Insurance: Mathematics and Economics 72 (2017) 21–35

Contents lists available at ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

A factor model for joint default probabilities. Pricing of CDS, index swaps and index tranches

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HIGHLIGHTS

- We propose a factor model for valuation of CDS, CDSI and CDO.
- The model is a first-passage distribution of Brownian motion continuous time-changed.
- The credit quality process is driven by a mean reverting Levy OU volatility process.
- The model is capable to reproduce the credit gap risk.
- FFT computational tools are developed to calculate the distribution of losses.

ARTICLE INFO

Article history:
Received April 2016
Received in revised form
October 2016
Accepted 6 October 2016
Available online 18 October 2016

JEL classification:
G12
C51
C63

Keywords:
Time-change
Mean-reverting process with jumps
CDS pricing
Credit index pricing
Tranche pricing

ABSTRACT

A factor model is proposed for the valuation of credit default swaps, credit indices and CDO contracts. The model of default is based on the first-passage distribution of a Brownian motion time modified by a continuous time-change. Various model specifications fall under this general approach based on defining the credit-quality process as an innovative time-change of a standard Brownian motion where the volatility process is mean reverting Lévy driven OU type process. Our models are bottom-up and can account for sudden moves in the level of CDS spreads representing the so-called credit gap risk. We develop FFT computational tools for calculating the distribution of losses and we show how to apply them to several specifications of the time-changed Brownian motion. Our line of modelling is flexible enough to facilitate the derivation of analytical formulae for conditional probabilities of default and prices of credit derivatives.

1. Introduction

One of the most blamed financial instruments in the aftermath of the subprime crisis is the Collateralized Debt Obligation (CDO), largely associated with the last financial crisis. Next to it, the Gaussian copula model for pricing CDOs is argued to be the formula that killed[1] the Wall-Street (MacKenzie and Spears, 2014). Due to this unfavourable perception of the instrument and tougher regulation, the issuances of such instruments were greatly diminished during and after the crisis. However, most of the negative consequences associated with this instrument are born from a lack of understanding and misuse combined with simplistic modelling approaches. We argue that the instrument itself has many merits and, if used and understood properly, it can improve diversification, customised risk transfer and hedging for credit portfolios.

While the subprime crisis affected negatively the issuances of CDOs, there are still important outstanding CDO contracts on the market that were issued before 2007 and that need to be properly evaluated. The low interest rate levels revived the interest of investors into the CDO market which responded with a substantial increase[2] of issuances both in Europe and US.

Given the renewed interest in this class of instruments there is a need for further improvements to the current models for credit

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1 It was the article "Recipe for Disaster: The Formula that Killed Wall Street" from February 2009 issue of Wired Magazine which popularised first this idea.

2 See the Online Appendix for the total value of outstanding CDOs and the volumes of new issuance (see Appendix C).

http://dx.doi.org/10.1016/j.insmatheco.2016.10.004
0167-6687/

risk and especially portfolio credit risk. In this paper, we aim to propose a modelling methodology that allows pricing of single-name credit contracts such as credit derivative swaps as well as multi-name contracts such as credit indices, tranches of a credit indices or CDOs. Investment banks routinely have to manage positions in both categories of credit instruments and managing portfolio credit risk requires the ability to calibrate on the individual components of the portfolio. Important recent contributions to credit risk modelling are (Blanchet-Scalliet et al., 2011; Hurd and Zhou, 2011; Cont and Minca, 2013; Gatarek and Jabłecki, 2013; Packham et al., 2013; Kijima and Siu, 2014; Ballestra and Pacelli, 2014; Hao and Li, 2015; Wei and Yuan, 2016).

The evolution of credit spreads for individual obligors and portfolio of obligors may be influenced by idiosyncratic effects as well as contagion sector or industry effects, and also economy wide events. Thus, several superimposed layers of information may determine the ebbs and flows of credit spreads. There is a clear empirical evidence that credit spreads exhibit jumps, see Dai and Singleton (2003), Tauchen and Zhou (2011), Zhang et al. (2009) and Schneider et al. (2010). These jumps are mostly positive being caused by the arrival of bad news and they impact CDS contracts across all maturities. Therefore, using Lévy processes will allow us to capture these jump effects. Early research modelling the credit quality process as a jump–diffusion or a Lévy process, see Baxter (2007) and Cariboni and Schoutens (2007), was hindered by the fact that computing first passage times was either intractable or computationally very demanding. Hao et al. (2013) obtained an analytical formula for the survival function and also for the single-name CDS and they showed why the par CDS spread is not negligible at very short maturities.

Our main contribution is an improved credit risk model that works well with single-name contracts as well as with multi-name contracts. Our techniques are based on defining the credit-quality process as an innovative time-change of a standard Brownian motion where the volatility process is mean reverting Lévy driven OU type process. The factor model we propose for the evolution of probability of default for single-names is a bottom-up approach to model the evolution of credit portfolios. Packham et al. (2013) conceptualised the default of a company as the first-passage time of a process modelling the credit worthiness of the company, being able in this way to capture credit gap risk and to provide an intuitive understanding of the hedging. We develop in this paper a multivariate extension, preserving the properties of the univariate model while adding the capability of modelling the time evolution of dependence between the defaults of different obligors, which is important for pricing multi-name credit contracts.

Our second contribution is to improve the computational tools that are necessary to calculate the distribution of losses at given maturity. Our FFT approach is better suited for this type of calculations than Panjer recursion and it applies to several specifications for the time-changed Brownian motion. We are also able to derive analytical formulae for conditional probabilities of default and credit derivatives. The advantage here is that one can investigate easily the sensitivity of our formulae to various model parameters. This is not always possible in general with all credit risk models, see Cont and Savescu (2008) and Bielecki et al. (2010), where numerical methods are required.

The remaining of the article is structured as follows. The modelling set-up is described in Section 2. In Section 3 we derive the formulae for default probabilities and portfolio loss. A particular feature of our modelling, the volatility of the credit quality process, is discussed in Section 4. The credit derivatives prices formulae are detailed in Section 5 while the calibration is exemplified in Section 6. Last Section summarises our findings.

2. Default modelling

In this section we propose a factor extension for the model of default of a company proposed in Packham et al. (2013). The default is represented as the first passage time of a time-changed BM. The location of the BM represents the credit quality of the company while the time-change models the arrival of information on the market that are relevant for the survival of the company. The model has the capability of modelling credit gap risk, being useful for pricing exotic credit derivative contracts and provides an intuitive understanding of the hedging. The multivariate extension of the model, proposed in this section, maintains the properties of the univariate model while adding the capability of modelling the time evolution of dependence between the defaults of the names in the portfolio by introducing a common factor driving the arrival of information affecting all names, which is important for pricing multi-name credit derivative contracts.

2.1. Informational setting

We consider an economy represented by a stochastic basis $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ where $\mathbb{P}$ is a risk neutral probability measure. In order for our model to be well defined and to reflect reality, the filtration $\mathbf{F} = (\mathcal{F}_t, t \geq 0)$ has a multi-structure design. First, we have the sub-filtration $\mathbf{B} = (\mathcal{F}_\theta, \theta \geq 0)$ which is the filtration of a (possibly multivariate) Brownian Motion (BM) B_θ. The second sub-filtration $\mathbf{H} = (\mathcal{H}_t, t \geq 0)$ is incorporating information about the common factor $\{H_t\}_{t\geq 0}$, and the last sub-filtration $\mathbf{G} = (\mathcal{G}_t, t \geq 0)$ is the filtration incorporating information about the idiosyncratic name specific factors $\{G_t^i\}_{t\geq 0}$ where i denotes the name i. Hence, $\mathbf{F} = \mathbf{B} \vee \mathbf{G} \vee \mathbf{H}$ and we assume that all processes defined below are adapted with respect to at least one of the sub-filtrations above.

2.2. Default model

For modelling the default of companies in a portfolio of N obligors we start with three mutually independent processes $\{B_t\}_{t\geq 0}$, $\{\Sigma_t^G\}_{t\geq 0}$ and $\{\Sigma_t^H\}_{t\geq 0}$, where the first two processes are N-dimensional and the last one is a unidimensional process.[3] The process B is a standard Brownian Motion (BM) with independent components, while Σ^G and Σ^H are positive processes. For any $t \geq 0$ we define $\Sigma_t^\Gamma = \Sigma_t^G + \beta \Sigma_t^H$, with β a vector of factor loadings. Now, using B and Σ^Γ we can define the credit quality process $\{X_t\}_{t\geq 0}$ as the stochastic integral:

$$X_t = \int_0^t \Sigma_s^\Gamma dB_s = \int_0^t \Sigma_s^G dB_s + \beta \int_0^t \Sigma_s^H dB_s. \tag{2.1}$$

The default probabilities are driven not only by the location of the Brownian motion but also by the level of volatility.[4] In order to capture this salient feature of the credit quality process one key insight in our modelling is expressing X_t as a time changed BM:

$$X_t = W_{\Gamma_t} = W_{G_t+\beta H_t} \tag{2.2}$$

where $\{W_\theta\}_{\theta\geq 0}$ is a BM on the scale θ and Γ_t is a continuous time change;

$$\Gamma_t = \int_0^t \Sigma_s^\Gamma ds = \int_0^t \Sigma_s^G ds + \beta \int_0^t \Sigma_s^H ds = G_t + \beta H_t. \tag{2.3}$$

[3] Other combinations are possible where more factors are considered or homogeneous groups of names are modelled by the same model.

[4] We thank an anonymous referee for indicating this improved explanation.

The two processes G and H are integrated variance processes[5] and they capture the impact of company specific and respectively market specific information on the volatility of the credit quality of the names in the portfolio. Due to the common factor H the credit quality processes in the portfolio will move together when the market information creates volatility movements, generating dependence between the credit quality of the names.

Given the credit quality process X and its representation as a time changed BM we model as in Packham et al. (2013), henceforth the (PSS) model, the univariate default as a first-passage time over a fixed barrier b_n with $n = 1, 2, \dots, N$ of the individual credit quality process $\{X_t^n\}_{t\geq 0}$, where $\tau_n = \inf\{t \geq 0 : X_t^n \leq b_n\}$. Our first result gives the individual probability of default.

Proposition 1. *For the model described above the probability of default is given by the formula*

$$\mathbb{P}(\tau_n < s|\mathcal{F}_t) = \mathbb{E}\left[2\mathcal{N}\left(\frac{b_n - X_t^n}{\sqrt{\Gamma_s^n - \Gamma_t^n}}\right)\middle|\mathcal{F}_t\right]. \tag{2.4}$$

The proof is almost identical to the proof of Proposition 3.3 in (Packham et al., 2013) with the exception of the way condition on the filtration for the volatility takes place.

The above default model can be characterised as a hybrid "first passage time model" where the default mechanism relies on the first passage of a BM formula as in the Merton's structural model. However, X_t^n and b_n are not interpreted as asset and debt level of the company, which are assumed to be unobservable. Given the representation of the credit worthiness process as a time-change BM, employing the common volatility factors to introduce dependence in the multivariate context becomes the obvious choice and it brings economical interpretation as well as computational tractability to the multivariate model. The factors drive the uncertainty from various sources which adversely affects the probability of survival of a certain company (higher volatility implies less probability of survival). The formula (2.3) is similar to the one obtained in Hurd (2009) where a more relaxed definition of the first passage time of a time-changed BM is adopted. Due to this similarities, the multivariate extension discussed in this paper can be easily extended to such models.

3. Probability of default and portfolio loss

3.1. Computation of univariate probability of default

Without loss of generality we drop n from the notation G_t^n, τ_t^n and b^n when we only refer to one specific name. The default probability can be expressed as

$$\mathbb{P}(\tau < s|\mathcal{F}_t) = \mathbb{E}\left[\mathbb{E}\left[2\mathcal{N}\left(\frac{b - X_t}{\sqrt{\Gamma_s - \Gamma_t}}\right)\middle|\mathcal{F}_t \vee \mathcal{G}_s \vee \mathcal{H}_s\right]\middle|\mathcal{F}_t\right]$$
$$= \int_{\mathbb{R}_+}\int_{\mathbb{R}_+} 2\mathcal{N}\left(\frac{b - X_t}{\sqrt{G_s + \beta H_s - G_t - \beta H_t}}\right)\mathbb{P}_{G_s}(dz)\mathbb{P}_{H_s}(dy).$$

The computation of this probability requires the existence of closed form formulae for the densities of G_s and H_s, which are usually not available.

For the special case where the credit worthiness process $\{X_t\}_{t\geq 0}$ is driven by a Compound Poisson process as the Background Driving Lévy Process (BDLP), see definition later in Section 4), one can use Panjer recursions to compute the probability of G_s and H_s and then approximate the integrals by some quadrature methods as described in Packham et al. (2013). However, this approach does not work for more general processes of the variance and hence, we propose here a faster and more general technique based on the Fourier transform. Fast computational methods are an imperative requirement for multi-name credit derivatives, like derivative contracts on credit indices which have regularly more than 100 names in their structure (iTraxx for example has 125 names).

We start by observing that the probability of default (2.3) can be expressed in terms of a standard normal variable Z:

$$\begin{aligned}\mathbb{P}(\tau < s|\mathcal{F}_t) &= \mathbb{E}[2\mathbb{P}(Z\sqrt{\Gamma_s - \Gamma_t} - (b - X_t) \leq 0|\Gamma_s)|\mathcal{F}_t]\\ &= \mathbb{E}[2\mathbb{P}(Y - K \leq 0|H_s, G_s)|\mathcal{F}_t]\\ &= \mathbb{E}[2\mathbb{P}(V \leq 0|H_s, G_s)|\mathcal{F}_t]\end{aligned} \tag{3.1}$$

where $Z \sim N(0, 1)$, $Y = Z\sqrt{G_s - G_t}$, $K = b - X_t$ and $V = Y - K$.

Since we are interested only in companies that are not in default at the time of computation of the probabilities of default, we focus on the case where the credit worthiness process X starts from above the barrier b, which implies that initially $K < 0$. The characteristic function of V is just $\phi_V(u) = \mathbb{E}[e^{iu(Y-K)}|\mathcal{F}_t] = e^{-iuK}\phi_Y(u)$ and the characteristic function of Y can be calculated analytically as

$$\begin{aligned}\phi_Y(u) &= \mathbb{E}[e^{iuY}|\mathcal{F}_t] = \mathbb{E}[e^{iuZ\sqrt{\Gamma_s-\Gamma_t}}|\mathcal{F}_t]\\ &= \mathbb{E}[\mathbb{E}[e^{iuZ\sqrt{\Gamma_s-\Gamma_t}}|\mathcal{G}_s \vee \mathcal{H}_s]|\mathcal{F}_t]\\ &= \mathbb{E}\left[\exp\left\{-\frac{u^2(\Gamma_s - \Gamma_t)}{2}\right\}\middle|\mathcal{F}_t\right]\\ &= e^{u^2\Gamma_t/2}\mathbb{E}\left[\exp\left\{-\frac{u^2}{2}\Gamma_s\right\}\middle|\mathcal{F}_t\right]\\ &= e^{u^2\Gamma_t/2}\phi_{\Gamma_s}(-u^2/2)\\ &= e^{u^2(G_t-\beta H_t)/2}\phi_{G_s}(-u^2/2)\phi_{H_s}(-\beta u^2/2)\end{aligned} \tag{3.2}$$

where $\phi_{G_t}(u)$, the moment generating function of G_t, and $\phi_{H_t}(u)$, the moment generating function of H_t, are assumed to have analytical expressions. The computation of the portfolio loss requires the computation of the probability of default conditional on the factor H_s. The conditional (on the factor H_t) characteristic function can be obtained by observing that H_s is a known value:

$$\begin{aligned}\phi_{Y|H}(u) &= e^{u^2\Gamma_t/2}\mathbb{E}\left[\exp\left\{-\frac{u^2}{2}\Gamma_s\right\}\middle|\mathcal{F}_t\right]\\ &= e^{u^2(G_t-\beta H_t)/2}\mathbb{E}\left[\exp\left\{-\frac{u^2}{2}(G_s - \beta H_s)\right\}\middle|\mathcal{F}_t\right]\\ &= e^{u^2(\beta H_s+G_t-\beta H_t)/2}\phi_{G_s}(-u^2/2).\end{aligned} \tag{3.3}$$

By inverting (3.3) and integrating with respect to the distribution of the factor H_t one can obtain the probability of default $\mathbb{P}(\tau < s|\mathcal{F}_t)$. However, while the conditional characteristic function (3.3) has a simple form, its inversion leads to a Laplace transform which is known to be difficult to implement and expensive to evaluate (Epstein and Schotland, 2008). Therefore, we advocate using a simple Fourier inversion that is implementable by Fast Fourier Transform algorithms which are possible due to the representation in the following proposition.

Proposition 2. *Given the multivariate default model described in Section 2.2, any single name conditional probability of default given the factor H_s can be computed as :*

$$\mathbb{P}(\tau < s|\mathcal{H}_s \vee \mathcal{F}_t)$$

[5] Note that the mutual independence assumed between B, Σ^G and Σ^H implies that also W, G and H are mutually independent (see Thm. 2.6 of Barndorff-Nielsen and Shiryaev, 2010). Moreover, the processes W, G and H are adapted to the filtrations **B**, **G** and **H** respectively.

$$= \mathbb{E}\left[\mathbb{P}\left(\frac{1}{(b-X_t)^{-2}\Upsilon} - (G_s - G_t) - \beta(H_s - H_t) \le 0\right)\middle| \mathcal{H}_s \vee \mathcal{F}_t\right]$$
$$\doteq \mathbb{E}[\mathbb{P}(Q \le 0)|\mathcal{H}_s \vee \mathcal{F}_t] \tag{3.4}$$

where Υ is a chi-squared variable.

Proof. See Appendix A. □

3.2. Inversion formulae

If both $\phi_{G_t}(u)$ and $\phi_{H_t}(u)$ are known the probability of default (and the conditional probability of default) can be obtained by inverting the matching characteristic functions. This can be done in three ways. The first[6] is the Gil-Pelaez formula (Gil-Pelaez, 1951) for the cumulative distribution which in our context gives:

$$F_Q(q) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{e^{iuq}\phi_Q(-u) - e^{-iuq}\phi_Q(u)}{2iu}du.$$

To get $\mathbb{P}[Q \le 0]$ we need to evaluate $F_Q(q)$ at zero and obtain:

$$\mathbb{P}[Q \le 0] = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{e^{iuq}\phi_Q(-u) - e^{-iuq}\phi_Q(u)}{2iu}du$$
$$= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{e^{iu\times 0}\phi_Q(-u) - e^{-iu\times 0}\phi_Q(u)}{2iu}du$$
$$= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{\phi_Q(-u) - \phi_Q(u)}{2iu}du. \tag{3.5}$$

Because $\overline{\phi_Q(u)} = \phi_Q(-u)$ and $\Re(\frac{\phi_Q(u)}{iu}) = \frac{\phi_Q(u) - \overline{\phi_Q}(u)}{2iu}$:

$$\mathbb{P}[Q \le 0] = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{\phi_Q(-u) - \phi_Q(u)}{2iu}du$$
$$= \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \Re\left(\frac{\phi_Q(u)}{iu}\right)du.$$

This method has the disadvantage that the above integrand has a singularity at zero which could create numerical problems. Since the singularity is at the lower limit of the integration domain this can be dealt with by choosing the lower limit close enough to zero.

A second useful formula has been described in Kim et al. (2010). Using that $F_X(x) = \frac{e^{x\rho}}{\pi}\Re\left(\int_0^\infty e^{-iux}\frac{\phi_X(u+i\rho)}{\rho - iu}du\right)$ implies that:

$$\mathbb{P}\left(Q \le 0\right) = \frac{1}{\pi}\Re\left(\int_0^\infty \frac{\phi_Q(u+i\rho)}{\rho - iu}du\right). \tag{3.6}$$

The advantage of the above formula over (3.5) is the lack of singularity given by division by zero but care is needed especially when dealing with the evaluation of the exponential function at high negative powers.

A third formula is from Feng and Lin (2013) and it is based on the Hilbert transform representation of a cumulative distribution function, expressed as a Cauchy principal value integral, $\mathbb{H}(f(x)) = \frac{1}{\pi}\text{p.v}\int_{\mathbb{R}}\frac{f(y)}{x-y}dy$ and exploit the relation between the Hilbert and Fourier transform $\mathbb{F}(\cdot)$:

$$\mathbb{F}(\mathbb{1}_{(-\infty,l]}f)(\eta) = \frac{1}{2}\phi_Q(\eta) - \frac{i}{2}e^{i\eta l}\mathbb{H}(e^{iul}\phi_Q(u))(\eta) \tag{3.7}$$

in order to write the probability distribution as:

$$\mathbb{P}(Q \le 0) = \int_{-\infty}^0 p_Q(x)dx = \int_{\mathbb{R}} \mathbb{1}_{(-\infty,0)}p_Q(x)dx$$
$$= \int_{\mathbb{R}} \mathbb{1}_{(-\infty,0)}e^{ix0}p_Q(x)dx = \mathbb{F}(\mathbb{1}_{(-\infty,0]}f)(0)$$
$$= \frac{1}{2}\phi_Q(0) - \frac{i}{2}e^{i00}\mathbb{H}(e^{iu0}\phi_Q(u))(0)$$
$$= \frac{1}{2} - \frac{i}{2}\mathbb{H}(\phi_Q(u))(0) \tag{3.8}$$

where the known relation $\phi_Q(0) = 1$ is used. An approximation of Hilbert transform as a truncated infinite series is available in the form:

$$\mathbb{H}(f(x)) \approx \sum_{m=-M}^{M} f(mh)\frac{1-\cos(\pi(x-mh)/h)}{\pi(x-mh)/h}$$

for a step size $h > 0$ and M a large positive integer.

All three formulae of the probability distribution above lead to Fast Fourier Transform implementation of the conditional probability of default which we summarise in the following proposition:

Proposition 3. *The conditional probability of default formula in Proposition 2 has the following alternative representations with direct* FFT *implementations:*

- $\mathbb{P}^{Gill}[\tau < s|\mathcal{H}_s \vee \mathcal{F}_t] =$
$$\frac{1}{2} - \frac{1}{\pi}\Re\left(\int_0^\infty e^{-iu(\beta H_s)}\frac{\phi_{i\Gamma}(u)\phi_{G_s}(-u)e^{iu(\beta H_t + G_t)}}{iu}du\right).$$
- $\mathbb{P}^{Kim}[\tau < s|\mathcal{H}_s \vee \mathcal{F}_t] =$
$$\frac{1}{\pi}\Re\left(\int_0^\infty e^{-iu(\beta H_s)} \times \frac{\phi_{i\Gamma}(u+i\rho)\phi_{G_s}(-u-i\rho)e^{iu(G_t+\beta H_t)+\rho(\beta(H_t-H_s)+G_t)}}{\rho - iu}du\right).$$
- $\mathbb{P}^{Hilb}[\tau < s|\mathcal{H}_s \vee \mathcal{F}_t] \approx$
$$\frac{1}{2} - \frac{i}{2}\sum_{m=0}^{M-1} e^{-2imh(\beta H_s)} \times \frac{2\phi_{i\Gamma}((2m+1-M)h)\phi_{G_s}((M-2m-1)h)e^{ih\Lambda}}{\pi(M-2m-1)}$$

where we used the notation $\Lambda = (2m+1-M)((G_t + \beta H_t) - \beta H_s(1-M))$.

Proof. See Appendix B. □

When compared to the Panjer recursion based methodology our Fourier transform methods have the advantage of being fast and accurate. The above formulae only require the characteristic functions of the variance variables to be available analytically, which makes the applicability of the above methodology very large when compared to the Panjer recursion methodology limited to specifications of the variance in the Compound Poisson class.

3.3. Portfolio cumulative loss

Since the multidimensional credit quality process $\{X_t\}_{t\ge 0}$ defined in (2.1) has a factor structure with the common variance process $\{H_t\}_{t\ge 0}$ driving the common informational shocks, we can use the fact that the univariate probabilities of default conditional on the common factor are independent. Due to this property one can compute the joint probability of default by first computing the conditional default probabilities and then just integrating their product with respect to the distribution of the common factor:

$$\mathbb{P}[\tau_t^1 \le s, \ldots, \tau_t^N \le s] = \int_{\mathbb{R}_+}\prod_{j=1}^N \mathbb{P}[\tau_t^j \le s|H_s]\mathbb{P}_{H_s}(dh) \tag{3.9}$$

6 An earlier formula due to Lévy is also known but not useful in our context.

For pricing most liquid existent multi-name contracts (like synthetic CDOs or index tranches) it is sufficient to know the *loss process* $\{L_t\}_{t\geq 0}$ defined as $L_t = \sum_{j=1}^{n}(1-R^j)\mathbb{1}_{\{\tau_t^j \leq s\}}$ where $\mathbb{1}_{\{\tau_t^j \leq s\}}$ is an indicator variable signalling whether the name j is in default or not and R^j is the associated recovery rate which can be random. For computing the conditional distribution of the cumulative loss of the portfolio $L(s)$ one can employ the so-called ASB algorithm proposed in Andersen and Sidenius (2004) and Andersen et al. (2003), see also Hull and White (2004).

Here we present a slight modification of the algorithm, working with the cumulative loss expressed as percentage instead of dollar losses. We start from the conditional (on the common factor) loss distribution of a single-name which we discretise. Under the assumption that the recovery rate R^n corresponding to the name n is an independent random variable with a known distribution which can be discretised by using the relation $\mathbb{P}[R^n \in (q_{k-1}, q_k]] = \mathbb{P}[R^n \leq q_k] - \mathbb{P}[R^n \leq q_{k-1}]$ for $u = q_k - q_{k-1}$ and $k \in \{1, 2, \dots, k_{max}\}$.[7] Next, the conditional distribution of $\ell_n = (1 - R^n)\mathbb{1}_{\{\tau_n \leq T\}}$ can be computed for a generic q:

$$\begin{aligned}
&\mathbb{P}[\ell_n(s) \leq q|H_s] \\
&\quad = \mathbb{P}[(1-R^n) \leq q|H_s]\mathbb{P}[\tau_n \leq T|H_s] + \mathbb{P}[\tau_n > T|H_s] \\
&\quad = \mathbb{P}[(1-R^n) \leq q]p_n(H_s) + (1 - p_n(H_s))
\end{aligned}$$

because R^n is an independent variable and where $p_n(H_s)$ denoting the conditional probability of default (3.4) for the name n.

Now the computation for the probability distribution of the portfolio loss is based the observation that given the conditional probability distribution $\mathbb{P}[L^m(s) \leq q|H_s]$ for a portfolio made of m credit names $(1 < m \leq n)$ we can compute the conditional distribution of a portfolio with an additional credit name by:

$$\begin{aligned}
&\mathbb{P}[L^{m+1}(s) \leq K|H_s] \\
&\quad = \sum_{k=1}^{k_{max}} \mathbb{P}[L^m(s) \leq K - q_k|H_s]\mathbb{P}[\ell_{m+1}(s) \leq q_k|H_s]. \qquad (3.10)
\end{aligned}$$

The conditional portfolio loss can be computed by starting from the initial case with zero companies in the portfolio $\mathbb{P}[L^0(s) \leq K|H_s] = \mathbb{1}_{\{K=0\}}$ and applying the recursion relation above. The computation of the unconditional portfolio loss distribution requires to integrate $\mathbb{P}[L^{m+1}(s) \leq K|H_s]$ with respect to the density of the factor H_s. Because the density of the factor H_s is not usually known, we need to use the Fourier inversion to compute it from the characteristic function.

We follow this approach for the case of the IG-OU model and generate the density of the factor H by inverting the associated characteristic function for the most liquid maturities on the CDS market (see Fig. 1). One can see from Fig. 1 that a decrease in θ has the effect of flattening the distribution of the factor. Remember that the factor is an integrated variance process with θ controlling the speed of mean reverting. As a result, the lower the θ is the slower is the mean reversion implying that the jumps in the variance process will have a persistent impact leading to the flattened distributions observed in Fig. 1(b). The role of parameter a is to control the shape of the distribution of increments for the BDLP driving the variance process while b is controlling the mean. These two parameters are closely related to the shape and scale parameters of Inverse Gaussian distributions. Fig. 1(c) suggests that a lower value for a increases the peak of the distributions while from Fig. 1(d) it is obvious the impact of b on the location of the mean for the distribution of the factor for each of the five maturities analysed. For the Gamma-OU model similar comments can be made.

7 Alternatively one can assume a discrete distribution for R^n.

4. Variance modelling

The volatility of the credit quality of a company may exhibit jumps that can lead to sudden moves in the probability of default. Therefore, for the volatility we select a positive process in the class of mean reverting Lévy-driven OU type process introduced by Barndorff-Nielsen (2001) and Barndorff-Nielsen and Shephard (2001). Packham et al. (2013) showed that this choice can be beneficial when modelling jumps in credit spreads for some exotic univariate derivative contracts like credit-linked notes. There are several possible specifications for variance or volatility processes.

4.1. Lévy-OU model

The model for Σ_t that was discussed in Norberg (2004) has the form:

$$d\Sigma_t = \theta(\mu(t) - \Sigma_{t-})dt + dZ_t, \quad \Sigma_0 > 0 \qquad (4.1)$$

where $\{Z_t\}_{t\geq 0}$ is the so-called *background driving Lévy process* (BDLP). This is the model of choice in Packham et al. (2013), where the long term mean parameter $\mu(t)$ is specified as a piecewise constant function that takes different values for various maturities and plays a special role in the calibration of the univariate model.

The integrated variance process which plays the role of the time change is then obtained by integrating the variance process $\{\Sigma_t\}_{t\geq 0}$ over time.

$$\begin{aligned}
G_t &= \int_0^t \Sigma_s ds = \int_0^t \left(e^{-\theta s}\Sigma_0 + \int_0^s e^{-\theta(s-u)}\theta\mu(u)du \right. \\
&\quad \left. + \int_0^s e^{-\theta(s-u)}\Sigma_u du \right) ds \\
&= (1 - e^{-\theta t})\frac{\Sigma_0}{\theta} + \int_0^t (1 - e^{-\theta(t-u)})\mu(u)du \\
&\quad + \frac{1}{\theta}\int_0^t 1 - e^{-\theta(t-s)} dZ_s.
\end{aligned}$$

Denoting $\epsilon(t) = \frac{(1-e^{-\theta t})}{\theta} + \int_0^t (1 - e^{-\theta(t-u)})\mu(u)du$ and taking $f(s) = iu\epsilon(T-s)$ (with $\Re(f) = 0$) the characteristic function of G_T becomes:

$$\phi_{G_T}(u) = \exp\left\{ iu[\epsilon(T-t)\Sigma_0] + \int_t^T \theta\kappa_Z(iu\epsilon(T-s))ds \right\} \qquad (4.2)$$

where κ_Z is the cumulant of the distribution of Z. Our models are spanned by specifications for the processes driving the randomness of the variance process, both individual and common factors.

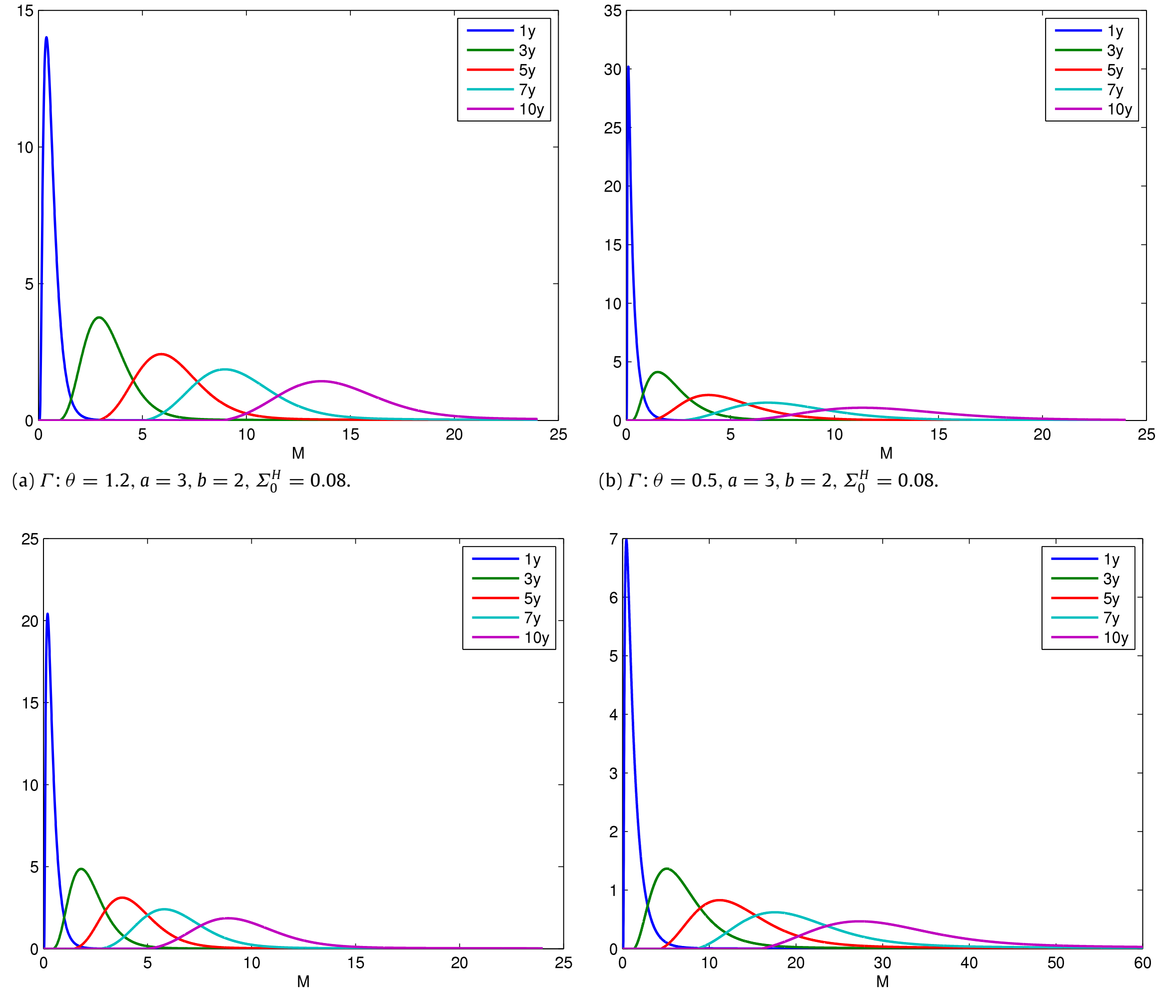
4.1.1. Compound Poisson BDLP

The compound Poisson process is specified by the intensity of the Poisson process denoted by λ and the jumps Y assumed to be only positive in order to guarantee the positiveness of the time change. The choice of distribution for Y is restricted to the class of distributions with positive support.

The cumulant function of a compound Poisson process driven by a Poisson process $\{N_t\}_{t\geq 0}$ with intensity λ and having jumps Y, with κ_Y being the cumulant of the distribution of Y, is $\kappa_{X_t} = t\lambda(e^{\kappa_Y(u)} - 1)$. When the jumps in a Compound Poisson process are Gamma distributed we call it a CPG process, with the cumulant function $\kappa_{X_t} = t\lambda(e^{(1-\beta u)^{-\alpha}} - 1)$.

The characteristic function of the one factor integrated variance process G with Compound Poisson processes as BDLP can be calculated easily.

$$\phi_G(u) = \exp\left\{ iu\epsilon(T-t)\Sigma_0 + \int_t^T \kappa_{X_1}(iu\epsilon(T-s))ds \right\}$$



(a) $\Gamma: \theta = 1.2, a = 3, b = 2, \Sigma_0^H = 0.08$.

(b) $\Gamma: \theta = 0.5, a = 3, b = 2, \Sigma_0^H = 0.08$.

(c) $\Gamma: \theta = 1.2, a = 2, b = 2, \Sigma_0^H = 0.08$.

(d) $\Gamma: \theta = 1.2, a = 3, b = 1, \Sigma_0^H = 0.08$.

Fig. 1. Density of the common factor. *Note*: To generate the graphs we used the IG-OU specification for the factor H_s with parameters set Γ specified under each graph. The parameters θ and a affect the variance of the factor distribution while b affects the location.

$$= \exp\left\{iu\epsilon(T-t)\Sigma_0 + \int_t^T \lambda(e^{\kappa_Y(iu\epsilon(T-s))} - 1)ds\right\}$$

$$= \exp\left\{iu\epsilon(T-t)\Sigma_0 + \lambda(T-t) \times \left(\int_t^T e^{\kappa_Y(iu\epsilon(T-s))}\frac{1}{T-t}ds - 1\right)\right\} \quad (4.3)$$

$$= \exp\{iu\epsilon(T-t)\Sigma_0 + \lambda(T-t)(\mathbb{E}[e^{iu\epsilon(S)Y}] - 1)\}$$

$$= \exp\{iu\epsilon(T-t)\Sigma_0 + \lambda(T-t)(\phi_{\epsilon(S)Y}(u) - 1)\}. \quad (4.4)$$

The integral in (4.3) can be interpreted as an expectation with respect to a uniform density on $[t, T]$. As in Norberg (2004), recognising the last part in (4.4) as the characteristic function of the compound Poisson process variable $\mathbf{CPO}(\lambda(T-t), \epsilon(S)Y)$ we can conclude that the integrated variance process for the case of a compound Poisson BDLP is a compound Poisson with drift and characteristic function as in (4.4). Given the integrated variance process is a compound Poisson process, its distribution at time T can be computed by the means of Panjer recursion technique. We note that (4.4) is not an analytic characteristic function for the integrated variance process for the cases of CPO considered above.

4.2. Gamma-OU and IG-OU specifications

For $\mu = 0$ and BDLP $\{Z_{\theta t}\}_{t\geq 0}$, a subordinator defined on a deterministic time change $s = \theta t$, we have the case of the Lévy driven OU process

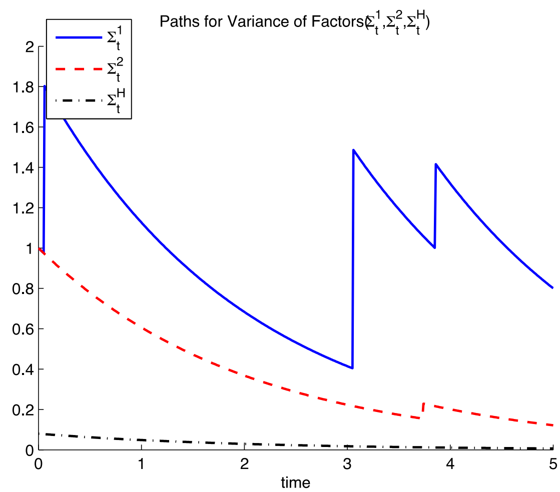
$$d\Sigma_t = -\theta\Sigma_{t-}dt + dZ_{\theta t}, \quad \Sigma_0 > 0 \quad (4.5)$$

with the known solution $\Sigma_t = e^{-\theta t}\Sigma_0 + e^{-\theta t}\int_0^{\theta t} e^s dZ_s$.

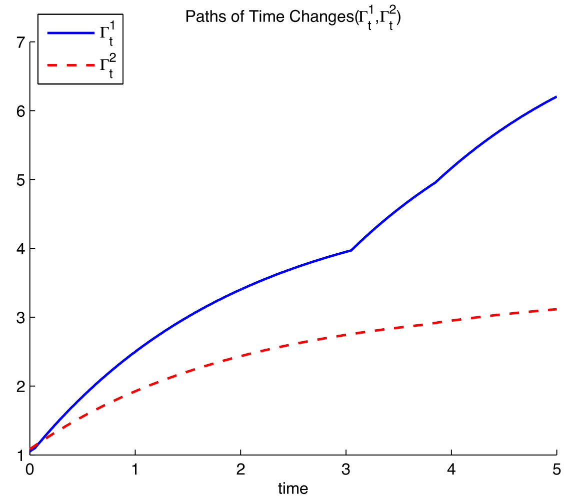
For these processes the BDLP can be specified such that the marginal law of Σ_s is a given distribution. In addition, the choice of the time scale λt guarantees that for any λ, the process $\{\Sigma_t\}_{t\geq 0}$ is a stationary process, meaning that the marginal distribution of Σ_t remains unchanged. Two very tractable specifications are the IG-OU and Gamma-OU specifications studied in Barndorff-Nielsen and Shephard (2001). These processes have as an approximate stationary distribution the IG and Gamma distributions, respectively. In the sequel we follow the parametrisation in Cariboni and Schoutens (2009) of these processes. Starting from the cumulant of a Gamma distribution $\Gamma(a, b)$ respectively IG distribution $IG(a, b)$:

$$\kappa_\Gamma(u) = \frac{u\nu}{\alpha - u} \quad \alpha, \nu > 0.$$

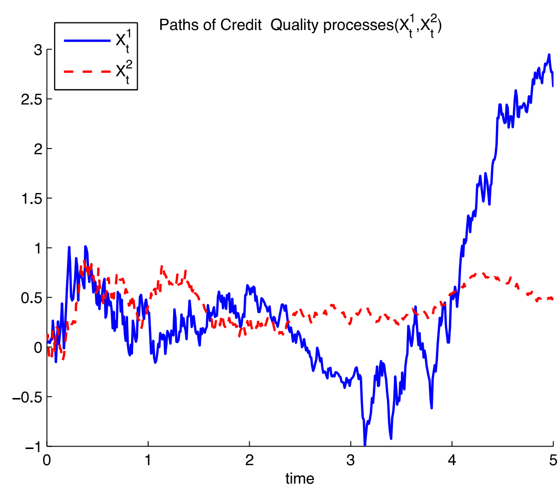
$$\kappa_{IG}(u) = \frac{u\delta}{\sqrt{\gamma^2 - 2u}} \quad \gamma, \delta > 0$$



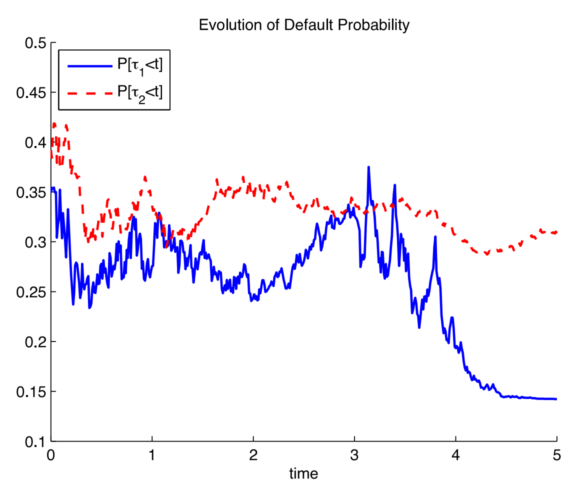
(a) Simulated paths of Variance processes (Σ^1_t, Σ^2_t, Σ^H_t).



(b) Evolution of time changes (Γ^1_t, Γ^2_t).



(c) Evolution of credit quality (X^1_t, X^2_t).



(d) Evolution of default probability ($P[\tau_1 < t]$, $P[\tau_2 < t]$).

Fig. 2. Path Simulation (no factor jump). *Note*: Simulations of Gamma-OU model with parameters set $\theta_1 = \theta_2 = \theta_H = 0.5$, $a_1 = a_2 = a_H = 1$, $b_1 = b_2 = b_H = 0.7$, $\Sigma^1_0 = \Sigma^2_0 = 1$, $\Sigma^H_0 = 0.08$, $\beta_1 = 0.5$, $\beta_2 = 0.9$ The presence of **no jump** in the market factor H_s leaves the probabilities of default driven by only intrinsic factors.

one can use the link between a self-decomposable distribution[8] D with cumulant function $\kappa_D(u) = \mathbb{E}[e^{uD}]$ and the cumulant function of the BDLP $\kappa_X(u) = \mathbb{E}[e^{uX_1}]$ at time $t = 1$ $\kappa_X(u) = u\frac{d\kappa_D}{du}(u)$ to derive the cumulant function at time $t = 1$ of the BDLP processes.

The characteristic functions of the intOU processes modelling the integrated variance are analytic and this is a major advantage of the approach presented in this paper. The Laplace transform of the Integrated Gamma-OU and IG-OU processes has been derived in Nicolato and Venardos (2003) (see also Cariboni and Schoutens, 2009) and they are:

$$\psi_{IG^*}(u) = \exp\left(\frac{iu\Sigma_0}{\theta}(1 - e^{-\theta t}) + \frac{2aiu}{b\theta}B\right) \tag{4.6}$$

$$\psi_{\Gamma^*}(u) = \exp\left(\frac{iu\Sigma_0}{\theta}(1 - e^{-\theta t}) + \frac{\theta a}{iu - \theta b}C\right) \tag{4.7}$$

where

$$B = +\frac{1}{\sqrt{1+\nu}}\left(\operatorname{arctanh}\left(\frac{\sqrt{1+\nu(1-e^{-\theta t})}}{\sqrt{1+\nu}}\right) - \operatorname{arctanh}\left(\frac{1}{\sqrt{1+\nu}}\right)\right) + \frac{1-\sqrt{1+\nu(1-e^{-\theta t})}}{\nu}$$

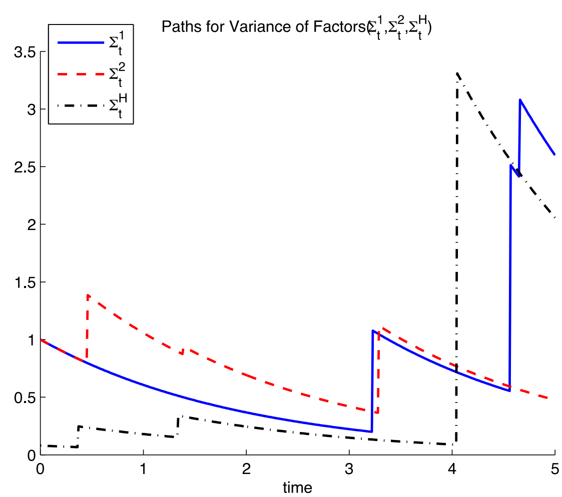
$$C = b\log\left(\frac{b}{b - iu\theta^{-1}(1-e^{-\theta t})}\right) - iut, \quad \nu = \frac{-2iu}{\theta b^2}.$$

This feature creates an advantage over models of Compound Poison OU type discussed in the previous section and provided us with a strong motivation to choose this class of models for the pricing of credit default swap, credit index and CDO later on in this paper.

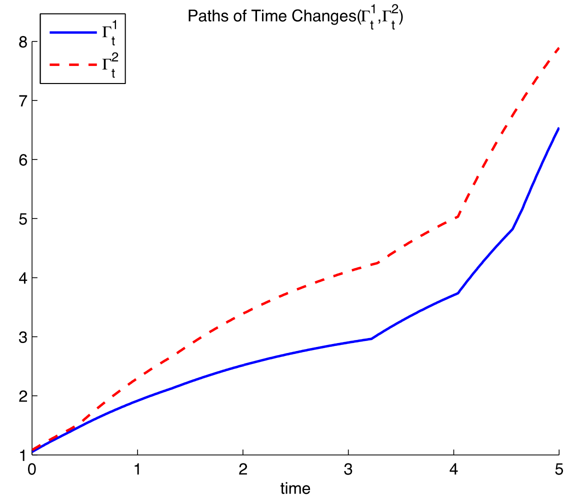
4.3. Properties related to dependence and contagion

The evolution of the variance in CP-OU, Gamma-OU and IG-OU specifications are driven by pure jump processes, allowing the model a fast precipitation to default. One peculiarity of the IG-OU process when compared to Gamma-OU and CP-OU specifications is that the first has an infinite number of jumps per time interval while the last two have only a finite number of jumps per time interval. To highlight how the superposition of the OU processes driving the variance factors introduces credit dependence we simulated the factors in a two factor model and show how this translates into dependence between the integrated variance of the two names (Figs. 2 and 3). The variance process of each of the two

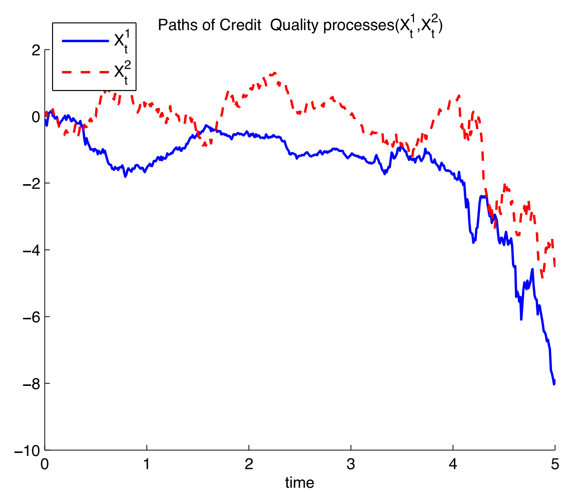
[8] A random variable X is said to have a self-decomposable distribution if for a constant $0 < c < 1$ there exist an independent random variable $X^{(c)}$ such that $X = cX + X^{(c)}$. The self-decomposable random variables are infinitely divisible.



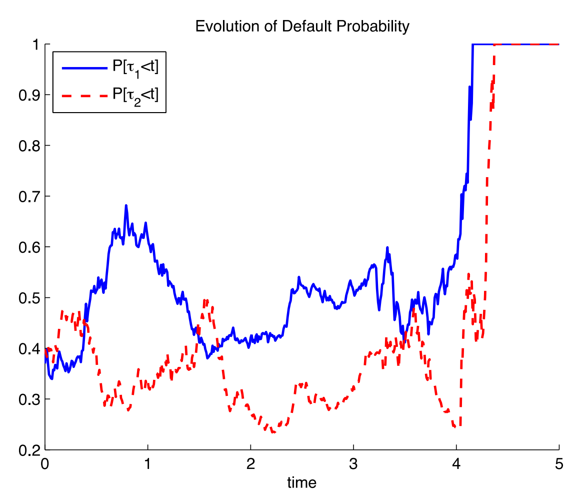
(a) Simulated paths of variance processes (Σ^1_t, Σ^2_t, Σ^H_t).



(b) Evolution of time changes (Γ^1_t, Γ^2_t).



(c) Evolution of credit quality (X^1_t, X^2_t).



(d) Evolution of default probability ($P[\tau_1 < t]$, $P[\tau_2 < t]$).

Fig. 3. Path simulation (with factor jump). *Note*: Simulations of Gamma-OU model with parameters set $\theta_1 = \theta_2 = \theta_H = 0.5, a_1 = a_2 = a_H = 1, b_1 = b_2 = b_H = 0.7$, $\Sigma_0^1 = \Sigma_0^2 = 1$, $\Sigma_0^H = 0.08$, $\beta_1 = 0.5$, $\beta_2 = 0.9$ The presence of a big jump in the market factor H_s after time 4 increases the probability of default of the two companies and eventually leads to their default.

factors has paths characterised by jumps and exponential decays and an increase in the variance translates into a faster growth of the integrated variance process. The dependence in this setting is produced by the effect that the jumps in the common factor has on the default probability of all the names in the portfolio. When the integrated variance driving the common factor rises, this increases the probability of default of all the names in the portfolio. In Fig. 2 we illustrate the paths of the variance, integrated variance(factors) and credit quality processes for the case when no jump takes place on the path of the variance driving in the common factor. In this instance the evolution of the credit quality processes and the default probabilities of the two names evolve independently of each-other. When a jump in the variance of the market factor occurs, as is illustrated in Fig. 3, this leads to dependence between the credit quality and the probability of default of the two companies. As illustrated in Fig. 3(c) this could lead to the default of both companies.

As depicted in Fig. 2, the OU process driving the variance is characterised by sudden jumps followed by exponential decay periods. The decay in the *common factor* produces autocorrelation of the variance and, due to the fact that the jump in the business time persists for some period, the probability that the company will default is increasing. More importantly, two companies affected by a common shock such as the market factor will be exposed to this shock for as long as the decay continues. This can generate a contagion effect in which the default of a company due to a jump in the market factor can be followed by the default of another company which at first has not defaulted but later on it does because of the prolonged decay period. While the default of the first company does not cause the default of the second (due to the conditional independence assumption) as in the standard contagion models, see Davis and Lo (2001), the important feature of the domino effect is captured in our model.

5. Credit derivatives pricing

5.1. Single-name credit default swaps

Under ISDA 2009 documentation which is known as the "Big Bang" specification, see MARKIT (2009) the CDS contracts have fixed coupons and at the cash settlement date the difference between the premium leg (computed with fixed coupons) and the protection leg is paid upfront. The formulae needed for the pricing of CDS contracts are based on assuming a unit notional and denoting by R the (random) recovery, $r_t(v)$ the discount rate[9] at the present time t for maturity v and by T the expiration of the CDS contract, the present value of the dirty protection leg is:

$$\begin{aligned} V_t^{Prot,dirty} &= \mathbb{E}[e^{-\int_t^\tau r_t(v)dv}(1-R)\mathbb{1}_{\{\tau\le T\}}|\mathcal{F}_t] \\ &= \mathbb{E}[\mathbb{E}[B_t(\tau)(1-R)\mathbb{1}_{\{\tau\le T\}}|\mathcal{H}_T]|\mathcal{F}_t] \\ &= \mathbb{E}[B_t(\tau^*)(1-\bar{R})p^*(T)|\mathcal{F}_t] \end{aligned} \quad (5.1)$$

where $B_t(\tau) = e^{-\int_t^\tau r_t(v)dv}$ is the discount factor, $\bar{R}$ is expected recovery rate and $p^*(T) = \mathbb{E}[\mathbb{1}_{\{\tau\le T\}}|\mathcal{H}_T]$ is the conditional (on the H) probability of default. The present value of the accrued coupon at default ($Accr_t^{def}$) is detracted from $V_t^{Prot,dirty}$ to obtain the clean protection leg value $V_t^{Prot,clean} = V_t^{Prot,dirty} - Accr_t^{def}$ where

$$\begin{aligned} Accr_t^{def} &= \mathbb{E}\left[B_t(\tau)\mathbb{1}_{\{\tau\le T\}}\frac{C(\tau-\max(t_c : t_c\le\tau))}{360}\middle|\mathcal{F}_t\right] \\ &= \mathbb{E}\left[B_t(\tau^*)p^*(T)\frac{C(\tau-\max(t_c : t_c\le\tau^*))}{360}\middle|\mathcal{F}_t\right] \end{aligned} \quad (5.2)$$

where C denotes the fixed coupon and t_c the coupon dates. Similarly, the present value of the dirty premium leg is:

$$\begin{aligned} V_t^{Prem,dirty} &= \mathbb{E}\left[\sum_{t_c>t} B_t(t_c)C\mathbb{1}_{\{\tau\ge t_c\}}\middle|\mathcal{F}_t\right] \\ &= \mathbb{E}\left[\sum_{t_c>t} B_t(t_c)C(1-p^*(t_c))\middle|\mathcal{F}_t\right]. \end{aligned} \quad (5.3)$$

Thus $V_t^{Prem,clean} = V_t^{Prem,dirty} - Accr_t^{init}$ where $Accr_t^{init} = C\frac{(t-\max(t_c:t_c\le t))}{360}$ and then $CDS_t = V_t^{Prot,clean} - V_t^{Prem,clean}$ is the upfront (in bps) premium.

Standard practice assumes that default occurs half-way between coupon payments, see OKane and Turnbull (2003). Approximating (5.1)–(5.3) by discretisation of the time line at the coupon payment nodes t_c gives:

$$\begin{aligned} V_t^{Prot,dirty} &\approx (1-\bar{R})\sum_{t_c>t} B_t\left(\frac{\max(t_{c-1},t)}{2}\right)\mathbb{E}_{H_{t_c}} \\ &\quad\times[\mathbb{P}^*(\tau\in(\max(t_{c-1},t),t_c])] \\ Accr_t^{def} &\approx \sum_{t_c>t} B_t\left(\frac{\max(t_{c-1},t)}{2}\right)\frac{C[t_c-\max(t_{c-1},t)]}{2*360}\mathbb{E}_{H_{t_c}} \\ &\quad\times[\mathbb{P}^*(\tau\in(\max(t_{c-1},t),t_c])] \\ V_t^{Prem,dirty} &\approx C\sum_{t_c>} B_t(t_c)\frac{t_c-t_{c-1}}{360}\mathbb{E}_{H_{t_c}}[\mathbb{P}^*(\tau>t_c)] \end{aligned}$$

where $\mathbb{P}^*(\tau > t_c) = 1 - p^*(t_c)$ and $\mathbb{P}^*(\tau\in(t_{c-1},t_c]) = p^*(t_c) - p^*(t_{c-1})$ that can be easily computed by the formulae introduced in Section 4. The notation $\mathbb{E}_{H_{t_c}}[\cdot]$ shows calculation based on the density of the factor H_{t_c}.

In Fig. 4 we present the resulting default probability curve and the associated spreads term structure for various sets of parameters. The model can generate various curve shapes, from the normal CDS curve (graphs 4(a)–(d)) in which the spreads increase with the time to maturity to inverted CDS curves (graph 4(f)) characterised by lower spreads for higher maturities.

[9] We assume independence between the interest rate and default or recovery rate. This can be relaxed.

5.2. Credit default swaps index

If we denote by $H_t = \sum_{j=1}^K n_j\mathbb{1}_{\{\tau_j>t\}}$ the unit notional at time t, by n_j the percentage exposure[10] to the name j, by R_j the respective recovery rate and by $B_t(s)$ the discount factor, then the discounted cash-flows corresponding to *Protection Leg* payments can be written as:

$$\begin{aligned} W_t^{Prot,dirty} &= \mathbb{E}\left[\sum_{j=1}^K n_jB_t(\tau_j)(1-R_j)\mathbb{1}_{\{\tau_j\le T\}}\middle|\mathcal{F}_t\right] \\ &= \mathbb{E}\left[\mathbb{E}\left[\sum_{j=1}^K n_jB_t(\tau_j)(1-R_j)\mathbb{1}_{\{\tau_j\le T\}}\middle|\mathcal{H}_T\right]\middle|\mathcal{F}_t\right] \\ &= \mathbb{E}\left[\sum_{j=1}^K n_jB_t(\tau_j^*)(1-\bar{R}_j)p_j^*(T)\middle|\mathcal{F}_t\right]. \end{aligned} \quad (5.4)$$

Again there may be a possible accrued coupon at the time of default. Denoting by C_{idx} the fixed coupon we have:

$$\begin{aligned} A_t^{def} &= \mathbb{E}\left[\sum_{j=1}^K n_jB_t(\tau_i)\mathbb{1}_{\{\tau_j\le T\}}\frac{C_{idx}(\tau-\max(t_c : t_c\le\tau))}{360}\middle|\mathcal{F}_t\right] \\ &= \mathbb{E}\left[\sum_{j=1}^K n_jB_t(\tau^*)p_j^*(T)\frac{C_{idx}(\tau-\max(t_c : t_c\le\tau))}{360}\middle|\mathcal{F}_t\right]. \end{aligned} \quad (5.5)$$

The present value of the cash-flows corresponding to the *Premium Leg* is:

$$\begin{aligned} W_t^{Prem,dirty} &= \mathbb{E}\left[\sum_{t_c>t} B_t(t_c)C_{idx}\sum_{j=1}^K n_j\mathbb{1}_{\{\tau_j>t_c\}}\middle|\mathcal{F}_t\right] \\ &= \mathbb{E}\left[\sum_{t_c>t} B_t(t_c)C_{idx}\sum_{j=1}^K n_j(1-p^*(t_c))\middle|\mathcal{F}_t\right]. \end{aligned} \quad (5.6)$$

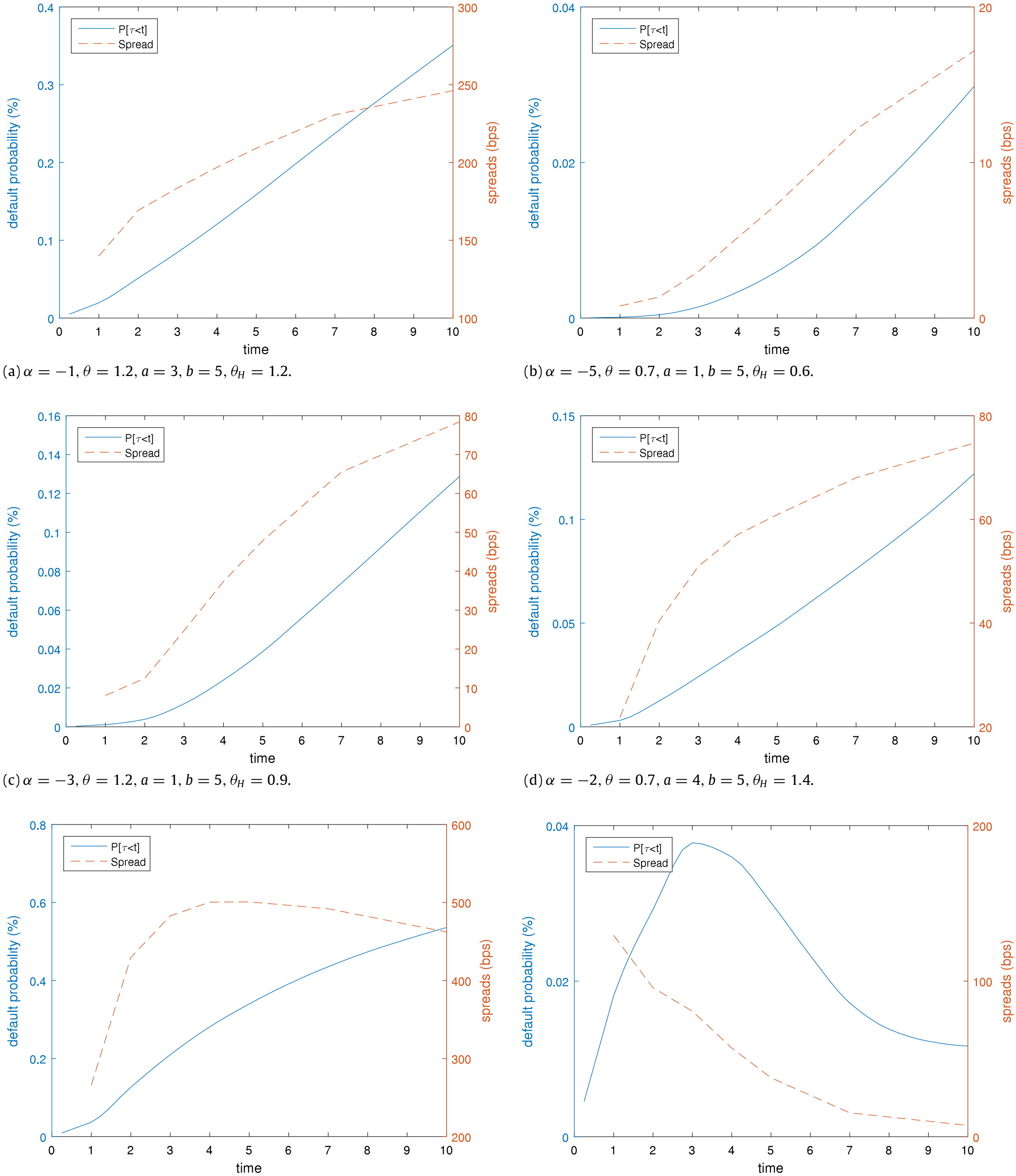
An initial correction corresponding to the accrued coupon up to the trade date is computed similar as for the CDS contract $A_t^{init} = C_{idx}\frac{(t-\max(t_c:t_c\le t))}{360}$ which leads to the index swap price $CDSI_t = W_t^{Prot,clean} - W_t^{Prem,clean}$. With the assumption that defaults occur half-way between coupon payments:

$$\begin{aligned} W_t^{Prot,dirty} &\approx \sum_{t_c>t} B_t\left(\frac{\max(t_{c-1},t)+t_c}{2}\right)\sum_{j=1}^K n_j(1-\bar{R}_j)\mathbb{E}_{H_{t_c}} \\ &\quad\times[\mathbb{P}^*(\tau_j\in(\max(t_{c-1},t),t_c])] \\ A_t^{def} &\approx \sum_{t_c>t} B_t\left(\frac{\max(t_{c-1},t)+t_c}{2}\right)\frac{C_{idx}[t_c-\max(t_{c-1},t)]}{2*360} \\ &\quad\times\sum_{j=1}^K n_j\mathbb{E}_{H_{t_c}}[\mathbb{P}^*(\tau_j\in(\max(t_{c-1},t),t_c])] \\ W_t^{Prem,dirty} &\approx C_{idx}\sum_{t_c>t} B_t(t_c)\frac{t_c-t_{c-1}}{360}\sum_{j=1}^K n_j\mathbb{E}_{H_{t_c}}[\mathbb{P}^*(\tau_j>t_c)] \end{aligned}$$

where $\mathbb{P}^*(\tau > t_c) = 1 - p^*(t_c)$ and $\mathbb{P}^*(\tau\in(t_{c-1},t_c]) = p^*(t_c) - p^*(t_{c-1})$ are computed as for the CDS pricing.

In Fig. 5 we present an analysis of the parameter's impact on the shape of the spread curve for the CDSI of iTraxx Europe index. We fix the univariate parameters and focus on the multivariate parameters with impact on the CDSI price. Similar to Eckner (2009)

[10] Usually computed as $1/K$. For iTraxx $n_j = 0.08$.



(a) $\alpha = -1$, $\theta = 1.2$, $a = 3$, $b = 5$, $\theta_H = 1.2$.

(b) $\alpha = -5$, $\theta = 0.7$, $a = 1$, $b = 5$, $\theta_H = 0.6$.

(c) $\alpha = -3$, $\theta = 1.2$, $a = 1$, $b = 5$, $\theta_H = 0.9$.

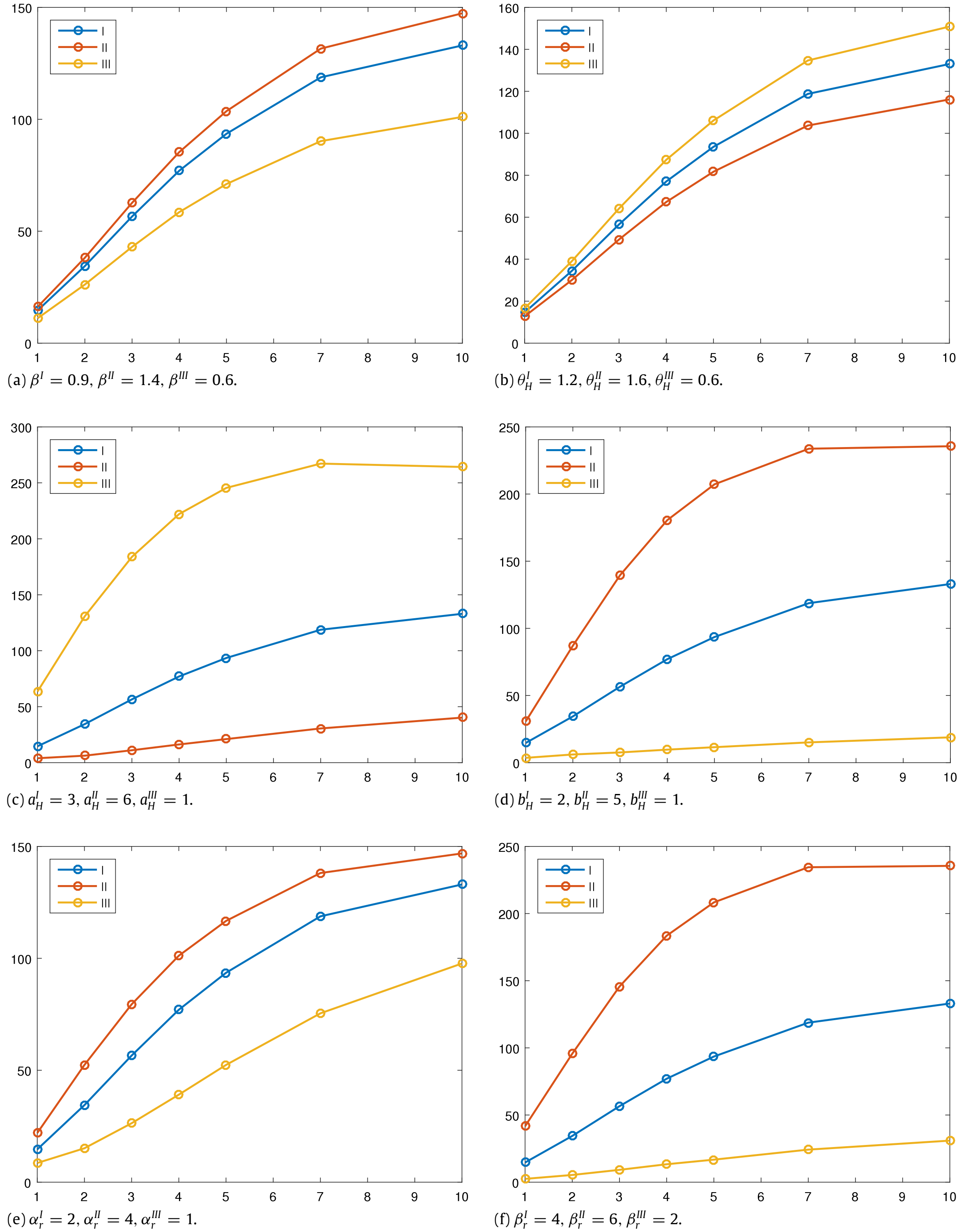
(d) $\alpha = -2$, $\theta = 0.7$, $a = 4$, $b = 5$, $\theta_H = 1.4$.

(e) $\alpha = -1$, $\theta = 1.2$, $a = 1$, $b = 5$, $\theta_H = 1.2$.

(f) $\alpha = -1$, $\theta = 0.2$, $a = 2$, $b = 2$, $\theta_H = 0.4$.

Fig. 4. Term structure of default probabilities and spreads. *Note*: The graphs above are generated under Gamma specification both the idiosyncratic and common factor. The parameters that varies have their values specified under each graph while the rest of the parameters are kept constant ($\beta = 0.7$, $a_H = 2$, $b_H = 3$, $H_0 = G_0 = \Sigma_0^G = \Sigma_0^H = 0.02$, $R = 0.4$). The default probabilities are computed for every quarter maturity starting with $T = 0.25$ until $T = 10$ while the spreads are computed for $T = 1, 3, 5, 7, 10$. The variety of term structure shapes possible shows the flexibility of the model.

we used a common level α (estimated in our example at -5.5) which we multiply by the weight ω_c computed as the ratio of the average 5y spreads over the entire portfolio and the 5y spread of the company ($\alpha_c = -5.5\omega_c$). The weight ω_c represents the relative riskiness of the company when compared with the rest of the names in the index. The parameter β controls the dependence between the default probabilities for the obligors in the index. An increase in β produces a rise in the spreads for the CDSI contract



(a) $\beta^{I} = 0.9$, $\beta^{II} = 1.4$, $\beta^{III} = 0.6$.

(b) $\theta_H^{I} = 1.2$, $\theta_H^{II} = 1.6$, $\theta_H^{III} = 0.6$.

(c) $a_H^{I} = 3$, $a_H^{II} = 6$, $a_H^{III} = 1$.

(d) $b_H^{I} = 2$, $b_H^{II} = 5$, $b_H^{III} = 1$.

(e) $\alpha_r^{I} = 2$, $\alpha_r^{II} = 4$, $\alpha_r^{III} = 1$.

(f) $\beta_r^{I} = 4$, $\beta_r^{II} = 6$, $\beta_r^{III} = 2$.

Fig. 5. Parameter sensitivity of the CDSI curve. *Note*: Term structure of CDSI spreads at the most important maturities for the iTraxx index with various parameter sets. The model used is IG-OU for both idiosyncratic and common factor. The initial parameters $\alpha_c = -5.5\omega_c$, $\theta = 1.2$, $a = 2$, $b = 3$, $\beta = \beta_s = 0.9$, $\theta_H = 1.2$, $a_H = 3$, $b_H = 2$, $\alpha_r = 2$, $\beta_r = 4$, $H_0 = G_0 = \Sigma_0^G = \Sigma_0^H = 0.02$ are kept constant in all the graphs (model I) while on e of the parameters that affect the entire index are changed one at a time with values specified under each graph (model II and III).

(see Fig. 5(a)). This effect is usually more pronounced at longer maturities since for longer maturities there is time for the common factor to jump and create multiple defaults.

An opposite effect is observed in Fig. 5(b) for the parameter θ_H. This parameter controls the speed of the mean reversion of the common factor. One expects that for higher values of θ_H the variance of the common factor will reverse faster to the long-run mean value, reducing the impact of the jumps in the common factor on the probability of default of the names in the portfolio. The same conclusion can be drawn for the parameter a_H which controls the shape of the increments of the BDLP driving the variance of the common factor. Lower values of this parameter lead to a distribution of increments which is peaked closer to the origin implying smaller jump sizes of the common factor, and consequently a smaller importance of the common factor. This effect explains the lower spreads observed in Fig. 5(c). On the contrary, the effect of an increase in b_H, the scale parameter of the distribution of common factor's jumps, is to increase the level of spreads of the CDSI contract, as illustrated in Fig. 5(d). An explanation is that increasing b_H implies a distribution for jumps of the common factor centred at higher values, creating more possibilities of default in the index portfolio.

The last two analysed parameters control the distribution of the recovery rate. Here we used a discretisation of a $Beta(\alpha_r, \beta_r)$ distribution to model the random recovery rate, assumed to be the same for all the names in the portfolio. Since the mean of a Beta distribution is $\frac{\alpha_r}{\alpha_r+\beta_r}$ the expected recovery rate is positively related to α_r and negatively related to β_r. There is a negative linkage between the spreads of a CDSI contract and the expected recovery rate $\bar{R}$ so an increase in α_r leads to lower spreads, as in Fig. 5(e), while an increase in β_r implies higher spreads as in Fig. 5(f). Similar observations can be made in the case of the IG-OU specification.

5.3. CDO tranches

A CDSI tranche is an option contract on the index, with the attachment A_l and detachment points B_l of a tranche l representing the interval of the cumulative loss process $\{L_t\}_{t\geq 0}$ to which the investors in that specific tranche are exposed. For pricing CDO tranches we consider the cumulative loss $L_s = \sum_{j=1}^{K} n_j(1-R_j)\mathbb{1}_{\{\tau_j\leq s\}}$. Then the notional of a tranche l can be written as $N_s^{tr(l)} = f_l(L_s) = (B_l - A_l) - (L_s - A_l)\mathbb{1}_{\{L_s>A_l\}} + (L_s - B_l)\mathbb{1}_{\{L_s>B_l\}}$. The Premium Leg is

$$Tr_{t,l}^{Prem,dirty} = \mathbb{E}\left[\sum_{t_c>t} B_t(t_c)C_{tr}^l N_{t_c}^{tr(l)}\middle|\mathcal{F}_t\right]$$
$$= \sum_{t_c>t} B_t(t_c)C_{tr}^l \mathbb{E}[N_{t_c}^{tr(l)}|\mathcal{F}_t] \quad (5.7)$$

and applying the market correction $A_t^{init,l} = C_{tr}^l \frac{(t-\max(t_c:t_c\leq t))}{360}$ leads to the clean Premium price $Tr_{t,l}^{Prem,clean} = Tr_{t,l}^{Prem,dirty} - A_t^{init,l}$. With defaults halfway between coupon payments, the value of the *Protection Leg* and the accrued coupon at the time of default are:

$$Tr_{t,l}^{Prot,dirty} \approx \sum_{t_c>t} B_t\left(\frac{\max(t_{c-1},t)+t_c}{2}\right)\mathbb{E}[N_{t_c}^{tr(l)} - N_{\max(t_{c-1}t)}^{tr(l)}]$$

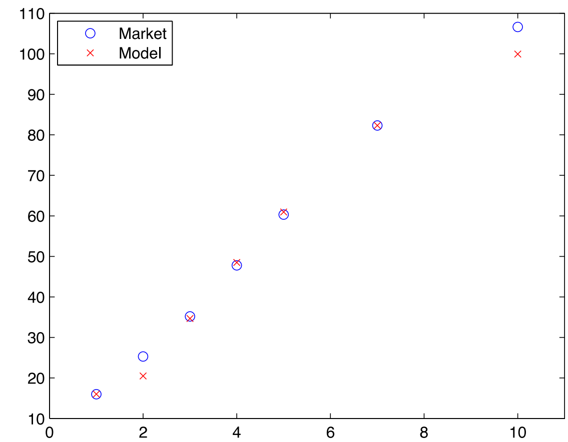
$$A_t^{def} \approx \sum_{t_c>t} B_t\left(\frac{\max(t_{c-1},t)+t_c}{2}\right) \times \frac{C_{tr}^l[t_c-\max(t_{c-1},t)]}{2*360}\mathbb{E}[N_{t_c}^{tr(l)} - N_{\max(t_{c-1},t)}^{tr(l)}].$$

The applicability of the formulae above depend on computing the values $\mathbb{E}[N_{t_c}^{tr(l)} - N_{t_{c-1}}^{tr(l)}] = \mathbb{E}[N_{t_c}^{tr(l)}] - \mathbb{E}[N_{t_{c-1}}^{tr(l)}]$ and $\mathbb{E}[N_{t_c}^{tr(l)}]$. Since $N_s^{tr(l)} = f_l(L_s)$ is a function of L_s for which we showed in Section 3.3 how to obtain its distribution, the computation of $\mathbb{E}[N_{t_c}^{tr(l)}]$ is determined by the integral $\int_{R^+} f_l(L_s)g_{L_s}(x)dx$.

6. Calibration methodology

The calibration of the model will use the information encoded in the term structure of CDS, CDSI and CDO tranches spreads. This requires the computation of the conditional probabilities of default, the conditional distribution of portfolio losses and the factor density for all the payment dates of the coupons.[11] For reducing the computational burden we follow Eckner (2009) and (Mortensen, 2006) and compute the above value for intervals of one year length and use a cubic spline to interpolate for values at coupon dates in-between.

[11] For a 10 year maturity there are 40 payment dates for which these calculations need to be performed.



RRMSE	0.0761	α_c	-2.0685
APE	0.0358	β_c	0.6860
ARPE	0.0419	θ_c	1.0166
RMSE	3.1308	a_c	2.6900
n.a	n.a	b_c	5.8324

Fig. 6. Calibration of LafargeHolcim Ltd CDS term structure. *Note*: The Gamma-OU model is calibrated on LafargeHolcim Ltd CDS term structure data available in Bloomberg for the date 18 Feb. 2015. The parameters of the common factor are fixed ($\theta_H = 0.6$, $a_H = 2$, $b_H = 3$, $\bar{R} = 0.4$) while the free parameters ($\alpha_c, \beta_c, \theta_c, a_c, b_c$) are implied from the calibration with the goodness-of-fit measured by RRMSE, APE, ARPE and RMSE.

6.1. Univariate calibration

The calibration of the default probability model for a specific name is based on CDS spreads or defaultable bond prices available on the market. The input observations can be either the market CDS rates or the implied probabilities of default bootstrapped from these spreads, see OKane and Turnbull (2003) for the bootstrapping procedure. In order to find the set of parameters Γ_c one could minimise the average relative percentage error:

$$ARPE = \frac{1}{M}\sum_{n=1}^{M}\frac{|S_{market}^n - S_{model}^n(\Gamma)|}{S_{market}^n} \quad (6.1)$$

where M is the number of available market spreads/upfronts. Other measures of goodness-of-fit that are routinely used, see (Schoutens et al., 2004), include RRMSE, APE and RMSE.

The example described here is for LafargeHolcim Ltd on the date 18 Feb 2015, with CDS market prices downloaded from Bloomberg. For the interest rate we used the swap curve data available in Bloomberg which consists of Euribor rates for maturities under one year and rates striped form interest rate swap prices for longer maturities.

Since we are interested in the performance of the univariate default probability model, we fix the parameters of the common factor to $\theta_H = 0.6$, $a_H = 2$, $b_H = 3$, $\bar{R} = 0.4$ while letting the other parameters free ($\alpha_c, \beta_c, \theta_c, a_c, b_c$). The results of the univariate calibration presented in Fig. 6 indicate that the model fits well the data.

Table 1
Parameters from the multivariate calibration. *Note*: The parameters resulted from the calibration of the multivariate model on the 22nd series of the iTraxx index CDSI term structure data available in Bloomberg for the date 18 Feb. 2015. The model used for the factors is Gamma-OU.

α	β_{AI}	β_C	β_E	β_{TMT}	β_F	α_r
−1.5337	0.7573	0.6675	1.1043	0.6693	1.2044	2.2394
θ	a	b	θ_H	a_H	b_H	
2.9999	2.8597	0.4301	0.2081	2.1495	0.8116	

6.2. Multivariate calibration

The multivariate calibration of the model requires the joint calibration of CDS, CDSI and index tranches spreads. The market data used for calibration purpose regards the trading date 18 Feb. 2015 and was downloaded from Bloomberg. It corresponds to the CDS spreads for maturities $(3y, 5y, 7y, 10y)$ for each of the 125 components of the iTraxx Series 22 index, the CDSI prices on the iTraxx Series 22 index for maturities $(3y, 5y, 7y, 10y)$ and the spreads for the 0%–3%, 3%–6%, 6%–12% and 12%–100% tranches on the index for maturities $(3y, 5y)$.[12] The total error to minimise is the (weighted) sum of the errors on CDS spreads, CDSI spreads and tranches spreads:

$$\min_{\Gamma}\left\{ARPE_{CDS}(\Gamma) + 5ARPE_{CDSI}(\Gamma) + 5ARPE_{tr}(\Gamma)\right\} \tag{6.2}$$

where

$$ARPE_{CDS}(\Gamma) = \frac{1}{N}\frac{1}{M}\sum_{n=1}^{N}\sum_{m=1}^{M}\frac{|S^{n,m}_{market,CDS} - S^{n,m}_{model,CDS}(\Gamma)|}{S^{n,m}_{market,CDS}} \tag{6.3}$$

$$ARPE_{CDSI}(\Gamma) = \frac{1}{M}\sum_{m=1}^{M}\frac{|S^{m}_{market,CDSI} - S^{m}_{model,CDSI}(\Gamma)|}{S^{n}_{market,CDSI}} \tag{6.4}$$

$$ARPE_{tr}(\Gamma) = \frac{1}{P}\frac{1}{M}\sum_{p=1}^{P}\sum_{m=1}^{M}\frac{|S^{m,p}_{market,tr} - S^{m,p}_{model,tr}(\Gamma)|}{S^{m,p}_{market,tr}} \tag{6.5}$$

for $n = 1, 2, 3, \ldots, N = 125$ names in the index portfolio, $m = 1, 2, \ldots, M$ maturities and $p = 1, 2, \ldots, P$ tranches available.

The general model employed here has $(N \times 5) + 5$ parameters, five univariate parameters corresponding to each obligor in the credit portfolio of N names and five common parameters. For the purposes of having a more parsimonious model we adopt a series of simplifying assumptions. The first assumption is that the individual barrier level α_c can be expressed as $\alpha_c = \alpha\omega_c$ for a general level α and a weight $\omega_c = \frac{CDS^c_5}{avgCDS_5}$ obtained as the quotient between the $5y$ CDS level of individual names and the average $5y$ CDS for the entire portfolio of credit names. The second assumption is that each obligor in a specific sector responds in the same way to the shocks from the market factor.

As a result we reduced the number of β_c parameters from $N = 125$ to only 5 beta values $\beta_{AI}, \beta_C, \beta_E, \beta_{\beta_{AI}}, \beta_F$ corresponding to the five sectors division of the names in the iTraxx index (Autos & Industrials, Consumers, Energy, Technology & Media & Telecommunications and Financials). The last assumption is that the parameters θ, a and b which control the distribution of the company specific factor are common to all the companies in the index. The results of the multivariate calibration are presented in Table 1. As can be seen from the measure of goodness of fit presented in Fig. 7, the multivariate model fits well the market data.

[12] Some of the tranches are quoted in upfront points and have been transformed to spreads by using the Bloomberg calculator available in the CDSW screen.

7. Conclusions

In this paper we developed a factor structure of the time-changes driving the impact of the information arrival on the credit worthiness process. Moreover we proposed a new FFT based general methodology for the computation of the probability of default which allows the extension of the univariate model to specifications not possible under the Panjer recursion technology used in recent credit risk literature.

Under our framework it is feasible to price single-name CDS, CDSI and index tranches contracts or CDOs. A useful and interesting characteristic of our proposed multivariate model, that stems from the use of mean reverting models for the variance of the common factor, is the ability of producing "contagion-like" effects observed in the market.

This paper described the set-up of the credit modelling framework based on Brownian time-changed processes with volatility belonging to the class of mean reverting Lévy driven OU type process. Within this framework flexible formulae were derived for CDS prices, credit index prices and CDO/tranche prices. The next challenge would be to link the drivers of this models to sectoral and macroeconomic effects such as described by Chava et al. (2011).

Appendix A. Proof Proposition 2

Because Υ has a chi-squared distribution with one degree of freedom so $\Upsilon \sim \Gamma(1/2, 2)$ and $k^{-2}\Upsilon \sim \Gamma(1/2, 2a^{-2})$. Then $\frac{1}{k^{-2}\Upsilon}$ has an *inverted Gamma* (or *inverted chi-squared*) distribution with density:

$$f(1/x) = \frac{k^n}{2^{n/2}\Gamma(n/2)}x^{(n+2)/2}e^{-\frac{xk^2}{2}} \tag{A.1}$$

and the characteristic function as given in Witkovský (2001),with $n = 1$ in our case, where $\Gamma(x)$ is the Gamma function and $K_\alpha(x)$ is the Bessel function of the second kind:

$$\phi_{i\Gamma}(u) = \frac{2(-2iuk^{-2})^{n/4}K_{n/2}[k^2(-2iuk^{-2})^{1/2}]}{(2k^{-2})^{n/2}\Gamma(n/2)}. \tag{A.2}$$

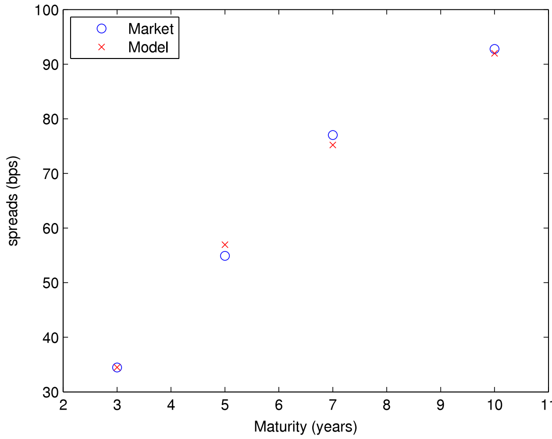
Taking the square of a standard normal variable by $\Upsilon = Z^2$ and denoting $\upsilon = \frac{k^2}{\gamma - c}$ for constants $k < 0, \gamma > 0, c >$ given that $\gamma - c > 0$, we can write:

$$\begin{aligned}\mathbb{P}\left(\Upsilon \le \upsilon\right) &= \mathbb{P}\left(Z^2 \le \frac{k^2}{\gamma - c}\right) = \mathbb{P}\left(|y| \le \sqrt{z}\right)\\ &= \mathbb{P}\left(Z \le \frac{|k|}{\sqrt{\gamma - c}}\right) - \left[1 - \mathbb{P}\left(Z \le \frac{|k|}{\sqrt{\gamma - c}}\right)\right]\\ &= 2\mathbb{P}\left(Z \le \frac{|k|}{\sqrt{\gamma - c}}\right) - 1.\end{aligned} \tag{A.3}$$

Standard probability calculus gives $\mathbb{P}\left(Z \le \frac{-k}{\sqrt{\gamma - c}}\right) = 1 - \mathbb{P}\left(Z \le \frac{k}{\sqrt{\gamma - c}}\right)$ so $2\mathbb{P}\left(Z \le \frac{a}{\sqrt{\gamma - c}}\right) = 1 - \mathbb{P}\left(\Upsilon \le \frac{k^2}{\gamma - c}\right) = \mathbb{P}\left(\frac{1}{k^{-2}\Upsilon} \le \gamma - c\right)$.

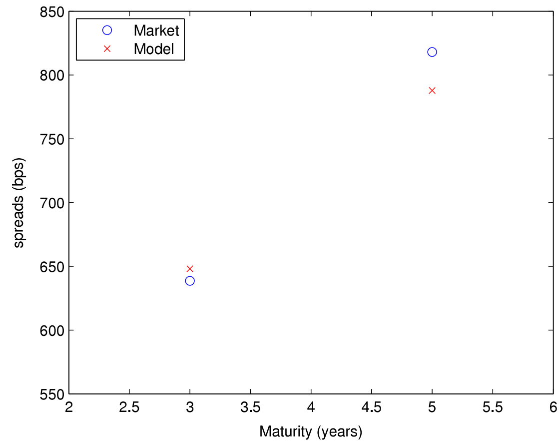
Therefore we can write the default probability (3.1) as

$$\begin{aligned}\mathbb{P}(\tau < s|\mathcal{F}_t) &= \mathbb{E}[2\mathbb{P}(Z\sqrt{\Gamma_s - \Gamma_t} - (b - X_t) \le 0|\Gamma_s)|\mathcal{F}_t]\\ &= \mathbb{E}\left[2\mathbb{P}\left(Z \le \frac{(b - X_t)}{\sqrt{\Gamma_s - \Gamma_t}}\middle|\Gamma_s\right)\middle|\mathcal{F}_t\right]\\ &= \mathbb{E}\left[\mathbb{P}\left(\frac{1}{(b - X_t)^{-2}\Upsilon} \le (\Gamma_s - \Gamma_t)\middle|\Gamma_s\right)\middle|\mathcal{F}_t\right]\end{aligned}$$



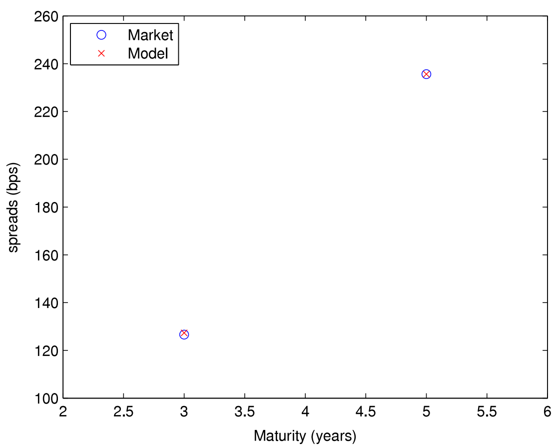
RRMSE	APE	ARPE	RMSE
0.0141	0.0098	0.0096	1.0401

(a) CDSI.



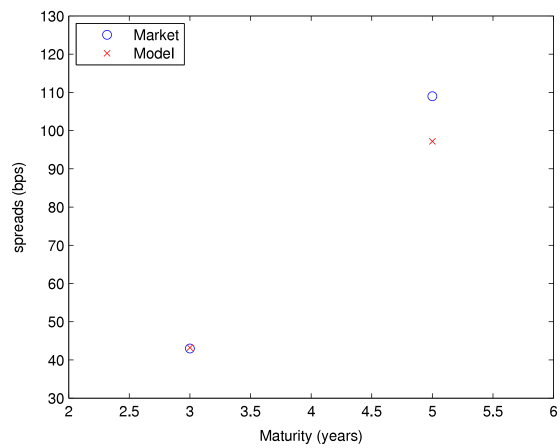
RRMSE	APE	ARPE	RMSE
0.0282	0.0273	0.0259	22.3948

(b) 0%–3% tranche.



RRMSE	APE	ARPE	RMSE
0.0040	0.0005	0.0029	0.5018

(c) 3%–6% tranche.



RRMSE	APE	ARPE	RMSE
0.0766	0.0082	0.0567	8.3408

(d) 6%–12% tranche.

Fig. 7. Multivariate calibration results. *Note*: The multivariate model is calibrated on CDSI and index tranches data on iTraxx index available from Bloomberg for the date 18 Feb. 2015. The model used for the factors is Gamma-OU. The parameters from the calibration are presented in Table 1. Four measures of goodness of fit (RRMSE, APE, ARPE and RMSE) are also displayed in the tables below each graph.

$$= \mathbb{E}\left[\mathbb{P}\left(\frac{1}{(b-X_t)^{-2}\Upsilon} - (\Gamma_s - \Gamma_t) \le 0 \middle| \Gamma_s\right)\middle| \mathcal{F}_t\right]. \tag{A.4}$$

The variable inside (A.4) is the difference of two independent variables: the first is an inverted Gamma with characteristic function given by (A.2) and the second is just the distribution of the integrated variance for which the characteristic function of various specifications will be discussed in Section 4. Now the conditional probability of default given the factor H_s is:

$$\begin{aligned}
&\mathbb{P}(\tau < s|\mathcal{H}_s \vee \mathcal{F}_t) \\
&= \mathbb{E}\left[\mathbb{P}\left(\frac{1}{(b-X_t)^{-2}\Upsilon} - (G_s - G_t) - \beta(H_s - H_t) \le 0\right)\middle| \mathcal{H}_s \vee \mathcal{F}_t\right] \\
&= \mathbb{E}[\mathbb{P}(Q \le 0)|\mathcal{H}_s \vee \mathcal{F}_t] \qquad \text{(A.5)}
\end{aligned}$$

where $Q = \frac{1}{(b-X_t)^{-2}\Upsilon} - (G_s - G_t) - \beta(H_s - H_t)$.

Appendix B. Proof Proposition 3

The formulae follow by just writing the characteristic function of $Q = \frac{1}{(b-X_t)^{-2}\Upsilon} - (G_s - G_t) - \beta(H_s - H_t)$ as function of the characteristic functions of the inverse Gaussian variable $\frac{1}{(b-X_t)^{-2}\Upsilon}$ and of the variables G_s and H_s:

$$\begin{aligned}
\mathbb{P}^{Gill}[\tau < s|\mathcal{H}_s \vee \mathcal{F}_t] &= \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \Re\left(\frac{\phi_{Q|H}(u)}{iu}\right)du \\
&= \frac{1}{2} - \frac{1}{\pi}\Re\left(\int_0^\infty e^{-iu(\beta H_s)}\frac{\phi_{i\Gamma}(u)\phi_{G_s}(-u)e^{iu(\beta H_t + G_t)}}{iu}du\right) \qquad \text{(B.1)} \\
\mathbb{P}^{Kim}[\tau < s|\mathcal{H}_s \vee \mathcal{F}_t] &= \frac{1}{\pi}\int_0^\infty \Re\left(\frac{\phi_{Q|H}(u+i\rho)}{\rho - iu}\right)du
\end{aligned}$$

$$= \frac{1}{\pi}\Re\Bigg(\int_0^\infty e^{-iu(\beta H_s)} \times \frac{\phi_{i\Gamma}(u+i\rho)\phi_{G_s}(-u-i\rho)e^{iu(G_t+\beta H_t)+\rho(\beta(H_t-H_s)+G_t)}}{\rho - iu}du\Bigg) \quad (B.2)$$

$$\mathbb{P}^{Hilb}[\tau < s|\mathcal{H}_s \vee \mathcal{F}_t] = \frac{1}{2} - \frac{i}{2}\mathbb{H}[\phi_Q(0)](0)$$

$$= \frac{1}{2} - \frac{i}{2}\mathbb{H}[\phi_{i\Gamma}(u)\phi_{G_s}(-u)e^{-iu(G_t+\beta H_t-\beta H_s)}](0)$$

$$\approx \frac{1}{2} - \frac{i}{2}\sum_{m=-M,m\neq 0}^{M}\phi_{i\Gamma}(mh)\phi_{G_s}(-mh)e^{imh(G_t+\beta H_t-\beta H_s)} \times \frac{1-\cos(-\pi m)}{-\pi m}$$

$$= \frac{1}{2} - \frac{i}{2}\sum_{m=-M,m\neq 0}^{M}\phi_{i\Gamma}(mh)\phi_{G_s}(-mh)e^{imh(G_t+\beta H_t-\beta H_s)} \times \frac{1-(-1)^{-m}}{-\pi m}$$

$$= \frac{1}{2} - \frac{i}{2}\sum_{m=0}^{M-1}e^{-2imh(\beta H_s)} \times \frac{2\phi_{i\Gamma}((2m+1-M)h)\phi_{G_s}((M-2m-1)h)e^{ih\Lambda}}{\pi(M-2m-1)} \quad (B.3)$$

where we used the notation $\Lambda = (2m+1-M)((G_t+\beta H_t) - \beta H_s(1-M))$.

Appendix C. Size of CDO Market and Parameter Sensitivity Analysis

Supplementary material related to this article can be found online at http://dx.doi.org/10.1016/j.insmatheco.2016.10.004.

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