On bifurcation for semilinear elliptic Dirichlet problems on geodesic balls

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Abstract

We study bifurcation from a branch of trivial solutions of semilinear elliptic Dirichlet boundary value problems on a geodesic ball, whose radius is used as the bifurcation parameter. In the proof of our main theorem we obtain in addition a special case of an index theorem due to S. Smale.

1 Introduction

Let (M, g) be an oriented Riemannian manifold of dimension n and let $\Delta = \operatorname{div}\operatorname{grad} : C^\infty(M) \to C^\infty(M)$ denote the associated Laplace-Beltrami operator. Let $V : M \times \mathbb{R} \to \mathbb{R}$ be a smooth function such that $V(p, 0) = 0$ for all $p \in M$ and

$$|V(p,\xi)| \leq C(1+|\xi|^\alpha), \quad \left|\frac{\partial V}{\partial \xi}(p,\xi)\right| \leq C(1+|\xi|^\beta), \quad (p,\xi) \in M\times\mathbb{R}, \tag{1}$$

for some $C > 0$ and constants $\alpha, \beta \geq 0$ depending on the dimension n of M (cf. [AP93, §1.2]). In this paper we deal with local solutions of the semilinear equation

$$-\Delta u(p) + V(p, u(p)) = 0, \quad p \in M, \tag{2}$$

under Dirichlet boundary conditions. Note that many equations from geometric analysis are of the type (2). Let us refer to [Au82], [Be87] and just mention as an example on compact manifolds of dimension $n \geq 3$ the equation

$$4\,\frac{n-1}{n-2}\,\Delta u(p) + s(p)u(p) = \mu\, u(p)^{\frac{n+2}{n-2}}, \quad p \in M, \tag{3}$$

where $s : M \to \mathbb{R}$ denotes the scalar curvature function and μ the Yamabe invariant of the metric g on M. Positive solutions $u \in C^\infty(M)$ of (3) give rise to metrics $\tilde{g}$ of constant scalar curvature on M by $\tilde{g} = u^{\frac{4}{n-2}} g$.

We now fix a point $p_0 \in M$ and assume that the unit ball $B(0, 1) \subset T_{p_0}M$ is contained in the

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maximal domain on which the exponential map $\exp_{p_0}$ at p_0 is an embedding. Let us denote by $B(p_0, r) = \exp_{p_0}(B(0, r))$ the geodesic ball of radius $0 < r \leq 1$ around p_0 and consider the Dirichlet boundary value problems

$$\begin{cases} -\Delta u(p) + V(p, u(p)) = 0, & p \in B(p_0, r) \\ \qquad\qquad\qquad u(p) = 0, & p \in \partial B(p_0, r). \end{cases} \tag{4}$$

We call $r^* \in (0, 1]$ a *bifurcation radius* for the boundary value problems (4) if there exists a sequence of radii $r_n \to r^*$ and functions $u_n \in H^1_0(B(p_0, r_n))$ such that u_n is a non-trivial weak solution of (4) on $B(p_0, r_n)$ and $\|u_n\|_{H^1_0(B(p_0,r_n))} \to 0$. Note that we exclude from the definition the limiting case $r^* = 0$ in which the domain degenerates to a point. The reason is that $\|u_n\|_{H^1_0(B(p_0,r_n))} \to 0$ for $r_n \to 0$ holds, for example, for any sequence of functions $u_n \in C^1(B(p_0, r_n))$, $n \in \mathbb{N}$, such that all u_n and their weak derivatives are bounded uniformly. Consequently, a bifurcation radius $r^* = 0$ would not imply the existence of non-trivial solutions of (4) for small $r > 0$ which are arbitrarily close to the trivial solution $u \equiv 0$ in a suitable sense. Let us now consider the linearised boundary value problems

$$\begin{cases} -\Delta u(p) + f(p)u(p) = 0, & p \in B(p_0, r) \\ \qquad\qquad\qquad u(p) = 0, & p \in \partial B(p_0, r), \end{cases} \tag{5}$$

where $f(p) = \frac{\partial V}{\partial \xi}(p, 0)$, $p \in M$. We call $r^* \in (0, 1]$ a *conjugate radius* for (5) if

$$m(r^*) := \dim\{u \in C^2(B(p_0, r^*)) : u \text{ solves } (5)\} > 0,$$

and from now on we assume that $m(1) = 0$. Our main result reads as follows:

Theorem 1.1. *The bifurcation radii of* (4) *are precisely the conjugate radii of* (5).

We explain below in the proof of Theorem 1.1 that we obtain from our methods a new proof of the Morse-Smale index theorem [Sm65] (cf. also [Sm67]) for the linearised equations (5). As a consequence, we conclude that $m(r) = 0$ for almost all radii $r \in (0, 1)$, and moreover, we derive from Theorem 1.1 the following corollary:

Corollary 1.2. *Let μ denote the Morse index of* (5) *on $B(p_0, 1)$, i.e. the number of negative eigenvalues counted according to their multiplicities. If $\mu \neq 0$, then there exist at least*

$$\left\lfloor \frac{\mu}{\max_{0<r<1} m(r)} \right\rfloor$$

distinct bifurcation radii in $(0, 1)$*, where* $\lfloor \cdot \rfloor$ *denotes the integral part of a real number.*

Let us point out that a proof of Theorem 1.1 and Corollary 1.2 for the special case that M is a star-shaped domain in $\mathbb{R}^n$ can be found in [PW13]. The following section is devoted to the more general setting which we consider here.

2 The proof

Our main reference for the Laplace-Beltrami operator on manifolds with boundary is [Ta96, §2.4]. Let us recall at first that in local coordinates

$$\Delta u = \sum_{j,k=1}^{n} |g|^{-\frac{1}{2}} \frac{\partial}{\partial x^j} \left(g^{jk} |g|^{\frac{1}{2}} \frac{\partial u}{\partial x^k} \right),$$

where g^{jk}, $1 \leq j,k \leq n$, are the components of the inverse of the metric tensor $g = \{g_{jk}\}$ and $|g| := |\det\{g_{jk}\}|$ is the absolute value of its determinant. Denoting by dvol_g the volume form of g, we find for $v \in H^1_0(B(p_0,r))$, $0 < r \leq 1$,

$$\begin{aligned}
&-\int_{B(p_0,r)} (\Delta u)(p) v(p)\ \mathrm{dvol}_g + \int_{B(p_0,r)} V(p,u(p))\, v(p)\ \mathrm{dvol}_g \\
&= -\int_{B(0,r)} v(x) \sum_{j,k=1}^{n} \frac{\partial}{\partial x^j} \left(g^{jk}(x) |g(x)|^{\frac{1}{2}} \frac{\partial u}{\partial x^k}(x) \right) dx + \int_{B(0,r)} |g(x)|^{\frac{1}{2}} V(x,u(x)) v(x)\, dx \\
&= \int_{B(0,r)} \sum_{j,k=1}^{n} g^{jk}(x) |g(x)|^{\frac{1}{2}} \frac{\partial u}{\partial x^k}(x) \frac{\partial v}{\partial x^j}(x)\, dx + \int_{B(0,r)} |g(x)|^{\frac{1}{2}} V(x,u(x)) v(x)\, dx \\
&= r \int_{B(0,1)} \sum_{j,k=1}^{n} g^{jk}(r\cdot x) |g(r\cdot x)|^{\frac{1}{2}} \frac{\partial u}{\partial x^k}(r\cdot x) \frac{\partial v}{\partial x^j}(r\cdot x)\, dx \\
&+ r \int_{B(0,1)} |g(r\cdot x)|^{\frac{1}{2}} V(r\cdot x, u(r\cdot x)) v(r\cdot x)\, dx,
\end{aligned}$$

and analogously

$$\begin{aligned}
&-\int_{B(p_0,r)} (\Delta u)(p) v(p)\ \mathrm{dvol}_g + \int_{B(p_0,r)} f(p) u(p) v(p)\ \mathrm{dvol}_g \\
&= r \int_{B(0,1)} \sum_{j,k=1}^{n} g^{jk}(r\cdot x) |g(r\cdot x)|^{\frac{1}{2}} \frac{\partial u}{\partial x^k}(r\cdot x) \frac{\partial v}{\partial x^j}(r\cdot x)\, dx \\
&+ r \int_{B(0,1)} |g(r\cdot x)|^{\frac{1}{2}} f(r\cdot x) u(r\cdot x) v(r\cdot x)\, dx.
\end{aligned}$$

We now set $B := B(0,1)$ and define for $0 \leq r \leq 1$ a functional $q_r : H^1_0(B) \times H^1_0(B) \to \mathbb{R}$ by

$$q_r(u,v) = \int_B \sum_{j,k=1}^{n} g^{jk}(r\cdot x) |g(r\cdot x)|^{\frac{1}{2}} \frac{\partial u}{\partial x^k}(x) \frac{\partial v}{\partial x^j}(x)\, dx + r^2 \int_B |g(r\cdot x)|^{\frac{1}{2}} V(r\cdot x, u(x)) v(x)\, dx$$

as well as a quadratic form $h_r : H^1_0(B) \to \mathbb{R}$ by

$$h_r(u) = \int_B \sum_{j,k=1}^{n} g^{jk}(r\cdot x) |g(r\cdot x)|^{\frac{1}{2}} \frac{\partial u}{\partial x^k}(x) \frac{\partial u}{\partial x^j}(x)\, dx + r^2 \int_B |g(r\cdot x)|^{\frac{1}{2}} f(r\cdot x) u(x)^2\, dx.$$

From the computations above we conclude that:

i) $r^* \in (0,1]$ is a bifurcation radius for (4), if and only if there exist a sequence $\{r_n\}_{n\in\mathbb{N}} \subset (0,1]$, $r_n \to r^*$, and a sequence of non-trivial functions $\{u_n\}_{n\in\mathbb{N}} \subset H^1_0(B)$, $u_n \to 0$, such that $q_{r_n}(u_n, \cdot) = 0 \in (H^1_0(B))^*$ for all $n \in \mathbb{N}$.

ii) $r^* \in (0,1]$ is a conjugate radius for (5) if and only if h_r is degenerate.

We now define a function $\psi : [0,1] \times H^1_0(B) \to \mathbb{R}$ by

$$\psi(r,u) = \int_B \sum_{j,k=1}^n g^{jk}(r\cdot x)|g(r\cdot x)|^{\frac{1}{2}} \frac{\partial u}{\partial x^k}(x)\frac{\partial u}{\partial x^j}(x)\,dx + r^2\int_B |g(r\cdot x)|^{\frac{1}{2}}\,G(r\cdot x, u(x))\,dx,$$

where

$$G(x,t) = \int_0^t V(x,\xi)\,d\xi.$$

It is a standard result that ψ is C^2-smooth under the growth conditions (1), and $D_u\psi_r = q_r(u,\cdot)$, $u \in H^1_0(B)$. Moreover, $0 \in H^1_0(B)$ is a critical point of all functionals ψ_r and the corresponding Hessians are given by $D^2_0\psi_r = h_r$. From the compactness of the inclusion $H^1_0(B) \hookrightarrow L^2(B)$, we see at once that the Riesz representation of the quadratic form h_r is a selfadjoint Fredholm operator. In particular, it is invertible if h_r is non-degenerate.
Let us now assume at first that $r^* \in (0,1]$ is not a conjugate radius. Then h_{r^*} is non-degenerate and we conclude by the implicit function theorem [AP93, §2.2] that the equation $q_r(u,\cdot) = 0$ has no other solutions than $(r,0) \in [0,1] \times H^1_0(B)$ in a neighbourhood of $(r^*,0)$. Consequently, $(r^*,0)$ is not a bifurcation radius, and we have shown that every bifurcation radius in $(0,1]$ is a conjugate radius.
In order to prove the remaining implication of Theorem 1.1, we make use of the bifurcation theory for critical points of smooth functionals developed in [FPR99]. Accordingly, we consider for $r_0 \in (0,1)$ the quadratic forms

$$\Gamma(h,r_0) : \ker h_{r_0} \to \mathbb{R}, \quad \Gamma(h,r_0)[u] = \left(\frac{d}{dr}\,|_{r=r_0}\, h_r\right) u.$$

By [FPR99, Thm. 1& Thm. 4.1], r_0 is a bifurcation radius if $\Gamma(h,r_0)$ is non-degenerate and has a non-vanishing signature (cf. also Section 2.1 in [PW13]). Consequently, we now assume that $r_0 \in (0,1)$ is a conjugate radius and our aim is to compute $\Gamma(h,r_0)$. Let us write for simplicity of notation

$$\begin{aligned} a^{jk}(x) &= g^{jk}(x)|g(x)|^{\frac{1}{2}}, \quad x \in B,\ 1 \le j,k \le n,\\ \tilde{f}(x) &= |g(x)|^{\frac{1}{2}} f(x), \quad x \in B. \end{aligned}$$

For $u \in \ker h_{r_0}$ we have by definition

$$\Gamma(h,r_0)[u] = \int_B \sum_{j,k=1}^n \langle \nabla a^{jk}(r_0\cdot x), x\rangle \frac{\partial u}{\partial x^k}\frac{\partial u}{\partial x^j}\,dx + \int_B \frac{d}{dr}\,|_{r=r_0}\,(r^2\tilde{f}(r\cdot x))u(x)^2\,dx. \quad (6)$$

Let us now introduce a new function by $v_r(x) := u(\frac{r}{r_0}\cdot x)$, $r \in (0,r_0]$, $x \in B$, and denote

$$\dot{u}(x) := \frac{d}{dr}\,|_{r=r_0}\, v_r(x) = \frac{1}{r_0}\langle \nabla u(x), x\rangle. \quad (7)$$

It is readily seen that v_r satisfies

$$-\sum_{j,k=1}^{n} \frac{\partial}{\partial x^j}\left(a^{jk}(r\cdot x)\frac{\partial v_r}{\partial x^k}\right) + r^2\tilde{f}(r\cdot x)v_r(x) = 0,$$

and by differentiating with respect to r at $r = r_0$ we have

$$\begin{aligned} 0 = &-\sum_{j,k=1}^{n} \frac{\partial}{\partial x^j}\left(\langle \nabla a^{jk}(r_0\cdot x), x\rangle \frac{\partial u}{\partial x^k}\right) - \sum_{j,k=1}^{n} \frac{\partial}{\partial x^j}\left(a^{jk}(r_0\cdot x)\frac{\partial \dot{u}}{\partial x^k}\right) \\ &+ \frac{d}{dr}\mid_{r=r_0} (r^2\tilde{f}(r\cdot x))u(x) + r_0^2\tilde{f}(r_0\cdot x)\dot{u}(x). \end{aligned} \tag{8}$$

We multiply (8) by u and integrate over B:

$$\begin{aligned} 0 = &-\int_B \sum_{j,k=1}^{n} \frac{\partial}{\partial x^j}\left(\langle \nabla a^{jk}(r_0\cdot x), x\rangle \frac{\partial u}{\partial x^k}\right)u(x)\,dx - \int_B \sum_{j,k=1}^{n} \frac{\partial}{\partial x^j}\left(a^{jk}(r_0\cdot x)\frac{\partial \dot{u}}{\partial x^k}\right)u(x)\,dx \\ &+ \int_B \frac{d}{dr}\mid_{r=r_0} (r^2\tilde{f}(r\cdot x))u(x)^2\,dx + \int_B r_0^2\tilde{f}(r_0\cdot x)\dot{u}(x)u(x)\,dx. \end{aligned}$$

Let $\nu(x) = (\nu_1(x), \ldots, \nu_n(x))$, $x \in \partial B$, denote the outward pointing unit normal to the boundary of B. Using $u \mid_{\partial B} = 0$, we obtain from integration by parts

$$\begin{aligned} 0 = &\int_B \sum_{j,k=1}^{n} \langle \nabla a^{jk}(r_0\cdot x), x\rangle \frac{\partial u}{\partial x^k}\frac{\partial u}{\partial x^j}\,dx - \int_{\partial B}\left(\sum_{j,k=1}^{n} \langle \nabla a^{jk}(r_0\dot{x}), x\rangle \nu_j(x)\frac{\partial u}{\partial x^k}\right)u(x)\,dS \\ &+ \int_B \sum_{j,k=1}^{n} a^{jk}(r_0\cdot x)\frac{\partial \dot{u}}{\partial x^k}\frac{\partial u}{\partial x^j}\,dx - \int_{\partial B}\left(\sum_{j,k=1}^{n} a^{jk}(r_0\cdot x)\nu_j(x)\frac{\partial \dot{u}}{\partial x^k}\right)u(x)\,dS \\ &+ \int_B \frac{d}{dr}\mid_{r=r_0} (r^2\tilde{f}(r\cdot x))u(x)^2\,dx + \int_B r_0^2\tilde{f}(r_0\cdot x)\dot{u}(x)u(x)\,dx \\ = &\int_B \sum_{j,k=1}^{n} \langle \nabla a^{jk}(r_0\cdot x), x\rangle \frac{\partial u}{\partial x^k}\frac{\partial u}{\partial x^j}\,dx - \int_B \sum_{j,k=1}^{n} \frac{\partial}{\partial x^j}\left(a^{jk}(r_0\cdot x)\frac{\partial u}{\partial x^k}\right)\dot{u}(x)\,dx \\ &+ \int_{\partial B}\left(\sum_{j,k=1}^{n} a^{jk}(r_0\cdot x)\nu_j(x)\frac{\partial u}{\partial x^k}\right)\dot{u}(x)\,dS \\ &+ \int_B \frac{d}{dr}\mid_{r=r_0} (r^2\tilde{f}(r\cdot x))u(x)^2\,dx + \int_B r_0^2\tilde{f}(r_0\cdot x)\dot{u}(x)u(x)\,dx. \end{aligned}$$

Since $u \in \ker h_{r_0}$,

$$-\sum_{j,k=1}^{n} \frac{\partial}{\partial x^j}\left(a^{jk}(r_0\cdot x)\frac{\partial u}{\partial x^k}\right) + r_0^2\tilde{f}(r_0\cdot x)u(x) = 0, \quad x \in B, \tag{9}$$

and it follows from (6) and (7) that

$$\Gamma(h,r_0)[u] = -\frac{1}{r_0}\int_{\partial B}\left(\sum_{j,k=1}^{n} a^{jk}(r_0\cdot x)\nu_j(x)\frac{\partial u}{\partial x^k}\right)\langle\nabla u(x), x\rangle\, dS. \tag{10}$$

If we set $A(x) := \{a_{jk}(x)\}$, $x \in B$, and use that $\nu(x) = x$ for all $x \in \partial B$, we can rewrite (10) as

$$\Gamma(h,r_0)[u] = -\frac{1}{r_0}\int_{\partial B}\langle A(r_0\cdot x)x, \nabla u(x)\rangle\langle\nabla u(x), x\rangle\, dS.$$

Denoting by $(A(r_0\cdot x)x)^T$, $x \in \partial B$, the tangential component of the vector $A(r_0\cdot x)x$, we have

$$\langle A(r_0\cdot x)x, \nabla u(x)\rangle = \langle\nabla u(x), x\rangle\langle A(r_0\cdot x)x, x\rangle + \langle\nabla u(x), (A(r_0\cdot x)x)^T\rangle$$

and hence

$$\begin{aligned}\Gamma(h,r_0)[u] = &-\frac{1}{r_0}\int_{\partial B}\langle\nabla u(x), x\rangle^2\langle A(r_0\cdot x)x, x\rangle\, dS\\ &-\frac{1}{r_0}\int_{\partial B}\langle\nabla u(x), x\rangle\langle\nabla u(x), (A(r_0\cdot x)x)^T\rangle\, dS.\end{aligned}$$

Since

$$\langle\nabla u(x), x\rangle\langle\nabla u(x), (A(r_0\cdot x)x)^T\rangle = \operatorname{div}(u(x)\langle x, \nabla u(x)\rangle(A(r_0\cdot x)x)^T), \quad x \in \partial B,$$

we finally get by Stokes' theorem

$$\Gamma(h,r_0)[u] = -\frac{1}{r_0}\int_{\partial B}\langle\nabla u(x), x\rangle^2\langle A(r_0\cdot x)x, x\rangle\, dS \leq 0, \tag{11}$$

where we use that $A(x)$ is positive definite for all $x \in B$.
Moreover, we obtain from (11) that if $\Gamma(h,r_0)[u] = 0$ for some $u \in \ker h_{r_0}$, then

$$\langle\nabla u(x), x\rangle = \langle\nabla u(x), \nu(x)\rangle = \frac{\partial u}{\partial\nu}(x) = 0$$

for all $x \in \partial B$ which implies $u \equiv 0$ by the unique continuation property.
In summary, we have shown that $\Gamma(h,r_0)$ is negative definite, and so in particular non-degenerate with the non-vanishing signature

$$\operatorname{sgn}\Gamma(h,r_0) = m(r_0). \tag{12}$$

Consequently, r_0 is a bifurcation radius and Theorem 1.1 is proven.
Let us now prove Corollary 1.2. We note at first that the Morse index μ of (5) on the full domain $B(p_0,1)$ is given by the Morse index $\mu(h_1)$ of the quadratic form h_1. Moreover, since h_0 is positive, we see that $\mu(h_0) = 0$. It is shown in [FPR99, Prop. 3.9& Thm. 4.1] that if $\Gamma(h,r)$ is non-degenerate for all $r \in (0,1)$, then $\ker h_r = 0$ for almost all $r \in (0,1)$ and

$$\mu(h_1) - \mu(h_0) = \sum_{0<r<1} \operatorname{sgn}\Gamma(h,r).$$

Consequently, we conclude from (12) that $m(r) = \dim\ker h_r = 0$ for almost all $r \in (0,1)$ and

$$\mu = \sum_{0<r<1} m(r). \tag{13}$$

Let us point out that (13) was obtained by Smale in [Sm65] by studying the monotonicity of eigenvalues under shrinking of domains. Hence we have obtained a new proof of Smale's theorem for the boundary value problem (5), and moreover, Corollary 1.2 is now an immediate consequence of (13) and Theorem 1.1.

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