Adaptive Robust Fault-Tolerant Control for Linear MIMO Systems with Unmatched Uncertainties

Kangkang Zhang[1,2], Bin Jiang[1,2,*], Xing-Gang Yan[3], Zehui Mao[1,2]

Abstract

In this paper, two novel fault-tolerant control design approaches are proposed for linear MIMO systems with actuator additive faults, multiplicative faults and unmatched uncertainties. For time-varying multiplicative and additive faults, new adaptive laws and additive compensation functions are proposed. A set of conditions is developed such that the unmatched uncertainties are compensated by actuators in control. On the other hand, for unmatched uncertainties with their projection in unmatched space being not zero, based on a (vector) relative degree condition, additive functions are designed to compensate for the uncertainties from output channels in presence of actuator faults. The developed fault-tolerant control schemes are applied to two aircraft systems to demonstrate the efficiency of the proposed approaches.

Index Terms

Fault-tolerant control, adaptive and robust control, loss of effectiveness faults, stuck faults, unmatched uncertainties.

I. Introduction

Modern control systems have become more complex in order to meet the increasing requirements of system performances. Control engineers are faced with increasingly complex systems, for which both reliability and safety are very important. However, system faults, such

[1]College of Automation Engineering, Nanjing University of Aeronautics and Astronautics, Nanjing210016, China, (e-mail: KangZhang359@163.com; binjiang@nuaa.edu.cn; zehuimao@nuaa.edu.cn).

[2]Jiangsu Key Laboratory of Internet of Things and Control Technologies (Nanjing Univ. of Aeronautics and Astronautics).

[3]School of Engineering and Digital Arts, University of Kent, Canterbury, Kent CT2 7NT, United Kingdom, (e-mail: x.yan@kent.ac.uk).

[*] Corresponding author. Email: binjiang@nuaa.edu.cn

as actuator faults, sensor faults, structural damages and uncertainties may induce drastically changes of system dynamics, and result in undesirable performance degradation, even instability. To overcome such a weakness, robust fault-tolerant controls (FTCs) for uncertain systems have been developed to tolerate component malfunctions while maintaining desirable stability and system performances. This is particularly important for safety and actuate critical systems, such as aircrafts, spacecrafts, nuclear power plants, chemical plants processing hazardous materials and high-speed railways.

It should be pointed out that some robust control methods can be applied to FTC design. However, FTC is different from robust control. Generally speaking, FTC can be separated into two types: AFTC (active FTC) and PFTC (passive FTC) [1]. In PFTC systems, controller structures are fixed and designed to be against presumed faults, which need neither fault diagnosis schemes nor reconfiguration controllers [1]. Therefore, PFTC can also be considered as a special robust control. This paper focuses on FTC by design adaptive actuator faults compensation schemes, taking into account types and features of actuator faults, which belongs to PFTC.

Unmatched uncertainties are inevitable in practical control systems, and have being widely studied in recent years. Typically, adaptive and robust controllers are powerful to stabilize uncertain systems, and several design procedures for systems with matched and unmatched uncertainties have been proposed in [2], [3], [4], [5], [6], [7], [8]. These uncertain systems may experience faults which may further result in performance degradation. FTCs for systems with uncertainties not only eliminate the effect of faulty actuators, but also reject the effect of uncertainties on the systems, which are full of challenges. Therefore, it is significant to study FTC for systems with uncertainties, especially, unmatched uncertainties. Last decades, great achievement has been made in this area, and most of them belong to the following categories: adaptive control [9], [10], [11], [12], multiple-model control [13], integrated diagnosis and control [14], [15], [16], [17], sliding mode variable structure control [18], [19], and robust $\mathcal{H}_\infty$ control [20], [21].

In much existing literature for uncertain systems with additive and multiplicative faults, the FTCs are designed using compensation method through reconstructing actuators in control [15], [16], [22], [23], [24]. The key technologies are to develop conditions under which the left actuators in control can tolerate faults and compensate for unmatched uncertainties, moreover, to construct corresponding control functions. An on-line multiplicative fault estimation module is provided in [9], and a FTC structure is proposed such that the optimal robustness to $\mathcal{L}_2$ disturbances is still maintained in presence of faults. In reference [21], a reliable control system is designed based on new proposed adaptive $\mathcal{H}_\infty$ performance index. An actuator redundancy condition is derived in [24], and a direct adaptive control law, aiming at compensating for actuator faults, is proposed. Nevertheless, [9], [21] do not consider the worst case when stuck faults occur on some actuators, [24] requires that stuck faults can be parameterized linearly, and all of the above three papers do not consider time-varying multiplicative faults and system uncertainties. The FTC design for systems with matched uncertainties is studied in [10], and for systems with only specific unmatched uncertainties is considered in [12], both of which motivate FTC design for systems with more general uncertainties.

In this paper, firstly, built on the work in [24], an adaptive and robust FTC is proposed for faulty systems with uncertainties satisfying a set of conditions, such that the closed-loop systems are asymptotically stable. Secondly, for MIMO faulty systems with uncertainties projection in unmatched space being not zero, a (vector) relative degree condition is developed. Then a novel adaptive robust FTC design approach is proposed, which guarantees that all the signals in the closed-loop system are bounded, and that the outputs go to zero asymptotically. The main contribution of this paper is summarized as follows. A new set of sufficient FTC conditions for systems with unmatched uncertainties are developed. And novel adaptive robust FTC for systems with time varying multiplicative faults, stuck faults and unmatched uncertainties are designed.

The remaining parts of this paper are organized as follows: In Section II, the system is formulated, and assumptions are presented. In Section III, adaptive robust FTC is designed for systems with "equivalent matched" uncertainties and "exactly unmatched" uncertainties, respectively. Simulation results are shown to verify the effectiveness of the designed controllers in Section IV. Finally, comments are presented to conclude this paper in Section V.

II. Problems Formulation And Assumptions

A. System Description

Consider a class of linear systems described by

$$\begin{aligned} \dot{x} &= Ax + Bu, x(t_0) = x_0, \\ y &= Cx \end{aligned} \tag{1}$$

where $x \in \mathcal{R}^n$ is state vector, $u \in \mathcal{R}^{m_1}$ is control input vector, $y \in \mathcal{R}^p$ is output vector, $A \in \mathcal{R}^{n\times n}$, $B \in \mathcal{R}^{n\times m_1}$ and $C \in \mathcal{R}^{p\times n}$ are known system matrices.

B. Fault Model

Actuator faults considered in this paper include outage faults, loss of effectiveness faults and stuck faults. A unified model of actuator faults is given by

$$\begin{aligned} u_i^f &= \rho_i\,(t) u_i + \sigma_i \psi_i\,(t)\,, \\ \rho_i(t)\sigma_i &= 0,\ i = 1, \cdots, m_1 \end{aligned} \tag{2}$$

where ρ_i, $i = 1, \cdots, m_1$ are unknown time-varying efficiency factors satisfying $\underline{\rho}_i \leq \rho_i\,(t) \leq \bar{\rho}_i$ with $\underline{\rho}_i$ and $\bar{\rho}_i$ being upper bound and lower bound of $\rho_i\,(t)$, respectively. $\sigma_i, i = 1, \cdots, m_1$ are unknown scalars. $\psi_i\,(t)\,, i = 1, \cdots, m_1$ represent un-parameterizable time-varying actuator stuck fault values.

Note that, there is no fault on the actuator u_i when $\underline{\rho}_i = \overline{\rho}_i = 1$ and $\sigma_i = 0$. When $\bar{\rho}_i = \underline{\rho}_i = 0$ and $\sigma_i = 1$, a stuck fault occurs on the actuator u_i. The case of $\bar{\rho}_i = \underline{\rho}_i = 0$ and $\sigma_i = 0$ means that the actuator u_i is outage. When $0 < \underline{\rho}_i \leq \overline{\rho}_i < 1$, it corresponds to the case that a loss of effectiveness fault occurs on the actuator u_i. Table I is given to illustrate the fault modes.

TABLE I
FAULT MODEL

$\underline{\rho}_i$	$\overline{\rho}_i$	σ_i	fault mode
1	1	0	normal
0	0	1	stuck
> 0	< 1	0	loss of effectiveness
0	0	0	outage

Then (2) can be written in a compact form

$$u^f = \left[u_1^f, \ldots, u_{m_1}^f\right]^T = \Lambda\,(t)\,u + \Sigma\psi\,(t) \text{ with } \Lambda\,(t)\,\Sigma = \mathbf{0} \tag{3}$$

where $\psi(t) = \mathrm{col}(\psi_1(t), \psi_2(t), \cdots, \psi_{m_1}(t))$,
$\Lambda\,(t) = \mathrm{diag}\{\rho_1\,(t)\,, \rho_2\,(t)\,, \cdots, \rho_{m_1}\,(t)\},\ \rho_i(t) \in \left[\bar{\rho}_i, \underline{\rho}_i\right],\ i = 1, \cdots, m_1,$

$$\sigma_i = \begin{cases} 0, \text{ if a stuck fault or outage fault occurs on the actuator } u_i, \\ 1, \text{ otherwise, } i = 1, \cdots, m_1, \end{cases}$$

and $\Sigma = \mathrm{diag}\{\sigma_1, \sigma_2, \cdots, \sigma_{m_1}\}$.
Define the following sets

$$\begin{aligned} &\triangle_\Lambda\,(t) = \left\{\Lambda\,(t)\,|\Lambda\,(t) = \mathrm{diag}\left\{\rho_1\,(t)\,, \rho_2\,(t)\,, \cdots, \rho_{m_1}\,(t)\right\},\ \rho_i\,(t) \in \left[\bar{\rho}_i, \underline{\rho}_i\right]\right\}, \\ &\triangle_\Sigma = \left\{\Sigma | \Sigma = \mathrm{diag}\left\{\sigma_1, \sigma_2, \cdots, \sigma_{m_1}\right\},\ \sigma_i = 0 \text{ or } 1\right\}. \end{aligned}$$

Then a fault mode set is described by

$$\triangle = \left\{ (\Lambda(t)\,,\ \Sigma) | \Lambda(t)\, \Sigma = \mathbf{0},\ \Lambda(t) \in \triangle_{\Lambda(t)},\ \Sigma \in \triangle_{\Sigma} \right\}. \tag{4}$$

Remark 1. There are different ways to deal with actuator loss of effectiveness faults and stuck faults. The fault model compact form (3) includes normal, loss of effectiveness faults and stuck faults (outage fault is a special stuck fault). In this paper, the inputs of the loss of effectiveness actuators can be adjusted adaptively to keep the outputs of faulty actuators unchanged. However, the output signals of the stuck actuators are considered as external disturbances, and compensated by the partial operational actuators through designed additive functions. In addition, all the fault modes considered in this paper belong to the set $\triangle$ given in (4). ∇

C. Assumptions

To achieve FTC objective, some assumptions for system (1) and fault model (3) are needed.

Assumption 1. The pair (A, B) is stabilizable.

Assumption 2. For all considered fault modes $(\Lambda(t), \Sigma) \in \Delta$ in (4), the following equation holds

$$\text{rank}(B\Lambda(t)) = \text{rank}(B). \tag{5}$$

Remark 2. Assumption 1 is a basic assumption for linear systems, and Assumption 2 is a sufficient FTC condition about actuator redundancy [10], [23]. ∇

Based on Assumptions 1 and 2, the following results are ready to be presented.

Lemma 1. [23] The rank relation (5) is a necessary and sufficient condition for the existence of a function $K_2(t)$ such that $B\Lambda(t)\, K_2(t) = -B\Sigma\psi(t)$.

Proposition 1. The matrix rank relation (5) holds if and only if there exists a matrix $K^*(t) \in \mathcal{R}^{m_1 \times m_1}$ satisfying

$$B\Lambda(t)\, K^*(t) = B. \tag{6}$$

Proof: (Necessary) From basic matrix theory, $\text{rank}(B\Lambda(t)) \leq \text{rank}(B)$. If $\text{rank}(B\Lambda(t)) < \text{rank}(B)$, there exists at least one column of B which cannot be expressed as a linear combination of the columns of $B\Lambda(t)$. This implies that there exists no such a $K^*(t)$ satisfying (6). Therefore, the equation (5) is necessary for (6).

(Sufficiency) Note that

$$\operatorname{rank}(B\Lambda(t)) \leq \operatorname{rank}([B\Lambda(t), BI]), \tag{7}$$

$$\operatorname{rank}([B\Lambda(t), BI]) = \operatorname{rank}(B[\Lambda(t), I]) \tag{8}$$

always hold. From the fact that $\operatorname{rank}(B[\Lambda(t), I]) \leq \min\{\operatorname{rank}(B), \operatorname{rank}([\Lambda(t), I])\}$ and equation (5), it follows that $\operatorname{rank}(B) = \operatorname{rank}(B\Lambda(t)) \leq \min\{\operatorname{rank}(B), \operatorname{rank}(\Lambda(t))\}$. Then, $\operatorname{rank}([\Lambda(t), I]) \geq \operatorname{rank}(\Lambda(t)) \geq \operatorname{rank}(B)$. It can be concluded that

$$\operatorname{rank}(B\Lambda(t)) \leq \operatorname{rank}([B\Lambda(t), B]) \leq \operatorname{rank}(B). \tag{9}$$

Since $\operatorname{rank}(B\Lambda(t)) = \operatorname{rank}(B)$, $\operatorname{rank}(B\Lambda(t)) = \operatorname{rank}([B\Lambda(t), B])$. Hence there exists $K^*(t)$ satisfying equation (6).

Thus, the result follows. ■

Remark 3. Proposition 1 can be satisfied for the case when actuator stuck faults occur. A simple example is shown as follows. Let

$$B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \Lambda(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This implies that the first and second actuators are healthy, and an actuator stuck fault occurs on the third actuator. It is clear to see that the matrix $\Lambda(t)$ is not regular, and rank condition (5) is satisfied. Further, there exists a matrix K^*, given by

$$K^* = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

satisfying (6). This example shows that the matrix $\Lambda(t)$ is not required to be regular in this paper. ▽

Under Assumption 1, for any given symmetric positive definite (SPD) matrix $Q \in \mathcal{R}^{n\times n}$, there exist a solution $K \in \mathcal{R}^{m_1\times n}$ and SPD matrix $P \in R^{n\times n}$ such that

$$P(A + BK) + (A + BK)^T P = -Q. \tag{10}$$

From Proposition 1, for any $\Lambda(t)$ satisfies Assumption 2, there exist a solution $K_{\Lambda}(t) = K^*(t)K \in \mathcal{R}^{m_1 \times n}$ and **a common** SPD matrix $P \in R^{n \times n}$ given in (10) such that

$$P(A + B\Lambda(t)K_{\Lambda}(t)) + (A + B\Lambda(t)K_{\Lambda}(t))^T P = -Q. \tag{11}$$

Assumption 3. The un-parameterized time-varying stuck fault vector $\psi(t)$ is assumed to be bounded by an unknown constant $\bar{\psi}$, i.e., $\|\psi(t)\| \leq \bar{\psi}$.

Remark 4. In [24], the time-varying stuck fault is parameterized and can be compensated by additive input control signals directly. For un-parameterized time-varying stuck faults considered in this paper, same control objective as in [24] can be achieved by using the bounds of stuck fault values. ∇

Suppose that actuator stuck fault happens at time instant t_k, with $t_k < t_{k+1}$ for $k = 1, 2, \ldots, N$. As in [22], the fault mode ($\Lambda(t), \Sigma$) is fixed in $(t_k,\ t_{k+1}),\ k = 1, 2, \ldots, N$, i.e., the elements of $\Lambda(t)$ in $(t_k,\ t_{k+1})$ are always zero or not zero, and Σ is fixed.

Assumption 4. In the interval $(t_k,\ t_{k+1})$, the efficiency factors $\rho_i(t), i = 1, \cdots, m_1$ are unknown continuous time-varying functions and their time derivatives satisfy

$$|\dot{\rho}_i(t)| \leq \vartheta_0(t)\rho_i(t),\ i = 1, \cdots, m_1 \tag{12}$$

where $\vartheta_0(t)$ satisfies that

$$0 < \lim_{t \to \infty} \int_{t_0}^{t} \vartheta_0(\tau)\, d\tau \leq \vartheta_0 \leq \infty.$$

Remark 5. The efficiency factors $\rho_i(t), i = 1, \cdots, m_1$ satisfying (12) can be used to model many time-varying multiplicative faults, for example, for the fault considered in [25],

$$\rho(t) = \begin{cases} 1, & \text{if } t < t_k, \\ e^{-\alpha(t-t_k)}, & \text{if } t_k \leq t < t_{k+1} \end{cases}$$

where the constant scalar $\alpha > 0$ denotes the fault evolution rate of ρ. Similar to [25], the efficiency factors ρ_i, $i = 1, \cdots, m_1$ can be described by

$$\rho_i(t) = \begin{cases} 1, & \text{if } t < t_k, \\ e^{(\beta_i/\alpha_i e^{-\alpha_i(t-t_k)} - a_i)} - 1, & \text{if } t_k \leq t < t_{k+1} \end{cases}$$

where α_i and β_i determine the evolution rate of the efficiency factors ρ_i, and a_i is used to adjust the value of ρ_i such that $\rho_i(t_k) = 1$. Then

$$\dot{\rho}_i(t) = \begin{cases} 0, & \text{if } t < t_k, \\ -\beta_i e^{-\alpha_i(t-t_k)} e^{\beta_i/\alpha_i e^{-\alpha_i(t-t_k)} - a_i}, & \text{if } t_k \leq t < t_{k+1}, i = 1, \cdots, m_1. \end{cases}$$

Therefore, the $\vartheta_0(t)$ satisfying (12) can be chosen as

$$\vartheta_0(t) = \frac{\bar{\bar{\rho}}+1}{\underline{\underline{\rho}}}\beta e^{-\underline{\alpha}t}$$

where $\bar{\bar{\rho}} = \max_{i=1,\cdots,m_1}\{\rho_i\}$, $\underline{\underline{\rho}} = \min_{i=1,\cdots,m_1,\rho_i\neq 0}\{\rho_i\}$ and $\underline{\alpha} = \min_{i=1,\cdots,m_1}\{\alpha_i\}$. ∇

III. Adaptive Robust FTC Design for MIMO Systems

A. With "Equivalent Matched" Uncertainties

Consider the following uncertain faulty system

$$\dot{x} = Ax + f(x,\omega(t),t) + B\Lambda(t)u + B\Sigma\psi(t), \;\; x(t_0) = x_0 \tag{13}$$

where $\omega(t) \in \mathcal{R}$ represents disturbance, and the unknown nonlinear vector $f(\cdot) : \mathcal{R}^n \times \mathcal{R} \times \mathcal{R}^+ \to \mathcal{R}^n$ represents system lump uncertainty.

Remark 6. In system (13), the unknown nonlinear vector $f(x,\omega(t),t)$ represents the lumped uncertainty, which is a generalized concept, possibly including disturbances, un-modelled dynamics, parameter variations, and complex nonlinear dynamics. ∇

The following assumption for $f(x,\omega(t),t)$ is given.

Assumption 5. The uncertainty vector $f(x,\omega(t),t)$ satisfies that

$$\|x^T P f(x,\omega(t),t)\| \leq \alpha(x,t)\|x^T PB\| \tag{14}$$

where $\alpha(x,t)$ is a known continuous and locally bounded function for all $x \in \mathcal{R}^n$ and $t \in \mathcal{R}^+$, and the SPD matrix P satisfies equation (11).

Remark 7. For matrix B in system (13), there exist matrices $\bar{B}$ and W such that $[B, \bar{B}]\text{col}(W_1, W_2) = I_{n\times n}$. Then it follows that $\left\|x^T P f(\cdot)\right\| = \left\|x^T PBW_1 f(\cdot) + x^T P\bar{B}W_2 f(\cdot)\right\| \leq \left\|x^T PB\right\| \|W_1 f(\cdot)\| + \left\|x^T P\bar{B}W_2 f(\cdot)\right\|$. Assumption 5 implies that $\frac{\left\|x^T P\bar{B}W_2 f(\cdot)\right\|}{\|x^T PB\|}$ is bounded when $\left\|x^T PB\right\| \neq 0$, and that $\lim_{\|x^T PB\|\to 0} \frac{\left\|x^T P\bar{B}W_2 f(\cdot)\right\|}{\|x^T PB\|}$ is bounded. For matched uncertainties, $x^T P\bar{B}W_2 f(\cdot) = 0$, which implies that Assumption 5 holds automatically. In fact, uncertainties satisfying Assumption 5 are called "equivalent matched" uncertainties, which include the matched uncertainties considered in [3] and [10], and partial of unmatched uncertainties in [4] as special cases. ∇

Remark 8. The function $\alpha(x,t)$ is kind of bound on the uncertainty vector $f(x,\omega(t),t)$. It is not a design parameter ,and thus it is not chosen by us. For a specific real system, $\alpha(x,t)$

may be obtained from the characteristics of the real system and historical statistical information. In addition, from reference [15], if the nonlinear vector $f(x,\omega(t),t)$ is Lipschitz (including ΔAx), then there exists a known nonlinear function $\alpha(x,t)$ such that $\left\|x^T Pf(x,\omega(t),t)\right\| \leq \alpha(x,t)\left\|x^T PB\right\|$. ∇

The objective is to design a class of adaptive robust state feedback FTC for system (13) to guarantee that all the signals in the closed-loop system are bounded, and the states go to zero asymptotically. Then the following controller is constructed

$$u = u_f + u_u \tag{15}$$

where u_u is an auxiliary control function to compensate for uncertainties, and u_f is the fault compensation function, described by

$$u_f = \hat{K}_\Lambda x + K_2(t) \tag{16}$$

where $\hat{K}_\Lambda \in \mathcal{R}^{m_1\times n}$ is the estimation of $K_\Lambda(t) \in \mathcal{R}^{m_1\times n}$ defined in (11) and $K_2(t)$ is also an auxiliary control function to compensate for actuator stuck faults.

Remark 9. It should be noted the FTC structure u in (15) is fixed. However, the parameter $\hat{K}_\Lambda$ and other parameters in u_f and $K_2(t)$ are to be estimated by adaptive technique later. ∇

The adaptive law of $\hat{K}_\Lambda$ is given by

$$\dot{\hat{K}}_\Lambda = -\Gamma((xx^T PB)^T + \epsilon\vartheta_0(t)\hat{K}_\Lambda) \tag{17}$$

where $\Gamma = \Gamma^T > 0$ is a constant matrix, and $\epsilon \geq 6$ is a constant scalar.

Before constructing auxiliary control functions $K_2(t)$ and u_u, the following lemmas are needed.

Lemma 2. [11] If the fault mode considered $(\Lambda(t),\ \Sigma) \in \triangle$ in (4) satisfies Assumption 2, there exists a positive constant $\mu > 0$ such that

$$x^T PB\Lambda(t) B^T Px(t) \geq \mu\left\|x^T PB\right\|^2 \tag{18}$$

where the SPD matrix P is defined in (10).

Lemma 3. [26] For any square matrices X and Y with appropriate dimensions, the following inequality

$$X^T Y + Y^T X \leq \alpha X^T X + \alpha^{-1} Y^T Y \tag{19}$$

holds, where α is a positive scalar.

Lemma 4. Denote $\hat{K} = \tilde{K} + K$ with $\hat{K},\ \tilde{K},\ K$ having appropriate dimensions. Then

$$tr(\tilde{K}^T\Gamma\tilde{K}) - tr(K^T\Gamma K) \leq 2tr(\hat{K}^T\Gamma\tilde{K}) \tag{20}$$

holds, where $\Gamma = \Gamma^T > 0$ is a constant matrix.

Proof: For any $\alpha > 0$,

$$\begin{aligned} & tr(\tilde{K}^T\Gamma\tilde{K}) - tr(K^T\Gamma K) \\ & = (1+\alpha)tr(\tilde{K}^T\Gamma\tilde{K}) - (1-\alpha^{-1})tr(K^T\Gamma K) - \alpha tr(\tilde{K}^T\Gamma\tilde{K}) - \alpha^{-1}tr(K^T\Gamma K), \\ & \leq (1+\alpha)tr(\tilde{K}^T\Gamma\tilde{K}) - (1-\alpha^{-1})tr(K^T\Gamma K) + 2tr(K^T\Gamma\tilde{K}). \end{aligned} \tag{21}$$

Let $\alpha = 1$. Then it follows from (21) that $tr(\tilde{K}^T\Gamma\tilde{K}) - tr(K^T\Gamma K) \leq 2tr(\hat{K}^T\Gamma\tilde{K})$. ■

Based on Assumption 3, there exists a positive constant k_3 such that $\|\Sigma\psi(t)\| \leq \|\Sigma\|\bar{\psi} \leq \mu k_3$, where μ is positive unknown scalar defined in (18). Based on Assumption 5, there exists a positive constant k_4 such that $\left\|x^T Pf\left(x, \omega\left(t\right), t\right)\right\| \leq \mu k_4 \alpha\left(x, t\right)\left\|x^T PB\right\|$ where $k_4 = \frac{1}{\mu}$. Here, it is worth pointing out that since the fault parameters $\Lambda(t)$, Σ and $\bar{\psi}$ are unknown, the associated constant parameters μ, k_3 and k_4 are unknown. Adaptive laws are to be designed to identify the parameters k_3 and k_4.

The auxiliary control functions $K_2(t)$ and u_u are defined by

$$K_2\left(t\right) = -\frac{B^T Px\hat{k}_3^2}{\|x^T PB\|\,\hat{k}_3 + \vartheta(t)}, \tag{22}$$

$$u_u = -\frac{B^T Px\alpha^2\left(x,t\right)\hat{k}_4^2}{\|x^T PB\|\,\alpha\left(x,t\right)\hat{k}_4 + \vartheta\left(t\right)} \tag{23}$$

where the SPD matrix $P \in \mathcal{R}^{n\times n}$ is given in (10), and $\vartheta(t)$ is any positive uniformly continuous and bounded function, satisfying that

$$0 < \lim_{t\to\infty}\int_{t_0}^{t}\vartheta\left(\tau\right)d\tau \leq \vartheta < \infty. \tag{24}$$

The estimations $\hat{k}_3$ and $\hat{k}_4$ are updated by

$$\dot{\hat{k}}_3 = \gamma_1\left\|x^T PB\right\| - \gamma_1\vartheta(t)\hat{k}_3, \tag{25}$$

$$\dot{\hat{k}}_4 = \gamma_2\left\|x^T PB\right\|\alpha\left(x,t\right) - \gamma_2\vartheta\left(t\right)\hat{k}_4 \tag{26}$$

where γ_1 and γ_2 are positive scalars.

Let $\tilde{K}_\Lambda = \hat{K}_\Lambda - K_\Lambda(t)$, $\tilde{k}_3 = \hat{k}_3 - k_3$ and $\tilde{k}_4 = \hat{k}_4 - k_4$. Then the error dynamics of (17), (25) and (26) are described by

$$\begin{aligned} \dot{\tilde{K}}_\Lambda &= -\Gamma\left((xx^T(t)PB)^T + \epsilon\vartheta_0(t)\tilde{K}_\Lambda + \epsilon\vartheta_0(t)K_\Lambda(t)\right), \\ \dot{\tilde{k}}_3 &= \gamma_1\left\|x^TPB\right\| - \gamma_1\vartheta(t)\tilde{k}_3 - \gamma_1\vartheta(t)k_3, \\ \dot{\tilde{k}}_4 &= \gamma_2\left\|x^TPB\right\|\alpha(x,t) - \gamma_2\vartheta(t)\tilde{k}_4 - \gamma_2\vartheta(t)k_4. \end{aligned} \tag{27}$$

Therefore, the closed-loop system by applying control (15) to system (13) is obtained and described by

$$\dot{x}(t) = \left(A + B\Lambda(t)\hat{K}_\Lambda\right)x + B\Lambda(t)u_u + B\Lambda(t)K_2(t) + B\Sigma\psi(t) + f(x,\omega(t),t) \tag{28}$$

where the unknown nonlinear vector $f(\cdot) : \mathcal{R}^n \times \mathcal{R} \times \mathcal{R}^+ \to \mathcal{R}^n$ is system lump uncertainty.

Remark 10. Both the error dynamics (27) and the closed-loop system (28) are continuous in any time intervals $(t_k,\ t_{k+1})$. The existence of the solution to differential equation (27) and (28) in the usual sense can be guaranteed. Therefore, the controller (15) with the continuous auxiliary control functions (22), (23) and the continuous adaptive laws (17), (25), (26) can be easily implemented in practical problems. ∇

Remark 11. The proposed $\sigma-$modification adaptive laws (17), (25) and (26), like in [8] and [27], are capable of avoiding high gain effectively. Moreover, from auxiliary control functions (22) and (23), it is straight forward to see that $\|K_2(t)\| \leq \hat{k}_3$ and $\|u_u(t)\| \leq \alpha(x,t)\hat{k}_4$. ∇

Denote $\left(x,\ \tilde{k}_\Lambda,\ \tilde{k}_3,\ \tilde{k}_4\right)$ as the solution of the closed-loop system (28) and the error dynamics (27). Then the following theorem is ready to present.

Theorem 1. *For the error dynamics (27) and the closed-loop system (28), supposing that Assumptions 1-5 are satisfied, then the solution* $\left(x,\ \tilde{k}_\Lambda,\ \tilde{k}_3,\ \tilde{k}_4\right)$ *to the error dynamics (27) and the closed-loop system (28) is bounded. Furthermore,*

$$\lim_{t\to\infty} x(t;\ t_0,\ x_0) = 0. \tag{29}$$

Proof: For the error dynamics (27) and the closed-loop system (28), a Lyapunov function candidate is chosen as

$$V(x, \sqrt{\Lambda}\tilde{k}_\Lambda, \tilde{k}_3, \tilde{k}_4) = x^TPx + tr(\Lambda(t)\tilde{k}_\Lambda^T\Gamma^{-1}\tilde{k}_\Lambda) + \mu\gamma_1^{-1}\tilde{k}_3^2 + \mu\gamma_2^{-1}\tilde{k}_4^2. \tag{30}$$

Then the time derivative $V(\cdot)$ in each time interval ($t_k,\ t_{k+1}$) along the trajectories of (27) and (28) is given by

$$\begin{aligned}
&\frac{dV_k(x,\sqrt{\Lambda}\tilde{K}_\Lambda,\tilde{k}_3,\tilde{k}_4)}{dt}\\
&= -x^T\left((A+B\Lambda\left(t\right)K_\Lambda(t))\,P+P(A+B\Lambda\left(t\right)K_\Lambda(t))^T\right)x\\
&+2x^TPB\Lambda\left(t\right)\tilde{K}_\Lambda x+2tr(\Lambda\left(t\right)\tilde{K}_\Lambda^T\Gamma^{-1}\dot{\hat{K}}_\Lambda)-2tr(\Lambda\left(t\right)\tilde{K}_\Lambda^T\Gamma^{-1}\dot{K}_\Lambda(t))\\
&+tr(\dot{\Lambda}\left(t\right)\tilde{K}_\Lambda^T\Gamma^{-1}\tilde{K}_\Lambda)-3\vartheta_0(t)tr(\Lambda\left(t\right)K_\Lambda^T(t)\Gamma^{-1}K_\Lambda(t))+3\vartheta_0(t)tr(\Lambda\left(t\right)K_\Lambda^T(t)\Gamma^{-1}K_\Lambda(t))\\
&+2x^TPB\Lambda\left(t\right)K_2\left(t\right)+2x^TPB\Sigma\psi\left(t\right)+2\mu\gamma_1^{-1}\tilde{k}_3\dot{\tilde{k}}_3\\
&+2x^TPB\Lambda\left(t\right)u_u+2x^TPf\left(x,\omega(t),t\right)+2\mu\gamma_2^{-1}\tilde{k}_4\dot{\tilde{k}}_4
\end{aligned}\tag{31}$$

where

$$\begin{aligned}
&2x^TPB\Lambda(t)\tilde{K}_\Lambda x+2tr(\Lambda(t)\tilde{K}_\Lambda^T\Gamma^{-1}\dot{\hat{K}}_\Lambda)\\
&\quad-2tr(\Lambda(t)\tilde{K}_\Lambda^T\Gamma^{-1}\dot{K}_\Lambda(t))-2tr(\dot{\Lambda}(t)\tilde{K}_\Lambda^T\Gamma^{-1}\tilde{K}_\Lambda)\\
&\quad+3tr(\dot{\Lambda}(t)\tilde{K}_\Lambda^T\Gamma^{-1}\tilde{K}_\Lambda)-3\vartheta_0(t)tr(\Lambda(t)K_\Lambda^T(t)\Gamma^{-1}K_\Lambda(t))\\
&\quad+3\vartheta_0(t)tr(\Lambda(t)K_\Lambda^T(t)\Gamma^{-1}K_\Lambda(t))\\
=&2x^TPB\Lambda(t)\tilde{K}_\Lambda x+2tr(\Lambda(t)\tilde{K}_\Lambda^T\Gamma^{-1}\dot{\hat{K}}_\Lambda)\\
&\quad+2tr(\dot{\Lambda}(t)\tilde{K}_\Lambda^T\Gamma^{-1}K_\Lambda(t))-2tr(\dot{\Lambda}(t)\tilde{K}_\Lambda^T\Gamma^{-1}\tilde{K}_\Lambda)\\
&\quad+3tr(\dot{\Lambda}(t)\tilde{K}_\Lambda^T\Gamma^{-1}\tilde{K}_\Lambda)-3\vartheta_0(t)tr(\Lambda(t)K_\Lambda^T(t)\Gamma^{-1}K_\Lambda(t))\\
&\quad+3\vartheta_0(t)tr(\Lambda(t)K_\Lambda^T(t)\Gamma^{-1}K_\Lambda(t)).
\end{aligned}\tag{32}$$

From Lemma 4, the two terms in Eq. (32) can be enlarge into that

$$\begin{aligned}
&3tr(\dot{\Lambda}(t)\tilde{K}_\Lambda^T\Gamma^{-1}\tilde{K}_\Lambda)-3\vartheta_0(t)tr(\Lambda(t)K_\Lambda^T(t)\Gamma^{-1}K_\Lambda(t))\\
&\leq 3\vartheta_0(t)tr(\Lambda(t)\tilde{K}_\Lambda^T\Gamma^{-1}\tilde{K}_\Lambda)-3\vartheta_0(t)tr(\Lambda(t)K_\Lambda^T(t)\Gamma^{-1}K_\Lambda(t))\\
&\leq 6\vartheta_0(t)tr(\Lambda(t)\tilde{K}_\Lambda^T\Gamma^{-1}\hat{K}_\Lambda).
\end{aligned}\tag{33}$$

Suppose that $\dot{\rho}_i(t)\geq 0,\ i=1,\cdots,m_1$. Then

$$tr\left(\dot{\Lambda}\left(t\right)\tilde{K}_\Lambda^T\Gamma^{-1}\tilde{K}_\Lambda\right)=\sum_{i=1}^{m_1}\sum_{j=1}^{n}\tilde{K}_{\Lambda ij}^2\dot{\rho}_i\Gamma_i^{-1}\geq\left\|\tilde{K}_\Lambda'\right\|^2\min\left(\Gamma_i^{-1}\underline{\dot{\rho}}\right)\tag{34}$$

where $\tilde{K}_\Lambda'=[\tilde{K}_{\Lambda i}]$, $\dot{\rho}_i\neq 0$, $i=1,\cdots m_1$, and $\underline{\dot{\rho}}$ is the minimum value of $\dot{\rho}_i$. From Assumption 3, there exists a positive scalar $\underline{\rho}$ such that $\underline{\dot{\rho}}\geq\underline{\rho}\vartheta_0(t)$. Then it follows from (34) that

$$tr\left(\dot{\Lambda}\left(t\right)\tilde{K}_\Lambda^T\Gamma^{-1}\tilde{K}_\Lambda\right)\geq\left\|\tilde{K}_\Lambda'\right\|^2\min\left(\Gamma_i^{-1}\right)\underline{\rho}\vartheta_0(t).\tag{35}$$

Differentiating (11) on both sides, it follows that $B(\dot{\Lambda}(t)K_\Lambda(t)+\Lambda(t)\dot{K}_\Lambda(t)) = \mathbf{0}$, $\dot{\Lambda}(t)K_\Lambda(t) = -\Lambda(t)\dot{K}_\Lambda(t)$. Then, $-2tr\left(\Lambda(t)\tilde{K}_\Lambda^T\Gamma^{-1}\dot{K}_\Lambda(t)\right) = 2tr\left(\dot{\Lambda}(t)\tilde{K}_\Lambda^T\Gamma^{-1}K_\Lambda(t)\right)$. Using Schwarz inequality, it has that

$$\begin{aligned}\left|tr\left(\dot{\Lambda}(t)\tilde{K}_\Lambda^T\Gamma^{-1}K_\Lambda(t)\right)\right| &\le \left\|\dot{\Lambda}(t)\tilde{K}_\Lambda^T K_\Lambda(t)\right\|\max\left(\Gamma_i^{-1}\right)\\ &\le \left\|\tilde{K}'_\Lambda\right\|\left\|K_\Lambda(t)\right\|\max\left(\Gamma_i^{-1}\right)|\bar{\dot{\rho}}|\end{aligned} \tag{36}$$

where $\bar{\dot{\rho}}$ is the maximum value of $\dot{\rho}_i$. Since there exists a positive scalar $\bar{\rho}$ such that $|\bar{\dot{\rho}}| \le \vartheta_0(t)\bar{\rho}$, it can be obtained from (35) and (36) that

$$\begin{aligned}&tr\left(\dot{\Lambda}(t)\tilde{K}_\Lambda^T\Gamma^{-1}K_\Lambda(t)\right) - tr\left(\dot{\Lambda}(t)\tilde{K}_\Lambda^T\Gamma^{-1}\tilde{K}_\Lambda\right)\\ &\le -\vartheta_0(t)\left\|\tilde{K}'_\Lambda\right\|^2\min\left(\Gamma_i^{-1}\right)\underline{\rho} + \vartheta_0(t)\left\|\tilde{K}'_\Lambda\right\|\left\|K_\Lambda(t)\right\|\max\left(\Gamma_i^{-1}\right)\bar{\rho}\\ &\le \vartheta_0(t)\tfrac{\|K_\Lambda(t)\|^2\max\left(\Gamma_i^{-1}\right)\bar{\rho}}{4\min\left(\Gamma_i^{-1}\right)\underline{\rho}} = \vartheta_0(t)\delta_0\end{aligned} \tag{37}$$

where $\delta_0 = \frac{\|K_\Lambda(t)\|^2\max\left(\Gamma_i^{-1}\right)\bar{\rho}}{4\min\left(\Gamma_i^{-1}\right)\underline{\rho}}$.

Then substituting (33), (37) and adaptive law in (17) into (32), it follows that when $\epsilon \ge 6$,

$$\begin{aligned}&2x^TPB\Lambda(t)\tilde{K}_\Lambda x + 2tr(\Lambda(t)\tilde{K}_\Lambda^T\Gamma^{-1}\dot{\hat{K}}_\Lambda) - 2tr(\Lambda(t)\tilde{K}_\Lambda^T\Gamma^{-1}\dot{K}_\Lambda(t))\\ &-2tr(\dot{\Lambda}(t)\tilde{K}_\Lambda^T\Gamma^{-1}\tilde{K}_\Lambda) + 3tr(\dot{\Lambda}(t)\tilde{K}_\Lambda^T\Gamma^{-1}\tilde{K}_\Lambda)\\ &-3\vartheta_0(t)tr(\Lambda(t)K_\Lambda^T(t)\Gamma^{-1}K_\Lambda(t)) + 3\vartheta_0(t)tr(\Lambda(t)K_\Lambda^T(t)\Gamma^{-1}K_\Lambda(t))\\ &\le \vartheta_0(t)(2\delta_0 + 3\delta_1)\end{aligned} \tag{38}$$

where $\delta_1 = tr(\Lambda(t)K_\Lambda^T(t)\Gamma^{-1}K_\Lambda(t))$.

For other cases that $\rho_i(t) \le 0, i = 1, \cdots, m_1$ and $\rho_i(t) \ge 0, \rho_j(t) \le 0,\ i \ne j,\ i,j = 1, \cdots, m_1$, the results are similar to (38) and omitted here.

Substituting auxiliary function $K_2(t)$ in (22) and the adaptive law $\hat{k}_3$ in (25) into (31),

$$\begin{aligned}&2x^TPB\Lambda(t)K_2(t) + 2\mu\left\|x^TPB\right\|k_3 + 2\mu\gamma_1^{-1}\tilde{k}_3\dot{\hat{k}}_3\\ &\le -\frac{2\mu\left\|x^TPB\right\|^2\hat{k}_3^2}{\|x^TPB\|\hat{k}_3 + \vartheta(t)} + 2\mu\left\|x^TPB\right\|k_3 + 2\mu\gamma_1^{-1}\tilde{k}_3\dot{\hat{k}}_3\\ &= \frac{2\mu\left\|x^TPB\right\|\hat{k}_3\vartheta(t)}{\|x^TPB\|\hat{k}_3 + \vartheta(t)} - \mu\vartheta(t)\tilde{k}_3\left(\tilde{k}_3 + k_3\right).\end{aligned} \tag{39}$$

Substituting auxiliary function u_u in (23) and the adaptive law $\hat{k}_4$ in (26) into (31),

$$\begin{aligned}&2x^T PB\Lambda(t)\,u_u + 2x^T Pf(x,\omega(t),t) + 2\mu\gamma_2^{-1}\tilde{k}_4\dot{\tilde{k}}_4\\ &\leq -\frac{2\mu\left\|x^T PB\right\|^2\alpha^2(x,t)\,\hat{k}_4^2}{\|x^T PB\|\,\alpha(x,t)\,\hat{k}_4+\vartheta(t)} + 2\left\|x^T PB\right\|\alpha(x,t)\,\mu k_4 + 2\mu\gamma_2^{-1}\tilde{k}_4\dot{\tilde{k}}_4\\ &= \frac{2\mu\left\|x^T PB\right\|\alpha(x,t)\,\hat{k}_4\vartheta(t)}{\|x^T PB\|\,\alpha(x,t)\,\hat{k}_4+\vartheta(t)} - \mu\vartheta(t)\,\tilde{k}_4\left(\tilde{k}_4+k_4\right).\end{aligned} \tag{40}$$

Notice the fact that for any positive constant $c > 0$, $0 \leq \frac{ab}{a+b} \leq a$, $\forall a,\ b > 0$ and that $-\tilde{k}_3\left(\tilde{k}_3+k_3\right) \leq \frac{1}{4}k_3^2$, $-\tilde{k}_4\left(\tilde{k}_4+k_4\right) \leq \frac{1}{4}k_4^2$, it follows from (39) and (40) that

$$2x^T PB\Lambda(t)K_2(t) + 2x^T PB\Sigma\psi(t) + 2\mu\gamma_1^{-1}\tilde{k}_3\dot{\tilde{k}}_3 \leq \mu\vartheta(t)\,(1+\frac{1}{4}k_3^2), \tag{41}$$

$$2x^T PB\Lambda(t)u_u(t) + 2x^T Pf(x,\omega(t),t) + 2\mu\gamma_2^{-1}\tilde{k}_4\dot{\tilde{k}}_4 \leq \mu\vartheta(t)\,(1+\frac{1}{4}k_4^2). \tag{42}$$

Thus, from (38), (41) and (42), it can be concluded that for any $t \in (t_k,\ t_{k+1})$,

$$\frac{dV_k(x,\sqrt{\Lambda}\tilde{K}_\Lambda,\tilde{k}_3,\tilde{k}_4)}{dt} = -\lambda_{min}(Q)\|x\|^2 + (2\delta_0+3\delta_1)\vartheta_0(t) + \delta_2\vartheta(t) \tag{43}$$

where $\lambda_{min}(Q)$ represents the minimum eigenvalue of Q, $\delta_2 = \mu(2+\frac{1}{4}k_3^2+\frac{1}{4}k_4^2)$.

Let $\tilde{x} = \left(x,\ \sqrt{\Lambda}\tilde{K}_\Lambda,\ \tilde{k}_3,\ \tilde{k}_4\right)$. Then there exists a class $\mathcal{K}_\infty$ function $\gamma_1(\cdot)$ such that

$$0 < \gamma_1(\|\tilde{x}\|) \leq V(\tilde{x}(t)). \tag{44}$$

Thus for any $t \in (t_k,\ t_{k+1})$,

$$\begin{aligned}0 \leq \gamma_1(\|\tilde{x}\|) \leq V(\tilde{x}(t)) \leq &V(\tilde{x}(t_k)) + \int_{t_k}^{t_{k+1}}(-\lambda_{\min}(Q)\,\|x(\tau)\|^2)d\tau\\ &+ \int_{t_k}^{t_{k+1}}(2\delta_0{+}3\delta_1)\,\vartheta_0(\tau)\,d\tau + \int_{t_k}^{t_{k+1}}\delta_2\vartheta(\tau)\,d\tau.\end{aligned} \tag{45}$$

Note that for any $t > t_0$, it has that

$$\sup_{t\in[t_0,\infty)}\left(\int_{t_0}^{t}(\delta_0{+}3\delta_1)\,\vartheta_0(\tau)\,d\tau + \int_{t_0}^{t}\delta_2\vartheta(\tau)\,d\tau\right) \leq (2\delta_0{+}3\delta_1)\,\vartheta_0 + \delta_2\vartheta. \tag{46}$$

Consequently,

$$0 \leq \gamma_1(\|\tilde{x}(t)\|) \leq V(\tilde{x}(t_k)) + (2\delta_0{+}3\delta_1)\,\vartheta_0 + \delta_2\vartheta, \tag{47}$$

which implies that if the initial value $V(t_k^+)$ is finite, $\tilde{x} \in \mathcal{L}_\infty$, then $x \in \mathcal{L}_\infty$, $\left\|\sqrt{\Lambda}\hat{K}_\Lambda\right\| \in \mathcal{L}_\infty, \hat{k}_3 \in \mathcal{L}_\infty, \hat{k}_4 \in \mathcal{L}_\infty$ in each time interval (t_k, t_{k+1}). Note that the Lyapunov function $V(\cdot)$ is not continuous and has a jump with a finite value, at each time instant t_k. If $V(t_0)$ is finite, then $V(\cdot) \in \mathcal{L}_\infty$, $\forall t \geq 0$ with several jumps of finite values. Consequently, $\tilde{x} \in \mathcal{L}_\infty,\ x \in \mathcal{L}_\infty$, $\left\|\sqrt{\Lambda}\hat{K}_\Lambda\right\| \in \mathcal{L}_\infty, \hat{k}_3 \in \mathcal{L}_\infty, \hat{k}_4 \in \mathcal{L}_\infty$ for all $t \geq 0$.

Consider the Lyapunov candidate function

$$V_{I-\Lambda} = \frac{1}{2}(I-\Lambda)\hat{K}_\Lambda^T\Gamma^{-1}\hat{K}_\Lambda.$$

It can be proved that there is a constant $\kappa > 0$ such that $\dot{V}_{I-\Lambda} < 0$ for $\left\|\sqrt{I-\Lambda}\hat{K}_\Lambda\right\| > \kappa$, which implies $\left\|\sqrt{I-\Lambda}\hat{K}_\Lambda\right\| \in \mathcal{L}_\infty$. Since it has proved that $\left\|\sqrt{\Lambda}\hat{K}_\Lambda\right\| \in \mathcal{L}_\infty$, $\left\|K_\Lambda(t)\right\| \in \mathcal{L}_\infty$.

Therefore, it can be concluded that $\hat{K}_\Lambda(t)\,x \in \mathcal{L}_\infty$, $K_2(t) \in \mathcal{L}_\infty$, $u_u \in \mathcal{L}_\infty$, $u \in \mathcal{L}_\infty$, $\dot{x}(t) \in \mathcal{L}_\infty$ and x is uniformly continuous. When t approaches infinity on both sides of (45), it follows that

$$\begin{aligned}
&\lim_{t\to\infty}\int_{t_0}^{t}\lambda_{\min}(Q)\left\|x(\tau)\right\|^2 d\tau \\
&\quad\leq V(\tilde{x}(t_0)) + \lim_{t\to\infty}\left(\int_{t_0}^{t}(2\delta_0+3\delta_1)\,\vartheta_0(\tau)\,d\tau + \int_{t_0}^{t}\delta_2\vartheta(\tau)\,d\tau\right) \\
&\quad\leq V(\tilde{x}(t_0)) + (2\delta_0+3\delta_1)\,\vartheta_0 + \delta_2\vartheta. \qquad (48)
\end{aligned}$$

Applying Barbălat lemma [8] to (48) yields $\lim_{t\to\infty}\lambda_{\min}(Q)\left\|x(t)\right\|^2 = 0$, which implies that (29) is satisfied.

Hence, the result follows. ■

This section is studied under Assumption 5, which includes matched and part of unmatched uncertainties. In the next section, the rest part of unmatched uncertainties, and a new control objective will be considered.

B. With "Exactly Unmatched" Uncertainties

Consider the following uncertain faulty system

$$\begin{aligned}
&\dot{x} = Ax + Df(x, \omega(t), t) + Bu^f,\ x(t_0) = x_0, \\
&y = Cx
\end{aligned} \qquad (49)$$

where matrices A, B and C are the same as system (1). The fault model u^f is described by (3), and considered fault mode $(\Lambda(t), \Sigma) \in \triangle$ satisfies Assumption 2. The unknown nonlinear term $Df(x, \omega(t), t)$ represents lumped uncertainties with $D \in \mathcal{R}^{n\times m_2}$. Without loss of generality, the matrix D is assumed to be full column rank and $Im(D) \not\subset Im(B)$, i.e., $Df(\cdot)$ is unmatched. The following lemmas are introduced to project $Df(\cdot)$ into matched space and unmatched space.

Lemma 5. For any matrix $S \in \mathcal{R}_r^{m\times n}$ with rank $r > 0$, there exists a decomposition

$$S = QR \tag{50}$$

where $Q \in \mathcal{R}^{m\times r}$ with $Q^T Q = I_r$, and $R \in \mathcal{R}^{r\times n}$ with R being full row rank, i.e., $RR^T > 0$.

Proof: The matrix S can be decomposed as $S = FG$ where $F \in \mathcal{R}_r^{m\times r}$ is full column rank, and $G \in R_r^{r\times n}$ is full row rank. Then F can be decomposed as $F = QR_1$ where R_1 is full rank nonsingular matrix, and $Q \in \mathcal{R}_r^{m\times r}$, $Q^T Q = I_r$. Therefore, $S = QR_1 G = QR$ where $R = R_1 G$ being full row rank, i.e., $RR^T > 0$. ■

Lemma 6. [7] For any matrix $Q \in \mathcal{R}^{n\times m}$ with $\operatorname{rank}(Q) = m$, the identity matrix

$$I_n = QQ^+ + Q^\perp Q^{\perp +}$$

holds, where Q^+ represents the left inverse of Q, that is $Q^+ = \left(Q^T Q\right)^{-1} Q^T$, and the columns of $Q^\perp \in \mathcal{R}^{n\times(n-m)}$ span the null space of Q^T.

Based on Lemma 5, the matrix B can be decomposed as $B = Q_B R_B$ with rank(Q_B) =rank(B) and R_B being full row rank. Based on Lemma 6, the identity matrix $I_n = Q_B Q_B^+ + Q_B^\perp Q_B^{\perp +}$. Now we can project $Df(\cdot)$ into the matched and unmatched spaces, $Df(\cdot) = f_m(\cdot) + f_u(\cdot)$, $f_m(\cdot) \triangleq Q_B Q_B^+ Df(\cdot)$ and $f_u(\cdot) \triangleq Q_B^\perp Q_B^{\perp +} Df(\cdot)$, where $f_m(\cdot)$ and $f_u(\cdot)$ are matched and unmatched uncertainties, respectively. $f_u(\cdot)$ is called "exactly unmatched" uncertainties. Similar discussion is available in [28].

Assumption 6. The uncertainty vector $f\left(x, \omega(t), t\right)$ satisfies

$$\|f\left(x, \omega(t), t\right)\| \leq \beta(x, t) \tag{51}$$

where $\beta(x, t)$ is known continuous and locally bounded function in $x \in \mathcal{R}^n$ and $t \in \mathcal{R}^+$.

Remark 12. Assumption 5 implies that the unmatched component $f_u(\cdot)$ goes to zero when $\|x^T PB\|$ goes to zero. However, there is no such requirement in Assumption 6. Therefore, the limitation in Assumption 6 is more relaxed than that in Assumption 5. ▽

The FTC objective for system (49) is to tolerate the actuator faults and compensate for the uncertainties $Df\left(\cdot\right)$ from output channels such that the outputs go to zero asymptotically and all the signals in the closed-loop system are bounded.

Remark 13. One of the most important problems in linear multivariable control theory is to control a fixed plant such that its outputs track reference signals and reject disturbance produced by an external generator (the exosystem) [29]. In engineering practice, it is interesting to keep the outputs as zero in the absence of subsequent disturbances, and control the outputs to respond in a desired way, such as in aircraft system [30] and electro-magnetic suspension system [5]. ∇

Definiton 1. [31] The MIMO linear time-invariant systems (1) are said to have a (vector) relative degree $\{\nu_1, \nu_2, \cdots, \nu_p\}$ at equilibrium point if

- $c_i A^{k_i} B = 0_{1\times m_1},$ for all $0 \le k_i < \nu_i - 1,\ 1 \le i \le p$;
- $K_B = \begin{bmatrix} c_1 A^{\nu_1 - 1} B \\ \vdots \\ c_p A^{\nu_p - 1} B \end{bmatrix}$ has rank equal to the number of its rows (i.e. to the number of output channels), where $c_i,\ i = 1, \cdots, p$ are the rows of matrix C.

Remark 14. By Definition 1, for all $0 \le k_i < \nu_i - 1$, the row vector $c_i A^{k_i} B$ is zero, and for $k_i = \nu_i - 1$, it is nonzero (i.e. has at least a nonzero element) since the matrix K_B is full row rank. In view of condition $c_i A^{k_i} B = 0_{1\times m_1},$ for all $0 \le k_i < \nu_i - 1,\ 1 \le i \le p$, we see that for each output channel c_i, there is at least one input channel b_j such that $c_i A^{\nu_i - 1} b_j \neq 0$, i.e. the triple (A, b_j, c_i) has exactly relative degree ν_i, while for any other b_j, the corresponding relative degree is necessarily higher than or equal to ν_i.

Assumption 7. Suppose that the triples (A, B, C) and (A, D, C) have (vector) relative degrees $\{\nu_1, \nu_2, \cdots \nu_p\}$ and $\{\upsilon_1, \upsilon_2, \cdots \upsilon_p\}$, respectively. It is assumed that $\nu_i \le \upsilon_i,\ i = 1, \cdots, p$.

Remark 15. The reference [31] uses the pole placement method to design robust controller, and removes disturbances from output channels in the steady state based on Assumption 7. However, this issue becomes more complex when actuators faults are considered. ∇

Denoting that the $k_i th$ order time derivative of $y_i(t), i = 1, 2, \cdots, p$ as $y_i^{(k_i)}(t)$, we have

$$y_i^{(k_i)}(t) = \begin{cases} c_i A^{k_i} x(t), \quad k_i = 0, 1, \cdots, \nu_i - 1, \\ c_i A^{k_i} x(t) + c_i A^{k_i - 1} B u^f + c_i A^{k_i - 1} D f(x, \omega(t), t), \ \ k_i = \nu_i. \end{cases}$$

Consider the differential equation

$$y_i^{(\nu_i)}(t) = c_i A^{\nu_i} x(t) + c_i A^{\nu_i - 1} B u^f + c_i A^{\nu_i - 1} D f(x, \omega(t), t). \tag{52}$$

For the case that $r = \nu_1 + \nu_2 + \cdots + \nu_p$ is strictly less than n, setting

$$\zeta^i = \begin{bmatrix} \zeta_1^i \\ \zeta_2^i \\ \vdots \\ \zeta_{\nu_i-1}^i \end{bmatrix} = \begin{bmatrix} y_i \\ \dot{y}_i \\ \vdots \\ y_i^{\nu_i-1} \end{bmatrix} = \begin{bmatrix} c_i x \\ c_i A x \\ \vdots \\ c_i A^{\nu_i-1} x \end{bmatrix}$$

and $\zeta = \mathrm{col}(\zeta^1, \zeta^2, \cdots \zeta^p)$, then there exists a vector $\eta = \mathrm{col}(\eta_1, \eta_2, \cdots \eta_{n-r})$ such that system (49) can be transformed into a normal form in new coordinates $z = \mathrm{col}(\zeta, \eta) = Tx$ with T being invertible, described by

$$\dot{\zeta}^i = A_{\zeta^i}\zeta^i + F_{\zeta^i}x + B_{\zeta^i}u^f + D_{\zeta^i} f\left(x, \omega(t), t\right), \tag{53}$$

$$\dot{\eta} = R\zeta + Q\eta + D_\eta f\left(x, \omega(t), t\right), \tag{54}$$

$$y_i = \zeta_1^i, \;\; i = 1, 2, \cdots, p \tag{55}$$

where $x = T^{-1}z$,
$A_{\zeta^i} = \begin{bmatrix} 0 & I_{(\nu_i-1)} \\ 0 & 0 \end{bmatrix}$, $B_{\zeta^i} = \begin{bmatrix} 0 \\ c_i A^{\nu_i-1} B \end{bmatrix}$, $F_{\zeta^i} = \begin{bmatrix} 0 \\ c_i A^{\nu_i} \end{bmatrix}$ and $D_{\zeta^i} = \begin{bmatrix} 0 \\ c_i A^{\nu_i-1} D \end{bmatrix}$.

It can be seen that subsystems (53)- (55) are controlled by the input u^f and uncertainties $f\left(x, \omega(t), t\right)$. From [29] and [32], the state vector η is completely unobservable, and the subsystem

$$\dot{\eta} = Q\eta + D_\eta f\left(T^{-1}\left[0, \eta^T\right]^T, \omega(t), t\right) \tag{56}$$

is the zero dynamics.

Assumption 8. The triple (A, B, C) is minimum phase.

Remark 16. From [32], under Assumption 8, Q is a Hurwize matrix. Therefore, based on Assumption 6, state vector η in (56) is bounded. ∇

Denote $\xi = S\zeta$ where S is described by

$$S = \begin{bmatrix} I_{\nu_1-1} & & & & & \\ & & I_{\nu_2-1} & & & \\ \vdots & \vdots & \vdots & \ddots & & \\ & & & & I_{\nu_p-1} & \\ & 1 & & \cdots & & \\ & & & \ddots & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & & & & & 1 \end{bmatrix}.$$

Then system (53) can be written as

$$\dot{\xi} = A_\xi \xi + F_\xi x + B_\xi \Lambda(t) u + B_\xi \Sigma \psi(t) + D_\xi f(x, \omega(t), t) \tag{57}$$

where $x = T^{-1}\mathrm{col}(S^{-1}\xi, \eta)$,

$$A_\xi = S \begin{bmatrix} \begin{matrix} \mathbf{0}_{(\nu_1-1)\times 1} & I_{\nu_1-1} \\ 0 & 0_{1\times(\nu_1-1)} \end{matrix} & & \\ & \ddots & \\ & & \begin{matrix} \mathbf{0}_{(\nu_p-1)\times 1} & I_{\nu_p-1} \\ 0 & 0_{1\times(\nu_p-1)} \end{matrix} \end{bmatrix} S^{-1},\ B_\xi = \begin{bmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \\ K_B \end{bmatrix}, F_\xi = \begin{bmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \\ K_F \end{bmatrix} \text{ and } D_\xi = \begin{bmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \\ K_D \end{bmatrix}$$

with $K_B = \begin{bmatrix} c_1 A^{\nu_1} B \\ \vdots \\ c_p A^{\nu_p} B \end{bmatrix}$, $K_F = \begin{bmatrix} c_1 A^{\nu_1} \\ \vdots \\ c_p A^{\nu_p} \end{bmatrix}$ and $K_D = \begin{bmatrix} c_1 A^{\nu_1-1} D \\ \vdots \\ c_p A^{\nu_p-1} D \end{bmatrix}$. The fault mode $(\Lambda(t),\ \Sigma) \in \triangle$ satisfies Assumption 2.

It is easy to see that all the eigenvalues of A_ξ are zero and $\mathrm{rank}(-A_\xi, B_\xi)$ is $\nu_1+\nu_2+\cdots+\nu_p$, which means that the system (57) is controllable based on the PBH stability criterion. Thus for any give SPD matrix Φ, there is a unique SPD matrix P_ξ such that

$$P_\xi(A_\xi + B_\xi K) + (A_\xi + B_\xi K)^T P_\xi = -\Phi. \tag{58}$$

Under Assumption 2 and based on Proposition 1, there exists a time varying matrix function $K^*(t) \in \mathcal{R}^{m_1\times m_1}$ such that $B\Lambda(t) K^*(t) = B$. Then for the given SPD matrix Φ in (58), there exist $K_\Lambda(t) = K^*(t)K$ and a common SPD matrix P_ξ in (58) such that

$$P_\xi(A_\xi + B_\xi \Lambda(t) K_\Lambda(t)) + (A_\xi + B_\xi \Lambda(t) K_\Lambda(t))^T P_\xi = -\Phi. \tag{59}$$

Also, from that $B\Lambda(t) K^*(t) = B$, $c_i A^{k_i} B\Lambda(t) K^*(t) = c_i A^{k_i} B = 0_{1\times m_1}$, for all $0 \le k_i < \nu_i - 1$, $1 \le i \le p$, and $K_B \Lambda(t) K^*(t) = K_B$, which means that the (vector) relative degree of the triple (A,B,C) is equal to the (vector) relative degree of the triple $(A,B\Lambda(t)K^{*}(t),C)$ for all $(\Lambda(t),\ \Sigma)\in\triangle$ satisfying Assumption 2. Therefore, after actuator faults satisfying Assumption 2 occur, Assumption 7 is also satisfied.

Moreover, the fact that the rows of K_B are linearly independent implies that there exists a matrix K_B^{*} such that $I-K_BK_B^{*}=\mathbf{0}$. Then, for $K^{*}(t)$ satisfying that $B\Lambda(t)K^{*}(t)=B$, $I-K_B\Lambda(t)K^{*}(t)K_B^{*}=\mathbf{0}$. From Assumption 2 that $\mathrm{rank}(B\Lambda(t))=\mathrm{rank}(B)$, $\mathrm{rank}(K_B\Lambda(t))=\mathrm{rank}(K_B)$. then it follows from Lemma 2 that there exists a $K_2(t)$ such that $K_B\Lambda(t)K_2(t)=-K_B\Sigma\psi(t)$.

Based on the above hypothesis and analysis, under Assumption 2, the undesirable terms in (57), $F_\xi x$, $B_\xi\Sigma\psi(t)$ and $D_\xi f(x,\omega(t),t)$ can be compensated after a fault mode $(\Lambda(t),\Sigma)\in\Delta$ satisfying Assumption 2 occur. The FTC structure is given by

$$u=u_1+\hat{K}_\Lambda\xi+K_2(t)+u_u \tag{60}$$

where $\hat{K}_\Lambda$ is the estimation of $K_\Lambda(t)$. The auxiliary controller u_1 is given by

$$u_1=-\hat{K}^{*}K_B^{*}K_Fx \tag{61}$$

where $\hat{K}^{*}$ is the estimation of $K^{*}(t)$.

Under Assumption 4, there exist positive scalars μ and k_3 such that $\|\Sigma\psi(t)\|\le\mu k_3$. Since $\mathrm{Im}(D_\xi)\subset\mathrm{Im}(B_\xi)$, there exists a positive scalar k_4 such that $\left\|\xi^TP_\xi D_\xi f(x,\omega(t),t)\right\|\le\mu k_4\beta(x,t)\left\|\xi^TP_\xi B_\xi\right\|$. Note that the parameters μ, k_3 and k_4 are unknown. The two auxiliary control functions $K_2(t)$ and u_u are given by

$$\begin{aligned}K_2(t)&=-\frac{B_\xi^TP_\xi\xi\hat{k}_3^2}{\|\xi^TP_\xi B_\xi\|\hat{k}_3+\vartheta(t)},\\ u_u&=-\frac{B_\xi^TP_\xi\xi\hat{k}_4^2\beta^2(x,t)}{\|\xi^TP_\xi B_\xi\|\hat{k}_4\beta(x,t)+\vartheta(t)}\end{aligned} \tag{62}$$

where $\hat{k}_3$ and $\hat{k}_4$ are the estimations of k_3 and k_4 respectively, and $\vartheta(t)$ satisfies (24).

The parameters $\hat{K}_\Lambda$, $\hat{K}^{*}$, $\hat{k}_3$ and $\hat{k}_4$ are updated by

$$\begin{aligned}\dot{\hat{K}}_\Lambda&=-\Gamma_1\left((\xi\xi^TP_\xi B_\xi)^T+\epsilon_1\vartheta_0(t)\hat{K}_\Lambda\right),\\ \dot{\hat{K}}^{*}&=-\Gamma_2\left((K_B^{*}K_Fx\xi^TP_\xi B_\xi)^T+\epsilon_2\vartheta_0(t)\hat{K}^{*}\right),\\ \dot{\hat{k}}_3&=\gamma_1\left\|\xi^TP_\xi B_\xi\right\|-\gamma_1\vartheta(t)\hat{k}_3,\\ \dot{\hat{k}}_4&=\gamma_2\left\|\xi^TP_\xi B_\xi\right\|\beta(x,t)-\gamma_2\vartheta(t)\hat{k}_4\end{aligned} \tag{63}$$

where $\Gamma_1 = \Gamma_1^T > 0$ and $\Gamma_2 = \Gamma_2^T > 0$ are constant matrices. $\epsilon_1 \geq 6$, $\epsilon_2 \geq 6$, $\gamma_1 > 0$ and $\gamma_2 > 0$ are constant scalars. The matrix P_ξ is the solution of (59).

Let $\tilde{K}_\Lambda = \hat{K}_\Lambda - K_\Lambda(t)$, $\tilde{K}^* = \hat{K}^* - K^*(t)$, $\tilde{k}_3 = \hat{k}_3 - k_3$, $\tilde{k}_4 = \hat{k}_4 - k_4$. Then the error dynamics are described by

$$\begin{aligned}
\dot{\tilde{K}}_\Lambda &= -\Gamma_1 \left((\xi\xi^T P_\xi B_\xi)^T + \epsilon_1 \vartheta_0(t) K_\Lambda(t) + \epsilon_1 \vartheta_0(t) \tilde{K}_\Lambda \right), \\
\dot{\tilde{K}}^* &= -\Gamma_2 \left((K_B^* K_F x \xi^T P_\xi B_\xi)^T + \epsilon_2 \vartheta_0(t) K^*(t) + \epsilon_2 \vartheta_0(t) \tilde{K}^* \right), \\
\dot{\tilde{k}}_3 &= \gamma_1 \left\| \xi^T P_\xi B_\xi \right\| - \gamma_1 \vartheta(t) k_3 - \gamma_1 \vartheta(t) \tilde{k}_3, \\
\dot{\tilde{k}}_4 &= \gamma_2 \left\| \xi^T P_\xi B_\xi \right\| \beta(x,t) - \gamma_2 \vartheta(t) k_4 - \gamma_2 \vartheta(t) \tilde{k}_4.
\end{aligned} \tag{64}$$

the closed-loop system is described by

$$\dot{\xi} = A_\xi \xi + B_\xi \Lambda(t) \left(u_1 + \hat{K}_\Lambda \xi + K_2(t) + u_u \right) + F_\xi x + B_\xi \Sigma \psi(t) + D_\xi f(x, \omega(t), t), \tag{65}$$

and the subsystem (54) is described by

$$\dot{\eta} = RS^{-1}\xi + Q\eta + D_\eta f \left(T^{-1} \left[0, \eta^T \right]^T, \omega(t), t \right). \tag{66}$$

Remark 17. From (62) and (63), it can be seen that the auxiliary functions $K_2(t)$, u_u are continuous, and adaptive laws $\hat{K}_\Lambda$, $\hat{K}^*$, $\hat{k}_3$ and $\hat{k}_4$ are also continuous. Moreover, $\|K_2(t)\| \leq \hat{k}_3$ and $\|u_u(t)\| \leq \beta(x,t) \hat{k}_4$. ∇

The following theorem is ready to present.

Theorem 2. *For the error dynamics (64), the closed-loop system (65) and the subsystem (66), supposing that Assumptions 1-4 and Assumptions 6-8 are satisfied, then, the solution* $(\xi,\ \tilde{K}_\Lambda,\ \tilde{K}^*,\ \tilde{k}_3,\ \tilde{k}_4)$ *to the error dynamics (64) and the closed-loop system (65) is bounded, and the state vector* x *in (49) is bounded. Furthermore,*

$$\lim_{t \to \infty} y(t; t_0, x_0) = 0. \tag{67}$$

Proof: For the error dynamics (64) and the closed-loop system (65), a Lyapunov function candidate is chosen as

$$V\left(\xi, \tilde{K}_\Lambda, \tilde{K}^*, \tilde{k}_3, \tilde{k}_4\right) = \xi^T P_\xi \xi + tr\left(\Lambda(t) \tilde{K}_\Lambda^T \Gamma_1^{-1} \tilde{K}_\Lambda\right) + tr\left(\Lambda(t) \tilde{K}^{*T} \Gamma_2^{-1} \tilde{K}^*\right) + \mu\gamma_1 \tilde{k}_3^2 + \mu\gamma_2 \tilde{k}_4^2. \tag{68}$$

Then the time derivative of $V(\cdot)$ along the trajectories of (64) and (65) is

$$\dot{V} = \dot{V}_1 + \dot{V}_2 + \dot{V}_3 + \dot{V}_4 \tag{69}$$

where

$\dot{V}_1 = \xi^T \left(P_\xi A_\xi + A_\xi^T P_\xi\right)\xi + 2\xi^T P_\xi B_\xi \Lambda(t)\hat{K}_\Lambda \xi + 2tr\left(\Lambda(t)\tilde{K}_\Lambda^T \Gamma_1^{-1}\dot{\tilde{K}}_\Lambda\right) + tr\left(\dot{\Lambda}(t)\tilde{K}_\Lambda^T \Gamma_1^{-1}\tilde{K}_\Lambda\right),$

$\dot{V}_2 = 2\xi^T P_\xi F_\xi x + 2\xi^T P_\xi B_\xi \Lambda(t) u_1 + 2tr\left(\Lambda(t)\tilde{K}^{*T}\Gamma_2^{-1}\dot{\tilde{K}}^*\right) + tr\left(\dot{\Lambda}(t)\tilde{K}^{*T}\Gamma_2^{-1}\dot{\tilde{K}}^*\right),$

$\dot{V}_3 = 2\xi^T P_\xi B_\xi \Sigma\psi(t) + 2\xi^T P_\xi B_\xi K_2(t) + 2\mu\gamma_1 \tilde{k}_3 \dot{\tilde{k}}_3,$

$\dot{V}_4 = 2\xi^T P_\xi D_\xi f(x, \omega(t), t) + 2\xi^T P_\xi B_\xi u_u + 2\mu\gamma_2 \tilde{k}_4 \dot{\tilde{k}}_4.$

Substituting the auxiliary control functions (62) and the adaptive laws (63) into (69),

$$\begin{aligned} \dot{V}_1 &\le -\xi^T \Phi \xi + \vartheta_0(t)(3\delta_0' + 2\delta_1'), \\ \dot{V}_2 &\le \vartheta_0(t)(3\delta_0'' + 2\delta_1''), \\ \dot{V}_3 &\le \mu\vartheta(t)\left(1 + \tfrac{k_3^2}{4}\right), \\ \dot{V}_4 &\le \mu\vartheta(t)\left(1 + \tfrac{k_4^2}{4}\right) \end{aligned} \tag{70}$$

where δ_0', δ_1', δ_0'' and δ_1'' are positive scalars. Then, it can be concluded that

$$\dot{V} \le -\xi^T \Phi \xi + \kappa_0 \vartheta_0(t) + \kappa_1 \vartheta(t) \tag{71}$$

where $\kappa_0 = 3\delta_0' + 2\delta_1' + 3\delta_0'' + 2\delta_1''$ and $\kappa_1 = \mu\frac{k_3^2}{4} + \mu\frac{k_4^2}{4} + 2$.

Let $\bar{\xi} = \left(\xi, \tilde{K}_\Lambda, \tilde{K}, \tilde{k}_3, \tilde{k}_4\right)$. It follows from (71) that $\bar{\xi} \in L_\infty$. Then $\xi \in L_\infty$, $\left\|\hat{K}_\Lambda\right\| \in \mathcal{L}_\infty$, $\left\|\hat{K}\right\| \in \mathcal{L}_\infty$, $\left\|\hat{k}_3\right\| \in \mathcal{L}_\infty$, $\left\|\hat{k}_4\right\| \in \mathcal{L}_\infty$. It follows that $x \in \mathcal{L}_\infty$, $u \in L_\infty$ and $\dot{\xi} \in L_\infty$ and $\xi(t)$ is uniformly continuous. Therefore, using Barbălat Lemma, it can be obtained that $\lim_{t\to\infty} \xi(t) = 0$, and $\lim_{t\to\infty} y(t; t_0, x(t_0)) = 0$.

The proof is completed. ■

IV. Simulation

Two simulation examples will be presented to verify the results developed in this paper.

Example 1: Consider the nonlinear model of F-16 aircraft [30] (trimmed conditions are velocity=400ft/s, altitude=300000ft, cg=0.3$\bar{C}$, pitch rate=0deg/s, angle of attack=13.1deg, pitch angle=13.1deg, elevator deflection=0.4deg, throttle position=0.5227) given by

$$\dot{x} = Ax + Bu + f(x, \omega(t), t)$$

where $x = \mathrm{col}(\alpha, q, \theta)$, $u = \delta_e$ and

$$A = \begin{bmatrix} -0.29114 & 0.96353 & 0 \\ 0.41357 & -0.39716 & 0 \\ 0 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} -0.000602 \\ -0.034 \\ 0 \end{bmatrix}, f(x, \omega(t), t) = \begin{bmatrix} 0.426\alpha^2 + 0.024\alpha q + 0.161\alpha\theta - 0.08\theta^2 \\ -0.4191\alpha^2 + 0.9\alpha q \\ 0.000301\alpha^2 + 0.017\alpha q + 0.000124\alpha\theta + 0.0068 q\theta \end{bmatrix},$$

α is the angle of attack (deg), q is the pitch rate (deg/sec), θ is the pitch angle and δ_e is the elevator deflection (deg).

The elevator is assumed to be double-redundant such that the redundancy condition (5) is satisfied, i.e., for $u = [\delta_{e_1}, \delta_{e_2}]$, the input matrix B becomes

$$B = [b_1, b_2] = \begin{bmatrix} -0.000603 & -0.000603 \\ -0.035 & -0.035 \\ 0 & 0 \end{bmatrix}.$$

Then there exist a vector $\bar{B} = \mathrm{col}(0, 0, 1)$, matrices $W_1 = \left[\begin{smallmatrix} -1024 & -16 & 0 \\ -1024 & -16 & 0 \end{smallmatrix}\right]$ and $W_2 = [0, 0, 1]$ such that $[B, \bar{B}]\mathrm{col}(W_1, W_2) = I_{3\times 3}$.

For the given

$$Q = \begin{bmatrix} 0.0092 & 0.0018 & 0.0050 \\ 0.0018 & 0.0184 & 0.0070 \\ 0.0050 & 0.0070 & 0.0109 \end{bmatrix},$$

the solution P to Lyapunov equation (10) is given by

$$P = \begin{bmatrix} 0.0135 & 0.0004 & 0.0011 \\ 0.0004 & 0.0166 & 0.0042 \\ 0.0011 & 0.0042 & 0.0167 \end{bmatrix}.$$

From the expression of $f(x, \omega(t), t)$, it follows that

$$\begin{aligned} \frac{\left\|x^T P \bar{B} W_2 f(\cdot)\right\|}{\|x^T P B\|} &\leq \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)} \frac{\left\|x^T \bar{B} W_2 f(\cdot)\right\|}{\|x^T B\|} \\ &\leq \frac{\lambda_{\max}(P)}{\sqrt{2}\lambda_{\min}(P)} \frac{|0.000603x_1 + 0.035x_2|\,|0.5x_1 + 0.2x_2|}{|0.000603x_1 + 0.035x_2|} \\ &= \frac{\lambda_{\max}(P)}{\sqrt{2}\lambda_{\min}(P)} |0.5x_1 + 0.2x_2|, \end{aligned}$$

which implies that Assumption 5 is satisfied. Thus,

$$\begin{aligned} \left\|x^T P f(\cdot)\right\| &= \left\|x^T P B W_1 f(\cdot)\right\| + \left\|x^T P \bar{B} W_2 f(\cdot)\right\| \\ &\leq \left\|x^T P B\right\| \left(\|W_1 f(\cdot)\| + \frac{\lambda_{\max}(P)}{\sqrt{2}\lambda_{\min}(P)} |0.5x_1 + 0.2x_2| \right), \end{aligned}$$

and $\alpha(x, t)$ satisfying Assumption 5 can be chosen as

$$\alpha(x, t) = \|W_1 f(\cdot)\| + 1.2134\,|0.5x_1 + 0.2x_2|. \tag{72}$$

Therefore, the Assumption 5 holds if the uncertain $f(\cdot)$ experienced by the aircraft, satisfies (14) with $\alpha(x,t)$ given in (72). Here, $\alpha(x,t)$ shows the admissible bounds on uncertainty $f(\cdot)$, which is calculated for the specific systems from mathematical point of view.

In this example, the considered fault mode is that δ_{e_1} loses of effectiveness, and the efficient factor $\rho_1(t)$ is given by

$$\rho_1(t) = \begin{cases} 1, & \text{if } 0 \le t \le 20, \\ e^{(e^{-0.02(t-20)} - \ln 2)} - 1, & \text{if } 20 \le t \le 24, \\ 0.2586, & \text{if } 24 \le t < \infty, \end{cases}$$

and δ_{e_2} is stuck at $\sin(0.5t)$ after $24s$.

It can be verified that Assumptions 1-5 for Theorem 1 are satisfied. The control parameters are chosen as $\Gamma = 10^7 I$, $\gamma_1 = \gamma_2 = 5 \times 10^3$, $\vartheta_0(t) = 0.2e^{(-0.02t)}$, $\vartheta(t) = 5e^{(-0.02t)}$. The simulation results are shown in Figs. 1 and 2.

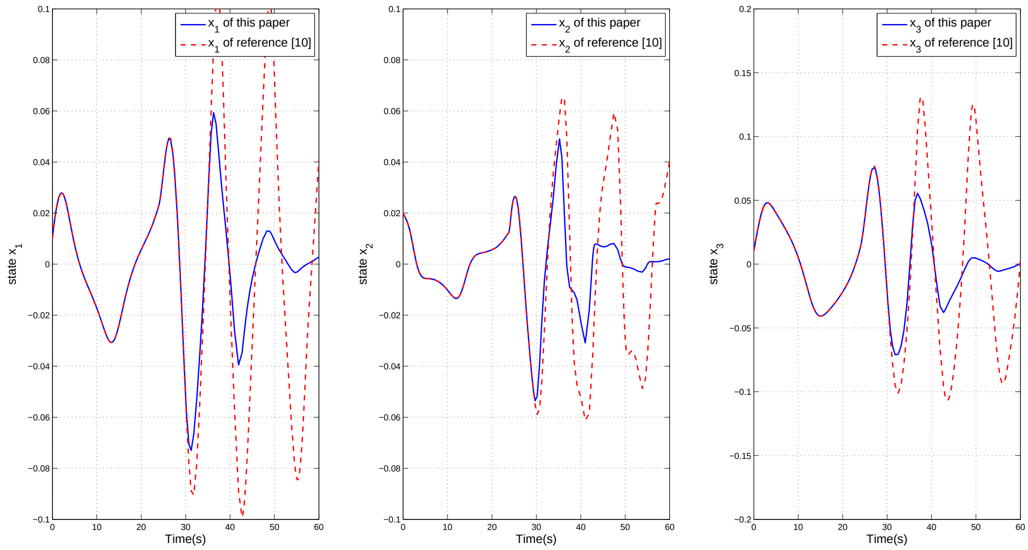

Fig. 1. Time responses of system states x

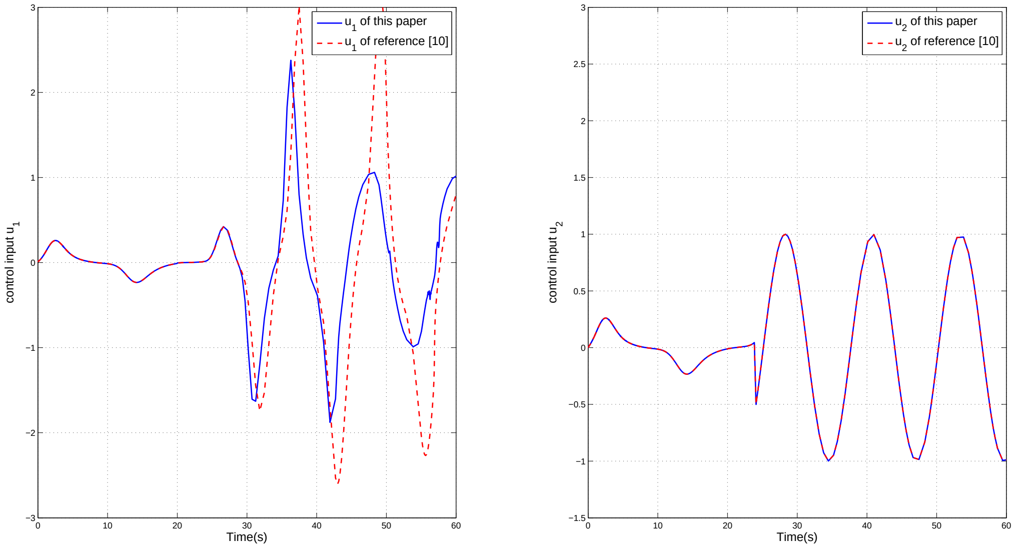

Fig. 2. Time response of control signals u

It can be seen from the solid blue lines in Fig. 1 that all the states in the closed-loop system are asymptotically stable before and after considered faults occur under the designed controller (15). Comparing the solid blue lines with the dashed red lines in Fig. 1, it can be seen that, after faults occur, all the states converge to zero faster under the controller designed in this paper than that in reference [10]. In addition, comparing the solid blue lines with the dashed red lines in Fig. 2, it can be seen that after faults occur, the amplitude of the actuator u_1 is smaller than that in [10].

Example 2: In reference [33], the aircraft Boeing747 lateral motion is described by $\dot{x} = Ax + Bu$, where $x = \mathrm{col}(v_b, p_b, r_b, \phi, \varphi)$, $u = \mathrm{col}(d_r, d_a)$. The five state variables are: lateral velocity v_b, roll rate p_b, yaw rate r_b, roll angle ϕ and yaw angle φ. The rudder position d_r and aileron position d_a are chosen as outputs $y = Cx$. In the case of landing, the matrices A, B and C are described by

$$A = \begin{bmatrix} -0.13858 & 14.326 & -219.04 & 32.167 & 0 \\ -0.02073 & -2.1692 & 0.91315 & 0.000256 & 0 \\ 0.00289 & -0.16444 & -0.15768 & -0.00489 & 0 \\ 0 & 1 & 0.000618 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

$$B = [b_1, b_2]\,,\ b_1 = \begin{bmatrix} 0.15935 \\ 0.01264 \\ -0.12879 \\ 0 \\ 0 \end{bmatrix},\ b_2 = \begin{bmatrix} 0.00211 \\ 0.21326 \\ 0.00171 \\ 0 \\ 0 \end{bmatrix},\ C = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Suppose that both rudder and aileron are double-redundant. Then there are four actuators such that $u = \mathrm{col}(d_{r_1}, d_{r_2}, d_{a_1}, d_{a_2})$ and $B = [b_{11}, b_{12}, b_{21}, b_{22}]$. Hence the matrix B becomes

$$b_{11} = \begin{bmatrix} 0.15935 \\ 0.01264 \\ -0.12879 \\ 0 \\ 0 \end{bmatrix}, b_{12} = \begin{bmatrix} 0.16 \\ 0.012 \\ -0.13 \\ 0 \\ 0 \end{bmatrix} b_{21} = \begin{bmatrix} 0.00211 \\ 0.21326 \\ 0.00171 \\ 0 \\ 0 \end{bmatrix}, b_{22} = \begin{bmatrix} 0.002 \\ 0.02 \\ 0.0015 \\ 0 \\ 0 \end{bmatrix}.$$

Consider the influence of the turbulence to the aircraft. The lumped disturbance $f\,(x, \omega(t), t)$ is given by $f(x, \omega(t), t) = 0.5\mathrm{sin}(\upsilon_b) + 0.5$. Then $\beta(\cdot)$ in Assumption 6 can be chosen as $\|f(x, \omega(t), t)\| + \pi$ with $\pi > 0$ being scalar. The distribution matrix D is chosen as [31], i.e., $D = \mathrm{col}(013858, 0.02073, -0.00289, 0, 0)$. Then it has that $c_1B = [0.1593, 0.1600, 0.0021, 0.0020]$, i.e., $\nu_1 = 1$, $c_2B = [0, 0, 0, 0]\,, c_2AB = [-0.1288, -0.1300, 0.0017, 0.0015]$, i.e., $\nu_2 = 2$ and $c_1D = 0.1386$, i.e., $\upsilon_1 = 1$, $c_2D = 0, c_2AD = -0.0029$, i.e., $\upsilon_2 = 2$. Therefore, $\nu_1 = \upsilon_1 = 1$ and $\nu_2 = \upsilon_2 = 2$ satisfy Assumption 7.

The simulated fault mode is that a loss of effectiveness fault occurs on the second actuator b_{12}, and the efficiency factor is given by

$$\rho_2\,(t) = \begin{cases} 1, & \text{if } 0 \le t \le 30, \\ e^{(e^{-0.02(t-30)} - \ln 2)} - 1, & \text{if } 30 \le t \le 34, \\ 0.2586, & \text{if } 34 \le t < \infty, \end{cases}$$

the third actuator is stuck at $2\mathrm{sin}(t)$. The simulation results are shown in Figs. 3-5.

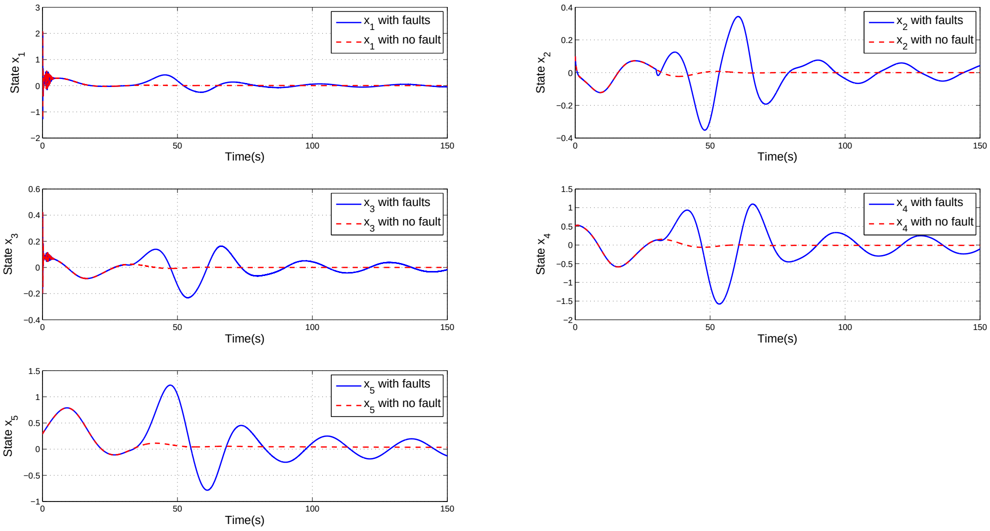

Fig. 3. Time responses of system states x

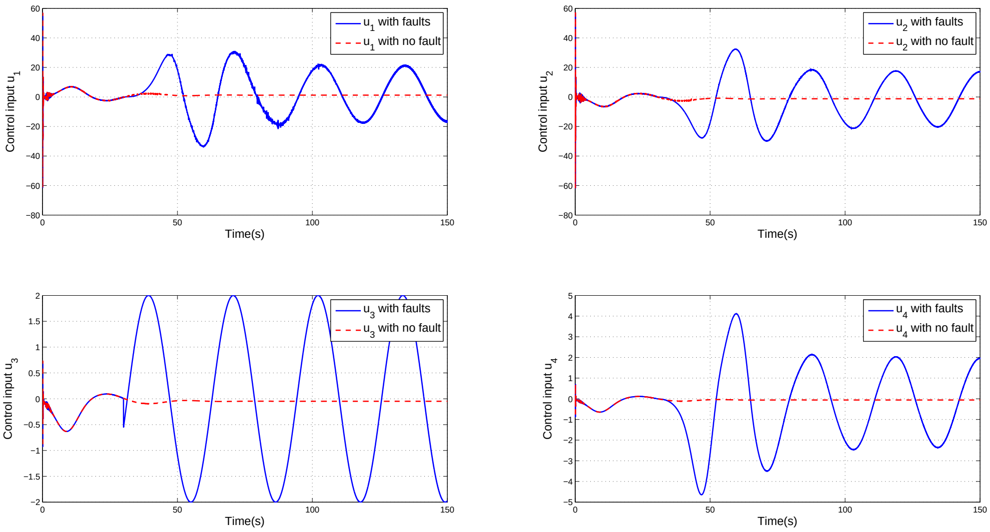

Fig. 4. Time response of control signals u

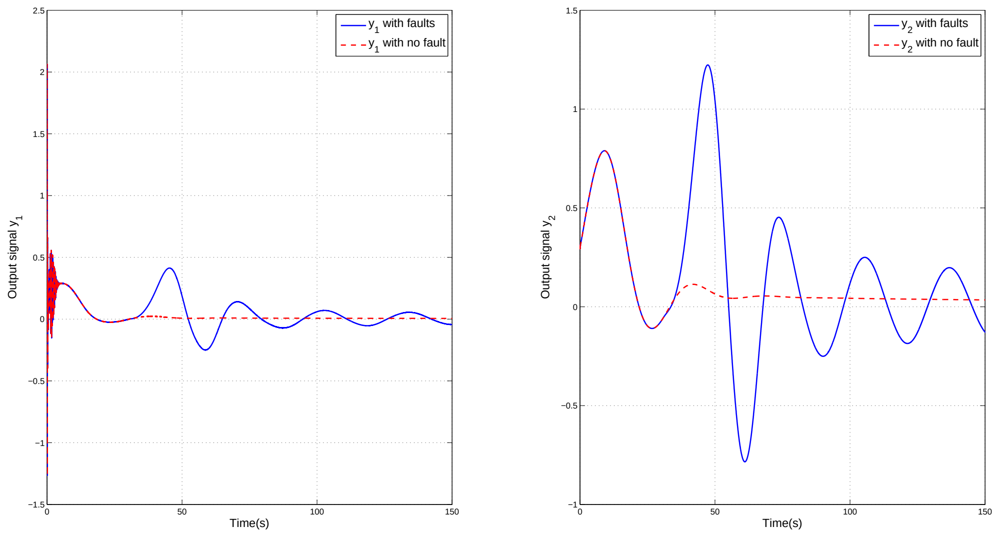

Fig. 5. Time response of outputs y

From the dashed red lines in Figs. 3 and 5, it can be seen that without faults, the states x are bounded, the outputs y are asymptotically stabilized by the designed controller (60) and go to zero asymptotically. Moreover, from the solid blue lines in Fig. 3 and 5, it can be seen that after actuator faults occur, the designed controller (60) can asymptotically stabilize the outputs y, and ensure that the states x are bounded simultaneously. However, the outputs y go to zero slower than that without faults. From Fig. 4, it can be seen that when actuator faults occur, actuators in control produce stronger control signals to compensate for the uncertainties and faults.

V. Conclusion

Two novel adaptive and robust FTC schemes have been proposed for linear faulty MIMO systems with unmatched uncertainties under a set of conditions developed in this paper. The $\sigma-$ modification adaptive laws have been used to estimate the values of time-varying fault parameters. Based on matched and unmatched characteristic of the uncertainties, two adaptive and robust FTC design approaches have been proposed with different control objectives. The future work will focus on development of new adaptive robust FTC methodology for more general nonlinear systems.

VI. ACKNOWLEDGEMENTS

This work is supported in part by the National Natural Science Foundation of China (Grant 61533008 and 61573180), by the Priority Academic Program Development of Jiangsu Higher Education Institutions, Fundamental Research Funds for the Central Universities (NO. NE2014202).

REFERENCES

[1] Y. Zhang and J. Jiang, "Bibliographical review on reconfigurable fault-tolerant control systems," *Annu. Rev. Control*, vol. 32, no. 2, pp. 229-252, 2008.

[2] H. S. Wu, "Continuous adaptive robust controllers guaranteeing uniformultimate boundedness for uncertain nonlinear systems," *Int. J. Control*, vol. 72, no. 2, 115-122, 1999.

[3] H. S. Wu, "Adaptive Robust Tracking and Model Following of Uncertain Dynamical Systems With Multiple Time Delays," *IEEE Trans. Autom. Control*, vol. 49, no. 4, 611-616, 2004.

[4] Z. Qu, " Global stabilization of nonlinear systems with a class of unmatched uncertainties," *Syst. Contr. Lett.*, vol. 18, no. 4, pp. 301-307, 1992.

[5] S. H. Li, J. Yang and W. H. Chen, "Generalized extended state observer based control for systems with mismatched uncertainties," *IEEE Trans. Ind. Electron.*, vol. 12, no. 59, pp. 4792-4802, 2012.

[6] W. J. Cao and J. X. Xu, " Nonlinear integral-type sliding surface for both matched and unmatched uncertain systems," *IEEE Trans. Autom. Control*, vol 8, no. 49, pp. 1355-1360, 2004.

[7] F. Castaños and L. Fridman, " Analysis and design of integral sliding manifolds for systems with unmatched perturbations," *IEEE Trans. Autom. Control*, vol 5, no. 51, pp. 853-858, 2006.

[8] P. A.Ioannou and J. Sun, *Robust adaptive control*, Courier Dover Publications, 2012.

[9] D. Ye and G. H. Yang, "Adaptive fault-tolerant tracking control against actuator faults with application to flight control," *IEEE Trans. Contr. Syst. Technol.*, vol. 14, no. 6, pp. 1088-1096, 2006.

[10] X. J. Li and G. H. Yang, " Robust adaptive fault-tolerant control for uncertain linear systems with actuator failures," *IET Control Theory Appl.*, vol 6, no. 10, pp. 1544-1551, 2012.

[11] L. Y. Hao and G. H. Yang, " Robust fault tolerant control based on sliding mode method for uncertain linear systems with quantization," *ISA transactions*, vol 5, no. 5, pp. 600-610, 2013.

[12] L. B. Wu, G. H. Yang and D. Ye, " Robust adaptive fault-tolerant control for linear systems with actuator failures and mismatched parameter uncertainties," *IET Control Theory Appl.*, vol 6, no. 8, pp. 441-449, 2014.

[13] Y. Ma, B. Jiang and G. Tao, "Adaptive actuator failure identification for microsatellites under closed-loop control," on-line availiable, *IEEE Trans. Contr. Syst. Technol.*, 2015, in press.

[14] B. Jiang and F. N. Chowdhury, "Fault estimation and accommodation for linear MIMO discrete-time systems," *IEEE Trans. Contr. Syst. Technol.*, vol. 13, no. 3, pp. 493-499, 2005.

[15] B. Jiang, M. Staroswiecki and V. Cocquempot, "Fault accommodation for nonlinear dynamic systems," *IEEE Trans. Autom. Control*, vol. 51, no. 9, pp. 1578-1583, 2006.

[16] Z. Gao and S. X. Ding, "Actuator fault robust estimation and fault-tolerant control for a class of nonlinear descriptor systems," *Automatica*, vol. 43, no. 5, pp. 912-920, 2007.

[17] S. Yin, H. Luo and S. X. Ding, "Real-time implementation of fault-tolerant control systems with performance optimization," *IEEE Trans. Ind. Electron.*, vol 47, no. 4, pp. 2402-2411, 2014.

[18] X. G. Yan and C. Edwards, "Adaptive sliding mode observer based fault reconstruction for nonlinear systems with parametric uncertainties," *IEEE Trans. Ind. Electron.*, vol. 55, no. 11, pp. 4029-4036, 2008.

[19] M. Liu and P. Shi, "Sensor fault estimation and tolerant control for Itô stochastic systems with a descriptor sliding mode approach," *Automatica*, Vol. 49, no. 5, pp. 1242-1250, 2013.

[20] M. Zhong, D. Zhou and S. X. Ding, "On designing fault detection filter for linear discrete time-varying systems," *IEEE Trans. Autom. Control*, vol. 55, no 7, pp. 1689-1695, 2010.

[21] G. H. Yang and D. Ye, "Reliable control of linear systems with adaptive mechanism," *IEEE Trans. Autom. Control*, vol. 55, no. 1, pp. 242-247, 2010.

[22] G. Tao, S. M. Joshi and X. Ma, " Adaptive state feedback and tracking control of systems with actuator failures," *IEEE Trans. Autom. Control*, vol. 46, no. 1, pp. 78-95, 2001.

[23] S. Chen, G. Tao and S. M. Joshi, " On matching conditions for adaptive state tracking control of systems with actuator failure," *IEEE Trans. Autom. Control*, vol. 47, no. 3, pp. 473-478, 2002.

[24] X. Tang, G. Tao and L. Wang, "Robust and adaptive actuator failure compensation designs for a rocket fairing structural-acoustic model," *IEEE Trans. Aerosp. Electron. Syst.*, vol. 40, no. 4, pp, 1359-1366, 2004.

[25] X. Zhang, M. M. Polycarpou and T. Parisini, "A robust detection and isolation scheme for abrupt and incipient faults in nonlinear systems," *IEEE Trans. Autom. Control*, vol 47, no. 4, pp. 576-593, 2002.

[26] Francis B A. *A course in* H_∞ *control theory*. 1987.

[27] Lavretsky E and Wise K. *Robust and adaptive control: with aerospace applications*. Springer Science Business Media, 2012.

[28] X. G. Yan, S. K. Spurgeon and C. Edwards, "Global decentralised static output feedback slidingmode control for interconnected time-delay systems," *IET Control Theory Appl.*, vol 2, no. 6, pp. 192-202, 2012.

[29] A. Isidori, *Nonlinear control systems*, Springer, 1995.

[30] B. L. Stevens and F. L. Lewis, *Aircraft control and simulation*, John Wiley Sons, Inc., 1992.

[31] L. Y. Wen, G. Tao and H. Yang, "Aircraft turbulence compensation using adaptive multivariable disturbance rejection techniques," *J. Guid. Control Dyn.*, pp. 1-9, 2014.

[32] H. K. Khalil and J. W. Grizzle, "Nonlinear systems," New Jersey: Prentice hall, 1996.

[33] S. Chen, G. Tao and S. M. Joshi, "Adaptive actuator failure compensation control for MIMO systems," *Int. J. Control*, vol. 77, no. 15, pp. 1307-1317, 2004.