A Solution Method for Linear Rational Expectation Models under Imperfect Information*

Katsuyuki Shibayama
UKC/LSE

May 4, 2007

Abstract

This paper has developed a solution algorithm for linear rational expectation models under imperfect information. Imperfect information in this paper means that some decision makings are based on smaller information sets than others.

The algorithm generates the solution in the form of

$$\begin{aligned} \kappa_{t+1} &= H\kappa_t + J\xi^{t,S} \\ \phi_t &= F\kappa_t + G\xi^{t,S} \end{aligned}$$

where $\xi^{t,S} \equiv \begin{pmatrix} \xi_t^T & \cdots & \xi_{t-S}^T \end{pmatrix}^T$. The technical breakthrough in this article is made by expanding the innovation vector, rather than expanding the set of crawling variables.

Perhaps surprisingly, the H and F matrices are the same as those under the corresponding perfect information models. This implies that if the corresponding perfect information model is saddle path stable (sunspot, explosive), the imperfect model is also saddle-path stable (sunspot, explosive, respectively). Moreover, if the minimum information set in the model has all the information up to time $t-S-1$, then the direct effects on the impulse response functions last for only the first S periods after the impulse. In the subsequent dates, impulse response functions follow essentially the same process as in the perfect information counterpart.

However, imperfect information can significantly alter the quantitative properties of a model, though it does not drastically change its qualitative nature. This article demonstrates, as an example, that adding imperfect information to the standard RBC models remarkably improves the correlation between labour productivity and output. Hence, a robustness check for information structure is recommended.

Keywords: Linear rational expectations models, Imperfect Information
JEL classification codes: C63, C65, C68

*I would like to thank my colleagues and seminar participants at the University of Kent at Canterbury and London School of Economics and Political Science. This article is based on one chapter of my Ph.D. thesis at LSE.

1 Introduction

This paper has developed a solution algorithm for linear rational expectation models under imperfect information. Imperfect information in this paper signifies that some decisions may be made before observing some shocks, while others may be made after observing them. For example, we can consider a variant of RBC model, in which labour supply is decided before observing today's innovation on productivity. In this variant, apart from the information structure (i.e., the FOC w.r.t labour supply has an expectation operator), the equations that defines the equilibrium is the same as in the standard RBC model.

Imperfect information is important for several reasons. First, imperfect information plays an important role in many important classes of models such as sticky information models by Mankiw and Reis (2001). Second, researchers often do not know a priori what information is available when each decision is made. Hence, they may want to estimate the information structure by parameterizing it, or may want to experiment on a model under several patterns of information structure. It is easy to implement such robustness checks with the algorithm. Once structural equations are obtained, then the additional input to the algorithm is only information structure. Third, the obtained numerical result may not be robust for a small change in information structure. Indeed, imperfect information may significantly alter the second moments and the shapes of impulse response functions.

This paper offers an easy-to-use MATLAB code to solve a general class of linear models under imperfect information.[1] The solution of imperfect information models has the form of

$$\begin{aligned}
\kappa_{t+1} &= H\kappa_t + J\xi^{t,S} \\
\phi_t &= F\kappa_t + G\xi^{t,S} \\
\xi^{t,S} &\equiv \begin{pmatrix} \xi_t^T & \cdots & \xi_{t-S}^T \end{pmatrix}^T
\end{aligned}$$

where κ_t and ϕ_t are the vectors of crawling and jump variables, respectively, and ξ_{t-s} is the vector of innovations at time $t-s$, for $s = 0, \cdots, S$, where S is such that the minimum information set in the model includes all information up to time $t-S-1$. $\xi^{\tau,S}$ is the vertical concatenation of $\{\xi_{\tau-s}\}_{s=0}^{S}$. H, J, F and G are the solution matrices that are provided by the algorithm. The algorithm is an extension of the QZ method by Sims (2002).

The most important breakthrough in this paper is the choice of state variables. The state variables in this solution are κ_t and $\xi^{t,S}$. Imperfect information requires the expansion of the state space, but this can be done either by expanding innovation vector or by expanding the set of crawling variables. Note that the representation of state space is not necessarily unique. Our choice of state variables works intuitively because, if past innovations are recorded, we can recover the past crawling variables, and hence can recover the information available in past periods.[2]

By keeping the number of crawling variables unchanged, it can be shown that the dynamic parts of the solution (i.e., H and F matrices) are the same as in the corresponding perfect information model. Thus, it is clear that if the corresponding perfect model is saddle-path stable (sunspot, explosive), then an imperfect information model is also saddle-path stable (sunspot, explosive, respectively). That is to say, the information structure does not alter the dynamic stability property.

Moreover, invariant H and F matrices imply that the direct effects of imperfect information on impulse response functions last for only S after an impulse, if the minimum information set at

[1] The set of Matlab codes is available upon request: k.shibayama@kent.ac.uk

[2] Hence, even though some decisions are made without observing κ_t, for example, economic models can be formulated as in (2).

time t in a model has all the information up to time $t-S-1$. In subsequent periods, the impulse response functions follow essentially the same process as in the perfect information counterpart. More specifically, if, S period after an impulse, the values of the crawling variables are κ_S, then the following impulse response functions are the exactly same as those of the perfect information counterpart that starts with κ_S and zero innovations. One such example can be found in Dupor and Tsuruga (2005). They argue that the hump-shaped impulse response functions found in Mankiw and Reis (2001) critically hinge on the assumption of Calve style information updating: some agents, though their population decreases over time, cannot renew their information forever. By constructing Taylor style staggered information renewal, Dupor and Tsuruga (2005) show that impulse response functions jump to zero right after the last cohort renews its information set.

There are, at least allegedly, two existing treatments of imperfect information.[3] The first remedy for imperfect information is to define *dummy* variables. For example, consider a variant of the standard RBC model, in which labour supply L_t is determined without observing today's innovations. Then, the optimal labour supply is determined by

$$0 = E_{t-1}\left[\eta L_t + \sigma C_t - W_t\right] \tag{1}$$

where C_t and W_t are consumption and wage at time t, η and σ are parameters provided by a theory, and $E_{t-1}[\,]$ is the expectation operator with all information up to time $t-1$. Define dummy variable L_t^* such that

$$\begin{aligned} 0 &= E_t\left[\eta L_{t+1}^* + \sigma C_{t+1} - W_{t+1}\right] \\ L_{t+1} &= L_t^* \end{aligned}$$

In this method, having additional crawling variable L_t, the set of crawling variables is expanded. The problem of this method is that it cannot solve the model if some endogenous variables are determined before observing some (not all) of today's innovations but after observing the others.

The other possibility is a modification of method of undetermined coefficients. According to Christiano (1998), his version of method of undetermined coefficients, like ours, can deal with models in which some endogenous variables are determined before observing some (not all) of today's innovations are observed but after observing the others. The most salient difference between his method and ours is in the specification of information structure. Christiano (1998) requires a user to provide only one matrix R that specifies which innovations are included in the information set of each expectation operator. Roughly speaking, matrix R relates equations to observable innovations (i.e., information). In contrast, in the algorithm developed in this article, a researcher must specify two matrices: one relates innovations to equations, and the other relates

[3] There are three types of methods for perfect information models.

1. King and Watson's method (1998 and 2002) (see also Woodford (undated)) implements two stage substitution. First non-dynamic jump variables are substituted out, and then dynamic jump variables are substituted out from the system of equations.

2. In the QZ method by Sims (2002) (see also Klein (2000)) the QZ decomposition is applied to matrices on endogenous variables. Recognizing that (1) roots that correspond to non-dynamic jump variables are infinite, and (2) roots that correspond to dynamic jump variables are larger than one in absolute terms, the transversality conditions eliminates both types of jump variables at once.

3. The method of undetermined coefficients by Uhlig (1999) (see also Christiano (1998)) substitutes a guess solution into the given system of equations; the resulting matrix polynomial is solved directly. In principle, this method does not require that the give equations are first order difference equations. Higher order matrix polynomials can be numerically solved (see Appendix).

innovations to variables. This difference is crucial. To understand it, consider the above example (1). It is clear that a researcher has to specify the information set of the expectation operator in (1). However, in a given information set, there are generically three possibilities, namely that (a) the representative household fixes labour supply before observing some of today's innovations, (b) it determines wage before innovations (sticky wage), or (c) it decides consumption before innovations. Hence, one more matrix is necessary in our algorithm to specify which of C_t, W_t or H_t is chosen without having full information. In general, the quantitative behaviour of a model is totally different, depending on which variable are assumed to be decided before observing some information. Indeed, in following section, it is demonstrated that the difference between (a) and (b) is very crucial.

The plan of this paper is as follows. In Section 2, we define the problem and derive the solution. There are two key observations. First, if the k-th time t variable $y_{k,t}$ is determined without observing the i-th time $t-s$ innovations $\xi_{i,t-s}$, then $y_{k.t}$ cannot respond to $\xi_{i,t-s}$, given κ_{t-S}. Second, if the expectation operator in the j-th equation has an information set that includes $\xi_{i,t-s}$, $\xi_{i,t-s}$ cannot be the source of the expectation error in the j-th equation. It turns out that these two restrictions are enough to determine the unique solution coefficients. In Section 3, we discuss the assumptions that are necessary to guarantee the existence of a solution. Each of them has some economic meaning. The existence condition is slightly tighter under imperfect information than under perfect information. In Section 4, the main features of the solution of imperfect information models are briefly discussed. Most of them are direct consequences of the invariant H and F matrices. In Section 5, we demonstrate the effects of imperfect information on the otherwise standard RBC model as an example. The final section concludes the discussion.

2 Derivation of the Solution

Essentially, our algorithm is an extension of the QZ method used in Sims (2002). Our problem is to obtain the state space representation of a solution that satisfies two key zero restrictions. For the details of matrix notation, see Appendix.

2.1 Definition of the problem

The inputs and outputs of the algorithm are defined.

2.1.1 Given models

Following Sims (2002), we formulate the linear rational models with expectation errors as follows.

$$0 = Ay_{t+1} + By_t + C\xi_t + D\xi_{t+1} + E\xi^{t,S} \tag{2}$$

where

$$
\begin{aligned}
E &= \begin{bmatrix} E_0 & E_1 & \cdots & E_s & \cdots & E_S \end{bmatrix} \\
&= \begin{bmatrix} E_{0,11} & \cdots & E_{0,1N} & & E_{s,11} & \cdots & E_{s,1N} & & E_{S-1,11} & \cdots & E_{S,1N} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ E_{0,M1} & \cdots & E_{0,MN} & & E_{s,M1} & \cdots & E_{s,MN} & & E_{S-1,M1} & \cdots & E_{S,MN} \end{bmatrix} \\
y &= \begin{pmatrix} \kappa_t \\ \phi_t \end{pmatrix}, \ \xi^{t,S} = \begin{pmatrix} \xi_t \\ \vdots \\ \xi_{t-S} \end{pmatrix}
\end{aligned}
$$

y_t is the vector of all endogenous variables, in which κ_t is the vector of crawling variables and ϕ_t is that of jump variables. Stock variables are all recorded at the beginning of each period. M is the number of equations, which is equal to the number of endogenous variables, N is the number of innovations, and S is such that the minimum information set includes ξ_{t-S-1}.

ξ_{t-s} is a column vector of *iid* innovations at time $t-s$. Limiting ξ_t to *iid* is not restrictive since we can add the law of motions of serially correlated shocks to the system of equations and treat the shocks themselves as crawling variables.[4].

A, B and C are proper coefficient matrices, which are provided by an economic theory. D and E represent the expectation errors. D is non-zero even for the perfect information models, because of the dynamic jump variables (e.g., expectation error in the Euler equation). An economic theory must specify the positions of zero elements in D and E, while the values of non-zero elements are calculated by the algorithm. ξ_{t-s} can be the source of expectation errors because some endogenous variables are decided without observing it.

2.1.2 Goal of the algorithm

Our objective is to obtain the state space representation of (2).[5]

$$
\begin{aligned}
\kappa_{t+1} &= H\kappa_t + J\xi^{t,S} \qquad (3a) \\
\phi_t &= F\kappa_t + G\xi^{t,S} \qquad (3b)
\end{aligned}
$$

where

$$
\begin{aligned}
J &\equiv \begin{bmatrix} J_0 & J_1 & \cdots & J_s & \cdots & J_S \end{bmatrix} \\
&\equiv \begin{bmatrix} J_{0,11} & \cdots & J_{0,1N} & & J_{s,11} & \cdots & J_{s,1N} & & J_{S,11} & \cdots & J_{S,1N} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ J_{0,M_\kappa 1} & \cdots & J_{0,M_\kappa N} & & J_{s,M_\kappa 1} & \cdots & J_{s,M_\kappa N} & & J_{S,M_\kappa 1} & \cdots & J_{S,M_\kappa N} \end{bmatrix} \\
G &\equiv \begin{bmatrix} G_0 & G_1 & \cdots & G_s & \cdots & G_S \end{bmatrix} \\
&\equiv \begin{bmatrix} G_{0,11} & \cdots & G_{0,1N} & & G_{s,11} & \cdots & G_{s,1N} & & G_{S,11} & \cdots & G_{S,1N} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ G_{0,M_\phi 1} & \cdots & G_{0,M_\phi N} & & G_{s,M_\phi 1} & \cdots & G_{s,M_\phi N} & & G_{S,M_\phi 1} & \cdots & G_{S,M_\phi N} \end{bmatrix}
\end{aligned}
$$

[4] See Woodford (undated). This simplifies the algebra and computation significantly.

[5] $S = 0$ does *not* imply perfect information. If an endogenous variable does not observe only today's innovation ξ_t, then $S = 0$. In a sense, J_0 and G_0 consist of the direct effects of ξ_t through $C\xi_t$ in (2) as under perfect information models, and the effects of expectation errors due to ξ_t through $E\xi_t$.

2.2 Two key observations

This subsection shows two key zero restrictions. The algorithm seeks the solution that satisfies them.

2.2.1 Repeated substitutions

To obtain the representation of κ_t and ϕ_t as functions of κ_{t-S} and $\xi_{t-\tau}$ for $\tau = 0, \cdots, 2S-1$, repeat the substitution of vertically concatenated "guess solution" (3) into itself.

$$
\begin{aligned}
\begin{pmatrix} \kappa_{t+1} \\ \phi_t \end{pmatrix} &= \begin{bmatrix} H \\ F \end{bmatrix} \kappa_t + \tilde{\Gamma}\xi^{t,S} = \begin{bmatrix} H \\ F \end{bmatrix} \left(H^S \kappa_{t-S} + \textstyle\sum_{k=1}^{S} H^{k-1} J \xi^{t-k,S} \right) + \tilde{\Gamma}\xi^{t,S} \\
&= \begin{bmatrix} H \\ F \end{bmatrix} H^S \kappa_{t-S} + \left(\Gamma_0 \xi_{t-0} + \Gamma_1 \xi_{t-1} + \cdots + \Gamma_S \xi_{t-S} \right) \\
&\quad + \begin{bmatrix} H \\ F \end{bmatrix} \begin{pmatrix} H^1 \left(J_0 \xi_{t-1} + J_1 \xi_{t-2} + \cdots + J_S \xi_{t-1-S} \right) \\ +H^1 \left(J_0 \xi_{t-2} + J_1 \xi_{t-3} + \cdots + J_S \xi_{t-2-S} \right) + \cdots \\ +H^{s-1} \left(J_0 \xi_{t-s} + J_1 \xi_{t-s-1} + \cdots + J_S \xi_{t-s-S} \right) + \cdots \\ +H^{S-1} \left(J_0 \xi_{t-S} + J_1 \xi_{t-S-1} + \cdots + J_S \xi_{t-S-S} \right) \end{pmatrix} \\
&= \begin{bmatrix} H \\ F \end{bmatrix} H^S \kappa_{t-S} + \Pi_0 \xi_t + \Pi_1 \xi_{t-1} + \cdots + \Pi_s \xi_{t-s} + \cdots + \Pi_S \xi_{t-S} \\
&\quad + \text{terms with } \xi_{t-\tau} \text{ for } \tau \geq S+1
\end{aligned} \tag{4}
$$

where $\tilde{\Gamma} \equiv \begin{bmatrix} \Gamma_0 & \cdots & \Gamma_s & \cdots & \Gamma_S \end{bmatrix}$ with $\Gamma_s \equiv \begin{bmatrix} J_s^T & G_s^T \end{bmatrix}^T$, and

$$
\begin{aligned}
\Pi_0 &\equiv \Gamma_0 = \begin{bmatrix} J_0 \\ G_0 \end{bmatrix} \\
\Pi_1 &\equiv \Gamma_1 + \begin{bmatrix} H \\ F \end{bmatrix} J_0 = \begin{bmatrix} J_1 + HJ_0 \\ G_1 + FJ_0 \end{bmatrix} \\
\Pi_2 &\equiv \Gamma_2 + \begin{bmatrix} H \\ F \end{bmatrix} (J_1 + HJ_0) = \begin{bmatrix} J_2 + H(J_1 + HJ_0) \\ G_2 + F(J_1 + HJ_0) \end{bmatrix}, \cdots \\
\Pi_s &\equiv \Gamma_s + \begin{bmatrix} H \\ F \end{bmatrix} \left(\textstyle\sum_{k=0}^{s-1} H^{s-1-k} J_k \right) = \begin{bmatrix} J_s + H \sum_{k=0}^{s-1} H^{s-1-k} J_k \\ G_s + F \sum_{k=0}^{s-1} H^{s-1-k} J_k \end{bmatrix}, \cdots \\
\Pi_S &\equiv \Gamma_S + \begin{bmatrix} H \\ F \end{bmatrix} \left(\textstyle\sum_{k=0}^{S-1} H^{S-1-k} J_k \right) = \begin{bmatrix} J_S + H \sum_{k=0}^{S-1} H^{S-1-k} J_k \\ G_S + F \sum_{k=0}^{S-1} H^{S-1-k} J_k \end{bmatrix}
\end{aligned}
$$

In the recursive representation,

$$
\begin{aligned}
\Pi_0 &= \Gamma_0 = \begin{bmatrix} J_0 \\ G_0 \end{bmatrix} \\
\Pi_s &= \Gamma_s + \tilde{H}\Pi_{s-1} \text{ for } s = 1, \cdots, S
\end{aligned}
$$

where

$$
\tilde{H} \equiv \begin{bmatrix} H & 0 \\ F & 0 \end{bmatrix} \tag{6}
$$

Intuitively, the j,k-th element of Π_s is the effect of $\xi_{k,t-s}$ (the k-th innovation at time $t-s$) on $y_{j,t}$ (the j-th endogenous variable at time t). Thus, given κ_{t-S}, $\Pi_{s,jk}$, which is defined as the j,k-th element of Π_s, is zero if $y_{j,t}$ is determined without observing $\xi_{k,s}$.

In the matrix representation

$$\Gamma = M_{\Gamma\Pi}\Pi \tag{7}$$

where

$$\begin{aligned}
\Gamma &\equiv \begin{bmatrix} \Gamma_0^T & \cdots & \Gamma_s^T & \cdots & \Gamma_S^T \end{bmatrix}^T && \text{(8a)}\\
\Pi &\equiv \begin{bmatrix} \Pi_0^T & \cdots & \Pi_s^T & \cdots & \Pi_S^T \end{bmatrix}^T && \text{(8b)}\\
M_{\Gamma\Pi} &\equiv \begin{bmatrix} I & & & 0 \\ -\tilde{H} & I & & \\ & \ddots & \ddots & \\ 0 & & -\tilde{H} & I \end{bmatrix} && \text{(8c)}
\end{aligned}$$

This matrix plays a key role in the following. $M_{\Gamma\Pi}$ is clearly invertible.

2.2.2 Zero Restrictions

Throughout this paper, we exploit the following two observations.

1. If the k-th set of variables $y_{k,t}$ does not observe the i-th set of time $t-s$ innovations $\xi_{i,t-s}$, given κ_{t-S} and $\xi_{t-\tau}$ for $\tau = s+1, \cdots$, $\partial y_{k,t}/\partial \xi_{t-s} = 0$. Simply put, any decision cannot respond to unobserved innovations. Hence, $\Pi_{s,ki} = 0$.

2. If the information set of the expectation operator in the j-th set of equations includes the i-th set of time $t-s$ innovations $\xi_{i,t-s}$, then the realization of the j-th set of equations must hold for any realization of the i-th set of innovations. The expectation errors only occur only due to innovations that are not included in the information set. Thus, $E_{s,ji} = 0$.

For example, suppose that labour supply L_t (k-th variable, $y_{k,t}$) is decided on before observing today's technology shock (i-th shock, $\xi_{i,t}$), but after today's preference shock (l-th shock, $\xi_{l,t}$), both of which are *iid*. If the FOC w.r.t. L_t is the j-th equation,

$$\begin{aligned}
\Pi_{0,ki} &= 0 \ (\xi_{0,i} \text{ does not affect } y_{k,t})\\
E_{0,jl} &= 0 \ (\xi_{0,l} \text{ does not cause expectation error in } j\text{-th eqn})
\end{aligned}$$

Roughly speaking, $E_{0,jl} = 0$ means that if the expectation operator of the j-th equation is eliminated, still it holds in terms of $\xi_{0,l}$.

It is the duty of a user to specify the positions of these zero elements in Π and E.[6]

2.3 Sketch of Derivation and Key Equations for Computation

The fully detailed derivation is given in Appendix. This subsection briefly describes the skeleton of the derivation and lists the minimum results necessary for computation.

[6]The algorithm automatically finds the positions of zero elements in D matrix based on the specification of jump and crawling variables (Only dynamic jump variables can be the sources of expectation errors). A user is required to specify which variables are crawling variables and which are jump.

2.3.1 QZ Decomposition

In order to introduce notations, this subsection briefly reviews the QZ decomposition (or generalized Schur decomposition). For matrices A and B ($\in \mathbb{C}^{n\times n}$), there exist unitary matrices Q and Z such that

$$\begin{aligned} Q^H A Z &= \Omega_A \\ Q^H B Z &= \Omega_B \end{aligned}$$

where Ω_A and Ω_B are both upper triangular matrices, and superscript H indicates a conjugate transpose. A unitary matrix U satisfies $U^H U = U U^H = I$. Let a_{kk} and b_{kk} be the k-th diagonal elements in Ω_A and Ω_B, respectively. Assuming that a_{kk} and b_{kk} are not zero at the same time, then $\lambda_k \equiv b_{kk}/a_{kk}$ for $k = 1, \cdots, n$ are the generalized eigenvalues of the matrix pencil $B - \lambda_k A$.[7]

The basic idea is that by applying the QZ decomposition to (2) as in Sims (2002), the algorithm separates unstable roots from stable roots.

$$\begin{aligned} 0 &= A y_{t+1} + B y_t + C\xi_t + D\xi_{t+1} + E\xi^{t,S} \\ &= \Omega_A Z^H y_{t+1} + \Omega_B Z^H y_t + Q^H C\xi_t + Q^H D\xi_{t+1} + Q^H E\xi^{t,S} \\ &= \begin{bmatrix} \Omega^A_{ss} & \Omega^A_{su} \\ 0 & \Omega^A_{uu} \end{bmatrix} \begin{pmatrix} s_{t+1} \\ u_{t+1} \end{pmatrix} + \begin{bmatrix} \Omega^B_{ss} & \Omega^B_{su} \\ 0 & \Omega^B_{uu} \end{bmatrix} \begin{pmatrix} s_t \\ u_t \end{pmatrix} \\ &\quad + \begin{bmatrix} Q^H_{s.} \\ Q^H_{u.} \end{bmatrix} C\xi_t + \begin{bmatrix} Q^H_{s.} \\ Q^H_{u.} \end{bmatrix} D\xi_{t+1} + \begin{bmatrix} Q^H_{s.} \\ Q^H_{u.} \end{bmatrix} E\xi^{t,S} \end{aligned}$$

where

$$\begin{pmatrix} s_t \\ u_t \end{pmatrix} \equiv Z^H \begin{pmatrix} \kappa_t \\ \phi_t \end{pmatrix}$$

By transversality conditions (TVCs), all unstable roots are set to be equal to zero (Remember that all innovations are assumed to be *iid*). All unstable roots must be zero under imperfect information as under perfect information, which is guaranteed by the iterated linear projection.

2.3.2 Notations for the Outputs of QZ Decomposition

For later use, we define submatrices as follows

$$Z^H \equiv \begin{bmatrix} Z^H_{s.} \\ Z^H_{u.} \end{bmatrix} \equiv \begin{bmatrix} Z^H_{s\kappa} & Z^H_{s\phi} \\ Z^H_{u\kappa} & Z^H_{u\phi} \end{bmatrix}, Z \equiv \begin{bmatrix} Z_{\kappa s} & Z_{\kappa u} \\ Z_{\phi s} & Z_{\phi u} \end{bmatrix}, Q^H \equiv \begin{bmatrix} Q^H_{s.} \\ Q^H_{u.} \end{bmatrix} \tag{9a}$$

$$\Omega^A \equiv \begin{bmatrix} \Omega^A_{ss} & \Omega^A_{su} \\ 0 & \Omega^A_{uu} \end{bmatrix}, \Omega^B \equiv \begin{bmatrix} \Omega^B_{ss} & \Omega^B_{su} \\ 0 & \Omega^B_{uu} \end{bmatrix} \tag{9b}$$

where subscripts u and s implies unstable and stable roots. Note that Ω^A_{ss} and Ω^B_{uu} are both invertible by construction.

[7] See Appendix for a brief review of the relationship between the system of first order difference equations and generalized eigenvalues.

Also, we define four matrices as

$$
\begin{aligned}
\Lambda^A_{s\kappa} &\equiv \Omega^A_{ss} Z^H_{s\kappa} + \Omega^A_{su} Z^H_{u\kappa} && \text{(10a)}\\
\Lambda^A_{s\phi} &\equiv \Omega^A_{ss} Z^H_{s\phi} + \Omega^A_{su} Z^H_{u\phi} && \text{(10b)}\\
\Lambda^B_{s\kappa} &\equiv \Omega^B_{ss} Z^H_{s\kappa} + \Omega^B_{su} Z^H_{u\kappa} && \text{(10c)}\\
\Lambda^B_{s\phi} &\equiv \Omega^B_{ss} Z^H_{s\phi} + \Omega^B_{su} Z^H_{u\phi} && \text{(10d)}
\end{aligned}
$$

Note that all the matrices defined here are obtained from the outputs of the QZ decomposition.

2.3.3 Matrix Subscripts

We introduce the following notation rule for subscripts on matrices. For a matrix A,

- $A_{.x}$ is columns x of a matrix A.
- $A_{x.}$ is rows x of a matrix A.
- $A_{.\neg x}$ is the columns remaining after the elimination of columns x.
- $A_{\neg x.}$ is the rows remaining after the elimination of rows x.

where x is the name of a set of columns or rows. This notation makes certain matrix operations extremely simple. See Appendix for further details.

2.3.4 Zero Restrictions

As a result of some manipulation of matrix equations, it is shown that

$$
\begin{aligned}
0 &= \Pi + M_{\Pi E}\left(E + \mathbf{C}\right) && \text{(11)}\\
M_{\Pi E} &\equiv \left(M_{y\Gamma} M_{\Gamma\Pi}\right) \backslash \mathbf{Q} && \text{(12)}
\end{aligned}
$$

where

$$
\Gamma \equiv \begin{pmatrix} \Gamma_0 \\ \vdots \\ \Gamma_{S-1} \end{pmatrix}, E \equiv \begin{pmatrix} E_0 \\ \vdots \\ E_{S-1} \end{pmatrix}, \mathbf{C} \equiv \begin{pmatrix} C_0 \\ 0 \end{pmatrix}, \mathbf{Q} \equiv \begin{bmatrix} Q & & 0 \\ & \ddots & \\ 0 & & Q \end{bmatrix} \quad \text{(13a)}
$$

$$
M_{y\Gamma} \equiv \begin{bmatrix} \Phi & \Lambda^{0A} & & & 0 \\ & \Phi & \Lambda^{0A} & & \\ & & \ddots & \ddots & \\ 0 & & & \Phi & \Lambda^{0A} \\ & & & & \Phi \end{bmatrix}, \Phi \equiv \begin{bmatrix} \Omega^A_{ss}/Z_{\kappa s} & \Lambda^B_{s\phi} \\ 0 & \Omega^B_{uu} Z^H_{u\phi} \end{bmatrix}, \Lambda^{0A} \equiv \begin{bmatrix} 0 & \Lambda^A_{s\phi} \\ 0 & \Omega^A_{uu} Z^H_{u\phi} \end{bmatrix} \quad \text{(13b)}
$$

and $X \backslash Y = X^{-1}Y$. Bear in mind that Π and E are still "undetermined coefficients," while, once H and F are obtained, $M_{\Pi E}$ is computable from the outputs of the QZ decomposition.

E and Π are computed column by column. It is important to remember due to zero restrictions some elements in Π and E are zero. Thus, for the i-th column (or equivalently for the i-th innovation),

$$0=\begin{pmatrix}\Pi_{1,i}\\ \vdots\\ \Pi_{k,i}\,(=0)\\ \vdots\\ \Pi_{M(S+1),i}\end{pmatrix}+M_{\Pi E}\left(\begin{pmatrix}0\\ \vdots\\ E_{ji}\\ \vdots\\ 0\end{pmatrix}+\begin{pmatrix}C_{.i}\\ 0\\ \vdots\\ \vdots\\ 0\end{pmatrix}\right) \tag{14}$$

where M is the number of equations and hence $M\,(S+1)$ is the number of rows in Π.

From the k-th set of equations in (14)

$$0=\left[\ M_{\Pi E}\ \right]_{kj}E_{ji}+\left[\ M_{\Pi E}\ \right]_{kj}\mathbf{C}_{ji}+\left[\ M_{\Pi E}\ \right]_{k\neg j}\mathbf{C}_{\neg ji} \tag{15}$$

which gives the values of non-zero elements of E. From the rest of the equations in (14),

$$\begin{aligned}0\ &=\ \Pi_{\neg ki}+\left[\ M_{\Pi E}\ \right]_{\neg kj}\mathbf{C}_{ji}+\left[\ M_{\Pi E}\ \right]_{\neg k\neg j}\mathbf{C}_{\neg ji}\\ &\quad-\left[\ M_{\Pi E}\ \right]_{\neg kj}\left(\left[\ M_{\Pi E}\ \right]_{kj}\backslash\left[\ M_{\Pi E}\ \right]_{k\neg j}\mathbf{C}_{\neg ji}+\mathbf{C}_{ji}\right)\end{aligned} \tag{16}$$

which gives the non-zero elements of Π.

The existence of the inverse of $[M_{\Pi E}]_{kj}$ is assumed. In general, however, it is not necessarily invertible and an economic meaning of its invertibility is discussed later.

2.3.5 Solution

The solution algorithm computes key matrices sequentially. The basic structure is as follows.

1. Obtain submatrices form the outputs of QZ decomposition (9 and 10).
2. Obtain H and F from (17).
3. Obtain $M_{\Gamma\Pi}$, $M_{y\Gamma}$ and $M_{\Pi E}$ from (8c, 13b and 12).
4. Obtain E and Π from (18 and 19)
5. Obtain G and J from (20).

H and F: As in (Sims (2002)), the H and F matrices are derived independently from the G and J matrices based on the coefficient on κ_t (see Appendix for details). Therefore, they are exactly the same as in perfect information models.

$$F\ =\ -Z^H_{u\phi}\backslash Z^H_{u\kappa}=Z_{\phi s}/Z_{\kappa s} \tag{17a}$$

$$H\ =\ -Z_{\kappa s}\left(\Omega^A_{ss}\backslash\Omega^B_{ss}\right)/Z_{\kappa s} \tag{17b}$$

E and Π: From (15) and (16), the non-zero elements of E and Π are

$$E_{ji}\ =\ -\left[\ M_{\Pi E}\ \right]_{kj}\backslash\left[\ M_{\Pi E}\ \right]_{k\neg j}\mathbf{C}_{\neg ji}-\mathbf{C}_{ji} \tag{18}$$

$$\Pi_{\neg ki}\ =\ -\left[\ M^{-1}_{\Pi E}\ \right]_{\neg j\neg k}\backslash\mathbf{C}_{\neg ji} \tag{19}$$

where $M_{\Pi E}$ can be obtained from (8c) and (13) with the solution of H and F. Note that H and F can be computed without referring E, Π or $M_{\Pi E}$. Since $[M_{\Pi E}]_{kj}$ is assumed to be invertible, $\left[M^{-1}_{\Pi E}\right]_{\neg j\neg k}$ is also invertible (see Appendix for the proof).

J and G: From the definition of Γ (8a),

$$\Gamma \equiv \begin{bmatrix} J_0 \\ G_0 \\ \vdots \\ J_S \\ G_S \end{bmatrix} = M_{\Gamma\Pi}\Pi \tag{20}$$

Non-zero elements of $M_{\Gamma\Pi}$ are recovered from (19).

D: From a given economic model (2) it is obvious that

$$D = -A \begin{bmatrix} 0 \\ G_0 \end{bmatrix}$$

3 Assumptions

In this section, we discuss three assumptions. Assumptions 1 and 2 in the following are the same as in the solution method for perfect information models, while assumption 3 is specific to imperfect information models. This subsection omits discussion about Blanchard-Kahn condition, which is automatically satisfied by assumption 1.

3.1 Assumption1: $Z_{u\phi}^H$ is invertible

Klein (2000) shows that this assumption is a generalization of the condition derived in Blanchard and Kahn (1980). Boyd and Dotsey (1990) makes clear that the Blanchard-Kahn condition which counts and compares the numbers of unstable roots and jump variables is a necessary (but not sufficient) condition for the existence of a unique solution; they provide a counter example that satisfies the Blanchard-Kahn counting condition but does not have a stable solution. Intuitively, invertible $Z_{u\phi}^H$ means that we can always find the values of jump variables that guarantee $u_{t+1} = 0$ (TVCs) in any states. Remember that $Z_{u\phi}^H$ maps jump variables to unstable roots, and its inverse maps unstable roots to jump variables. See King and Watson (1995) for an intuitive exposition.

The existence of the right inverse of $Z_{u\phi}^H$ entails the existence of jump variables, but the non-existence of its left inverse implies non-uniqueness of jump variables. See Uhlig (2000) for a treatment of non-uniqueness.

3.2 Assumption2: a_{kk} and b_{kk} are not zero at the same time

If a_{kk} and b_{kk} are zero at the same time, it implies that there exist row vectors X such that $0 = X\xi$. One of such examples is the k-th row of Q. The existence of such row vectors generically implies either of the following.

(a) If $X\xi$ is indeed zero, then some equations are not linearly independent from the others. Essentially there are fewer equations than endogenous variables. At least one of equation can be expressed as a linear combination of others, and such a linear combination is shown by X.

(b) If $X\xi$ is not zero, clearly there is an internal contradiction (i.e., a system of equations (2) is internally inconsistent). One of such examples is 2 equation and 2 variables non-dynamic model with no state variables

$$\begin{aligned}\phi_{1,t} &= \alpha\phi_{2,t} + \xi_t \\ \phi_{1,t} &= \alpha\phi_{2,t} + \xi_t + \eta_t\end{aligned}$$

Obviously both do not hold at the same time for non-zero η_t. Since QZ decomposition is merely a linear transformation, this implies that there is an internal inconsistency in the original system of equations (2).

Rephrasing the above arguments as follows might be more intuitive:

> If a_{kk} and b_{kk} are zero at the same time, there are fewer equations than endogenous variables in the non-stochastic steady state.

This is simply because in the non-stochastic steady state it must be the case that $X\xi = 0$. See Sims (2002) as well.

3.3 Assumption3: $[M_{\Pi E}]_{kj}$ is invertible

This condition is specific to imperfect information models, though it is analogous to the equation (40) in Sims (2002).[8] Intuitively, if it is not invertible, then the information structure is not consistent. Note that the inverse of $[M_{\Pi E}]_{kj}$, if it exists, maps expectation errors to innovations that some endogenous variables cannot respond. Hence, if the inverse of $[M_{\Pi E}]_{kj}$ exists, then expectation errors can equate both sides of the equations for any realization of innovations.

A non-invertible $[M_{\Pi E}]_{kj}$ appears in the following example. Suppose that all production factors and all demand components are decided before observing today's technology shock. In this case, output varies depending on the realization of technology, while demand cannot respond to it. Thus, the goods market does not clear at any price. One important lesson is that a researcher has to construct consistent models. An arbitrarily specified information structure may have internal inconsistency.

4 Properties of the Solution

The solutions computed by the algorithm have the following properties. Properties 1 and 2 are simply the basis of the algorithm and properties 3 and 4 are the direct consequences of invariant H and F.

1. If a variable $y_{k,t}$ is decided without observing an innovation $\xi_{i,t-s}$, then $\xi_{i,t-s}$ does not affect $y_{k,t}$. I.e., $\partial x_t/\partial\xi_{i,t-s} = 0$, given crawling variables κ_{t-S}.

2. If $\xi_{i,t}$ *are* included in the information set of expectation operators in the j-th equation, then $\xi_{i,t}$ cannot be the source of the expected error in the j-th equation.

3. The dynamic parts of the solution (H and F) are the same as in the perfect information models. Thus, imperfect information does not change the number of stable and unstable roots. As a consequence, if a model under imperfect information exhibits saddle-path stability, for example, then the corresponding model under perfect information must also exhibit saddle-path stability.

[8] However, note that Sims' condition is related to time $t+1$ expectation errors, while our discussion in the following is related to time τ expectation errors ($\tau < t$).

4. Invariant dynamic parts also imply that the direct effects of imperfect information last only S period after an impulse. In the subsequent periods, they essentially follow the same dynamics as under perfect information. More specifically, let $\tilde{\kappa}_{t+S}$ be the values of crawling variables S period after an impulse. Then subsequent impulse response functions are the exactly same as those under perfect information starting with $\tilde{\kappa}_{t+S}$ with setting all innovations equal to zero.

The properties 3 and 4 show that *qualitatively* an imperfect information model inherits key properties of the corresponding perfect information model. But as shown in the next section, imperfect information can have *quantitatively* significant effects.

The following points are also important.

- In our representation of solutions, the set of state variables at time t is $\left\{\kappa_t, \xi_t, \xi_{t-1}, \cdots, \xi_{t-S}\right\}$. Namely, today's crawling variables and current and past innovations. Roughly speaking, crawling variables correspond to state variables under perfect information. Past innovations are necessary to describe the economy, because they recover past information sets.

- If κ_{t-j} and $\xi^{t-j,S}$ are observable, from (3a), the economic agents in a model can infer κ_{t-j+1}. The algorithm does not accept illogical information structure. For example, information set $\left\{\kappa_{t-1}, \kappa_{t-2}, \cdots, \xi_t, \xi_{t-1}, \cdots\right\}$ (without κ_t) is not allowed because, observing κ_{t-1} and $\xi^{t,S}$, economic agents must know κ_t. Similarly, $\left\{\kappa_t, \kappa_{t-1}, \cdots, \xi_t, \xi_{t-3}, \xi_{t-4}, \cdots\right\}$ (without ξ_{t-1} or ξ_{t-2}) is not acceptable. On the other hand, the algorithm can deal with information set $\left\{\kappa_{t-2}, \kappa_{t-3}, \cdots, \xi_t, \xi_{t-3}, \xi_{t-4}, \cdots\right\}$, though it is hard to interpret economically.

  The algorithm only requires the positions of zero elements in Π and E matrices, both of which are coefficient matrices on $\xi^{t,S}$. This means that the algorithm detects S from the zero elements of Π and E. The minimum information is deemed to be $\left\{\kappa_{t-2}, \kappa_{t-3}, \cdots, \xi_t, \xi_{t-3}, \cdots\right\}$ if $S = 2$.

- The maximum possible information set at time t (perfect information) is $\left\{\kappa_{t-j}, \xi_{t-j}\right\}_{j=0}^{\infty}$, though some of these elements are redundant (i.e., some of them are not state variables). This implies that the algorithm does not allow inference.

  If the information set of economic agents in a model includes all current and past variables $\left\{y_{t-j}, \xi_{t-j}\right\}_{j=0}^{\infty}$, then the economic agents can infer most hidden information, which reduces an imperfect model to the corresponding perfect information model in most cases. For example, if households observe all production factors and output, they can infer today's productivity correctly.

  One natural interpretation of imperfect information is that agents have to make future decision in the current period, like in sticky price models.

- The algorithm cannot deal with parameter uncertainty.

- The algorithm can deal with noisy information models easily. Suppose an AR(1) shock process A_t follows

$$\ln A_{t+1} = \rho \ln A_t + \sqrt{1-\eta}\,\xi_t^{ob} + \sqrt{\eta}\,\xi_t^{uo} \tag{21}$$

  where ξ_t^{ob} and ξ_t^{uo} are the observable and unobservable components of innovation, and $(1-\eta)/\eta$ is the signal to noise ratio. This technique allows us to parameterise the extent of imperfect information.

5 An Example

5.1 Standard RBC Model

To demonstrate the quantitative effects of imperfect information, we consider the standard RBC model under imperfect information, focussing on impulse response functions and second moments.

The main economic motivation is to address an overly high $Corr\left(Y_t - H_t, Y_t\right)$ in the standard RBC model. Under the plausible parameter range, the standard RBC model predicts almost perfect correlation between labour productivity $Y_t - H_t$ and output Y_t, but in the data the correlation is only slightly positive.

Hence, we modify the standard RBC model by adding imperfect information related to the labour market. The relevant equations are

$$0 = bH_t - W_t - \lambda_t \tag{22a}$$
$$0 = Y_t - H_t - W_t \tag{22b}$$

where Y_t, H_t, W_t, λ_t are output, working hours, wage and the marginal utility of consumption, respectively. All endogenous variables are measured as deviations from their steady state values in % terms. b is a constant, which represents (a multiple of) the elasticity of marginal disutility of labour. The first equation is the FOC w.r.t. labour supply; the second shows that the marginal product of labour $(Y_t - H_t)$ is equal to wage.[9] The set of state variables under perfect information is $\{K_t, A_t, \xi_t\}$, where K_t and A_t are capital and technology at the beginning of time t, respectively, and ξ_t represents the innovation on technology. Note that A_t is regarded as an endogenous crawling variable, and there is only one *iid* exogenous variable ξ_t. That is to say, A_t is treated as a stock variable.

Assuming that today's innovation affects today's output,

$$\begin{aligned} Y_t &= A_{t+1} K_t^{\alpha} H_t^{1-\alpha} \\ \ln A_{t+1} &= \rho \ln A_t + \xi_t \end{aligned}$$

where ρ is a parameter that governs the persistence of technology shock.

5.1.1 Case I: HH decides labour supply before observing innovations

In this case (22a) does not hold. Instead, the labour supply decision is governed by

$$0 = E\left[\ bH_t - W_t - \lambda_t \mid \left\{K_{t-j}, A_{t-j}, \xi_{t-j}\right\}_{j=S+1}^{\infty}\right]$$

Since H_t cannot react to past innovations, for $s = 0, 1, \cdots, S$,

$$\frac{\partial H_t}{\partial \xi_{t-s}} = 0 \text{ given } K_{t-S}, A_{t-S}$$

[9]Note that since all endogenous variables are represented as log-deviations from their steady state, $Y_t - H_t$ is the deviation of "output divided by labour hour" (i.e., labour productivity). The Cobb-Douglas production function implies that the marginal product of labour is $(1-\alpha)$ times labour productivity, which means that the % change of labour productivity is exactly the same as that of the marginal product of labour. In other words, in the Cobb-Douglas production function, $Y_t - H_t$ represents both the % deviation of labour productivity and the marginal product of labour.

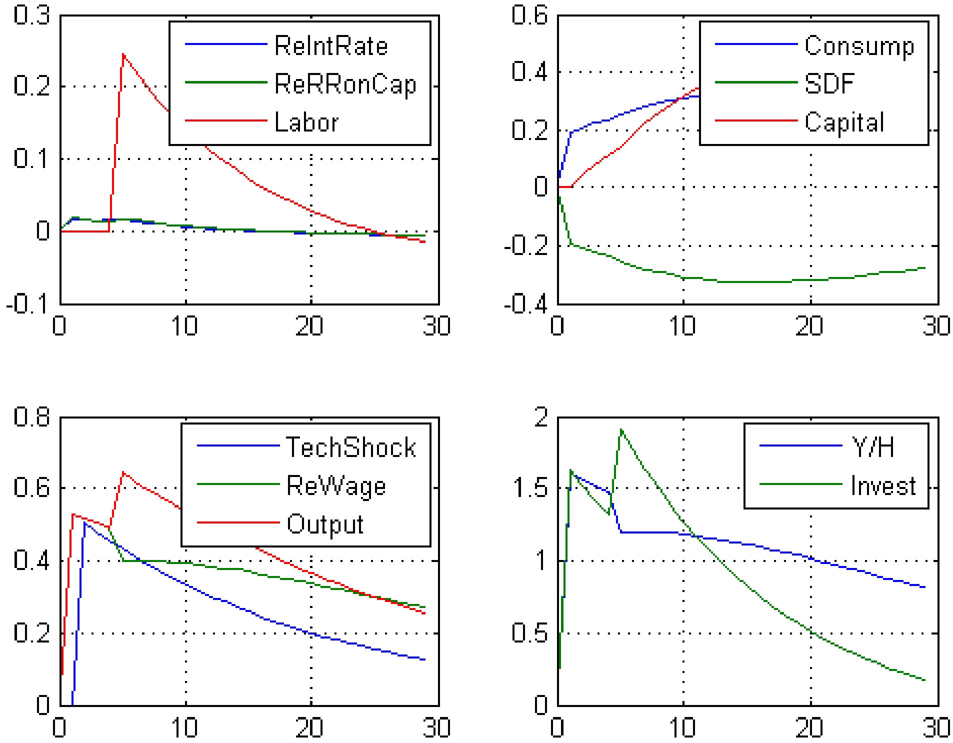

Figure 1: Impulse response functions to a positive technology innovation of the standard RBC model in which labour supply is determined 5 periods in advance.

Figure (1) shows the impulse response functions with $S = 5$, which means that the household decides its labour supply 5 quarters in advance.

There are several of points worth noting here.

- Labour hours do not move for the first S periods. That is, $\partial H_t / \partial \xi_{t-s} = 0$ for $s = 0, 1, \cdots, S$.
- Labour productivity $(Y_t - H_t)$ and investment show "unusual" movements for the first S periods. However, after $S + 1$ periods all endogenous variables follow (a linear combinations of) AR(1) processes. This is one example of the proposition that *the direct effect of imperfect information lasts only S period after an impulse.*
- $Corr\,(Y_t - H_t, Y_t)$ is lower than under perfect information, but only slightly (exact numbers are not shown).

5.1.2 Case II: Firm decides labour demand before observing innovations

In this case (22b) does not hold. Instead, the labour supply decision is governed by

$$0 = E\left[\; Y_t - H_t - W_t \mid \{K_{t-j}, A_{t-j}, \xi_{t-j}\}_{j=S+1}^{\infty}\right]$$

Since H_t cannot react to the innovations, for $s = 0, 1, \cdots, S$,

$$\frac{\partial H_t}{\partial \xi_{t-s}} = 0 \text{ given } K_{t-S}, A_{t-S}$$

The results are not very interesting in terms of economics.

- The impulse response functions are almost the same as in the case I, except for wage (hence, the figure is omitted).
- $Corr\,(Y_t - H_t, Y_t)$ is lower than under perfect information, but only slightly.

However, the important message in this experiment lies in computation: Simply specifying the endogenous variables that are determined without observing perfect information is not enough to find a solution. This is evident in that the results of Case I and Case II are not the same.

5.1.3 Case III: HH decides wage before observing innovations but accommodates labour demand

This case can be regarded as a version of the sticky wage model. The representative household fixes wages before observing innovations, and commits itself to supplying labour to accommodate labour demand.

In this case (22a) does not hold. Instead, the labour supply decision is governed by

$$0 = E\left[\; bH_t - W_t - \lambda_t \mid \{K_{t-j}, A_{t-j}, \xi_{t-j}\}_{j=S+1}^{\infty}\right]$$

Since W_t cannot react to the innovations, for $s = 0, 1, \cdots, S$,

$$\frac{\partial W_t}{\partial \xi_{t-s}} = 0 \text{ given } K_{t-S}, A_{t-S}$$

The results are very interesting.

Cooley and Prescott (1995)

	Output	Hours	Consumption	Investment	Corr(Output,Outpu/Hours)
		Data			
s.d	1.72	1.59	0.86	8.24	0.41
relative	1.00	0.92	0.50	4.79	
		Standard RBC			
s.d	1.35	0.47	0.33	5.95	0.98
relative	1.00	0.35	0.24	4.41	

Imperfect information (RBC with Prefixed Wage)

	Output	Hours	Consumption	Investment	Corr(Output,Outpu/Hours)
s.d	2.15	2.10	0.53	7.92	0.25
relative	1.00	0.98	0.25	3.69	

Figure 2: Comparison among data, standard RBC and RBC with sticky wages.

- The volatility of labour is much higher.
- $Corr\left(Y_t - H_t, Y_t\right)$ is much lower than under perfect information.
- Given standard deviation of the innovation, both output and labour are more volatile.
- The behaviours of most variables other than labour and labour productivity do not change significantly.

The intuition behind these results is quite simple. Without imperfect information, when there is a positive productivity innovation, wage increases, which discourages firms to hire more labour. As a result, labour does not increase very much. Indeed, another failure of the standard RBC model is that its prediction of labour volatility relative to output volatility is too small. During a boom both Y_t and H_t increase, while $Y_t - H_t$ increases because the increase in H_t is not large enough. Consequently, both Y_t and $Y_t - H_t$ increase in a boom, which is the (one possible) mechanism behind a high $Corr\left(Y_t - H_t, Y_t\right)$ in the standard RBC model.

However, if wage is determined without seeing a positive innovation, it does not change quickly. Hence, firms are not discouraged from using more labour. The effect of imperfect information is larger on labour than on output. Thus, in a boom both Y_t and H_t increase, while $Y_t - H_t$ does not increase very much because the increase in H_t is large enough. Indeed, the standard RBC model with one-period wage stickiness predicts labour volatility relative to output volatility that almost matches the data. Consequently, in a boom Y_t increases but $Y_t - H_t$ does not, which leads to low $Corr\left(Y_t - H_t, Y_t\right)$. Indeed, in our parameter set, $Corr\left(Y_t - H_t, Y_t\right)$ is negative if S is larger than 2.

Figure (2) shows the summary table of the selected second moments for one-period wage stickiness ($S = 1$). One-period wage stickiness improves the labour volatility and correlation between labour productivity and output, while it slightly deteriorates the model performance in terms of the relative volatility of investment.

Figure (3) shows the comparison of selected impulse response functions between perfect and imperfect information models. The salient differences appear only in the first period. In the sticky wage model, both labour and output jump in the first period, and the size of the jumps

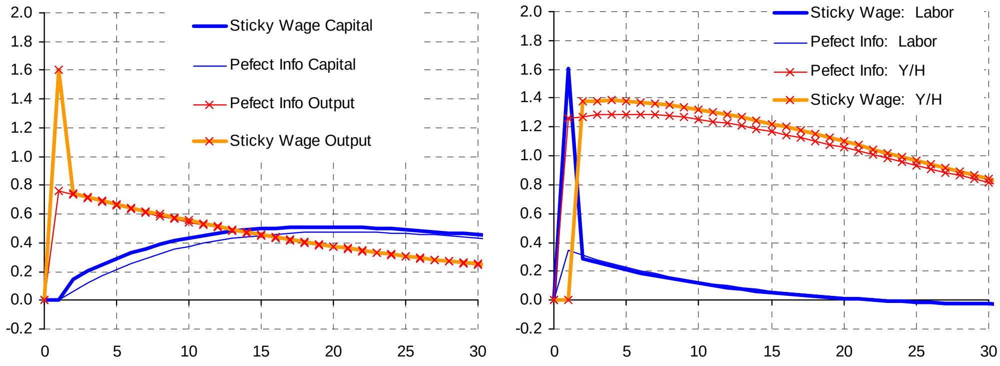

Figure 3: Comparison of selected impulse response functions to a positive technology innovation between standard RBC and RBC with wage stickiness.

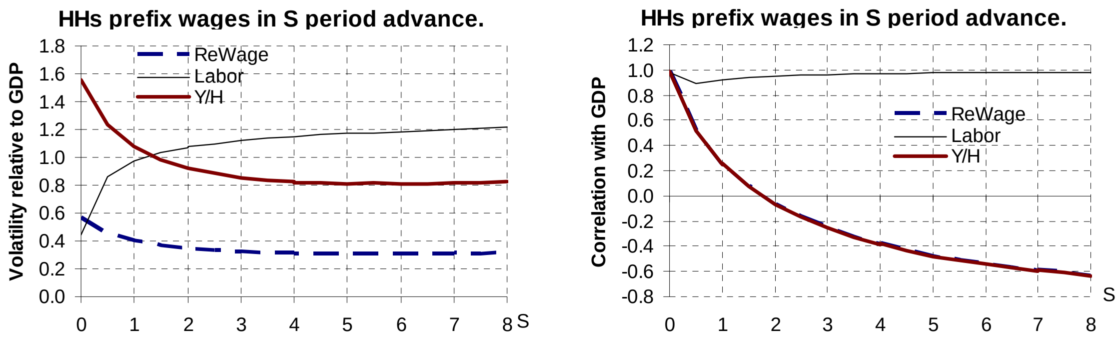

Figure 4: Effect of different degrees of imperfect information on selected second moments.

are the same. Hence, the labour productivity does not change in the first period. Note that (22b) shows that the labour productivity is always equal to wage.

Figure (4) shows the relative volatilities and correlations for different degrees of imperfect information (i.e., for different values of S). As S increases, $Corr\,(Y_t - H_t, H_t)$ decreases.

Case III also reveals one computational requirement: Simply specifying the information set in each equation is not enough to find a solution. A researcher also has to specify which variables are determined without observing perfect information. This is evident in that the results of Case I and III are not the same.

5.1.4 Conclusion for RBC with Sticky Wage

Adding one-period wage stickiness is quantitatively enough to overcome the two drawbacks of the standard RBC model, (a) labour volatility is too small and (b) the correlation between labour productivity and output is too high, without significantly deteriorating other dimensions of the model performance. This example shows the possibility that the information structure of a model has significant quantitative effects.

6 Conclusion

This paper has developed an algorithm for linear rational models under imperfect information. Imperfect information is important because it includes many interesting classes of models such as sticky information and noisy signal models.

The algorithm exploits two observations: (1) if an endogenous variable $y_{k,t}$ is decided without observing an innovation $\xi_{i,t-s}$, then $y_{k,t}$ is not affected by $\xi_{i,t-s}$ (i.e., $\partial y_{k,t}/\partial \xi_{i,t-s} = 0$); (2) if the information set in the j-th equation includes $\xi_{i,t-s}$, then $\xi_{i,t-s}$ cannot be the source of expectation error in the j-th equation ($E_{s,ji} = 0$). The solution is defined by these two zero restrictions, and it turns out that they are enough to determine solutions.

The state space representation chosen in this algorithm is the set of crawling variables at the beginning of the current period and current and past innovations. This representation reveals that the dynamic parts of the solution (H and F matrices) are the same as under the corresponding perfect information models. Invariant H and F imply that (a) the dynamic property, such as sunspot or saddle-path stability is not altered by information structure, and (b) impulse response functions are not (directly) affected by the information structure after the first S periods, where S is such that the minimum information set in a model has all the information up to time S. These findings show that *qualitatively* imperfect information models inherit properties of their perfect information counterparts.

However, as the RBC example demonstrates, *quantitatively* imperfect information may be important. Hence, it is desirable to check robustness in terms of the information structure, and our MATLAB algorithm offers an easy way to conduct such experiments. Once structural equations are obtained, then the additional inputs to the algorithm are only two zero restrictions.

Appendix

A Generalized Eigenvalues and Linear Difference Equations

The generalized eigenvalues λ_i and eigenvectors v_i satisfy

$$\lambda_i A v_i = B v_i \tag{23}$$

Suppose we have homogenous (i.e., no exogenous shocks) linear difference equations such as

$$A x_{t+1} = B x_t \tag{24}$$

One possible solution is[10]

$$x_t = v_i \lambda_i^t$$

Note that for a stable solution $|\lambda_i| < 1$. Also, note that $x_{t+1} = v_i \lambda_i^{t+1} = \lambda_i v_i \lambda^t = \lambda_i x_t$.[11] Hence, substituting this solution back into (24), we can confirm that our solution is

$$\lambda_i A v_i = B v_i$$

Indeed, the primary motivation for studies on linear matrix pencils (23) is the dynamic system such as (24). This is in parallel with the relationship between eigenvalues and difference equations such as $x_{t+1} = B x_t$. See the Wilkinson's (1979) Introduction for linear differential equations.

B Extension of Uhlig's Theorem 3

Proposition 1 *(Extension of Uhlig's Theorem 3) To find a $m \times m$ matrix X that solves the matrix polynomial*

$$\Theta_n X^n - \Theta_{n-1} X^{n-1} - \cdots - \Theta_1 X - \Theta_0 = 0$$

Given $m \times m$ coefficient matrices $\{\Theta_{n'}\}_{n'=0}^{n}$, define the $nm \times nm$ matrices Ξ and Δ by

$$\Xi = \begin{bmatrix} \Theta_{n-1} & \cdots & \Theta_1 & \Theta_0 \\ I & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & I & 0 \end{bmatrix}$$

$$\Delta = \begin{bmatrix} \Theta_n & 0 & \cdots & 0 \\ 0 & I & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & I \end{bmatrix}$$

[10] In general, a solution is a linear combination of $v_j \lambda_j^t$, and such a linear combination is specified by the initial condition.

[11] This implies that the generalized eigenvalue λ_i has an effect analogous to the lead operator.

and obtain the generalized eigenvalues λ and the generalized eigenvector s such that $\lambda\Delta s = \Xi s$. Then, s can be written as

$$s = \begin{pmatrix} \lambda^{n-1}x \\ \vdots \\ \lambda x \\ x \end{pmatrix}$$

for some $x \in \mathbb{R}^m$, and

$$X = \Omega\Lambda\Omega^{-1}$$

where $\Omega = [x_1, \cdots, x_m]$ and $\Lambda = diag(\lambda_1, \cdots, \lambda_m)$.

Proof. Almost the same as Uhlig (1999). ■

C Matrix Operations

To pick up and drop out columns and rows from a matrix, we define

- $[A]_{.x}$ as columns x of a matrix A.
- $[A]_{x.}$ as rows x of a matrix A.
- $[A]_{.\neg x}$ as the columns remaining after the elimination of columns x.
- $[A]_{\neg x.}$ as the rows remaining after the elimination of rows x.

where x is the name of a set of columns or rows. The brackets are used simply because they often clarify the notation, and often can be omitted, i.e., $[B]_{.\neg y} = B_{.\neg y}$. The dot "." implies all rows or columns, e.g., $B_{..} = B$. It is quite easy to show the following formulae.

$$\begin{aligned}
[AB] &= [A]_{.\neg x}[B]_{\neg x.} + [A]_{.x}[B]_{x.} \\
[AB]_{.\neg y} &= [A][B]_{.\neg y} \\
[AB]_{\neg x.} &= [A]_{\neg x.}[B] \\
[AB]_{\neg x \neg y} &= [A]_{\neg x.}[B]_{.\neg y}
\end{aligned}$$

An example for the first formula is

$$\begin{aligned}
\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} &= \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \end{bmatrix} + \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}\begin{bmatrix} b_{21} & b_{22} \end{bmatrix} \\
&= \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} \\ a_{21}b_{11} & a_{21}b_{12} \end{bmatrix} + \begin{bmatrix} a_{12}b_{21} & a_{12}b_{22} \\ a_{22}b_{21} & a_{22}b_{22} \end{bmatrix} \\
&= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}
\end{aligned}$$

where $x = 2$.

Note that this notation is consistent with other matrix subscripts; for example, the rows of $Z_{s\kappa}$ is are related to stable roots s and its columns are to crawling variables κ.

D Invertible $Z_{u\phi}^H$ Implies Invertible $Z_{s\kappa}^H$

Proposition 2 *For an invertible matrix Z, which is partitioned as*

$$Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}$$

if Z_{11} is invertible, then $[Z^{-1}]_{22}$ is also invertible.

Proof. Define

$$\begin{aligned} Z_L &\equiv \begin{bmatrix} I & 0 \\ -Z_{21}Z_{11}^{-1} & I \end{bmatrix} \\ Z_R &\equiv \begin{bmatrix} I & -Z_{11}^{-1}Z_{12} \\ 0 & I \end{bmatrix} \end{aligned}$$

Note that $Z_L Z Z_R$ has full rank because all of Z_L, Z and Z_R have full rank, and note that

$$\begin{bmatrix} I & 0 \\ -Z_{21}Z_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I & -Z_{11}^{-1}Z_{12} \\ 0 & I \end{bmatrix} = \begin{bmatrix} Z_{11} & 0 \\ 0 & Z_{22} - Z_{21}Z_{11}^{-1}Z_{12} \end{bmatrix}$$

Hence, $G \equiv Z_{22} - Z_{21}Z_{11}^{-1}Z_{12}$ must have full rank.
For a full rank matrix with an invertible upper left submatrix, the well-known formula tells us

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}^{-1} = \begin{bmatrix} Z_{11}^{-1} + Z_{11}^{-1}Z_{12}G^{-1}Z_{21}Z_{11}^{-1} & -Z_{11}^{-1}Z_{12}G^{-1} \\ -G^{-1}Z_{21}Z_{11}^{-1} & G^{-1} \end{bmatrix}$$

Note that the RHS exists since we know that both Z_{11} and G are invertible. Thus, $[Z^{-1}]_{22}$ is invertible. ■

Since Z is unitary, $Z^{-1} = Z^H$, which implies $G^{-1} = [Z^{-1}]_{22} = Z_{22}^H$. Since Z_{22}^H has full rank, its conjugate transpose $Z_{22} \left(= \left[Z_{22}^H \right]^H \right)$ also has full rank.

E Invertibility of Block Triangular Matrices

Due to the following introductory result, we know that Ω_{ss}^A, Ω_{uu}^B, Φ, $M_{y\Gamma}$ and $M_{\Gamma\Pi}$ are all invertible.

Consider a block triangular Δ which has invertible block diagonal submatrices Δ_{dd}

$$\Delta = \begin{bmatrix} \Delta_{11} & & & & \\ & \ddots & & & \\ & & \Delta_{dd} & & \\ & & & \ddots & \\ & & & & \Delta_{DD} \end{bmatrix}$$

Δ is either an upper or lower block diagonal. Then, Δ is invertible.

To show this, focus on the upper left part first

$$\det \begin{bmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{bmatrix} = \left(\det \Delta_{11} \right) \left(\det \left(\Delta_{22} - \Delta_{21} \left(\Delta_{11} \backslash \Delta_{12} \right) \right) \right)$$
$$= \left(\det \Delta_{11} \right) \left(\det \Delta_{22} \right) \neq 0$$

Note that $\Delta_{21} \left(\Delta_{11} \backslash \Delta_{12} \right) = 0$ since either Δ_{21} or Δ_{12} is zero. We can repeat this process until it shows $\det \Delta \neq 0$.

F Rank Deficient H Matrix and Expansion of Innovation Vector

The representation of a solution under imperfect information is not necessarily unique. This section shows the equivalence of two representations.

Consider the RBC model, in which labour supply is decided without observing today's innovation. The vector of crawling variables is

$$\kappa_t = \begin{bmatrix} K_t \\ H_t^S \end{bmatrix}$$

where K_t and H_t^S are capital and labour supply at time t, respectively. Then, the solution has an H matrix that is rank deficient.

We can decompose such an H matrix by using eigenvalue-eigenvector decomposition

$$V \kappa_{t+1} = \lambda V \kappa_t + V J \xi_t$$
$$\lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & 0 \end{bmatrix}$$
$$V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}$$

From the first row

$$V_{11} K_{t+1} + V_{12} H_{t+1}^S = \lambda_1 \left(V_{11} K_t + V_{12} H_t^S \right) + \left(V_{11} J_1 + V_{12} J_2 \right) \xi_t$$

where J is 2x1. From the second row

$$V_{21} K_{t+1} + H_{t+1}^S = \left(V_{21} J_1 + V_{22} J_2 \right) \xi_t$$
$$H_{t+1}^S = \left(V_{22} \backslash V_{21} J_1 + J_2 \right) \xi_t - V_{22} \backslash V_{21} K_{t+1}$$

Under our assumption 3 (invertible $[M_{\Pi E}]_{kj}$), V_{11} and V_{22} are non-zero. Hence,

$$K_{t+1} = \lambda_1 K_t + J_1 \xi_t + \lambda_1 \left(\tfrac{V_{12}(V_{22} \backslash V_{21}) J_1 + V_{12} J_2}{V_{11} - V_{12}(V_{22} \backslash V_{21})} \right) \xi_{t-1}$$

Thus, it is shown that with a rank deficient H matrix we can reduce the set of crawling variables by increasing the number of innovations.

G Full Derivation

This section provides the full derivation. For the notation, see the main text.

G.1 QZ Decomposition

Applying the QZ decomposition to (2)

$$\begin{aligned} 0 &= \Omega_A Z^H y_{t+1} + \Omega_B Z^H y_t + Q^H C\xi_t + Q^H D\xi_{t+1} + Q^H E\xi^{t,S} \\ &= \begin{bmatrix} \Omega^A_{ss} & \Omega^A_{su} \\ 0 & \Omega^A_{uu} \end{bmatrix} \begin{pmatrix} s_{t+1} \\ u_{t+1} \end{pmatrix} + \begin{bmatrix} \Omega^B_{ss} & \Omega^B_{su} \\ 0 & \Omega^B_{uu} \end{bmatrix} \begin{pmatrix} s_t \\ u_t \end{pmatrix} \\ &\quad + \begin{bmatrix} Q^H_{s.} \\ Q^H_{u.} \end{bmatrix} C\xi_t + \begin{bmatrix} Q^H_{s.} \\ Q^H_{u.} \end{bmatrix} D\xi_{t+1} + \begin{bmatrix} Q^H_{s.} \\ Q^H_{u.} \end{bmatrix} E\xi^{t,S} \end{aligned} \tag{25}$$

where s_t and u_t are stable and unstable roots, respectively, such that

$$\begin{pmatrix} s_t \\ u_t \end{pmatrix} \equiv \begin{bmatrix} Z^H_{s\kappa} & Z^H_{s\phi} \\ Z^H_{u\kappa} & Z^H_{u\phi} \end{bmatrix} \begin{pmatrix} \kappa_t \\ \phi_t \end{pmatrix}$$

G.1.1 Unstable Roots and Transversality Conditions (TVCs)

Imperfect information requires a slightly careful treatment of TVCs. Focusing on the lower half of (25)

$$0 = \Omega^A_{uu} u_{t+1} + \Omega^B_{uu} u_t + Q^H_{u.} C\xi_t + Q^H_{u.} D\xi_{t+1} + Q^H_{u.} E\xi^{t,S} \tag{26}$$

Iterating it forward

$$\begin{aligned} &\lim_{l\to\infty} \left\{ \left(-\Omega^B_{uu} \backslash \Omega^A_{uu} \right)^l u_{t+l} + \sum_{s=1}^{l-1} \left(-\Omega^B_{uu} \backslash \Omega^A_{uu} \right)^s \left(\Omega^B_{uu} \backslash Q^H_{u.} \right) \left(C\xi_{t+s} + D\xi_{t+1+s} + E\tilde{\xi}^{t+s,S} \right) \right\} \\ =\ & -u_t - \left(\Omega^B_{uu} \backslash Q^H_{u.} \right) C\xi_t - \sum_{l=0}^{S} \left(-\Omega^B_{uu} \backslash \Omega^A_{uu} \right)^l \left(\Omega^B_{uu} \backslash Q^H_{u.} \right) E\hat{\xi}^{t+l,S} \end{aligned} \tag{27}$$

where

$$\xi^{t+l,S} = \begin{pmatrix} \xi_{t+l} \\ \vdots \\ \xi_{t+1} \\ \xi_t \\ \vdots \\ \xi_{t+l-S} \end{pmatrix} = \hat{\xi}^{t+l,S} + \tilde{\xi}^{t+l,S} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \xi_t \\ \vdots \\ \xi_{t+l-S} \end{pmatrix} + \begin{pmatrix} \xi_{t+l} \\ \vdots \\ \xi_{t+1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

where $A\backslash B = A^{-1}B$, and $A/B = AB^{-1}$. There are many information sets, under each of which TVCs must be satisfied. That is, TVCs are (seemingly) tighter under imperfect information. However, if the full information counterpart satisfies TVCs, corresponding imperfect information models also satisfy them automatically due to the law of iterated linear projection.[12]

[12] There are two comments. First, (27) must hold for *any* realization of κ_{t-1} and ξ_{t-s} for $s = 0, 1, \cdots$. Hence, it is *not* possible that TVCs hold under imperfect information but not under perfect information. Second, if an information set does not include, for example, $\xi_{i,t-s}$ then relevant expected value of u_{t+s} is the RHS with setting $\xi_{i,t-s} = 0$. Hence, if TVCs hold for the full information set, they hold for non-full information sets as well.

Thus, consider the perfect information case. Because $\left(-\Omega^B_{uu}\backslash\Omega^A_{uu}\right)^l \to 0$ as $l \to 0$ by construction, the expected value of u_{t+l} explodes for any non-zero value of the RHS of (27), which contradicts the TVCs. Note that inside the limit operator in the LHS shows the expected value of u_{t+l} (the realization of u_{t+l} plus expectation errors) times $\left(-\Omega^B_{uu}\backslash\Omega^A_{uu}\right)^l$. Hence, the RHS of (27) must be zero.

Therefore,

$$\begin{aligned} -\Omega^B_{uu}u_t &= -\Omega^B_{uu}Z^H_{u\kappa}\kappa_t - \Omega^B_{uu}Z^H_{u\phi}\phi_t \\ &= Q^H_{u.}C\xi_t + \Omega^B_{uu}\sum_{l=0}^{S}\left(-\Omega^B_{uu}\backslash\Omega^A_{uu} \right)^l \left(\Omega^B_{uu}\backslash Q^H_{u.} \right) E\hat{\xi}^{t+l,S} \\ &= Q^H_{u.}C\xi_t + \sum_{l=0}^{S}\left(-\Omega^A_{uu}/\Omega^B_{uu} \right)^l Q^H_{u.}E\hat{\xi}^{t+l,S} \end{aligned} \tag{28}$$

Substituting our "guess solution" (3) into (28),

$$\begin{aligned} 0 &= \left(\Omega^B_{uu}Z^H_{u\kappa} + \Omega^B_{uu}Z^H_{u\phi}F \right)\kappa_t + \Omega^B_{uu}Z^H_{u\phi}G\xi^{t,S} + Q^H_{u.}C\xi_t \\ &\quad + \sum_{l=0}^{S}\left(-\Omega^A_{uu}/\Omega^B_{uu} \right)^l Q^H_{u.}E\hat{\xi}^{t+l,S} \end{aligned} \tag{29}$$

G.1.2 Stable Roots

Similarly, from the upper half,

$$\begin{aligned} 0 &= \Omega^A_{ss}\left(Z^H_{s\kappa}\kappa_{t+1} + Z^H_{s\phi}\phi_{t+1}\right) + \Omega^A_{su}\left(Z^H_{u\kappa}\kappa_{t+1} + Z^H_{u\phi}\phi_{t+1}\right) \\ &\quad + \Omega^B_{ss}\left(Z^H_{s\kappa}\kappa_t + Z^H_{s\phi}\phi_t\right) + \Omega^B_{su}\left(Z^H_{u\kappa}\kappa_t + Z^H_{u\phi}\phi_t\right) \\ &\quad + Q^H_{s.}C\xi_t + Q^H_{s.}D\xi_{t+1} + Q^H_{s.}E\xi^{t,S} \end{aligned} \tag{30}$$

Again substituting (3) into (30), after some manipulation

$$\begin{aligned} 0 &= \left(\Lambda^A_{s\phi}FH + \Lambda^A_{s\kappa}H + \Lambda^B_{s\phi}F + \Lambda^B_{s\kappa} \right)\kappa_t \\ &\quad + \Lambda^A_{s\phi}G\xi^{t+1,S} + Q^H_{s.}D\xi_{t+1} + Q^H_{s.}C\xi_t \\ &\quad + \left(\Lambda^A_{s\phi}FJ + \Lambda^A_{s\kappa}J + \Lambda^B_{s\phi}G + Q^H_{s.}E \right)\xi^{t,S} \end{aligned} \tag{31}$$

Though the definitions of $\Lambda^A_{s\kappa}$, $\Lambda^A_{s\phi}$, $\Lambda^B_{s\kappa}$ and $\Lambda^B_{s\phi}$ are (10) in the main text, the following result may be more useful.

$$\begin{bmatrix} \Lambda^A_{s\kappa} & \Lambda^A_{s\phi} \\ \Lambda^B_{s\kappa} & \Lambda^B_{s\phi} \end{bmatrix} = \begin{bmatrix} \Omega^A_{ss} & \Omega^A_{su} \\ \Omega^B_{ss} & \Omega^B_{su} \end{bmatrix} \begin{bmatrix} Z^H_{s\kappa} & Z^H_{s\phi} \\ Z^H_{u\kappa} & Z^H_{u\phi} \end{bmatrix} \tag{32}$$

G.2 Expansion of $\xi^{t+1,S}$ and $\xi^{t,S}$

Expanding $\xi^{t+1,S}$ and $\xi^{t,S}$ in (31) and (29),

$$
\begin{aligned}
0 \;=\; & \left(\ \Lambda^A_{s\phi} F H + \Lambda^A_{s\kappa} H + \Lambda^B_{s\phi} F + \Lambda^B_{s\kappa}\ \right) \kappa_t \\
& + \left(\ \Lambda^A_{s\phi} G_0 + Q^H_{s.} D\ \right) \xi_{t+1} \\
& + \left(\ \Lambda^A_{s\phi} G_1 + \left(\Omega^A_{ss}/Z_{\kappa s}\right) J_0 + \Lambda^B_{s\phi} G_{0.} + Q^H_{s.} E_{0.} + Q^H_{s.} C\ \right) \xi_t \\
& + \left(\ \Lambda^A_{s\phi} G_2 + \left(\Omega^A_{ss}/Z_{\kappa s}\right) J_1 + \Lambda^B_{s\phi} G_{1.} + Q^H_{s.} E_{1.}\ \right) \xi_{t-1} + \cdots \\
& + \left(\ \Lambda^A_{s\phi} G_S + \left(\Omega^A_{ss}/Z_{\kappa s}\right) J_{S-1} + \Lambda^B_{s\phi} G_{S-1.} + Q^H_{s.} E_{S-1.}\ \right) \xi_{t-(S-1)} \\
& + \left(\ \left(\Omega^A_{ss}/Z_{\kappa s}\right) J_S + \Lambda^B_{s\phi} G_{S.} + Q^H_{s.} E_{S.}\ \right) \xi_{t-S}
\end{aligned}
$$

$$
\begin{aligned}
0 \;=\; & \left(\ \Omega^B_{uu} Z^H_{u\kappa} + \Omega^B_{uu} Z^H_{u\phi} F\ \right) \kappa_t \\
& + \sum_{s=1}^{S} \left(\ \Omega^B_{uu} Z^H_{u\phi} G_s + \left(\textstyle\sum_{k=0}^{S-s} \left(\ -\Omega^A_{uu}/\Omega^B_{uu}\ \right)^k Q^H_{u.} E_{k+s}\right)\ \right) \xi_{t-s} \\
& + \left(\ Q^H_{u.} C + \Omega^B_{uu} Z^H_{u\phi} G_0 + \left(\textstyle\sum_{k=0}^{S} \left(\ -\Omega^A_{uu}/\Omega^B_{uu}\ \right)^k Q^H_{u.} E_k\right)\ \right) \xi_t
\end{aligned}
$$

Since these matrix equations must hold for any realization of κ_t, $\xi_{t-\tau}$ for $\tau = -1, 0, 1, \cdots, S$,

$$
\begin{aligned}
0 &= \Lambda^A_{s\phi} F H + \Lambda^A_{s\kappa} H + \Lambda^B_{s\phi} F + \Lambda^B_{s\kappa} && \text{(33a)} \\
0 &= \Omega^B_{uu} Z^H_{u\kappa} + \Omega^B_{uu} Z^H_{u\phi} F && \text{(33b)}
\end{aligned}
$$

$$
\begin{aligned}
0 &= \Lambda^A_{s\phi} G_{0.} + Q^H_{s.} D && \text{(34a)} \\
0 &= 0 && \text{(34b)}
\end{aligned}
$$

$$
\begin{aligned}
0 &= \Lambda^A_{s\phi} G_1 + \left(\Omega^A_{ss}/Z_{\kappa s}\right) J_0 + \Lambda^B_{s\phi} G_0 + Q^H_{s.} E_{S.} + Q^H_{s.} C && \text{(35a)} \\
0 &= \Omega^B_{uu} Z^H_{u\phi} G_0 + \left(\sum_{s=0}^{S} \left(\ -\Omega^A_{uu}/\Omega^B_{uu}\ \right)^s Q^H_{u.} E_s\right) + Q^H_{u.} C && \text{(35b)}
\end{aligned}
$$

$$
\begin{aligned}
0 &= \Lambda^A_{s\phi} G_{s+1} + \left(\Omega^A_{ss}/Z_{\kappa s}\right) J_s + \Lambda^B_{s\phi} G_s + Q^H_{s.} E_s \text{ for } s = 1, \cdots, S-1 && \text{(36a)} \\
0 &= \Omega^B_{uu} Z^H_{u\phi} G_s + \left(\sum_{k=0}^{S-s} \left(\ -\Omega^A_{uu}/\Omega^B_{uu}\ \right)^k Q^H_{u.} E_{k+s}\right) \text{ for } s = 1, \cdots, S-1 && \text{(36b)}
\end{aligned}
$$

$$
\begin{aligned}
0 &= \left(\Omega^A_{ss}/Z_{\kappa s}\right) J_S + \Lambda^B_{s\phi} G_S + Q^H_{s.} E_S && \text{(37a)} \\
0 &= \Omega^B_{uu} Z^H_{u\phi} G_S + Q^H_{u.} E_S && \text{(37b)}
\end{aligned}
$$

G.3 Dynamic Parts (H and F)

Since (33a) and (33b) do not include G, J, D, E or Π, these two matrix equations can be solved for H and F independently. Thus, assuming $Z^H_{u\phi}$ has a (right) inverse,[13]

$$\begin{aligned} F &= -Z^H_{u\phi} \backslash Z^H_{u\kappa} = Z_{\phi s} / Z_{\kappa s} \\ H &= -Z_{\kappa s} \left(\Omega^A_{ss} \backslash \Omega^B_{ss} \right) / Z_{\kappa s} \end{aligned}$$

Note that the H and F matrices are the same as in the corresponding perfect information model.[14]

G.4 Zero Restrictions on E and Π

Vertically concatenating matrix equations (35a)-(37b) in pairs,

$$0 = \begin{bmatrix} 0 & \Lambda^A_{s\phi} \\ 0 & 0 \end{bmatrix} \Gamma_1 + \begin{bmatrix} \Omega^A_{ss}/Z_{\kappa s} & \Lambda^B_{s\phi} \\ 0 & \Omega^B_{uu} Z^H_{u\phi} \end{bmatrix} \Gamma_0 + \sum_{k=0}^{S} \begin{bmatrix} 0 & 0 \\ 0 & -\Omega^A_{uu}/\Omega^B_{uu} \end{bmatrix}^k Q^H E_k + Q^H C \tag{38a}$$

$$0 = \begin{bmatrix} 0 & \Lambda^A_{s\phi} \\ 0 & 0 \end{bmatrix} \Gamma_{s+1} + \begin{bmatrix} \Omega^A_{ss}/Z_{\kappa s} & \Lambda^B_{s\phi} \\ 0 & \Omega^B_{uu} Z^H_{u\phi} \end{bmatrix} \Gamma_s + \sum_{k=0}^{S-s} \begin{bmatrix} 0 & 0 \\ 0 & -\Omega^A_{uu}/\Omega^B_{uu} \end{bmatrix}^k Q^H E_{k+s} \tag{38b}$$

for $s = 1, \cdots, S-1$

$$0 = \begin{bmatrix} \Omega^A_{ss}/Z_{\kappa s} & \Lambda^B_{s\phi} \\ 0 & \Omega^B_{uu} Z^H_{u\phi} \end{bmatrix} \Gamma_S + Q^H E_S \tag{38c}$$

Note that

$$\begin{aligned} 0 &= \begin{bmatrix} 0 & 0 \\ 0 & -\Omega^A_{uu}/\Omega^B_{uu} \end{bmatrix} \left(\begin{array}{c} \begin{bmatrix} 0 & \Lambda^A_{s\phi} \\ 0 & 0 \end{bmatrix} \Gamma_{s+2} + \begin{bmatrix} \Omega^A_{ss}/Z_{\kappa s} & \Lambda^B_{s\phi} \\ 0 & \Omega^B_{uu} Z^H_{u\phi} \end{bmatrix} \Gamma_{s+1} \\ + \sum_{k=0}^{S-(s+1)} \begin{bmatrix} 0 & 0 \\ 0 & -\Omega^A_{uu}/\Omega^B_{uu} \end{bmatrix}^k Q^H E_{k+s+1} \end{array} \right) \\ &= \begin{bmatrix} 0 & 0 \\ 0 & -\Omega^A_{uu} Z^H_{u\phi} \end{bmatrix} \Gamma_{s+1} + \sum_{k=1}^{S-s} \begin{bmatrix} 0 & 0 \\ 0 & -\Omega^A_{uu}/\Omega^B_{uu} \end{bmatrix}^k Q^H E_{k+s} \end{aligned} \tag{39}$$

[13] Remember that an invertible $Z^H_{u\phi}$ implies an invertible $Z^H_{s\kappa}$.

[14] For the F matrix, note

$$Z^H Z = \begin{bmatrix} Z^H_{s\kappa} & Z^H_{s\phi} \\ Z^H_{u\kappa} & Z^H_{u\phi} \end{bmatrix} \begin{bmatrix} Z_{\kappa s} & Z_{\kappa u} \\ Z_{\phi s} & Z_{\phi u} \end{bmatrix} = \begin{bmatrix} Z^H_{s\kappa} Z_{\kappa s} + Z^H_{s\phi} Z_{\phi s} & Z^H_{s\kappa} Z_{\kappa u} + Z^H_{s\phi} Z_{\phi u} \\ Z^H_{u\kappa} Z_{\kappa s} + Z^H_{u\phi} Z_{\phi s} & Z^H_{u\kappa} Z_{\kappa u} + Z^H_{u\phi} Z_{\phi u} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

Looking at the lower left element

$$\begin{aligned} Z^H_{u\kappa} Z_{\kappa s} + Z^H_{u\phi} Z_{\phi s} &= 0 \\ -Z^H_{u\kappa} Z_{\kappa s} &= Z^H_{u\phi} Z_{\phi s} \\ -Z^H_{u\phi} \backslash Z^H_{u\kappa} &= Z_{\phi s} / Z_{\kappa s} \end{aligned}$$

Also, remember that

$$Z^{-1}_{\kappa s} = Z^H_{s\kappa} - Z^H_{s\phi} \left(Z^H_{u\phi} \backslash Z^H_{u\kappa} \right)$$

and that Ω^A_{ss} is invertible by the reordering of QZ decomposition.

Subtracting (39) from each of (38),[15]

$$
\begin{aligned}
0 &= \begin{bmatrix} 0 & \Lambda^A_{s\phi} \\ 0 & \Omega^A_{uu} Z^H_{u\phi} \end{bmatrix} \Gamma_1 + \begin{bmatrix} \Omega^A_{ss}/Z_{\kappa s} & \Lambda^B_{s\phi} \\ 0 & \Omega^B_{uu} Z^H_{u\phi} \end{bmatrix} \Gamma_0 + Q^H E_k + Q^H C && \text{(40a)} \\
0 &= \begin{bmatrix} 0 & \Lambda^A_{s\phi} \\ 0 & \Omega^A_{uu} Z^H_{u\phi} \end{bmatrix} \Gamma_{s+1} + \begin{bmatrix} \Omega^A_{ss}/Z_{\kappa s} & \Lambda^B_{s\phi} \\ 0 & \Omega^B_{uu} Z^H_{u\phi} \end{bmatrix} \Gamma_s + Q^H E_{k+s} && \text{(40b)} \\
& \quad \text{for } s = 1, \cdots, S-1 \\
0 &= \begin{bmatrix} \Omega^A_{ss}/Z_{\kappa s} & \Lambda^B_{s\phi} \\ 0 & \Omega^B_{uu} Z^H_{u\phi} \end{bmatrix} \Gamma_S + Q^H E_S && \text{(40c)}
\end{aligned}
$$

and again vertically concatenating these equations,

$$
\begin{aligned}
0 &= M_{y\Gamma}\Gamma + \mathbf{Q}\left(E + \mathbf{C}\right) \\
\Gamma &\equiv \begin{pmatrix} \Gamma_0 \\ \vdots \\ \Gamma_S \end{pmatrix}, E \equiv \begin{pmatrix} E_0 \\ \vdots \\ E_S \end{pmatrix}, \mathbf{C} \equiv \begin{pmatrix} C_0 \\ 0 \end{pmatrix}, \mathbf{Q} \equiv \begin{bmatrix} Q & & 0 \\ & \ddots & \\ 0 & & Q \end{bmatrix} \\
M_{y\Gamma} &\equiv \begin{bmatrix} \Phi & \Lambda^{0A} & & & & \\ & \Phi & \Lambda^{0A} & & 0 & \\ & & \Phi & \Lambda^{0A} & & \\ & & & \ddots & \ddots & \\ & 0 & & & \Phi & \Lambda^{0A} \\ & & & & & \Phi \end{bmatrix} \\
\Phi &\equiv \begin{bmatrix} \Omega^A_{ss}/Z_{\kappa s} & \Lambda^B_{s\phi} \\ 0 & \Omega^B_{uu} Z^H_{u\phi} \end{bmatrix}, \Lambda^{0A} \equiv \begin{bmatrix} 0 & \Lambda^A_{s\phi} \\ 0 & \Omega^A_{uu} Z^H_{u\phi} \end{bmatrix}
\end{aligned}
$$

Note that since Φ is invertible, $M_{y\Gamma}$ is also clearly invertible. Hence,

$$
\begin{aligned}
0 &= \Gamma + M_{y\Gamma} \backslash \mathbf{Q}\left(E + \mathbf{C}\right) \\
&= M_{\Gamma\Pi}\Pi + M_{y\Gamma} \backslash \mathbf{Q}\left(E + \mathbf{C}\right)
\end{aligned}
$$

where (7) is used to derive the second line. Hence,

$$
\begin{aligned}
0 &= \Pi + M_{\Pi E}\left(E + \mathbf{C}\right) && \text{(41a)} \\
M_{\Pi E} &\equiv \left(M_{y\Gamma} M_{\Gamma\Pi}\right) \backslash \mathbf{Q} && \text{(41b)}
\end{aligned}
$$

In the following, we compute E and Π *column by column.*

$$
\Pi_{.i} = M_{\Pi E}\left(E_{.i} + \mathbf{C}_{.i}\right)
$$

Remember that some elements in $\Pi_{.i}$ are zero due to imperfect information, while some elements in $E_{.i}$ are non-zero. For example,

[15] Though this process is not necessary, it reduces the computational burden.

$$0=\begin{pmatrix}\Pi_{1,i}\\ \vdots\\ \Pi_{k,i}\,(=0)\\ \vdots\\ \Pi_{M(S+1),i}\end{pmatrix}+M_{\Pi E}\left(\begin{pmatrix}0\\ \vdots\\ E_{ji}\\ \vdots\\ 0\end{pmatrix}+\begin{pmatrix}C_{.i}\\ 0\\ \vdots\\ \vdots\\ 0\end{pmatrix}\right)\tag{42}$$

G.4.1 E matrix

From the k-th set of equations in (42)

$$0=\left[\ M_{\Pi E}\ \right]_{kj}E_{ji}+\left[\ M_{\Pi E}\ \right]_{kj}\mathbf{C}_{ji}+\left[\ M_{\Pi E}\ \right]_{k\neg j}\mathbf{C}_{\neg ji}$$

Hence, assuming $\left[\ M_{\Pi E}\ \right]_{kj}$ is invertible,

$$E_{ji}=-\left[\ M_{\Pi E}\ \right]_{kj}\backslash\left[\ M_{\Pi E}\ \right]_{k\neg j}\mathbf{C}_{\neg ji}-\mathbf{C}_{ji}$$

G.4.2 Π matrix

From the other equations in (42), we eliminate the expectation errors E_{ji}.

$$\begin{aligned}\Pi_{\neg ki} &= \left[\ M_{\Pi E}\ \right]_{\neg kj}\Big(\left[\ M_{\Pi E}\ \right]_{kj}\backslash\left[\ M_{\Pi E}\ \right]_{k\neg j}\mathbf{C}_{\neg ji}+\mathbf{C}_{ji}\Big)\\ &\qquad -\left[\ M_{\Pi E}\ \right]_{\neg kj}\mathbf{C}_{ji}-\left[\ M_{\Pi E}\ \right]_{\neg k\neg j}\mathbf{C}_{\neg ji}\\ &= \Big(\left[\ M_{\Pi E}\ \right]_{\neg kj}\Big(\left[\ M_{\Pi E}\ \right]_{kj}\backslash\left[\ M_{\Pi E}\ \right]_{k\neg j}\Big)-\left[\ M_{\Pi E}\ \right]_{\neg k\neg j}\Big)\mathbf{C}_{\neg ji}\\ &= -\left[\ M_{\Pi E}^{-1}\ \right]_{\neg j\neg k}\backslash\mathbf{C}_{\neg ji}\end{aligned}$$

The vector $\Pi_{\neg ki}$ and $\Pi_{ki}=0$ can be vertically merged to recover $\Pi_{.i}$, and the vector $\Pi_{.i}$ are horizontally concatenated to recover full Π matrix. Note that invertible $\left[\ M_{\Pi E}\ \right]_{kj}$ implies invertible $\left[\ M_{\Pi E}^{-1}\ \right]_{\neg j\neg k}$. Not surprisingly, $\mathbf{C}_{ji}$ does not affect the coefficient matrix $\Pi_{.i}$, because the j-th set of equations does not hold for the i-th innovation anyway. It only affects the expectation error E_{ji}.

G.5 Other Matrices (J, G and D)

G.5.1 J and G matrices

To obtain the J and G matrices, from (7),

$$\Gamma\equiv\begin{bmatrix}J_0\\ G_0\\ \vdots\\ J_S\\ G_S\end{bmatrix}=M_{\Gamma\Pi}\Pi$$

G.5.2 D matrix

From the A matrix in a given model (2),

$$D = -A \begin{bmatrix} 0 \\ G_0 \end{bmatrix}$$

which always satisfies (34a). It can be shown that the j-th rows in D are zeros if the j-th equation does not include $t+1$ dynamic jump variable (see the next section).

H The D Matrix

The derivation of the D matrix is a bit tricky, and requires careful attention concerning non square matrices $\Lambda^A_{s\phi}$ and $Q^H_{s.}$. We do not show the straightforward derivation of D, which is perhaps not intuitive, but instead we simply shows our solution always satisfy (34a), which reveals an important intuition.

First, we define dynamic and non-dynamic jump variables.

$$\phi_{t+1} = \begin{bmatrix} \phi^d_{t+1} \\ \phi^n_{t+1} \end{bmatrix}$$

Note that the coefficients of the non-dynamic jump variables ϕ^n_{t+1} in A matrix must be zero by the definition of "non-dynamic".

$$Ay_{t+1} \equiv \begin{bmatrix} A_{\kappa\kappa} & A_{\kappa\phi^d} & 0 \\ A_{\phi^d\kappa} & A_{\phi^d\phi^d} & 0 \\ A_{\phi^n\kappa} & A_{\phi^n\phi^d} & 0 \end{bmatrix} \begin{pmatrix} \kappa_{t+1} \\ \phi^d_{t+1} \\ \phi^n_{t+1} \end{pmatrix}$$

where ϕ^d_{t+1} is the vector of dynamic variables, such as consumption in the Euler equation. The submatrices in G_0 and Q^H are defined as

$$\tilde{G}_0 \equiv \begin{bmatrix} 0 \\ G_0 \end{bmatrix} \equiv \begin{bmatrix} 0 \\ G_{0,\phi^d.} \\ G_{0,\phi^n.} \end{bmatrix}$$

$$Q^H \equiv \begin{bmatrix} Q^H_{s.} \\ Q^H_{\phi.} \end{bmatrix}, Q^H_{s.} \equiv \begin{bmatrix} Q^H_{s\kappa} & Q^H_{s\phi^d} & Q^H_{s\phi^n} \end{bmatrix}, Q^H_{u.} \equiv \begin{bmatrix} Q^H_{u^f\kappa} & Q^H_{u^f\phi^d} & Q^H_{u^f\phi^n} \\ Q^H_{u^i\kappa} & Q^H_{u^i\phi^d} & Q^H_{u^i\phi^n} \end{bmatrix}$$

where u^f and u^i imply finite and infinite unstable roots, respectively.

Focusing on the second term of (34a)

$$\begin{aligned} Q^H_{s.}D &= Q^H_{s.}A\tilde{G}_0 = \begin{bmatrix} Q^H_{s\kappa} & Q^H_{s\phi^d} & Q^H_{s\phi^n} \end{bmatrix} \begin{bmatrix} A_{\kappa\kappa} & A_{\kappa\phi^d} & 0 \\ A_{\phi^d\kappa} & A_{\phi^d\phi^d} & 0 \\ A_{\phi^n\kappa} & A_{\phi^n\phi^d} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ G_{0,\phi^d.} \\ G_{0,\phi^n.} \end{bmatrix} \\ &= \left(Q^H_{s\kappa}A_{\kappa\phi^d} + Q^H_{s\phi^d}A_{\phi^d\phi^d} + Q^H_{s\phi^n}A_{\phi^n\phi^d} \right) G_{0,\phi^d.} \end{aligned} \quad (43)$$

For the first term of (34a) note that $\Lambda^A_{s\phi}$ is the $s\phi$-th elements in $\Omega^A Z^H$, i.e.,

$$\begin{aligned}
\Lambda^A_{s\phi} &= \left[\ \Omega^A Z^H \ \right]_{s\phi} \\
&= \left[\ QQ^H \Omega^A Z^H \ \right]_{s\phi} = \left[\ QA \ \right]_{s\phi} \\
&= \left[\begin{bmatrix} Q^H_{s\kappa} & Q^H_{s\phi^d} & Q^H_{s\phi^n} \\ Q^H_{u^f\kappa} & Q^H_{u^f\phi^d} & Q^H_{u^f\phi^n} \\ Q^H_{u^i\kappa} & Q^H_{u^i\phi^d} & Q^H_{u^i\phi^n} \end{bmatrix} \begin{bmatrix} A_{\kappa\kappa} & A_{\kappa\phi^d} & 0 \\ A_{\phi^d\kappa} & A_{\phi^d\phi^d} & 0 \\ A_{\phi^n\kappa} & A_{\phi^n\phi^d} & 0 \end{bmatrix} \right]_{s\phi} \\
&= \left[\begin{bmatrix} * & \left(Q^H_{s\kappa} A_{\kappa\phi^d} + Q^H_{s\phi^d} A_{\phi^d\phi^d} + Q^H_{s\phi^n} A_{\phi^n\phi^d} \right) & 0 \\ * & * & 0 \\ * & * & 0 \end{bmatrix} \right]_{s\phi} \\
&= \left[\ \left(Q^H_{s\kappa} A_{\kappa\phi^d} + Q^H_{s\phi^d} A_{\phi^d\phi^d} + Q^H_{s\phi^n} A_{\phi^n\phi^d} \right) \ \ 0 \ \right]
\end{aligned}$$

where $*$ elements are irrelevant for our current interest. Hence,

$$\begin{aligned}
\Lambda^A_{s\phi} G_0 &= \left[\ \left(Q^H_{s\kappa} A_{\kappa\phi^d} + Q^H_{s\phi^d} A_{\phi^d\phi^d} + Q^H_{s\phi^n} A_{\phi^n\phi^d} \right) \ \ 0 \ \right] \begin{bmatrix} G_{0,\phi^d\cdot} \\ G_{0,\phi^n\cdot} \end{bmatrix} \\
&= \left(Q^H_{s\kappa} A_{\kappa\phi^d} + Q^H_{s\phi^d} A_{\phi^d\phi^d} + Q^H_{s\phi^n} A_{\phi^n\phi^d} \right) G_{0,\phi^d\cdot} \qquad (44)
\end{aligned}$$

(43) and (44) show that (34a) holds. The key to the solution is a sort of a zero restriction: A matrix has zero columns by the definition of "non-dynamic" variables.

A further question is the consistency of D, i.e. whether the computed D always has zeros at the proper positions? Specifically, if the j-th equation does not have ϕ^d_{t+1}, it should not have an expectation error due to ξ_{t+1}, and hence the row vector $D_{j\cdot}$ must be zero. This zero restriction on D is analogous to that on E. For example, in the standard RBC model, all but the Euler equation have zero rows in D. However, the j-th row of $D = A\tilde{G}_0$ is always zero simply because, by the construction of A, the j-th row in A is zero if the j-th equation does not include dynamic jump variables ϕ^d_{t+1}.

What this section discusses is the correspondence between expectation errors and the source of such errors. If, for example, expectation errors with respect to full information up to time ι_t appears in the equations without dynamic jump variables, then it is a logical contradiction (which variable makes an expectational mistake?), and hence (34a) is not satisfied. Conceptually, the consistency of the D matrix is parallel to the invertibility of $[M_{\Pi E}]_{kj}$. As mentioned in the main text, the non-invertibility of $[M_{\Pi E}]_{kj}$ implies an incorrect specification of the information structure with respect to $\xi_{t+\tau}$ $(\tau = 0, 1, \cdots, S)$. Similarly, an inconsistent D (or non-existence of a consistent D) implies an incorrect specification of the information structure with respect to ξ_{t+1}. Such inconsistency/non-existence happens, for example, if a researcher puts an expectation operator on the evolution of capital, rather than on the consumption Euler equation.

Finally, note that

$$\text{"A consistent } D \text{ matrix exists"} \Leftrightarrow \text{"Equation (40) in Sims (2002) holds"}$$

Thus, it is now clear that equation (40) in Sims (2002) must always be satisfied if expectation errors appear only in the equations with dynamic jump variables, *regardless of the dynamic property such as saddle-path stable, sunspot, or explosive*!

References

Blanchard, Olivier Jean and Charles M. Kahn, "The Solution of Linear Difference Models under Rational Expectations," *ECM*, July 1980, *48* (5).

Boyd, John H. and Michael Dotsey, "Interest Rate Rues and Nominal Determinacy," *Federal Reserve Bank of Richmond, Working Paper*, 1990.

Christiano, Lawrence J., "Solving Dynamic Equilibrium Models by a Method of Undetermined Coefficients," *Working paper*, 1998.

Dupor, Bill and Takayuki Tsuruga, "Sticky Information: The Impact of Different Information Updating Assumptions," *Journal of Money, Credit and Banking*, dec 2005.

Golub, Gene H. and Charles F. Van Loan, *Matrix Computations, 3rd ed.*, Baltimore: Johns Hopkins University Press, 1996.

King, Robert G. and Mark W. Watson, "The Solution of Singular Linear Difference Systems under Rational Expectations," *International Economic Review*, November 1998, *39* (4).

_ **and** _ , "System Reduction and Solution Algorithms for Singular Linear Difference Systems under Rational Expectations," *Computational Economics*, October 2002, *20* (1).

Klein, Paul, "Using the generalized Schur form to solve a multivariate linear rational expectations model," *Journal of Economic Dynamics and Control*, September 2000, *24* (10).

Mankiw, Gregory N. and Ricardo Reis, "Sticky Information Versus Sticky Prices: A Proposal to Replace the New Keynesian Phillips Curve," *NBER working paper*, 2001.

Sims, Christopher A., "Solving Linear Rational Expectations Models," *Computational Economics*, October 2002, *20* (1-2).

Strang, Gilbert, *Linear Algebra and Its Applications, 3rd ed.*, Orlando, Florida: Harcourt Brace Jovanovich, Inc., 1988.

Uhlig, Harald, "A Toolkit for Analyzing Nonlinear Dynamic Stochastic Models Easily," *Ch3. in "Computational Methods for the Study of Dynamic Economics"*, 1999.

Wilkinson, J. H., "Kronecker's Canonical Form and the QZ Algorithm," *Linear Algebra and Its Applications*, 1979, *28.*

Woodford, Michael, "REDS-SOLDS USER'S GUIDE," undated.