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Song, Dandan, Wang, Huamao and Yang, Zhaojun (2014) *Learning, pricing, timing and hedging of the option to invest for perpetual cash flows with idiosyncratic risk.* Journal of Mathematical Economics, 51 . pp. 1-11. ISSN 0304-4068.

Learning, pricing, timing and hedging of the option to invest for perpetual cash flows with idiosyncratic risk

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Abstract

The paper considers the option of an investor to invest in a project that generates perpetual cash flows, of which the drift parameter is unobservable. The investor invests in a liquid financial market to partially hedge cash flow risk and estimation risk. We derive two 3-dimensional non-linear free-boundary PDEs satisfied by the utility-based prices of the option and the cash flows. We provide an approach to measure the information value. A numerical procedure is developed. We show that investors have not only idiosyncratic-risk-induced but also estimation-risk-induced precautionary saving demands. A growth of estimation risk, risk aversion or project risk delays investment, but it is accelerated if the project is more closely correlated with the market. Partial information results in a considerable loss, which reaches the peak value at the exercising time and increases with project risk and estimation risk. The more risk-averse the investor or the weaker the correlation, the larger the loss.

Keywords: Partial information, Hedging, Real options, Precautionary

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savings, Information value, Non-linear PDEs

JEL: C61, G11, G31

1. Introduction

The paper considers the option of an investor to invest in a non-tradable irreversible project that generates perpetual cash flows, of which the drift parameter is unobservable (partial information). We assume the investor hedges the estimation risk and cash flow risk by investing in a liquid financial market. However, generally speaking, an investor is still exposed to considerable unhedged idiosyncratic risk and therefore, we price the real option and cash flows by consumption utility indifference pricing approach.

We study the investor's joint decisions of investment for perpetual cash flows, consumption/savings, and portfolio selection when he cannot fully insure the cash flow shocks and needs to learn about the uncertain drift parameter. As a result, risk attitude, idiosyncratic risk, and the subjective estimate of the drift parameter have substantial effects on the decisions.

Applying consumption utility indifference pricing method, continuous-time stochastic control and filtering theory, we derive a system of high-dimensional non-linear free-boundary PDEs (Partial Differential Equations) for the implied values of the real option and cash flows. We develop an effective finite difference procedure, which allows us to present an extensive analysis with regard to the impact of learning about uncertainty on the pricing, timing and hedging of the option to invest.

Our contributions. Intuitively, unlike the full information case (i.e. the drift is observable), an investor with partial information would very likely make

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a "wrong" decision and thus incur a loss. It is interesting to measure the loss since it can be considered as the maximum cost an investor would like to pay in order to obtain the full information. Naturally, we call the loss the implied information value. However, there are almost no papers to measure the quantity of the loss in real options literature including Décamps et al. (2005) among others. In this paper, we provide an approach to quantify the loss. It surprises us that the partial information leads to a considerable decrease (loss) in the implied value of the option to invest, i.e. the implied information value is significant relative to investment cost (sunk cost).

Our analysis indicates that the implied information value reaches the maximum value at investment threshold and it increases with risk aversion, project risk, and prior variance. Their growths also considerably raise the precautionary saving motive, decrease the certainty-equivalent wealth of cash flows, and delay real investment. Investors, particularly ones with partial information, are still exposed to the idiosyncratic risk of cash flows after investment, though the systematic risk can be hedged away by investing in a liquid financial market. Consequently, in contrast to Décamps et al. (2005), learning about the uncertain drift parameter is valuable all the time no matter whether the option is exercised or not, and a more effective estimate of the drift remarkably increases the implied values of both the option and cash flows, speeding up investment.

Unlike Miao and Wang (2007) and Décamps et al. (2005), our results provide four new insights into the irreversible investment for perpetual cash flows with an unobservable drift and idiosyncratic risk. Firstly, investors have not only idiosyncratic-risk-induced but also have estimation-risk-induced precau-

tionary saving demands both before and after investment. Secondly, a growth of estimation risk, risk aversion or project risk delays investment. Thirdly, investment is accelerated if the project is more closely correlated with the market. Last but more importantly, partial information results in a considerable loss, which reaches the peak value around the investment threshold and increases with project risk and estimation risk. The more risk-averse the investor or the weaker the correlation, the larger the loss.

Furthermore, we develop an efficient finite difference procedure to solve the system of three-dimensional non-linear free-boundary PDEs that characterize the model solutions. This is numerically more challenging than the one-dimensional problem considered by Miao and Wang (2007), and the two-dimensional problems discussed by Décamps et al. (2005), Yang and Yang (2012) and Yang et al. (2011) due to the high non-linearity in two spatial dimensions. The additional complexity arises from the dependence of the filtering estimate of the drift not only on cash flows, but also on the market portfolio.

Literature review. Real investment decisions play a fundamental role in entrepreneurial activities and modern economics. A real investment is typically irreversible with uncertain future rewards and flexible investment time. The right to decide when to invest in a project is analogous to an American style financial option and so it is called real option. The real options approach to investment under uncertainty originates from the work of Myers (1977) and presently becomes more popular. Major contributions along this research line are McDonald and Siegel (1986) and Dixit and Pindyck (1994) among others. Recently, Henderson and Hobson (2002), Miao and Wang

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(2007), Henderson (2007) and Ewald and Yang (2008) study the real options problem under incomplete markets by utility indifference pricing approach. However, almost all papers including Miao and Wang (2007) in the literature assume that an investor has access to full information. Under this assumption, the mean appreciation rate of the value or cash flows of a project and the driving Brownian motion are observable, which is of course unrealistic. Following Yang and Yang (2012) and Song and Yang (2013), the feature of this paper is that in contrast to the above papers, we relax this assumption to suppose that the investor can not observe the drift parameter and the Brownian motion appearing in the stochastic differential equation describing the cash flows. In other words, we assume an investor has only access to partial information, as argued by Gennotte (1986), Lakner (1998), Brennan (1998), Yang and Ma (2001), Xiong and Zhou (2007), Monoyios (2007), Monoyios (2008), Wang (2009) among others.

The "partial information" assumption in our model is quite realistic since the drift parameter and the paths of Brownian motions are fictitious mathematical tools, which are of course not observable. On the contrary, the volatility/dispersion parameter for the cash flows will be observable since one can prove that the volatility is adapted to the filtration generated by the cash flows.

Our model is closely related with Décamps et al. (2005) and Klein (2009) since the two papers also discuss the real options problems with partial information. But the distinction between them and our paper are also evident: First, we suppose that the drift parameter follows a normal distribution other than a two-point distribution as assumed by them. Second and the

most importantly, we solve the real options problem based on consumption utility indifference pricing approach, while they assume that investors are risk-neutral. Taking into account that a real investment is generally exposed to considerable idiosyncratic risk, this difference makes our problem more interesting and naturally more challenging as well. We fill the gap by developing a comprehensive model and a numerical method for an investor who has to deal with both estimation risk and idiosyncratic risk resulting from cash flows.

To the best of our knowledge, this paper is most related with Yang et al. (2011), Yang and Yang (2012) and Song and Yang (2013). However, we assume in this paper that an investor obtains stochastic cash flows rather than a lump-sum payoff upon investment as assumed by Yang and Yang (2012) and Song and Yang (2013). This distinction is trivial in a risk neutral world but significant in our model since we suppose that the investor is risk-averse and thus, one can not get an equivalent lump-sum payment simply by discounting future cash flows. On the other hand, although Yang et al. (2011) do consider the situation where the project generates cash flows, they make the assumption that the investor has only assess to one risk-free asset. In our paper, aside from the risk-free asset, there exists another tradable risky asset (e.g. market portfolio) in a liquid financial market which can partially hedge the cash flow risk and estimation risk. This difference is fundamental because the problem we discuss here is more realistic, interesting and challenging. In this way, we can obtain more insightful conclusions. For example, once the absolute value of the correlation between the market portfolio and the project goes up, the investor will hedge more risk of the cash flows and the

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remaining idiosyncratic risk become less, which decreases his precautionary savings demand, naturally raises the implied value of the option and speeds up the real investment as well.

In fact, our paper undertakes a systematic investigation of learning and hedging and many conclusions here have not been addressed yet. Unlike the previous papers, for instance, the learning in the present paper has implications not only on the implied option value of waiting, but also on the certainty-equivalent wealth of perpetual cash flows after investment. More importantly, the estimation risk from learning and the idiosyncratic risk left after partial hedging have effects on hedging and precautionary saving demands, which affect investment decisions and the implied information value.

Furthermore, cash flow volatility has three effects on the investor's decisions in this study. First, the volatility increases the option value because of the standard asymmetric convex payoff of an option. The second effect is that idiosyncratic volatility instead of project volatility induces the standard precautionary saving demand against cash flow fluctuations. Third, the idiosyncratic volatility rather than project volatility *increases* the estimation risk induced by learning about the drift parameter for a given fixed prior variance. This finding is implied by our filtering results and is also confirmed by the effects of the project volatility and prior variance as is evident from our numerical results. The interpretation is that the realized cash flows with higher idiosyncratic volatility make it more difficult to estimate the drift effectively and the estimation risk is increased accordingly, which naturally induces a larger precautionary saving demand.

Finally, the study incorporating learning and hedging is also particularly

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challenging. The main additional difficulty is that, the filtering estimate of the drift parameter depends not only on the observations of cash flows, but also on the value of the market portfolio. Consequently, solving a combined stochastic control and optimal stopping problem of investing for cash flows leads to a system of three-dimensional non-linear free-boundary PDEs. This is more complicated than a two-dimensional PDE in Yang and Yang (2012) and Yang et al. (2011). The system of PDEs appears too complex to derive a closed form solution and numerical technique is required to approximate the solution, which brings intensive computational demands due to the high non-linearity in two spatial dimensions.

The remainder of the paper proceeds as follows. Section 2 presents an optimization investment model under uncertainty and partial information. In Section 3, we derive a system of non-linear PDEs with free-boundary conditions based on the filtering theory and utility indifference pricing approach. The results under the full information case and the implied information value are introduced as well. Section 4 analyzes the economic implications. Section 5 concludes. Appendix A provides computational details, Appendix B derives model solutions and Appendix C discusses the smooth-pasting conditions and presents a verification theorem.

2. Model setup

This section establishes an investment model with uncertainty under partial information and incomplete markets. The investment payoff is continuous-time cash flows with infinite horizon rather than a lump-sum payment.

Given a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we consider two standard

Brownian motions B and Z defined on it with correlation coefficient $\rho \in [-1, 1]$. An investor may borrow or lend one risk-free asset at a constant risk-free $r > 0$ and invest in a liquid risky asset (e.g. market portfolio) with the price process P following the geometric Brownian motion (GBM) below:

$$dP_t/P_t = \alpha_p dt + \sigma_p dB_t, \tag{1}$$

where α_p and σ_p are known positive constants. Let $\eta = (\alpha_p - r)/\sigma_p$ denote the sharp ratio of the market portfolio.

In addition, the investor can choose to invest in an irreversible investment project by paying sunk cost $I > 0$ at an arbitrary time τ. After investment, the investor obtains a perpetual stream of payoffs X, which is observable and follows the arithmetic Brownian motion (ABM):[1]

$$dX_t = \mu dt + \sigma_x dZ_t, \tag{2}$$

where the project volatility σ_x is a known positive constant and the drift parameter μ is an unobservable Gaussian random variable with the prior mean m_0 and the variance v_0. The market is incomplete when $|\rho| \neq 1$ and the volatility σ_x can be decomposed into the diversifiable systematic volatility $\rho \sigma_x$ and the undiversifiable idiosyncratic volatility $\sigma_x \sqrt{1 - \rho^2}$.

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[1]Other dynamics processes can describe the payoff as well, e.g. a GBM in Yang and Yang (2012) and a mean-reversion process in Ewald and Yang (2008). ABM implies that payoffs can take negative values as losses. This specification is assumed by Miao and Wang (2007) as well and their analysis shows that the main implications are robust to the GBM because the precautionary savings effect and the option effect are independent of the specifications of the payoff process.

We denote by $\mathcal{F} \equiv \{\mathcal{F}_t\}_{t\geq 0}$ the filtration (*full information*) generated by the unobservable random variable μ, processes B and Z; and by $\mathcal{G} \equiv \{\mathcal{G}_t\}_{t\geq 0}$ the filtration (*partial information*) generated by the observable price process P and the cash flow process X. This setting is obviously more reasonable than the full-information assumption in Miao and Wang (2007) among others.

We denote by $\mathcal{T}$ the set of all stopping times with respect to $\mathcal{G}$ taking values in $[0, \infty)$. Let $\mathcal{C}$ be the space of $\mathcal{G}$-progressively measurable process C, taking value on $[0, \infty)$, such that $\int_0^\infty C_t dt < \infty$ (a.s.), where C_t represents the consumption rate selected by the investor at time $t \in [0, \infty)$. We call a consumption plan C is *admissible*, if $C \in \mathcal{C}$. Denote by π_t the amount of wealth allocated to the risky asset at time t, and let Π be the set of $\mathcal{G}$-adapted process π which satisfies the integrability condition $\int_0^\infty \sigma_p^2 \pi_t^2 dt < \infty$ (a.s.).

An investor is characterized by his initial wealth W_0, a time-discount rate β and his preference $U(\cdot)$. At any time $t \geq 0$, if the option to invest is not exercised, he must choose a stopping time $\tau \in \mathcal{T}$, a consumption plan $C \in \mathcal{C}$ and an investment portfolio $\pi \in \Pi$ to maximize his following expected lifetime utility of consumption conditional on partial information:

$$\sup_{(\tau,C,\pi)\in\mathcal{T}\times\mathcal{C}\times\Pi} J(\tau,C,\pi) \equiv \mathbb{E}\left[\int_t^\infty \exp\left(-\beta s\right) U(C_s)ds \,|\mathcal{G}_t\right], \qquad (3)$$

subject to the budget constraint:

$$\begin{cases} dW_s = (rW_s + \pi_s(\alpha_p - r) - C_s)ds + \pi_s\sigma_p dB_s, t \leq s < \tau, \\ W_\tau = W_{\tau-} - I, \\ dW_s = (rW_s + \pi_s(\alpha_p - r) + X_s - C_s)ds + \pi_s\sigma_p dB_s, s > \tau, \\ W_t \text{ given}, W_s > 0 \text{ for } s \geq t, \end{cases}$$

where the process X is given by (2). We assume in this paper the preference

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$U(\cdot)$ is a constant absolute risk aversion (CARA) utility function given by[2]

$$U(c) = -\exp(-\gamma c)/\gamma, c \in \Re, \qquad (4)$$

where $\gamma > 0$ is the absolution risk aversion parameter.

Remark 1. *The problem (3) we consider here is a combined stochastic control and optimal stopping problem under partial information, which leads to a free boundary problem for two 3-dimensional non-linear PDEs. It is much more challenging than that discussed by Miao and Wang (2007), who need only to solve an ODE (Ordinary Differential Equation). It is also much more complicated than the pure optimal stopping problem considered by Décamps et al. (2005), who consider a lump-sum payoff model without hedging opportunities in a risk-neutral world mainly with two possible values of the drift parameter.*

3. Model solutions

We summarize model solutions in this section and present derivation in Appendix B. According to the separation theorem (e.g. Gennotte (1986)), we deal with the optimization problem by first deriving the filtering estimate

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[2]CARA utility is widely applied in utility-based studies of real options to reduce the PDE's dimension by exploiting the property of wealth independence, see, e.g. Miao and Wang (2007) who point out that the homogeneity property of the constant relative risk aversion (CRRA) utility function does not hold for the real options problem because of the jump of wealth at investment time. Since our three-dimensional PDEs thanks to the CARA utility already requires intensive efforts in computation, the model with the CRRA utility may impose too many difficulties to find the solution.

for the drift parameter and then solving an equivalent optimization problem conditional on this estimate.

Denote $m_t \equiv \mathbb{E}(\mu|\mathcal{G}_t)$ and $v_t \equiv \mathbb{E}[(\mu - m_t)^2|\mathcal{G}_t]$. The conditional mean m_t and the conditional variance v_t represent the estimate of μ and estimation risk (error) respectively. Filtering theory ensures that the conditional distribution of μ is Gaussian and hence m_t is the optimal estimate. An application of filtering technique (Theorem 12.1 in Liptser and Shiryayev (1977)) leads to

$$\begin{cases} \frac{m_t}{v_t} = \frac{m_0}{v_0} - \frac{\rho}{\sigma_x \sigma_p (1-\rho^2)} \left[\log(\frac{P_t}{P_0}) + (\frac{1}{2}\sigma_p^2 - \alpha_p)t \right] + \frac{1}{\sigma_x^2(1-\rho^2)}(X_t - X_0) \\ v_t = \frac{v_0 \sigma_x^2 (1-\rho^2)}{v_0 t + \sigma_x^2(1-\rho^2)}. \end{cases} \tag{5}$$

As expected, $v_t \to 0$ and $m_t \to \mu$ for each $t > 0$ if $|\rho| \to 1$. Therefore, when the tradable risky asset and the investment project are perfectly correlated, the investor's problem is simplified into an optimization problem with full information. Furthermore, the expression of v_t in (5) implies that the idiosyncratic volatility $\epsilon = \sigma_x \sqrt{1 - \rho^2}$ instead of the systematic volatility $\rho \sigma_x$ results in estimation risk. This is because the change resulting from systematic volatility is not a noise to estimate the state variable μ. In fact, the change due to systematic volatility can be accurately measured from (1) and so it will not increase the estimation risk.

Define the innovation process $\tilde{Z}$ by

$$d\tilde{Z}_t = \frac{1}{\sigma_x} \left(dX_t - m_t dt \right), \tag{6}$$

and then the dynamics of the cash flow process is transformed to

$$dX_t = m_t dt + \sigma_x d\tilde{Z}_t. \tag{7}$$

According to the filtering estimate and the separation theorem, we can restate

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the optimization problem (3) as an equivalent optimization problem with (7) replacing (2).

Thanks to CARA utility (4) and the consumption utility indifference pricing approach, we denote the implied value (i.e. certainty-equivalent wealth) of the option (resp. of cash flows) by $y = g(t, X_t, m_t)$ (resp. by $z = f(t, X_t, m_t)$). Solving the stochastic control problems for them leads to the following two second-order non-linear PDEs, of which functions $f(t, x, m)$ and $g(t, x, m)$ are solutions respectively.

$$
\begin{aligned}
rf &= x + f_t + (m - \rho\sigma_x\eta)f_x + \tfrac{\sigma_x^2}{2}[f_{xx} - \gamma r(1 - \rho^2)f_x^2] \\
&\quad + \tfrac{v_t^2}{2\sigma_x^2(1-\rho^2)}(f_{mm} - \gamma r f_m^2) + v_t(f_{mx} - \gamma r f_m f_x),
\end{aligned}
\tag{8}
$$

$$
\begin{aligned}
rg &= g_t + (m - \rho\sigma_x\eta)g_x + \tfrac{\sigma_x^2}{2}[g_{xx} - \gamma r(1 - \rho^2)g_x^2] \\
&\quad + \tfrac{v_t^2}{2\sigma_x^2(1-\rho^2)}(g_{mm} - \gamma r g_m^2) + v_t(g_{mx} - \gamma r g_m g_x),
\end{aligned}
\tag{9}
$$

subject to the no-bubble condition $\lim_{x\to-\infty} g(t, x, m) = 0$ and the free-boundary conditions:[3]

$$
\begin{cases}
g(\tilde{t}, \tilde{x}, \tilde{m}) = f(\tilde{t}, \tilde{x}, \tilde{m}) - I, \\
g_t(\tilde{t}, \tilde{x}, \tilde{m}) = f_t(\tilde{t}, \tilde{x}, \tilde{m}), \\
g_x(\tilde{t}, \tilde{x}, \tilde{m}) = f_x(\tilde{t}, \tilde{x}, \tilde{m}), \\
g_m(\tilde{t}, \tilde{x}, \tilde{m}) = f_m(\tilde{t}, \tilde{x}, \tilde{m}),
\end{cases}
\tag{10}
$$

where the subscript of functions f and g denotes the differentiation with respect to that variable. Formally, we obtain the following theorem:

Theorem 3.1. *Suppose that $f(t, x, m)$ and $g(t, x, m)$ are solutions of the PDEs formulated by (8), (9) and (10). Define stopping time τ^* by*

$$
\tau^* = \inf \left\{ t \geq 0 : g(t, X_t, m_t) \leq f(t, X_t, m_t) - I \right\},
$$

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[3]The discussion about the smooth-pasting conditions is left to Appendix C.

and then τ^* is the optimal exercising time of the option to invest. The optimal consumption rate is given by

$$
\begin{cases}
C_t^* = \frac{\beta - r}{\gamma r} + r[W_t + g(t, X_t, m_t) + \frac{\eta^2}{2\gamma r^2}], & 0 \leq t < \tau^*; \\
C_t^* = \frac{\beta - r}{\gamma r} + r[W_t + f(t, X_t, m_t) + \frac{\eta^2}{2\gamma r^2}], & t \geq \tau^*.
\end{cases}
$$

The optimal portfolio rule is given by

$$
\begin{cases}
\pi_t^* = \frac{\eta}{\sigma_p \gamma r} - \frac{\rho \sigma_x}{\sigma_p} g_x(t, x, m), & 0 \leq t < \tau^*; \\
\pi_t^* = \frac{\eta}{\sigma_p \gamma r} - \frac{\rho \sigma_x}{\sigma_p} f_x(t, x, m), & t \geq \tau^*.
\end{cases}
$$

The implied value $F(t, X_t, m_t)$ of the option to invest is given by

$$
F(t, X_t, m_t) = \max \left\{ g(t, X_t, m_t), f(t, X_t, m_t) - I \right\}. \tag{11}
$$

This theorem shows that the investor has not only idiosyncratic-risk-induced precautionary saving demand as explained by Miao and Wang (2007), but also has estimation-risk-induced precautionary saving demand. Thanks to (8) and (9), such two demands disappear once the investor is risk-neutral, i.e. $\gamma = 0$. However, in sharp contrast to Miao and Wang (2007), even in a risk-neutral world, the estimation risk and idiosyncratic risk must be still taken into account while pricing the cash flows and the option to invest.

According to the theorem, the time-discount rate β has impact on consumption choice and the total consumption utility, but it is independent of the portfolio rules, implied value and the investment time. This conclusion is reasonable since we expect that two investors with different time-discount rates have identical prices and exercising times, and it is similar to that obtained by Yang and Yang (2012) and Song and Yang (2013), but differs from Dixit and Pindyck (1994), Décamps et al. (2005) and many others who use

the ordinary utility indifference pricing instead of the superior consumption utility indifference pricing.

Furthermore, in sharp contrast to Miao and Wang (2007), the investment threshold $\tilde{x}$ here is no longer constant and it depends on time (estimation risk in essence) and the estimate of the drift parameter or equivalently the price level of the liquid risky asset. The conclusion is also significantly different from Décamps et al. (2005), who conclude that the investment threshold is independent of time and depend only on the decision maker's beliefs.

Next, we quantify the loss resulting from partial information, which is measured by the *implied information value* (IV) since it represents the maximum cost an investor would like to pay in return for getting full information.

In our model, the drift parameter μ of the cash flows is randomly selected by nature from the distribution $\mathcal{N}(m_0, v_0)$, but its realization (μ_0) becomes known immediately to the investor who has full information. As far as an investor with full information, the drift parameter can be therefore considered as a known constant, which corresponds to the special case of Theorem 3.1: $m_0 = \mu_0$ and $v_0 = 0$. Under this situation, the state variables in our optimization model exclude time t and filtering estimate m and we can directly get the following results as a corollary of Theorem 3.1, which correspond those derived by Miao and Wang (2007) in their model IV where the time discount rate β takes the special value r, i.e. the risk-free interest rate.

Corollary 3.2. *Suppose that for a given value μ_0, i.e. a sample point from the distribution $\mathcal{N}(m_0, v_0)$, of the drift parameter, functions $f(x; \mu_0)$ and $g(x; \mu_0)$ satisfy*

$$rf = x + (\mu_0 - \rho\sigma_x\eta)f_x(x) + \frac{1}{2}\sigma_x^2\left[f_{xx} - \gamma r(1 - \rho^2)f_x^2\right],$$

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and

$$rg = (\mu_0 - \rho\sigma_x\eta)g_x(x) + \frac{1}{2}\sigma_x^2\left[g_{xx} - \gamma r(1-\rho^2)g_x^2\right]$$

subject to the no-bubble condition $\lim_{x\to-\infty} g(x) = 0$ *and the free-boundary conditions*

$$\begin{cases} g(\tilde{x}) = f(\tilde{x}) - I, \\ g_x(\tilde{x}) = f_x(\tilde{x}) = \frac{1}{r}, \end{cases}$$

and define stopping time τ^* *by*

$$\tau^* = \inf\left\{t \geq 0 : g(X_t;\mu_0) \leq f(X_t;\mu_0) - I\right\},$$

then τ^* *is the optimal exercising time of the option to invest with full information. The optimal consumption rate is given by*

$$\begin{cases} C_t^* = \frac{\beta-r}{\gamma r} + r[W_t + g(X_t;\mu_0) + \frac{\eta^2}{2\gamma r^2}], & 0 \leq t < \tau^*; \\ C_t^* = \frac{\beta-r}{\gamma r} + r[W_t + f(X_t;\mu_0) + \frac{\eta^2}{2\gamma r^2}], & t \geq \tau^*. \end{cases}$$

And the optimal portfolio rule is given by

$$\begin{cases} \pi_t^* = \frac{\eta}{\sigma_p\gamma r} - \frac{\rho\sigma_x}{\sigma_p}g_x(X_t;\mu_0), & 0 \leq t < \tau^*; \\ \pi_t^* = \frac{\eta}{\sigma_p\gamma r} - \frac{\rho\sigma_x}{\sigma_p}\frac{1}{r}, & t \geq \tau^*. \end{cases}$$

The implied value of the option to invest with full information depends on the sample point μ_0 *and is given by*

$$F^{Full}(X_t;\mu_0) = \max\left\{g(X_t;\mu_0), f(X_t;\mu_0) - I\right\},$$

where, following Miao and Wang (2007), *we have*

$$f(X_t;\mu_0) = \frac{X_t}{r} + \frac{\mu_0 - \rho\sigma_x\eta}{r^2} - \frac{\gamma\sigma_x^2(1-\rho^2)}{2r^2}.$$

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From this corollary, with full information, the implied value of the option is actually a function of the random variable μ and thus, we naturally define the implied information value IV as its expected value minus the corresponding value with partial information, i.e. at any time $t \geq 0$, the implied information value is given by

$$IV(t, X_t, m_t) \equiv \int_{-\infty}^{\infty} F^{Full}(X_t; u)\varphi(u)du - F(t, X_t, m_t), \qquad (12)$$

where $\varphi(\cdot)$ is the normal probability density function with mean m_0 and variance v_0. Intuitively, the implied information value must decrease with time since learning will continuously make partial information become closer to full information.

4. Implications: pricing, timing, hedging and learning

In this section, we discuss the implications of our model by numerical simulations with regard to learning about uncertainty, pricing option, timing investment, hedging against risk, and implied information value under partial information in an incomplete market.

Our numerical results are based on the following annualized baseline parameter values unless otherwise stated: risk-free interest rate $r = 0.05$, risk aversion $\gamma = 1$, initial project value $X_0 = 1$, investment cost $I = 1$, and the volatility of cash flows $\sigma_x = 0.3$. The prior mean m_0 and variance v_0 of the drift parameter of cash flows are 0.06 and 0.03^2 respectively. The mean return rate α_p and volatility σ_p of the tradable risky asset are 0.06 and 0.5 respectively. The correlation coefficient between the tradable asset and the cash flows is $\rho = 0.8$.
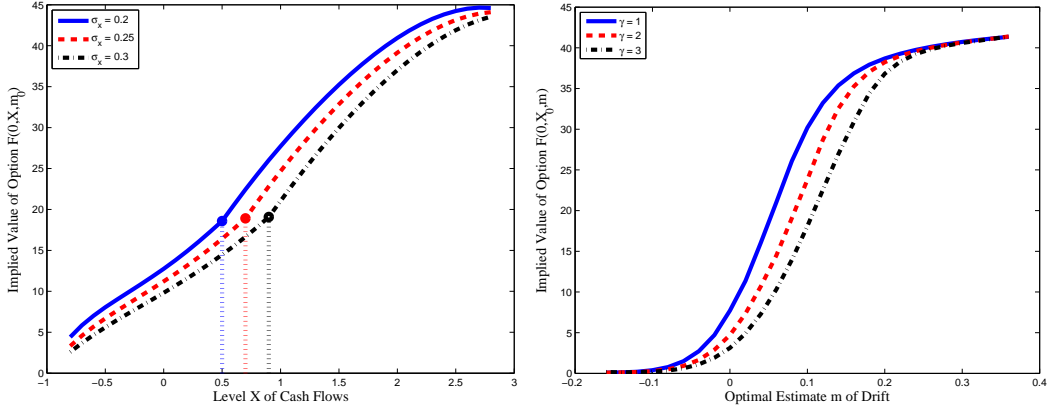
(a) $F(t = 0, X, m_0)$ for three levels of σ_x (b) $F(t = 0, X_0, m)$ for three levels of γ

Figure 1: The figure explains the implications for pricing and timing under partial information by displaying the implied value of option $F(t = 0, X, m)$ (a) against the level X of cash flows for three levels of volatility σ_x; and (b) against the estimate m of the drift for three levels of risk aversion γ with baseline parameter values.

## 4.1. Implications for pricing and timing

We begin with the numerical illustration of the investment decision characterized by Theorem 3.1. Figure 1 depicts the implied value $F(t = 0, X, m)$ of the option given by (11). Not surprisingly, Figure 1(a) says that the implied value rises with the level X of cash flows. Figure 1(b) shows that the implied value increases quickly with the estimate m of the drift, i.e. the option is more valuable if its holder is optimistic about the return of the project's cash flows. However, conversely to standard real options theory, Figure 1(a) says that the implied value of the option decreases with the project volatility σ_x. We note that as expected, the larger the volatility the higher the investment threshold but somewhat surprisingly, the less the implied value, the larger the investment threshold. This is because a less implied

value results from a larger project risk, which also leads to a even less value of cash flows. For example, Figure 1(a) plots that when cash flows reach the investment threshold $\tilde{X}(0, m) = 0.7$ (resp. 0.9) for $\sigma_x = 0.25$ (resp. 0.3), the option is exercised.

Figure 1(b) also states that the implied value of the option deceases with the risk aversion γ. Particularly, we find from most of the figures in the text that the effects of the parameters on the implied value of the option reach a maximum when the level of cash flows is close to the investment threshold. In addition, we emphasize that in sharp contrast to Décamps et al. (2005) and Miao and Wang (2007), the implied value $F(t, X_t, m_t)$ of the option to invest depends on the level X of cash flows, the estimate m of the drift and time t and for this reason, to compute the implied values, we need to solve the high-dimension PDEs (8) and (9) with a free boundary.

4.2. Implications for hedging and learning

Figure 2(a) shows the effects of the estimate m of the drift and the correlation ρ between the market portfolio and the cash flows on the implied value of the option to invest. It is shown that the implied option value $F(t, X, m)$ increases with the absolute value $|\rho|$ of the correlation. There are two reasons why this happens: First, a strong correlation means a less idiosyncratic risk since more risks are hedged, which reduces the idiosyncratic-risk-induced precautionary saving demand; Second, a strong correlation leads to a more effective estimate for the unobservable drift and so the estimation risk is declined, which reduces the estimation-risk-induced precautionary saving demand. Obviously, the former is ignored by Décamps et al. (2005) and the latter is not discussed by Miao and Wang (2007). In addition, Figure 2(a)
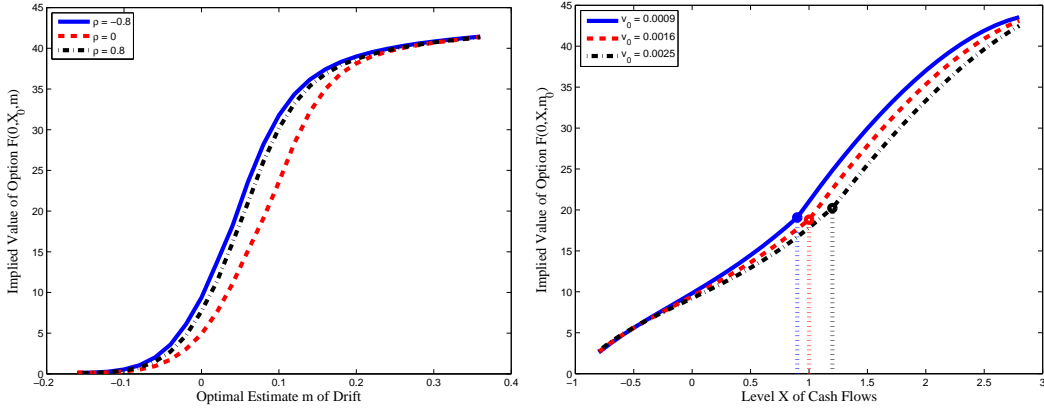
(a) $F(t = 0, X_0, m)$ for three levels of ρ (b) $F(t = 0, X, m_0)$ for three levels of v_0

Figure 2: The figure illustrates the implications for hedging and learning under partial information by depicting the implied value of option $F(t = 0, X, m)$ (a) against m for three levels of correlation coefficient ρ; and (b) against X for three levels of prior variance v_0 with baseline parameter values.

states that the option to invest under a negative rather than positive correlation case is more valuable. This quite accords with the corresponding conclusion from the standard equilibrium pricing theory.

Since the estimate m of the drift has considerable impact on the implied option value as seen in Figures 1 and 2, the learning by filtering techniques is economically valuable. In fact, learning can reduce the uncertainty of the drift or estimation risk, as shown by the expression of the posterior variance v_t in (5). To highlight the effect of estimation risk, Figure 2(b) displays the effects of the prior variance v_0 on the implied value $F(t = 0, X, m)$ of the option. It turns out that the implied value descends with the prior variance v_0. The reason is that a higher variance of the drift makes it more difficult for an investor to estimate effectively the drift and thus, the investor

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is more likely to make a "wrong" decision. In sharp contrast to Décamps et al. (2005), who conclude that an investor should accelerate investment in a more uncertain situation, Figure 2(b) states that the investor should reversely delay investment if the prior variance is increased. It turns out that the distinction results from the two different assumptions about project payoff: We address cash flows while Décamps et al. (2005) consider a lump-sum payoff case. A large uncertainty of the drift increases the precautionary saving demands, which, in our model, also reduces the certainty-equivalent wealth $f(t, X, m)$ of cash flows after investment. The decline is even more than the decline of the implied option value $g(t, X, m)$ before investment and putting both together, an investor should finally postpone investment.

Figure 3 exhibits the effects of project volatility, prior variance and the estimate of the drift on investment threshold. In short, the estimation risk and project risk almost have the same effect on investment threshold: The larger the risk, the higher the investment threshold, i.e. investment should be delayed. In addition, Figure 3 indicates that the higher the posterior estimate of the drift, the less the investment threshold, i.e. investment should be accelerated. This result is reasonable but opposite to Décamps et al. (2005), who derive that a higher estimate of drift raises the option value and delays investment as well. The distinction is due to the fact that in our model, a higher estimate m of the drift not only increases the implied option value $g(t, X, m)$ before investment but also raises the implied value $f(t, X, m)$ of the cash flows after investment. The rise of the latter is more than the rise of the former.
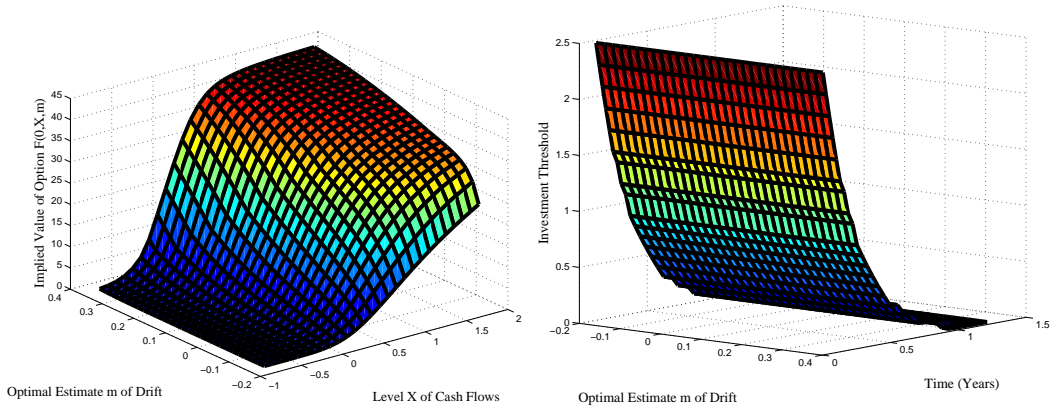
Furthermore, the three-dimensional Figure 4(a) plots the implied value

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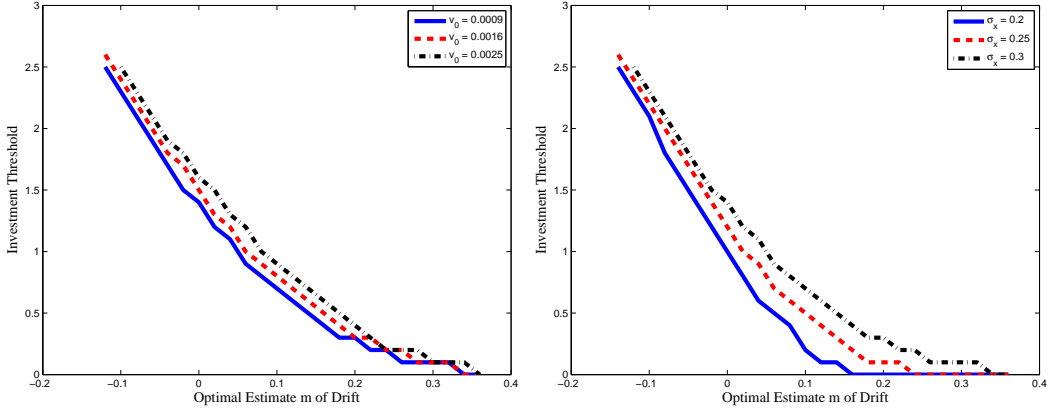
(a) $\tilde{X}(t = 0, m)$ for three levels of v_0 (b) $\tilde{X}(t = 0, m)$ for three levels of σ_x

Figure 3: The figure shows the impact of learning on the investment threshold $\tilde{X}(t = 0, m)$ against m under partial information for (a) three levels of prior variance v_0; and (b) three levels of volatility σ_x with baseline parameter values.

(a) implied value of option $F(t = 0, X, m)$ (b) investment threshold $\tilde{X}(t, m)$

Figure 4: The figure plots (a) the implied value of option $F(t = 0, X, m)$; and (b) the investment threshold $\tilde{X}(t, m)$ under partial information with baseline parameter values.

of the option at varying levels of cash flows X and the estimate m of the drift keeping time $t = 0$, while the three-dimensional Figure 4(b) displays the investment threshold at varying levels of the estimate m of the drift and time t. Evidently, the figures tell us the same but clearer story with the preceding text and so the analysis is omitted.

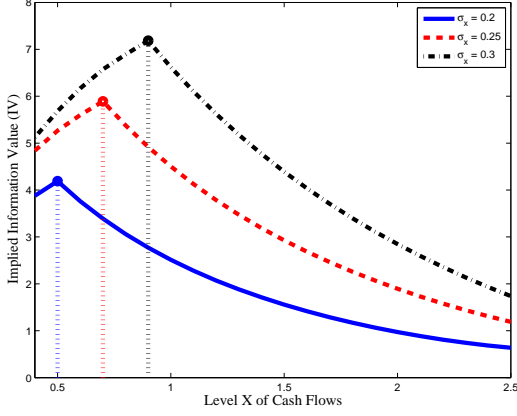### 4.3. The effects of parameters on the implied information value

According to our previous analysis, learning is able to reduce parameter uncertainty and increase the implied values of a project and the option to invest in the project. Unfortunately, the uncertainty cannot be completely removed in a finite time horizon and consequently, investors would incur a loss since he might often make a "wrong" decision based on his partial information. Naturally, we wonder how much the loss is. This is an interesting problem since the loss can be considered as a reasonable cost investors would like to pay in order to obtain full information. However, to the best of our knowledge, there are almost no papers to address the problem in the real options literature including Décamps et al. (2005) and Miao and Wang (2007) among others.

In this subsection, based on our formula (12) for measuring the loss due to partial information, we examine how the loss (i.e. implied information value) is linked to project risk, estimation risk, risk aversion and the correlation between the project and the market portfolio. Strikingly, the loss is considerable.

Figures 5 and 6 plot the loss measured by the implied information value IV at varying levels of the project volatility σ_x, prior variance v_0, risk aversion γ and correlation ρ. The figures show that the loss is significant relative to the

sunk cost $I = 1$. The implied information value experiences a steady growth with cash flows X within the waiting region and peaks at the exercising time. After investment has taken place, it declines gradually. The conclusions are quite in agreement with intuition. In fact, when the economic state varies around the exercising threshold, an important decision will be made soon. At this critical period, sufficient information is especially valuable to make a right decision. On the contrary, if the state is far away from the investment threshold, it should be quite easy to determine whether the investment should be exercised or postponed and naturally, the information is not so important. In particular, after investment has taken place, investors are not bothered about when to exercise the investment option and thus the value of information moves downward, although information is still valuable since it helps make a good hedging strategy. Actually, this phenomenon is similar with the fact seen in the previous figures that the effects of parameters on the implied value of the option peak at the exercising time.

Specifically, Figures 5(a) and 5(b) show that the implied information value IV increases with project volatility σ_x and the prior variance v_0. This is intuitively obvious because a larger project risk and estimation risk represent a higher uncertainty about the investment and drift parameter and therefore information is more valuable to make a good decision. Figure 6(a) explains that the more risk-averse the investor, the larger the implied information value. This is reasonable since investors with higher risk aversion levels are more desirous of information to reduce uncertainties. Figure 6(b) indicates that the stronger the correlation between the project and the market, the less the implied information value. This is also expected because a strong
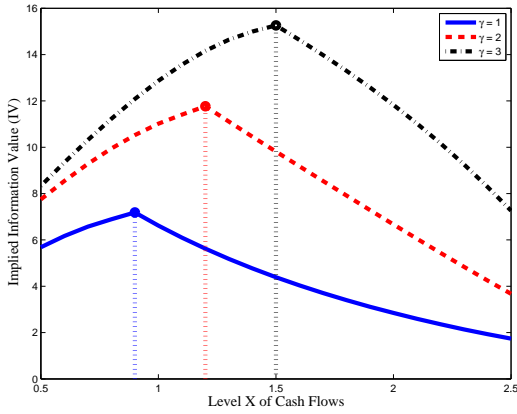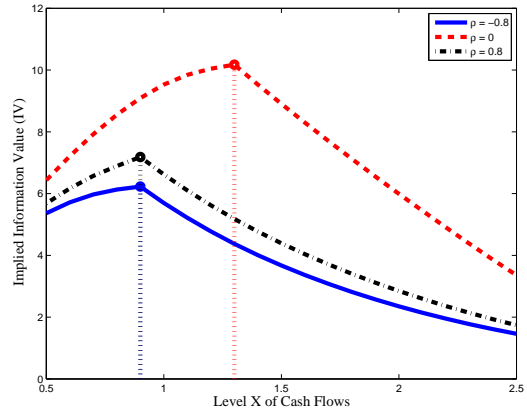
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(a) The effects of volatility σ_x
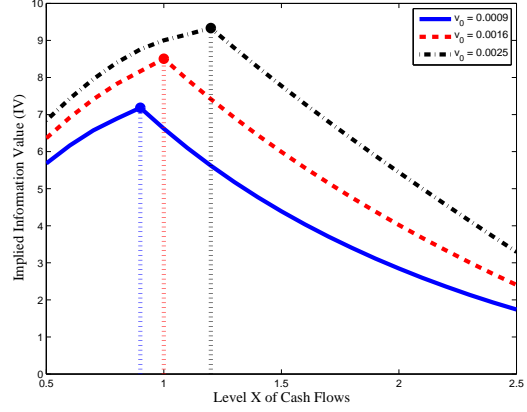
(b) The effects of prior variance v_0

Figure 5: The figure demonstrates the effects of three levels of (a) volatility σ_x; and (b) prior variance v_0 on the implied information value $IV(X)$ against X with baseline parameter values.
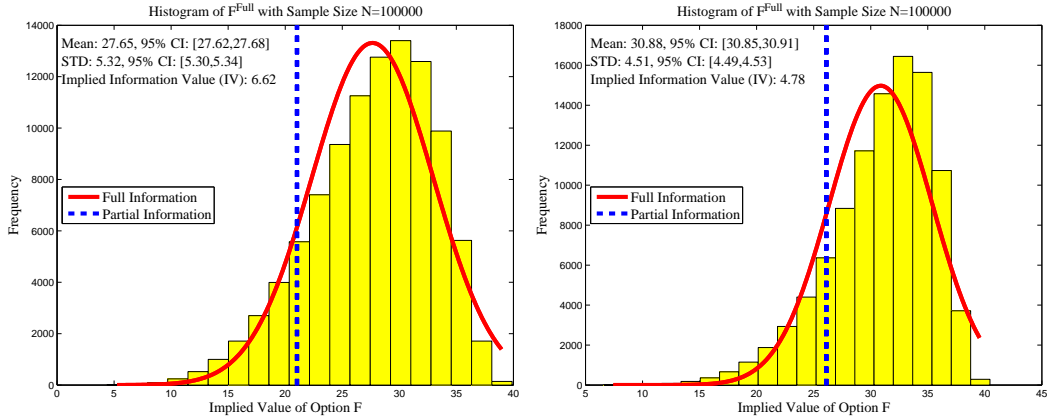

(a) The effects of risk averse γ

(b) The effects of correlation ρ

Figure 6: The figure demonstrates the effects of (a) three levels of risk averse γ; and (b) three levels of correlation coefficient ρ on the implied information value $IV(X)$ against X with baseline parameter values.

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(a) Histogram of $F^{Full}(X_0; \mu_0)$ (b) Histogram of $F^{Full}(X_0; \mu_0)$ with $m_0 = 8\%$

Figure 7: The figure displays the histogram of $F^{Full}(X_0; \mu_0)$ with the density from the fitted normal distribution under (a) the prior mean $m_0 = 6\%$; and (b) the prior mean $m_0 = 8\%$ with baseline parameter values.

correlation allows investors to make an efficient estimate of the drift based on the observable process of the market return and accordingly, information is not so valuable.

Figure 7 plots the histogram of the implied option value $F^{Full}(X_0; \mu_0)$ under full information situation and its density fitted from the corresponding normal distribution. The figure says that in contrast with Yang and Ma (2001), who conclude that the information value in their model is independent of the estimate of the drift parameter, the implied information value IV here decreases with the estimate. It turns out that after the option has been exercised, the information in our model is still important unlike Décamps et al. (2005) but it is not so valuable as it was before investment since it can only help investors to hedge the cash flow risk of the project by investing in a liquid financial market. Additionally, a large estimate m leads to an early

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investment and thus, the implied information value decreases naturally with the estimate.

In addition, as shown in Figure 7, it is not uncommon but somewhat counter-intuitive at first sight that the implied option values with full information for some sample points from the prior normal distribution $\mathcal{N}(m_0, v_0)$ are less than the implied option value $F(0, X_0, m)$ with partial information.

5. Conclusions

Based on consumption utility indifference pricing approach, this paper deals with the learning, pricing, timing, and hedging of the option to invest for perpetual cash flows, of which the drift parameter is unobservable. We implement a computational procedure to solve the system of high-dimensional non-linear PDEs with a free boundary for the investment problem. We present a method to quantify the loss resulting from replacing full information with partial information. Numerical results reveal that the loss due to partial information is substantial in terms of the implied information value. The loss reaches the maximum value at investment threshold. The risk aversion, project risk, and estimation risk considerably raise precautionary saving motive and delay the real investment. These results indicate that learning about the uncertainty is important and valuable all the time. We believe that our investment model taking into account both hedging and learning and the numerical methods developed here are applicable to finance theory and entrepreneurial activities, such as evaluating natural resource investments, valuing alternative market entry and exit strategies, and examining agency conflicts and financial policy.

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Acknowledgments

Appendices

Appendix A Computation methods

We solve the highly non-linear system of free-boundary PDEs (8)-(10) under partial information in the following way. We build an equally spaced lattice in (t, x, m)-space within the domain $[0, t_h] \times [x_l, x_h] \times [m_l, m_h]$ defined by the grid points

$$\{(t_k, x_i, m_j) \,|\, k = 1, \cdots, N_t + 1, \ i = 1, \cdots, N_x + 1, \ j = 1, \cdots, N_m + 1\},$$

where t_h, x_l, x_h, m_l and m_h are artificial (lower and upper) bounds, $x_i = x_l + (i-1)\Delta x$, $m_i = m_l + (j-1)\Delta m$, and $t_i = (k-1)\Delta t$ for fixed positive spacing parameters $\Delta x = \frac{x_h - x_l}{N_x}$, $\Delta m = \frac{m_h - m_l}{N_m}$ and $\Delta t = \frac{t_h}{N_t}$. We denote the approximated implied values of $f(t, x, m)$ and $g(t, x, m)$ by $f_{i,j}^k$ and $g_{i,j}^k$ respectively.

For each $g_{i,j}^k$ in the interior of the grid, we use the central difference to approximate the derivatives in space given below. The approximations of the derivatives of $f_{i,j}^k$ are similar.

$$\frac{\partial g_{i,j}^k}{\partial x} = D_x^0 g_{i,j}^k = \frac{g_{i+1,j}^k - g_{i-1,j}^k}{2\Delta x},$$

$$\frac{\partial g_{i,j}^k}{\partial m} = D_m^0 g_{i,j}^k = \frac{g_{i,j+1}^k - g_{i,j-1}^k}{2\Delta m},$$

$$\frac{\partial^2 g_{i,j}^k}{\partial x^2} = D_{xx}^2 g_{i,j}^k = \frac{g_{i+1,j}^k - 2g_{i,j}^k + g_{i-1,j}^k}{\Delta x^2},$$

$$\frac{\partial^2 g_{i,j}^k}{\partial m^2} = D_{mm}^2 g_{i,j}^k = \frac{g_{i,j+1}^k - 2g_{i,j}^k + g_{i,j-1}^k}{\Delta m^2}.$$

$$\frac{\partial^2 g_{i,j}^k}{\partial x \partial m} = D_{xm}^2 g_{i,j}^k = \frac{g_{i+1,j+1}^k - g_{i+1,j-1}^k + g_{i-1,j-1}^k - g_{i-1,j+1}^k}{4\Delta x \Delta m}.$$

We apply a backward procedure starting at the time t_h. The algorithm for computing $g_{i,j}^k$ is outlined as follows and the computation of $f_{i,j}^k$ is similar and so omitted. For convenience we rewrite the PDE (9) in short notation as

$$rg = g_t + \nu g_x + a_1 g_{xx} - b_1 g_x^2 + d_1 g_{mm} - e_1 g_m^2 + v_t g_{mx} - h_1 g_m g_x, \qquad (13)$$

where $\nu = m - \rho \sigma_x \eta$, $a_1 = \frac{\sigma_x^2}{2}$, $b_1 = a_1 \gamma r (1 - \rho^2)$, $d_1 = \frac{v_t^2}{2\sigma_x^2(1-\rho^2)}$, $e_1 = d_1 \gamma r$, and $h_1 = v_t \gamma r$.

$Step\,1$: At time t_h, i.e. $k = N_t + 1$, and for a given m_j, the PDE (13) is degenerated into a non-linear ODE in finite difference terms

$$rg_{i,j}^k = \nu_j D_x^0 g_{i,j}^k + a_1 D_{xx}^2 g_{i,j}^k - b_1 D_x^0 g_{i,j}^k D_x^0 g_{i,j}^k. \qquad (14)$$

We guess a investment threshold $\tilde{x}$ and obtain the corresponding solution of $g_{i,j}^k$ for all $[x_l, \tilde{x}]$ by iteratively solving the non-linear system of finite difference equations (14).

$Step\,2$: We adjust $\tilde{x}$ and repeat $Step\,1$ until the free-boundary condition (10) is approximately satisfied. Then we repeat the above-mensioned calculation for all m_j, $j = 1, ..., N_m + 1$.

$Step\,3$: At any earlier time t_k, $k = 1, ..., N_t$, we know the value of $g_{i,j}^{k+1}$ for all i and j. Due to the mixed derivatives g_{mx} in the highly non-linear

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PDE (9), the standard Alternative Direction Implicit (ADI) method does not work well but the splitting method provides stable and convergent solutions to such problem (Yanenko, 1971).

Specifically, we first approximate the PDE (13) within the whole domain of $[x_l, x_h] \times [m_l, m_h]^4$ by

$$
\begin{aligned}
rg_{i,j}^{\tilde{k}} &= \frac{g_{i,j}^{k+1} - g_{i,j}^{\tilde{k}}}{\Delta t} + \nu_j D_x^0 g_{i,j}^{\tilde{k}} + a_1 D_{xx}^2 g_{i,j}^{\tilde{k}} - b_1 D_x^0 g_{i,j}^{\tilde{k}} D_x^0 g_{i,j}^{\tilde{k}} \\
&\quad + 0.5 v_k D_{xm}^2 g_{i,j}^{k+1} - 0.5 h_1 D_m^0 g_{i,j}^{k+1} D_x^0 g_{i,j}^{\tilde{k}},
\end{aligned}
$$

where $g_{i,j}^{\tilde{k}}$ represents the value of g on the node (i,j) at the artificial auxiliary time layer $\tilde{k}$, whose values are unknown at this step.[5] Hence, we use the implicit method to solve $g_{i,j}^{\tilde{k}}$, which is unconditionally stable.[6] The non-linear system of equations of $g_{i,j}^{\tilde{k}}$ can be solved by standard numerical methods such as Levenberg-Marquardt algorithm.

$Step\,4$: For the given $g_{i,j}^{\tilde{k}}$ we have computed in $Step\,3$, we turn to approximate the PDE (13) by[7]

$$
0 = \frac{g_{i,j}^{\tilde{k}} - g_{i,j}^{k}}{\Delta t} + d_1 D_{mm}^2 g_{i,j}^{\tilde{k}} - e_1 D_m^0 g_{i,j}^{\tilde{k}} D_m^0 g_{i,j}^{\tilde{k}} + 0.5 v_k D_{xm}^2 g_{i,j}^{\tilde{k}} - 0.5 h_1 D_m^0 g_{i,j}^{\tilde{k}} D_x^0 g_{i,j}^{\tilde{k}},
$$

where $g_{i,j}^{\tilde{k}}$ is known at this step and $g_{i,j}^{k}$ is unknown. Namely, we use the explicit method to solve $g_{i,j}^{k}$ without numerically solving a non-linear system of equations. It produces stable solutions in our numerical example.

$Step\,5$: At any state (t_k, m_j), by comparing the implied option value $g_{i,j}^{k}$ and the certainty-equivalent wealth $f_{i,j}^{k}$ of cash flows, we can determine the

---

[4]The approach of guessing a threshold $\tilde{x}$ used in $Step\,1$ cannot be applied in the splitting method since the values of $g_{i,j}^{\tilde{k}}$ for all $x \in [x_l, x_h]$ need to be known in $Step\,4$.

[5]The splitting method uses the whole step-size Δt in the finite difference equation.

[6]It is unstable to use the explicit method at this step in our numerical example.

[7]The item $rg_{i,j}^{\tilde{k}}$ does not appear in $Step\,4$.

investment threshold $\tilde{x}(t, m)$ such that the free-boundary condition (10) is approximately satisfied.

Appendix B   The derivation of model solutions

Lemma B.1. *If the conditional distribution $\Psi_{\mathcal{G}_0}(x) = \mathbb{P}(\mu \leq x | \mathcal{G}_0)$ is normal with mean m_0 and variance v_0, a.s., then the conditional distribution $\Psi_{\mathcal{G}_t}(x) = \mathbb{P}(\mu \leq x | \mathcal{G}_t)$ is normal with mean m_t and variance v_t, a.s..*

This lemma follows from Theorem 11.1 in Liptser and Shiryayev (1977) and we obtain the next lemma from Theorem 12.1 in Liptser and Shiryayev (1977). After a careful calculation, we obtain the explicit solution (5) to the following (15).

Lemma B.2. *Let $\{X_t\}_{t \geq 0}$ be stochastic process given by (2). Suppose that $\mathbb{P}(\mu \leq x | \mathcal{G}_0)$ is Gaussian with mean m_0 and variance v_0. Then m_t and v_t satisfy the following dynamics:*

$$
\begin{cases}
dm_t &= -\frac{\rho v_t}{\sigma_x \sigma_p (1-\rho^2)} \left(\frac{dP_t}{P_t} - \alpha_p dt \right) + \frac{v_t}{\sigma_x^2 (1-\rho^2)} \left(dX_t - m_t dt \right), \\
\dot{v}_t &= -\left(\frac{v_t}{\sigma_x \sqrt{1-\rho^2}} \right)^2,
\end{cases}
\tag{15}
$$

Filtering theory ensures that the process $\tilde{Z}$ defined by (6) is a standard Brownian motion with respect to the stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{G}_t\}_{t \geq 0})$. According to (1) and (6), the correlation coefficient between processes $\tilde{Z}$ and B is ρ. Then, we obtain from (15) and (6) the following expressions:

$$
\begin{aligned}
dm_t &\equiv -\frac{\rho v_t}{\sigma_x (1-\rho^2)} dB_t + \frac{v_t}{\sigma_x (1-\rho^2)} d\tilde{Z}_t \\
&= -\frac{\rho v_t}{\sigma_x (1-\rho^2)} dB_t + \frac{v_t}{\sigma_x (1-\rho^2)} (\rho dB_t + \sqrt{1-\rho^2} d\tilde{Z}_t^1) \\
&= \frac{v_t}{\sigma_x \sqrt{1-\rho^2}} d\tilde{Z}_t^1,
\end{aligned}
$$

where $\tilde{Z}^1$ is a Brownian motion, independent of B but correlated with $\tilde{Z}$ with correlation coefficient $\sqrt{1 - \rho^2}$.

In contrast to Yang and Yang (2012) and Song and Yang (2013), the value function of (3) depends on the filtering estimate no matter whether investment has taken place or not. Therefore, we denote the value function of (3) after and before investment by $V^0(t, W_t, X_t, m_t)$ and $V(t, W_t, X_t, m_t)$ respectively.

To apply the consumption utility indifference pricing method, we first introduce the following optimization problem without investment project:

$$\sup_{(C,\pi)\in\mathcal{C}\times\Pi} J^0((C_s)_{s\geq t}, (\pi_s)_{s\geq t}) \equiv \mathbb{E}\left[\int_t^\infty \exp\left(-\beta(s-t)\right) U(C_s) ds \,|\mathcal{G}_t\right],$$

subject to

$$dW_s = (rW_s + \pi_s(\alpha_p - r) - C_s)ds + \pi_s\sigma_p dB_s.$$

Similar to Merton (1971), we obtain the following explicit solution by dynamic programming:

$$G(W_t) \equiv J^0((C_s^*)_{s\geq t}, (\pi_s^*)_{s\geq t}) = -\frac{1}{\gamma r} \exp(1 - \beta/r - \gamma r(W_t + \frac{\eta^2}{2\gamma r^2})), \quad (16)$$

where C_s^* and π_s^* are the optimal consumption and portfolio rules selected at time s respectively, which are given by

$$\begin{cases} C_s^* = \frac{\beta-r}{\gamma r} + r(W_s + \frac{\eta^2}{2\gamma r^2}), \\ \pi_s^* = \frac{\eta}{\sigma_p \gamma r}. \end{cases}$$

Following Hodges and Neuberger (1989), we define the consumption utility-based indifference price or implied value of the option to invest at time t by y, which satisfies

$$V(t, W_t, X_t, m_t) = G(W_t + y). \quad (17)$$

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where $V(t, W_t, X_t, m_t)$ is the value function defined by (3). Similarly, we define the indifference price or implied value of the cash flows after investment at time t by z, which satisfies

$$V^0(t, W_t, X_t, m_t) = G(W_t + z). \tag{18}$$

By the standard argument of dynamic programming, $V^0(t, W_t, X_t, m_t)$ satisfies the following Hamilton-Jacobi-Bellman equation:

$$\sup_{c,\pi}(rw + \pi(\alpha_p - r) + x - c)V_w^0 + U(c) + \frac{(\pi\sigma_p)^2}{2}V_{ww}^0 + \rho\pi\sigma_p\sigma_x V_{wx}^0 \tag{19}$$
$$+V_t^0 + mV_x^0 + \frac{\sigma_x^2}{2}V_{xx}^0 + v_t V_{mx}^0 + \frac{v_t^2}{2\sigma_x^2(1-\rho^2)}V_{mm}^0 - \beta V^0 = 0,$$

where the subscript of V^0 denotes the differentiation with respect to that variable. We assume the usual transversality condition

$$\lim_{t\to\infty} \mathbb{E}[\exp(-\beta t)V^0(t, W_t, X_t, m_t)] = 0$$

is satisfied. The first-order conditions for the optimal consumption and portfolio rule after exercising the option are given by:

$$U'(c) = V_w^0 \quad \text{and} \quad \pi = -\frac{\eta}{\sigma_p}\frac{V_w^0}{V_{ww}^0} - \frac{\rho\sigma_x}{\sigma_p}\frac{V_{wx}^0}{V_{ww}^0}. \tag{20}$$

Then, we turn to the case before the option is exercised. By Bellman principle, we have

$$\begin{aligned}V(t, W_t, X_t, m_t) &= \sup_{(\tau,C,\pi)\in\mathcal{T}\times\mathcal{C}\times\Pi} \mathbb{E}\left[\int_t^\tau \exp\left(-\beta(s-t)\right)U(C_s)ds\right. \\ &\quad \left. + \exp(-\beta(\tau-t))V^0(\tau, W_{\tau-} - I, X_\tau, m_\tau)\,|\mathcal{G}_t\right].\end{aligned} \tag{21}$$

This is a combined stochastic control and optimal stopping problem and the Hamilton-Jacobi-Bellman (HJB) equation has the form

$$\sup_{c,\pi}(rw + \pi(\alpha_p - r) - c)V_w + U(c) + \frac{(\pi\sigma_p)^2}{2}V_{ww} + \rho\pi\sigma_p\sigma_x V_{wx} \tag{22}$$
$$+V_t + mV_x + \frac{\sigma_x^2}{2}V_{xx} + v_t V_{mx} + \frac{v_t^2}{2\sigma_x^2(1-\rho^2)}V_{mm} - \beta V = 0.$$

The first-order conditions are similar to (20) and given by

$$U'(c) = V_w, \quad \text{and} \quad \pi = -\frac{\eta}{\sigma_p} \frac{V_w}{V_{ww}} - \frac{\rho \sigma_x}{\sigma_p} \frac{V_{wx}}{V_{ww}}. \tag{23}$$

We enforce the no-bubble condition

$$\lim_{x \to -\infty} V(t, W_t, X_t, m_t) = G(W_t)$$

as an economically sensible solution refinement in the class of fundamental solutions. At the investment boundary $(\tilde{t}, \tilde{w}, \tilde{x}, \tilde{m})$, the value matching condition

$$V(\tilde{t}, \tilde{w}, \tilde{x}, \tilde{m}) = V^0(\tilde{t}, \tilde{w} - I, \tilde{x}, \tilde{m}) \tag{24}$$

and the smooth-pasting conditions

$$\begin{cases} V_t(\tilde{t}, \tilde{w}, \tilde{x}, \tilde{m}) = V_t^0(\tilde{t}, \tilde{w} - I, \tilde{x}, \tilde{m}), \\ V_w(\tilde{t}, \tilde{w}, \tilde{x}, \tilde{m}) = V_w^0(\tilde{t}, \tilde{w} - I, \tilde{x}, \tilde{m}), \\ V_x(\tilde{t}, \tilde{w}, \tilde{x}, \tilde{m}) = V_x^0(\tilde{t}, \tilde{w} - I, \tilde{x}, \tilde{m}), \\ V_m(\tilde{t}, \tilde{w}, \tilde{x}, \tilde{m})) = V_m^0(\tilde{t}, \tilde{w} - I, \tilde{x}, \tilde{m}) \end{cases} \tag{25}$$

are imposed, see Section 1.G of Chapter 4 in Dixit and Pindyck (1994) and Section 9 in Peskir and Shiryaev (2006) for details.

On account of (16), (17) and (18), we can conclude that the value functions take the following forms respectively:

$$\begin{align} V^0(t, w, x, m) &= -\tfrac{1}{\gamma r} \exp(1 - \beta/r - \gamma r(w + f(t, x, m) + \tfrac{\eta^2}{2\gamma r^2})), \\ V(t, w, x, m) &= -\tfrac{1}{\gamma r} \exp(1 - \beta/r - \gamma r(w + g(t, x, m) + \tfrac{\eta^2}{2\gamma r^2})). \end{align} \tag{26}$$

Substituting (26) into the HJB equations (19) and (22), we get the PDEs (8) and (9) for the functions f and g subject to the no-bubble condition $\lim_{x \to -\infty} g(t, x, m) = 0$ and the free-boundary conditions (10).

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Appendix C The smooth-fit principle and a verification theorem

The smooth-pasting conditions shown in (25) is generally called "smooth-fit or high contact principle". Roughly speaking, it states that under certain conditions, the value function like $V(t, w, x, m)$ here is differentiable on the free boundary. Peskir and Shiryaev (2006) (see Theorem 9.5 in their book) present a general result under one-dimensional situation. Aliev (2007) provide sufficient conditions for this principle to hold under multi-dimensional case. The principle is necessary for the optimality of the value function like V here. As pointed out by Øksendal (2003) (see Section 10.4 of their book), this principle is so useful that it is frequently applied in the literature, although under many cases, especially in economic studies, its validity has not been rigorously proved, e.g. Miao and Wang (2007) under one-dimensional situation and Alvarez and Lippi (2012) under multi-dimensional case among others. We believe our model satisfies the sufficient conditions demanded by Aliev (2007) and so the smooth-pasting conditions shown in (25) must hold. However, it is a challenging task to produce a valid formal proof for this and thus we leave it for future research.

The following verification theorem presents in essence that if a function $\Gamma(t, w, x, m)$ satisfying (22)$\sim$(25) is regular enough so that Itô's stochastic differential rule holds for $\Gamma(t, W_t, X_t, m_t)$, then $\Gamma(t, w, x, m) = V(t, w, x, m)$.

Theorem C.1. *For a given consumption plan $C \in \mathcal{C}$ and an investment portfolio $\pi \in \Pi$, let*

$$
\begin{aligned}
\mathcal{L}_{C,\pi} =\ & \frac{\partial}{\partial t} + [rw + \pi_t(\alpha_p - r) - C_t]\frac{\partial}{\partial w} + m\frac{\partial}{\partial x} + \frac{1}{2}(\sigma_p \pi_t)^2 \frac{\partial^2}{\partial w^2} \\
& + \frac{1}{2}\sigma_x^2 \frac{\partial^2}{\partial x^2} + \frac{1}{2}\frac{v_t^2}{\sigma_x^2(1-\rho^2)} \frac{\partial^2}{\partial m^2} + \rho\pi_t \sigma_p \sigma_x \frac{\partial^2}{\partial w \partial x} + v_t \frac{\partial^2}{\partial x \partial m}
\end{aligned}
$$

be the partial differential operator of the diffusion process (t, W_t, X_t, m_t) and let $\mathcal{O} \equiv (0, \infty) \times \Re^2$. Suppose we can find a function $\Gamma \colon [0, \infty) \times \overline{\mathcal{O}} \to \Re$, which is regular enough so that for each $(C, \pi) \in \mathcal{C} \times \Pi$, Itô's stochastic differential rule and Dynkin's formula hold for $\Gamma(t, W_t, X_t, m_t)$ and it is a solution of the following variational inequality

$$
\begin{cases}
\sup_{C_t, \pi_t} \mathcal{L}_{C, \pi}[\Gamma](t, w, x, m) \leq 0, \ \Gamma(t, w, x, m) \geq V^0(t, w, x, m), \\
(\Gamma(t, w, x, m) - V^0(t, w, x, m)) \sup_{C_t, \pi_t} \mathcal{L}_{C, \pi}[\Gamma](t, w, x, m) = 0,
\end{cases}
$$

then we have $\Gamma(t, w, x, m) = V(t, w, x, m)$.

The proof of the theorem can be finished by using a similar structure with the proof of the corresponding assertions in Rishel and Helmes (2006) or the proof of Theorem 10.4.1 in Øksendal (2003) and it is therefore omitted.

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