surface $Q_{n,X}$ to the hypertube $\delta_{n,X}$ that surrounds $\chi_{S_{n,X}}$; this collapse is shown in Figure 20. We are thus able to write

$$\oiint_{Q_{n,X}} d\boldsymbol{z} \frac{g(\boldsymbol{z})}{(f(\boldsymbol{z}) - f_n)^{\frac{r+q_n}{\omega_n}}} = \oiint_{\delta_{n,X}} d\boldsymbol{z} \frac{g(\boldsymbol{z})}{(f(\boldsymbol{z}) - f_n)^{\frac{r+q_n}{\omega_n}}} \tag{5.9}$$

and it is clear from Figure 20 that for all values of X, the surface $Q_{n,X}$ will collapse to the same hypertube $\delta_{n,X} \equiv \delta_n$. Similar to the result in §4, the value of the residue type quantity (5.9) is therefore the same for all X. This allows us to write

$$T^{(n)}_{(X)}(k,N) = \sum_{r=0}^{N-1} e^{\frac{2\pi i X(r+q_n)}{\omega_n}} \frac{\Gamma\left(\frac{r+q_n}{\omega_n}\right)}{\omega_n k^{\frac{r}{\omega_n}}} \operatorname*{Res}_{\boldsymbol{z}\in\chi_n}\left(\frac{g(\boldsymbol{z})}{(f(\boldsymbol{z})-f_n)^{\frac{r+q_n}{\omega_n}}}\right) + R^{(n)}_{(X)}(k,N), \tag{5.10}$$

so that the coefficients are given by

$$T^{(n)}_{r,X} = e^{\frac{2\pi i X(r+q_n)}{\omega_n}} \frac{\Gamma\left(\frac{r+q_n}{\omega_n}\right)}{\omega_n} \operatorname*{Res}_{\boldsymbol{z}\in\chi_n}\left(\frac{g(\boldsymbol{z})}{(f(\boldsymbol{z})-f_n)^{\frac{r+q_n}{\omega_n}}}\right). \tag{5.11}$$

The full expansion and coefficients are thus given by

$$\begin{aligned} T^{(n)}(k,N;\alpha_n,\beta_n) = &\sum_{r=0}^{N-1}\left(e^{\frac{2\pi i\beta_n(r+q_n)}{\omega_n}} - e^{\frac{2\pi i\alpha_n(r+q_n)}{\omega_n}}\right) \frac{\Gamma\left(\frac{r+q_n}{\omega_n}\right)}{\omega_n k^{\frac{r}{\omega_n}}} \operatorname*{Res}_{\boldsymbol{z}\in\chi_n}\left(\frac{g(\boldsymbol{z})}{(f(\boldsymbol{z})-f_n)^{\frac{r+q_n}{\omega_n}}}\right) \\ &+ R^{(n)}(k,N;\alpha_n,\beta_n), \end{aligned} \tag{5.12}$$

$$T^{(n)}_{r}(\alpha_n,\beta_n) = \left(e^{\frac{2\pi i\beta_n(r+q_n)}{\omega_n}} - e^{\frac{2\pi i\alpha_n(r+q_n)}{\omega_n}}\right) \frac{\Gamma\left(\frac{r+q_n}{\omega_n}\right)}{\omega_n} \operatorname*{Res}_{\boldsymbol{z}\in\chi_n}\left(\frac{g(\boldsymbol{z})}{(f(\boldsymbol{z})-f_n)^{\frac{r+q_n}{\omega_n}}}\right). \tag{5.13}$$

Looking back at the original integral (5.1), we see that the powers of k appearing in the expansion for $I^{(n)}(k;\alpha_n,\beta_n)$ are thus

$$\frac{1}{k^{\frac{r+q}{\omega_n}}}$$

as deduced in §3.3. Note that similar to in §4, whenever a value of r satisfies the congruence equation

$$(\beta_n-\alpha_n)r \equiv -q_n(\beta_n-\alpha_n) \mod \omega_n, \tag{5.14}$$

it means that the r-th coefficient is identically zero.

In equations (5.10) - (5.13) we have written the residue as being simply on χ_n due to the following argument (which was also used in §3.1). The critical component χ_n will always be fully contained within the steepest descent space $\sigma^{(n)}_f$, so by Cauchy's integral theorem the value of $T^{(n)}_{(X)}(k,N)$ is the same regardless of which integration surface $S_{n,X}$ we choose, so long as it remains wholly within $\sigma^{(n)}_f$. Therefore, the residue along $\chi_{S_{n,X}}$ must be the same regardless of the choice of $S_{n,X}$, implying that the residue of

$$\frac{g(\boldsymbol{z})}{(f(\boldsymbol{z})-f_n)^{\frac{r+q_n}{\omega_n}}}$$

is the same everywhere on χ_n and thus allowing us to simply consider the residue on χ_n rather than on more complicated surfaces $\chi_{S_{n,X}}$. The exact meaning of the residue operator used in these equations will be discussed in detail in §5.2.

Before moving on, we point out that an alternate schematic visualisation of the problem so far and its important component parts is shown at the end of this section in Figures 21, 22, and 23 and we describe them here briefly to streamline the associated captions. These figures focus on the problem from the point of view of the $\boldsymbol{z}$-plane in the special case where $\omega_n = 2$, essentially making them higher dimensional analogues of Figure 1 from Berry and Howls (Figure 1 in this thesis). In these figures, the vertical axis schematically represents the u variable, with the other two axes schematically representing any slice $\mathbb{R}^2$ of $\mathbb{C}^d$. This case affords many algebraic and conceptual simplifications, the most prominent being that since χ_n only has two legs, the only possibilities for α_n and β_n are zero and one, allowing us to consider the fully infinite integration surface all at once without the complications present in the general order case. This case also visually simplifies the problem when viewed from the $\boldsymbol{z}$-plane, which is the rationale for using this case in these figures. We believe the u-plane focused representation of the problem displayed in Figures 18, 19, and 20 is more intuitive, but the $\boldsymbol{z}$-plane representations are more familiar due to the similarities to Figure 1; it is for this reason we include both sets of figures.

Returning to the derivation, our task now is to manipulate the remainder

$$\begin{aligned} &R^{(n)}_{(X)}(k,N) \qquad (5.15)\\ &= \frac{1}{2\pi i k^{\frac{N}{\omega_n}}} \int_0^{\infty e^{\frac{2\pi i X}{\omega_n}}} du\, e^{-u^{\omega_n}} u^{N+q_n-1} \oiint_{Q_{n,X}} \frac{d\boldsymbol{z}\, g(\boldsymbol{z})}{(f(\boldsymbol{z})-f_n)^{\frac{N+q_n}{\omega_n}}} \frac{1}{\left(1-\frac{u}{k^{\frac{1}{\omega_n}}(f(\boldsymbol{z})-f_n)^{\frac{1}{\omega_n}}}\right)} \end{aligned}$$

into a resurgence relation involving the asymptotic expansion of critical components χ_{m_j} of $f(\boldsymbol{z})$ and then write the full remainder using

$$R^{(n)}(k,N;\alpha_n,\beta_n) = R^{(n)}_{(\beta_n)}(k,N) - R^{(n)}_{(\alpha_n)}(k,N)$$

as we did in §4. We do this by deforming $Q_{n,X}$ within $\sigma_f^{(n)}$ into a union of steepest descent surfaces at infinity (between two valleys and not intersecting any critical component) and integration surfaces $S_{m_j}(\alpha_n,\beta_n)$ through critical components χ_{m_j}.

The contributions from the surfaces at infinity will vanish provided that the condition

$$\left|\frac{g(\boldsymbol{z})}{(f(\boldsymbol{z})-f_n)^{\frac{q_n}{\omega_n}}}\frac{1}{\left(1-\frac{u}{k^{\frac{1}{\omega_n}}(f(\boldsymbol{z})-f_n)^{\frac{1}{\omega_n}}}\right)}\right| = o\left(\frac{1}{k^{\frac{1}{\omega_n}}(f(\boldsymbol{z})-f_n)^{\frac{1}{\omega_n}}}\right), \left|k^{\frac{1}{\omega_n}}(f(\boldsymbol{z})-f_n)^{\frac{1}{\omega_n}}\right| \to \infty$$

is satisfied, which can again be achieved by prior restriction of the integrand. As in §4, the surface $Q_{n,X}$ deforms as

$$\oiint_{Q_{n,X}} \equiv \sum_{j=1}^{\gamma} K_{nm_j}\left(\int_{S_{m_j,\beta_{nm_j}}} - \int_{S_{m_j,\alpha_{nm_j}}}\right),$$

along which we define

$$u^{\omega_n} = v^{\omega_{m_j}}\left(\frac{f(\boldsymbol{z})-f_n}{F_{nm_j}}\right) =: v^{\omega_{m_j}} V_j(\boldsymbol{z}) \tag{5.16}$$

$$\Rightarrow \frac{du}{dv} = \frac{\omega_{m_j}}{\omega_n}\frac{v^{\frac{\omega_{m_j}}{\omega_n}-1}(f(\boldsymbol{z})-f_n)^{\frac{1}{\omega_n}}}{F_{nm_j}^{\frac{1}{\omega_n}}}.$$

The quantities α_{nm_j} and β_{nm_j} are defined and determined in the same way as in §4.

As before, Figure 15 can be used to determine that the upper integration limit in our remainder integral (5.15) transforms as

$$u = \infty e^{\frac{2\pi i X}{\omega_n}} \Rightarrow v = \infty e^{\frac{2\pi i\left(X-\rho_{nm_j}\right)}{\omega_{m_j}}}$$

using the same arguments as in §4. We can now write (5.15) completely in terms of v as

$$R^{(n)}_{(X)}(k,N) = \sum_{j=1}^{\gamma}\frac{\omega_{m_j}K_{nm_j}}{2\pi i\omega_n k^{\frac{N}{\omega_n}}F_{nm_j}^{\frac{N+q_n}{\omega_n}}}\int_0^{\infty e^{\frac{2\pi i\left(X-\rho_{nm_j}\right)}{\omega_{m_j}}}}\frac{dv\, e^{-v^{\omega_{m_j}}}v^{\omega_{m_j}\left[\frac{N+q_n}{\omega_n}-\frac{1}{\omega_{m_j}}\right]}}{1-\left(\frac{v^{\omega_{m_j}}}{kF_{nm_j}}\right)^{\frac{1}{\omega_n}}}\cdots$$
$$\times\left(\int_{S_{m_j,\beta_{nm_j}}} - \int_{S_{m_j,\alpha_{nm_j}}}\right) d\boldsymbol{z}\, g(\boldsymbol{z}) e^{-\frac{v^{\omega_{m_j}}}{F_{nm_j}}\left(f(\boldsymbol{z})-f_{m_j}\right)}.$$

We make the further transformation

$$v = \tilde{v}e^{\frac{2\pi i\left(X-\rho_{nm_j}\right)}{\omega_{m_j}}} \iff \tilde{v} = ve^{-\frac{2\pi i\left(X-\rho_{nm_j}\right)}{\omega_{m_j}}}$$
$$\Rightarrow dv = d\tilde{v}\, e^{\frac{2\pi i\left(X-\rho_{nm_j}\right)}{\omega_{m_j}}},$$

so that the upper integration limit becomes real infinity, as well as noting that

$$\frac{v^{-q_n}}{F_{nm_j}^{-\frac{q_{m_j}}{\omega_{m_j}}}} T_{(Y)}^{(m_j)}\left(\frac{v^{\omega_{m_j}}}{F_{nm_j}}\right) = \int_{S_{m_j,Y}} d\boldsymbol{z}\, g(\boldsymbol{z}) e^{-\frac{v^{\omega_{m_j}}}{F_{nm_j}}\left(f(\boldsymbol{z})-f_{m_j}\right)},$$

so that our expression for the remainder integral - after dropping the tildes - becomes

$$R_{(X)}^{(n)}(k,N) = \sum_{j=1}^{\gamma} \frac{\omega_{m_j} K_{nm_j} e^{\frac{2\pi i\left(X-\rho_{nm_j}\right)(N+q_n)}{\omega_n}}}{2\pi i \omega_n k^{\frac{N}{\omega_n}} F_{nm_j}^{\frac{N+q_n}{\omega_n}-\frac{q_{m_j}}{\omega_{m_j}}}} \int_0^\infty \frac{dv\, e^{-v^{\omega_{m_j}}} v^{\omega_{m_j}\left[\frac{N+q_n}{\omega_n}-\frac{q_{m_j}}{\omega_{m_j}}\right]-1}}{1-\left(\frac{v^{\omega_{m_j}} e^{2\pi i\left(X-\rho_{nm_j}\right)}}{kF_{nm_j}}\right)^{\frac{1}{\omega_n}}} \cdots$$
$$\times T^{(m_j)}\left(\frac{v^{\omega_{m_j}}}{F_{nm_j}}; \alpha_{nm_j}, \beta_{nm_j}\right), \tag{5.17}$$

with the full remainder being given by

$$R^{(n)}(k,N;\alpha_n,\beta_n) = \sum_{j=1}^{\gamma} \frac{\omega_{m_j} K_{nm_j}}{2\pi i \omega_n k^{\frac{N}{\omega_n}} F_{nm_j}^{\frac{N+q_n}{\omega_n}-\frac{q_{m_j}}{\omega_{m_j}}}} \int_0^\infty dv\, e^{-v^{\omega_{m_j}}} v^{\omega_{m_j}\left[\frac{N+q_n}{\omega_n}-\frac{q_{m_j}}{\omega_{m_j}}\right]-1} \cdots \tag{5.18}$$
$$\times \left[\frac{e^{\frac{2\pi i\left(\beta_n-\rho_{nm_j}\right)(N+q_n)}{\omega_n}}}{1-\left(\frac{v^{\omega_{m_j}} e^{2\pi i\left(\beta_n-\rho_{nm_j}\right)}}{kF_{nm_j}}\right)^{\frac{1}{\omega_n}}} - \frac{e^{\frac{2\pi i\left(\alpha_n-\rho_{nm_j}\right)(N+q_n)}{\omega_n}}}{1-\left(\frac{v^{\omega_{m_j}} e^{2\pi i\left(\alpha_n-\rho_{nm_j}\right)}}{kF_{nm_j}}\right)^{\frac{1}{\omega_n}}}\right] T^{(m_j)}\left(\frac{v^{\omega_{m_j}}}{F_{nm_j}}; \alpha_{nm_j}, \beta_{nm_j}\right).$$

The complete semi and fully infinite resurgence relations are thus given by

$$\begin{aligned} &T_{(X)}^{(n)}(k,N) \\ &= \sum_{r=0}^{N-1} e^{\frac{2\pi i X(r+q_n)}{\omega_n}} \frac{\Gamma\left(\frac{r+q_n}{\omega_n}\right)}{\omega_n k^{\frac{r}{\omega_n}}} \operatorname*{Res}_{\boldsymbol{z}\in\chi_n}\left(\frac{g(\boldsymbol{z})}{(f(\boldsymbol{z})-f_n)^{\frac{r+q_n}{\omega_n}}}\right) \\ &+ \sum_{j=1}^{\gamma} \frac{\omega_{m_j} K_{nm_j} e^{\frac{2\pi i\left(X-\rho_{nm_j}\right)(N+q_n)}{\omega_n}}}{2\pi i \omega_n k^{\frac{N}{\omega_n}} F_{nm_j}^{\frac{N+q_n}{\omega_n}-\frac{q_{m_j}}{\omega_{m_j}}}} \int_0^\infty \frac{dv\, e^{-v^{\omega_{m_j}}} v^{\omega_{m_j}\left[\frac{N+q_n}{\omega_n}-\frac{q_{m_j}}{\omega_{m_j}}\right]-1}}{1-\left(\frac{v^{\omega_{m_j}} e^{2\pi i\left(X-\rho_{nm_j}\right)}}{kF_{nm_j}}\right)^{\frac{1}{\omega_n}}} \cdots \\ &\times T^{(m_j)}\left(\frac{v^{\omega_{m_j}}}{F_{nm_j}}; \alpha_{nm_j}, \beta_{nm_j}\right) \end{aligned} \tag{5.19}$$

and

$$\begin{aligned} &T^{(n)}(k,N;\alpha_n,\beta_n) \\ &= \sum_{r=0}^{N-1} \left(e^{\frac{2\pi i\beta_n(r+q_n)}{\omega_n}} - e^{\frac{2\pi i\alpha_n(r+q_n)}{\omega_n}}\right) \frac{\Gamma\left(\frac{r+q_n}{\omega_n}\right)}{\omega_n k^{\frac{r}{\omega_n}}} \operatorname*{Res}_{\boldsymbol{z}\in\chi_n}\left(\frac{g(\boldsymbol{z})}{(f(\boldsymbol{z})-f_n)^{\frac{r+q_n}{\omega_n}}}\right) \\ &+ \sum_{j=1}^{\gamma} \frac{\omega_{m_j} K_{nm_j}}{2\pi i \omega_n k^{\frac{N}{\omega_n}} F_{nm_j}^{\frac{N+q_n}{\omega_n}-\frac{q_{m_j}}{\omega_{m_j}}}} \int_0^\infty dv\, e^{-v^{\omega_{m_j}}} v^{\omega_{m_j}\left[\frac{N+q_n}{\omega_n}-\frac{q_{m_j}}{\omega_{m_j}}\right]-1} \cdots \end{aligned} \tag{5.20}$$

$$\times \left[\frac{e^{\frac{2\pi i\left(\beta_n - \rho_{nm_j}\right)(N+q_n)}{\omega_n}}}{1 - \left(\frac{v^{\omega_{m_j}} e^{2\pi i\left(\beta_n - \rho_{nm_j}\right)}}{kF_{nm_j}}\right)^{\frac{1}{\omega_n}}} - \frac{e^{\frac{2\pi i\left(\alpha_n - \rho_{nm_j}\right)(N+q_n)}{\omega_n}}}{1 - \left(\frac{v^{\omega_{m_j}} e^{2\pi i\left(\alpha_n - \rho_{nm_j}\right)}}{kF_{nm_j}}\right)^{\frac{1}{\omega_n}}} \right] T^{(m_j)} \left(\frac{v^{\omega_{m_j}}}{F_{nm_j}}; \alpha_{nm_j}, \beta_{nm_j}\right)$$

respectively, from which we can derive expressions for the 'late terms' and a full hyperasymptotic expansion of the original integral. Upon rationalising the denominator of the remainder, it is apparent that the Stokes phenomenon is incorporated into (5.20); this rationalisation will be carried out in §6.

Setting $d = 1$ in (5.20) means that we are in $\mathbb{C}$, so the only critical components are isolated critical points with codimension $q_n \equiv d = 1$. We recover (4.19) exactly from this scenario, as we would expect. If we instead set $\omega_n \equiv \omega_{m_j} = 2$, $q_n \equiv d$ and leave d general, then we are looking only at quadratic order isolated critical points in $\mathbb{C}^d$, the scenario explored in Howls (1997). After substituting in and rewriting the sum over r to skip every odd (identically zero) term, we exactly recover equation (5.5) in Howls (1997) for the coefficients. The remainder expression that we obtain will have a slightly different looking v-integrand to (5.9) in Howls (1997), but when they are evaluated and the rewriting of the sum over r is taken into account, the expressions will be the same.

The semi-infinite expansion (5.19) is displayed explicitly because as well as being a stepping stone towards (5.20), it also acts as an 'endpoint' expansion (for $d = 1$) as explored in Howls (1992). This paper discusses asymptotic contributions of quadratic isolated critical points that are also endpoints of integration ('quadratic endpoints'), as well as non-critical endpoints of integration ('linear endpoints') in $\mathbb{C}$ to the same class of integrals that we studied in §4. The way we have set up our problem in §4 means that the critical point is *always* an endpoint of integration, with the fully infinite expansion (4.19) being two semi-infinite expansions combined.

Comparing with §2 in Howls (1992), we can immediately see that - after setting $\omega_n = 2$ and $X = 0$ - integral (7) is identical to our integral (4.5), the formal expansion (14) is identical to our (4.8), and the coefficients (15) are identical to our (4.12). The remainder and complete resurgence relation look different since transformation (19) is not the same as our transformation (4.16) due to different set-ups, but - as mentioned in the previous paragraph when referring to Howls (1997) - after being evaluated and the sum appropriately rewritten, the expressions will be the same. Therefore, when $d = 1$ our semi-infinite resurgence relation (5.19) acts as an expression for the contribution of a general order critical endpoint of integration in $\mathbb{C}$. When d is general, we have to be more careful; an isolated critical point in $\mathbb{C}^d$

makes no sense as an integration boundary, but the semi-infinite expansion is of course still valid. Instead, when it just so happens that the integration boundary coincides entirely with χ_{S_n}, then the asymptotic contribution of that boundary will be (5.19).

An interesting observation is that our formulae can still be applied in the case of linear integration boundaries - denoted χ_e (with e replacing the subscript n whenever it appears) - provided only that f is constant on χ_e. When χ is a critical component this condition is automatically satisfied so we never had to worry about it, but for a general linear boundary this will not be the case; so long as this condition is enforced, the entire derivation of §5 can be repeated for such a linear boundary with little extra required clarification. We briefly discuss these clarifications now in the order they are required in the derivation.

For linear points we have $\omega_e = 1$, so χ_e only has one leg that goes off into an asymptotic valley, corresponding to $X = 0$ (so we omit the X from the notation). The integral (5.5) still only diverges at critical points of f, as F and hence the zeroes of $|J_F(\boldsymbol{z})|$ remain unchanged. Since S_e does not pass through any critical points of f, the quantity $f(\boldsymbol{z}) - f_e$ is only zero on χ_e, so the loop integral over Q_e is only singular on χ_e. We can deform Q_e the same way we deformed $Q_{n,X}$, so the remainder will take the same form in both cases. With these clarifications made, when faced with finding the asymptotic contribution of a linear boundary that is constant on f, we can simply use (5.19) with $\omega_e = 1$ and $X = 0$.

If we additionally set $d = 1$, we are in the same scenario as §3 in Howls (1992). The coefficients (34) match our (4.12) and once again the remainder in the resurgence relation (39) only looks different to ours because transformation (37) is different to our transformation (4.16); as before, evaluating and rewriting the sum appropriately matches up the expressions.

Other literature focusing on non-isolated critical points - namely that discussed in §2.6 - deals solely with integrals over bounded domains in real space. While our results cannot match exactly due to the presence of boundary terms in such literature, the forms of the coefficients bear clear similarity and the powers of the asymptotic parameter that appear in the expansions in each paper match exactly with ours for the specific scenario the author has chosen.

In the context of differential equations, the case handled here is not like one we have seen so far in the literature when considering exponentially improved solutions. For an nth order homogeneous linear ordinary differential equation with an irregular singularity of exponential rank $\omega - 1$ - such as that considered in Murphy (2001) - there are n asymptotic contributions arising from the irregular singularity (assumed to be at infinity) and the corresponding integral problem involves n order ω isolated critical points in $\mathbb{C}$. This implies that the general integral problem in $\mathbb{C}^d$ corresponds to a linear *partial* differential equation in d variables with analogous singularities. Relating this to the integral studied in this chapter, the corresponding differential equation problem would involve an nth order homogeneous linear partial differential equation with an analogous singularity set of codimension q and exponential rank $\omega - 1$. These codimensions and ranks would need to be 'mixed' with the possibility of non-constant exponential rank across the singularity set, but we are unsure what mixed singularity codimension would mean in this context.

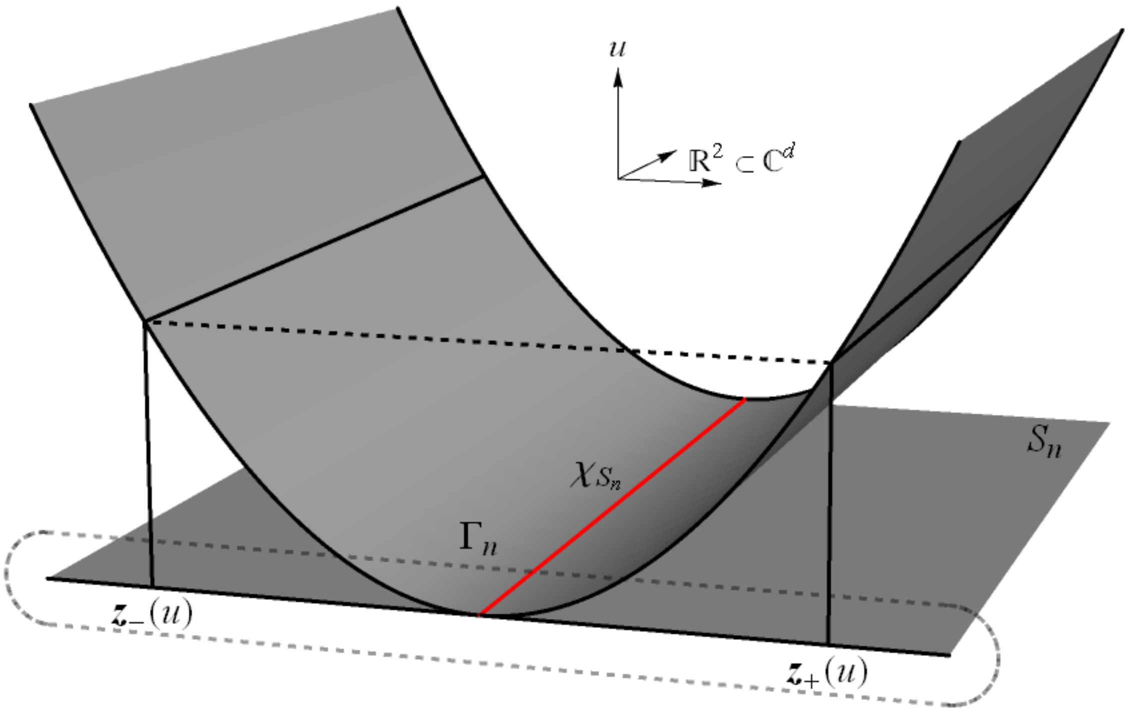


Figure 21: Schematic $\boldsymbol{z}$-plane representation of key components of the problem considered in this chapter for the case $\omega_n = 2$. The vertical axis schematically represents the u variable, with the other two axes schematically representing any slice $\mathbb{R}^2 \subset \mathbb{C}^d$. The faint dashed line is the sausage contour Γ_n that covers both legs of χ_n at once due to the simple nature of the $\omega_n = 2$ case. In relation to the general problem and an integral such as (5.6), Γ_n is the union of Γ_{n,α_n} and Γ_{n,β_n}.

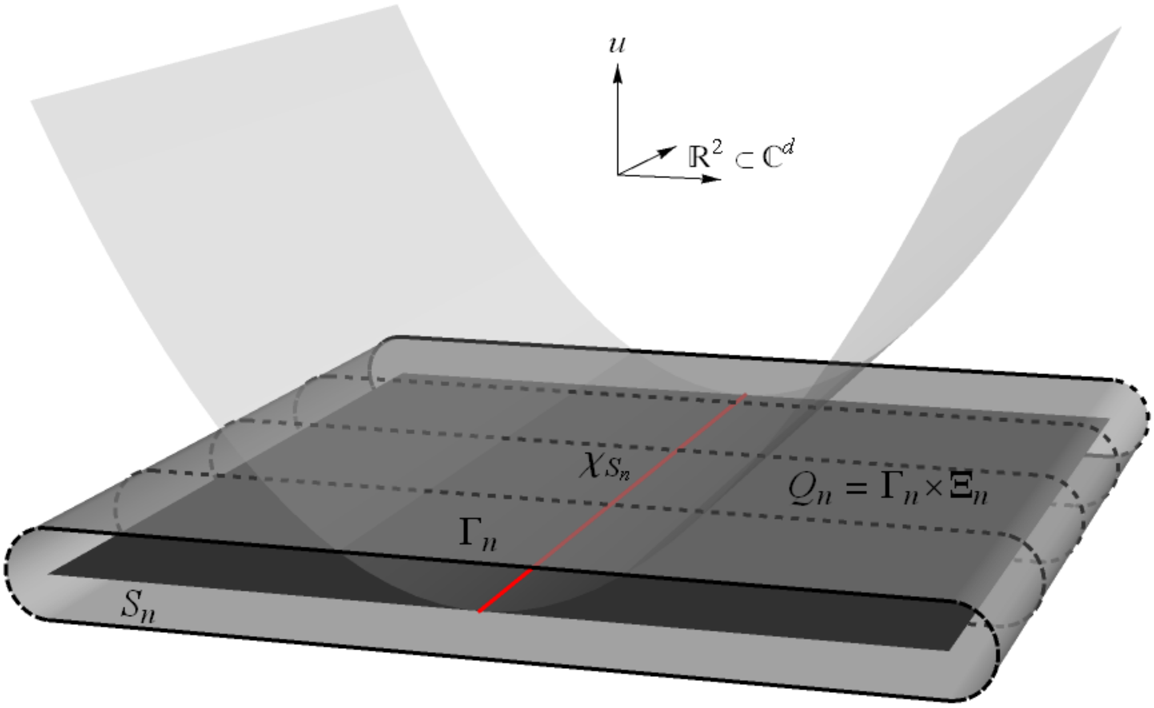


Figure 22: Schematic $\boldsymbol{z}$-plane representation of key components of the problem considered in this chapter for the case $\omega_n = 2$, focusing on the sausage hypersufrace Q_n. It illustrates the fact that Q_n is effectively the result when Γ_n is 'pushed out' over Ξ_n, resulting in the enclosure of the entire integration surface S_n. The axes represent the same quantities as in Figure 21 and once again the simplifications afforded by using the case $\omega_n = 2$ allow us to drop the subscript X denoting leg number. In relation to the general problem and an integral such as (5.7), Q_n is the union of Q_{n,α_n} and Q_{n,β_n}, with other similar quantities following the same pattern.

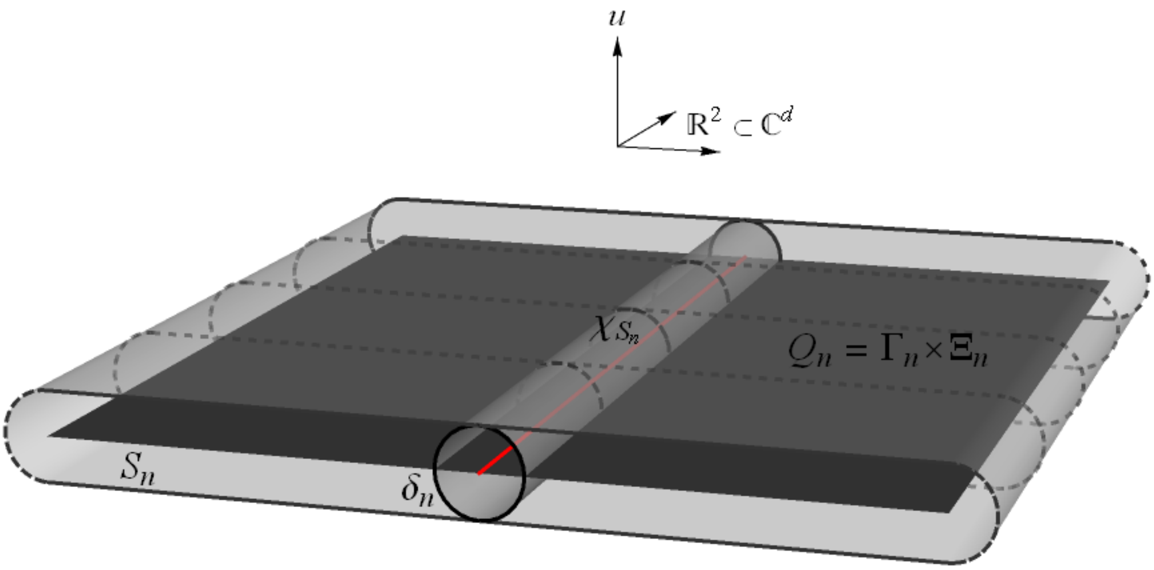


Figure 23: Schematic $\boldsymbol{z}$-plane representation of key components of the problem considered in this chapter for the case $\omega_n = 2$, focusing on the hypertube δ_n that encloses χ_{S_n}. The surface Q_n enclosing S_n has been collapsed to the hypertube δ_n surrounding χ_{S_n}, as the integrand analogous to (5.9) is only singular on χ_{S_n}. The axes represent the same quantities as in Figures 21 and 22 and once again the simplifications afforded by using the case $\omega_n = 2$ allow us to drop the subscript X denoting leg number. In relation to the general problem and an integral such as (5.9), $\delta_{n,X} \equiv \delta_n$ for all X as discussed earlier in the chapter.

5.2 Computing the Multidimensional Residues

The integral (5.9) is clearly a residue type quantity of the critical component χ_n, but what does the residue operator present in equations (5.10) - (5.13) actually represent?

When discussing residues in multidimensional complex space we need to be careful, as there are multiple definitions of and approaches towards them in the literature. These include Leray residues, Grothendieck residues, and global residue operators and they are often discussed in the framework of algebraic geometry and homology. The closest approach of relevance to our work is the local residue of a meromorphic form given in Cattani, Dickenstein, and Sturmfels (1994), which is discussed in a complex analytical setting (references within this paper give details of the algebraic geometry approach). However, our problem in this chapter is not generally covered by this definition (only extremely specific cases could be potentially computed in this way), so we need to carefully consider what exactly the residue operator in equations (5.10) - (5.13) means and how we can deal with it in a practical manner.

To do this, we recall the definitions of the surfaces

$$Q_{n,X} = \Gamma_{n,X} \times \Xi_{n,X} \quad \text{and} \quad S_{n,X} = L_{n,X} \times \Upsilon_{n,X}.$$

In (5.6), the ζ-plane loop integral is over the sausage contour $\Gamma_{n,X}$ enclosing $L_{n,X}$ and the ξ-plane surface integral is over 'the rest of the integration surface' $\Xi_{n,X} (\equiv \Upsilon_{n,X}$ locally). Just like in the one-dimensional case discussed in §4, the ζ-plane loop integral goes on to provide a traditional one-dimensional residue, but when considering the problem in higher dimensional complex space we will always have a 'left over' surface $\Xi_{n,X}$ that also needs to be integrated over.

Recall from §3.1 that $\dim_{\mathbb{R}}(S_{n,X}) = d$ in $\mathbb{C}^d$, meaning that $\dim_{\mathbb{R}}(\Xi_{n,X}) = d-1$. This implies that after computing the ζ-plane loop integral over $\Gamma_{n,X}$, we must still carry out the remaining $d-1$ integrations in the $\boldsymbol{\xi}$ coordinates. In the local coordinates $(\zeta, \boldsymbol{\xi})$, χ_n is given by $\zeta = 0$, but in the original $\boldsymbol{z}$ coordinates, χ_n will have a definition involving some or all of the variables z_j. To deal with this, for each critical component we choose one variable in which to take the traditional one-dimensional residue (often an obvious choice depending on the example and critical component in question) and then directly integrate the remaining $d-1$ variables. Although this does not provide a general theoretical result, we found that it is actually in fact possible to consistently and correctly compute these multidimensional residues for a wide range of examples; we know they are correct because upon substitution of these computed residues into the asymptotic coefficients (5.11) and

(5.13), the respective asymptotic expansions (5.10) and (5.12) do indeed produce a very good approximation of relevant numerically evaluated integrals as required. Interestingly, we observe through examples and by using some intuition that the order of this direct integration matters and we discuss this order now.

A critical component χ in $\mathbb{C}^d$ of complex co-dimension q (and hence complex dimension $\mu = d - q$) is defined by a set of q distinct equations

$$\begin{aligned} F_1(z_1, \ldots, z_d) &= 0, \\ &\vdots \\ F_q(z_1, \ldots, z_d) &= 0. \end{aligned}$$

We must carefully consider how each complex variable is involved in this definition for each critical component. While we do not have a complete rigorous knowledge of how to compute this type of residue in general, we do have a reasonably good working knowledge of how to do so in a variety of different cases. For clarity, we note that this system of equations is not in any way related to the derivatives of the function $f(\boldsymbol{z})$ from which the critical set $C(f)$ of critical components is derived; this system simply reflects the fact that any surface of codimension q in $\mathbb{C}^d$ is defined by q distinct equations.

When $q = 1$, we have a complete understanding of the meaning of the residue operator. The critical component is defined by the single equation $F_1(z_1, \ldots, z_d) = 0$, although not all d complex variables may feature in this equation. We will provide specific examples for both the subcases, but will not do so for the other cases as specific examples for other values of q will just be generic extensions of these examples.

Without loss of generality, assume F_1 depends on z_1 and further assume that we can rewrite $F_1 = 0$ as

$$z_1 = \tilde{F}_1(z_2, \ldots, z_d), \tag{5.21}$$

with all variables z_2 through z_d appearing in the expression on the right hand side. We can then take the one-dimensional residue in z_1 and integrate in the other $d-1$ complex variables directly, writing the integral operator in equation (5.9) as

$$\frac{1}{2\pi i} \oiint_{\delta} d\boldsymbol{z} \equiv \operatorname*{Res}_{\boldsymbol{z} \in \chi} \equiv \int_{\Upsilon} dz_2 \cdots dz_d \operatorname*{Res}_{z_1 = \tilde{F}_1(z_2, \ldots, z_d)},$$

where the integration over Υ is done directly in each complex variable separately. Note that for the rest of this section, we will always take the one-dimensional residue in the z_1 variable.

In this instance, we found that the residue must be taken *before* the direct integration; intuitively, this is because taking the residue injects the RHS of (5.21) into the integrand, which does not really make sense if we have already carried out the direct integration (or at least, requires complicated or unintuitive additional consideration that we have not looked at).

An example of this type of set-up appeared in §3.1, specifically for $f(\boldsymbol{z})$ given by (3.2), reproduced here as

$$f(z_1, z_2) = \frac{1}{8}(z_1^2 + z_2^2)^4 - \frac{17}{6}(z_1^2 + z_2^2)^3 + 22(z_1^2 + z_2^2)^2 - 72(z_1^2 + z_2^2).$$

The critical components of f (shown in Figure 14) are

$$\begin{aligned}
\chi_1 &= (0, 0), \\
\chi_2 &= \left\{ \boldsymbol{z} \in \mathbb{C}^2 \mid z_1^2 + z_2^2 = 4 \right\}, \\
\chi_3 &= \left\{ \boldsymbol{z} \in \mathbb{C}^2 \mid z_1^2 + z_2^2 = 9 \right\},
\end{aligned}$$

where χ_2 and χ_3 can be rewritten in the form of (5.21). Writing the equation defining χ_2 as $z_1 = \sqrt{4 - z_2^2}$ (ignoring the issue of $\pm$ for the sake of demonstration), then the residue quantity becomes

$$\operatorname*{Res}_{z \in \chi_2} \left(\frac{g(z_1, z_2)}{(f(z_1, z_2) - f_2)^{\frac{r+1}{3}}} \right) = \int_{\Upsilon_2} dz_2 \operatorname*{Res}_{z_1 = \sqrt{4 - z_2^2}} \left(\frac{g(z_1, z_2)}{(f(z_1, z_2) - f_2)^{\frac{r+1}{3}}} \right).$$

For $r = 0$, this residue is

$$\operatorname*{Res}_{z_1 = \sqrt{4 - z_2^2}} \left(\frac{g(z_1, z_2)}{(f(z_1, z_2) - f_2)^{\frac{1}{3}}} \right) = \frac{\left(\frac{3}{5}\right)^{1/3} g\left(\sqrt{4 - z_2^2}, z_2\right)}{2^{2/3}\left(-\left(4 - z_2^2\right)^{3/2}\right)^{1/3}}$$

and as r increases, the residue becomes increasing algebraically complicated, but still easily computable by a computer algebra package. For suitable choice of g and integration contour Υ_2, the z_2 integral will converge, enabling full computation of the two-dimensional residue. Conceptually, taking the residue using $z_1 = \sqrt{4 - z_2^2}$ hooks us onto a generic point on the complex circle χ_2 and then integrating with respect to z_2 pushes us fully around the circle, giving us the full residue (so long as the $\pm$ square root issue is handled correctly).

If $\tilde{F}_1$ does not depend on some of the complex variables, for example - without loss of generality - if $\tilde{F}_1$ did not depend on z_2 so that $z_1 = \tilde{F}_1(z_3, \dots, z_d)$, then we may perform the integration with respect to z_2 at *any point*. In terms of integral operators, this means that

$$\frac{1}{2\pi i}\oiint_{\delta} d\boldsymbol{z} \equiv \operatorname*{Res}_{\boldsymbol{z}\in\chi} \equiv \int_{\Upsilon} dz_2 \cdots dz_d \operatorname*{Res}_{z_1=\tilde{F}_1(z_2,\ldots,z_d)}$$
$$\equiv \int_{\tilde{\Upsilon}} dz_3 \cdots dz_d \operatorname*{Res}_{z_1=\tilde{F}_1(z_3,\ldots,z_d)} \left(\int_{\Upsilon_2} dz_2\right),$$

where $\tilde{\Upsilon}\times\Upsilon_2=\Upsilon$. Similarly, if $\tilde{F}_1$ did not depend on the complex variables z_2 through z_6 so that $z_1=\tilde{F}_1(z_6,\ldots,z_d)$, then we can integrate with respect to z_2 through z_6 at any point, so long as we integrate with respect to z_6 through z_d after taking the residue. One possible residue operator for this example would be

$$\frac{1}{2\pi i}\oiint_{\delta} d\boldsymbol{z} \equiv \operatorname*{Res}_{\boldsymbol{z}\in\chi} \equiv \int_{\Upsilon_5} dz_5 \int_{\tilde{\Upsilon}} dz_6 \cdots dz_d \int_{\Upsilon_3} dz_3 \operatorname*{Res}_{z_1=\tilde{F}_1(z_6,\ldots,z_d)} \left(\int_{\Upsilon_2} dz_2 \int_{\Upsilon_4} dz_4\right),$$

where $\tilde{\Upsilon}\times\Upsilon_2\times\cdots\times\Upsilon_6=\Upsilon$. We are free to move the z_2 and z_6 integral sign wherever we like within this operator expression, but the position of all other variables remains fixed.

An example of this type of set-up also appeared in §3.1, specifically for $f(\boldsymbol{z})=f(z_1,z_2)=z_2^2$. The only critical component of f is the complex line $z_2=0$, the definition of which does not depend on z_1. The residue quantity therefore becomes

$$\operatorname*{Res}_{z\in\chi}\left(\frac{g(z_1,z_2)}{\left(z_2^2\right)^{\frac{r+1}{2}}}\right) = \int_{\Upsilon_1} dz_1 \operatorname*{Res}_{z_2=0}\left(\frac{g(z_1,z_2)}{\left(z_2^2\right)^{\frac{r+1}{2}}}\right) = \operatorname*{Res}_{z_2=0}\left(\int_{\Upsilon_1} dz_1 \frac{g(z_1,z_2)}{\left(z_2^2\right)^{\frac{r+1}{2}}}\right),$$

which is far simpler than those in the previous specific example. The z_2 residue for general g and r is

$$\frac{1}{r!}\frac{\partial^r}{\partial z_2^r} g(z_1,z_2),$$

with the form of the full residue depending on the choice of g. If $g=e^{-z_1^2}$ and $\Upsilon_1=\bar{\mathbb{R}}$, then the residue is zero for all except $r=0$, where it is given by

$$\operatorname*{Res}_{z\in\chi}\left(\frac{e^{-z_1^2}}{\left(z_2^2\right)^{\frac{1}{2}}}\right) = \int_{-\infty}^{\infty} dz_1 \operatorname*{Res}_{z_2=0}\left(\frac{e^{-z_1^2}}{z_2}\right) = \int_{-\infty}^{\infty} dz_1 e^{-z_1^2} = \sqrt{\pi}.$$

This makes sense as f only has one critical component and so should converge for suitable choice of g and Υ_1, meaning that the asymptotic expansion terminates to provide an exact result.

When $q>1$, the situation is more complicated and is not yet completely understood. As an example, let $q=2$ and let χ be defined by

$$z_1 = \tilde{F}_1(z_3, \ldots, z_d), \quad z_2 = a, \tag{5.22}$$

where $a \in \mathbb{C}$. It is observed that the z_2 integration must be carried out *before* the residue is taken, along with the previously discussed requirement that the z_3 through z_d integration must take place *after* the residue is taken. The only residue operator that produces the correct result is thus

$$\frac{1}{2\pi i} \oiint_{\delta} d\boldsymbol{z} \equiv \operatorname*{Res}_{\boldsymbol{z}\in\chi} \equiv \int_{\tilde{\Upsilon}} dz_3 \cdots dz_d \operatorname*{Res}_{z_1=\tilde{F}_1(z_3,\ldots,z_d)} \left(\int_{\Upsilon_2} dz_2 \right),$$

where $\tilde{\Upsilon} \times \Upsilon_2 = \Upsilon$. The uncertainty arises as we are currently unsure if the integration contour Υ_2 must pass through the point a; intuitively it seems like it must, since the z_2 integration then correctly 'picks up' the value of the critical component in that variable and otherwise has no way of knowing what the value of a is. Regardless, we were unable to test this extensively, as the examples we studied all used integration surface that passed through points analogous to a and we did not have time to construct examples whose integration surface did not.

Similar to the $q = 1$ case, if $\tilde{F}_1$ in (5.22) did not depend on some of the complex variables, then integration in those variables could be carried out at any point. For example, define χ according to

$$z_1 = \tilde{F}_1(z_6, \ldots, z_d), \quad z_2 = a_2, \quad z_3 = a_3.$$

According to our discussion thus far, we take the one-dimensional residue in z_1, the z_2 and z_3 integration must be carried before the residue, the z_6 through z_d integration must be carried out after the residue, and the z_4 and z_5 integration can be carried out at any point in the integration process. One possible residue operator is therefore

$$\frac{1}{2\pi i} \oiint_{\delta} d\boldsymbol{z} \equiv \operatorname*{Res}_{\boldsymbol{z}\in\chi} \equiv \int_{\Upsilon_5} dz_5 \int_{\tilde{\Upsilon}} dz_6 \cdots dz_d \operatorname*{Res}_{z_1=\tilde{F}_1(z_6,\ldots,z_d)} \left(\int_{\Upsilon_2} dz_2 \int_{\Upsilon_3} dz_3 \int_{\Upsilon_4} dz_4 \right),$$

where $\tilde{\Upsilon} \times \Upsilon_2 \times \cdots \times \Upsilon_6 = \Upsilon$. We are free to move the z_4 and z_5 integral sign wherever we like within this operator expression, but the position of all other variables remains fixed.

The final case to consider is that of $q = d$; in this case, χ is an isolated critical point and so there is only one possible situation to consider. A specific example is not provided here and we instead refer ahead to §8, in which an example requires that we compute residues for isolated critical points in $\mathbb{C}^4$. Defining $\chi = \boldsymbol{a} = (a_1, \ldots, a_d)$ - namely, defining χ by $z_j = a_j$ for all j - the only operator producing the correct result would be

$$\frac{1}{2\pi i}\oiint_{\delta} d\boldsymbol{z} \equiv \operatorname*{Res}_{\boldsymbol{z}\in\chi} \equiv \operatorname*{Res}_{z_1=a_1}\left(\int_{\Upsilon} dz_2\cdots dz_d\right) \equiv \operatorname*{Res}_{z_1=a_1}\left(\int_{\Upsilon_2} dz_2\cdots\int_{\Upsilon_d} dz_d\right),$$

where $\Upsilon_2\times\cdots\times\Upsilon_d=\Upsilon$. This type of residue operator is the one appearing in equation (5.5) in Howls (1997) and its practical calculation is a long standing problem in the field. Nonetheless, so far we have used this method to successfully calculate the residue around an isolated critical point in $\mathbb{C}^2$, with one example being extended generically into $\mathbb{C}^3$ and $\mathbb{C}^4$. This example is provided later in the thesis in §8 in the context of non-uniformly degenerate critical components (namely, of non-constant order). We were required to compute the expansions (and hence the residues) around isolated critical points in $\mathbb{C}^4$ along with a non-uniformly degenerate critical component of codimension one; the residues were successfully computed for all three critical components, but there are additional complications for non-uniformly degenerate critical components that will be discussed within that chapter. We also note that generically extending this example into $\mathbb{C}^d$ would be trivial.

Although this discussion of multidimensional residues is non-rigorous, we hope that describing our working knowledge of these residues inspires a closer rigorous look at them.

Equation (5.20) embodies multiple different generalisations of pre-existing work, but there are still many more to explore. Before addressing any of these, we look at deriving a complete hyperasymptotic expansion for the integral (5.2). We derive explicit expressions for the hyperterminants - which will be subsequently defined - and show how they can be practically computed.

6 Hyperterminants and the Full Hyperasymptotic Expansion

In this chapter our goal is produce a complete general level hyperasymptotic expansion of integral (5.2) using our resurgence relation (5.20).

Using (5.20), we can 'iterate' our expression in the usual way by substituting it into itself. Each substitution gives us another 'level' of our asymptotic expansion (increasing the numerical accuracy) and a new remainder. We begin by calculating the first level hyperasymptotic expansion and late term coefficient expansion, before detailing the form of the complete hyperasymptotic expansion and deriving relevant general level expressions. Note that we will also have to slightly change some notation - such as changing which functional dependencies are explicitly shown and introducing new functions - to simplify the increasingly complicated higher level expressions.

6.1 Hyperasymptotic Expansion and its Component Expressions

We start by substituting (5.20) into itself once, which gives us

$$
\begin{aligned}
T^{(n)}(k;N_n;N_{nm_1};\alpha_n,\beta_n) &= \sum_{r_0=0}^{N_n-1}\frac{T_{r_0}^{(n)}(\alpha_n,\beta_n)}{k^{\frac{r_0}{\omega_n}}} \\
&+\sum_{X_f^{(n)}}\sum_{r_1=1}^{N_{nm_1}-1}K_{nm_1}\int_0^\infty\frac{dv_1\,\omega_{m_1}e^{-v_1^{\omega_{m_1}}}v_1^{\omega_{m_1}\left[\frac{N_n+q_n}{\omega_n}-\frac{r_1+q_{m_1}}{\omega_{m_1}}\right]-1}}{2\pi i\omega_n k^{\frac{N_n}{\omega_n}}F_{nm_1}^{\frac{N_n+q_n}{\omega_n}-\frac{r_1+q_{m_1}}{\omega_{m_1}}}}\cdots \\
&\times\left[\frac{e^{\frac{2\pi i(\beta_n-\rho_{nm_1})(N_n+q_n)}{\omega_n}}}{1-\left(\frac{v_1^{\omega_{m_1}}e^{2\pi i(\beta_n-\rho_{nm_1})}}{kF_{nm_1}}\right)^{\frac{1}{\omega_n}}}-\frac{e^{\frac{2\pi i(\alpha_n-\rho_{nm_1})(N_n+q_n)}{\omega_n}}}{1-\left(\frac{v_1^{\omega_{m_1}}e^{2\pi i(\alpha_n-\rho_{nm_1})}}{kF_{nm_1}}\right)^{\frac{1}{\omega_n}}}\right]T_{r_1}^{(m_1)}(\alpha_{nm_1},\beta_{nm_1}) \\
&+R^{(n)}(N_n;N_{nm_1};\alpha_n,\beta_n), \qquad (6.1)
\end{aligned}
$$

where $R^{(n)}(N_n;N_{nm_1};\alpha_n,\beta_n)$ is the second remainder. We can see some notational changes in effect in (6.1) and take some time now to explain them.

The set of critical components to be summed over in (5.20) is

$$X_f^{(n)} = X_f \setminus \{\chi_n\} = \{\chi_1,\ldots,\chi_{\gamma_n}\},$$

(similarly defined in Definition 1), namely "every critical component except χ_n". Each iteration of (5.20) produces an new asymptotic series of computable terms and a new remainder; the Mth iteration produces the Mth level asymptotic series of the hyperasymptotic expansion. In this new series is an extra sum over an analogous set to $X_f^{(n)}$, so that our Mth level asymptotic series is summed over M such sets, thus requiring M different sum indices. The most immediate solution is to write the Mth sum using the dummy variable j_M and the affected quantities with the subscripts m_{M,j_M} or something similar, but the expressions we will generate are already complicated, so we opt to use the simpler notation of m_M, with the sum written over $X_f^{(m_{M-1})}$. Under this notation, $M = 0$ refers to the superasymptotic series (which on its own would be the superasymptotic expansion of the integral), or simply the zeroth level series.

The quantity r_M is used as the dummy variable in the Mth level asymptotic series of our hyperasymptotic expansion (with r_0 replacing r in (5.20)) and the quantity $N_{nm_1\cdots m_M}$ is the truncation point of the Mth level series. Note that for the purposes of general formulae, we use the convention that $n = m_0$, so that when $M = 0$, the truncation point is N_n (replacing N in (5.20)). Other quantities using the m_M subscript will also follow this convention.

Lastly, we separate the truncation points N_n, N_{nm_1} etc. with semi colons rather than commas. This is to prevent confusion during examples when multiple truncation points are required at the same hyperasymptotic level.

To calculate the late term coefficient expansion, we isolate the coefficient $T_{N_n}(\alpha_n, \beta_n)$ using

$$T^{(n)}(k; N_n; N_{nm_1}; \alpha_n, \beta_n) - T^{(n)}(k; N_n + 1; N_{nm_1}; \alpha_n, \beta_n) = 0$$

and compute using (6.1). This gives the first level expansion as

$$\begin{aligned} T_{N_n}^{(n)}(N_{nm_1}; \alpha_n, \beta_n) = \sum_{X_f^{(n)}} \sum_{r_1=0}^{N_{nm_1}-1} & \left(e^{\frac{2\pi i(\beta_n - \rho_{nm_1})(N_n+q_n)}{\omega_n}} - e^{\frac{2\pi i(\alpha_n - \rho_{nm_1})(N_n+q_n)}{\omega_n}} \right) \cdots \\ & \times \frac{K_{nm_1}\Gamma\left(\frac{N_n+q_n}{\omega_n} - \frac{r_1+q_{m_1}}{\omega_{m_1}}\right)}{2\pi i \omega_n F_{nm_1}^{\frac{N_n+q_n}{\omega_n} - \frac{r_1+q_{m_1}}{\omega_{m_1}}}} T_{r_1}^{(m_1)}(\alpha_{nm_1}, \beta_{nm_1}) \\ & + k^{\frac{N_n}{\omega_n}} \left(R^{(nm_1)}(N_n; N_{nm_1}; \alpha_n, \beta_n) - R^{(nm_1)}(N_n + 1; N_{nm_1}; \alpha_n, \beta_n) \right), \end{aligned} \tag{6.2}$$

where the remainder is independent of k, as it will cancel out when the expression is explicitly computed in exactly the same way as it did in the rest of (6.2). This late term expression is clearly consistent with Darboux's theorem.

We denote the r_1th term of this expansion as $T_{N_n,r_1}^{(n)}(\alpha_n, \beta_n)$. The terms in (6.2) will follow the standard factorial-over-power behaviour up until the point where the argument of the gamma function inevitable becomes negative; namely, when

$$\frac{N_n+q_n}{\omega_n}-\frac{r_1+q_{m_1}}{\omega_{m_1}}<0 \Rightarrow r_1>\omega_{m_1}\left(\frac{N_n+q_n}{\omega_n}-\frac{q_m}{\omega_{m_1}}\right). \tag{6.3}$$

Note that we will only explicitly compute the first level late term expansion here.

The general structure of the full hyperasymptotic expansion remains unchanged from Howls (1997); at each level, we have an asymptotic series of the product of coefficients and hyperterminants, summed over the appropriate set of critical components. The full hyperasymptotic expansion of integral (5.2) is

$$\begin{aligned}
&T^{(n)}(k;N_n;\ldots;N_{nm_1\ldots m_M};\alpha_n,\beta_n) \\
&=\sum_{r_0=0}^{N_n-1} T^{(n)}_{r_0}(\alpha_n,\beta_n)K^{(n)}_{r_0}+\sum_{X^{(n)}_f} K_{nm_1}\sum_{r_1=0}^{N_{nm_1}-1} T^{(m_1)}_{r_1}(\alpha_{nm_1},\beta_{nm_1})K^{(nm_1)}_{r_1}(N_n;\alpha_n,\beta_n)+\ldots \\
&+\sum_{X^{(n)}_f}\cdots\sum_{X^{(m_{M-1})}_f} K_{nm_1}\cdots K_{m_{M-1}m_M}\cdots \\
&\times\sum_{r_M=0}^{N_{nm_1\cdots m_M}-1} T^{(m_M)}_{r_M}(\alpha_{m_{M-1}m_M},\beta_{m_{M-1}m_M})K^{(nm_1\cdots m_M)}_{r_M}(N_n,\ldots,N_{nm_1\cdots m_{M-1}};\alpha_n,\beta_n) \\
&+R^{(nm_1\cdots m_M)}(N_n;\ldots;N_{nm_1\ldots m_M};\alpha_n,\beta_n),
\end{aligned} \tag{6.4}$$

where $K^{(nm_1\cdots m_M)}_{r_M}$ is the Mth hyperterminant of the expansion. Comparing against (6.1), we can see that $K^{(n)}_{r_0}=k^{-\frac{r_0}{\omega_n}}$ and

$$\begin{aligned}
K^{(nm_1)}_{r_1}(N_n;\alpha_n,\beta_n)=\int_0^\infty &\frac{dv_1\,\omega_{m_1}e^{-v_1^{\omega_{m_1}}}v_1^{\omega_{m_1}\left[\frac{N_n+q_n}{\omega_n}-\frac{r_1+q_{m_1}}{\omega_{m_1}}\right]-1}}{2\pi i\omega_n k^{\frac{N_n}{\omega_n}}F_{nm_1}^{\frac{N_n+q_n}{\omega_n}-\frac{r_1+q_{m_1}}{\omega_{m_1}}}}\cdots \\
&\times\left[\frac{e^{\frac{2\pi i(\beta_n-\rho_{nm_1})(N_n+q_n)}{\omega_n}}}{1-\left(\frac{v_1^{\omega_{m_1}}e^{2\pi i(\beta_n-\rho_{nm_1})}}{kF_{nm_1}}\right)^{\frac{1}{\omega_n}}}-\frac{e^{\frac{2\pi i(\alpha_n-\rho_{nm_1})(N_n+q_n)}{\omega_n}}}{1-\left(\frac{v_1^{\omega_{m_1}}e^{2\pi i(\alpha_n-\rho_{nm_1})}}{kF_{nm_1}}\right)^{\frac{1}{\omega_n}}}\right].
\end{aligned} \tag{6.5}$$

Equation (6.5) for the first hyperterminant is already fairly complicated and as we compute higher level hyperterminants, the already complicated expressions will become even more so; we now introduce some new functions that will help simplify such expressions.

We define

$$P^{(n)}_s(a,b)=\frac{N_{nm_1\cdots m_{s-1}}+q_{m_{s-1}}+a}{\omega_{m_{s-1}}}-\frac{N_{nm_1\cdots m_s}+q_{m_s}+b}{\omega_{m_s}}$$

to simplify some exponents,

$$G_1^{(n)}(c;\alpha_n,\beta_n)$$
$$= \exp\left(\frac{2\pi i\left(\beta_n-\rho_{nm_1}\right)\left(N_n+q_n+c\right)}{\omega_n}\right)-\exp\left(\frac{2\pi i\left(\alpha_n-\rho_{nm_1}\right)\left(N_n+q_n+c\right)}{\omega_n}\right)$$

and

$$G_s^{(n)}(c)=\exp\left(\frac{2\pi i\left(\beta_{m_{s-2}m_{s-1}}-\rho_{m_{s-1}m_s}\right)\left(N_{nm_1\cdots m_{s-1}}+q_{m_{s-1}}+c\right)}{\omega_{m_{s-1}}}\right)$$
$$-\exp\left(\frac{2\pi i\left(\alpha_{m_{s-2}m_{s-1}}-\rho_{m_{s-1}m_s}\right)\left(N_{nm_1\cdots m_{s-1}}+q_{m_{s-1}}+c\right)}{\omega_{m_{s-1}}}\right)$$

to simplify certain complicated exponentials, and

$$Y^{(nm_1)}(N_n;\alpha_n,\beta_n)=\frac{e^{\frac{2\pi i(\beta_n-\rho_{nm_1})(N_n+q_n)}{\omega_n}}}{1-\left(\frac{v_1^{\omega_{m_1}}e^{2\pi i(\beta_n-\rho_{nm_1})}}{kF_{nm_1}}\right)^{\frac{1}{\omega_n}}}-\frac{e^{\frac{2\pi i(\alpha_n-\rho_{nm_1})(N_n+q_n)}{\omega_n}}}{1-\left(\frac{v_1^{\omega_{m_1}}e^{2\pi i(\alpha_n-\rho_{nm_1})}}{kF_{nm_1}}\right)^{\frac{1}{\omega_n}}}$$

and

$$Y^{(nm_1\cdots m_M)}(N_{nm_1\cdots m_{M-1}})=\frac{e^{\frac{2\pi i\left(\beta_{m_{M-2}m_{M-1}}-\rho_{m_{M-1}m_M}\right)\left(N_{nm_1\cdots m_M}+q_{m_{M-1}}\right)}{\omega_{m_{M-1}}}}}{1-\left(\frac{v_M^{\omega_{m_M}}F_{m_{M-2}m_{M-1}}}{v_{m_{M-1}}^{\omega_{m_{M-1}}}F_{m_{M-1}m_M}}e^{2\pi i\left(\beta_{m_{M-2}m_{M-1}}-\rho_{m_{M-1}m_M}\right)}\right)^{\frac{1}{\omega_{m_{M-1}}}}}$$

$$-\frac{e^{\frac{2\pi i\left(\alpha_{m_{M-2}m_{M-1}}-\rho_{m_{M-1}m_M}\right)\left(N_{nm_1\cdots m_M}+q_{m_{M-1}}\right)}{\omega_{m_{M-1}}}}}{1-\left(\frac{v_M^{\omega_{m_M}}F_{m_{M-2}m_{M-1}}}{v_{m_{M-1}}^{\omega_{m_{M-1}}}F_{m_{M-1}m_M}}e^{2\pi i\left(\alpha_{m_{M-2}m_{M-1}}-\rho_{m_{M-1}m_M}\right)}\right)^{\frac{1}{\omega_{m_{M-1}}}}}$$

to simplify terms such as the one in the square brackets in (6.1) and (6.5). We will see later in this chapter that the Mth hyperterminant will contain one copy of every Y-function up to $Y^{(nm_1\cdots m_M)}$.

With these new functions, we can write the first level late term (6.2) and first hyperterminant (6.5) as

$$T_{N_n}^{(n)}(\alpha_n,\beta_n)$$
$$=\sum_{X_f^{(n)}}\sum_{r_1=1}^{N_{nm_1}-1}G_1^{(n)}(0;\alpha_n,\beta_n)\frac{K_{nm_1}\Gamma\left(P_1^{(n)}(0,r_1-N_{nm_1})\right)}{2\pi i\omega_n F_{nm_1}^{P_1^{(n)}(0,r_1-N_{nm_1})}}T_{r_1}^{(m_1)}\left(\alpha_{nm_1},\beta_{nm_1}\right)+R, \quad (6.6)$$

where R is the remainder, and

$$K^{(nm_1)}_{r_1}(N_n;\alpha_n,\beta_n) = \int_0^\infty \frac{dv_1\,\omega_{m_1} e^{-v_1^{\omega_{m_1}}} v_1^{\omega_{m_1} P_1^{(n)}(0,r_1-N_{nm_1})-1}}{2\pi i\omega_n k^{\frac{N_n}{\omega_n}} F_{nm_1}^{P_1^{(n)}(0,r_1-N_{nm_1})}} Y^{(nm_1)}(N_n;\alpha_n,\beta_n). \quad (6.7)$$

Unfortunately, an issue arises when we try to compute integrals involving Y-functions such as (6.7) using computer algebra packages; the presence of fractional powers of the v variables in the denominator means that we have branch points in the various v-planes. It is well known that computer algebra packages cannot always choose the correct root (and hence Riemann surface and function value) when presented with such a system and this is demonstrably the case here. The various constants α and β tell us which Riemann surface we are on and if we look at the two terms in the definition of the Y-functions, it is clear that individually their values should change as we vary α and β, with each valid choice corresponding to a different result. However, when computed in their current form, it is possible to get the same result for multiple values and therefore we do not always get the correct asymptotic expansion.

To combat this, we rewrite the Y-functions by employing the expression

$$\frac{1}{1-x^{\frac{a}{b}}} = \sum_{K=0}^{b-1} \frac{x^{K\frac{a}{b}}}{1-x^a}, \quad (6.8)$$

where a and b are positive integers, which is a more general version of rationalising the denominator. This has the effect of removing the branch points and hence the Riemann surface structure, replacing them with poles and thus making the possibility for Stokes phenomena to occur more apparent. Taking

$$x = \left(\frac{v_1^{\omega_{m_1}} e^{2\pi i\left(\beta_n-\rho_{nm_1}\right)}}{kF_{nm_1}}\right)^{\frac{1}{\omega_{m_1}}}$$

and

$$x = \left(\frac{v_M^{\omega_{m_M}} F_{m_{M-2}m_{M-1}}}{v_{m_{M-1}}^{\omega_{m_{M-1}}} F_{m_{M-1}m_M}} e^{2\pi i\left(\alpha_{m_{M-2}m_{M-1}}-\rho_{m_{M-1}m_M}\right)}\right)^{\frac{1}{\omega_{m_M}}},$$

we can rewrite the Y-functions using (6.8) as

$$Y^{(nm_1)}(N_n;\alpha_n,\beta_n) = \sum_{K_1=0}^{\omega_n-1} G_1^{(n)}(K_1;\alpha_n,\beta_n) \frac{\left(\frac{v_1^{\omega_{m_1}}}{kF_{nm_1}}\right)^{\frac{K_1}{\omega_n}}}{1-\frac{v_1^{\omega_{m_1}}}{kF_{nm_1}}},$$

$$Y^{(nm_1\cdots m_M)}(N_{nm_1\cdots m_{M-1}}) = \sum_{K_M=0}^{\omega_{m_{M-1}}-1} G_M^{(n)}(K_M) \frac{\left(\frac{v_M^{\omega_{m_M}} F_{m_{M-2}m_{M-1}}}{v_{M-1}^{\omega_{m_{M-1}}} F_{m_{M-1}m_M}}\right)^{\frac{K_M}{\omega_{m_{M-1}}}}}{1-\frac{v_M^{\omega_{m_M}} F_{m_{M-2}m_{M-1}}}{v_{M-1}^{\omega_{m_{M-1}}} F_{m_{M-1}m_M}}} \quad (6.9)$$

respectively.

We can see that the power of v_M in the denominator of $Y^{(nm_1\cdots m_M)}$ is now an integer; this means that for all $j \in \{1,\ldots,M\}$, the dv_j integral in the hyperterminant $K^{(nm_1\cdots m_M)}$ will have poles (instead of branch points) in the v_j plane whenever a Stokes phenomenon occurs. Additionally, in the form of (6.9), the hyperterminants are now computable via computer algebra packages. Note that the exponentials that were in the denominators of the two terms of the Y-functions have been safely evaluated to one, as they are no longer being raised to any exponent. These two terms now have a common denominator, allowing us to pull the exponentials in the numerator out and write them in terms of G-functions.

The first hyperterminant can now be written as

$$K_{r_1}^{(nm_1)}(N_n;\alpha_n,\beta_n) = \sum_{K_1=0}^{\omega_n-1}\int_0^\infty \frac{dv_1\,\omega_{m_1} e^{-v_1^{\omega_{m_1}}} v_1^{\omega_{m_1}P_1^{(n)}(K_1,r_1-N_{nm_1})-1}}{2\pi i\omega_n k^{\frac{N_n+K_1}{\omega_n}} F_{nm_1}^{P_1^{(n)}(K_1,r_1-N_{nm_1})}} \frac{G_1^{(n)}(K_1;\alpha_n,\beta_n)}{1-\frac{v_1^{\omega_{m_1}}}{kF_{nm_1}}}, \tag{6.10}$$

which is vastly visually simpler than (6.5) while also being readily computable, unlike (6.7). In this form, it is a lot simpler to iterate remainder substitution to derive an expression for the general hyperterminant. Each subsequent hyperterminant contains an extra v-integral - with an extra similar exponential and v factor in the numerator - and an extra similar singulant and $2\pi i$ factor in the denominator. Additionally, it is multiplied overall by an extra Y-function (which we rewrite using (6.9)), the ω subscript in the numerator changes to match the iteration level M and the argument of the P-functions will change slightly. This simple iterative procedure allows us to readily write down the general results by computing the second hyperterminant (which we omit explicitly) and then continuing the substitution pattern. These results can then be verified by induction.

The general Mth level hyperterminant is thus given by

$$\begin{aligned}
&K_{r_M}^{(nm_1\cdots m_M)}(N_n,\ldots,N_{nm_1\cdots m_M};\alpha_n,\beta_n)\\
&= \sum_{K_1=0}^{\omega_n-1}\cdots\sum_{K_M=0}^{\omega_{m_{M-1}}-1}\int_0^\infty\cdots\int_0^\infty dv_1\cdots dv_M \frac{\omega_{m_M} e^{-\left(v_1^{\omega_{m_1}}+\cdots+v_M^{\omega_{m_M}}\right)}}{\omega_n(2\pi i)^M k^{\frac{N_n+K_1}{\omega_n}}}\cdots\\
&\times \frac{v_1^{\omega_{m_1}P_1^{(n)}(K_1,K_2)-1}\cdots v_{M-1}^{\omega_{m_{M-1}}P_{M-1}^{(n)}(K_{M-1},K_M)-1} v_M^{\omega_{m_M}P_M^{(n)}(K_M,r_M-N_{nm_1\cdots m_M})-1}}{F_{nm_1}^{P_1^{(n)}(K_1,K_2)}\cdots F_{m_{M-2}m_{M-1}}^{P_{M-1}^{(n)}(K_{M-1},K_M)} F_{m_{M-1}m_M}^{P_M^{(n)}(K_M,r_M-N_{nm_1\cdots m_M})}}\cdots\\
&\times\left(\frac{G_1^{(n)}(K_1;\alpha_n,\beta_n)}{1-\frac{v_1^{\omega_{m_1}}}{kF_{nm_1}}}\right)\left(\frac{G_2^{(n)}(K_2)}{1-\frac{v_2^{\omega_{m_2}}F_{nm_1}}{v_1^{\omega_{m_1}}F_{m_1m_2}}}\right)\cdots\left(\frac{G_M^{(n)}(K_M)}{1-\frac{v_M^{\omega_{m_M}}F_{m_{M-2}m_{M-1}}}{v_{M-1}^{\omega_{m_{M-1}}}F_{m_{M-1}m_M}}}\right),
\end{aligned} \tag{6.11}$$

with the remainder after M iterations given by

$$R^{(nm_1\cdots m_M)}(N_n;\ldots;N_{nm_1\cdots m_M};\alpha_n,\beta_n)=\sum_{X_f^{(n)}}\cdots\sum_{X_f^{(m_M)}}K_{nm_1}\cdots K_{m_Mm_{M+1}}\cdots$$
$$\times T^{(m_{M+1})}\left(\frac{v^{\omega_{m_{M+1}}}}{F_{m_Mm_{M+1}}};\alpha_{m_Mm_{M+1}},\beta_{m_Mm_{M+1}}\right)K_0^{(nm_1\cdots m_{M+1})}(N_n,\ldots,N_{nm_1\cdots m_{M+1}};\alpha_n,\beta_n). \tag{6.12}$$

Note that we call (6.12) the $(M+1)$th remainder as it is the remainder after M iterations, much like the first remainder $R^{(n)}(N_n;\alpha_n,\beta_n)$ is the remainder after zero iterations. The $(M+1)$th iteration turns the $(M+1)$th remainder into the $(M+1)$th hyperterminant $K_{r_{M+1}}^{(nm_1\cdots m_{M+1})}$, leaving us with the $(M+2)$th remainder $R^{(nm_1\cdots m_{M+1})}$.

By minimising the remainder expression (6.12) in the variables $N_n,\ldots,N_{nm_1\cdots m_M}$, we can derive the optimal truncation scheme for an Mth level hyperasymptotic expansion. We use hyperterminant expressions similar to (6.7) (that is, the form that includes the original Y-functions) and proceed in a similar manner to §6 in both Howls (1997) and Olde Daalhuis (1998a), as well as Murphy (2001). The optimal truncation points derived are analogous to equations (6.3), (6.5), and those displayed in §3.3.4 in the respective papers and the error estimates will also have a similar, analogous form. We omit the algebra in favour of brief details as the calculations are very similar, just with our more complicated hyperterminants.

When estimating the remainder $R^{(nm_1\cdots m_M)}$, we end up with an expression that contains the term

$$\frac{\Gamma\left(\frac{N_n+q_n}{\omega_n}-\frac{N_{nm_1}+q_{m_1}}{\omega_{m_1}}\right)}{|kF_{nm_1}|^{\left(\frac{N_n+q_n}{\omega_n}-\frac{N_{nm_1}+q_{m_1}}{\omega_{m_1}}\right)}}\cdots\frac{\Gamma\left(\frac{N_{nm_1\cdots m_{M-1}}+q_{m_{M-1}}}{\omega_{m_{M-1}}}-\frac{N_{nm_1\cdots m_M}+q_{m_M}}{\omega_{m_M}}\right)}{|kF_{m_{M-1}m_M}|^{\left(\frac{N_{nm_1\cdots m_{M-1}}+q_{m_{M-1}}}{\omega_{m_{M-1}}}-\frac{N_{nm_1\cdots m_M}+q_{m_M}}{\omega_{m_M}}\right)}}$$
$$\times\frac{\Gamma\left(\frac{N_{nm_1\cdots m_M}+q_{m_M}}{\omega_{m_M}}-\frac{q_{m_{M+1}}}{\omega_{m_{M+1}}}\right)}{|kF_{m_Mm_{M+1}}|^{\left(\frac{N_{nm_1\cdots m_M}+q_{m_M}}{\omega_{m_M}}-\frac{q_{m_{M+1}}}{\omega_{m_{M+1}}}\right)}}$$

containing all of remainder's dependency on $N_n,\ldots,N_{nm_1\cdots m_M}$. Using Stirling's approximation on each gamma function leads to the minimisation conditions

$$\begin{aligned}|kF_{nm_1}|&=\frac{N_n+q_n}{\omega_n}-\frac{N_{nm_1}+q_{m_1}}{\omega_{m_1}},\\|kF_{m_1m_2}|&=\frac{N_{nm_1}+q_{m_1}}{\omega_{m_1}}-\frac{N_{m_1m_2}+q_{m_2}}{\omega_{m_2}},\\&\vdots\qquad\vdots\end{aligned}$$

$$\begin{aligned}
\left|kF_{m_{M-1}m_M}\right| &= \frac{N_{nm_1\cdots m_{M-1}}+q_{m_{M-1}}}{\omega_{m_{M-1}}} - \frac{N_{nm_1\cdots m_M}+q_{m_M}}{\omega_{m_M}},\\
\left|kF_{m_M m_{M+1}}\right| &= \frac{N_{nm_1\cdots m_M}+q_{m_M}}{\omega_{m_M}} - \frac{q_{m_{M+1}}}{\omega_{m_{M+1}}},
\end{aligned} \tag{6.13}$$

as $k \to \infty$. The system of equations (6.13) relates each truncation point with the next one, meaning that once we have a value for N_n, we have a value for all of them.

Summing each equation in (6.13) yields the expression

$$\begin{aligned}
\frac{N_n+q_n}{\omega_n} - \frac{q_{m_{M+1}}}{\omega_{m_{M+1}}} &= |kF_{nm_1}| + \cdots + \left|kF_{m_M m_{M+1}}\right|\\
&\Rightarrow N_n = \omega_n\left(|kF_{nm_1}| + \cdots + \left|kF_{m_M m_{M+1}}\right| + \frac{q_{m_{M+1}}}{\omega_{m_{M+1}}} - \frac{q_n}{\omega_n}\right).
\end{aligned}$$

As mentioned in the three aforementioned papers, the best choice for N_n is when all of the singulants have minimal absolute value. Therefore, we choose

$$N_n = \omega_n\left(\left|kF_{nm_1^*}\right| + \cdots + \left|kF_{m_M^* m_{M+1}^*}\right| + \frac{q_{m_{M+1}^*}}{\omega_{m_{M+1}^*}} - \frac{q_n}{\omega_n}\right),$$

where a * denotes the closest critical component in relation to the previous one (so $\chi_{m_1}^*$ is the closest critical component to χ_n, $\chi_{m_2}^*$ is the closest critical component to $\chi_{m_1}^*$, and so on). The quantity

$$\left|kF_{nm_1^*}\right| + \cdots + \left|kF_{m_M^* m_{M+1}^*}\right|$$

is then the shortest directed M-step path between critical components in the u^{ω_n}-plane, starting at χ_n. Note that the quantities $q_{m_{M+1}*}$ and $\omega_{m_{M+1}^*}$ are marked with a * since they are related to the critical component $\chi_{m_{M+1}^*}$.

The optimal truncation scheme for an Mth level hyperasymptotic expansion can thus be derived from (6.13) as

$$\begin{aligned}
N_n &= \omega_n\left(\left|kF_{nm_1^*}\right| + \cdots + \left|kF_{m_M^* m_{M+1}^*}\right| + \frac{q_{m_{M+1}^*}}{\omega_{m_{M+1}^*}} - \frac{q_n}{\omega_n}\right),\\
N_{nm_1} &= \max\left[0, \omega_{m_1}\left(\frac{N_n+q_n}{\omega_n} - \frac{q_{m_1}}{\omega_{m_1}} - |kF_{nm_1}|\right)\right],\\
N_{nm_1m_2} &= \max\left[0, \omega_{m_2}\left(\frac{N_{nm_1}+q_{m_1}}{\omega_{m_1}} - \frac{q_{m_2}}{\omega_{m_2}} - |kF_{m_1m_2}|\right)\right],\\
&\;\;\vdots \qquad \vdots\\
N_{nm_1\cdots m_M} &= \max\left[0, \omega_{m_M}\left(\frac{N_{nm_1\cdots m_{M-1}}+q_{m_1}}{\omega_{m_{M-1}}} - \frac{q_{m_M}}{\omega_{m_M}} - \left|kF_{m_{M-1}m_M}\right|\right)\right].
\end{aligned} \tag{6.14}$$

If a truncation point is calculated to be a non-integer, we round to the nearest integer. If a truncation point is negative, the maximum value function selects zero and the scheme terminates.

These expressions correctly reduce down to those in the existing literature for appropriate values of q_j and ω_j. When $\omega_j = 2$ for all j and all values of q_j are identical, we correctly recover the results in Howls (1997) and Olde Daalhuis (1998a), modulo (once again) rewriting the asymptotic series to skip over the zero terms. When $\omega_j = a \in \mathbb{N}$ for all j (that is, all critical components have the same general order) and all values of q_j are identical, we correctly recover the results from Murphy (2001).

We now have all the expressions required to compute the full hyperasymptotic series for an integral of type (5.1); we write it as the T-integral (5.2) so that it has the complete hyperasymptotic expansion (6.4) and then substitute in (5.13), (6.11), and (6.12) as required.

Computing these hyperterminants can be a very computationally expensive task, so it will be convenient to rewrite them further into a form that is more practically computable. In Olde Daalhuis (1998b), hyperterminants are rewritten as convergent series expansions that are easily computable to arbitrary precision. Although the hyperterminants (6.11) are more complicated than in Olde Daalhuis (1998b), we can follow the same method to derive similar convergent series representations for them.

6.2 Rewriting the Hyperterminants

The hyperterminants handled in Olde Daalhuis (1998b) are integrals of the form

$$\begin{aligned} F^{(1)}\left(z;\begin{matrix}M_0\\ \sigma_0\end{matrix}\right) &= \int_0^{[\pi-\theta_0]} \frac{dt_0\, e^{\sigma_0 t_0} t_0^{M_0-1}}{z-t_0}, \\ F^{(l+1)}\left(z;\begin{matrix}M_0,\dots,M_l\\ \sigma_0,\dots,\sigma_l\end{matrix}\right) &= \int_0^{[\pi-\theta_0]} \cdots \int_0^{[\pi-\theta_l]} \frac{dt_0\cdots dt_l\ e^{\sigma_0 t_0+\cdots+\sigma_l t_l} t_0^{M_0-1}\cdots t_l^{M_l-1}}{(z-t_0)(t_0-t_1)\cdots(t_{l-1}-t_l)}, \end{aligned} \tag{6.15}$$

where $[\phi] = \infty e^{i\phi}$ for $\phi \in \mathbb{R}$, $l \in \mathbb{N}_0$, and for $j \in \{0,\dots,l\}$ we have $z, M_j, \sigma_j \in \mathbb{C}$ and $\theta_j = \arg(\sigma_j)$. We also have the conditions $\mathrm{Re}(M_j) > 1$ and $\sigma_j \neq 0$, and define $F^{(0)}(z) = 1$. Equations (6.15) are equations (2.2b) and (2.2c) in Olde Daalhuis (1998b), with the contents of Theorems 2 and 3 giving the convergent series representations of these integrals.

The goal of this section is to derive similar convergent series expansions for the similar but more general integrals

$$F^{(1)}\left(z_0;\begin{matrix}M_0\\ \sigma_0\end{matrix}\right) = \int_0^{[\pi-\theta_0]} \frac{dt_0\, e^{\sigma_0 t_0^{\omega_0}}\, t_0^{\omega_0 M_0-1}}{z_0-t_0^{\omega_0}},$$

$$F^{(l+1)}\left(z_0,\ldots,z_l;\begin{matrix}M_0,\ldots,M_l\\ \sigma_0,\ldots,\sigma_l\end{matrix}\right) \tag{6.16}$$
$$=\int_0^{[\pi-\theta_0]}\cdots\int_0^{[\pi-\theta_l]}\frac{dt_0\cdots dt_l\; e^{\sigma_0 t_0^{\omega_0}+\cdots+\sigma_l t_l^{\omega_l}}\, t_0^{\omega_0 M_0-1}\cdots t_l^{\omega_l M_l-1}}{(z_0-t_0^{\omega_0})(z_1 t_0^{\omega_0}-t_1^{\omega_1})\cdots(z_l t_{l-1}^{\omega_{l-1}}-t_l^{\omega_l})},$$

which will be referred to as 'F-functions', with $\omega_j\in\mathbb{N}$ and $z_j\in\mathbb{C}$ for all j, in addition to the same definitions and conditions as for (6.15). This derivation will closely follow the work done in and the results of Olde Daalhuis (1998b). After obtaining the desired convergent series expansions for (6.16), we will write the hyperterminants (6.11) in terms of these F-functions.

It is possible to write the hyperterminants (6.11) in terms of (6.15), but the required t_j transformations and subsequent M_j and σ_j definitions become increasingly convoluted to the point of ridiculous. However, it is also possible to write (6.16) in terms of (6.15) using the transformations and definitions

$$\tilde{z}=z_0,\quad \tilde{t}_0=t_0^{\omega_0},\quad \tilde{t}_j=\frac{t_j^{\omega_j}}{z_j},\quad \tilde{\sigma}_j=\sigma_j z_j\quad \text{for } j\in\{1,\ldots,l\},$$

where the presence and lack of a tilde denotes a quantity belonging to (6.15) and (6.16) respectively. This enables us to write

$$F^{(1)}_{(6.16)}\left(z_0;\begin{matrix}M_0\\ \sigma_0\end{matrix}\right)=\frac{1}{\omega_0}F^{(1)}_{(6.15)}\left(z_0;\begin{matrix}M_0\\ \sigma_0 z_0\end{matrix}\right), \tag{6.17}$$
$$F^{(l+1)}_{(6.16)}\left(z_0,\ldots,z_l;\begin{matrix}M_0,\ldots,M_l\\ \sigma_0,\ldots,\sigma_l\end{matrix}\right)=\frac{z_1^{M_1-1}\cdots z_l^{M_l-1}}{\omega_0\cdots\omega_l}F^{(l+1)}_{(6.15)}\left(z_0,\ldots,z_l;\begin{matrix}M_0,\ldots,M_l\\ \sigma_0 z_0,\ldots,\sigma_l z_l\end{matrix}\right),$$

with the transformed integration limit imposing the relationships

$$\pi-\tilde{\theta}_0=\omega_0(\pi-\theta_0),\quad \pi-\tilde{\theta}_j-\arg(z_j)=\omega_j(\pi-\theta_j),\quad \text{for}\quad j\in\{1,\ldots,l\}.$$

Despite the relation (6.17), we explicitly re-derive the convergent series representations for the directly applicable integrals (6.16) in order to avoid potential trouble when dividing by ω_j in the arguments of σ_j. Convergence of these series is not explicitly proven here as this is guaranteed by (6.17); the series representation for integrals (6.16) is just a constant multiple of the series representation for integrals (6.15), dictated by (6.17).

The derivation of the convergent series representation requires the two non-standard integral identities

$$\frac{1}{\Gamma(p+1)}\int_0^{[\theta_0]}ds_0\, e^{-z_0 s_0}s_0^p(s_0+\sigma_0)^{-M_0-p}=\sigma_0^{1-M_0}U(p+1,2-M_0,\sigma_0 z_0) \tag{6.18}$$

and

$$\int_0^{[\theta_1]} ds_1\, s_1^{p+q}(s_1+\sigma_1)^{-M_1-q}(s_1z_1-\sigma_0)^{-M_0-p} \tag{6.19}$$
$$= e^{(M_0+p)\pi i}\frac{\sigma_1^{1+p-M_1}}{\sigma_0^{M_0+p}}\frac{\Gamma(p+q+1)\Gamma(M_0+M_1-1)}{\Gamma(M_0+M_1+p+q)}\,{}_2F_1\left(\begin{matrix}M_0+p,p+q+1\\ M_0+M_1+p+q\end{matrix};1+\frac{\sigma_1z_1}{\sigma_0}\right).$$

Identity (6.18) is obtained simply by pulling out σ_0 from the bracket term in the integral and making the substitution $\tilde{s}_0 = s_0\sigma_0$; this gives us

$$\frac{\sigma_0^{1-M_0}}{\Gamma(p+1)}\int_0^\infty d\tilde{s}_0\, e^{-z_0\sigma_0\tilde{s}_0}\tilde{s}_0^p(\tilde{s}_0+1)^{-M_0-p} = \sigma_0^{1-M_0}U(p+1,2-M_0,\sigma_0z_0),$$

evaluating directly to the confluent hypergeometric function $U(a,b,z)$ given in (6.18) via the integral representation 13.4.4 in NIST. Note that in this section, we take $-1=e^{-i\pi}$, so that $(-1)^A = e^{-iA\pi}$.

Identity (6.19) requires slightly more work; we make the substitutions

$$s_1 = x\sigma_1 \Rightarrow ds_1 = \sigma_1 dx \quad \text{and} \quad x = \frac{t}{1-t} \Rightarrow dx = \frac{dt}{(1-t)^2},$$

so that the integral becomes

$$\sigma_1^{1-M_0-M_1}z_1^{-M_0-p}\int_0^\infty dx\, x^{p+q}(x+1)^{-M_1-q}\left(x-\frac{\sigma_0}{\sigma_1z_1}\right)^{-M_0-p}$$
$$= \sigma_1^{1-M_0-M_1}z_1^{-M_0-p}\int_0^1 dt\, t^{p+q}(1-t)^{M_1-p-2}\left(\frac{1}{1-t}\left(t-(1-t)\frac{\sigma_0}{\sigma_1z_1}\right)\right)^{-M_0-p}$$
$$= e^{(M_0+p)i\pi}\sigma_1^{1-M_1+p}\sigma_0^{-M_0-p}\int_0^1 dt\, t^{p+q}(1-t)^{M_0+M_1-2}\left(1-t\left(1+\frac{\sigma_1z_1}{\sigma_0}\right)\right)^{-M_0-p},$$

evaluating directly into the hypergeometric function ${}_2F_1(a,b;c;z)$ given in (6.19) via the integral representation 15.6.1 in NIST (note q is just a meaningless variable and is not related to critical component codimension). With these identities, we are able to begin rewriting the hyperterminants.

By writing

$$\frac{1}{z_0-t_0^{\omega_0}} = \int_0^{[\theta_0]} ds_0\, e^{-s_0(z_0-t_0^{\omega_0})} \quad \text{and} \quad \frac{1}{z_lt_{l-1}^{\omega_{l-1}}-t_l^{\omega_l}} = \int_0^{[\theta_l]} ds_l\, e^{-s_l(z_lt_{l-1}^{\omega_{l-1}}-t_l^{\omega_l})},$$

we are able to write the integrals (6.16) by evaluating the t-integrals as

$$F^{(1)}\left(z_0;\begin{matrix}M_0\\ \sigma_0\end{matrix}\right) = \frac{\Gamma(M_0)}{\omega_0}\int_0^{[\theta_0]} ds_0\, e^{-z_0s_0}(-s_0-\sigma_0)^{-M_0}$$

$$F^{(l+1)}\left(z_0,\ldots,z_l;\begin{matrix}M_0,\ldots,M_l\\ \sigma_0,\ldots,\sigma_l\end{matrix}\right)=\frac{\Gamma(M_0)\cdots\Gamma(M_l)}{\omega_0\cdots\omega_l}\int_0^{[\theta_0]}\cdots\int_0^{[\theta_l]}ds_0\cdots ds_l\,\cdots \tag{6.20}$$
$$\times e^{-z_0s_0}(s_1z_1-s_0-\sigma_0)^{-M_0}\cdots(s_lz_l-s_{l-1}-\sigma_{l-1})^{-M_{l-1}}(-s_l-\sigma_l)^{-M_l}.$$

Using (6.18) with $p=0$, we can evaluate $F^{(1)}$ as

$$F^{(1)}\left(z_0;\begin{matrix}M_0\\ \sigma_0\end{matrix}\right)=\frac{e^{iM_0\pi}\sigma_0^{1-M_0}\Gamma(M_0)}{\omega_0}U(1,2-M_0,\sigma_0z_0). \tag{6.21}$$

To derive the expression for $F^{(l+1)}$, we start with deriving one for $F^{(2)}$, which can be written using (6.20) as

$$F^{(2)}\left(z_0,z_1;\begin{matrix}M_0,M_1\\ \sigma_0,\sigma_1\end{matrix}\right)$$
$$=\frac{\Gamma(M_0)\Gamma(M_1)}{\omega_0\omega_1}\int_0^{[\theta_0]}\int_0^{[\theta_1]}ds_0ds_1\,e^{-z_0s_0}(s_1z_1-s_0-\sigma_0)^{-M_0}(-s_1-\sigma_1)^{-M_1}.$$

Rewriting the first bracket in the integrand as

$$(s_1z_1-s_0-\sigma_0)^{-M_0}=\left(\frac{(s_0+\sigma_0)(s_1z_1-\sigma_0)}{\sigma_0}\left(1-\frac{s_0s_1z_1}{(s_0+\sigma_0)(s_1z_1-\sigma_0)}\right)\right)^{-M_0} \tag{6.22}$$
$$=\sum_{p=0}^{\infty}\frac{\Gamma(M_0+p)}{\Gamma(M_0)\Gamma(p+1)}\frac{\sigma_0^{M_0}(s_0s_1z_1)^p}{(s_0+\sigma_0)^{M_0+p}(s_1z_1-\sigma_0)^{M_0+p}}$$

along with using identity (6.18) allows us to write $F^{(2)}$ as

$$F^{(2)}\left(z_0,z_1;\begin{matrix}M_0,M_1\\ \sigma_0,\sigma_1\end{matrix}\right)=\sum_{p=0}^{\infty}\frac{e^{iM_1\pi}\sigma_0z_1^p\Gamma(M_0+p)\Gamma(M_1)}{\omega_0\omega_1}U(p+1,2-M_0,\sigma_0z_0)\cdots$$
$$\times\int_0^{[\theta_1]}ds_1\,s_1^p(s_1+\sigma_1)^{-M_1}(s_1z_1-\sigma_0)^{-M_0-p}. \tag{6.23}$$

Applying identity (6.19) with $q=0$ to (6.23) yields the desired expression

$$F^{(2)}\left(z_0,z_1;\begin{matrix}M_0,M_1\\ \sigma_0,\sigma_1\end{matrix}\right)$$
$$=\sum_{p=0}^{\infty}\frac{e^{(M_0+M_1+p)i\pi}\sigma_0^{1-M_0-p}\sigma_1^{1-M_1+p}z_1^p}{\omega_0\omega_1}\frac{\Gamma(M_0+p)\Gamma(M_1)\Gamma(p+1)\Gamma(M_0+M_1-1)}{\Gamma(M_0+M_1+p)}\cdots$$
$$\times\,{}_2F_1\left(\begin{matrix}M_0+p,p+1\\ M_0+M_1+p\end{matrix};1+\frac{\sigma_1z_1}{\sigma_0}\right)U(p+1,2-M_0,\sigma_0z_0), \tag{6.24}$$

which - comparing against (3.2) and (3.3b) in Olde Daalhuis (1998b) - we can see satisfies (6.17).

Moving on to the general hyperterminant $F^{(l+1)}$, we substitute (6.22) into the integral

(6.20) and then use (6.18) to obtain

$$F^{(l+1)}\left(z_0,\ldots,z_l;\begin{matrix}M_0,\ldots,M_l\\ \sigma_0,\ldots,\sigma_l\end{matrix}\right) \tag{6.25}$$
$$=\sum_{p=0}^{\infty}\frac{\sigma_0 z_1^p\Gamma(M_0+p)\Gamma(M_1)\cdots\Gamma(M_l)}{\omega_0\cdots\omega_l}U(p+1,2-M_0,\sigma_0 z_0)\int_0^{[\theta_1]}\cdots\int_0^{[\theta_l]}ds_1\cdots ds_l\cdots$$
$$\times\, s_1^p(s_1z_1-\sigma_0)^{-M_0-p}(s_2z_2-s_1-\sigma_1)^{-M_1}\cdots(s_lz_l-s_{l-1}-\sigma_{l-1})^{-M_{l-1}}(-s_l-\sigma_l)^{-M_l}.$$

We proceed by writing

$$F^{(l+1)}\left(z_0,\ldots,z_l;\begin{matrix}M_0,\ldots,M_l\\ \sigma_0,\ldots,\sigma_l\end{matrix}\right)=\sum_{p=0}^{\infty}A^{(l+1)}\left(p;z_1,\ldots,z_l;\begin{matrix}M_0,\ldots,M_l\\ \sigma_0,\ldots,\sigma_l\end{matrix}\right)U(p+1,2-M_0,\sigma_0z_0) \tag{6.26}$$

and compare this against expression (6.25) to obtain

$$A^{(l+1)}\left(p;z_1,\ldots,z_l;\begin{matrix}M_0,\ldots,M_l\\ \sigma_0,\ldots,\sigma_l\end{matrix}\right)$$
$$=\frac{\sigma_0 z_1^p\Gamma(M_0+p)\Gamma(M_1)\cdots\Gamma(M_l)}{\omega_0\cdots\omega_l}\int_0^{[\theta_1]}\cdots\int_0^{[\theta_l]}ds_1\cdots ds_l\cdots \tag{6.27}$$
$$\times\, s_1^p(s_1z_1-\sigma_0)^{-M_0-p}(s_2z_2-s_1-\sigma_1)^{-M_1}\cdots(s_lz_l-s_{l-1}-\sigma_{l-1})^{-M_{l-1}}(-s_l-\sigma_l)^{-M_l}.$$

Note that $A^{(1)}$ is given by

$$A^{(1)}\left(p;\begin{matrix}M_0\\ \sigma_0\end{matrix}\right)=\delta_{p,0}\frac{e^{iM_0\pi}\sigma_0^{1-M_0}\Gamma(M_0)}{\omega_0}, \tag{6.28}$$

with the Kronecker delta ensuring that only the zeroth term is retained as required by (6.21), and $A^{(2)}$ can be directly identified from (6.24). Using a similar expression to (6.22) for the bracket $(s_2z_2-s_1-\sigma_1)^{-M_1}$, we can write (6.27) as

$$A^{(l+1)}\left(p;z_1,\ldots,z_l;\begin{matrix}M_0,\ldots,M_l\\ \sigma_0,\ldots,\sigma_l\end{matrix}\right)$$
$$=\sum_{q=0}^{\infty}\frac{\sigma_0\sigma_1^{M_1-1}z_1^p\Gamma(M_0+p)}{\omega_0\Gamma(q+1)}\int_0^{[\theta_1]}ds_1\, s_1^{p+q}(s_1+\sigma_1)^{-M_1-q}(s_1z_1-\sigma_0)^{-M_0-p}\cdots$$
$$\times\,\frac{\sigma_1 z_2^q\Gamma(M_1+q)\Gamma(M_2)\cdots\Gamma(M_l)}{\omega_1\cdots\omega_l}\int_0^{[\theta_2]}\cdots\int_0^{[\theta_l]}ds_2\cdots ds_l\cdots \tag{6.29}$$
$$\times\, s_2^q(s_2z_2-\sigma_1)^{-M_1-q}(s_3z_3-s_2-\sigma_2)^{-M_2}\cdots(s_lz_l-s_{l-1}-\sigma_{l-1})^{-M_{l-1}}(-s_l-\sigma_l)^{-M_l}$$

and observe that the final two lines can be identified as (6.27) with slightly different arguments. Applying identity (6.22) to the s_1 integral of (6.29) thus yields the recurrence relation

$$
\begin{aligned}
&A^{(l+1)}\left(p; z_1, \ldots, z_l; \begin{matrix} M_0, \ldots, M_l \\ \sigma_0, \ldots, \sigma_l \end{matrix}\right) \\
&= \sum_{q=0}^{\infty} \frac{e^{(M_0+p)\pi i}\sigma_0^{1-M_0-p}\sigma_1^p z_1^p}{\omega_0} \frac{\Gamma(M_0+p)\Gamma(M_0+M_1-1)\Gamma(p+q+1)}{\Gamma(M_0+M_1+p+q)\Gamma(q+1)} \cdots \\
&\times {}_2F_1\left(\begin{matrix} M_0+p, p+q+1 \\ M_0+M_1+p+q \end{matrix}; 1+\frac{\sigma_1 z_1}{\sigma_0}\right) A^{(l)}\left(q; z_2, \ldots, z_l; \begin{matrix} M_1, \ldots, M_l \\ \sigma_1, \ldots, \sigma_l \end{matrix}\right),
\end{aligned}
\tag{6.30}
$$

which combined with (6.26) will also satisfy (6.17). Therefore, (6.26) will converge for all l under similar conditions to those in Olde Daalhuis (1998b) due to (6.17). We now move on to expressing the hyperterminants (6.11) in terms of the F-Functions (6.16), enabling us to compute them numerically using the results derived above.

The first hyperterminant (6.10) in terms of the F-functions is

$$
\begin{aligned}
&K_{r_1}^{(nm_1)}(N_n; \alpha_n, \beta_n) \\
&= \sum_{K_1=0}^{\omega_n-1} \frac{\omega_{m_1} G_1^{(n)}(K_1; \alpha_n, \beta_n)}{2\pi i \omega_n k^{\frac{N_n+K_1}{\omega_n}-1} F_{nm_1}^{P_1^{(n)}(K_1, r_1-N_{nm_1})-1}} F^{(1)}\left(kF_{nm_1}; \begin{matrix} P_1^{(n)}(K_1, r_1-N_{nm_1}) \\ -1 \end{matrix}\right),
\end{aligned}
\tag{6.31}
$$

with the Mth hyperterminant (6.11) written as

$$
\begin{aligned}
&K_{r_M}^{(nm_1\cdots m_M)}(N_n, \ldots, N_{nm_1\cdots m_M}; \alpha_n, \beta_n) = \sum_{K_1=0}^{\omega_n-1} \cdots \sum_{K_M=0}^{\omega_{m_{M-1}}-1} \frac{\omega_{m_M}}{\omega_n (2\pi i)^M k^{\frac{N_n+K_1}{\omega_n}-1}} \cdots \\
&\times \frac{G_1^{(n)}(K_1; \alpha_n, \beta_n) G_2^{(n)}(K_2) \cdots G_M^{(n)}(K_M)}{F_{nm_1}^{P_1^{(n)}(K_1, K_2)} \cdots F_{m_{M-2}m_{M-1}}^{P_{M-1}^{(n)}(K_{M-1}, K_M)} F_{m_{M-1}m_M}^{P_M^{(n)}(K_M, r_M - N_{nm_1\cdots m_M})-1}} \cdots \\
&\times F^{(M)}\left(kF_{nm_1}, \frac{F_{m_1m_2}}{F_{nm_1}}, \ldots, \frac{F_{m_{M-1}m_M}}{F_{m_{M-2}m_{M-1}}}; \cdots\right. \\
&\qquad \left.\cdots \begin{matrix} P_1^{(n)}(K_1, K_2)+1, & \ldots, & P_{M-1}^{(n)}(K_{M-1}, K_M)+1, & P_M^{(n)}(K_M, r_M - N_{nm_1\cdots m_M}) \\ -1, & \ldots, & -1, & -1 \end{matrix}\right).
\end{aligned}
\tag{6.32}
$$

For clarification we remark that in (6.32), the F-function used is $F^{(l+1)}$ (as in (6.16)), but with $l = M-1$. We are now able to compute the complete hyperasymptotic expansion around a critical component of general dimension and general constant order to as many levels as we wish.

The next chapter contains examples that demonstrate the new theory developed in §§5 and 6. They will be simple in order to showcase the theory without worrying about any subtleties that force us to alter our approach. We compute the value of the example integrals and then compare these results to the numerical values of the relevant hyperasymptotic expansion at various levels.

7 Examples

In this chapter, we present examples to demonstrate the new theory developed in §§5 and 6.

We recall that the integrals we are studying are (5.1) and (5.2), given by

$$I^{(n)}(k;\alpha_n,\beta_n) = \int_{S_n(\alpha_n,\beta_n)} dz\, g(\boldsymbol{z}) e^{-kf(\boldsymbol{z})} = \frac{e^{-kf_n}}{k^{\frac{q_n}{\omega_n}}} T^{(n)}(k;\alpha_n,\beta_n),$$

$$T^{(n)}(k;\alpha_n,\beta_n) = k^{\frac{q_n}{\omega_n}} \int_{S_n(\alpha_n,\beta_n)} d\boldsymbol{z}\, g(\boldsymbol{z}) e^{-k(f(\boldsymbol{z})-f_n)}.$$

To notationally differentiate the formal series $T^{(n)}(k;\alpha_n,\beta_n)$ from the integral (5.2) that it represents, we denote the example integrals by $T_I^{(n)}(k;\alpha_n,\beta_n)$. Hence, in the new notation we have that

$$k^{\frac{q_n}{\omega_n}} \int_{S_n(\alpha_n,\beta_n)} d\boldsymbol{z}\, g(\boldsymbol{z}) e^{-k(f(\boldsymbol{z})-f_n)} =: T_I^{(n)}(k;\alpha_n,\beta_n) \sim T^{(n)}(k;\alpha_n,\beta_n) = \sum_{r=0}^{\infty} \frac{T_r^{(n)}(\alpha_n,\beta_n)}{k^{\frac{r}{\omega_n}}}.$$

In order to achieve sufficient decimal accuracy to properly showcase the effects of higher level hyperasymptotic expansions, we may choose different values of $|k|$ for each n. To this end, we will write k_n when computing during examples. As a final point of clarification before moving on to the first example, we recall the standard notation

$$\bar{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\} = (-\infty,\infty)$$

for the affinely extended real numbers $\bar{\mathbb{R}}$. We make this definition explicit to avoid potential confusion, as it will be used for brevity in some examples.

7.1 Example 1 - Critical component Hyperasymptotic Expansion

7.1.1 Set-up, Coefficients, and Level Zero Expansion

In this example, we will focus on computing high level hyperasymptotic expansions.

We look at the integral

$$I^{(n)}(k;\alpha_n,\beta_n) = \int_{S_n(\alpha_n,\beta_n)} dz_1\, dz_2\, e^{-z_2^2} e^{-k\left(\frac{15}{28}z_1^7 - 5z_1^6 + 18z_1^5 - 30z_1^4 + 20z_1^3\right)}, \tag{7.1}$$

where $\boldsymbol{z} = (z_1, z_2) \in \mathbb{C}^2$, $S_n \subset \mathbb{C}^2$, and $f, g : \mathbb{C}^2 \to \mathbb{C}$ with

$$f(\boldsymbol{z}) = \frac{15}{28}z_1^7 - 5z_1^6 + 18z_1^5 - 30z_1^4 + 20z_1^3,$$

$$g(\boldsymbol{z}) = e^{-z_2^2}.$$

The critical point set $C(f)$ is given by

$$C(f) = \left\{ \boldsymbol{z} \in \mathbb{C}^2 \mid (z_1 = 0) \vee (z_1 = 2) \right\}$$

(where $\vee$ is the logical or symbol) and we define the critical components

$$\begin{aligned}
\chi_1 &= \left\{ \boldsymbol{z} \in \mathbb{C}^2 \mid z_1 = 2 \right\} = (2, z_2),\\
\chi_2 &= \left\{ \boldsymbol{z} \in \mathbb{C}^2 \mid z_1 = 0 \right\} = (0, z_2),
\end{aligned}$$

so that

$$\chi_1 \cup \chi_2 = C(f) \quad \text{and} \quad \chi_1 \cap \chi_2 = \emptyset.$$

We can see that both critical components are complex critical lines in $\mathbb{C}^2$, as there is one dimension of freedom in their definitions above. Additionally, it can be shown that χ_1 and χ_2 are constant order 5 and 3 critical components respectively according to Definition 2. Two and three-dimensional contour plots of $f(\boldsymbol{z})$ in the z_1-plane are given in Figures 25 and 24 respectively and a summary of the important quantities for this problem are given below:

$$\begin{aligned}
f_1 = \frac{32}{7}, \quad \mu_1 = 1, \quad q_1 = 1, \quad \omega_1 = 5, \quad F_{12} = \frac{32}{7};\\
f_2 = 0, \quad \mu_2 = 1, \quad q_2 = 1, \quad \omega_2 = 3, \quad F_{21} = -\frac{32}{7}.
\end{aligned}$$

The integration surfaces S_j will be of the form $(V_{a_j}, V_{b_j}) \times \bar{\mathbb{R}}$, so that z_2 runs between real infinities and z_1 runs between asymptotic valleys V_{a_j} and V_{b_j}. These valleys are defined according to Figures 24 and 25. Note that with these integration surfaces, the z_2 variable can be entirely removed from the problem by integrating with respect to z_2, since

$$\int_{-\infty}^{\infty} dz_2\, e^{-z_2^2} = \sqrt{\pi}.$$

This was intentional to keep the example relatively simple, but we will not use this fact to re-frame the problem in a one-dimensional setting - where the critical components are simply isolated critical points - as this would defeat the point of the example.

For each critical component, we will compute the asymptotic coefficients $T_r^{(n)}(\alpha_n, \beta_n)$ using (5.12), use these to write down the zeroth level expansion (5.13), and then compare these expansion to the relevant numerical integral. The coefficients are given by (5.13) as

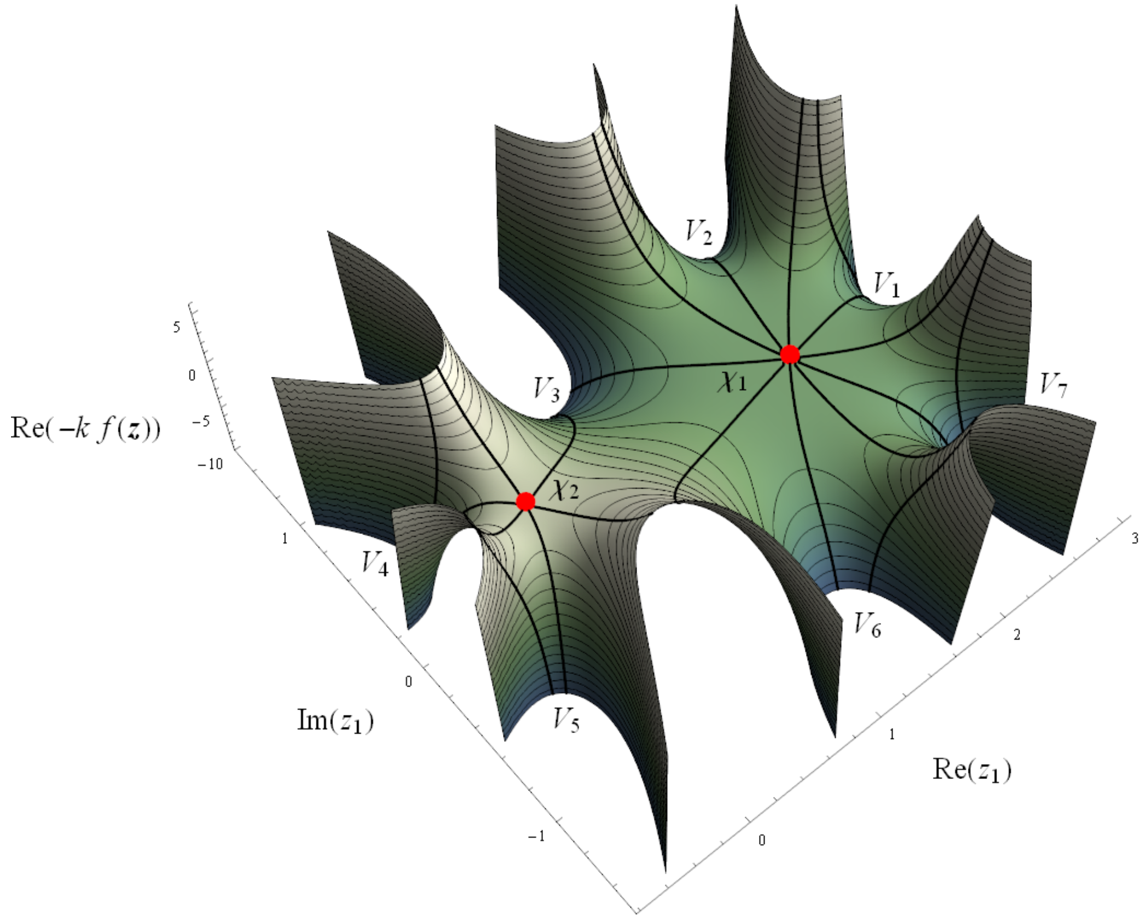


Figure 24: Three dimensional contour plot of $\mathrm{Re}(-kf(\boldsymbol{z}))$ in complex z_1 space, with $k = 1 - i$. This contour plot is entirely unaffected by the variable z_2, since $f(\boldsymbol{z})$ is not a function of z_2.

$$T_r^{(1)}(\alpha_1, \beta_1) = \left(e^{\frac{2\pi i \beta_1 (r+1)}{5}} - e^{\frac{2\pi i \alpha_1 (r+1)}{5}}\right) \frac{\Gamma\left(\frac{r+1}{5}\right)}{5} \operatorname*{Res}_{\boldsymbol{z}\in\chi_1} \left(\frac{e^{-z_2^2}}{\left(f(\boldsymbol{z}) - \frac{32}{7}\right)^{\frac{r+1}{5}}}\right),$$

$$T_r^{(2)}(\alpha_2, \beta_2) = \left(e^{\frac{2\pi i \beta_2 (r+1)}{3}} - e^{\frac{2\pi i \alpha_2 (r+1)}{3}}\right) \frac{\Gamma\left(\frac{r+1}{3}\right)}{3} \operatorname*{Res}_{\boldsymbol{z}\in\chi_2} \left(\frac{e^{-z_2^2}}{\left(f(\boldsymbol{z}) - 0\right)^{\frac{r+1}{3}}}\right),$$

where $f(\boldsymbol{z})$ remains unsubstituted and $f_2 = 0$ remains explicit for greater visual clarity. Both critical components are defined by $z_1 = a$ and $z_2 \in \mathbb{C}$, for $a \in \mathbb{C}$, and so as per the residue discussion in §5.2, we take the residue in z_1 and can integrate in z_2 at any stage. Thus, we calculate the coefficients using the expressions

$$T_r^{(1)}(\alpha_1, \beta_1) = \left(e^{\frac{2\pi i \beta_1 (r+1)}{5}} - e^{\frac{2\pi i \alpha_1 (r+1)}{5}}\right) \frac{\Gamma\left(\frac{r+1}{5}\right)}{5} \int_{-\infty}^{\infty} dz_2 \operatorname*{Res}_{z_1=2} \left(\frac{e^{-z_2^2}}{\left(f(\boldsymbol{z}) - \frac{32}{7}\right)^{\frac{r+1}{5}}}\right),$$

$$T_r^{(2)}(\alpha_2, \beta_2) = \left(e^{\frac{2\pi i \beta_2 (r+1)}{3}} - e^{\frac{2\pi i \alpha_2 (r+1)}{3}}\right) \frac{\Gamma\left(\frac{r+1}{3}\right)}{3} \int_{-\infty}^{\infty} dz_2 \operatorname*{Res}_{z_1=0} \left(\frac{e^{-z_2^2}}{\left(f(\boldsymbol{z}) - 0\right)^{\frac{r+1}{3}}}\right),$$

with the first few coefficients given explicitly as

$$
\begin{aligned}
\left\{T_j^{(1)}(\alpha_1,\beta_1)\right\}_{j\in\mathbb{N}_0} &= \Bigg\{\left(e^{\frac{2\pi i\beta_1}{5}}-e^{\frac{2\pi i\alpha_1}{5}}\right)\frac{\Gamma(\frac{1}{5})\sqrt{\pi}}{5\times 3^{\frac{1}{5}}}, -\left(e^{\frac{4\pi i\beta_1}{5}}-e^{\frac{4\pi i\alpha_1}{5}}\right)\frac{\Gamma(\frac{2}{5})\sqrt{\pi}}{15\times 3^{\frac{2}{5}}},\\
&\quad \left(e^{\frac{6\pi i\beta_1}{5}}-e^{\frac{6\pi i\alpha_1}{5}}\right)\frac{\Gamma(\frac{3}{5})19\sqrt{\pi}}{420\times 3^{\frac{3}{5}}}, -\left(e^{\frac{8\pi i\beta_1}{5}}-e^{\frac{8\pi i\alpha_1}{5}}\right)\frac{\Gamma(\frac{4}{5})11\sqrt{\pi}}{315\times 3^{\frac{4}{5}}}, 0, \dots\Bigg\},\\
\left\{T_j^{(2)}(\alpha_2,\beta_2)\right\}_{j\in\mathbb{N}_0} &= \Bigg\{\left(e^{\frac{2\pi i\beta_2}{3}}-e^{\frac{2\pi i\alpha_2}{3}}\right)\frac{\Gamma(\frac{1}{3})\sqrt{\pi}}{3\times 2^{\frac{2}{3}}5^{\frac{1}{3}}}, \left(e^{\frac{4\pi i\beta_2}{3}}-e^{\frac{4\pi i\alpha_2}{3}}\right)\frac{\Gamma(\frac{2}{3})\sqrt{\pi}}{6\times 2^{\frac{1}{3}}5^{\frac{2}{3}}}, 0,\\
&\quad \left(e^{\frac{8\pi i\beta_2}{3}}-e^{\frac{8\pi i\alpha_2}{3}}\right)\frac{\Gamma(\frac{4}{3})59\sqrt{\pi}}{1800\times 2^{\frac{2}{3}}5^{\frac{1}{3}}}, \left(e^{\frac{10\pi i\beta_2}{3}}-e^{\frac{10\pi i\alpha_2}{3}}\right)\frac{\Gamma(\frac{5}{3})4979\sqrt{\pi}}{201600\times 2^{\frac{1}{3}}5^{\frac{2}{3}}}, \dots\Bigg\}.
\end{aligned}
\tag{7.2}
$$

Recall that the r-th coefficient will be identically zero if r is a solution to the congruence equation (5.14), reproduced here for convenience as

$$(\beta_n-\alpha_n)r \equiv -q_n(\beta_n-\alpha_n) \mod \omega_n.$$

For χ_1 the only solution is

$$r\equiv 4 \mod 5 \Rightarrow r = 4+5x, \quad x\in\mathbb{Z}$$

for all values of α_n and β_n, and for χ_2 the only solution is

$$r\equiv 2 \mod 3 \Rightarrow r = 2+3x, \quad x\in\mathbb{Z}$$

for all values of α_n and β_n. This means that - counting zero as the 'first' coefficient - every fifth and third coefficient will be zero in the expansions around χ_1 and χ_2 respectively.

We now compare the asymptotic expansion generated by these coefficients against the numerical value of the integrals of interest

$$
\begin{aligned}
T_I^{(1)}(k_1;\alpha_1,\beta_1) &= k_1^{\frac{1}{5}}\int_{-\infty}^{\infty}dz_2\int_{V_{a_1}}^{V_{b_1}}dz_1 e^{-z_2^2}e^{-k_1\left(f(\boldsymbol{z})-\frac{32}{7}\right)},\\
T_I^{(2)}(k_2;\alpha_2,\beta_2) &= k_2^{\frac{1}{3}}\int_{-\infty}^{\infty}dz_2\int_{V_{a_2}}^{V_{b_2}}dz_1 e^{-z_2^2}e^{-k_2(f(\boldsymbol{z})-0)}.
\end{aligned}
$$

In accordance with Figure 25, we can see that $a_1,b_1\in\{1,2,3,6,7\}$ corresponds directly to $\alpha_1,\beta_1\in\{0,1,2,3,4\}$ and $a_2,b_2\in\{3,4,5\}$ corresponds directly to $\alpha_2,\beta_2\in\{0,1,2\}$, so that when compared to Figure 16,

$$
\begin{aligned}
\{V_1,V_2,V_3,V_6,V_7\} &= \{V_{1,0},V_{1,1},V_{1,2},V_{1,3},V_{1,4}\},\\
\{V_3,V_4,V_5\} &= \{V_{2,0},V_{2,1},V_{2,2}\}.
\end{aligned}
$$

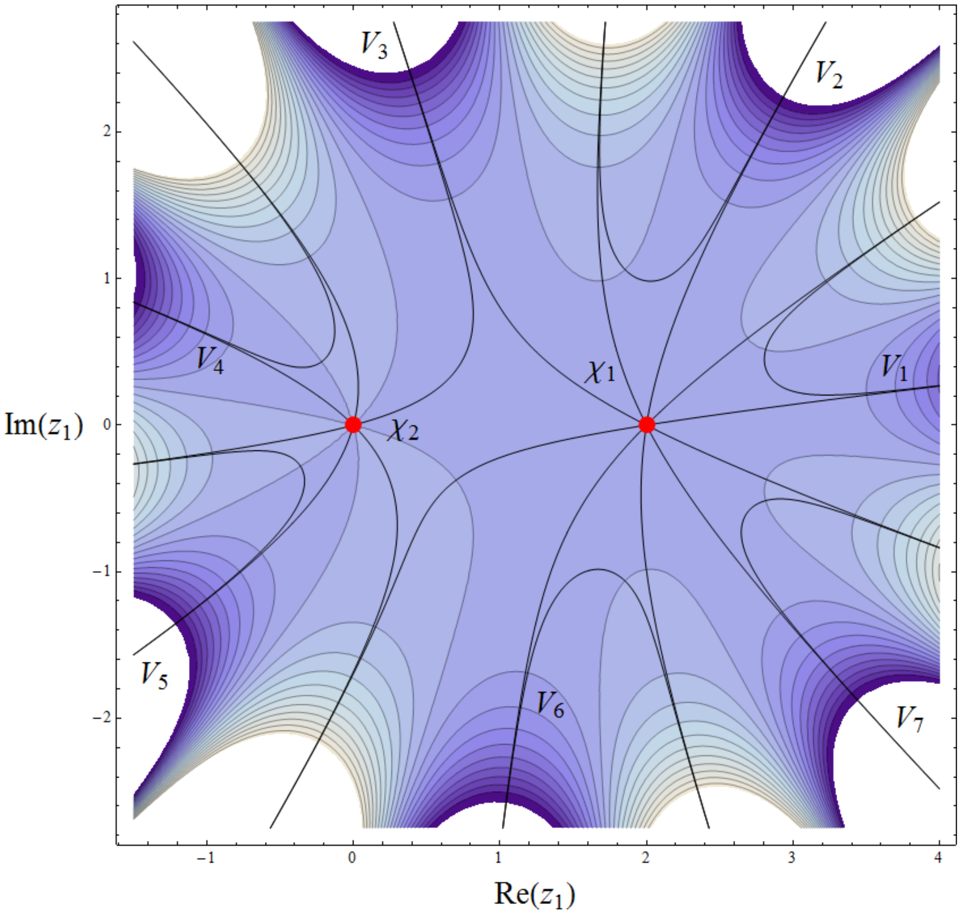


Figure 25: Contour plot of $\mathrm{Re}(-kf(\boldsymbol{z}))$ in complex z_1 space, with $k = 1 - i$. This contour plot is entirely unaffected by the variable z_2, since $f(\boldsymbol{z})$ is not a function of z_2.

Since there are ten and three possible unique integration surfaces through χ_1 and χ_2 respectively, we will focus only on the specific integrals

$$\begin{aligned}
T_I^{(1)}(k_1;1,0) &= k_1^{\frac{1}{5}} \int_{-\infty}^{\infty} dz_2 \int_{V_{1,1}}^{V_{1,0}} dz_1 e^{-z_2^2} e^{-k\left(f(\boldsymbol{z})-\frac{32}{7}\right)}, \\
T_I^{(1)}(k_1;2,0) &= k_1^{\frac{1}{5}} \int_{-\infty}^{\infty} dz_2 \int_{V_{1,2}}^{V_{1,0}} dz_1 e^{-z_2^2} e^{-k\left(f(\boldsymbol{z})-\frac{32}{7}\right)}, \\
T_I^{(2)}(k_2;1,0) &= k_2^{\frac{1}{3}} \int_{-\infty}^{\infty} dz_2 \int_{V_{2,1}}^{V_{2,0}} dz_1 e^{-z_2^2} e^{-k(f(\boldsymbol{z})-0)}.
\end{aligned} \tag{7.3}$$

For additional clarification on the valleys chosen for integration,

$$\begin{aligned}
(V_{1,1}, V_{1,0}) &= (V_2, V_1), \\
(V_{1,2}, V_{1,0}) &= (V_3, V_1), \\
(V_{2,1}, V_{2,0}) &= (V_4, V_3).
\end{aligned}$$

The reason for focusing on two integration surfaces for χ_1 is that the coefficients exhibit different behaviour based on the quantity

$$B^{(n)}_{\omega_n}(\alpha_n, \beta_n) := |\alpha_n - \beta_n \mod \omega_n|,$$

where the least absolute residue is taken. The coefficients of expansions that share the same value of $B^{(n)}_{\omega_n}(\alpha_n, \beta_n)$ will exhibit identical behaviour (but will in general still be numerically different). Two expansions with different values of $B^{(n)}_{\omega_n}(\alpha_n, \beta_n)$ will have coefficients exhibiting different behaviour. This will be discussed in greater detail in §7.3, but we discuss this effect for the current example below.

When $\omega_n = 3$,

$$\alpha_n - \beta_n \in \{\pm 1, \pm 2\},$$

so the least absolute residues are $\pm 1 \mod 3$. Here $B^{(n)}_{\omega_n}(\alpha_n, \beta_n) = 1$ always, so the coefficients of the three possible expansions all behave in the same way.

When $\omega_n = 5$,

$$\alpha_n - \beta_n \in \{\pm 1, \pm 2, \pm 3, \pm 4\},$$

so the least absolute residues are ± 1 and $\pm 2 \mod 5$. Hence, $B^{(n)}_{\omega_n}(\alpha_n, \beta_n) \in \{1, 2\}$, meaning there are two different sets of coefficient behaviour; the coefficients of five of the possible ten expansions will behave one way, and the coefficients of the other five will behave in a different way. Due to this, we have picked one expansion from each behavioural group for χ_1.

Note that this concept can be intuitively explained by looking at a plot similar to Figure 16 and counting the minimum number of legs separate the two legs in question. It is also worth noting that all coefficients will still follow the 'factorial-over-power' behaviour we come to expect, but each behavioural class will follow a slightly different sub-pattern.

The size of the coefficients $T^{(1)}_r(1, 0)$, $T^{(1)}_r(2, 0)$, and $T^{(2)}_r(1, 0)$ are plotted against their term number r in Figures 26a, 26b, and 28 respectively. The coefficient behaviour discussed so far is clearly displayed in these figures, namely which terms are identically zero and the differing behaviour of $T^{(1)}_r(1, 0)$ and $T^{(1)}_r(2, 0)$. Interestingly, when both behavioural patterns for χ_1 are plotted together (Figure 27), they form two perfect 'standard' factorial-over-power lines; this will also be discussed in §7.3. Note also that the figures displaying coefficient and series size and behaviour in all of our examples are large so that we can see the fine structure introduced by considering general order critical points. Visually, this structure is far harder to see when the image resolution is smaller.

We now numerically calculate the integrals of focus (7.3) and compare them to their respective asymptotic expansions using (5.20) and the calculated coefficients (7.2). For the level zero expansion and the late terms in this example, we will choose $k_1 = k_2 = k = \frac{2}{\sqrt{2}}(1-i)$, so that $|k| = 2$ but $\arg(k) \neq 0$, thus avoiding Stokes phenomena. For the higher level expansions we will choose smaller values of $|k|$ for numerical reasons.

For χ_1 and $|k| = 2$, the truncation scheme dictates that

$$N_1 = \omega_1 \left(|kF_{12}| + \frac{q_2}{\omega_2} - \frac{q_1}{\omega_1} \right) = 5 \left(\frac{64}{7} + \frac{1}{3} - \frac{1}{5} \right) = 46 + \frac{8}{21} \to 46.$$

Checking manually using the table Figure 31 (future tables will be omitted), we see that the most accurate zeroth level expansions are

$$T^{(1)}(k, 49; 1, 0) \quad \text{and} \quad T^{(1)}(k, 47; 2, 0),$$

with the different truncation point values stemming from the difference in the coefficient's behaviour. The expansions $T^{(1)}(k, N_1; 1, 0)$ and $T^{(1)}(k, N_1; 2, 0)$ are shown in Figures 29a and 29b respectively.

For χ_2 and $|k| = 2$, the truncation scheme dictates that

$$N_2 = \omega_2 \left(|kF_{21}| + \frac{q_1}{\omega_1} - \frac{q_2}{\omega_2} \right) = 3 \left(\frac{64}{7} + \frac{1}{5} - \frac{1}{3} \right) = 27 + \frac{1}{35} \to 27.$$

Again, checking manually reveals that the most accurate zeroth level asymptotic expansion is $T^{(2)}(k, 29; 1, 0)$. The expansion $T^{(2)}(k, N_2; 1, 0)$ is shown in Figure 30.

The exact numerical integrals compared to the optimal level zero expansions are given below as

$$T_I^{(1)}(k; 1, 0) = 0.6923782\ldots - 1.16730270\ldots i,$$
$$T^{(1)}(k, 49; 1, 0) = 0.6923771\ldots - 1.16730252\ldots i;$$

$$T_I^{(1)}(k; 2, 0) = 2.2918074\ldots - 0.9795519\ldots i,$$
$$T^{(1)}(k, 47; 2, 0) = 2.2918050\ldots - 0.9795560\ldots i;$$

$$T_I^{(2)}(k; 1, 0) = 0.995711\ldots - 0.384924\ldots i,$$
$$T^{(2)}(k, 29; 1, 0) = 0.995741\ldots - 0.384931\ldots i.$$

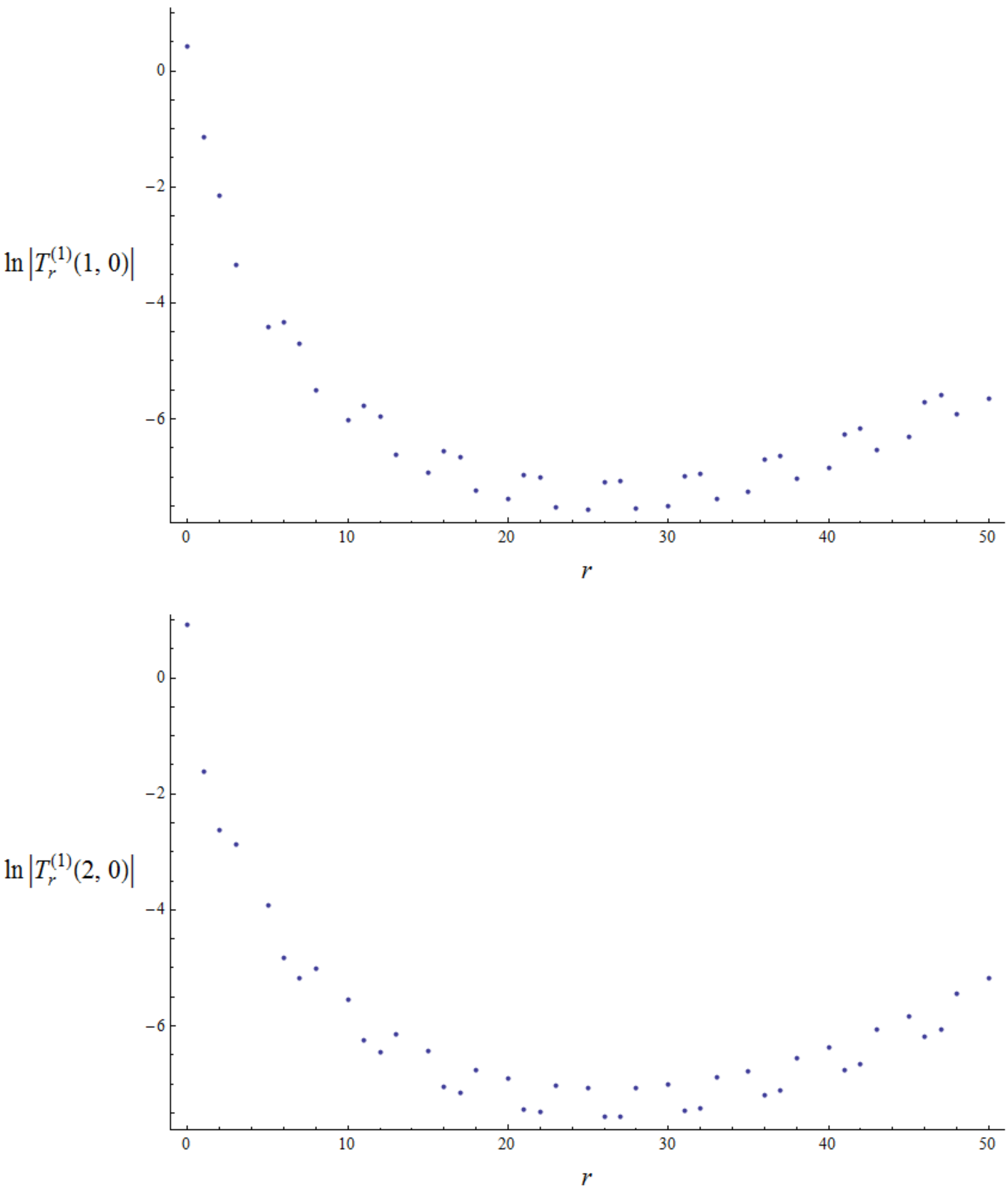


Figure 26: Coefficient size against term number for $T_r^{(1)}(1,0)$ and $T_r^{(1)}(2,0)$ respectively. Each plot represents a different coefficient behavioural pattern.

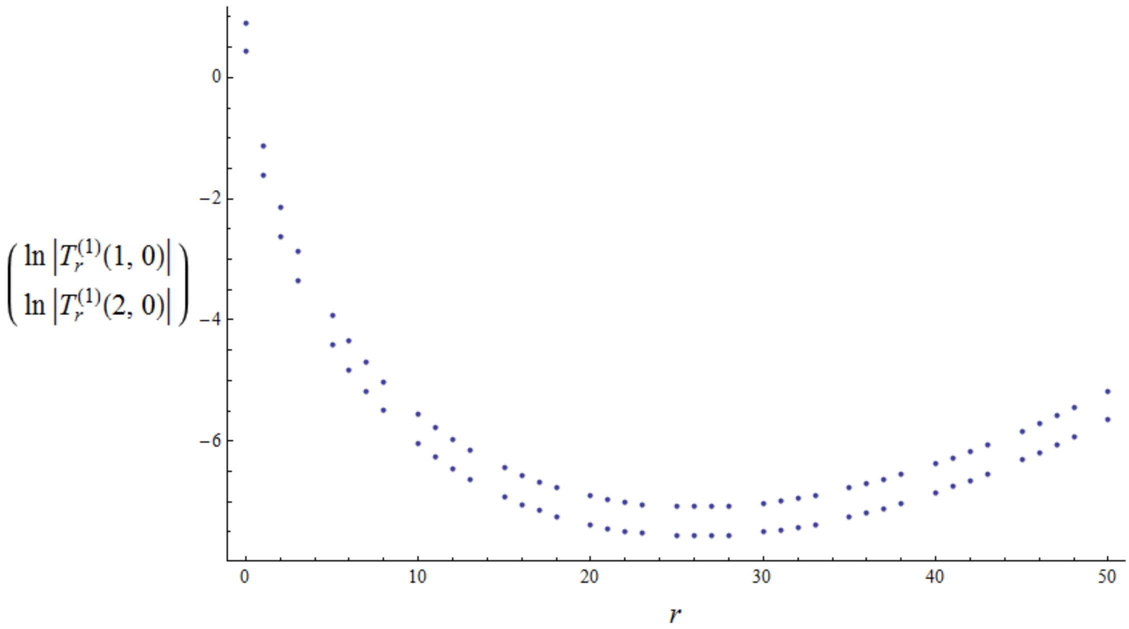


Figure 27: Coefficient size against term number for $T_r^{(1)}(1,0)$ and $T_r^{(1)}(2,0)$ displayed on the same plot.

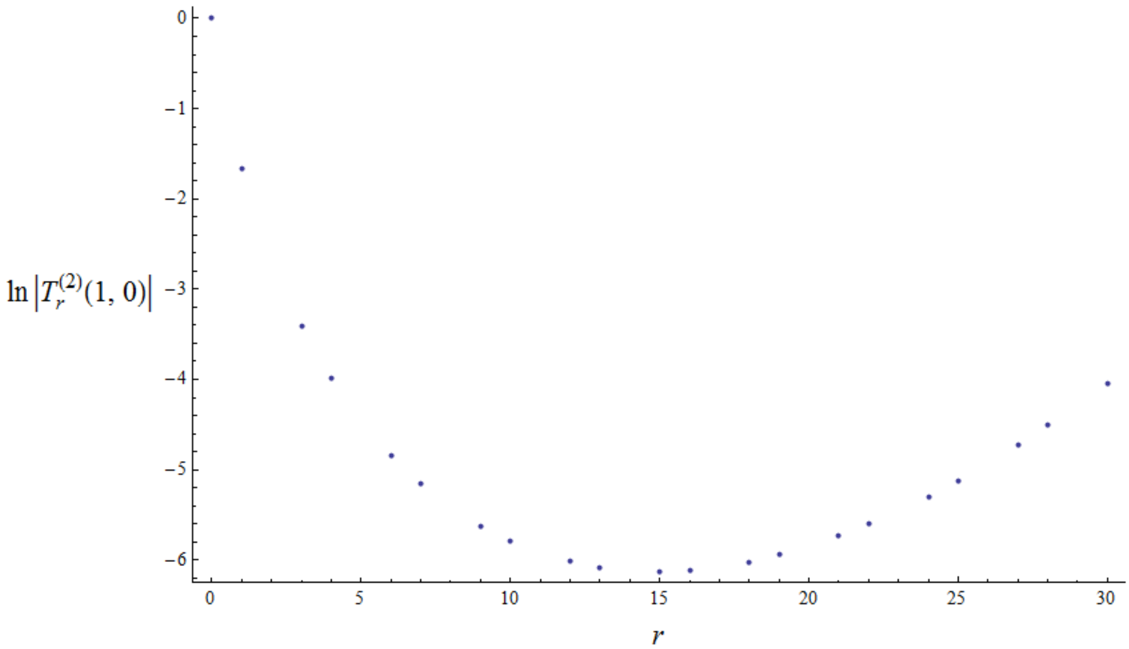


Figure 28: Coefficient size against term number for $T_r^{(2)}(1,0)$.

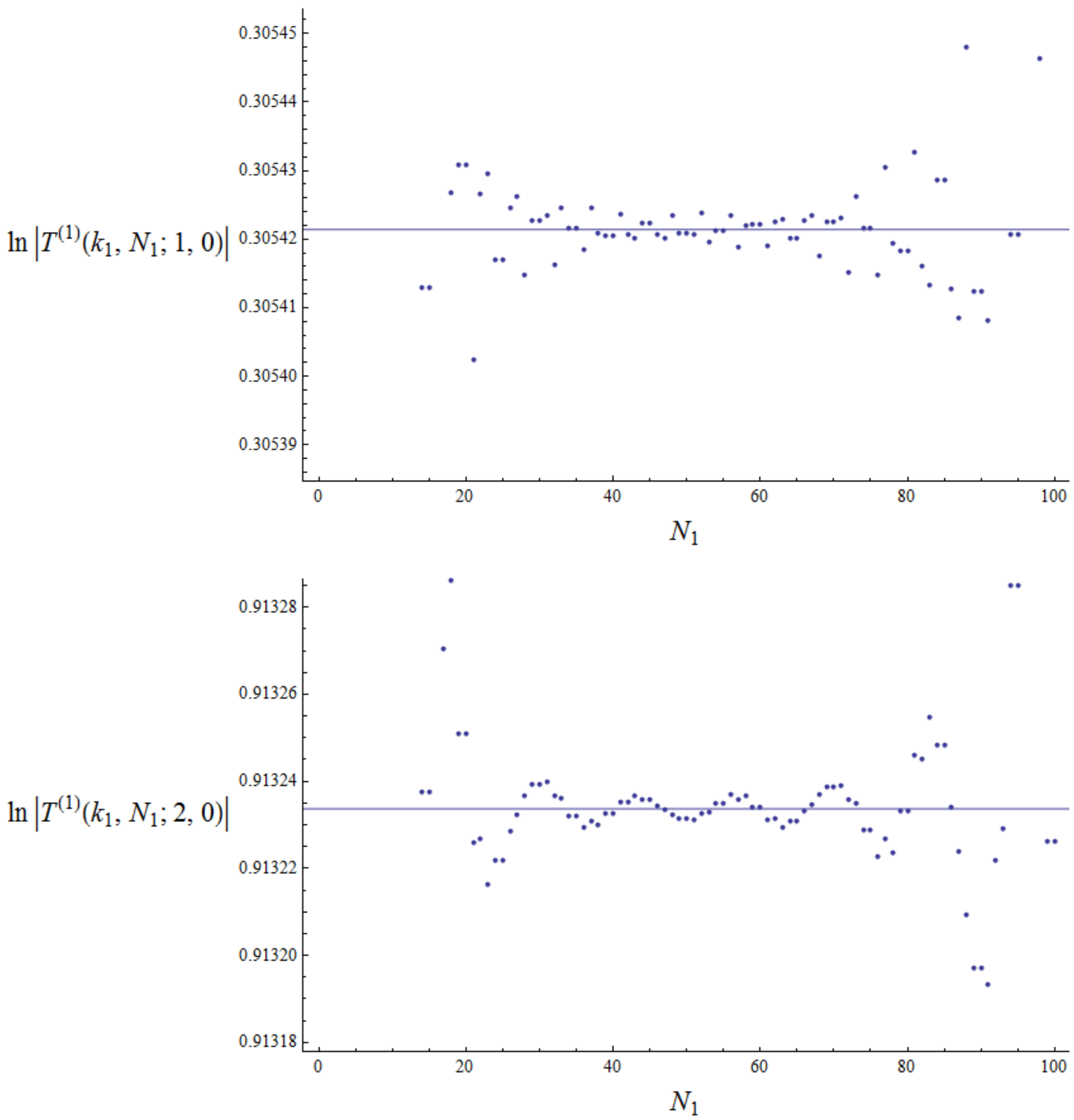


Figure 29: Expansion size against truncation point for $T^{(1)}(k, N_1; 1, 0)$ and $T^{(1)}(k, N_1; 2, 0)$ respectively. The solid line is the value of the exact integral.

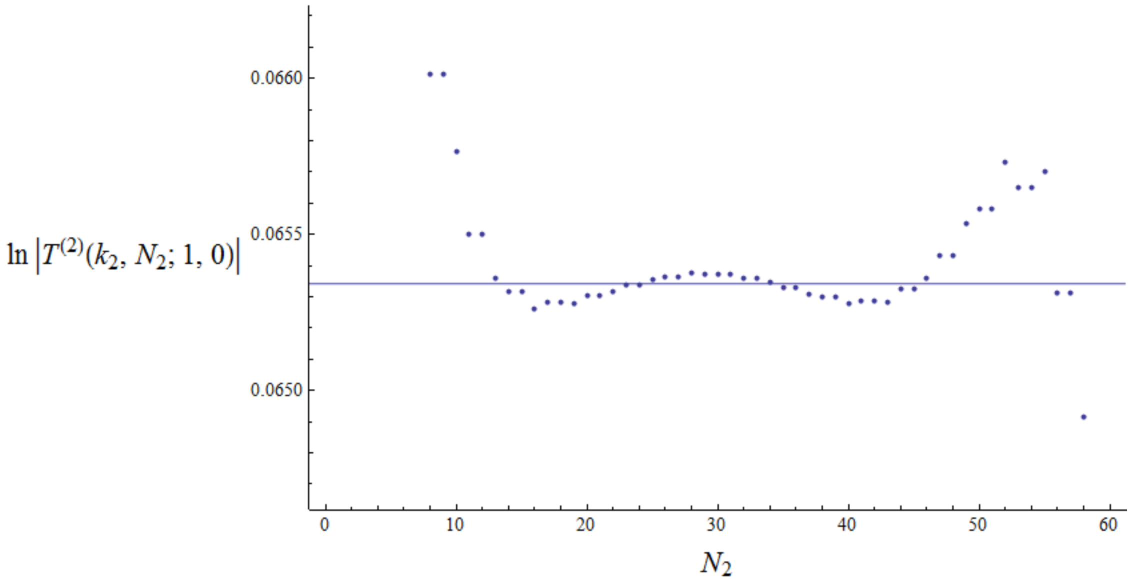


Figure 30: Expansion size against truncation point for $T^{(2)}(k, N_2; 1, 0)$. The solid line is the value of the exact integral.

7.1.2 Late Terms and Higher Level Expansions

Using the expression (6.2) or (6.6), we now compute late term expansions for coefficients and compare them to the actual coefficients using (5.13).

The late terms are given as

$$T^{(1)}_{N_1}(N_{12};\alpha_1,\beta_1) \sim \sum_{r_1=0}^{N_{12}-1}\left(e^{\frac{2\pi i(\beta_1-\rho_{12})(N_1+1)}{5}} - e^{\frac{2\pi i(\alpha_1-\rho_{12})(N_1+1)}{5}}\right)\cdots$$
$$\times\frac{K_{12}\Gamma\left(\frac{N_1+1}{5}-\frac{r_1+1}{3}\right)}{2\pi i\times 5F_{12}^{\frac{N_1+1}{5}-\frac{r_1+1}{3}}}T^{(2)}_{r_1}\left(\alpha_{12},\beta_{12}\right),$$

$$T^{(2)}_{N_2}(N_{21};\alpha_2,\beta_2) \sim \sum_{r_1=0}^{N_{21}-1}\left(e^{\frac{2\pi i(\beta_2-\rho_{21})(N_2+1)}{3}} - e^{\frac{2\pi i(\alpha_2-\rho_{21})(N_2+1)}{3}}\right)\cdots$$
$$\times\frac{K_{21}\Gamma\left(\frac{N_2+1}{3}-\frac{r_1+1}{5}\right)}{2\pi i\times 3F_{21}^{\frac{N_2+1}{3}-\frac{r_1+1}{5}}}T^{(1)}_{r_1}\left(\alpha_{21},\beta_{21}\right),$$

where the quantities K_{nm_1}, ρ_{nm_1}, α_{nm_1}, and β_{nm_1} are constants that can be calculated. We recall from §4 that α_{nm_1} and β_{nm_1} (and their analogous higher level quantities) are the 'open facing legs' of χ_{m_1}, from the point of view of χ_n. From Figure 25, we can read off the values as

$$(\alpha_{12},\beta_{12}) = (2,0); \quad (\alpha_{21},\beta_{21}) = (3,2).$$

The Stokes multipliers K_{12} and K_{21} must be non-zero else the asymptotic expansion would

```
Table[{N1, Abs[N[TI1aKF7[2/√2 (1 - ⅈ)] - Tr1KF7[2/√2 (1 - ⅈ), N1, 1, 0], 20]]}, {N1, 34, 65}]

Min[%]

Table[{N1, Abs[N[TI1bKF7[2/√2 (1 - ⅈ)] - Tr1KF7[2/√2 (1 - ⅈ), N1, 2, 0], 20]]}, {N1, 34, 65}]

Min[%]

{{34, 1.9573555286827×10^-6}, {35, 1.9573555286827×10^-6}, {36, 4.3059759866459×10^-6}, {37, 4.9102169202569×10^-6},
 {38, 3.7274879651530×10^-6}, {39, 1.4250221810540×10^-6}, {40, 1.4250221810540×10^-6}, {41, 3.2606255106838×10^-6},
 {42, 3.83967609576083×10^-6}, {43, 3.0072185280494×10^-6}, {44, 1.1924620854454×10^-6}, {45, 1.1924620854454×10^-6},
 {46, 2.8244820662943 3×10^-6}, {47, 3.4210909892494×10^-6}, {48, 2.7539951808303×10^-6}, {49, 1.12776898399926×10^-6}
 {50, 1.12776898399926×10^-6}, {51, 2.7546388066499×10^-6}, {52, 3.4213380883905×10^-6}, {53, 2.8225771151888×10^-6},
 {54, 1.1893817024482×10^-6}, {55, 1.1893817024482×10^-6}, {56, 2.9871024428176×10^-6}, {57, 3.7950171728473×10^-6},
 {58, 3.2007998524842×10^-6}, {59, 1.3840761273329×10^-6}, {60, 1.3840761273329×10^-6}, {61, 3.5652719749288×10^-6},
 {62, 4.62370038554070×10^-6}, {63, 3.9791982330193×10^-6}, {64, 1.7616049733367×10^-6}, {65, 1.7616049733367×10^-6}}

1.12776898399926×10^-6

{{34, 9.9560775960639×10^-6}, {35, 9.9560775960639×10^-6}, {36, 0.0000111432508017818}, {37, 6.6954380244152×10^-6},
 {38, 0.0000100880120437477}, {39, 7.3178270639565×10^-6}, {40, 7.3178270639565×10^-6}, {41, 8.5826777648957×10^-6},
 {42, 5.2771626221421×10^-6}, {43, 8.1176161471623×10^-6}, {44, 6.1585242792710×10^-6}, {45, 6.1585242792710×10^-6},
 {46, 7.5280209320084×10^-6}, {47, 4.7240150089173×10^-6}, {48, 7.4002723471198×10^-6}, {49, 5.8408147031962×10^-6},
 {50, 5.8408147031962×10^-6}, {51, 7.4093486392133×10^-6}, {52, 4.7352739759930×10^-6}, {53, 7.5393196986314×10^-6},
 {54, 6.1646841637655×10^-6}, {55, 6.1646841637655×10^-6}, {56, 8.0876151117194×10^-6}, {57, 5.2550674062270×10^-6},
 {58, 8.4904397483610×10^-6}, {59, 7.1678268718919×10^-6}, {60, 7.1678268718919×10^-6}, {61, 9.6975192718644×10^-6},
 {62, 6.3973549332667×10^-6}, {63, 0.0000104748324791111}, {64, 9.1044641333525×10^-6}, {65, 9.1044641333525×10^-6}}

4.7240150089173×10^-6
```

Figure 31: Tables of the difference between the numerical integral $T_I^{(1)}(k;\alpha_n,\beta_n)$ and the associated asymptotic expansion $T^{(1)}(k,N_1;\alpha_n,\beta_n)$ for varying N_1, for $(\alpha_n,\beta_n) = (1,0)$ and $(2,0)$ respectively. Here, 'TI1aKF7' and 'TI1bKF7' are the numerical integrals and 'Tr1KF7' is the asymptotic expansion.

converge and they must also satisfy $K_{12} = -K_{21}$. We can deduce the signs of the Stokes multipliers when we numerically calculate some late term expansions by checking if the overall sign is correct or not, and find that

$$K_{12} = -1, \quad K_{21} = 1.$$

The quantity ρ_{nm_1} and its higher level analogues generally requires detailed manual investigation to deduce, but in this simple topographical set-up (of critical components) it is simple to calculate.

For χ_1, we start at the leg corresponding to $\alpha_1 = 0$ and rotate around χ_1 anti-clockwise until we are between the two legs that face openly toward χ_2. The quantity ρ_{12} is then the number of legs that we passed through along the way, in this case $\rho_{12} = 2$. Rotating through those two legs of χ_1 in the z-plane corresponds to 'winding up' two Riemann surfaces in the u^{ω_1} plane in order to get onto the Riemann surface on which χ_1 can see χ_2. For χ_2, the leg corresponding to $\alpha_2 = 0$ is already openly facing χ_1; in this case, we simply have $\rho_{21} = 0$.

With these constants computed, we are now ready to compute some example late terms expansions, choosing to focus on $T_{47}^{(1)}(1,0)$ and $T_{27}^{(2)}(1,0)$. We leave out the second coefficient behavioural group for χ_1 for the rest of this example as we already know how it differs from the first group from when we looked at the coefficients in §7.1.1. It will not show us anything new, so is simply superfluous work.

For χ_1, the size of the coefficients $T_{47,r_1}^{(1)}(1,0)$ is shown in Figure 32 and the size of the expansion $T_{47}^{(1)}(N_{12};1,0)$ is shown in Figure 33. Equation (6.3) tells us that the coefficients $T_{47,r_1}^{(1)}(1,0)$ behave in a factorial-over-power way up until

$$r_1 = 3\left(\frac{47+1}{5} - \frac{1}{3}\right) = 27.8 \to 28$$

and this is reflected in Figure 32. By checking manually, we see that this late term expansion is most accurate when $N_{12} = 13$. Figures 34 and 35 show the ratio of the actual coefficient $T_{N_1}^{(1)}(1,0)$ against the first term in the late term expansion and the expansion summed to $N_{12} = \tilde{N}_{12}$ respectively. We would like these ratios to be as close to one as possible. The quantity $\tilde{N}_{12}$ is defined as

$$\tilde{N}_{12} = r_1 + 1 = \omega_2\left(\frac{N_1 + q_1}{\omega_1} - \frac{q_2}{\omega_2} - |F_{12}|\right) + 1,$$

which although does not minimise the error in late term expansion, is sufficient for the

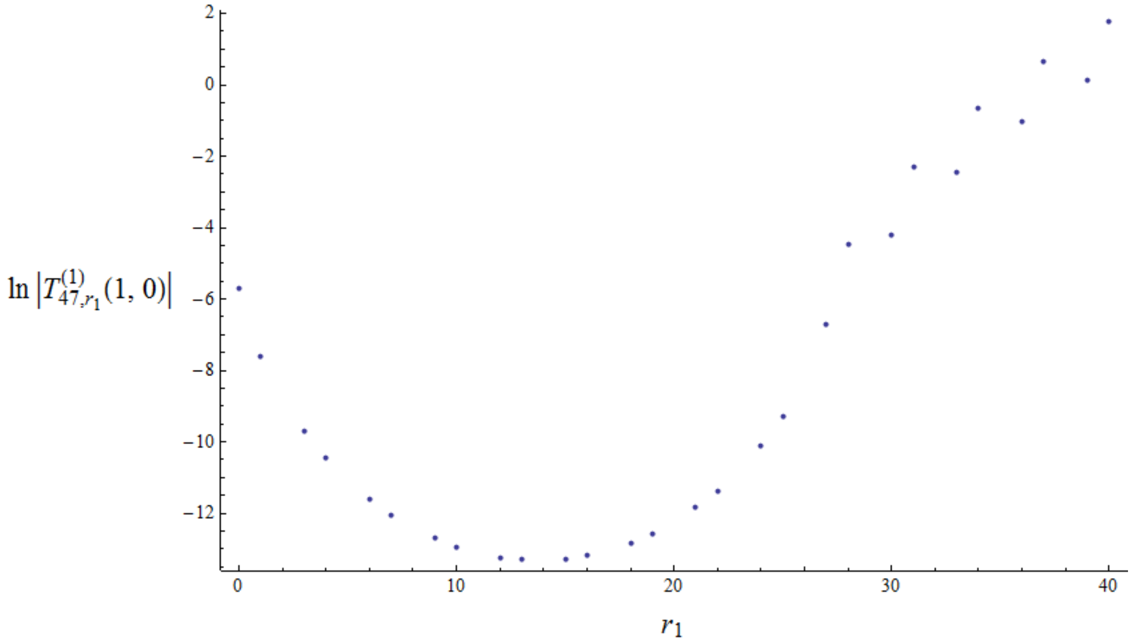


Figure 32: Late term coefficient size against term number for $T^{(1)}_{47,r_1}(1,0)$.

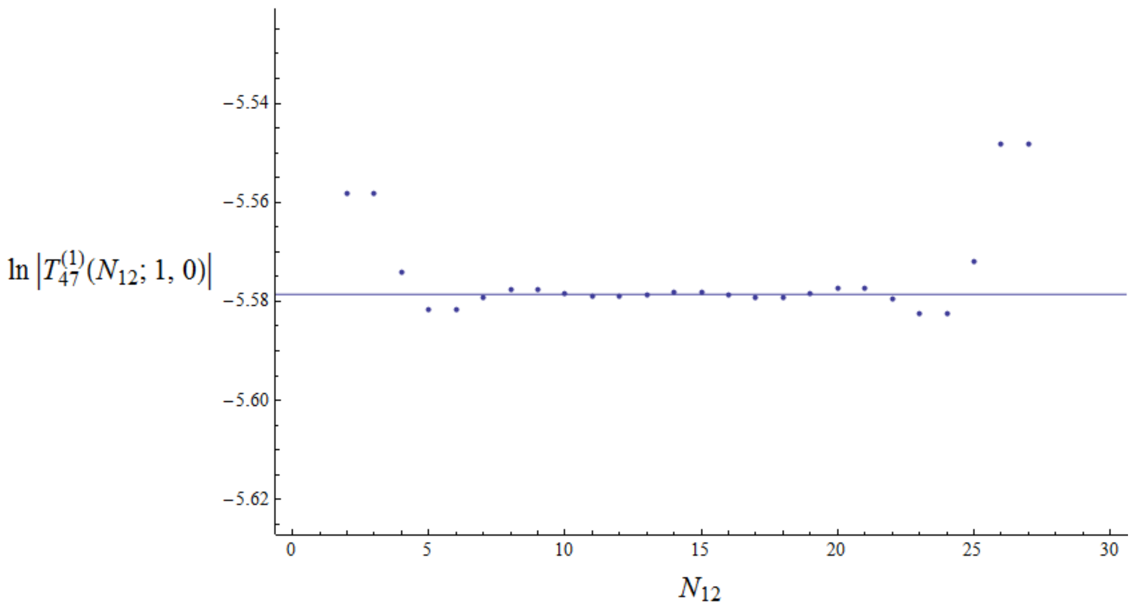


Figure 33: Late term expansion size against truncation point for $T^{(1)}_{47}(N_{12}; 1, 0)$. The solid line is the value of the actual coefficient $T^{(1)}_{47}(1, 0)$.

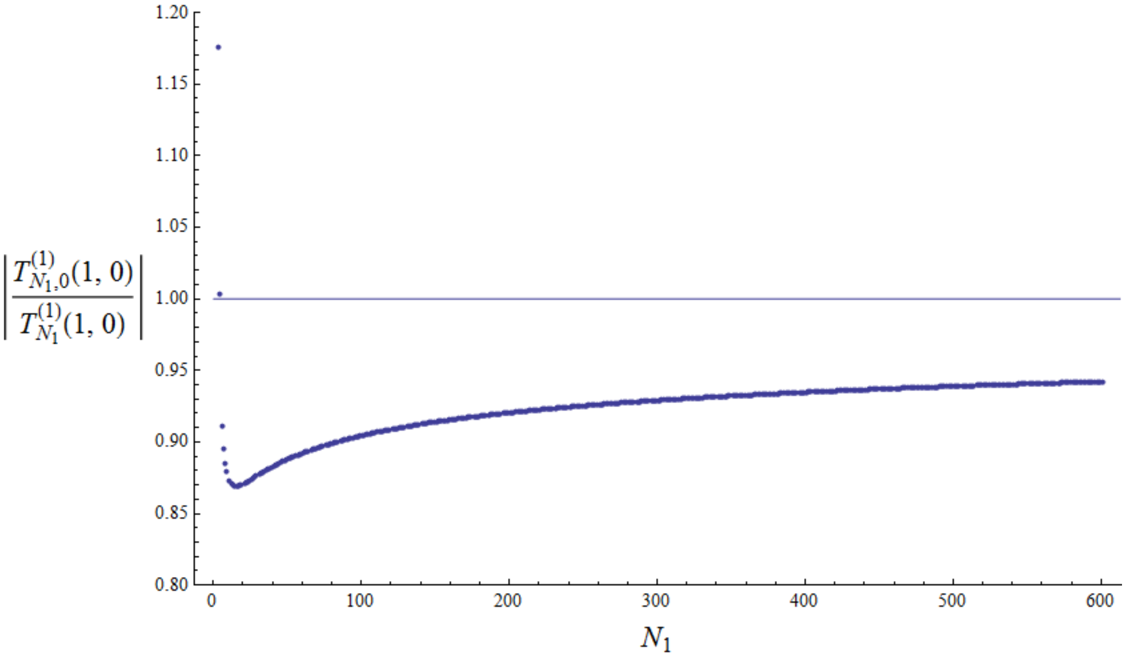


Figure 34: Size of ratio of first late term expansion coefficient and actual coefficient against actual coefficient number N_1 for $T^{(1)}_{N_1,0}(1,0)$ and $T^{(1)}_{N_1}(1,0)$ respectively.

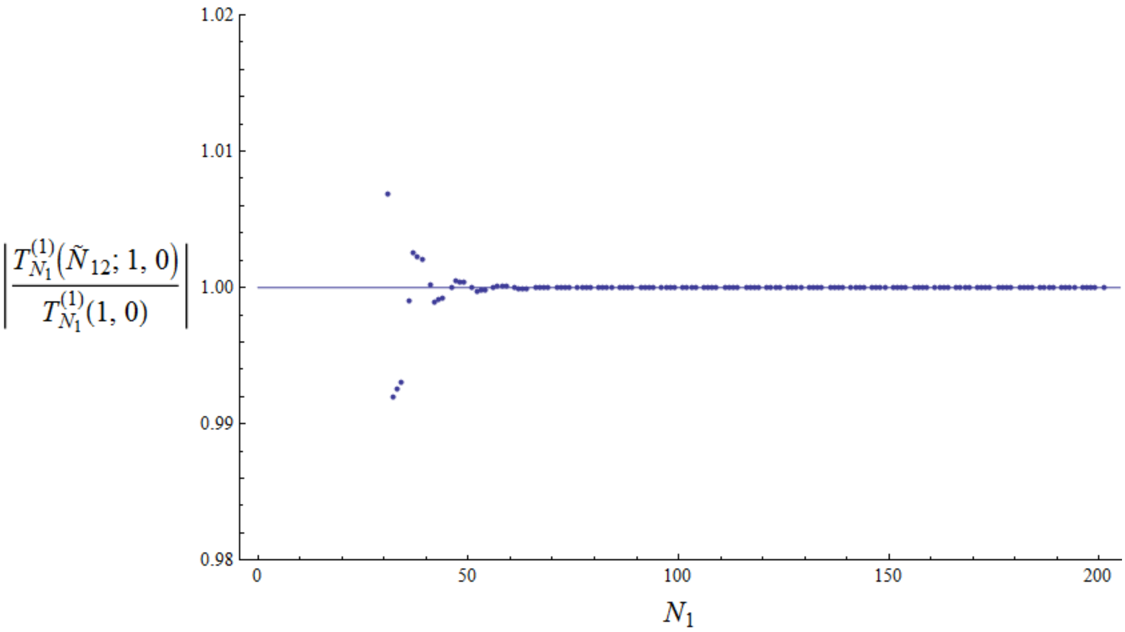


Figure 35: Size of ratio of late term expansion and actual coefficient against actual coefficient number N_1 for $T^{(1)}_{N_1}(\tilde{N}_{12};1,0)$ and $T^{(1)}_{N_1}(1,0)$ respectively. $\tilde{N}_{12}$ is defined in §7.1.2.

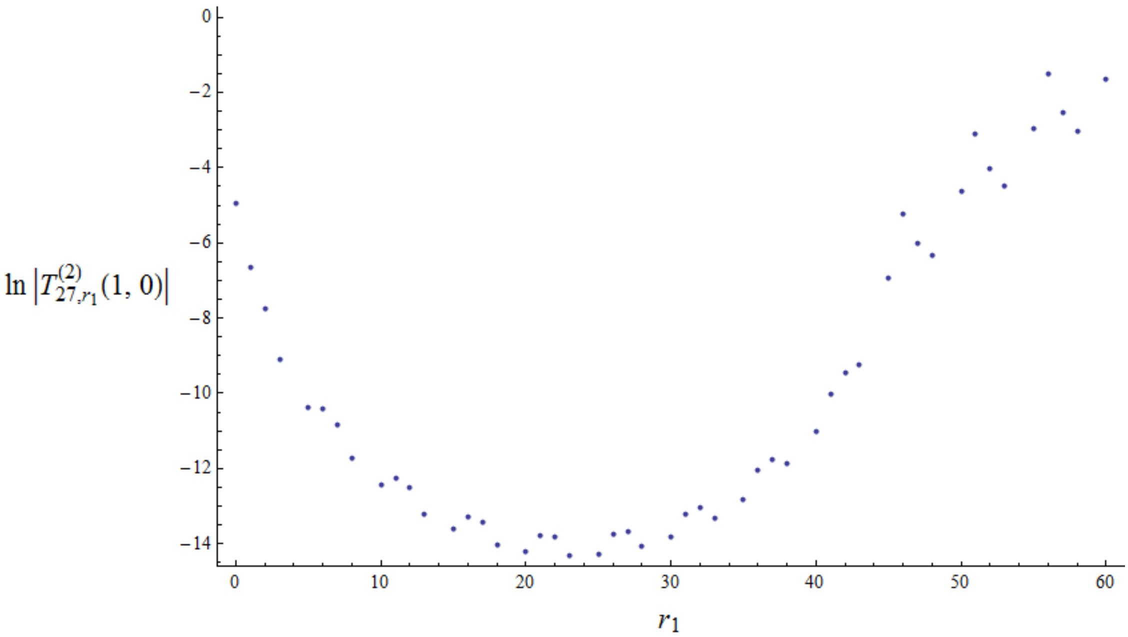


Figure 36: Late term coefficient size against term number for $T^{(2)}_{27,r_1}(1,0)$.

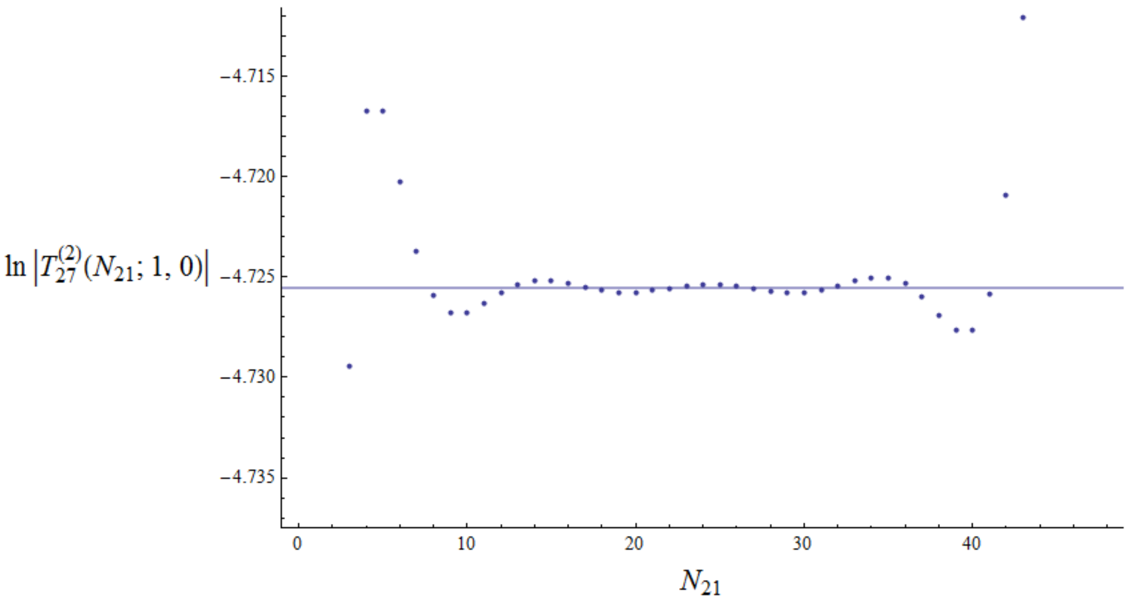


Figure 37: Late term expansion size against truncation point for $T^{(2)}_{27}(N_{21};1,0)$. The solid line is the value of the actual coefficient $T^{(2)}_{27}(1,0)$.

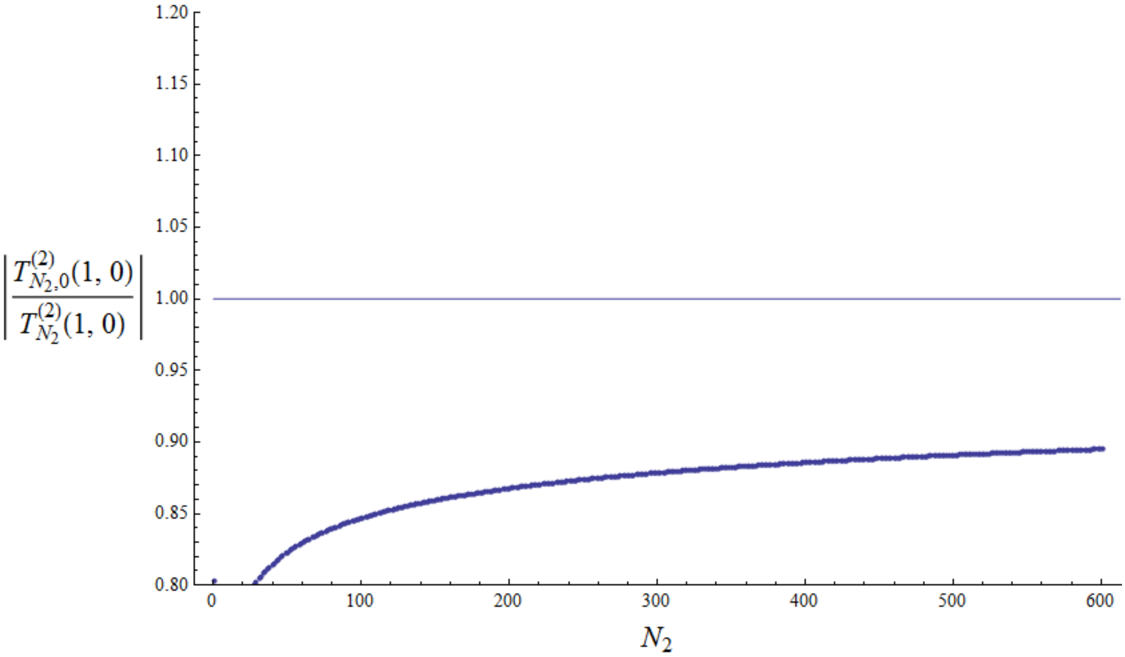


Figure 38: Size of ratio of first late term expansion coefficient and actual coefficient against actual coefficient number N_2 for $T^{(2)}_{N_2,0}(1,0)$ and $T^{(2)}_{N_1}(2,0)$ respectively.

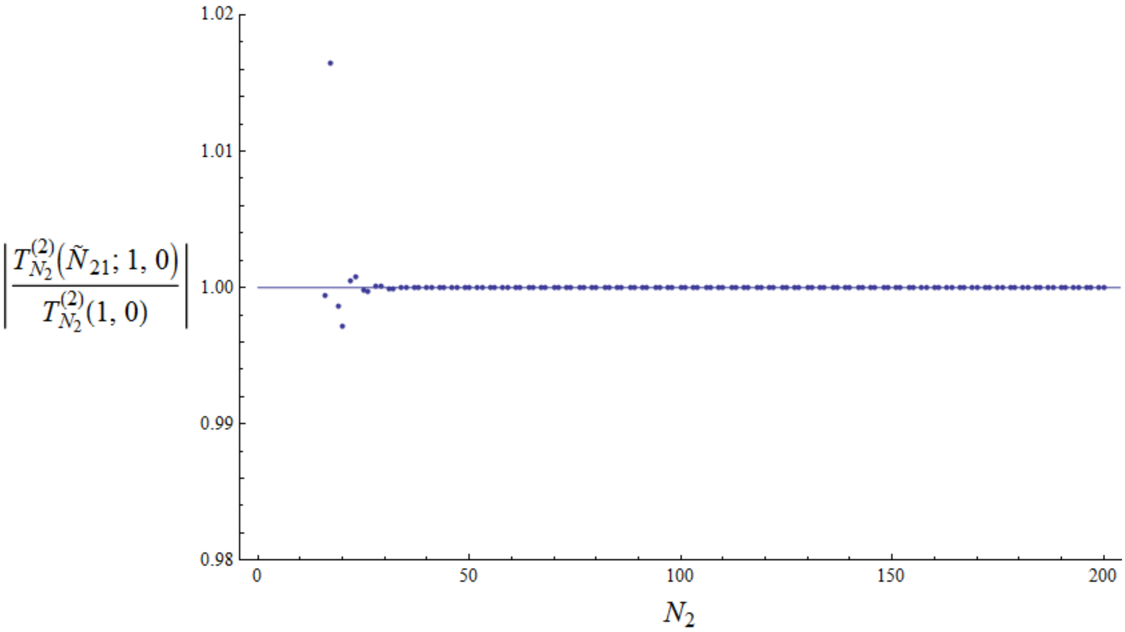


Figure 39: Size of ratio of late term expansion and actual coefficient against actual coefficient number N_2 for $T^{(2)}_{N_2}(\tilde{N}_{21};1,0)$ and $T^{(2)}_{N_2}(1,0)$ respectively. $\tilde{N}_{21}$ is defined in §7.1.2.

purposes of plotting Figure 35. Figure 34 shows the ratio tending to one (albeit very slowly) even when only the first term in the expansion is used, while Figure 35 shows the ratio very rapidly approaching one even when not using the optimal truncation point.

For χ_2, the size of the coefficients $T^{(2)}_{27,r_1}(1,0)$ is shown in Figure 36, the size of the expansion $T^{(2)}_{27}(N_{21};1,0)$ is shown in Figure 37, and ratios analogous to those looked at for χ_1 are shown in Figures 38 and 39. Equation (6.3) tells us that the coefficients $T^{(2)}_{27,r_1}(1,0)$ behave in a factorial-over-power way up until

$$r_1 = 5\left(\frac{27+1}{3} - \frac{1}{5}\right) = 45 + \frac{2}{3} \to 46$$

and this is reflected in Figure 36. By checking manually, we see that this late term expansion is most accurate when $N_{21} = 22$. Figures 38 and 39 show the ratio of the actual coefficient $T^{(2)}_{N_2}(1,0)$ against the first term in the late term expansion and the expansion summed to $N_{21} = \tilde{N}_{21}$ respectively, with

$$\tilde{N}_{21} = r_1 + 1 = \omega_1\left(\frac{N_2+q_2}{\omega_2} - \frac{q_1}{\omega_1} - |F_{21}|\right) + 1.$$

We again see that these tend to one in a similar fashion as those for χ_1.

The exact numerical coefficients compared to the optimal late term expansions are given below as

$$\begin{aligned} T^{(1)}_{47}(1,0) &= -0.00359303\ldots - 0.001167447\ldots i, \\ T^{(1)}_{47}(13;1,0) &= -0.00359312\ldots - 0.001167478\ldots i; \end{aligned}$$

$$\begin{aligned} T^{(2)}_{27}(1,0) &= 0.00767804\ldots - 0.004432923\ldots i, \\ T^{(2)}_{27}(22;1,0) &= 0.00767811\ldots - 0.004432962\ldots i. \end{aligned}$$

We now look at higher level hyperasymptotic expansions around both critical components. We compute the hyperterminants using the F-functions (6.31) and (6.32) derived in §6.2 and substitute them into the general hyperasymptotic expansion framework (6.4). All other quantities required for this computation were calculated when we looked at the late terms in the previous section, with the truncation points following the scheme (6.14).

For these higher level expansions we will choose different k values to those of the previous sections within this example, in order to sidestep numerical limitations that arise when Mathematica handles extremely small terms. For the remainder of this example, we choose

$$k_1 = \frac{5}{4}\frac{(1-i)}{\sqrt{2}} \Rightarrow |k_1| = \frac{5}{4} \quad \text{and} \quad k_2 = \frac{3}{2}\frac{(1-i)}{\sqrt{2}} \Rightarrow |k_2| = \frac{3}{2}.$$

We will also add a superscript to the truncation points to indicate which level of hyperasymptotics they are being used at; for example, $N_1^{(1)}$ and $N_1^{(3)}$ are the first truncation points in the first and third level expansion around χ_1 respectively. It is important to make this distinction clear, as these quantities will be very different.

We will compute expansions up to level three for both critical components and the size of the terms in these expansions is shown in Figures 40 through 45. We see that they follow the expected pattern, although there is an anomaly in Figures 42 and 44 where the coefficient size seems to warp downward slightly then recover back up to normal in the level three expansion. Through experimentation we deduced that this is Mathematica hitting some kind of internal numerical limit mid calculation and giving a slightly strange result, since if we reduce the value of $|k_1|$ sufficiently this no longer occurs. However, we lose a lot of accuracy by reducing $|k_1|$ so much and despite this slight anomaly we still get the extremely accurate numerical results that we expect from a level three expansion, so we decided to leave it as it is currently.

The truncation points for this problem are computed using (6.14) as

$$\begin{aligned} N_1^{(0)} &= 31, \quad \{N_1^{(1)}, N_{12}^{(1)}\} = \{51, 17\}, \quad \{N_1^{(2)}, N_{12}^{(2)}, N_{121}^{(2)}\} = \{86, 34, 29\}, \\ N_2^{(0)} &= 23, \quad \{N_2^{(1)}, N_{21}^{(1)}\} = \{41, 35\}, \quad \{N_2^{(2)}, N_{21}^{(2)}, N_{212}^{(2)}\} = \{61, 69, 20\}, \end{aligned}$$

$$\{N_1^{(3)}, N_{12}^{(3)}, N_{121}^{(3)}, N_{1212}^{(3)}\} = \{114, 51, 57, 17\}, \quad \{N_2^{(3)}, N_{21}^{(3)}, N_{212}^{(3)}, N_{2121}^{(3)}\} = \{82, 103, 41, 35\},$$

except for $N_1^{(0)}$ and $N_2^{(0)}$ where we manually checked the most accurate expansion. Below, the exact numerical integrals are compared to the respective hyperasymptotic level one, two, and three expansions. We take absolute values as it is then easier to see the improvement in accuracy at each level.

$$\begin{aligned} \left|T^{(1)}(k_1, 31; 1, 0)\right| &= 1.3422155783755304441\ldots, \\ \left|T^{(1)}(k_1; 57; 17; 1, 0)\right| &= 1.3421708062222489936\ldots, \\ \left|T^{(1)}(k_1; 86; 34; 29; 1, 0)\right| &= 1.3421708765130495174\ldots, \\ \left|T^{(1)}(k_1; 114; 51; 57; 17; 1, 0)\right| &= 1.3421708765129419140\ldots, \\ \left|T_I^{(1)}(k_1; 1, 0)\right| &= 1.3421708765129419079\ldots; \end{aligned}$$

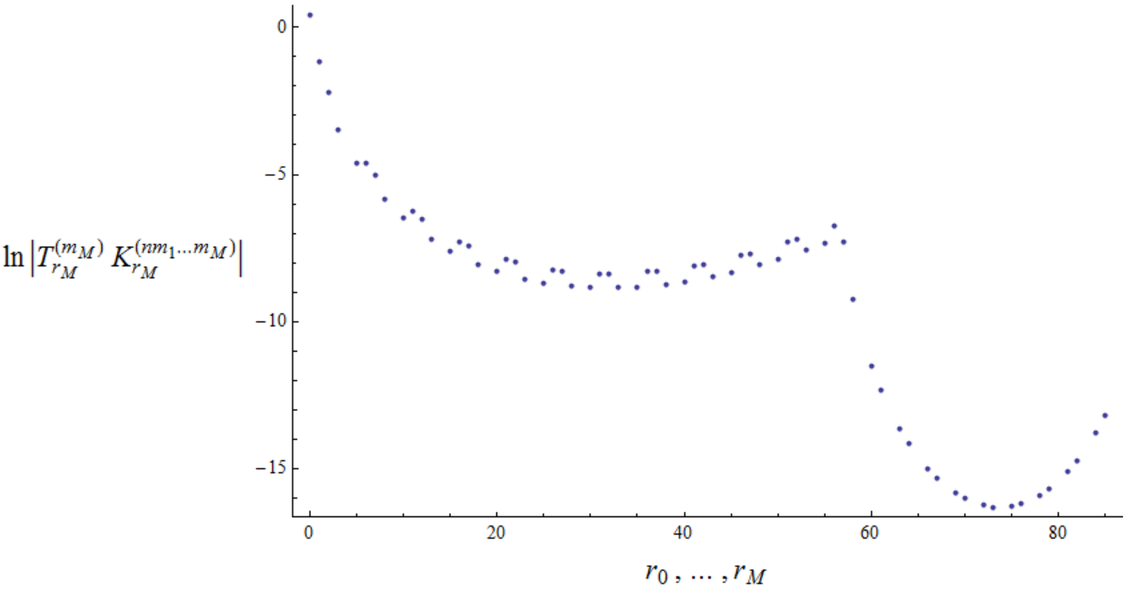


Figure 40: Size of the terms in the level one hyperasymptotic expansion $T^{(1)}(k; 57; 11; 1, 0)$ for $|k| = \frac{5}{4}$.

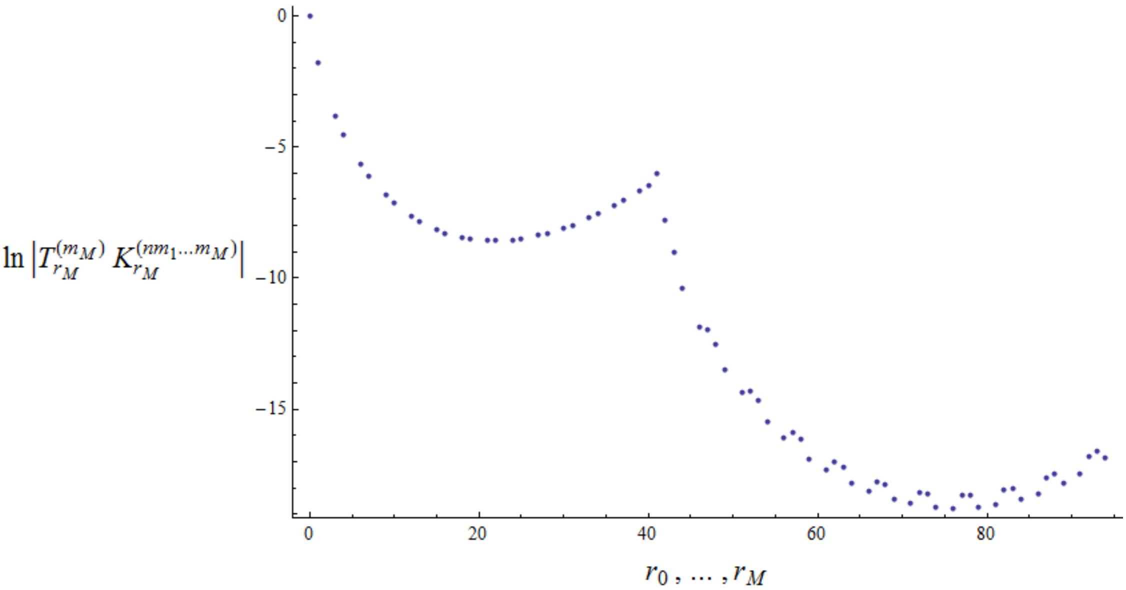


Figure 41: Size of the terms in the level one hyperasymptotic expansion $T^{(2)}(k; 41; 35; 1, 0)$ for $|k| = \frac{3}{2}$.

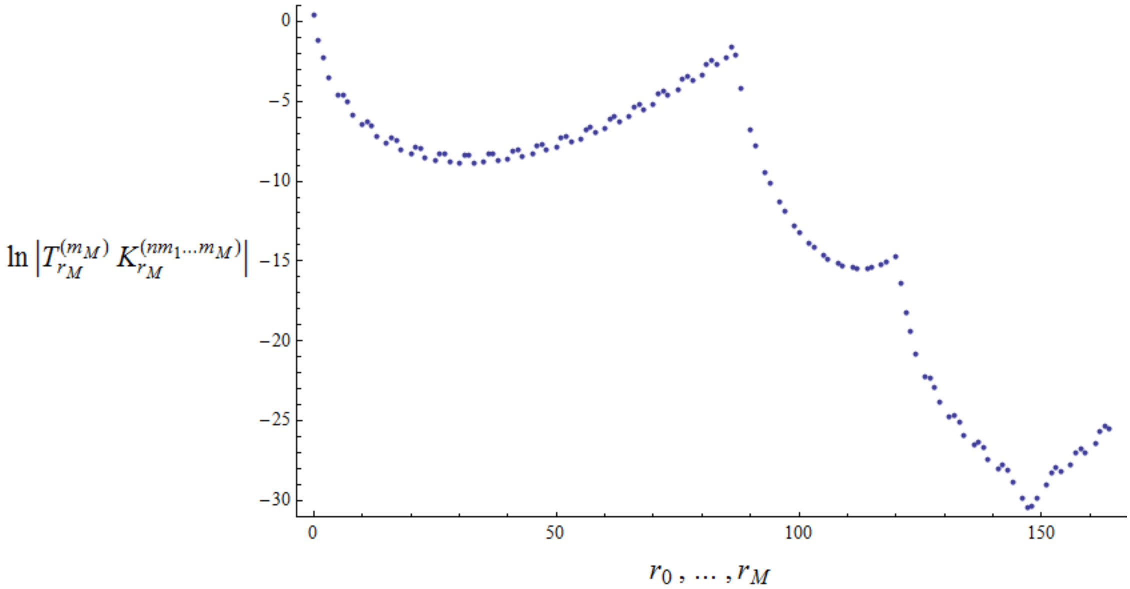


Figure 42: Size of the terms in the level two hyperasymptotic expansion $T^{(1)}(k;86;34;29;1,0)$ for $|k| = \frac{5}{4}$.

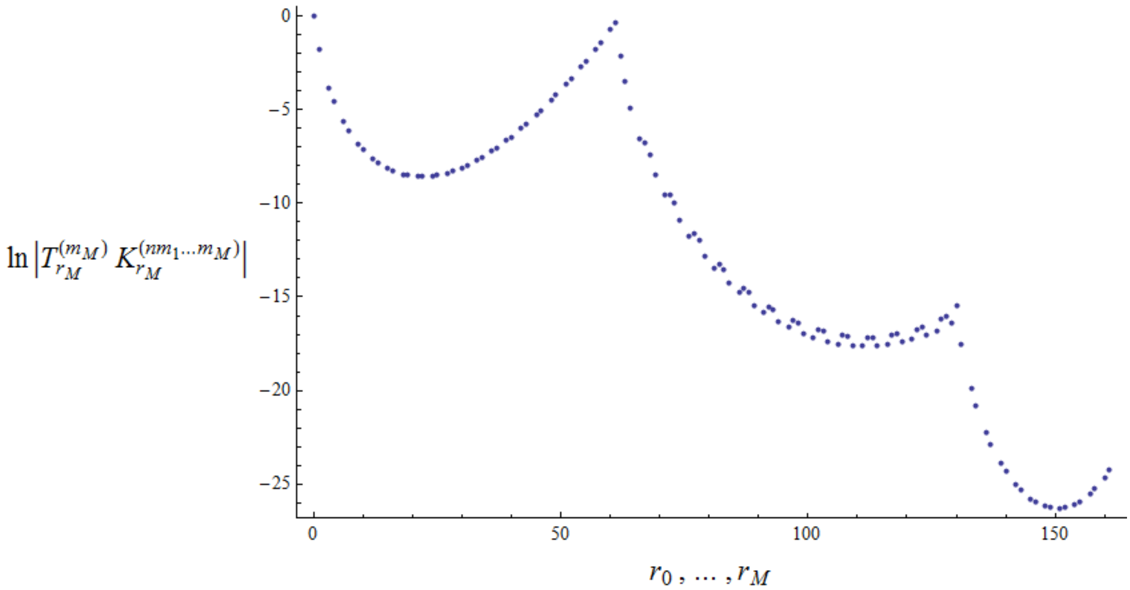


Figure 43: Size of the terms in the level two hyperasymptotic expansion $T^{(2)}(k;61;69;20;1,0)$ for $|k| = \frac{3}{2}$.

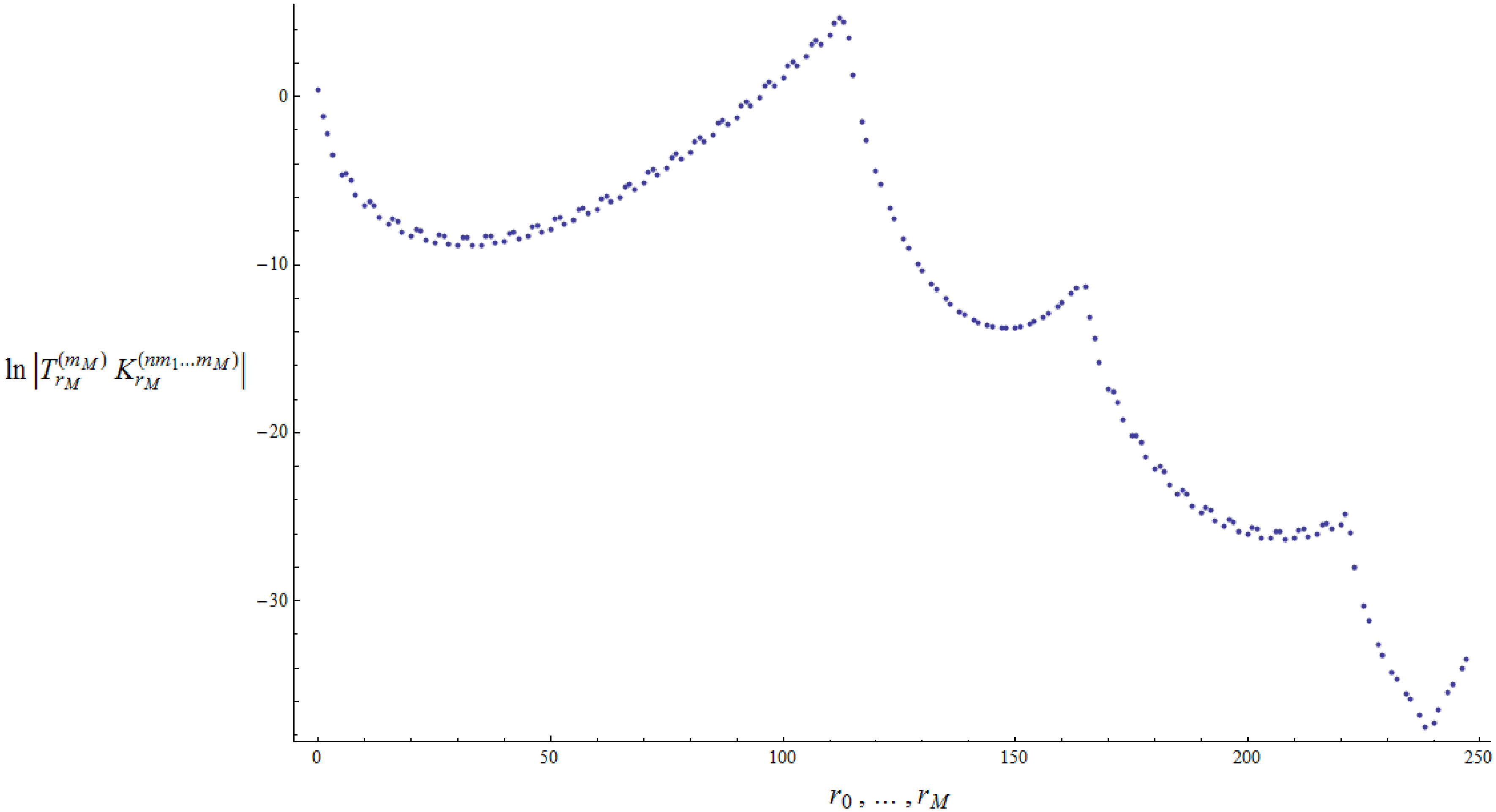


Figure 44: Size of the terms in the level three hyperasymptotic expansion $T^{(1)}(k;114;51;57;17;1,0)$ for $|k| = \frac{5}{4}$.

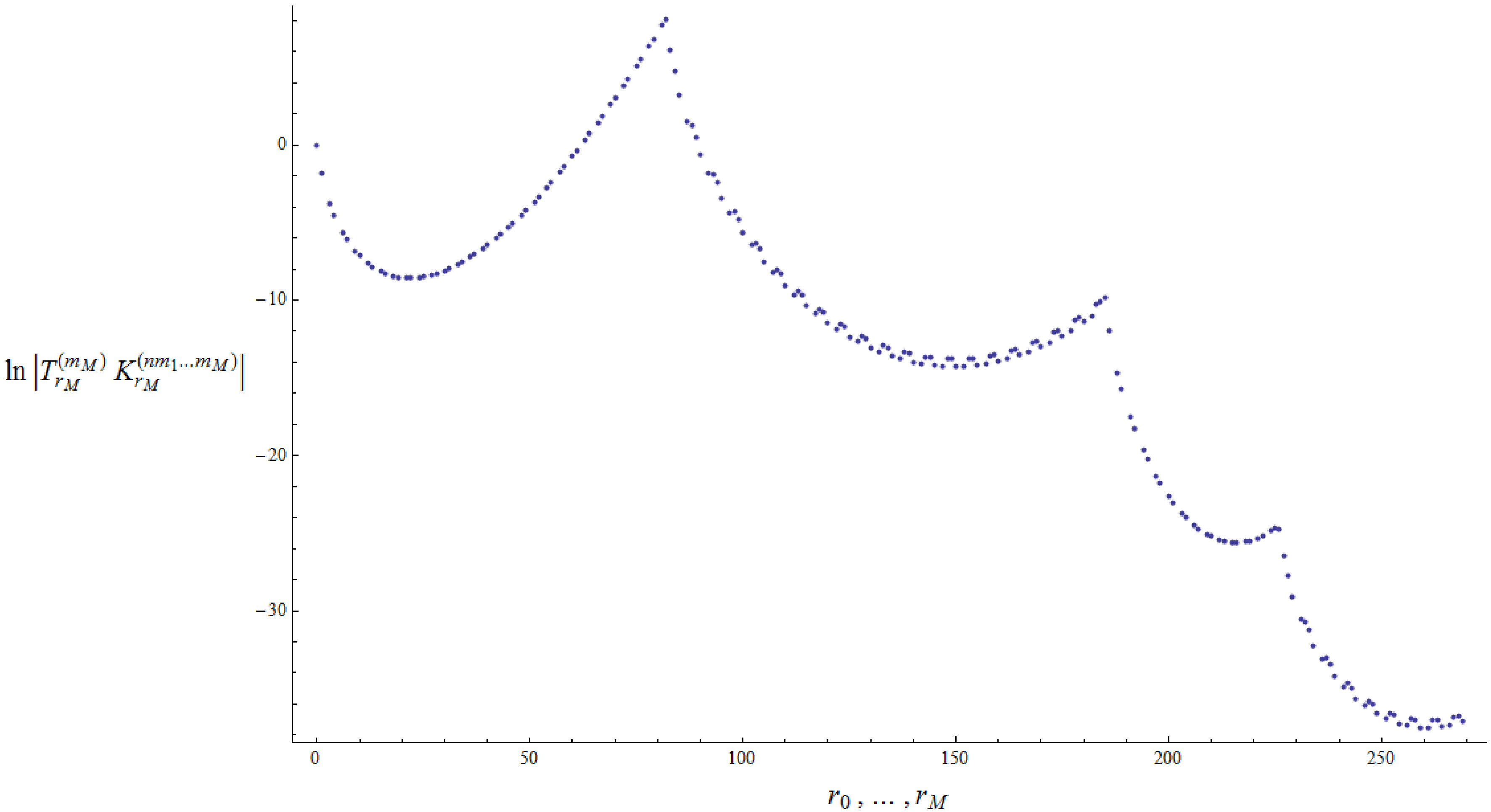


Figure 45: Size of the terms in the level three hyperasymptotic expansion $T^{(2)}(k; 82; 103; 41; 35; 1, 0)$ for $|k| = \frac{3}{2}$.

$$
\begin{aligned}
\left|T^{(2)}(k_2,23;1,0)\right| &= 1.07527154997422776\ldots,\\
\left|T^{(2)}(k_2;41;35;1,0)\right| &= 1.07521876639074055\ldots,\\
\left|T^{(2)}(k_2;61;69;20;1,0)\right| &= 1.07521874816278229\ldots,\\
\left|T^{(2)}(k_2;82;103;41;35;1,0)\right| &= 1.07521874816135257\ldots,\\
\left|T_I^{(2)}(k_2;1,0)\right| &= 1.07521874816135054\ldots.
\end{aligned}
$$

Displayed in this way (with the exact integral at the bottom), it is easy to see the improvement in the expansion accuracy as we increase the expansion level M.

7.2 Example 2 - Contour Boundary Hyperasymptotic Expansion

In this example, we will focus on computing an integral between two linear contour boundaries (constant order one surfaces along which $f(\boldsymbol{z})$ is constant). Since f is constant along these boundaries, there are no restricted critical points and they do not intersect with any unrestricted critical components. We use the same integrand as in Example 1 so we can focus solely on computing boundary asymptotics.

We look at the integral

$$
J(k) = \int_{\tilde{S}_J} dz_1\, dz_2\, e^{-z_2^2} e^{-k\left(\frac{15}{28}z_1^7 - 5z_1^6 + 18z_1^5 - 30z_1^4 + 20z_1^3\right)}, \tag{7.4}
$$

where

$$
\begin{aligned}
f(\boldsymbol{z}) &= \frac{15}{28}z_1^7 - 5z_1^6 + 18z_1^5 - 30z_1^4 + 20z_1^3,\\
g(\boldsymbol{z}) &= e^{-z_2^2},
\end{aligned}
$$

so that the integrand is the same as in Example 1. This time, we want to integrate over the surface

$$
\tilde{S}_J = (e_1, e_2) \times \bar{\mathbb{R}},
$$

where $e_1, e_2 \in \mathbb{C}$ and (e_1, e_2) is a straight line segment in $\mathbb{C}$. Defining $\chi_{e_j} := (e_j, z_2)$ with $e_j \in \mathbb{C}$ (and $q_j = 1$), then the integration surface $\tilde{S}_J$ is a subset of the most direct surface connecting χ_{e_1} and χ_{e_2}. We choose the specific boundary surfaces

$$
\chi_{e_1} = (1+i, z_2) \quad \text{and} \quad \chi_{e_2} = \left(1 - \tfrac{4}{5}i, z_2\right),
$$

with three and two-dimensional contour plots displayed in Figures 46 and 47 respectively.

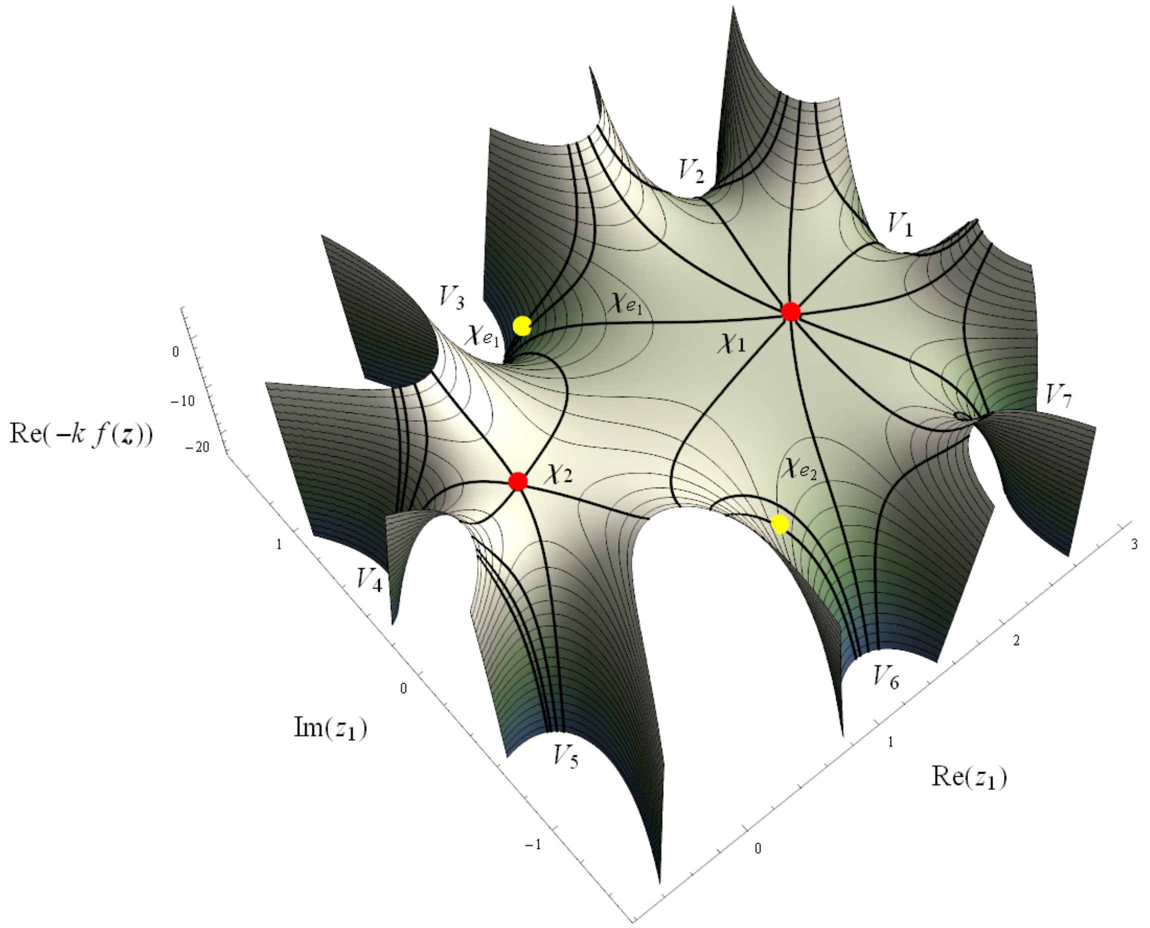


Figure 46: Three dimensional contour plot of $\mathrm{Re}(-kf(\boldsymbol{z}))$ in complex z_1 space, with $k = 1 - i$. This contour plot is entirely unaffected by the variable z_2, since $f(\boldsymbol{z})$ is not a function of z_2.

Within these two figures, the boundaries are represented by the yellow dots.

Assuming that we can deform $\tilde{S}_J$ into a union of steepest descent surfaces S_J based on the discussion in §3.2, we are able to deform $\tilde{S}_J$ - based on Figure 47 - into

$$S_J = (\chi_{e_1}, V_3, \chi_1, V_6, \chi_{e_2}),$$

where we travel along steepest descent paths only. Looking at Figure 47, the original integration surface $\tilde{S}_J$ is a straight line joining the yellow dots χ_{e_1} and χ_{e_2}, which is then deformed into S_J as detailed above. Starting at χ_{e_1}, we follow the only steepest path available that leads into a valley - specifically V_3 - and then travel up out of the valley along the steepest path leading to χ_1 (we could travel to χ_2, but cannot reach χ_{e_2} without retracing our steps). From χ_1 we choose to take the steepest path into the valley V_6 and then back up to the destination χ_{e_2}. We have therefore travelled from the start to the end point of integration along only steepest paths, whose asymptotic contribution can then be determined by the work done in this thesis.

Note that any surface $\chi_a = (a, z_2)$ with $a \in \mathbb{C}$ is a contour of f, since

$$f(\chi_a) = f(a, z_2) = f(a) = A \in \mathbb{C},$$

so that when χ_a is an integration boundary, we can use the resurgence relation (5.19) to calculate hyperasymptotic expansions around it.

We can now write the integral $J(k)$ as

$$J(k) = \left(\int_{\chi_{e_1}}^{V_3} + \int_{V_3}^{\chi_1} + \int_{\chi_1}^{V_6} + \int_{V_6}^{\chi_{e_2}} \right) d\boldsymbol{z}\, g(\boldsymbol{z}) e^{-kf(\boldsymbol{z})},$$

from which we can identify the integrals (5.1) and (5.4) and therefore write

$$\begin{aligned} J(k) &= I^{(e_1)}(k) + I^{(1)}(k; 2, 3) - I^{(e_2)}(k) \\ &= \frac{e^{-kf_{e_1}}}{k} T^{(e_1)}(k) + \frac{e^{-kf_1}}{k^{\frac{1}{5}}} T^{(1)}(k; 2, 3) - \frac{e^{-kf_{e_2}}}{k} T^{(e_2)}(k). \end{aligned} \tag{7.5}$$

Since we have already computed the hyperasymptotic expansion for $T^{(1)}(k; 2, 3)$ in Example 1, we now look at computing hyperasymptotic expansions around χ_{e_1} and χ_{e_2}. The process is identical to that of Example 1, so we will provide less detail.

The asymptotic coefficients $T_r^{(e_j)}$ are given by (5.19) with $\omega_{e_j} = 1$ and $X = 0$ as

$$\begin{aligned} T_r^{(e_1)} &= \Gamma(r+1) \operatorname*{Res}_{\boldsymbol{z} \in \chi_{e_1}} \left(\frac{e^{-z_2^2}}{(f(\boldsymbol{z}) - f_{e_1})^{r+1}} \right) = \Gamma(r+1) \int_{-\infty}^{\infty} dz_2 \operatorname*{Res}_{z_1 = 1+i} \left(\frac{e^{-z_2^2}}{(f(\boldsymbol{z}) - f_{e_1})^{r+1}} \right), \\ T_r^{(e_2)} &= \Gamma(r+1) \operatorname*{Res}_{\boldsymbol{z} \in \chi_{e_2}} \left(\frac{e^{-z_2^2}}{(f(\boldsymbol{z}) - f_{e_2})^{r+1}} \right) = \Gamma(r+1) \int_{-\infty}^{\infty} dz_2 \operatorname*{Res}_{z_1 = 1 - \frac{4}{5}i} \left(\frac{e^{-z_2^2}}{(f(\boldsymbol{z}) - f_{e_2})^{r+1}} \right), \end{aligned}$$

with none of the coefficients identically zero. Throughout this example, we will fix all values of k at

$$k_{e_1} = k_{e_2} = k_1 =: k = \frac{2}{\sqrt{2}}(1 - i),$$

with the most accurate level zero expansions given by $T^{(e_1)}(k, 17)$ and $T^{(e_1)}(k, 9)$. We compare the numerical value of these expansions against the integrals of interest

$$\begin{aligned} T_I^{(e_1)}(k) &= k \int_{-\infty}^{\infty} dz_2 \int_{1+i}^{V_3} dz_1 e^{-z_2^2} e^{-k\left(f(\boldsymbol{z}) - f_{e_1}\right)}, \\ T_I^{(e_2)}(k) &= k \int_{-\infty}^{\infty} dz_2 \int_{1-\frac{4}{5}i}^{V_6} dz_1 e^{-z_2^2} e^{-k\left(f(\boldsymbol{z}) - f_{e_2}\right)}, \end{aligned}$$

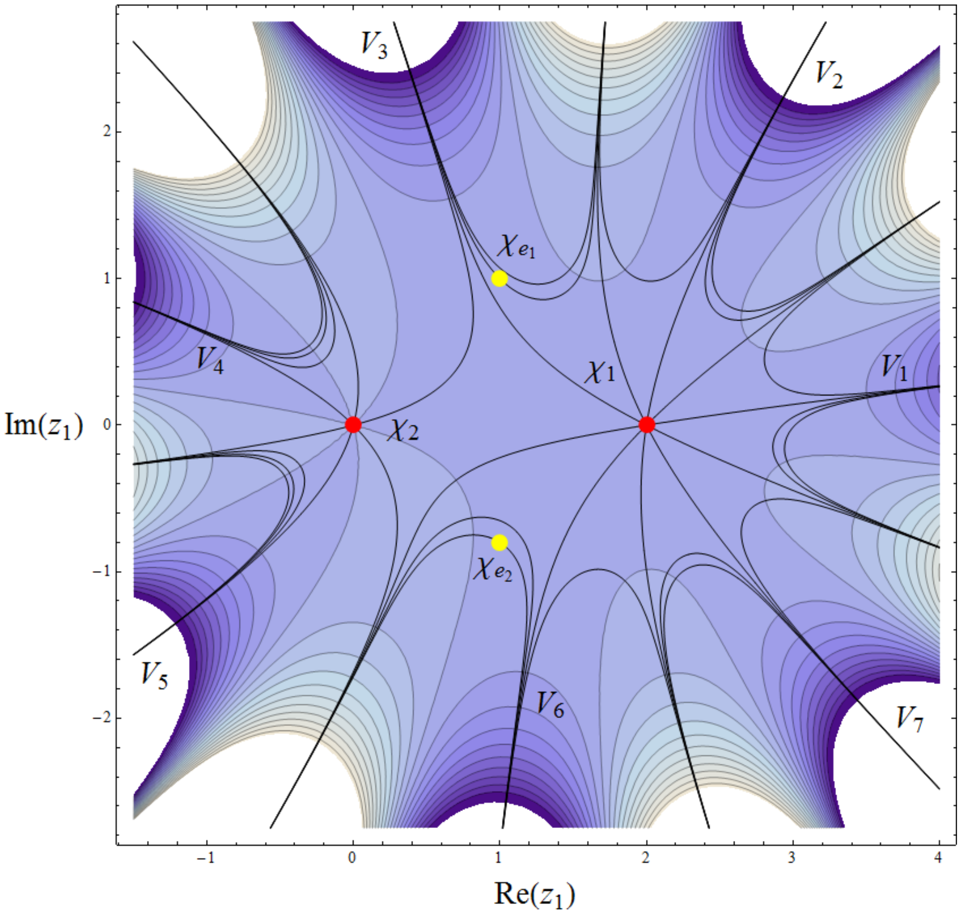


Figure 47: Contour plot of $\mathrm{Re}(-kf(\boldsymbol{z}))$ in complex z_1 space, with $k = 1 - i$. This contour plot is entirely unaffected by the variable z_2, since $f(\boldsymbol{z})$ is not a function of z_2.

to see that

$$T_I^{(e_1)}(k) = 0.0011735912\ldots - 0.056494277\ldots i,$$
$$T^{(e_1)}(k, 17) = 0.0011735928\ldots - 0.056494282\ldots i;$$

$$T_I^{(e_2)}(k) = 0.013839\ldots - 0.103311\ldots i,$$
$$T^{(e_2)}(k, 9) = 0.013858\ldots - 0.103334\ldots i.$$

We have now computed all of the components of $J(k)$ and can compare the result of the exact numerical integral (7.4) - denoted $J_I(k)$ - against the optimal level zero combination of expansions (7.5) - denoted $J_{(M)}(k)$ with $M = 0$; we have

$$J_I(k) = 0.001016935\ldots - 0.0024627114\ldots i,$$
$$J_{(0)}(k) = 0.001016947\ldots - 0.0024627171\ldots i.$$

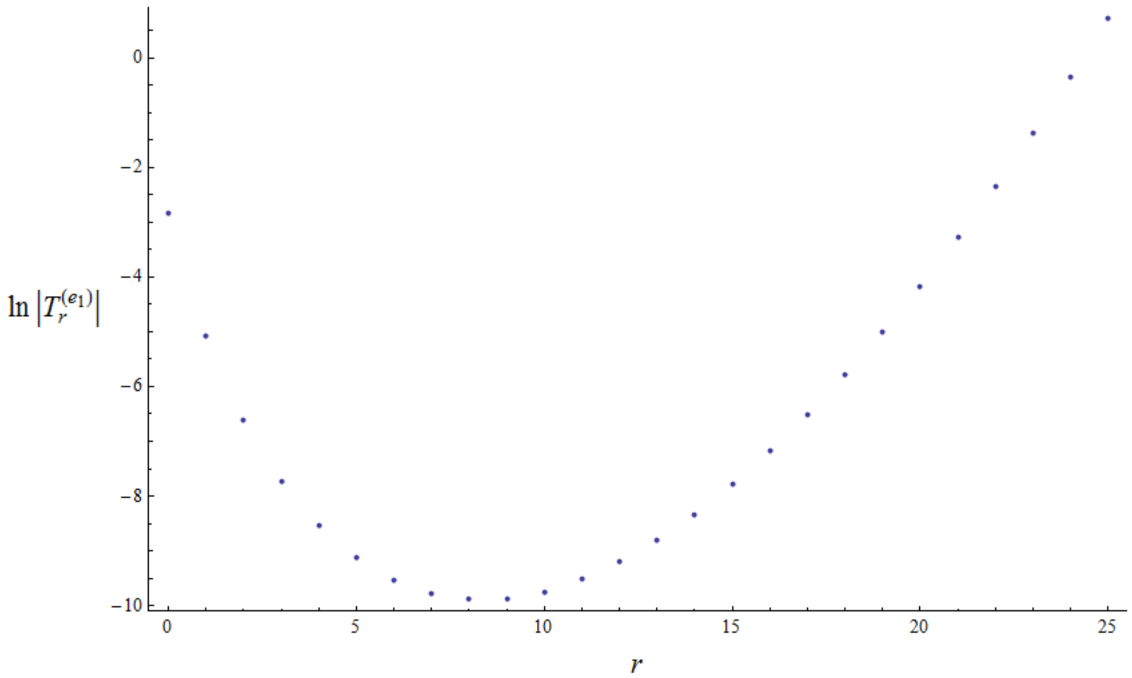


Figure 48: Coefficient size against term number for $T_r^{(e_1)}$.

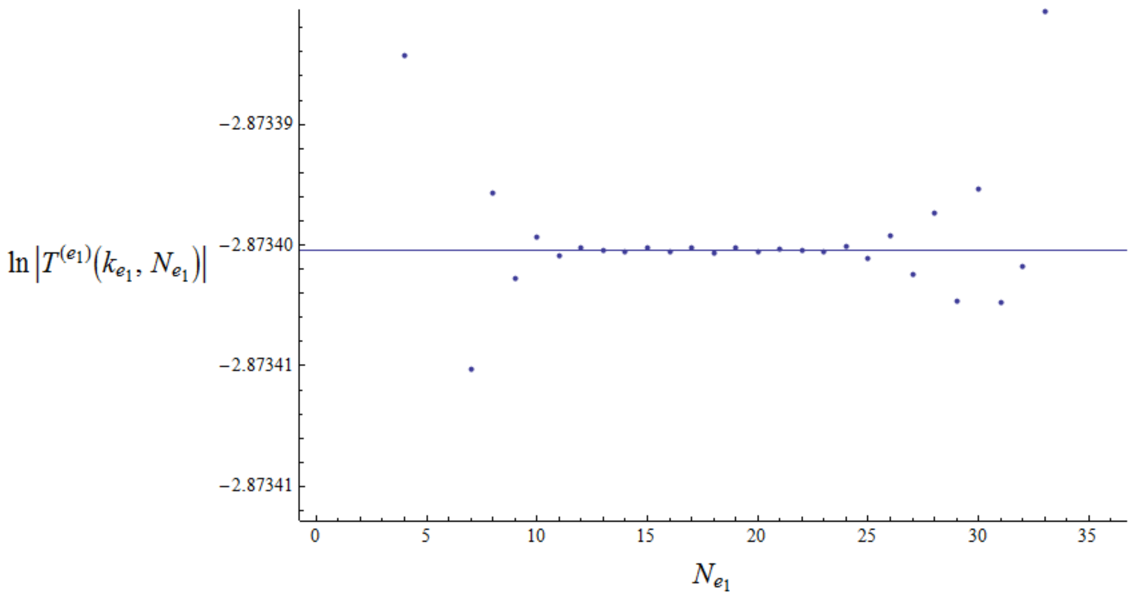


Figure 49: Expansion size against truncation point for $T^{(e_1)}(k_{e_1}, N_{e_1})$. The solid line is the value of the exact integral.

To compute higher level expansions, we need to discern the value of the quantities K_{em_1}, α_{em_1}, and β_{em_1} for both contour boundaries and we do this in the same way as for Example 1 (note that the quantities ρ_{em_1} and their higher level analogues do not feature in linear contour boundary expansions). We find that

$$K_{e_1 1} = 1,\ K_{e_1 2} = 0; \quad K_{e_2 1} = 1,\ K_{e_2 2} = -1;$$

$$(\alpha_{e_1 1}, \beta_{e_1 1}) = (\alpha_{e_1 2}, \beta_{e_1 2}) = (3, 2); \quad (\alpha_{e_2 2}, \beta_{e_2 2}) = (2, 0).$$

Curiously, $\alpha_{e_1 1}$ and $\beta_{e_1 1}$ are not as expected since by using the description on how to discern their values in §4, we should have that $(\alpha_{e_1 1}, \beta_{e_1 1}) = (2, 1)$. This is interesting as in all of the examples that we experimented with in which we computed critical component asymptotics (not all of which are featured in this thesis), we were able to correctly and consistently discern the values of α_{nm_1} and β_{nm_1} using the aforementioned description. Further investigation of this phenomenon in linear contour boundary asymptotics is warranted, but we did not have time to do so for this thesis.

Using (6.14), the truncation points for the level one contour boundary and critical component expansions are

$$\{N_1^{(1)}, N_{12}^{(1)}\} = \{91, 27\}, \quad \{N_{e_1}^{(1)}, N_{e_1 1}^{(1)}\} = \{26, 46\}, \quad \{N_{e_2}^{(1)}, N_{e_2 1}^{(1)}, N_{e_2 2}^{(1)}\} = \{18, 46, 4\}.$$

The truncation scheme (6.14) gives $N_{e_2 2}^{(1)} = 4$, but by manually checking we find that $N_{e_2 2}^{(1)} = 19$ is the optimal value for this truncation point, giving a very slightly more numerically accurate result. From Figure 51 we can see clearly see that in the level one expansion around χ_{e_2}, the terms $T_{r_1}^{(2)}(2,0)K_{r_1}^{(e_2 2)}(k, 18, r_1)$ are subdominant in this particular expansion as they are far smaller in size than the terms $T_{r_1}^{(1)}(3,2)K_{r_1}^{(e_2 1)}(k, 18, r_1)$. The fact that this branch of the expansion is subdominant means that the change in the truncation point is practically inconsequential, but we include it here for completeness.

The level one contour boundary expansions are compared against the level zero expansions and exact integrals below, where we use the absolute values for ease of comparison; we have

$$\begin{aligned} \left|T^{(e_1)}(k;17)\right| &= 0.05650647145623\ldots, \\ \left|T^{(e_1)}(k;26;46)\right| &= 0.05650646590974\ldots, \\ \left|T_I^{(e_1)}(k)\right| &= 0.05650646590967\ldots\ ; \end{aligned}$$

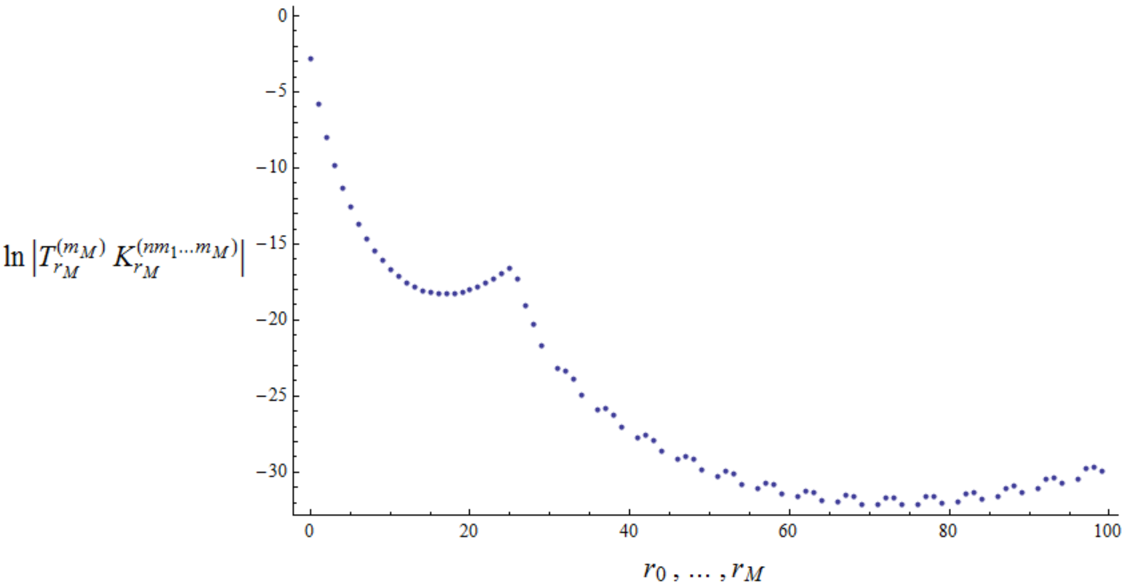


Figure 50: Size of the terms in the level one hyperasymptotic expansion $T^{(e_1)}(k; 26; 46)$ for $|k| = 2$.

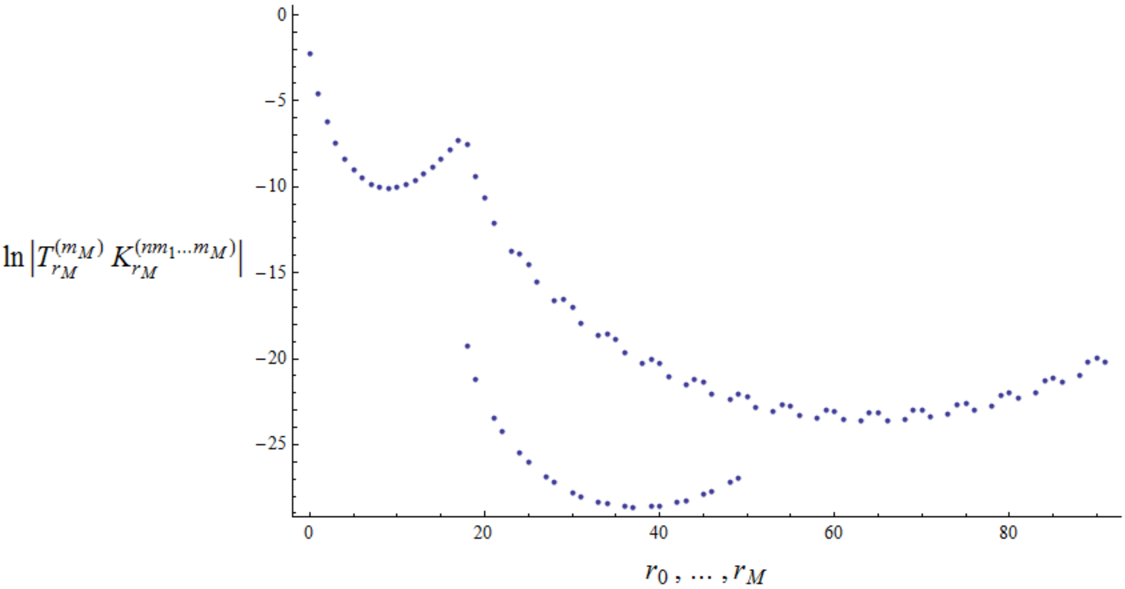


Figure 51: Size of the terms in the level one hyperasymptotic expansion $T^{(e_2)}(k; 18; 46, 19)$ for $|k| = 2$.

$$
\begin{aligned}
\left|T^{(e_2)}(k;9)\right| &= 0.104259471026\ldots,\\
\left|T^{(e_2)}(k;18;46,4)\right| &= 0.104234306292\ldots,\\
\left|T^{(e_2)}(k;18;46,19)\right| &= 0.104234306312\ldots,\\
\left|T_I^{(e_2)}(k)\right| &= 0.104234306495\ldots.
\end{aligned}
$$

The full level one expansion for $J(k)$ - along with the level zero expansion and exact integral for comparison - is then given by

$$
\begin{aligned}
\left|J_{(0)}(k)\right| &= 0.00266442459252\ldots,\\
\left|J_{(1)}(k)\right| &= 0.00266441472828\ldots,\\
\left|J_I(k)\right| &= 0.00266441472832\ldots,
\end{aligned}
$$

where we have again used the absolute value for ease of comparison.

7.3 Example 3 - Coefficient Analysis for Varying ω_n

In this example, we will focus on how the behaviour of the coefficients $T_r^{(n)}(\alpha_n,\beta_n)$ changes as ω_n varies. We have already encountered the fact that the coefficients have different behavioural patterns depending on α_n and β_n and we now investigate this in more detail.

Recall that in §7.1.1 we defined the quantity

$$
B_{\omega_n}^{(n)}(\alpha_n,\beta_n) := |\alpha_n - \beta_n \mod \omega_n|
$$

that takes values in the set $\{1,2,\ldots,\lfloor\frac{\omega_n}{2}\rfloor\}$, where $\lfloor..\rfloor$ is the floor function. For a given critical component χ_n of a function f, if we take the set of asymptotic expansions $T^{(n)}(k;\alpha_n,\beta_n)$ arising from considering all valid integration surfaces (namely, considering all valid combinations of α_n and β_n), then we can break this set down into equivalence classes based on the value of $B_{\omega_n}^{(n)}(\alpha_n,\beta_n)$. Defining the equivalence relation $\sim_R$ as "has the same coefficient behaviour", then

$$
T^{(n)}(k;\alpha_x,\beta_x) \sim_R T^{(n)}(k;\alpha_y,\beta_y) \iff B_{\omega_n}^{(n)}(\alpha_x,\beta_x) = B_{\omega_n}^{(n)}(\alpha_y,\beta_y).
$$

Hence, two expansions share the same coefficient behaviour if and only if they have the same value of $B_{\omega_n}^{(n)}(\alpha_n,\beta_n)$. This equivalent relation partitions the set of all possible asymptotic expansions of χ_n for a given f as

$$\left\{\left[T^{(n)}(k;1,0)\right], \left[T^{(n)}(k;2,0)\right], \ldots, \left[T^{(n)}\left(k; \left\lfloor \frac{\omega_n}{2} \right\rfloor, 0\right)\right]\right\}.$$

To explore these coefficient patterns, we investigate the one-dimensional complex function

$$f(z) = z^{\omega}\left(\frac{z}{\omega+1} - \frac{a}{\omega}\right), \tag{7.6}$$

where $a \in \mathbb{C}$, which has derivative

$$\frac{df}{dz} = z^{\omega-1}(z-a),$$

so that f has an order ω critical point z_1 at $z = 0$ and an order 2 critical point z_2 at $z = a$. We will focus solely on the order ω critical point z_1 at the origin and therefore drop all subscripts for cleaner expressions.

Note that the fact that we are in one complex dimension only is irrelevant for the purposes of this example, since it is clear from (5.13) that the co-dimension q has no effect on the coefficient behavioural patterns. From a behavioural point of view, increasing q essentially shifts each coefficient back q places, since we can replace r with $r - q$ in (5.13) (as q is always an integer).

The general asymptotic coefficients for this function are given by

$$T_r^{(1)}(\alpha,\beta) = \left(e^{\frac{2\pi i\beta(r+1)}{\omega}} - e^{\frac{2\pi i\alpha(r+1)}{\omega}}\right)\left(-\frac{a}{\omega}\right)^{-\frac{r+1}{\omega}} \frac{\Gamma\left(\frac{r+1}{\omega}+r\right)\omega^{r-1}}{\Gamma(r+1)(\omega+1)^r a^r} \tag{7.7}$$

and we now plot these coefficients for various ω. We will plot all the coefficient behavioural patterns separately and then together, for $\omega \in \{4, \ldots, 11\}$. We omit $\omega = 2$ and 3 as there is only one coefficient behavioural pattern for both cases and we already know what these patterns look like; the third order coefficient pattern can be seen in the previous examples (Figure 28 for example) and the second order pattern is simply the classic result from the literature (identical to the order three case but every second term is zero instead of every third term). These patterns are shown separately in Figures 52 - 79 and then together in Figures 80 - 87. In these figures, we omit the α and β dependency from $T_r^{(1)}$ to save space.

We can see that some of the higher order coefficient patterns are the same as some of the lower order patterns. For example, Figure 68 for $B_9^{(1)}(\alpha,\beta) = 3$ has the same behaviour as that of $B_3^{(1)}(\alpha,\beta) = 1$. Although the latter is not displayed explicitly in this section, it is in Figure 28 as the coefficient behaviour of the expansion around χ_2 in Example 1. We can also see that Figures 71 and 54 - for $B_{10}^{(1)}(\alpha,\beta) = 2$ and $B_5^{(1)}(\alpha,\beta) = 1$ respectively - have the same behaviour, as well as Figures 73 and 55 - for $B_{10}^{(1)}(\alpha,\beta) = 4$ and $B_5^{(1)}(\alpha,\beta) = 2$ respectively.

In general, we can deduce by inspection that the pattern will be the same as long as the quantity

$$\frac{B_{\omega_n}^{(n)}(\alpha_n,\beta_n)}{\omega_n}$$

is the same. This is also intuitive as the above quantity is related to the phase of the coefficients, but we can cancel any common factors to reduce it to its simplest form (as it's a rational number, since $B_{\omega_n}^{(1)}$ and ω_n are natural numbers). If any factors can be cancelled, the behaviour will naturally change to reflect the new denominator's value and we can see this in action for the examples discussed earlier; we have

$$\frac{3}{9} = \frac{1}{3}, \quad \frac{2}{10} = \frac{1}{5}, \quad \text{and} \quad \frac{4}{10} = \frac{2}{5},$$

and can see that this order nine case behaves like order three and these order ten cases behave like order five.

When ω_n is prime, it will always be coprime to $B_{\omega_n}^{(n)}(\alpha_n,\beta_n)$ meaning no cancellation can take place. We can see the effect of this in Figures 52 - 79 for prime ω, as we get the full $(\omega-1)/2$ unique coefficient patterns. It is worth pointing out that none of the coefficients $T_{r,X}^{(n)}$ along a single leg of χ_n are identically zero in general; the only reason some of the coefficients $T_r^{(n)}(\alpha_n,\beta_n)$ are identically zero is due to the phase factors cancelling for certain r, governed by equation (5.14).

Having demonstrated our new theory with a range of examples, we move on to discuss the work required to progress toward a hyperasymptotic expansion for integral (1.1) in the presence of a general boundary.

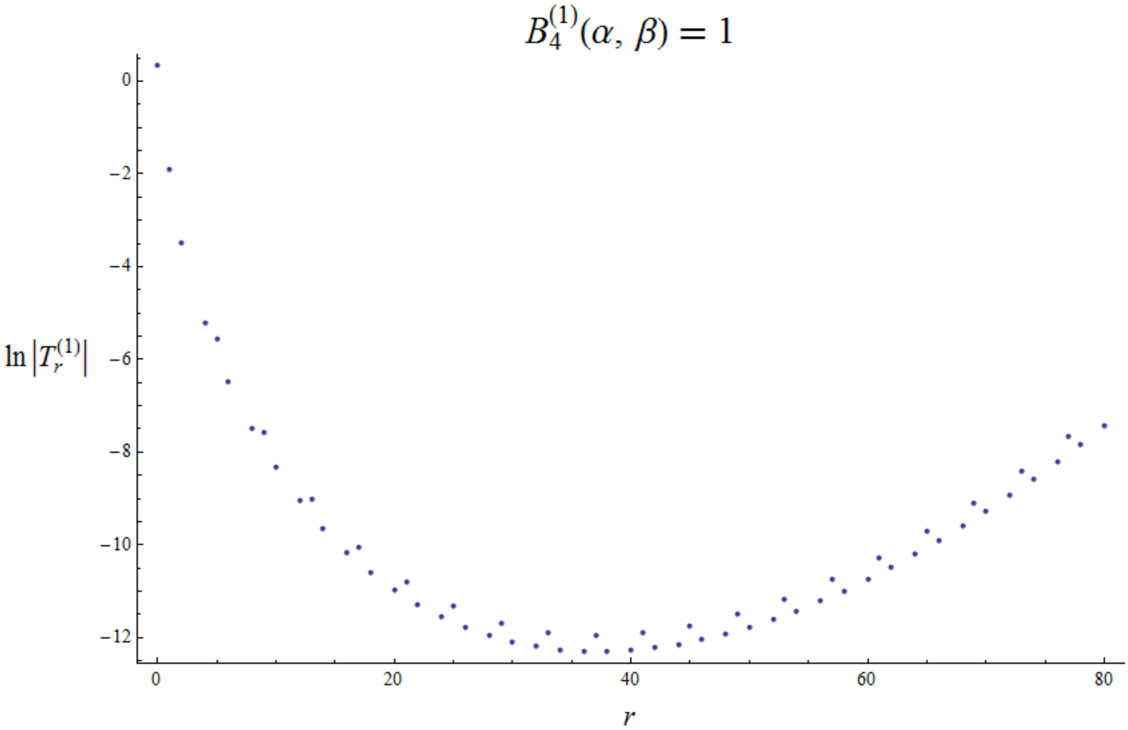


Figure 52: Coefficient size for $B_4^{(1)}(\alpha, \beta) = 1$.

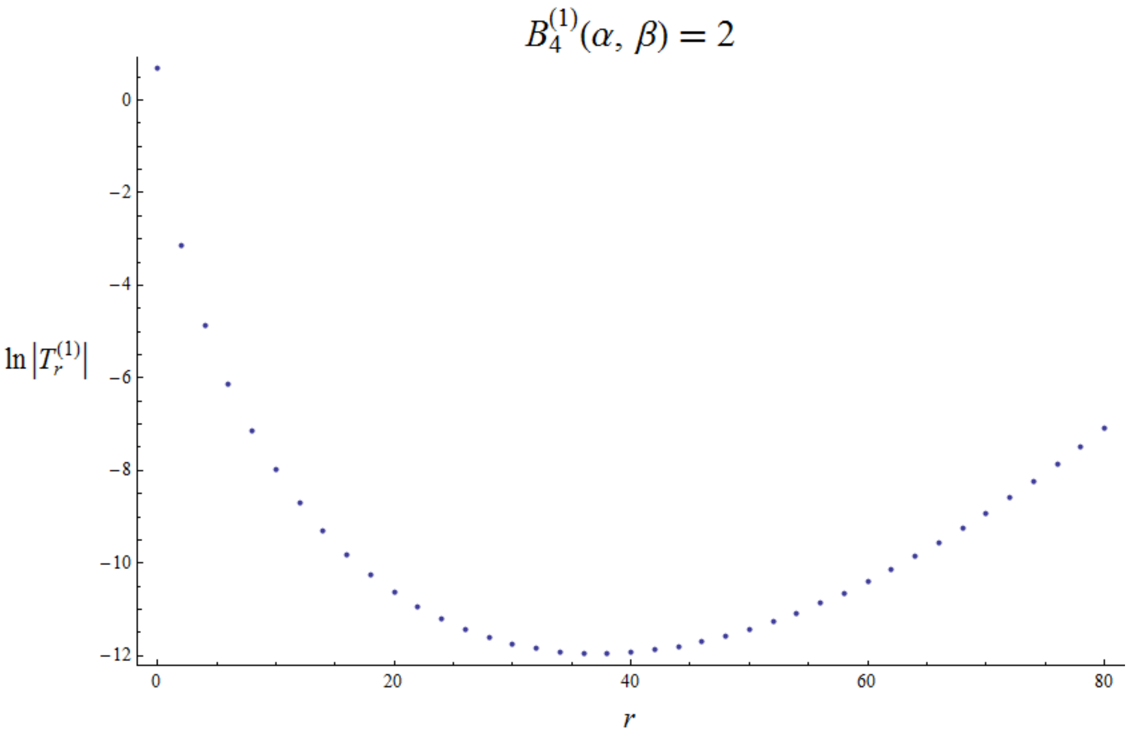


Figure 53: Coefficient size for $B_4^{(1)}(\alpha, \beta) = 2$.

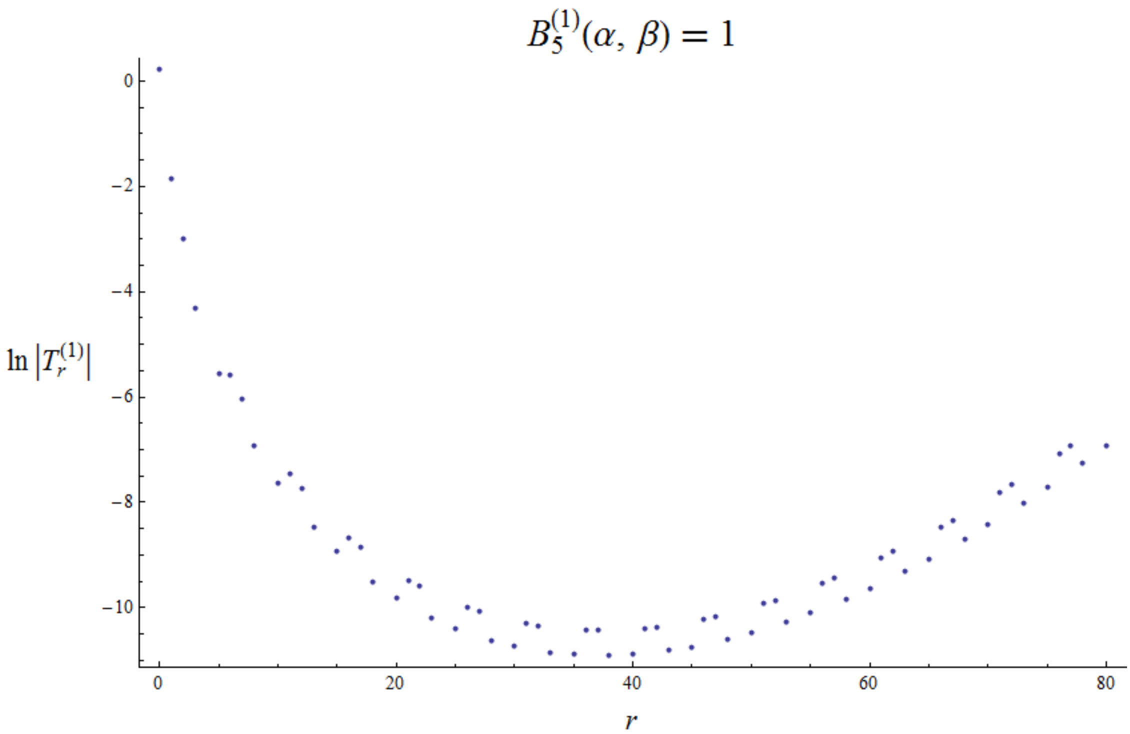


Figure 54: Coefficient size for $B_5^{(1)}(\alpha, \beta) = 1$.

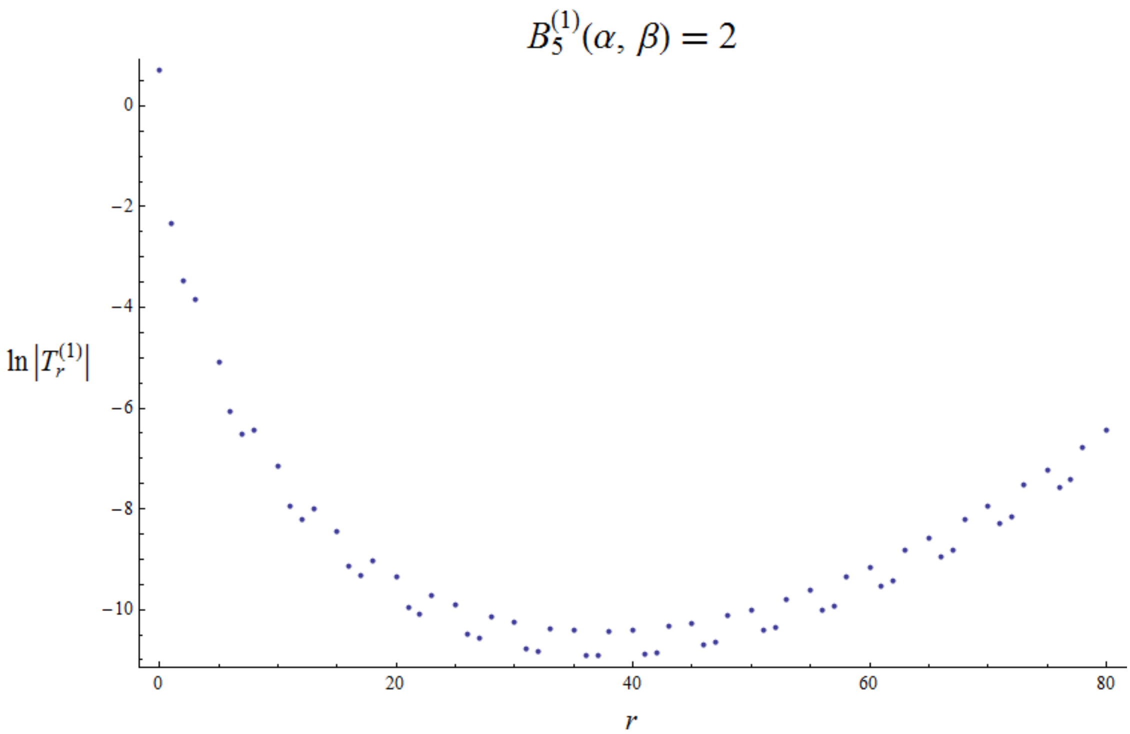


Figure 55: Coefficient size for $B_5^{(1)}(\alpha, \beta) = 2$.

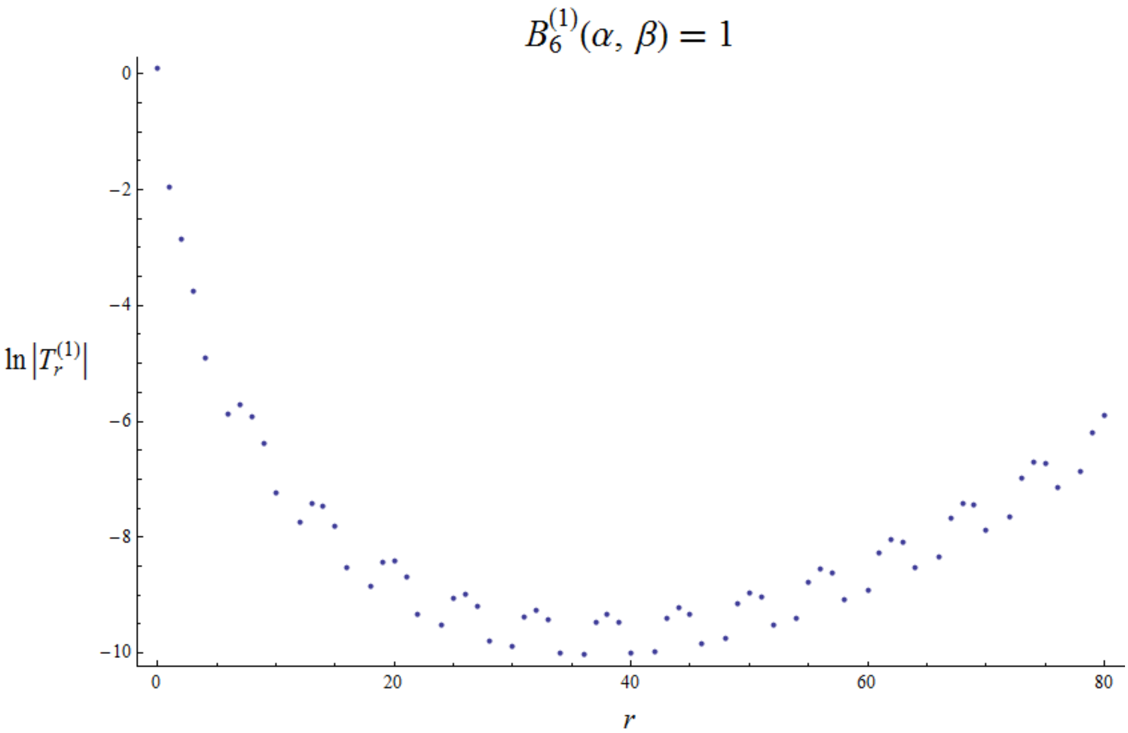


Figure 56: Coefficient size for $B_6^{(1)}(\alpha, \beta) = 1$.

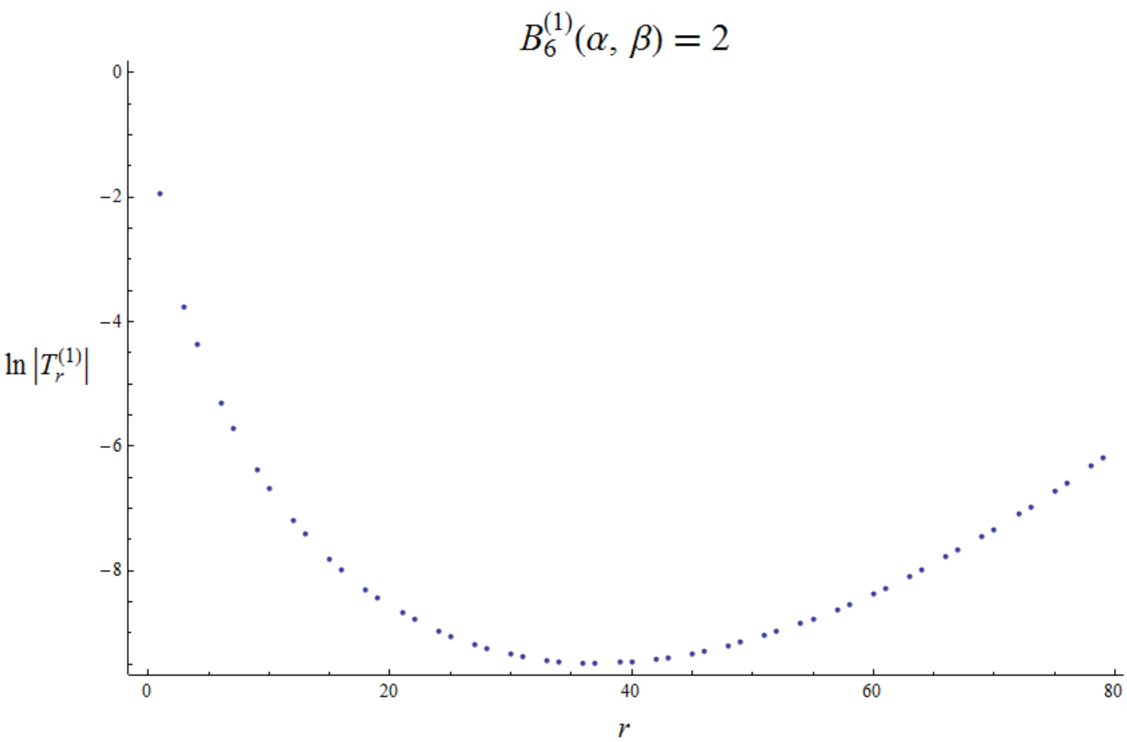


Figure 57: Coefficient size for $B_6^{(1)}(\alpha, \beta) = 2$.

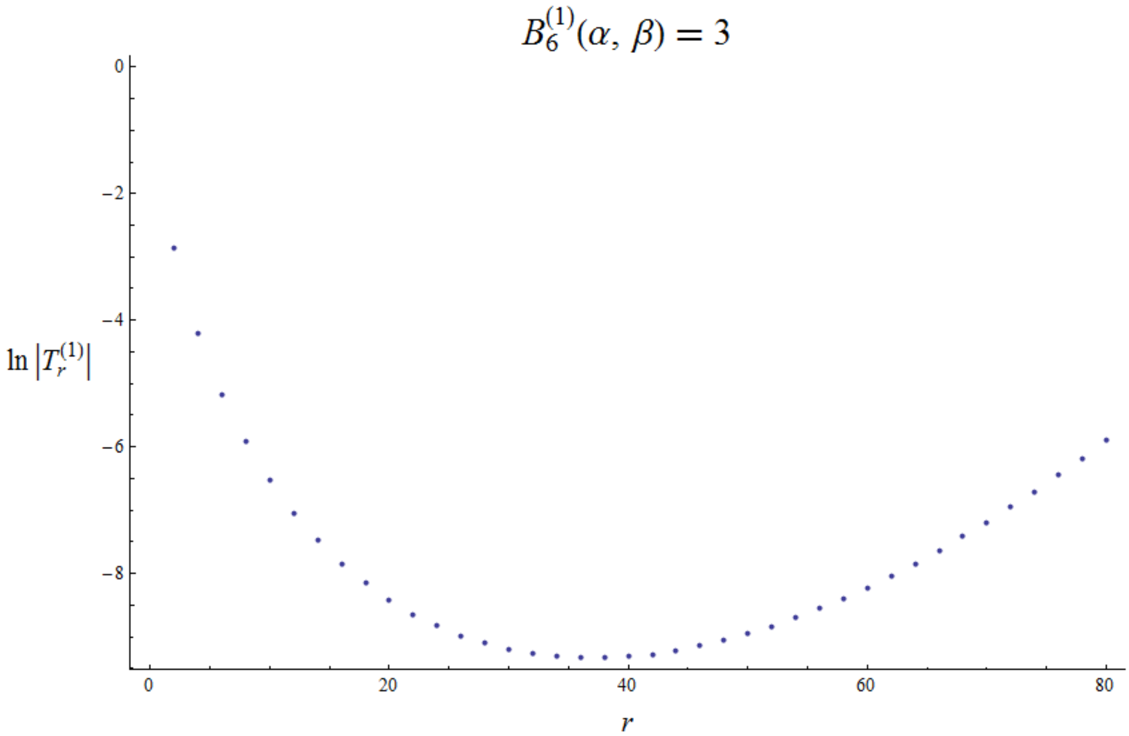


Figure 58: Coefficient size for $B_6^{(1)}(\alpha, \beta) = 3$.

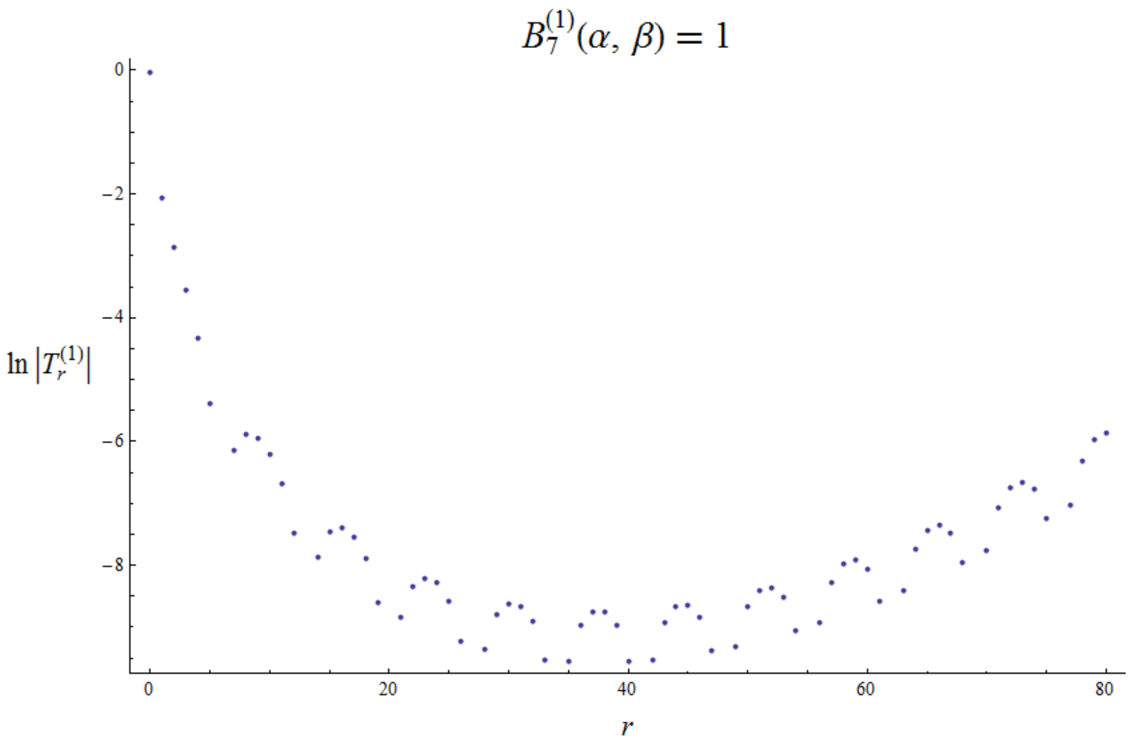


Figure 59: Coefficient size for $B_7^{(1)}(\alpha, \beta) = 1$.

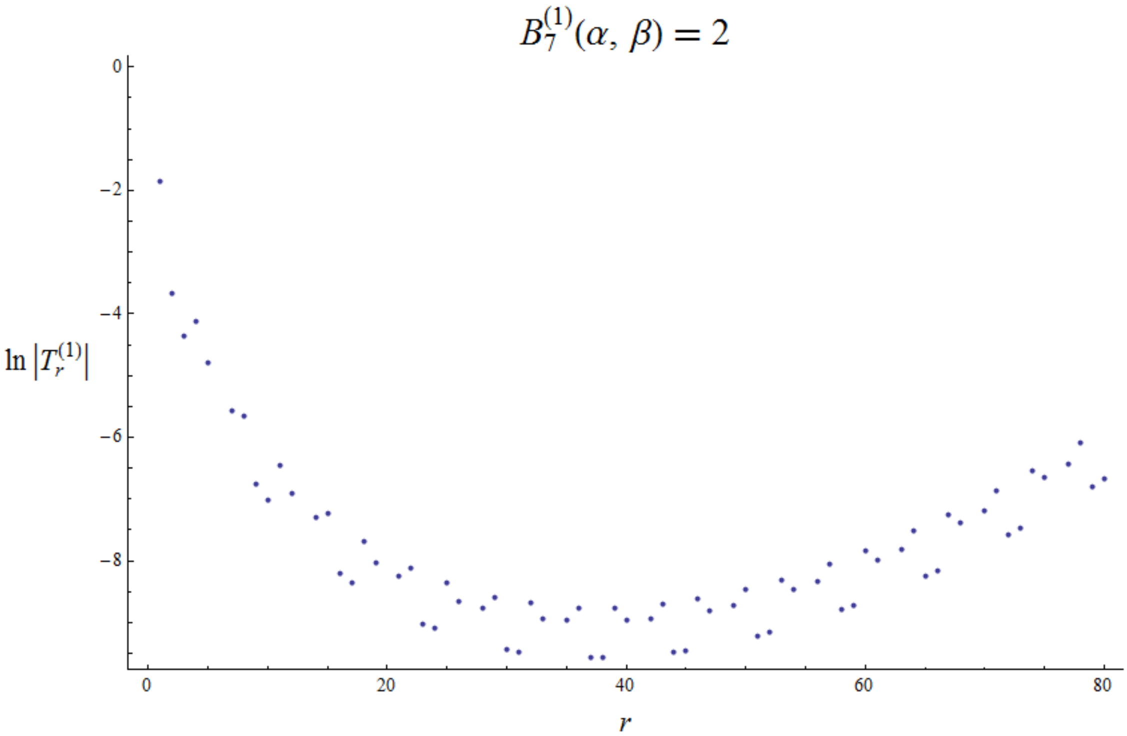


Figure 60: Coefficient size for $B_7^{(1)}(\alpha, \beta) = 2$.

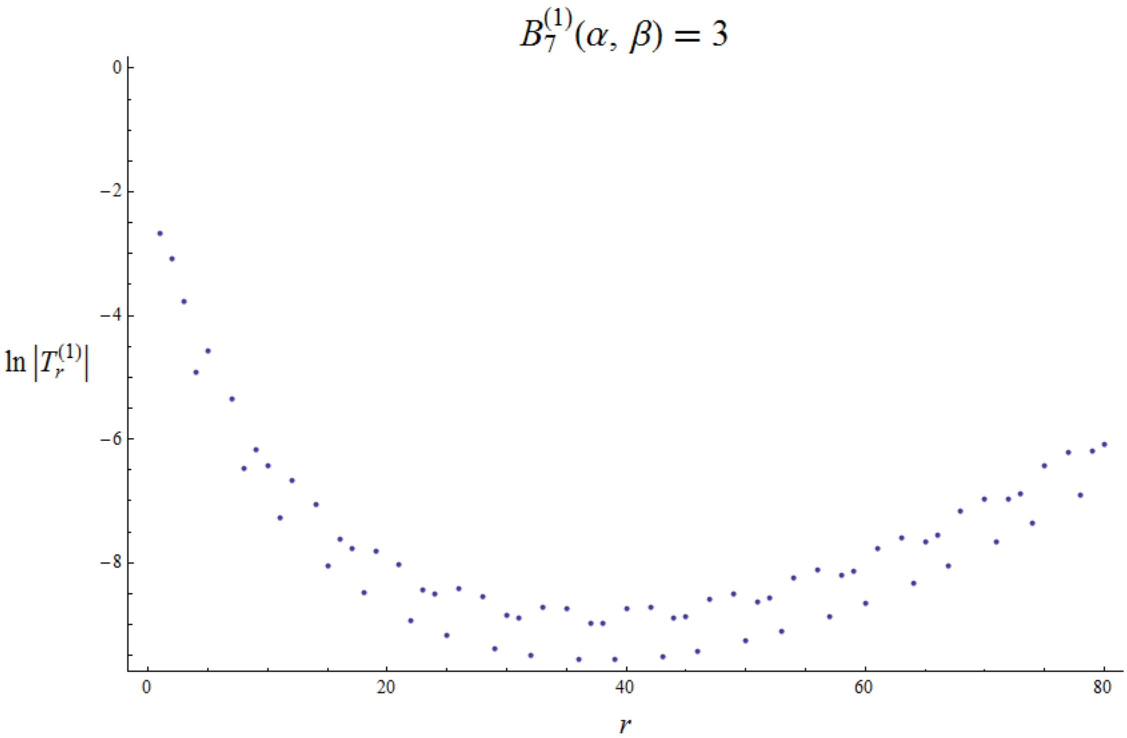


Figure 61: Coefficient size for $B_7^{(1)}(\alpha, \beta) = 3$.

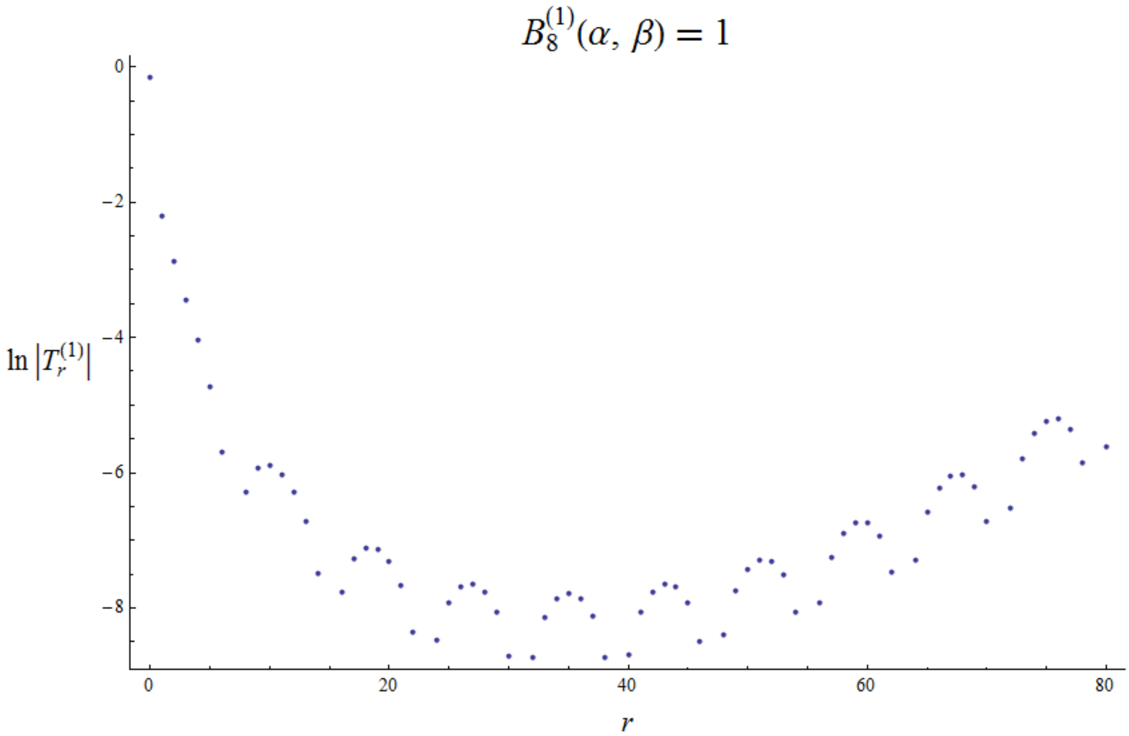


Figure 62: Coefficient size for $B_8^{(1)}(\alpha,\beta) = 1$.

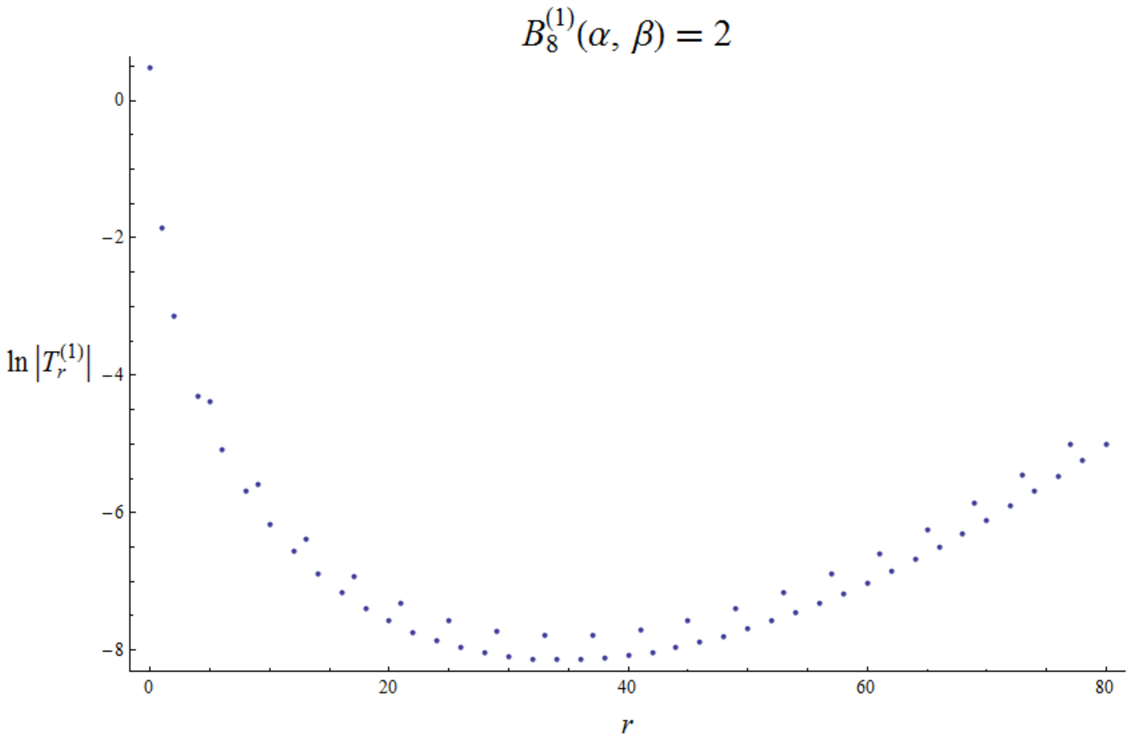


Figure 63: Coefficient size for $B_8^{(1)}(\alpha,\beta) = 2$.

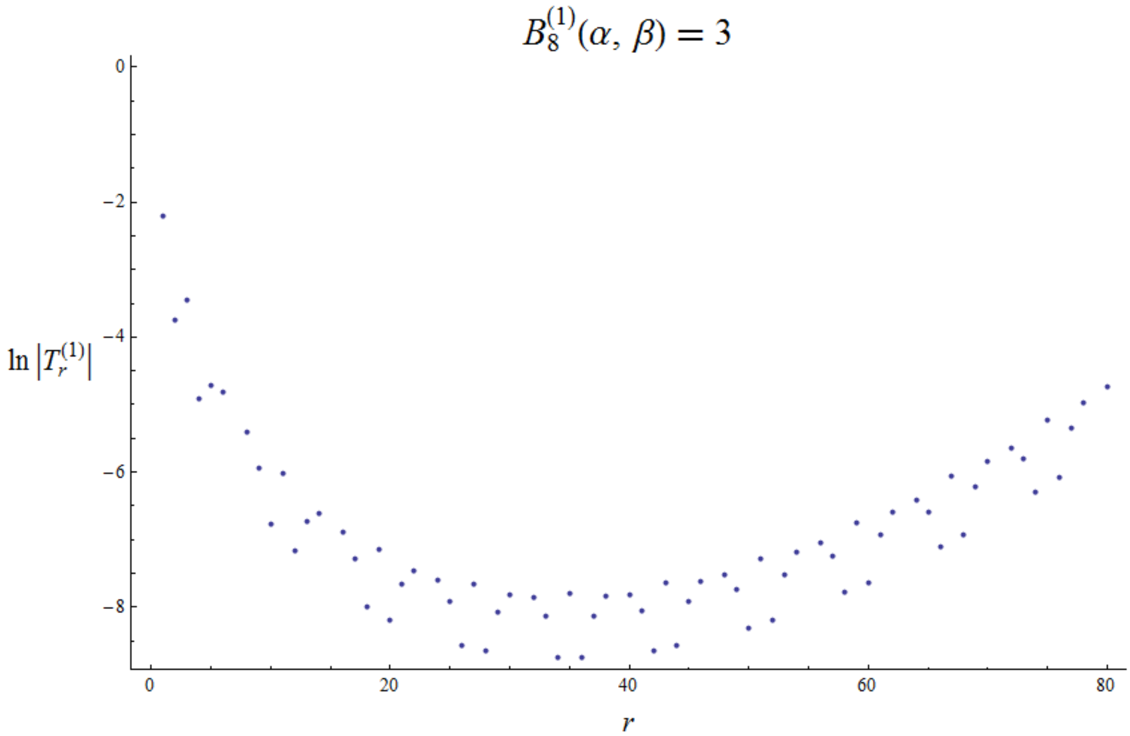


Figure 64: Coefficient size for $B_8^{(1)}(\alpha, \beta) = 3$.

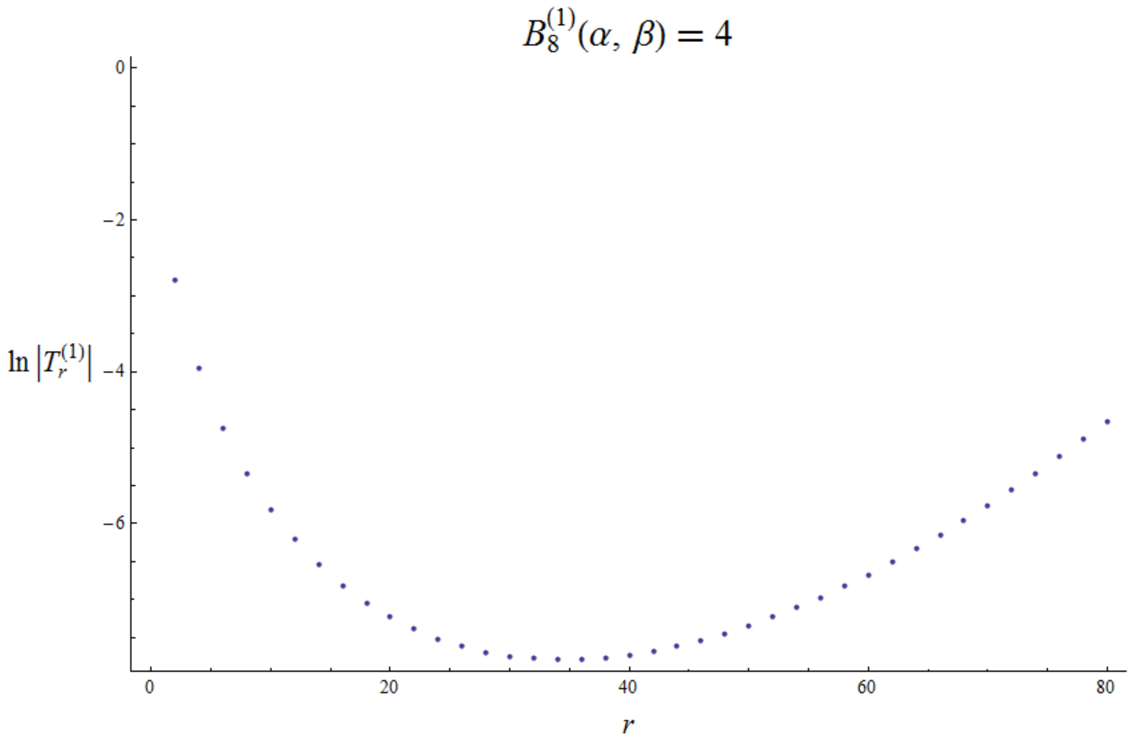


Figure 65: Coefficient size for $B_8^{(1)}(\alpha, \beta) = 4$.

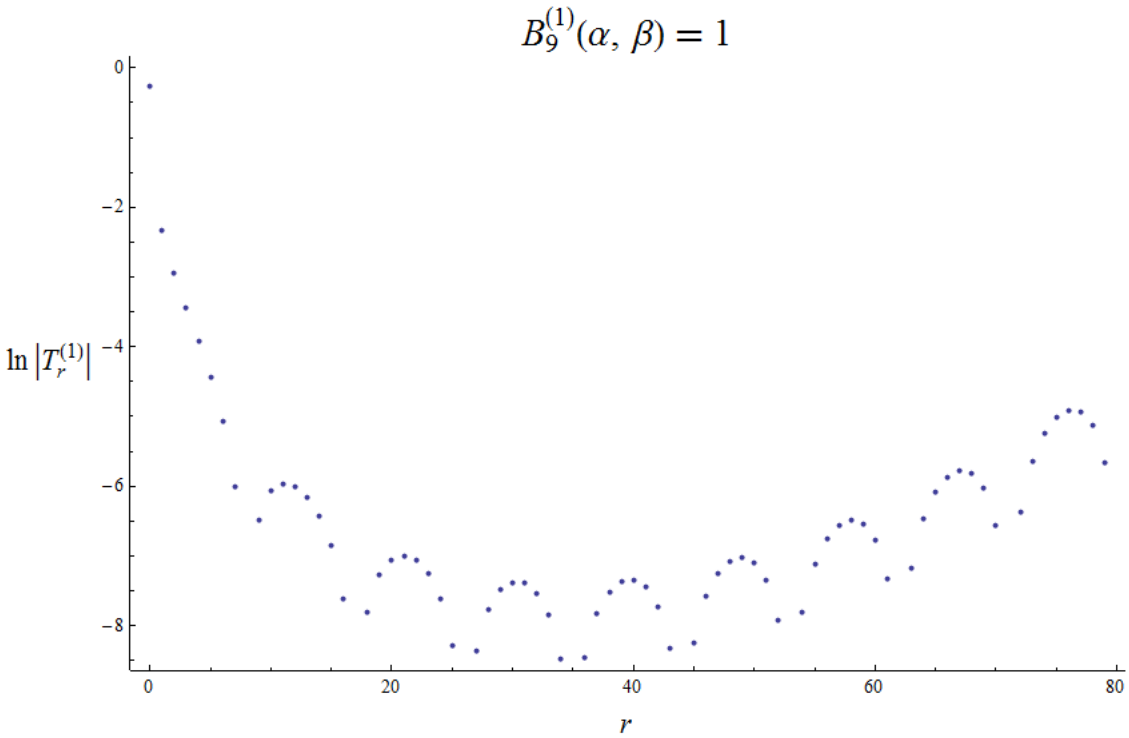


Figure 66: Coefficient size for $B_9^{(1)}(\alpha, \beta) = 1$.

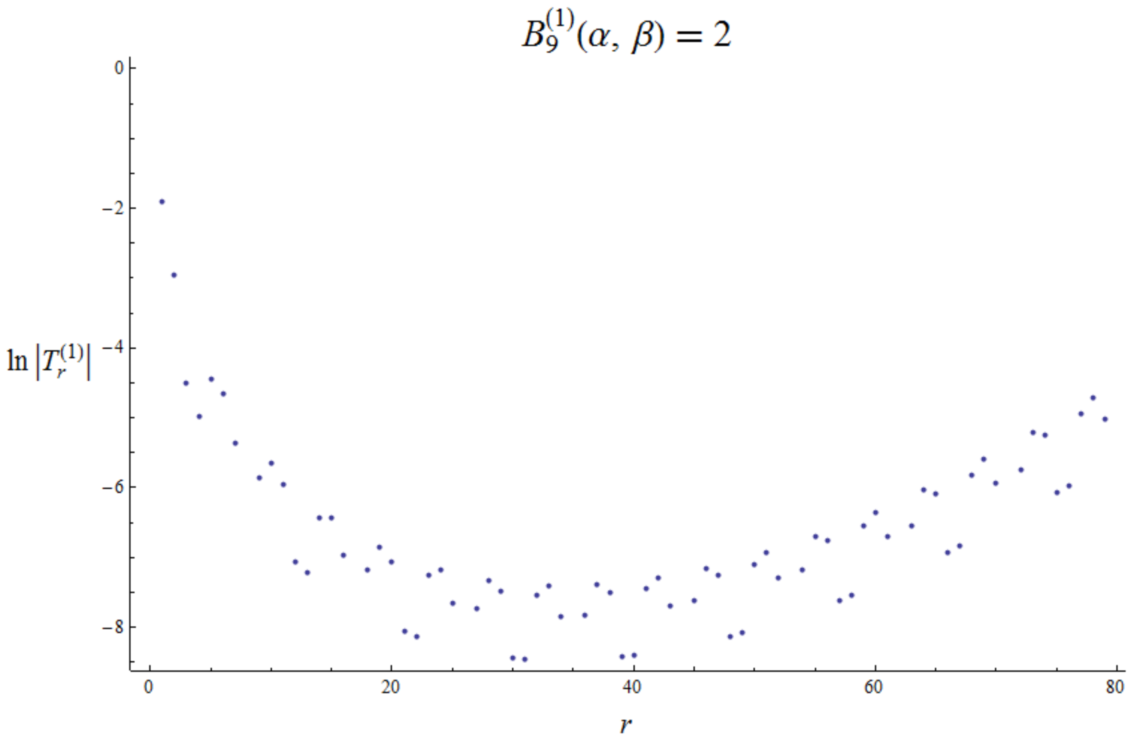


Figure 67: Coefficient size for $B_9^{(1)}(\alpha, \beta) = 2$.

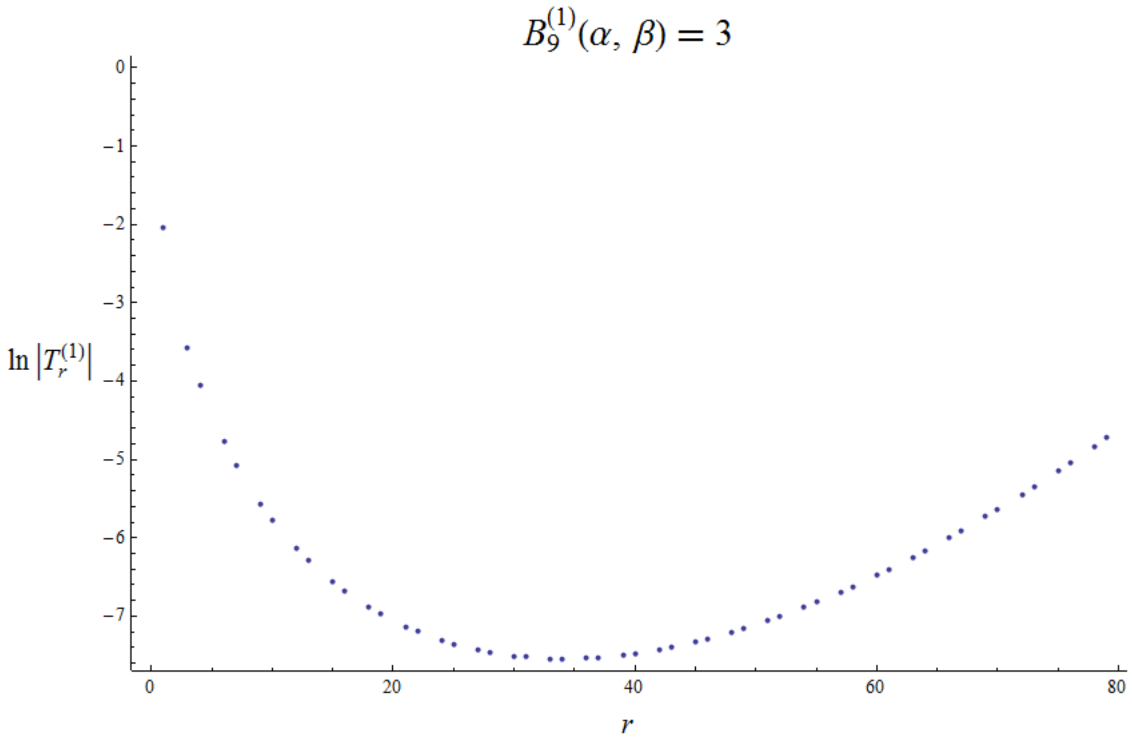


Figure 68: Coefficient size for $B_9^{(1)}(\alpha, \beta) = 3$.

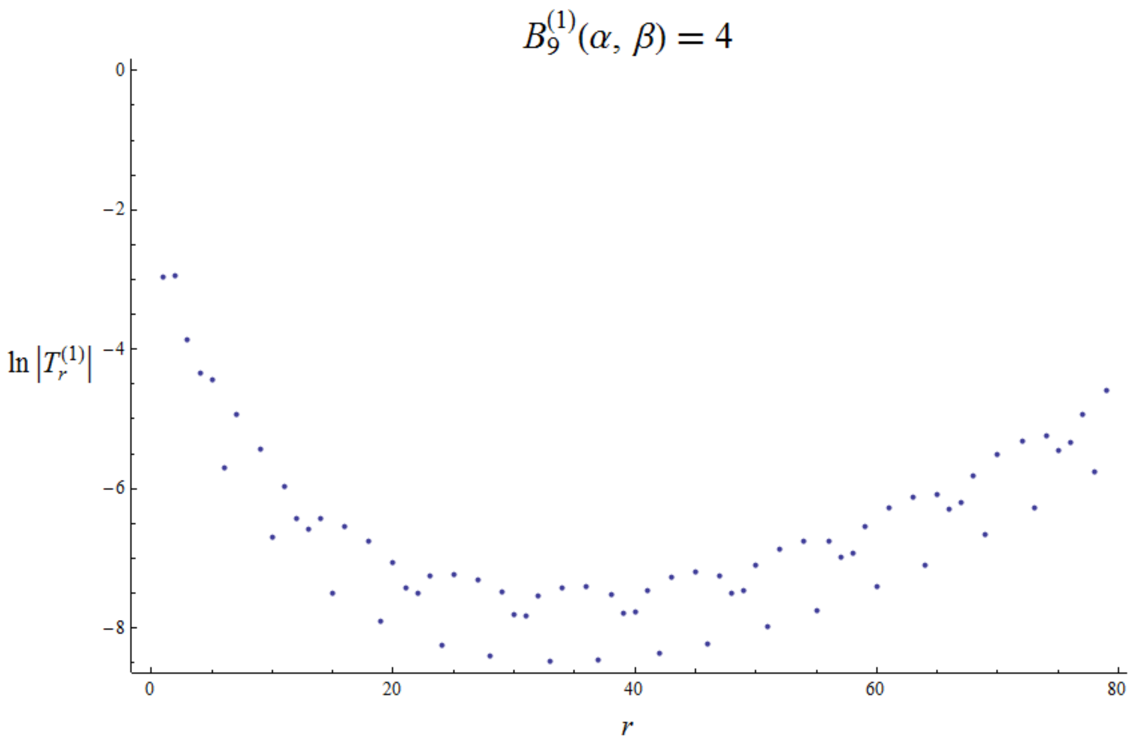


Figure 69: Coefficient size for $B_9^{(1)}(\alpha, \beta) = 4$.

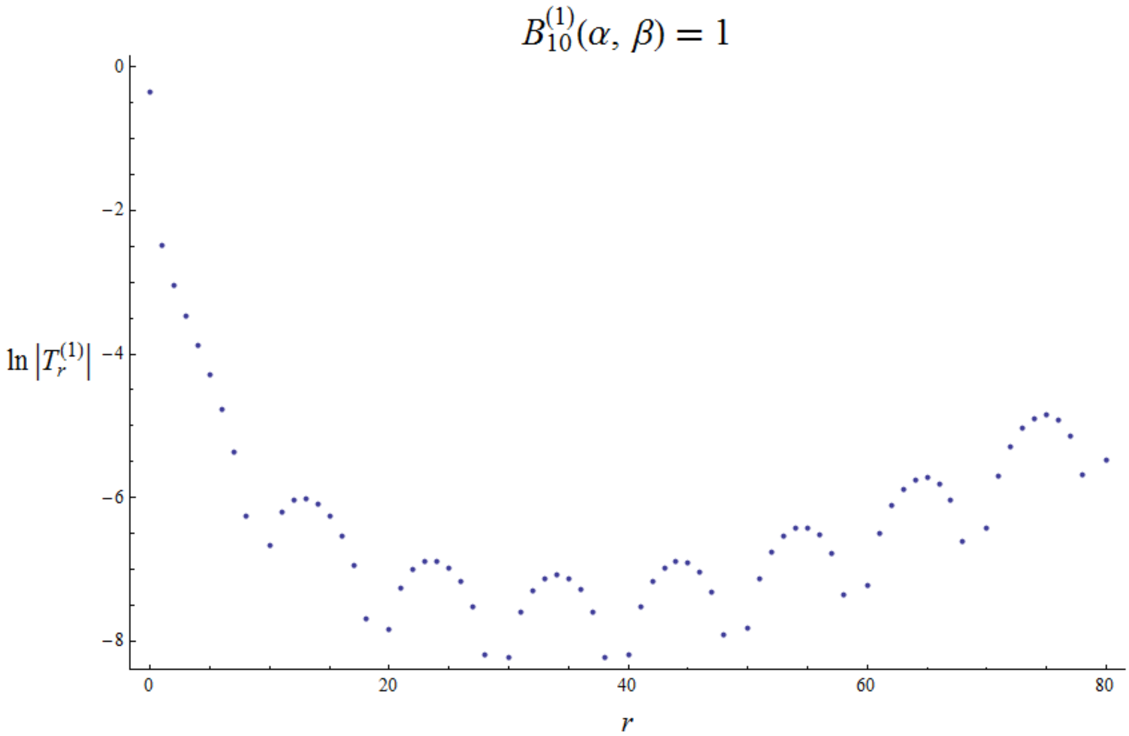


Figure 70: Coefficient size for $B_{10}^{(1)}(\alpha, \beta) = 1$.

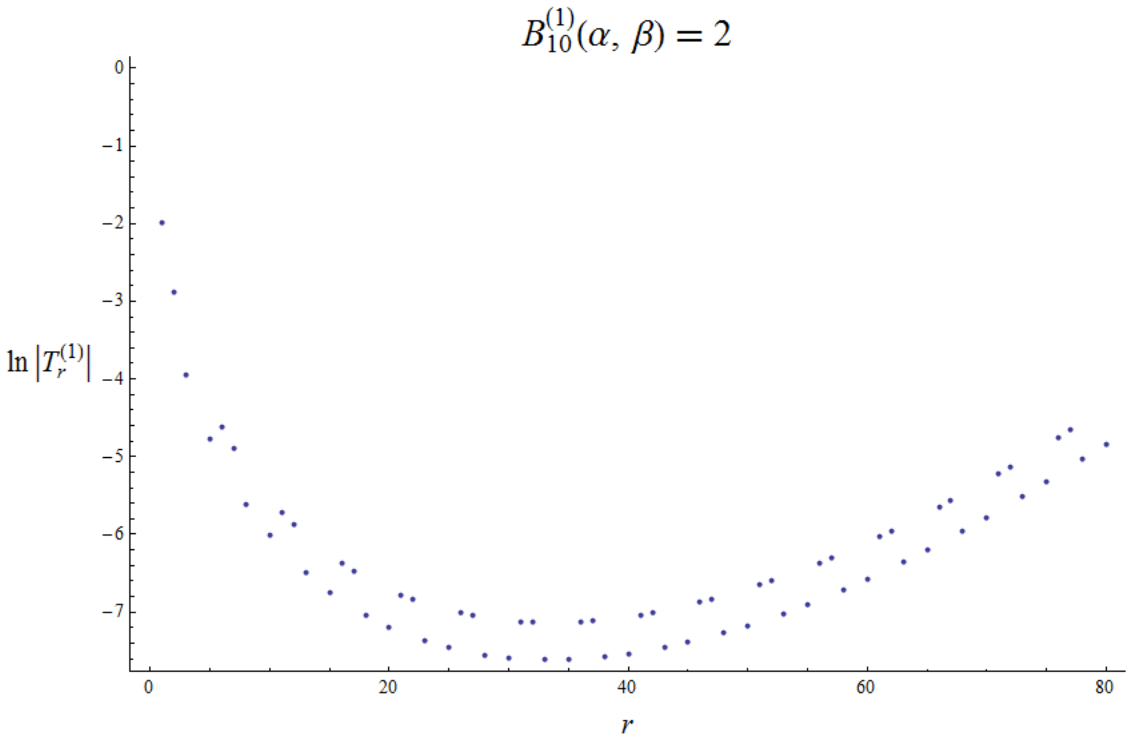


Figure 71: Coefficient size for $B_{10}^{(1)}(\alpha, \beta) = 2$.

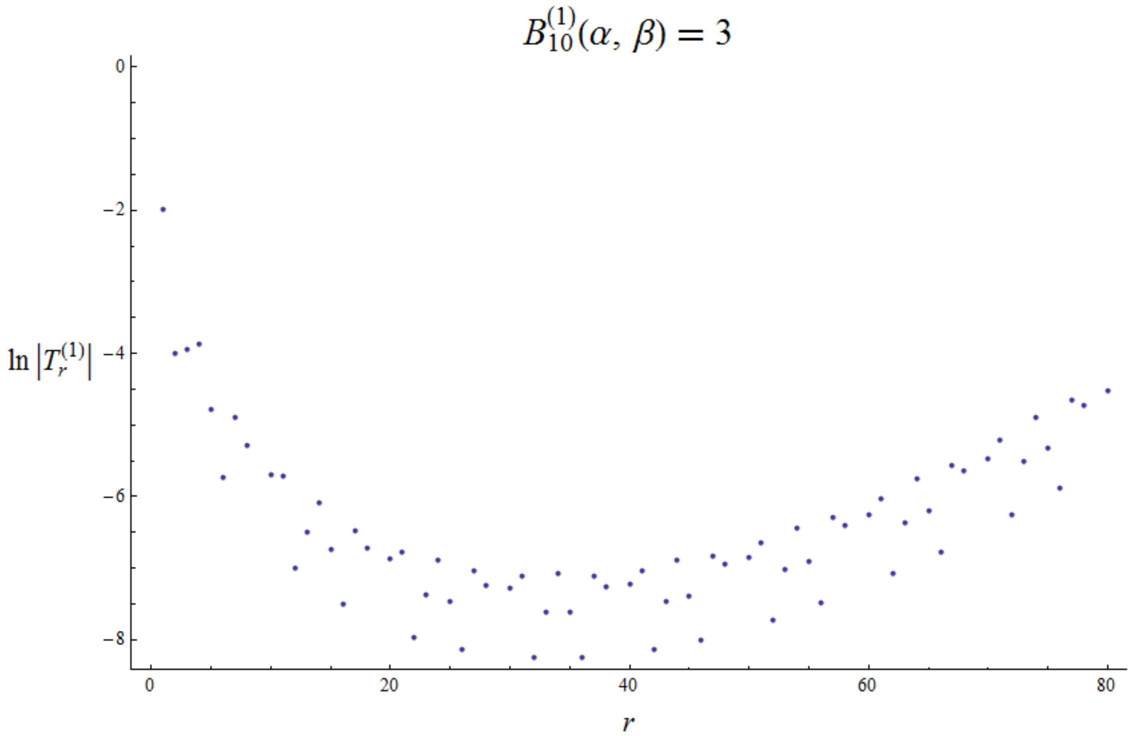


Figure 72: Coefficient size for $B_{10}^{(1)}(\alpha, \beta) = 3$.

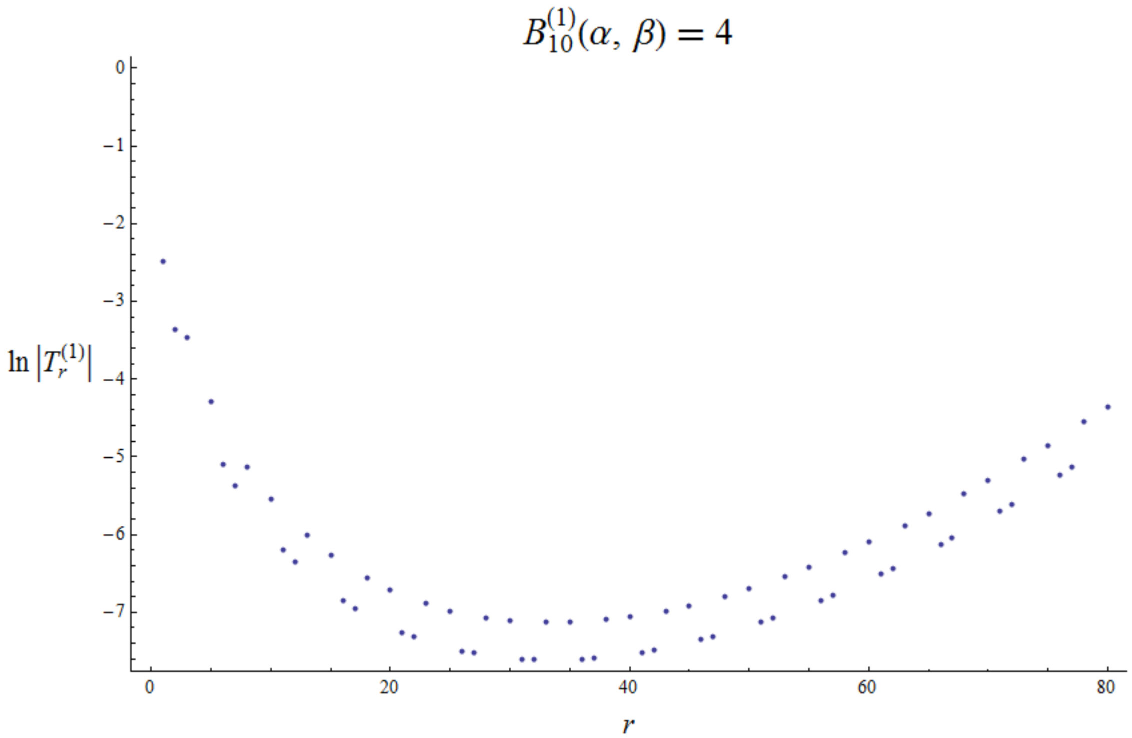


Figure 73: Coefficient size for $B_{10}^{(1)}(\alpha, \beta) = 4$.

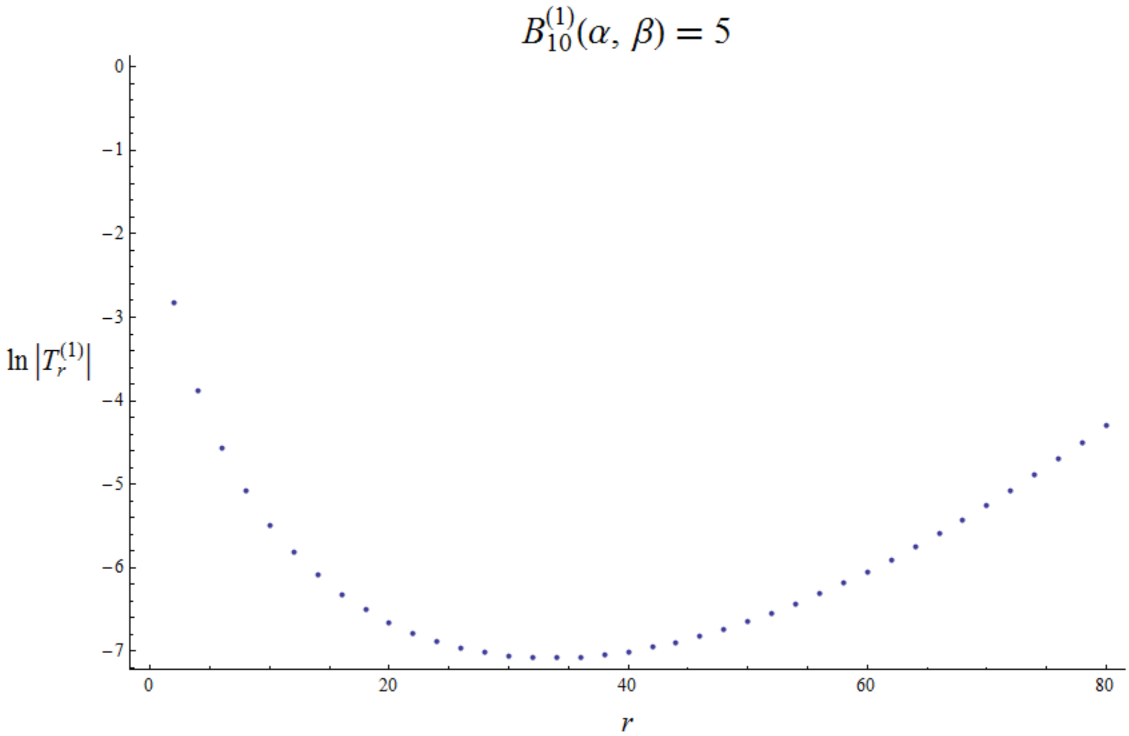


Figure 74: Coefficient size for $B_{10}^{(1)}(\alpha, \beta) = 5$.

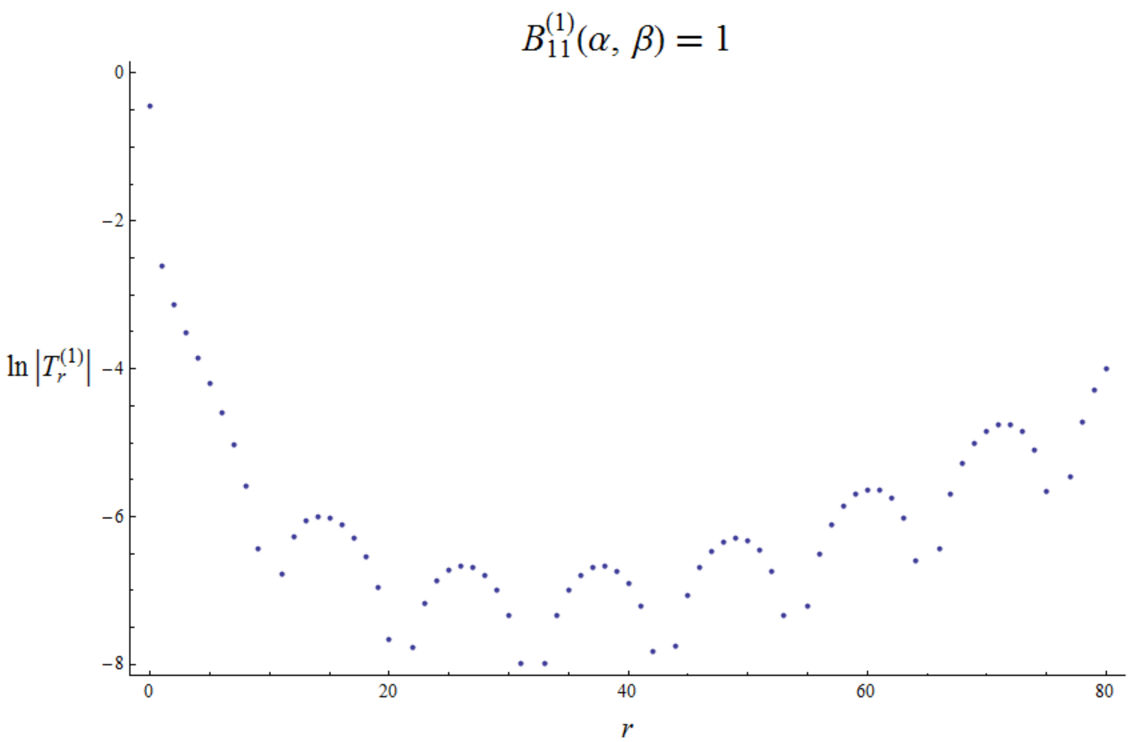


Figure 75: Coefficient size for $B_{11}^{(1)}(\alpha, \beta) = 1$.

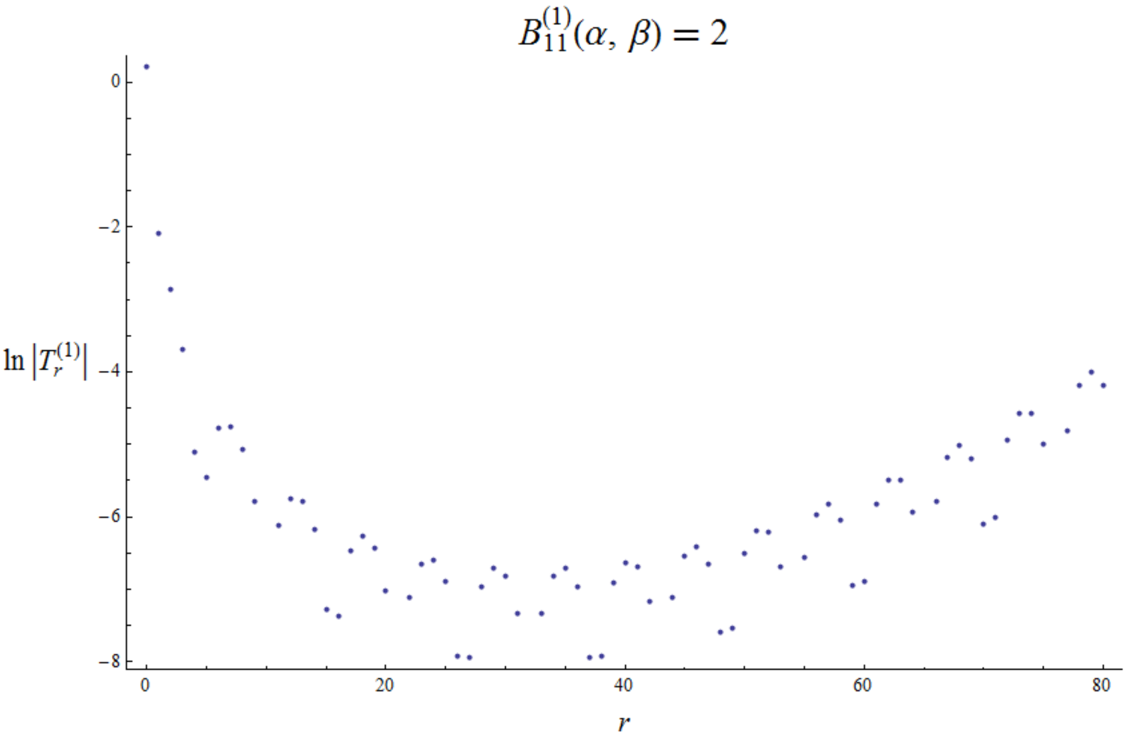


Figure 76: Coefficient size for $B_{11}^{(1)}(\alpha, \beta) = 2$.

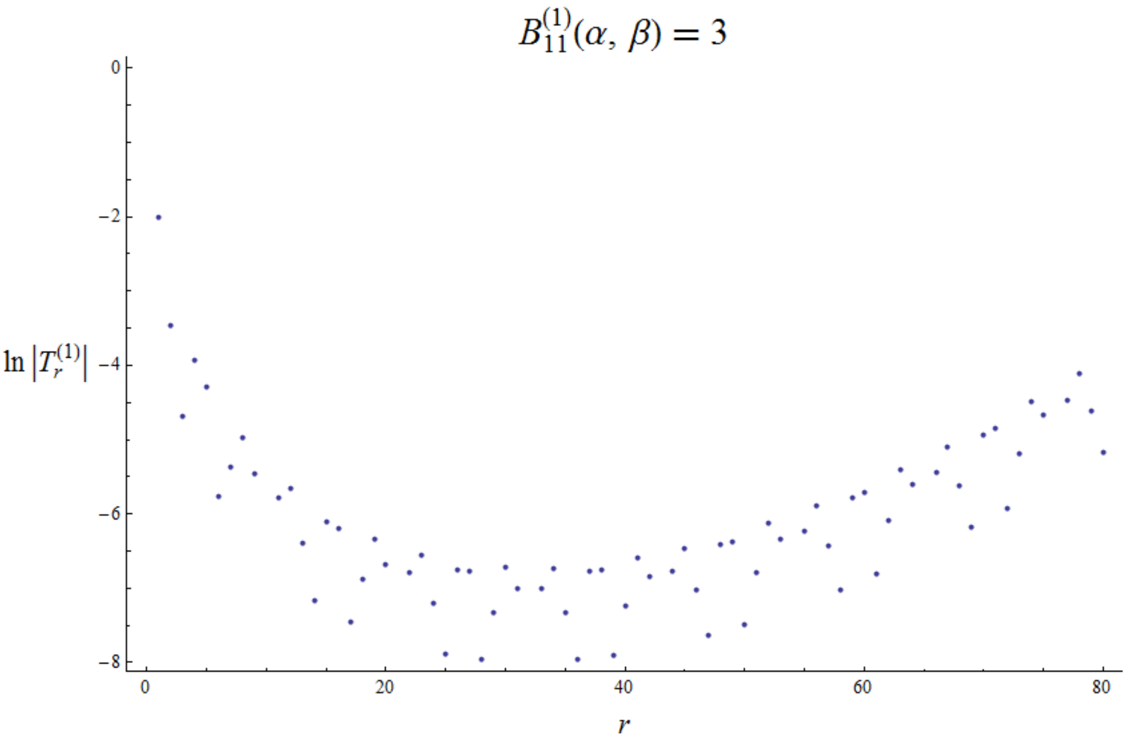


Figure 77: Coefficient size for $B_{11}^{(1)}(\alpha, \beta) = 3$.

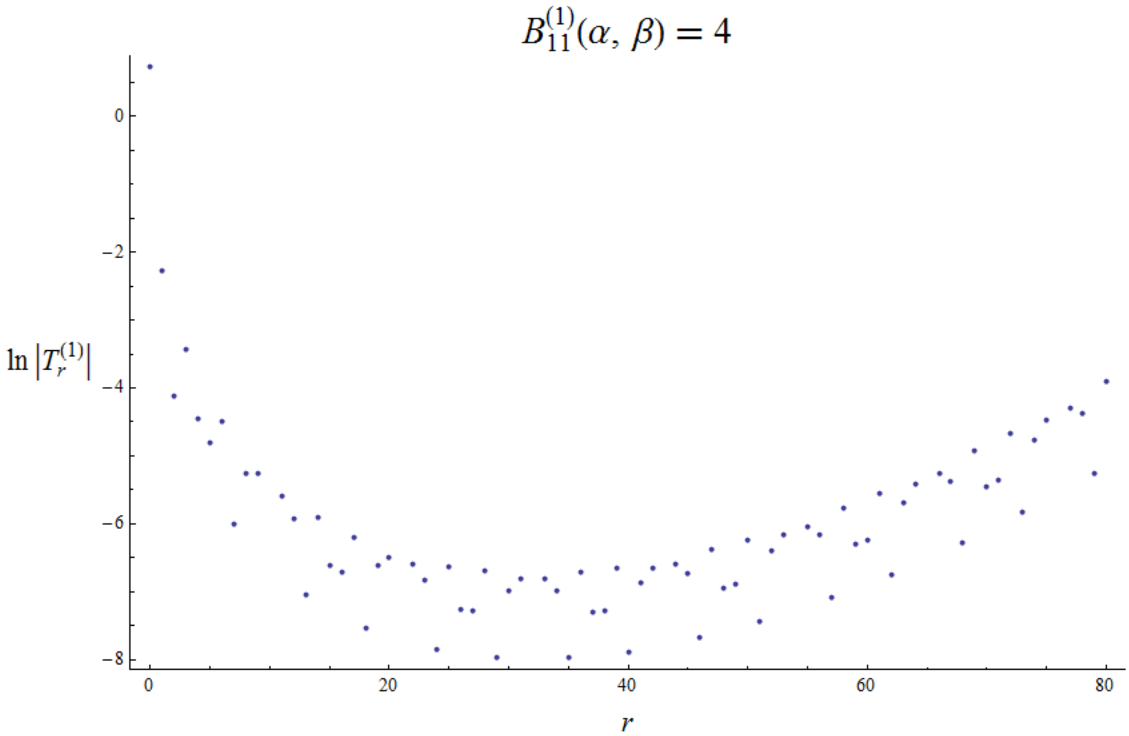


Figure 78: Coefficient size for $B_{11}^{(1)}(\alpha, \beta) = 4$.

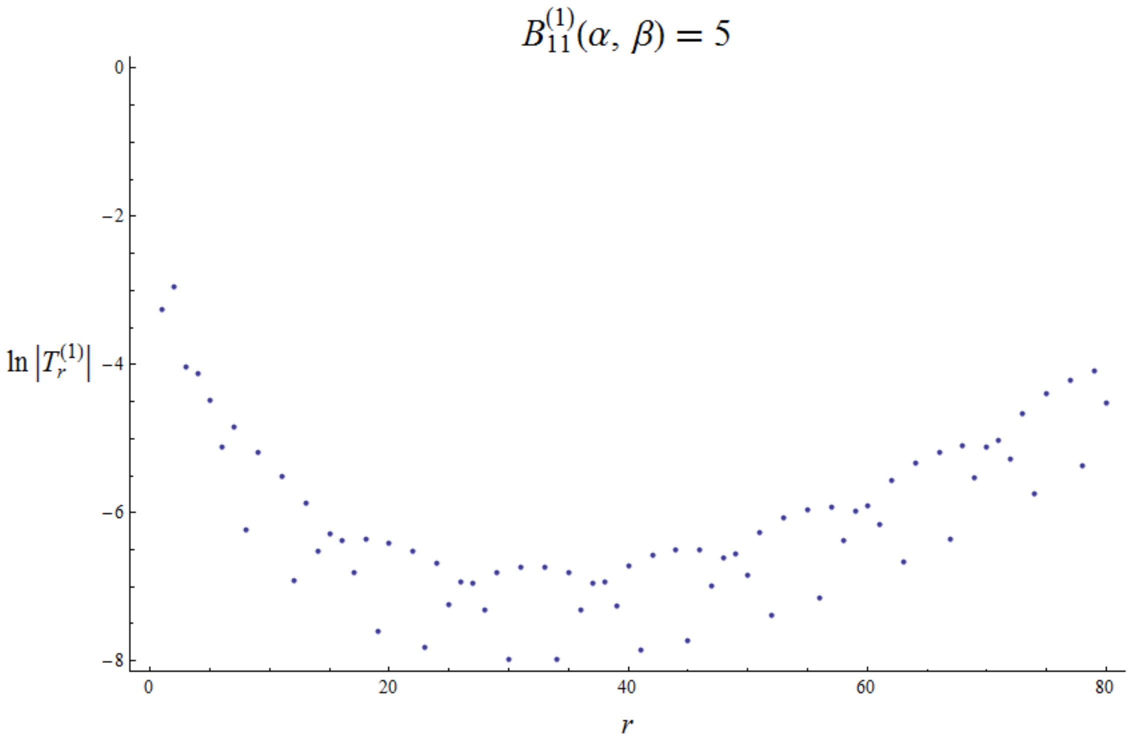


Figure 79: Coefficient size for $B_{11}^{(1)}(\alpha, \beta) = 5$.

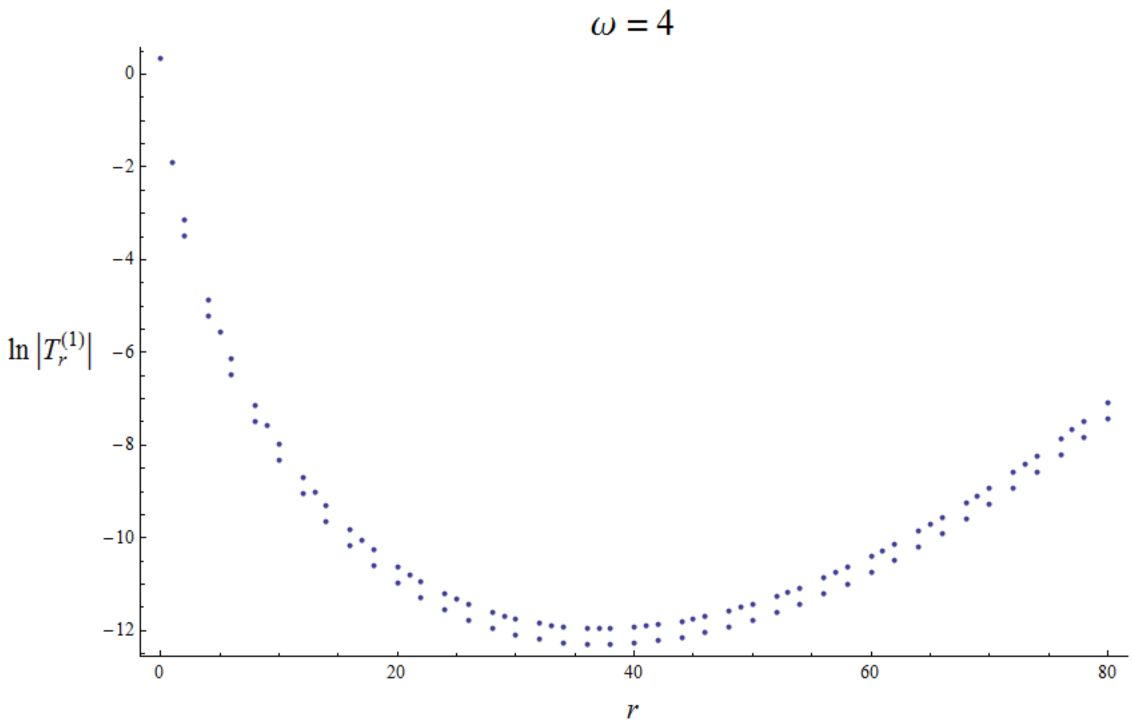


Figure 80: All coefficient patterns for $\omega = 4$.

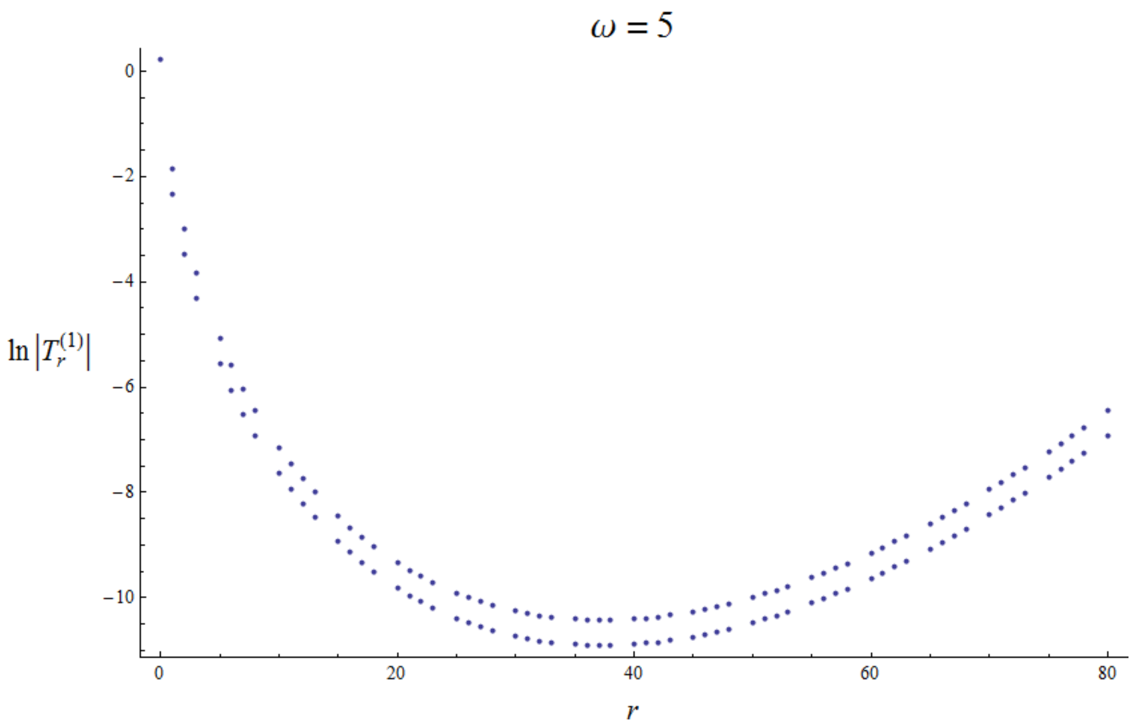


Figure 81: All coefficient patterns for $\omega = 5$.

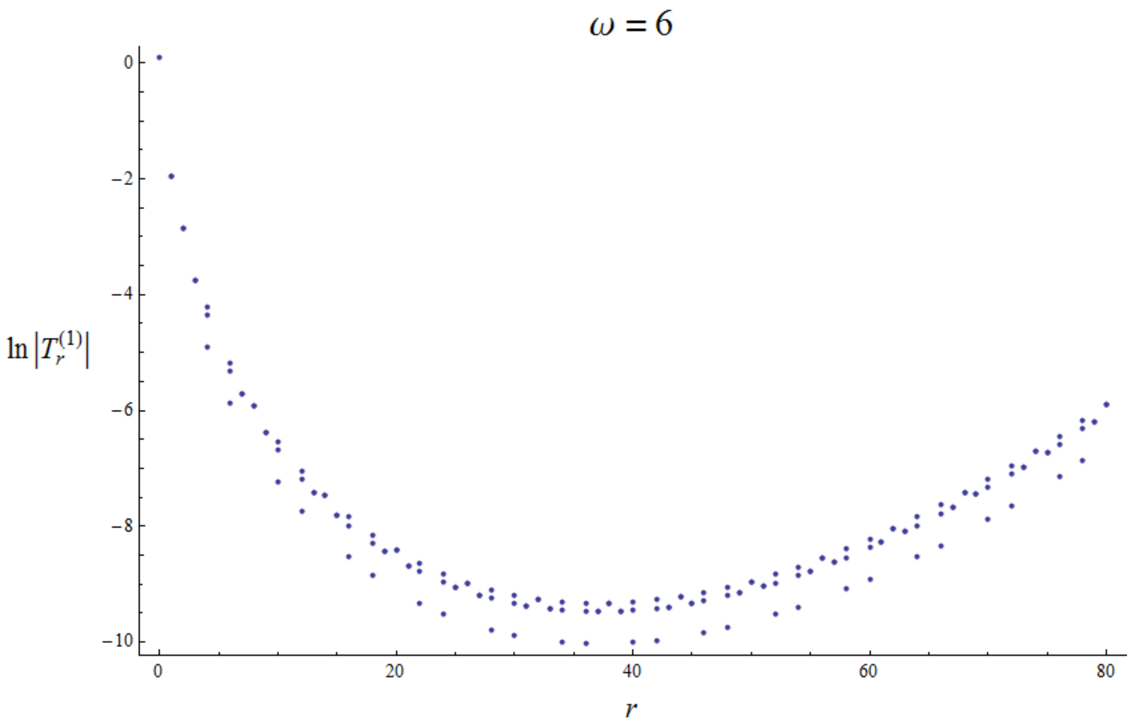


Figure 82: All coefficient patterns for $\omega = 6$.

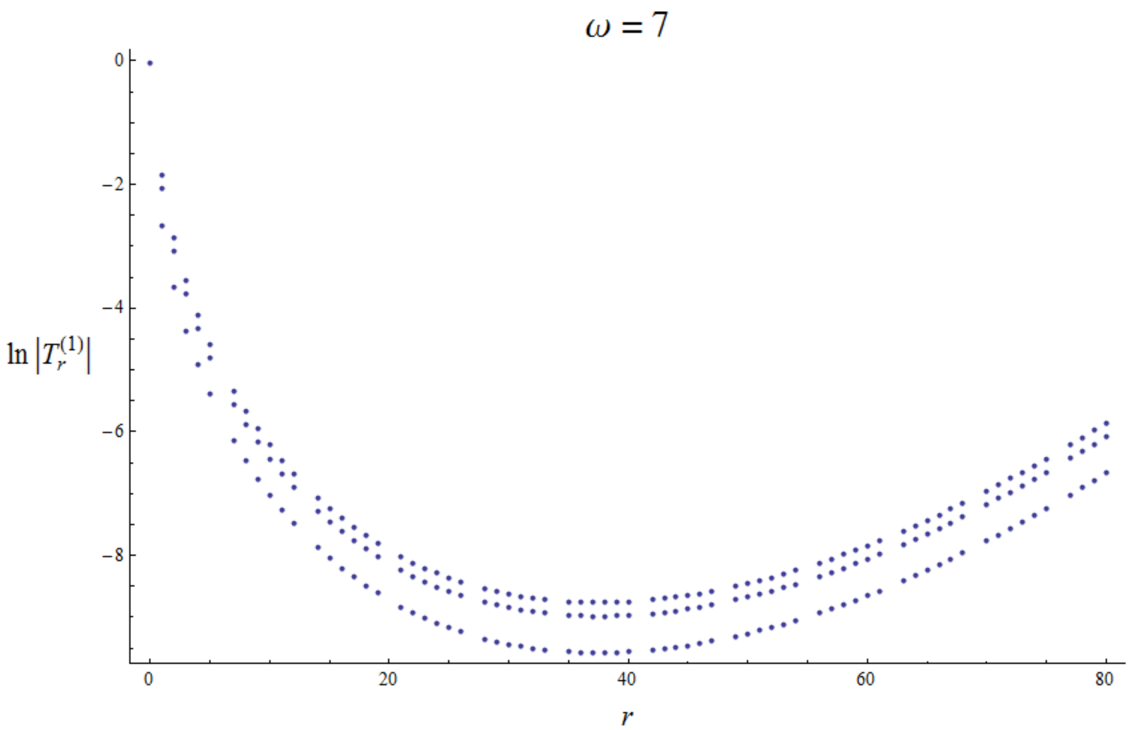


Figure 83: All coefficient patterns for $\omega = 7$.

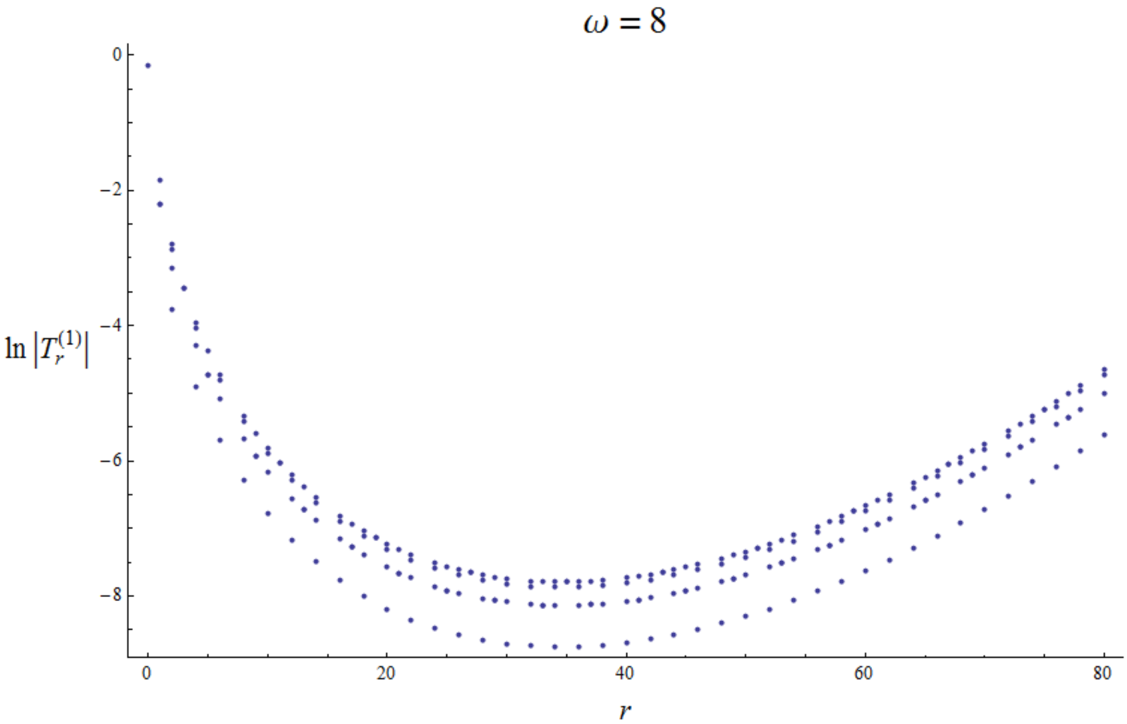


Figure 84: All coefficient patterns for $\omega = 8$.

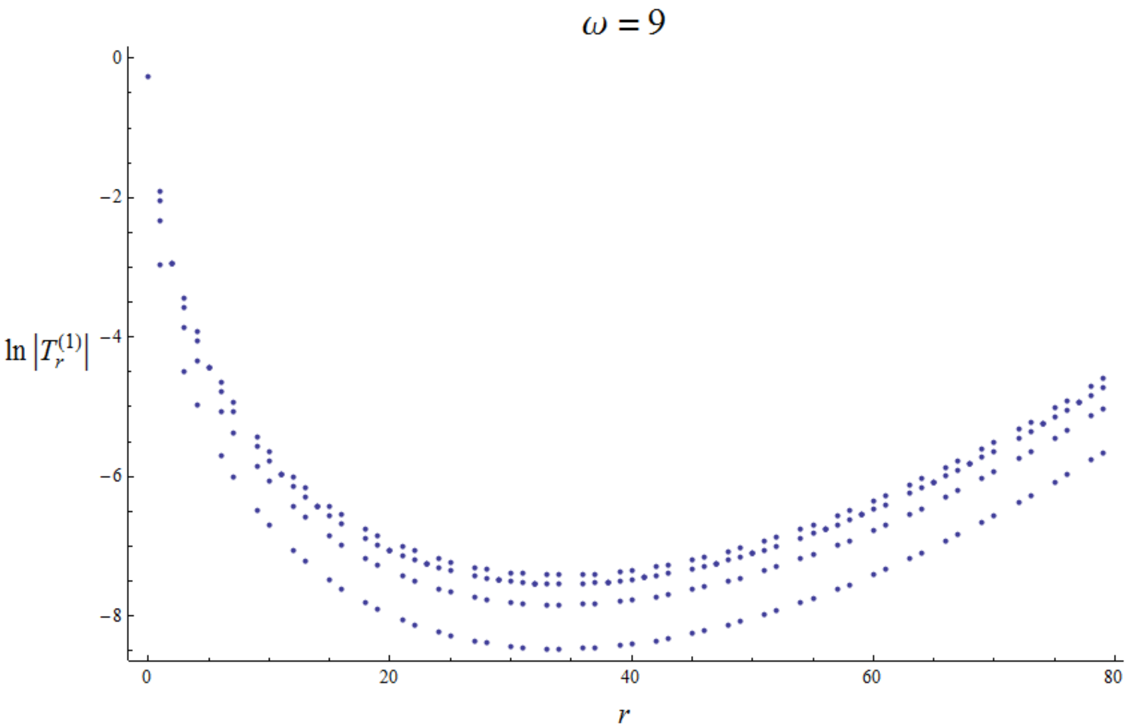


Figure 85: All coefficient patterns for $\omega = 9$.

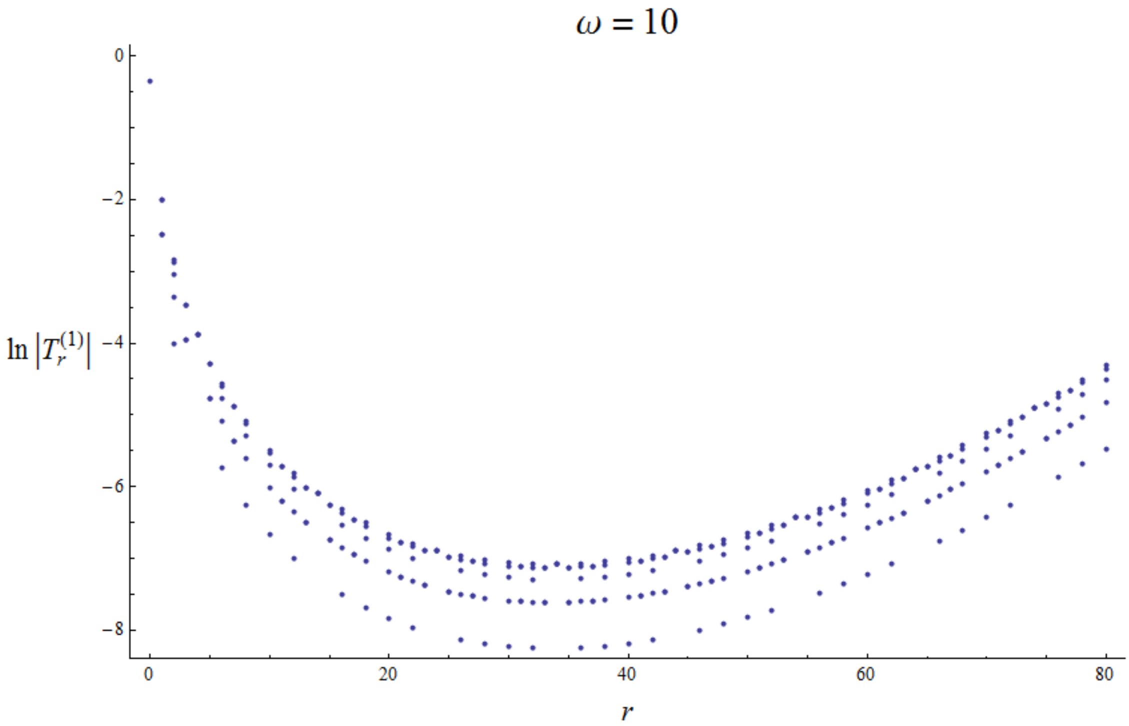


Figure 86: All coefficient patterns for $\omega = 10$.

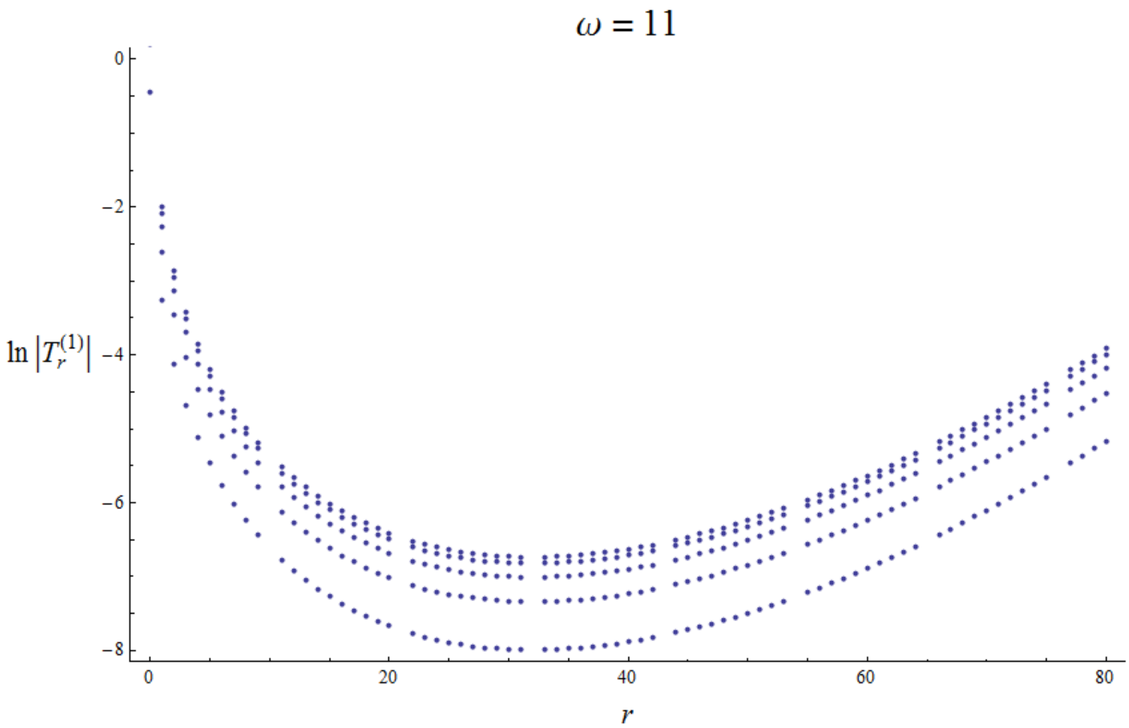


Figure 87: All coefficient patterns for $\omega = 11$.

8 Towards a General Boundary Expansion and Miscellaneous Work

In this chapter, we discuss the work that we believe needs to be done in order to produce a generalisation to general critical components of the general boundary case for isolated critical points found in Delabaere and Howls (2002), as well as briefly mention other miscellaneous but related work that was not entirely brought to completion.

Towards the end of this degree, we worked on many smaller problems with the view to produce a hyperasymptotic expansion in the case of a general boundary. This threw up multiple obstacles - more than we foresaw - all of which must of course be tackled before we can solve the problem. The biggest of these obstacles is successfully handling the case of critical components with non-constant order of degeneracy.

We believe we require knowledge of this case due to comments in Delabaere and Howls (2002); considering integral (1.1) over a general bounded integration region, the authors assume that no critical points of the unrestricted f lie on the boundary (Hypothesis H5 in the text). However, critical components will generally not fit entirely within the integration region (recall our discussion in §3.1), meaning that in general we will have unrestricted critical points on the boundary, violating Hypothesis H5 in the aforementioned text. Critical points on the boundary may experience a jump in order, and so it is important that we first consider the non-constant order case. We present our exploration into the non-constant order case later in this chapter, but first we discuss other partially or uncompleted pieces of work that are not yet ready for formal presentation.

We looked at an integral in $\mathbb{C}^2$ that is bounded by a complex manifold and whose phase function has a non-degenerate complex critical line, and considered the asymptotic expansion at the point where the critical line intersected the boundary. We followed the uniform asymptotic approach in Berry and Howls (1993) and derived expressions for the components of the full expansions, but did not test them using an example. This line of work could be pursued more generally, where instead of considering the intersection of a linear order surface (boundary) and a critical component, we instead considered the intersection of two constant order critical components. This intersection essentially merges two constant order critical components into one critical component that has non-constant order, with the points of intersection having a higher order than that of the component critical components.

We have previously studied the asymptotic contribution from a linear boundary along which f is constant (theory developed in §5 and example discussed in §7.2), but if f is not constant along the boundary then the boundary contributions will not take such a simple form. We look at this problem in $\mathbb{C}^2$ and assume we can transform from (z_1, z_2) to (s_1, s_2) such that the non-degenerate critical line χ is given by $s_2 = 0$. Note that this case is essentially the complex analogue of that studied by McW and BR01. The integral then transforms as

$$I(k) = \iint_D dz_1 dz_2\, g(z_1, z_2) e^{-kf(z_1,z_2)} = \int_a^b ds_1 \int_{\gamma^-(s_1)}^{\gamma^+(s_1)} ds_2\, G(s_1, s_2) e^{-kF(s_1,s_2)},$$

where the integration region D is such that $\gamma^+(s_1) \geq 0$ and $\gamma^-(s_1) \leq 0$ for all s_1, $\gamma^+(s_1) \cup \gamma^-(s_1) = \partial D$, and we assume that the image of D under this transformation lies wholly in the infinite column $[a, b] \times \mathbb{C}$. Note that by the 'complex interval' $[a, b] \subset \mathbb{C}$ with a and b complex, we mean the (geodesic) line segment connecting a and b in the complex plane. This integral can broken up as

$$I(k) = \int_a^b ds_1 \left[\int_{-\infty}^{\infty} ds_2 - \int_{\gamma^+(s_1)}^{\infty} ds_2 + \int_{\gamma^-(s_1)}^{-\infty} ds_2 \right] G(s_1, s_2) e^{-kF(s_1,s_2)},$$

namely into a fully infinite integral that we already have a hyperasymptotic expansion for and two endpoint-type boundary integrals starting at $\gamma_\pm(s_1)$.

Assuming that formal asymptotic expansions for these boundary integrals exist, then we can write

$$I(k) \sim \int_a^b ds_1 \left[\frac{e^{-kF(s_1,0)}}{k^{\frac{1}{2}}} \sum_{r=0}^{\infty} \frac{t_r(s_1)}{k^{\frac{r}{2}}} - \frac{e^{-kF(s_1,\gamma^+(s_1))}}{k} \sum_{r=0}^{\infty} \frac{b_r^+(s_1)}{k^r} + \frac{e^{-kF(s_1,\gamma^-(s_1))}}{k} \sum_{r=0}^{\infty} \frac{b_r^-(s_1)}{k^r} \right]$$

and hence

$$I(k) \sim \frac{e^{-kF(s_1,0)}}{k^{\frac{1}{2}}} \sum_{r=0}^{\infty} \frac{T_r(s_1)}{k^{\frac{r}{2}}} - \Gamma^+(s_1) + \Gamma^-(s_1),$$

with

$$T_r(s) = \int_a^b ds_1\, t_r(s) \quad \text{and} \quad \Gamma^\pm(s_1) = \sum_{r=0}^{\infty} \frac{1}{k^{r+1}} \int_a^b ds_1\, b_r^\pm(s_1) e^{-kF(s_1,\gamma^\pm(s_1))},$$

The integrals $\Gamma^\pm(s_1)$ can essentially be treated as brand new one-dimensional integrals to which we can find an asymptotic expansion by studying the contributing points, most importantly the critical points of the restricted function $F(s_1, \gamma^\pm(s_1))$. Critical points of $F(s_1, \gamma^\pm(s_1))$ are solutions to

$$\frac{dF}{ds_1} = \frac{\partial F}{\partial s_1} + \left.\frac{\partial F}{\partial s_1}\right|_{\gamma_\pm(s_1)} \frac{\partial \gamma_\pm(s_1)}{\partial s_1} = 0 \qquad (8.1)$$

and we can see that there are many different cases that will provide solutions to this equation. Our results were primarily exploratory, but we did produce some interesting concrete results.

We can immediately see that since $(a, 0)$ and $(b, 0)$ are two points within χ, they are critical points of $F(s_1, \gamma^{\pm}(s_1))$ as well. Additionally, any critical points $(c_j^{\pm}, 0)$ of $\gamma^{\pm}(s_1)$ respectively that are also on χ will solve (8.1) since $\partial f/\partial s_1 = 0$ on all of χ. We also note that $\gamma^{\pm}(s_1)$ is tangential to χ at such critical points. Further, it can be shown (such as in BR01) that any derivative involving the $\partial/\partial s_1$ operator is identically zero on χ. This enables us to show that the critical points $(a, 0)$ and $(b, 0)$ have order two with respect to all of $f(z_1, z_2)$ and $F(s_1, \gamma^{\pm}(s_1))$ and that the critical points $(c_j^{\pm}, 0)$ have order four with respect to $F(s_1, \gamma^{\pm}(s_1))$ respectively, but are of course only order two with respect to $f(z_1, z_2)$.

This is easily extended to the general case where χ has general constant order ω; in this case, $(a, 0)$ and $(b, 0)$ have order ω with respect to all of $f(z_1, z_2)$ and $F(s_1, \gamma^{\pm}(s_1))$ and critical points $(c_j^{\pm}, 0)$ have order 2ω with respect to $F(s_1, \gamma^{\pm}(s_1))$ respectively. It is worth noting that if both of $\gamma^{\pm}(s_1)$ have a critical point at the same point on χ, the order remains 2ω at that point with respect to both restricted functions. Many more solutions to (8.1) exist, including solutions that aren't on χ, but our initial exploration focused only on the aforementioned case.

Our last miscellaneous result involved rewriting the asymptotic contribution of a non-degenerate isolated critical point to a two-dimension integral in $\mathbb{C}^2$ in an iterated integration form, similar to the deconstruction above. Although such contributions are very well known for $\mathbb{C}^d$ (Howls (1997) and §§4 or 5 of this thesis, at least), computation of the asymptotic coefficients in any dimension higher than one is generally a difficult task. Generalising further, it is usually very difficult and/or computationally expensive to compute the coefficients of any critical component whose codimension is greater than one. The aim of this work was to break down the two-dimensional expansion into two one-dimensional expansions, compute the coefficients, and then re-combine these one-dimensional coefficients to form the two-dimensional ones that we originally sought. We already know the form of the two-dimensional coefficients from the literature and we were successful in reproducing them using this method, although some aspects of the derivation lacked rigour.

Briefly, the result was achieved by rewriting the two-dimensional expansion as two one-dimensional expansions over two sums, indexed by a and b from zero to infinity. By rewriting the sum index by defining $r = a + b$, the sums can be rewritten as

$$\sum_{a=0}^{\infty}\sum_{b=0}^{\infty} \equiv \sum_{r=0}^{\infty}\sum_{a=0}^{r} \equiv \sum_{r=0}^{\infty}\sum_{b=0}^{r}.$$

It was then pointed out (Adri Olde Daalhuis, private correspondence) that it was possible to extend the sum over a (or b) all the way to infinity as the summand is zero for all integer $a > r$ (equivalently, $b > r$), enabling us to rewrite the sum as a hypergeometric function. After suitable algebraic manipulation, we arrive at exactly the result we know is the answer, but with an extra integral added on. This extra integral is very intuitively zero, but it is rigorously difficult to prove so. Once again, this whole method could be repeated for critical components of general codimension and order in $\mathbb{C}^d$, but the algebra involved would be substantially more complicated.

We end this section with an example by studying an example where the function f has a non-uniformly degenerate critical component; namely, that it has a non-constant order of degeneracy. We demonstrate what aspects of the constant order theory remain functional under this violation of assumption, and what aspects need modifying to account for it.

We look at the integral

$$I^{(n)}(k;\alpha_n,\beta_n) = \int_{S_n(\alpha_n,\beta_n)} d\boldsymbol{z}\, z_1^4 e^{-k\left(-\frac{1}{2}z_1^4 - z_1^3 + \frac{5}{2}z_1^2 + 2z_1^2\left(z_2^2+z_3^2+z_4^2\right)\right)}, \tag{8.2}$$

where $\boldsymbol{z} = (z_1, z_2, z_3, z_4) \in \mathbb{C}^4$, $S_n \subset \mathbb{C}^4$, and $f, g : \mathbb{C}^4 \to \mathbb{C}$ with

$$\begin{aligned} f(\boldsymbol{z}) &= -\frac{1}{2}z_1^4 - z_1^3 + \frac{5}{2}z_1^2 + 2z_1^2\left(z_2^2+z_3^2+z_4^2\right), \\ g(\boldsymbol{z}) &= z_1^4. \end{aligned}$$

The critical point set $C(f)$ is given by

$$C(f) = \left\{\boldsymbol{z} \in \mathbb{C}^4 \mid (z_1 = 0) \vee \left(z_2 = z_3 = z_4 = 0 \wedge z_1 = -\tfrac{5}{2}\right) \vee \left(z_2 = z_3 = z_4 = 0 \wedge z_1 = 1\right)\right\}$$

and we define the critical components

$$\begin{aligned} \chi_1 &= \left\{\boldsymbol{z} \in \mathbb{C}^4 \mid z_1 = 0\right\} = (0, z_2, z_3, z_4), \\ \chi_2 &= \left(-\tfrac{5}{2}, 0, 0, 0\right), \\ \chi_3 &= (1, 0, 0, 0), \end{aligned}$$

so that

$$\chi_1 \cup \chi_2 \cup \chi_3 = C(f) \quad \text{and} \quad \chi_1 \cap \chi_2 \cap \chi_3 = \emptyset.$$

We can already see that we have more interesting critical components than in §7.

A two-dimensional contour plot of $f(\boldsymbol{z})$ in the z_1-plane (for fixed $z_2 = z_3 = z_4 = 0$) is given in Figure 89, a three-dimensional plot of the slice $\{(\mathrm{Re}(z_1), \mathrm{Re}(z_j))\} \cong \mathbb{R}^2$ of $\mathbb{C}^4$ is given in Figure 88, and a summary of the important quantities for this problem is given below:

$$\begin{aligned} f_1 &= 0, \quad \mu_1 = 3, \quad q_1 = 1, \quad \omega_1 = \omega_1(\boldsymbol{z}); \\ f_2 &= \tfrac{375}{32}, \quad \mu_2 = 0, \quad q_2 = 4, \quad \omega_1 = 2; \\ f_3 &= 1, \quad \mu_3 = 0, \quad q_3 = 4, \quad \omega_1 = 2. \end{aligned}$$

The value of the singulants are given in the matrix

$$\boldsymbol{F} = \{F_{ij}\} = \begin{pmatrix} \cdot & \frac{375}{32} & 1 \\ -\frac{375}{32} & \cdot & -\frac{343}{32} \\ -1 & \frac{343}{32} & \cdot \end{pmatrix}.$$

By considering the Hessian matrix

$$H_f(\boldsymbol{z}) = \begin{pmatrix} 4(z_2^2 + z_3^2 + z_4^2) - 6z_1^2 - 6z_1 + 5 & 8z_1 z_2 & 8z_1 z_3 & 8z_1 z_4 \\ 8z_1 z_2 & 4z_1^2 & 0 & 0 \\ 8z_1 z_3 & 0 & 4z_1^2 & 0 \\ 8z_1 z_4 & 0 & 0 & 4z_1^2 \end{pmatrix}$$

of f, we can see that all second derivatives are zero on the set

$$\left\{ \boldsymbol{z} \in \mathbb{C}^4 \,\middle|\, (z_1 = 0) \wedge \left(z_2^2 + z_3^2 + z_4^2 = -\frac{5}{4} \right) \right\} \subset \chi_1$$

and at no other point. The order of degeneracy $\omega_1(\boldsymbol{z})$ is therefore non-constant, as this set is a subset of χ_1 (that is given by $z_1 = 0$). Looking at the tensor of third derivatives (which will not be given explicitly), we can see that it is not possible that all third derivatives of f are simultaneous zero at *any* point in $\mathbb{C}^4$; for example, $f_{z_1 z_1 z_1} = -6 - 12z_1$, but $f_{z_2 z_2 z_1} = 8z_1$. We therefore have a complex two-dimensional subset of order three within χ_1. This subset is readily identifiable as the boundary of a complex sphere of radius $i\sqrt{5}/2$. Hence,

$$\omega_1(\boldsymbol{z}) = \begin{cases} 3 & \text{if } (z_1 = 0) \wedge \left(z_2^2 + z_3^2 + z_4^2 = -\frac{5}{4} \right), \\ 2 & \text{otherwise.} \end{cases} \tag{8.3}$$

Figure 90 shows how the order changes between two and three in the complex z_1 plane as z_2 varies local to $\frac{i\sqrt{5}}{2}$, for $z_3 = z_4 = 0$. A similar phenomenon will occur near every other point

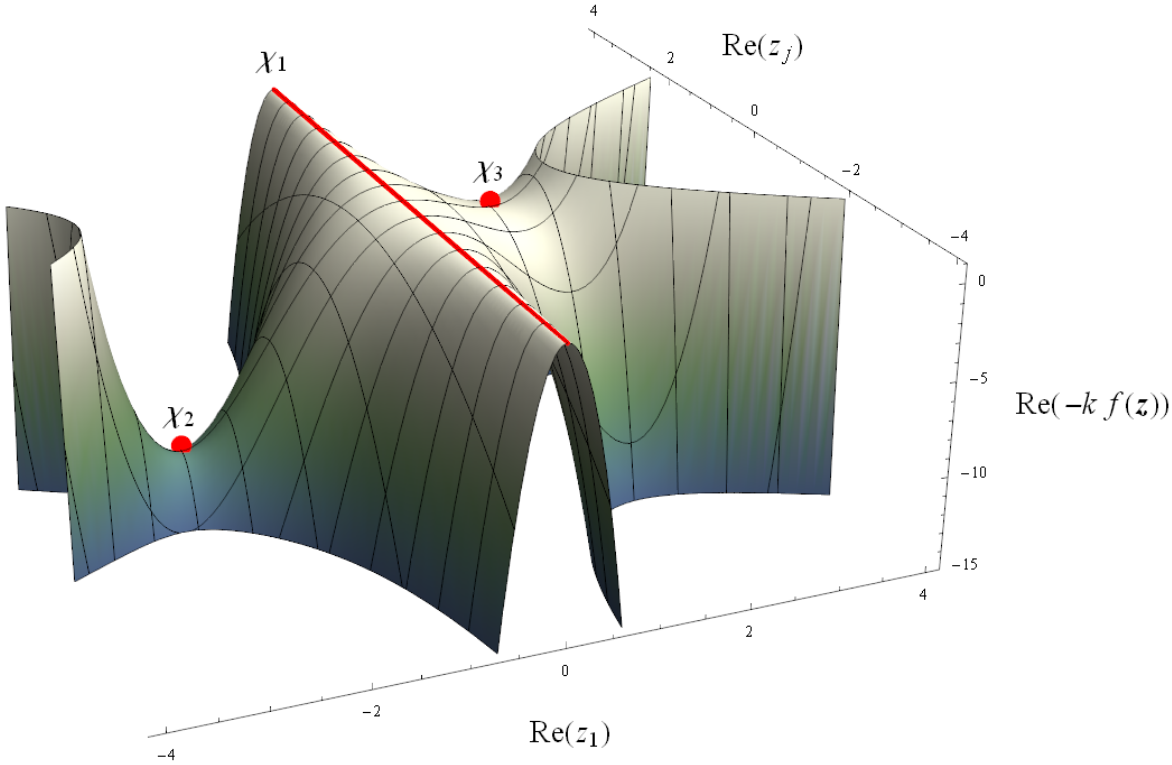


Figure 88: Three dimensional plot of $\mathrm{Re}(-kf(\boldsymbol{z}))$ for the slice $\{(\mathrm{Re}(z_1), \mathrm{Re}(z_j))\} \cong \mathbb{R}^2$ of $\mathbb{C}^4$. For $j \in \{2, \ldots, 4\}$, this plot will look identical due to the nature of the example. While this does not display the entirety of the critical component χ_1 (which is real six-dimensional!), it shows off its form.

of order three. The isolated critical points χ_2 and χ_3 only occur when $z_2 = z_3 = z_4 = 0$, but when we change these values their images propagate through. The coalescence of one of these images with the critical surface χ_1 is the reason for the increase in order.

The integration surfaces S_j will be of the form $(V_{a_j}, V_{b_j}) \times \bar{\mathbb{R}}^3$, so that z_2, z_3, and z_4 run between real infinities and z_1 runs between asymptotic valleys V_{a_j} and V_{b_j}. These valleys are defined according to Figures 88 and 89. For each critical component, we will compute the asymptotic coefficients $T_r^{(n)}(\alpha_n, \beta_n)$ using (5.12), use these to write down the zeroth level expansion (5.13), and then compare these expansions to the relevant numerical integral.

The coefficients are given by (5.13) as

$$T_r^{(1)} = \left(1 - (-1)^{r+1}\right) \frac{\Gamma\left(\frac{r+1}{2}\right)}{2} \int_{-\infty}^{\infty} dz_4 \int_{-\infty}^{\infty} dz_3 \int_{-\infty}^{\infty} dz_2 \operatorname*{Res}_{z_1=0} \left(\frac{z_1^4}{(f(\boldsymbol{z}) - 0)^{\frac{r+1}{2}}} \right),$$

$$T_r^{(2)} = \left(1 - (-1)^{r+4}\right) \frac{\Gamma\left(\frac{r+4}{2}\right)}{2} \operatorname*{Res}_{z_1=-\frac{5}{2}} \left(\int_{-\infty}^{\infty} dz_4 \int_{-\infty}^{\infty} dz_3 \int_{-\infty}^{\infty} dz_2 \frac{z_1^4}{\left(f(\boldsymbol{z}) - \frac{375}{32}\right)^{\frac{r+4}{2}}} \right),$$

$$T_r^{(3)} = \left(1 - (-1)^{r+4}\right) \frac{\Gamma\left(\frac{r+4}{2}\right)}{2} \operatorname*{Res}_{z_1=1} \left(\int_{-\infty}^{\infty} dz_4 \int_{-\infty}^{\infty} dz_3 \int_{-\infty}^{\infty} dz_2 \frac{z_1^4}{(f(\boldsymbol{z}) - 1)^{\frac{r+4}{2}}} \right),$$

with the first few given explicitly as

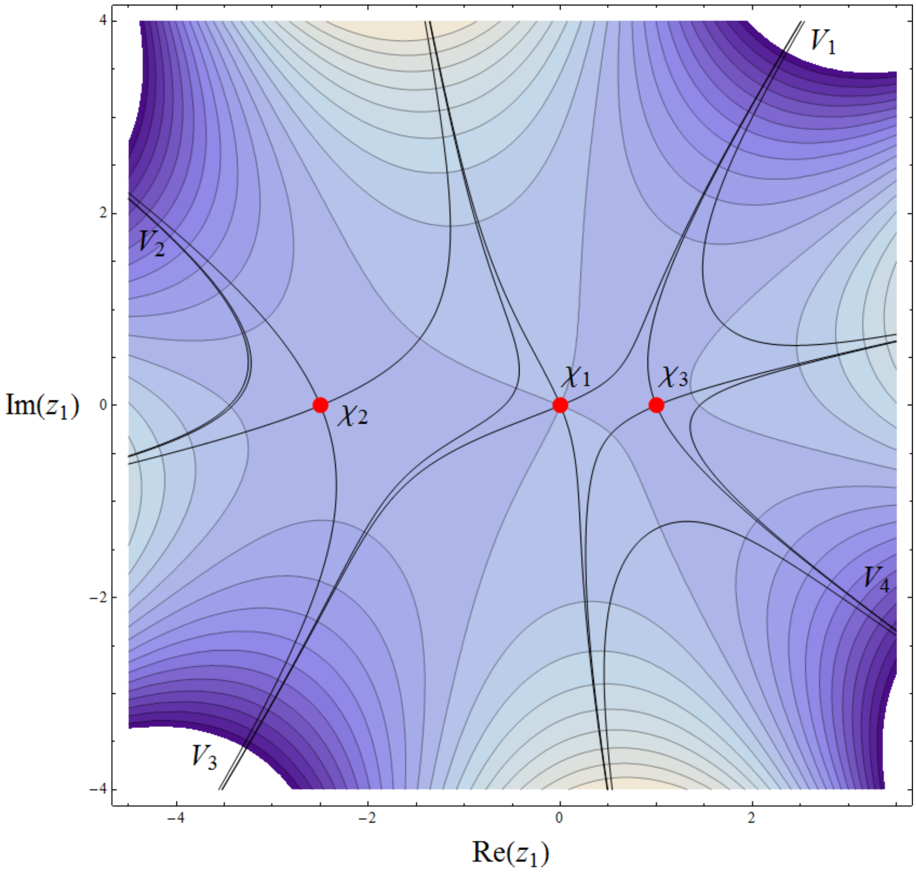


Figure 89: Contour plot of $\mathrm{Re}(-kf(\boldsymbol{z}))$ in complex z_1 space for fixed $z_2 = z_3 = z_4 = 0$, with $k = 1 - i$.

$$\left\{T_j^{(1)}\right\}_{j\in\mathbb{N}_0} = \left\{0, 0, 0, 0, \frac{\pi^{\frac{3}{2}}}{5\sqrt{2}}, 0, \frac{22\sqrt{2}\pi^{\frac{3}{2}}}{625}, 0, \frac{792\sqrt{2}\pi^{\frac{3}{2}}}{15625}, 0, \ldots\right\},$$

$$\left\{T_j^{(2)}\right\}_{j\in\mathbb{N}_0} = \left\{\frac{i\pi^2}{2}\sqrt{\frac{5}{14}}, 0, -\frac{39i\pi^2}{8575\sqrt{70}}, 0, \frac{3579i\pi^2}{10504375\sqrt{70}}, 0, \ldots\right\},$$

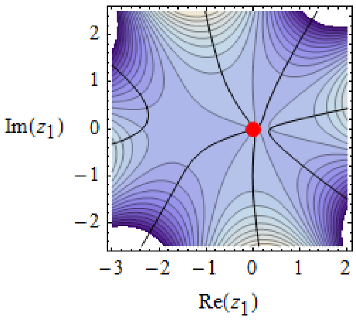
$$\left\{T_j^{(3)}\right\}_{j\in\mathbb{N}_0} = \left\{\frac{i\pi^2}{2\sqrt{7}}, 0, \frac{3i\pi^2}{343\sqrt{7}}, 0, -\frac{345i\pi}{67228\sqrt{7}}, 0, \ldots\right\}.$$

Figure 91 shows the size of these three sets of coefficients. Since all critical components are of order two (with χ_1 being considered 'mostly' order two), then the only integration surface corresponds to $(\alpha_j, \beta_j) = (1, 0)$ for all j, so we drop their functional dependency from the coefficients and expansions.

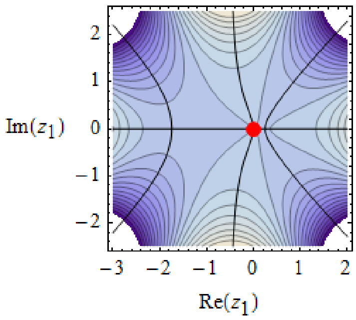
We now compare the asymptotic expansion generated by these coefficients against the numerical value of the integrals of interest

$$T_I^{(1)}(k) = k^{\frac{1}{2}} \int_{-\infty}^{\infty} dz_4 \int_{-\infty}^{\infty} dz_3 \int_{-\infty}^{\infty} dz_2 \int_{V_3}^{V_1} dz_1\, z_1^4 e^{-k(f(\boldsymbol{z})-0)},$$

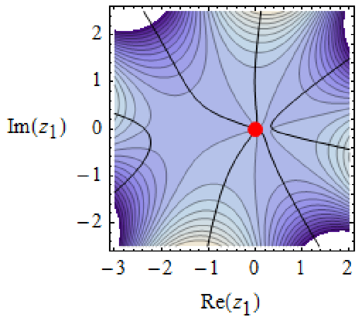
$$T_I^{(2)}(k) = k^{\frac{4}{2}} \int_{-\infty}^{\infty} dz_4 \int_{-\infty}^{\infty} dz_3 \int_{-\infty}^{\infty} dz_2 \int_{V_2}^{V_3} dz_1\, z_1^4 e^{-k\left(f(\boldsymbol{z})-\frac{375}{32}\right)},$$



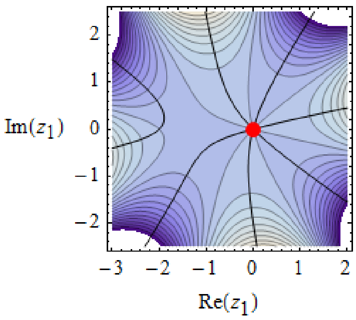
(a) $z_2 = i\left(\frac{\sqrt{5}}{2} - 0.1\right),\ k = 1 - i$



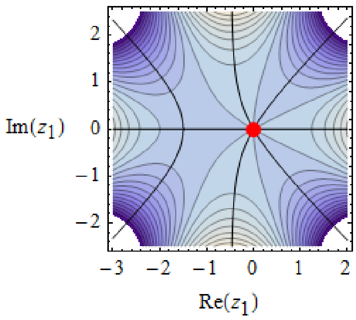
(b) $z_2 = i\left(\frac{\sqrt{5}}{2} - 0.1\right),\ k = 1$



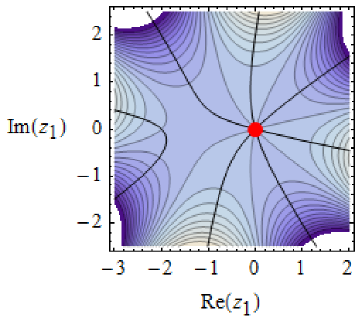
(c) $z_2 = i\left(\frac{\sqrt{5}}{2} - 0.1\right),\ k = 1 + i$



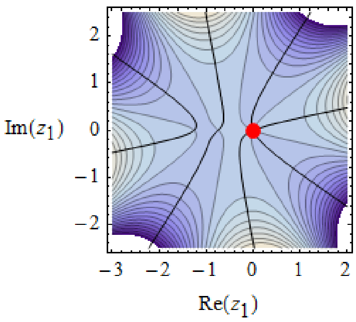
(d) $z_2 = i\frac{\sqrt{5}}{2},\ k = 1 - i$



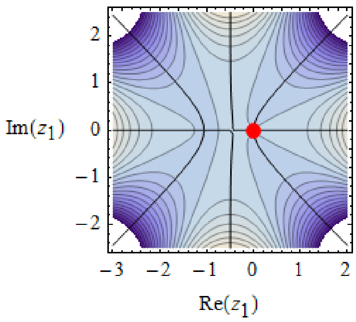
(e) $z_2 = i\frac{\sqrt{5}}{2},\ k = 1$



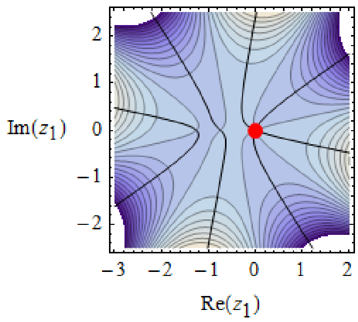
(f) $z_2 = i\frac{\sqrt{5}}{2},\ k = 1 + i$



(g) $z_2 = i\left(\frac{\sqrt{5}}{2} + 0.1\right),\ k = 1 - i$



(h) $z_2 = i\left(\frac{\sqrt{5}}{2} + 0.1\right),\ k = 1$



(i) $z_2 = i\left(\frac{\sqrt{5}}{2} + 0.1\right),\ k = 1 + i$

Figure 90: Contour plots of $\mathrm{Re}(-kf(\boldsymbol{z}))$ in the complex z_1 plane as z_2 passes through the third order point $\left(0, i\frac{\sqrt{5}}{2}, 0, 0\right)$ for different k. We can clearly see the order changing as z_2 passes through $i\frac{\sqrt{5}}{2}$.

$$T_I^{(3)}(k) = k^{\frac{4}{2}} \int_{-\infty}^{\infty} dz_4 \int_{-\infty}^{\infty} dz_3 \int_{-\infty}^{\infty} dz_2 \int_{V_4}^{V_1} dz_1\, z_1^4 e^{-k(f(\boldsymbol{z})-1)}.$$

The coefficients $T_r^{(1)}$ are extremely computationally intensive past $r = 17$, meaning that we cannot reasonably repeat coefficient computations past this point (for different values of k, for example). Additioanlly, to obtain reasonable accuracy to $T_I^{(1)}(k)$, we need to choose a k that yields an optimal truncation point greater than $N_1 = 17$ and thus it is not computationally reasonable to compute the optimal level zero expansion. Therefore, we simply choose k sufficiently large and truncate at $N_1 = 17$, in order to get a good (albeit non-optimal) approximation to demonstrate agreement between $T_I^{(1)}(k_1)$ and $T^{(1)}(k_1, N_1)$. The coefficients $T_r^{(2)}$ and $T_r^{(3)}$ are less computationally intensive, so we can easily compute the corresponding optimal level zero expansions. Taking

$$k_1 = \frac{15}{\sqrt{2}}(1-i), \quad k_2 = \frac{2}{\sqrt{2}}(1-i), \quad \text{and} \quad k_3 = \frac{6}{\sqrt{2}}(1-i),$$

we obtain the results

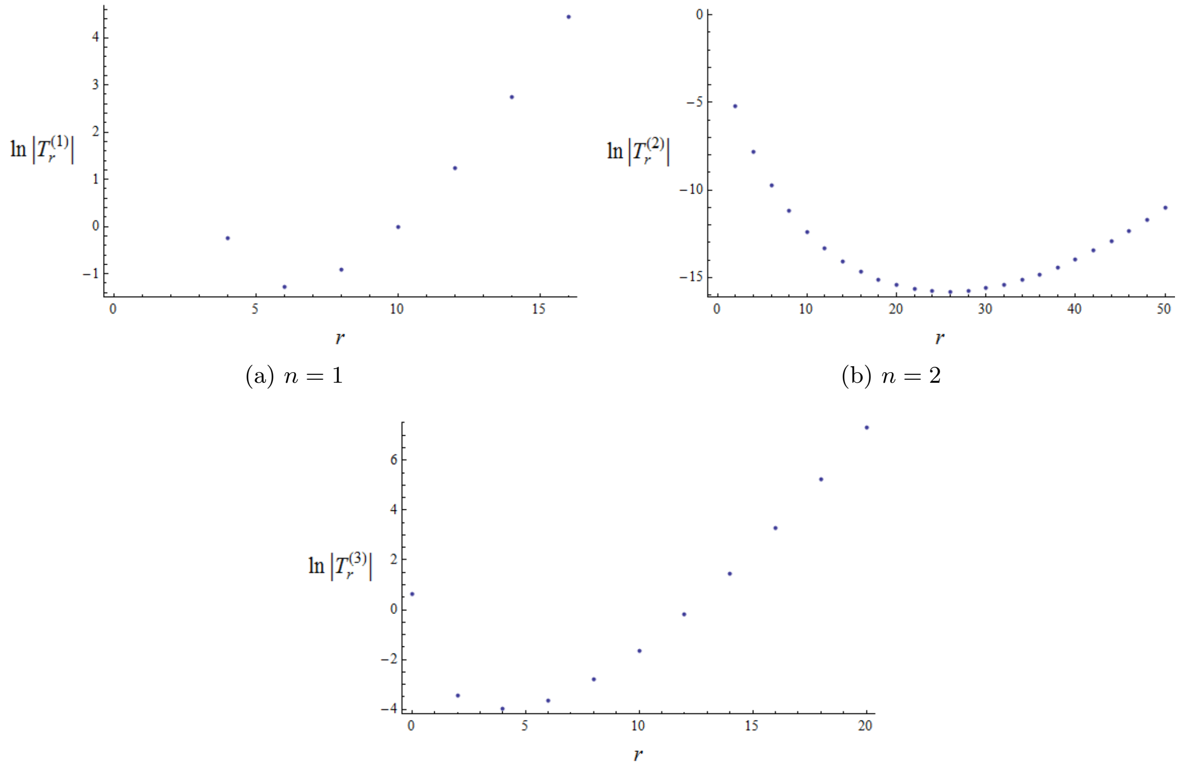
$$\begin{aligned} T_I^{(1)}(k_1) &= -0.0000667813\ldots + 0.00355671\ldots i, \\ T^{(1)}(k_1, 17) &= -0.0000667862\ldots + 0.00355669\ldots i; \end{aligned}$$

$$\begin{aligned} T_I^{(2)}(k_2) &= 0.00180162281932\ldots + 2.94721606707631\ldots i, \\ T^{(2)}(k_2, 49) &= 0.00180162281921\ldots + 2.94721606707656\ldots i; \end{aligned}$$

$$\begin{aligned} T_I^{(3)}(k_3) &= -0.003392\ldots + 1.8689705\ldots i, \\ T^{(3)}(k_3, 15) &= -0.003389\ldots + 1.8689778\ldots i. \end{aligned}$$

So far everything is working as expected, but the issue will reveal itself when we consider an exponentially improved expansion or the late terms. These two quantities will reveal the same information, as the late terms are derived from the exponentially improved expansion. We will compute the late terms for each critical component as they are simpler; they are given by (6.2) as

$$\begin{aligned} T_{N_1}^{(1)}(N_{12}, N_{13}) \sim &\sum_{r_1=0}^{N_{12}-1} \left(1-(-1)^{r+1}\right) \frac{K_{12}\Gamma\left(\frac{N_1+1}{2} - \frac{r_1+4}{2}\right)}{2\pi i \times 2\left(\frac{375}{32}\right)^{\frac{N_1+1}{2}-\frac{r_1+4}{2}}} T_{r_1}^{(2)} \\ &+ \sum_{r_1=0}^{N_{13}-1} \left(1-(-1)^{r+1}\right) \frac{K_{13}\Gamma\left(\frac{N_1+1}{2} - \frac{r_1+4}{2}\right)}{2\pi i \times 2(1)^{\frac{N_1+1}{2}-\frac{r_1+4}{2}}} T_{r_1}^{(3)}, \end{aligned}$$



(a) $n = 1$ (b) $n = 2$

(c) $n = 3$

Figure 91: Coefficients size against term number for $T_r^{(n)}$ for each n.

$$
\begin{aligned}
T_{N_2}^{(2)}(N_{21}, N_{23}) \sim & \sum_{r_1=0}^{N_{21}-1} \left(1 - (-1)^{r+4}\right) \frac{K_{21}\Gamma\left(\frac{N_2+4}{2} - \frac{r_1+1}{2}\right)}{2\pi i \times 2\left(-\frac{375}{32}\right)^{\frac{N_2+4}{2} - \frac{r_1+1}{2}}} T_{r_1}^{(1)} \\
& + \sum_{r_1=0}^{N_{23}-1} \left(1 - (-1)^{r+4}\right) \frac{K_{23}\Gamma\left(\frac{N_2+4}{2} - \frac{r_1+4}{2}\right)}{2\pi i \times 2\left(-\frac{343}{32}\right)^{\frac{N_2+4}{2} - \frac{r_1+4}{2}}} T_{r_1}^{(3)},
\end{aligned}
$$

$$
\begin{aligned}
T_{N_3}^{(2)}(N_{31}, N_{32}) \sim & \sum_{r_1=0}^{N_{31}-1} \left(1 - (-1)^{r+4}\right) \frac{K_{31}\Gamma\left(\frac{N_3+4}{2} - \frac{r_1+1}{2}\right)}{2\pi i \times 2(-1)^{\frac{N_3+4}{2} - \frac{r_1+1}{2}}} T_{r_1}^{(1)} \\
& + \sum_{r_1=0}^{N_{32}-1} \left(1 - (-1)^{r+4}\right) \frac{K_{32}\Gamma\left(\frac{N_3+4}{2} - \frac{r_1+4}{2}\right)}{2\pi i \times 2\left(\frac{343}{32}\right)^{\frac{N_3+4}{2} - \frac{r_1+4}{2}}} T_{r_1}^{(2)}.
\end{aligned}
$$

Note that $(\alpha_{nm_1}, \beta_{nm_1}) = (1, 0)$ always as each critical component is (mostly) order two, so there is only one possible integration surface for the expanded remainder contour to latch onto. We have that

$$
\begin{aligned}
T_{16}^{(1)} &= 85.7043\ldots, \\
T_{16}^{(1)}(12, 12) &= 85.4925\ldots
\end{aligned}
$$

with $K_{12} = K_{13} = 1$, which shows reasonable numerical agreement for low N_1, and we can

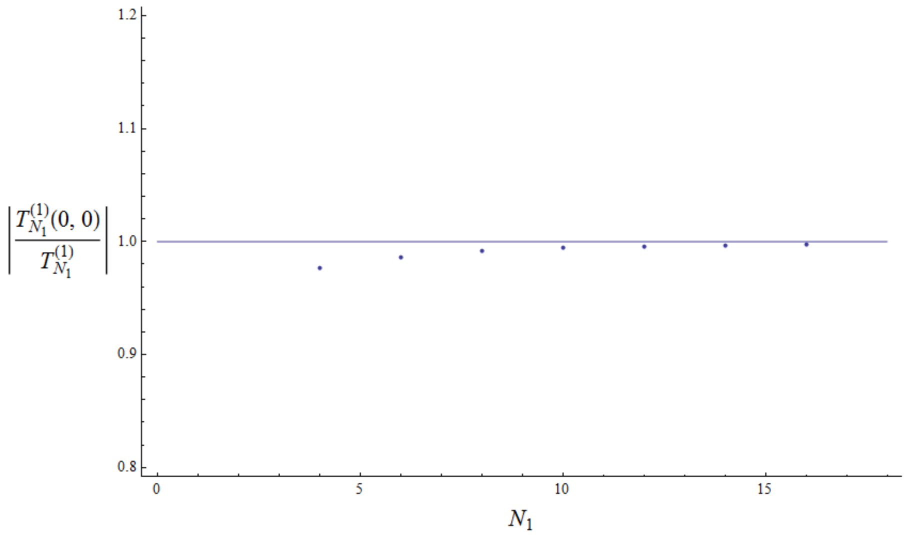


Figure 92: Size of ratio of late term expansion and actual coefficient against actual coefficient number N_1 for $T_{N_1}^{(1)}(0,0)$ and $T_{N_1}^{(1)}$ respectively.

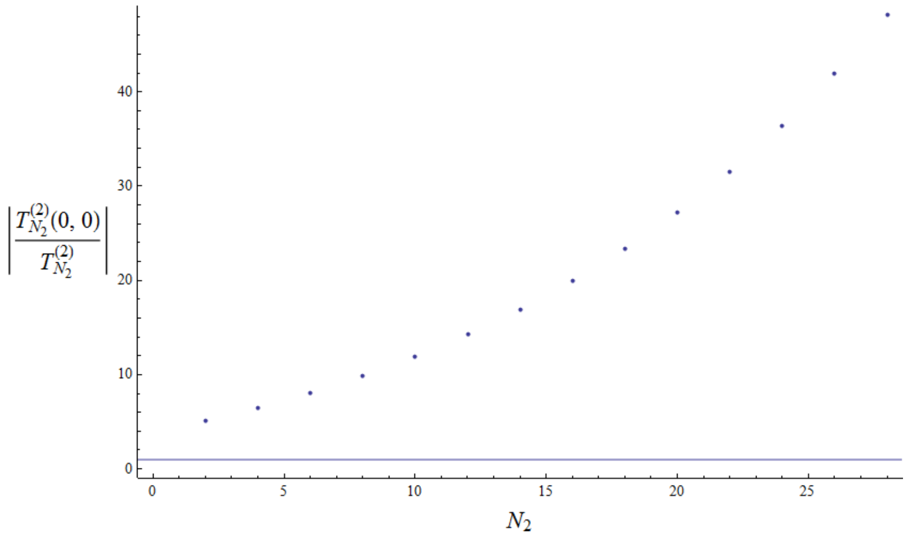


Figure 93: Size of ratio of late term expansion and actual coefficient against actual coefficient number N_2 for $T_{N_2}^{(2)}(0,0)$ and $T_{N_2}^{(2)}$ respectively.

see in Figure 92 that the ratio of the late term expression and the actual coefficient is tending to one very rapidly. This is not the case for χ_2 and χ_3, however; Figure 93 shows that in fact, the ratio is *diverging*, implying that the late term expression is incorrect. What we believe is happening is the following.

Since χ_2 and χ_3 are isolated critical points, they automatically have constant orders and so we have a working expression for the coefficients $T_r^{(2)}$ and $T_r^{(3)}$. This in turn allows us to calculate the late term expansions for $T_r^{(1)}$ in the usual way without hinderance, since they are expressed as functions of $T_r^{(2)}$ and $T_r^{(3)}$. There are no additional considerations to be made here, as when we deform S_1 through adjacent critical components χ_{m_j}, it just sees the two isolated critical points χ_2 and χ_3 and everything works out as expected.

The problem arises when we try to calculate the late terms of $T_r^{(2)}$ and $T_r^{(3)}$; no combination of possible K_{ij} values yield anywhere near reasonable accuracy. It may be thought that equation (5.20) and hence (6.2) is wrong in some way and can only handle the case where all critical components of f have the same codimension (equivalently, dimension), but this is immediately nullified by the good accuracy achieved by the late term approximations for $T_r^{(1)}$. We think that upon deformation of S_2 and S_3 through adjacent critical components, when they 'see' χ_1 they immediately 'notice' the higher order points within it and these higher order points modify the contribution to the late terms.

Interestingly, the regular expression for the coefficients gives accurate results when compared against the relevant integral; maybe because the higher order points are 'inside' χ_1 somehow so do not affect the coefficients, or maybe give a small contribution compared to the rest of the component and so is currently unnoticed in our numerics. Another possibility is that since we are not integrating through these higher order points in calculating $T_r^{(1)}$, they do not show their contribution until late terms are considered, although this is less likely due to Cauchy's integral theorem. Either way, this higher order subset leading to non-constant $\omega_1(\boldsymbol{z})$ warrants a deeper investigation.

One method of handling the case of non-constant order is described in §6 of Benaissa and Rogers (2013); assuming there are finitely many points of higher order, the methods described in Bleistein (1967) will provide the correct uniform asymptotic expansion around these points, with the standard expansion applying to all other points. However, the example we just considered had a non-isolated set of higher order critical points, rendering this method useless in this (and in the general) case. Following the ideas considered to initially handle sets of non-isolated critical points earlier in the thesis, it may be possible to derive a generic contribution from higher order subsets of critical points and add this onto the expansion when required.

9 Conclusion

In this thesis we have extended the results of Howls (1997) to include the case that the phase function f of integral (1.2) has critical components χ_j of general constant order of degeneracy ω_j and general dimension and codimension μ_j and q_j respectively. These critical components are sets of non-isolated critical points - connected components of the critical set $C(f)$ of f - and are defined in §3.1. We derived an expression for the hyperasymptotic contribution of such critical components to (1.2) using (5.20) and this is given by equation (6.4), with component expressions (5.13), (6.11), and (6.12).

After carrying out a review of literature for a variety of areas relating to exponential asymptotic analysis and asymptotic contributions of non-isolated critical points in §2, we built up to the main results described above by defining and discussing critical components in great detail in §3. By defining the concepts of order of degeneracy and critical component concretely (in Definitions 1 and 2 respectively), we were able to discuss and prove what kind of critical components are possible (§3.1 and Propositions 1 and 2) and what powers of the asymptotic parameter would appear in an expansion around such a component (§3.3). Morse and Morse-Bott theory were discussed in §3.2 and we used ideas from these areas to justify reducing the study of the asymptotic behaviour of (1.1) to (1.2) in both the isolated and non-isolated case respectively. Specifically, we respectively invoked the Morse and Morse-Bott lemma to express and hence suitably parameterise the integration surface as a Lefschetz thimble and pencil in the presence of isolated and non-isolated critical points. In the isolated case, the relevant results from homology rigorously justify the decomposition of the integration surface and hence the asymptotics into individual contributions and it is additionally shown that the asymptotics are Borel summable. In the non-isolated case, neither of these two results are shown; the homological result is beyond the scope of this thesis and the Borel summability is not an immediate concern as we deal with finitely truncated series. Currently, neither of these results exist in the literature.

The work in §4 produced a resurgence relation (4.19) for the asymptotic contribution of general order isolated critical points to the one-dimensional version of integral (1.2), from which it is possible to derive a complete and explicit hyperasymptotic expansion. This work was done as a lead-in to §5, so we deferred the hyperasymptotic derivation until §6. The far more general case where f has sets of non-isolated critical points of general constant order and general dimension was dealt with in §5 and a resurgence relation (5.20) for its asymptotic contribution was again produced. The former result has not been explicitly written down before, but could be derived from the results in Murphy and Wood (1997) or Murphy (2001), while the latter is a completely new result, greatly generalising much of the existing literature. This resurgence relation (5.20) was then turned into the full hyperasymptotic expansion (6.4) with component expressions (5.13), (6.11), and (6.12) in §6. The optimal truncation scheme (6.14) was derived by estimating the remainder and applying Stirling's approximation, and the hyperterminants (6.11) were rewritten in a form suitable for numerical calculation, namely (6.32).

It was noted toward the end of §5 that our resurgence relation in this case also gave the asymptotic contribution of an integration boundary along which f is constant, such as part or all of a critical component or simply a linear boundary along which f has constant value. The corresponding problem in the context of hyperasymptotic solutions to differential equations was also discussed, which we believe is a partial linear differential equation in d variables that has sets χ_j of non-isolated irregular singularities of exponential rank ω_j and respective dimension and codimension μ_j and q_j. Based on the current literature, this problem would require much clarification and very careful consideration before it could be approached.

Examples demonstrating these new results were provided in §7. The first example in §7.1 considered an unbounded double integral (7.1) with two degenerate critical components of codimension one and constant orders three and five; hyperasymptotic expansions up to level three were numerically computed for both critical components and various graphs involving the sizes of the terms and expansion values are given in Figures 26 - 45. The second example in §7.2 considered a bounded double integral (7.4) with the same integrand as the first example; the boundaries were linear codimension one surfaces along which f was constant (namely linear contour boundaries, implying no restricted critical points). To solve the problem, we decomposed (7.4) into the two boundary contributions and the relevant sum of critical component contributions (one full contribution from the order five critical component) and computed the asymptotic expansion of each contributing point. A level one hyperasymptotic expansion was computed for all three contributing points and various graphs involving the sizes of the terms and expansion values are given in Figures 48 - 51. The third example in §7.3 considered the one-dimensional version of (1.2) for the general class of phase functions (7.6) with isolated critical points of orders two and ω and $z = 0$ and $z = a$ respectively. The coefficients (7.7) of the expansion representing the full contribution from $z = a$ are analysed in detail in order to explore how the coefficient's behaviour changes with ω. Figures 52 - 87 display the complete range of behavioural patterns for $\omega \in \{2, \dots, 11\}$.

To end the thesis, we discussed the path towards deriving a hyperasymptotic expansion for general critical components in the case of a general integration region in §8, along with miscellaneous smaller results involving an introduction to restricted critical points and combining coefficients of lower dimensional expansions into an expression for higher dimensional expansion coefficients. Mainly, we concern ourselves with the non-constant order case and we considered an example involving the complex four-dimensional integral (8.2). Here the phase function f has two non-degenerate isolated critical points and a critical component of codimension one that violates Definition 2; it has non-constant order of degeneracy - given by (8.3) - as it is non-degenerate everywhere except on a complex two-dimensional subset of order three. We show that the level zero expansions can be computed without additional consideration, but when late terms or exponentially improved expansions are considered, the results from §5 break down. We show that the late term expansions for the non-constant order critical component work correctly as they are functions of constant order isolated critical points, but the corresponding expansions for the isolated points yield unreasonable results. This is likely due to an underlying additional contribution from the higher order subset and methods to attempt to handle this issue are discussed towards the end of the chapter.

Until now, exponentially improved asymptotic contributions of non-isolated critical points had not been considered and such asymptotic contributions had not been considered in a complex setting at all. Therefore, this work not only generalises Howls (1997), but also generalises the current literature on non-isolated critical points in a multitude of ways as discussed above. Table 1 below compares the generality of different results from different literature by comparing the values of various quantities in the problems studied.

Once conceptual barriers had been broken down and the problem had been set-up correctly, these results followed quite naturally from the isolated case despite requiring a fair amount of additional clarification. In addition to the smaller results discussed in §8, we see that isolated and non-isolated critical points are not such different creatures after all, as essentially any result will appropriately generalise to the non-isolated case once the problem has been correctly set-up. The main difficulty lies in constructing such a correct set-up, limited somewhat by a lack of required preliminary results in abstract algebra (such as those discussed in §3.2).

Although many generalisations are presented in this thesis, there is still work to be done and there are still more generalisations that need completing or considering, as indicated by §§3.2 and 8 respectively. The discussion in §3.2 pointed out that the rigorous homological proof that allows us to decompose the integration surface for the non-isolated case has not yet been carried out and that the asymptotics have not been shown to be Borel summable, the former of which is well beyond the scope of this thesis and the latter of which is not

Literature / Quantity	μ_j	q_j	ω_j	d	M
Berry and Howls (1991)	0	1	2	1	M
Olde Daalhuis (1998a)	0	1	2	1	M
Howls (1997)	0	d	2	d	M
Murphy (2001)	0	1	ω	1	M
McClure and Wong (1987)	1	1	2	2 (real)	0
Kaminski (1992)	1	2	2	3 (real)	0
Benaissa and Rogers (2001)	1	1	ω_j	2 (real)	0
Benaissa and Rogers (2013)	$d-1$	1	ω_j	d (real)	0
This thesis (2014)	μ_j	q_j	ω_j	d	M

Table 1: Table comparing the results in various pieces of literature by comparing the quantities of the critical components χ_j under study. The quantities μ_j, q_j, and ω_j relate to the critical component χ_j, d is the complex dimension (unless stated otherwise) of the parent space in which the study took place, and M is the level of hyperasymptotic expansion derived.

of immediate concern to us as we deal with finitely truncated expansions. The derivation of hyperasymptotic contributions to general critical components in the presence of a general bounded integration region would be a major result generalising Delabaere and Howls (2002), and the allowance of critical point at infinity would be a theoretical step forward as well. From there, we can keep complicating the components of integral (1.1) until the next major generalisation is achieved.

Finally, we briefly mention some less directly related (but related nonetheless) developments in high energy physics. The first is Cherman, Dorigoni, and Ünsal (2014) that uses resurgence to analyse the perturbation structure of a given quantum field theory in order to relate the perturbative and non-perturbative data quantitatively. The authors make use of the 'principle chiral model', a two dimensional asymptotically free matrix field theory with trivial homotopy group, implying that there are no instantons. While non-isolated critical points are not (yet) involved, it is a noteworthy new development, with the reader referred to the text for full details.

The second is Witten (2010) that looks at the complexification of Chern-Simons theory. Witten's work reduces complicated matrix integrals to integrals of type (1.1) that represent knots, where f has non-isolated critical points. However, in order to determine the contributing critical components, the solutions of non-linear partial differential equations are required. Using our results in §§5 and 6, calculation of the Stokes multipliers for this problem should be possible, substantially reducing the difficulty of the problem (solving linear algebraic equations rather than non-linear partial differential equations). Additionally, the full treatment of a complexified Chern-Simons theory would involve field integrals over matrix integrands; no exponential asymptotic theory currently exists for these types of components.

Witten's work also raises interesting questions about Stokes' phenomenon and knot theory; the crossings in medial graphs that can be used to characterise knots bear resemblance to the adjacency diagrams in hyperasymptotics that describes the Stokes structure of the problem. It is potentially the case that the roots of knot polynomials correspond to Stokes multipliers of underlying characteristic asymptotic problems. Thus, we may be able to link certain equivalence classes of knots to certain integrals by relating the associated medial graphs and Stokes structures respectively, a problem that has also not yet been formally considered.

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