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Bifurcation results for critical points of families of functionals

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Abstract

Recently the first author studied in [Po11] the bifurcation of critical points of families of functionals on a Hilbert space, which are parametrised by a compact and orientable manifold having a non-vanishing first integral cohomology group. We improve this result in two directions: topologically and analytically. From the analytical point of view we generalise it to a broader class of functionals; from the topological point of view we allow the parameter space to be a metrisable Banach manifold. Our methods are in particular powerful if the parameter space is simply connected. As an application of our results we consider families of geodesics in (semi-) Riemannian manifolds.

Dedicated to our mentor Jacobo Pejsachowicz

1 Introduction

Let $I := [0,1]$ denote the unit interval and let H be a separable, infinite-dimensional real Hilbert space. Let $f : I \times H \to \mathbb{R}$ be a C^2 function such that $0 \in H$ is a critical point of all $f_\lambda = f(\lambda, \cdot) : H \to \mathbb{R}$, $\lambda \in I$. An instant $\lambda^* \in I$ is called a *bifurcation point* if every neighbourhood of $(\lambda^*, 0) \in I \times H$ contains points of the form (λ, u), where $u \neq 0$ is a critical point of f_λ. The existence of bifurcation points can be studied by considering the associated path of Hessians L_λ, $\lambda \in I$, of the functions f_λ at their critical point $0 \in H$. In many geometric variational problems these bounded selfadjoint operators are Fredholm.

As an example, if $L_\lambda = id - \lambda K$, where K is a selfadjoint compact operator on H, then every characteristic value (reciprocal of a non-zero eigenvalue) of K is a bifurcation point according to a classical result of Krasnosel'skii (cf. [Kr64]). This result has been improved over the years and a bifurcation theorem for general one-parameter families of functionals $f : I \times H \to \mathbb{R}$ as above was obtained by Fitzpatrick, Pejsachowicz and Recht in [FPR99]. They considered the spectral flow, which is an integer-valued homotopy invariant for paths of bounded selfadjoint Fredholm operators introduced by Atiyah, Patodi and Singer in [APS76]. Roughly speaking, the spectral flow of a path L of selfadjoint Fredholm operators is the number of negative eigenvalues of L_0 that become positive as the parameter λ travels from 0 to 1 minus the number of positive eigenvalues of L_0 that become negative; i.e. the net number of eigenvalues which cross zero. The main result of [FPR99] shows the existence of a bifurcation point $\lambda^* \in I$ for f, if the operators

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L_0, L_1 are invertible and the spectral flow of the path of Hessians L_λ, $\lambda \in I$, does not vanish (cf. theorem 2.2 below). Applications of their result were obtained for bifurcation of periodic orbits of Hamiltonian systems (cf. [FPR00]), bifurcation of families of geodesics in semi-Riemannian manifolds (cf. [MPP07]) and for the study of conjugate points in [PPT03] and [PP05].
Recently the first author has considered the more general situation of C^2 functions $f : X \times H \to \mathbb{R}$, where X is a compact, orientable smooth manifold of dimension at least 2 and $0 \in H$ is again a critical point of all functionals $f_\lambda : H \to \mathbb{R}$, $\lambda \in X$. In this case the associated family of Hessians L_λ, $\lambda \in X$, of f_λ at $0 \in H$ is a family of bounded selfadjoint operators on H parametrised by the compact space X. If these operators are in addition Fredholm and strongly indefinite (cf. section 2 below), then one can assign an odd K-theory class $\mathrm{ind}_s\, L \in K^{-1}(X)$, which was defined by Atiyah, Patodi and Singer in [APS76] and which can be interpreted as a generalisation of the spectral flow to families. The main theorem of [Po11] states that if there exists $\lambda_0 \in X$ such that L_{λ_0} is invertible and, moreover, the first Chern class $c_1(\mathrm{ind}_s\, L) \in H^1(X; \mathbb{Z})$ is non-trivial, then

i) $\dim B(f) \geq n - 1$, where dim denotes the Lebesgue covering dimension,

ii) either $B(f)$ disconnects X or it is not contractible to a point in X.

The proof of this result uses the bifurcation theorem [FPR99], Čech cohomology and Poincaré duality. Note that it is in particular required that $H^1(X; \mathbb{Z}) \neq 0$ and, accordingly, simply connected parameter spaces like S^n, $n \geq 2$, are excluded a priori. In the final section of [Po11] the bifurcation theorem is applied to families of geodesics in semi-Riemannian manifolds. However, since the operators L are assumed to be strongly indefinite, geodesics in Riemannian manifolds cannot be treated in this way.
The aim of this paper is to improve the results of [Po11] to more general functionals f and parameter spaces X. Our idea is based on the observation that the bifurcation theorem of [Po11] can be restated in terms of the spectral flow instead of K-theory classes. Indeed, it is easily seen that the assumptions on the family L in [Po11] are equivalent to the existence of a closed path $\gamma : S^1 \to X$, such that $\gamma(1)$ is invertible and the spectral flow of $L \circ \gamma$ is non-trivial.
Necessarily, our methods are completely different from the ones in [Po11]. We assume that X is a metrisable, connected and smooth Banach manifold, but we do not require it to be orientable nor do we make any restrictions on the cohomology groups of X. For each path $\gamma : I \to X$ in X, the composite $L \circ \gamma$ defines a path of selfadjoint Fredholm operators acting on H. If, moreover, $L_{\gamma(0)}$ and $L_{\gamma(1)}$ are invertible, then the spectral flow $\mathrm{sf}(L \circ \gamma)$ is defined and assigns to each such path an integer. Our first theorem states that if we can generate a non-zero integer in this way, then $B(f)$ either disconnects some open connected subset of X or it has interior points, which is equivalent to the above i) if X is of finite dimension n. Note that, since we make no assumptions on the cohomology of X, X can be contractible and hence the statement ii) above is obviously wrong in general. However, we show in a second theorem, whose argument is based on [Wa12], that $B(f)$ disconnects X if X is simply connected and there exists a path $\gamma : I \to X$ as above such that $\mathrm{sf}(L \circ \gamma) \neq 0$. Moreover, we conclude that $X \setminus B(f)$ has infinitely many path components if we can find an infinite amount of paths γ such that the spectral flows of $L \circ \gamma$ are pairwise distinct.
An important special case, which cannot be treated by the methods in [Po11], emerges if the Morse indices of the operators L are finite. Then $\mathrm{sf}(L \circ \gamma)$ is the difference of the Morse indices of $L_{\gamma(0)}$ and $L_{\gamma(1)}$, so that our invariant does no longer depend on the whole path γ but is actually defined for ordered pairs of points in X. We want to point out that Morse indices have been computed for the Hessians of many different types of functionals and it is not possible to give an exhaustive list of references here. However, we mention in passing that current work is underway regarding surface theory (cf. [Na90], [Ro02]), the Fermi-Pasta-Ulam problem

(cf.[MSTT06]), Stokes waves (cf. [BT03, §11.3]) and semilinear elliptic equations (cf. [GGPS11], [Da13]), where in particular families of functionals appear that have infinitely many different Morse indices. Following [Po11], we restrict in this paper to applications to families of geodesics in semi-Riemannian manifolds.
The paper is structured as follows: in the following section we recall the definition of the spectral flow, following the approach of [FPR99], and we introduce the main theorem of [FPR99] on the bifurcation of critical points of paths of functionals. In the third section we state our theorems and discuss some immediate examples. The fourth section is devoted to the proof of our theorems. In the final section we consider families of geodesics in semi-Riemannian manifolds, where we can also treat the case of Riemannian manifolds as well as non-orientable and non-compact parameter spaces, in contrast to [Po11]. Here we assume that the parameter space X is of finite dimension for technical reasons.

2 Preliminaries: spectral flow and bifurcation of critical points

Let H be a real, separable infinite-dimensional Hilbert space. We denote by $\mathcal{L}(H)$ the space of all bounded linear operators acting on H endowed with the topology induced by the operator norm, and by $\Phi_S(H) \subset \mathcal{L}(H)$ the subspace of all selfadjoint Fredholm operators. Recall that $\Phi_S(H)$ consists of three connected components. Two of them are given by

$$\Phi_S^+(H) = \{L \in \Phi_S(H) : \sigma_{ess}(L) \subset (0, \infty)\}, \qquad \Phi_S^-(H) = \{L \in \Phi_S(H) : \sigma_{ess}(L) \subset (-\infty, 0)\}$$

and their elements are called essentially positive or essentially negative, respectively. Both of them are contractible as topological spaces. Elements of the remaining component $\Phi_S^i(H) = \Phi_S(H) \setminus (\Phi_S^+(H) \cup \Phi_S^-(H))$ are called strongly indefinite. $\Phi_S^i(H)$ has the same homotopy groups as the stable orthogonal group and the *spectral flow*, which we now want to introduce briefly, induces an isomorphism between its fundamental group and the integers. We follow the approach developed by Fitzpatrick, Pejsachowicz and Recht in [FPR99].
If S, T are two selfadjoint invertible operators such that $S - T$ is compact, then the difference of their spectral projections with respect to any spectral subset is compact as well. Hence, denoting by $E_-(\cdot)$, $E_+(\cdot)$ the negative and positive subspaces of a selfadjoint operator, the *relative Morse index*

$$\mu_{rel}(S, T) = \dim(E_-(S) \cap E_+(T)) - \dim(E_+(S) \cap E_-(T))$$

is well defined and finite.
The group $GL(H)$ of all invertible operators on H acts on $\Phi_S(H)$ by *congredience*, sending $M \in GL(H)$ to M^*LM, $L \in \Phi_S(H)$. This induces an action of paths in $GL(H)$ on paths in $\Phi_S(H)$. One of the main theorems in [FPR99] states that for any path $L : I \to \Phi_S(H)$ there exist a path $M : I \to GL(H)$ and an invertible operator $J \in \Phi_S(H)$, such that $M_t^* L_t M_t = J + K_t$ with K_t compact for each $t \in [0, 1]$.

Definition 2.1. *Let $L : I \to \Phi_S(H)$ be a path such that L_0 and L_1 are invertible. Then the spectral flow of L is the integer*

$$\mathrm{sf}(L) = \mu_{rel}(J + K_0, J + K_1),$$

where $\{J + K_t\}_{t\in I}$ *is any path of compact perturbations of an invertible operator* $J \in \Phi_S(H)$ *which is cogredient with* $\{L_t\}_{t\in I}$ *in the sense above.*

It follows from general properties of the relative Morse index that the spectral flow does not depend on the choices of J and K. Moreover, it is obviously preserved by cogredience.
The main properties of the spectral flow are:

- if L_t is invertible for all $t \in I$, then $\operatorname{sf} L = 0$.
- If L^1, L^2 have invertible ends and the concatenation $L^1 * L^2$ is defined, then $\operatorname{sf}(L^1 * L^2) = \operatorname{sf}(L^1) + \operatorname{sf}(L^2)$.
- Let $h : I \times I \to \Phi_S(H)$ be a homotopy such that $h(s, 0)$ and $h(s, 1)$ are invertible for all $s \in I$. Then

$$\operatorname{sf}(h(0, \cdot)) = \operatorname{sf}(h(1, \cdot)).$$

- If $L_t \in \Phi^+(H)$, $t \in I$, and L_0, L_1 are invertible, then the spectral flow of L is the difference of the Morse indices at its endpoints:

$$\operatorname{sf}(L) = \mu_{Morse}(L_0) - \mu_{Morse}(L_1).$$

- If L has invertible ends and $\tilde{L}_t = L_{1-t}$, $t \in I$, then $\operatorname{sf} \tilde{L} = -\operatorname{sf} L$.

Finally, we introduce the main result of [FPR99] on bifurcation of critical points of paths of functionals.

Theorem 2.2. *Let* $f : I \times H \to \mathbb{R}$ *be a* C^2 *function such that for each* $\lambda \in I$, 0 *is a critical point of the functional* $f_\lambda = f(\lambda, \cdot)$. *Assume that the Hessians* L_λ *of* f_λ *at* 0 *are Fredholm operators and that* L_0 *and* L_1 *are invertible. If* $\operatorname{sf} L \neq 0$, *then every neighbourhood of* $I \times \{0\}$ *in* $I \times H$ *contains points of the form* (λ, u), *where* $u \neq 0$ *is a critical point of* f_λ.

As already mentioned in the introduction, the aim of this paper is to generalise theorem 2.2 to families of functionals which are parametrised by Banach manifolds instead of the unit interval.

3 The theorems

Let H be a Hilbert space and X a metrisable, connected and smooth Banach manifold (cf. [La95, II]). Let $f : X \times H \to \mathbb{R}$ be a C^2 function such that $0 \in H$ is a critical point of all functionals $f_\lambda = f(\lambda, \cdot) : H \to \mathbb{R}$, $\lambda \in X$. We refer to $X \times \{0\}$ as the trivial branch of critical points of the family $\{f_\lambda\}_{\lambda\in X}$ and we denote henceforth by L_λ the associated Hessian of f_λ at 0, $\lambda \in X$. Note that each L_λ is a bounded selfadjoint operator on the Hilbert space H and the family $L : X \to \mathcal{L}(H)$ depends continuously on $\lambda \in X$. In what follows we assume that all L_λ, $\lambda \in X$, are Fredholm operators.
We call a path $\gamma : I \to X$ admissible if $L_{\gamma(0)}$ and $L_{\gamma(1)}$ are invertible. In this case $L_{\gamma(t)}$, $t \in I$, defines a path in $\Phi_S(H)$ having invertible endpoints and, accordingly, the spectral flow of $L \circ \gamma$ is well defined, which we will denote henceforth by $\operatorname{sf}(\gamma, L)$.
We call $\lambda^* \in X$ a *bifurcation point of critical points* from the trivial branch if there exist a sequence $\lambda_n \to \lambda^*$ in X and a sequence $u_n \to 0$ in H such that u_n is a non-zero critical point of f_{λ_n}, $n \in \mathbb{N}$. We denote by $B(f) \subset X$ the set of all bifurcation points and observe that

$$B(f) \subset \Sigma(L) := \{\lambda \in X : \ker L_\lambda \neq \{0\}\} \tag{1}$$

by the inverse function theorem. It is a direct consequence of the definition that $B(f)$ is closed in X. With all this in place, our first theorem reads as follows:

Theorem 3.1. *Let X admit smooth partitions of unity. If there exists an admissible path γ in X such that $\mathrm{sf}(\gamma, L) \neq 0$, then either $B(f)$ has interior points or it disconnects some open connected subset of X.*

Note that by inclusion (1) we can possibly exclude one of these alternatives if $\Sigma(L)$ is known. Moreover, observe that every paracompact smooth Hilbert manifold admits smooth partitions of unity (cf. [La95, Corollary II.3.8]).
From elementary dimension theory (cf. e.g. [Fed90, 5,§2]) we obtain the following corollary:

Corollary 3.2. *If $\dim X = n < \infty$ and the assumptions of theorem 3.1 hold, then $\dim B(f) \geq n - 1$, where $\dim$ denotes the Lebesgue covering dimension.*

The assertion of the corollary is the main result of the first author's article [Po11], in which it is assumed in addition that X is compact and orientable, that the Hessians L_λ, $\lambda \in X$, are strongly indefinite and that the path γ is closed.
In case X is simply connected, theorem 3.1 can be improved like this:

Theorem 3.3. *Let X be simply connected.*

i) If there exists an admissible path γ such that $\mathrm{sf}(\gamma, L) \neq 0$, then $B(f)$ disconnects X.

ii) If there exists a sequence of admissible paths $\{\gamma_k\}_{k\in\mathbb{N}}$, such that

$$\lim_{k\to+\infty} |\mathrm{sf}(\gamma_k, L)| = +\infty,$$

then $X \setminus B(f)$ has infinitely many path components.

In case of a simply connected parameter space X, theorem 3.3 *i)* implies theorem 3.1. Moreover, note that we do not require X to admit smooth partitions of unity in theorem 3.3.
We close this section by providing two immediate examples. The first makes theorem 3.3 reminiscent of the classical Krasnosel'skii theorem for variational bifurcation (cf. [Kr64]), which we have already mentioned in the introduction. Let $\mathcal{K}_s(H)$ denote the closed subspace of $\mathcal{L}(H)$ consisting of all compact selfadjoint operators acting on H. Let $g : \mathcal{K}_s(H) \times H \to \mathbb{R}$ be a C^2 function such that $g(K, u) = o(\|u\|^2)$, $u \to 0$, for all $K \in \mathcal{K}_s(H)$. We consider the family of functionals

$$f : \mathcal{K}_s(H) \times H \to \mathbb{R}, \quad f(K, u) = \frac{1}{2}\langle (id + K)u, u\rangle_H + g(K, u)$$

and note that $0 \in H$ is a critical point of each f_K. The Hessian of f_K at $0 \in H$ is given by $L_K = id + K \in \Phi^+_S(H)$, $K \in \mathcal{K}_s(H)$.
Now let $\{e_i\}_{i\in\mathbb{N}}$ be a complete orthonormal system of H and define a sequence of compact selfadjoint operators by

$$K_n u = -2\sum_{i=1}^{n} \langle u, e_i\rangle e_i, \quad n \in \mathbb{N},$$

and a sequence of paths by

$$\gamma_n : I \to \mathcal{K}_S(H), \quad \gamma_n(t) = tK_n.$$

It is clear that each path γ_n is admissible with respect to L and we obtain from the very definition of the spectral flow that

$$\mathrm{sf}(\gamma_n, L) = \mu_{rel}(id + K_n, id) = \dim(E_-(id + K_n) \cap E_+(id)) - \dim(E_+(id + K_n) \cap E_-(id)) = n.$$

We conclude from theorem 3.3 ii) that $\mathcal{K}_s(H) \setminus B(f)$ has infinitely many path components.
Our second example shows that simply connectedness is a necessary assumption in theorem 3.3. It is easy to construct a family of functionals $f : S^1 \times H \to \mathbb{R}$ such that the associated family of Hessians L_λ is not invertible precisely at $1 \in S^1$ and $\mathrm{sf}\, L \neq 0$. Hence $B(f) = \{1\} \subset S^1$ by theorem 2.2. Now we consider the torus $T = S^1 \times S^1$ and the family of functionals

$$\tilde{f} : T \times H \to \mathbb{R}, \quad \tilde{f}(\lambda_1, \lambda_2, u) = f(\lambda_1, u).$$

The path $\gamma : S^1 \to T$, $\gamma(\lambda) = (\lambda, 1)$, is such that $\mathrm{sf}(\gamma, L) \neq 0$ and hence we infer from corollary 3.2 that $\dim B(\tilde{f}) \geq 1$, whereas theorem 3.3 cannot be applied to $\tilde{f}$ since T is not simply connected. Indeed, $B(\tilde{f}) = \{1\} \times S^1$ which does not disconnect T. Note, however, that $B(\tilde{f})$ is not contractible to a point in X, in accordance with [Po11].

4 The proofs

We fix a compatible metric ρ on the smooth Banach manifold X, which we have assumed to be metrisable, and denote henceforth by $B(\lambda, \varepsilon)$ the open ball of radius ε around $\lambda \in X$. The metric ρ induces a metric on the space $C(I, X)$ of all continuous paths in X by $d(\gamma_1, \gamma_2) = \sup_{t\in I} \rho(\gamma_1(t), \gamma_2(t))$. We will use several times the following elementary approximation result, whose proof we leave to the reader.

Lemma 4.1. *Let $\gamma_1 : I \to X$ be a continuous path and $\varepsilon > 0$. Then there exists a smooth path $\gamma_2 : I \to X$ such that $\gamma_2(0) = \gamma_1(0)$, $\gamma_2(1) = \gamma_1(1)$ and $d(\gamma_1, \gamma_2) < \varepsilon$.*

We now prove our theorems 3.1 and 3.3 in two separate sections.

4.1 Proof of theorem 3.1

We assume that $B(f)$ neither has interior points nor disconnects some connected open subset of X, and that $\gamma : I \to X$ is an admissible path such that $\mathrm{sf}(\gamma, L) \neq 0$. Our aim is to reach a contradiction.
We fix any spray on X, which exists since we assume that X admits smooth partitions of unity (cf. [La95, Theorem IV.3.1]). Let $\exp : \mathcal{D} \to X$ denote the associated exponential map of this spray, where $\mathcal{D} \subset TX$ denotes its open maximal domain. We obtain a smooth map

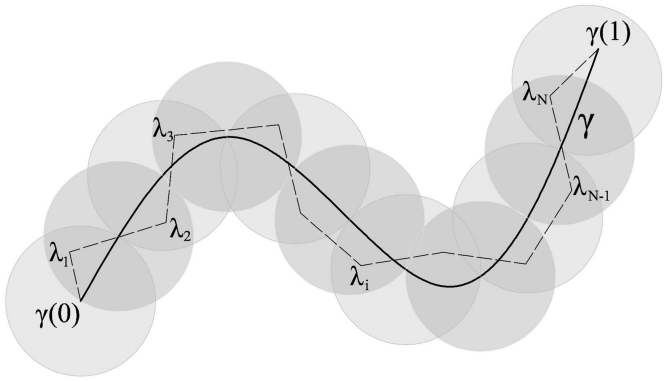


Figure 1: Construction of $\gamma_1 * \ldots * \gamma_{N+1}$.

$$F : \mathcal{D} \to X \times X, \quad F(v) = (\pi(v), \exp_{\pi(v)}(v)),$$

where $\pi : TX \to X$ is the projection of the tangent bundle. By [La95, Prop. VIII.5.1], F is a local diffeomorphism on the zero section of TX, and arguing as in [BJ73, §12] we can choose a subset $\tilde{D} \subset \mathcal{D}$ such that $\tilde{D}_\lambda = \mathcal{D} \cap T_\lambda X$ is star-shaped with respect to $0 \in T_\lambda X$, $\lambda \in X$, and F is a diffeomorphism from $\tilde{D}$ onto an open neighbourhood $\mathcal{U}$ of the diagonal in $X \times X$. Finally, we choose $\varepsilon > 0$ small enough so that $(\tilde{\gamma}(t), \gamma(t)) \in \mathcal{U}$, $t \in I$, for any path $\tilde{\gamma} : I \to X$ such that $d(\tilde{\gamma}, \gamma) < 3\varepsilon$.
Our next goal is to construct a path in X which is 2ε-close to γ but does not intersect $B(f)$. At first, if $\gamma(I) \subset B(\gamma(0), \varepsilon)$, there exists by assumption a path in $B(\gamma(0), \varepsilon)$ connecting $\gamma(0)$ and $\gamma(1)$ without intersecting $B(f)$. Hence let us assume that $\gamma(I) \not\subset B(\gamma(0), \varepsilon)$. We define

$$t_1 := \inf\{t \in I : \gamma(t) \in \partial B(\gamma(0), \varepsilon)\}.$$

By assumption there exists $\lambda_1 \in B(\gamma(0), \varepsilon) \cap B(\gamma(t_1), \varepsilon)$ such that $\lambda_1 \notin B(f)$, and moreover there exists a path

$$\gamma_1 : [0, t_1] \to B(\gamma(0), \varepsilon)$$

joining $\gamma(0)$ and λ_1 without intersecting $B(f)$. Note that

$$\sup_{t \in [0, t_1]} \rho(\gamma_1(t), \gamma(t)) < 2\varepsilon.$$

If $\gamma([t_1, 1]) \subset B(\gamma(t_1), \varepsilon)$, then we can again finish the construction by using the assumption that $B(f)$ does not disconnect $B(\gamma(t_1), \varepsilon)$. Otherwise, we define

$$t_2 := \inf\{t_1 \leq t \leq 1 : \gamma(t) \in \partial B(\gamma(t_1), \varepsilon)\}.$$

As before there are $\lambda_2 \in B(\gamma(t_1), \varepsilon) \cap B(\gamma(t_2), \varepsilon)$ such that $\lambda_2 \notin B(f)$ and a path

$$\gamma_2 : [t_1, t_2] \to B(\gamma(t_1), \varepsilon)$$

connecting λ_1 and λ_2 without intersecting $B(f)$ and such that

$$\sup_{t \in [t_1, t_2]} \rho(\gamma_2(t), \gamma(t)) < 2\varepsilon.$$

If we continue this process, we eventually arrive at an index $N \in \mathbb{N}$ such that the path

$$\gamma_N : [t_{N-1}, t_N] \to B(\gamma(t_{N-1}), \varepsilon)$$

does not intersect $B(f)$, plus $\gamma_N(t_{N-1}) = \gamma_{N-1}(t_{N-1})$, $\lambda_N := \gamma_N(t_N) \in B(\gamma(t_N), \varepsilon)$ and

$$\sup_{t \in [t_{N-1}, t_N]} \rho(\gamma_N(t), \gamma(t)) < 2\varepsilon,$$

but

$$\gamma([t_N, 1]) \cap \partial B(\gamma(t_N), \varepsilon) = \emptyset.$$

If this was not true, then we would obtain a sequence $\{t_n\}_{n \in \mathbb{N}} \subset I$, such that $t_n \to 1$ but $\gamma(t_n) \nrightarrow \gamma(1)$, $n \to \infty$, in contradiction to the continuity of γ.
Accordingly, $\gamma(1) \in B(\gamma(t_N), \varepsilon)$ and we can find a path

$$\gamma_{N+1} : I \to B(\gamma(t_N), \varepsilon),$$

which connects λ_N and $\gamma(1)$ without intersecting $B(f)$. Now the concatenation

$$\gamma_1 * \ldots * \gamma_{N+1} : I \to X$$

does not intersect $B(f)$ and

$$d(\gamma_1 * \ldots * \gamma_{N+1}, \gamma) < 2\varepsilon.$$

Since $B(f)$ is closed, we can use lemma 4.1 in order to choose a smooth path

$$\tilde{\gamma} : I \to X \text{ such that } \tilde{\gamma}(I) \cap B(f) = \emptyset \text{ and } d(\tilde{\gamma}, \gamma) < 3\varepsilon.$$

Now we consider the homotopy

$$h : I \times I \to X, \quad h(s,t) = \exp(s \cdot (F \mid_{\tilde{D}})^{-1}(\gamma(t), \tilde{\gamma}(t))).$$

Note that $h(s,0) = \gamma(0) = \tilde{\gamma}(0)$, $h(s,1) = \gamma(1) = \tilde{\gamma}(1)$ for all $s \in I$, and $h(0,\cdot) = \gamma$, $h(1,\cdot) = \tilde{\gamma}$ and hence $\mathrm{sf}(\tilde{\gamma}, L) = \mathrm{sf}(\gamma, L) \neq 0$ by the homotopy invariance of the spectral flow. We define

$$\tilde{f} : I \times H \to \mathbb{R}, \quad \tilde{f}(t,u) = f(\tilde{\gamma}(t), u),$$

which is a C^2 function. Each $\tilde{f}_t = \tilde{f}(t,\cdot) : H \to \mathbb{R}$ has $0 \in H$ as critical point and the associated Hessian is given by $L_{\tilde{\gamma}(t)}$, $t \in I$. According to theorem 2.2, there exist a sequence $\{t_n\}_{n\in\mathbb{N}} \subset I$ and a sequence $\{u_n\}_{n\in\mathbb{N}} \subset H$ such that $u_n \neq 0$ is a critical point of $\tilde{f}_{t_n}$, $n \in \mathbb{N}$, and $(t_n, u_n) \to (t^*, 0)$ for some $t^* \in I$. Now the sequence $\{(\tilde{\gamma}(t_n), u_n)\}_{n\in\mathbb{N}} \subset X \times H$ shows that $\gamma(t^*) \in X$ is a bifurcation point of f on $\tilde{\gamma}(I)$, contradicting the construction of $\tilde{\gamma}$, and this ends the proof.

4.2 Proof of theorem 3.3

At first we prove statement i) arguing by contradiction. Let γ be an admissible path such that

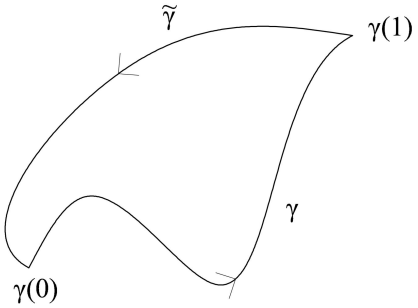


Figure 2: A closed path with vanishing spectral flow.

$\mathrm{sf}(\gamma, L) \neq 0$ and assume that $X \setminus B(f)$ is connected. Then we can find a path $\tilde{\gamma}$ such that $\tilde{\gamma}(0) = \gamma(1)$, $\tilde{\gamma}(1) = \gamma(0)$ and $\tilde{\gamma}(I) \cap B(f) = \emptyset$. Since $B(f)$ is closed we can assume without loss of generality that $\tilde{\gamma}$ is smooth by lemma 4.1. Arguing as in the last paragraph of the previous section, we conclude from theorem 2.2 that $\mathrm{sf}(\tilde{\gamma}, L) = 0$.

Now we consider the concatenation $\gamma * \tilde{\gamma}$ which is a closed path. Since X is simply connected, $\gamma * \tilde{\gamma}$ is homotopic to the constant path $\gamma'(t) \equiv \gamma(0)$, $t \in I$, by means of a homotopy leaving $\gamma(0)$ fixed. Hence

$$\mathrm{sf}(\gamma, L) = \mathrm{sf}(\gamma, L) + \mathrm{sf}(\tilde{\gamma}, L) = \mathrm{sf}(\gamma * \tilde{\gamma}, L) = \mathrm{sf}(\gamma', L) = 0$$

which is a contradiction to our assumption.

In order to prove ii) we first construct inductively a sequence $\tilde{\gamma}_k$, $k \in \mathbb{N}$, of admissible paths in X such that $\mathrm{sf}(\tilde{\gamma}_i, L) \neq \mathrm{sf}(\tilde{\gamma}_j, L)$ for all $i \neq j$ and $\tilde{\gamma}_k(0) = \lambda_0$, $k \in \mathbb{N}$, for some $\lambda_0 \in X$.
Let $\lambda_0 \in X$ be such that L_{λ_0} is invertible. Set $\tilde{\gamma}_1 \equiv \lambda_0$ and assume henceforth that we have already constructed paths $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_n$ such that $\mathrm{sf}(\tilde{\gamma}_i, L) \neq \mathrm{sf}(\tilde{\gamma}_j, L)$ for $i \neq j$ and $\tilde{\gamma}_i(0) = \lambda_0$ for all $1 \leq i \leq n$.
We choose an admissible path γ such that

$$|\,\mathrm{sf}(\gamma, L)| > \max_{1\leq i,j\leq n} |\,\mathrm{sf}(\tilde{\gamma}_i, L) - \mathrm{sf}(\tilde{\gamma}_j, L)|$$

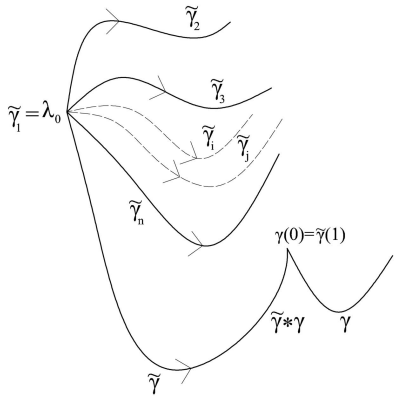


Figure 3: The inductive construction of $\tilde{\gamma}_k$.

and a path $\tilde{\gamma}$ such that $\tilde{\gamma}(0) = \lambda_0$ and $\tilde{\gamma}(1) = \gamma(0)$. If $\mathrm{sf}(\tilde{\gamma} * \gamma, L) \neq \mathrm{sf}(\tilde{\gamma}_i, L)$ for all $1 \leq i \leq n$, then we set $\tilde{\gamma}_{n+1} = \tilde{\gamma} * \gamma$. Otherwise we set $\tilde{\gamma}_{n+1} = \tilde{\gamma}$. In order to justify our choice, assume that $\mathrm{sf}(\tilde{\gamma} * \gamma, L) = \mathrm{sf}(\tilde{\gamma}_i, L)$ and $\mathrm{sf}(\tilde{\gamma}, L) = \mathrm{sf}(\tilde{\gamma}_j, L)$ for some $1 \leq i, j \leq n$. Then

$$\mathrm{sf}(\tilde{\gamma}_i, L) = \mathrm{sf}(\tilde{\gamma} * \gamma, L) = \mathrm{sf}(\tilde{\gamma}, L) + \mathrm{sf}(\gamma, L) = \mathrm{sf}(\tilde{\gamma}_j, L) + \mathrm{sf}(\gamma, L)$$

which is a contradiction to the choice of γ.
Hence there exists a sequence of admissible paths $\tilde{\gamma}_k$, $k \in \mathbb{N}$, such that $\mathrm{sf}(\tilde{\gamma}_i, L) \neq \mathrm{sf}(\tilde{\gamma}_j, L)$ for all $i \neq j$ and $\tilde{\gamma}_k(0) = \lambda_0$, $k \in \mathbb{N}$, for some $\lambda_0 \in X$. We claim that the endpoints $\tilde{\gamma}_i(1)$, $i \in \mathbb{N}$, all lie in different path components of $X \setminus B(f)$.
Assume on the contrary that there exists a path γ' connecting $\tilde{\gamma}_i(1)$ and $\tilde{\gamma}_j(1)$ and such that $\gamma'(I) \cap B(f) = \emptyset$. Again we can suppose by lemma 4.1 that γ' is smooth without loss of generality and we conclude that $\mathrm{sf}(\gamma', L) = 0$ by theorem 2.2. The concatenation $\tilde{\gamma}_i * \gamma' * \tilde{\gamma}_{-j}$ is a closed path, where we denote by $\tilde{\gamma}_{-j}(t) = \tilde{\gamma}_j(1-t)$, $t \in I$, the inverse path. We infer as in the proof of i) above that $\mathrm{sf}(\tilde{\gamma}_i * \gamma' * \tilde{\gamma}_{-j}, L) = 0$ and obtain

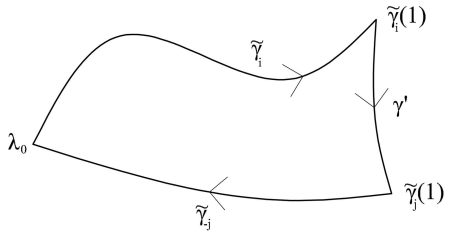


Figure 4: The construction of γ'.

$$0 = \mathrm{sf}(\tilde{\gamma}_i * \gamma' * \tilde{\gamma}_{-j}, L) = \mathrm{sf}(\tilde{\gamma}_i, L) + \mathrm{sf}(\gamma', L) + \mathrm{sf}(\tilde{\gamma}_{-j}, L) = \mathrm{sf}(\tilde{\gamma}_i, L) - \mathrm{sf}(\tilde{\gamma}_j, L),$$

which is a contradiction to the existence of the paths $\tilde{\gamma}_k$, $k \in \mathbb{N}$. This concludes the proof.

5 Families of geodesics

In this final section we apply our theory to families of geodesics in semi-Riemannian manifolds in order to improve the corresponding result from [Po11, §4]. We begin by recalling briefly some constructions from [MPP05]. The latter article is, together with [MPP07], the main reference for a more detailed introduction to the present setting.
Let M be a smooth connected manifold of dimension $n \in \mathbb{N}$. We denote by Ω the Hilbert manifold of all H^1 curves in M, which is modelled on the Sobolev space $H^1(I, \mathbb{R}^n)$. The tangent space $T_\gamma\Omega$ at an element $\gamma \in \Omega$ can be identified in a natural way with the space $H^1(\gamma)$ of all H^1 vector fields along γ. The endpoint map

$$\pi : \Omega \to M \times M, \quad \pi(\gamma) = (\gamma(0), \gamma(1))$$

is a submersion and hence for each $(p, q) \in M \times M$ the fibre

$$\Omega_{pq} = \{\gamma \in \Omega : \gamma(0) = p, \gamma(1) = q\}$$

is a submanifold of codimension $2n$ whose tangent space at $\gamma \in \Omega_{pq}$ is the subspace

$$H_0^1(\gamma) = \{\xi \in H^1(\gamma) : \xi(0) = 0,\, \xi(1) = 0\}$$

of $H^1(\gamma)$. Note that the family of spaces $H_0^1(\gamma)$, $\gamma \in \Omega$, are the fibres of the vertical bundle $TF(\pi)$ of the submersion π.
We have for each semi-Riemannian metric g on M the associated energy functional

$$f_g : \Omega \to \mathbb{R}, \quad f_g(\gamma) = \frac{1}{2} \int_0^1 g(\gamma', \gamma')\, dt.$$

The critical points of the restriction of f_g to the submanifold Ω_{pq} of Ω are precisely the geodesics with respect to g joining p and q.
Now let g and $p, q \in M$ be fixed and let $\gamma \in \Omega_{pq}$ be a geodesic. Then the second variation of f_g at γ is the quadratic form

$$h_\gamma(\xi) = \int_0^1 g\left(\frac{\nabla}{dt}\xi, \frac{\nabla}{dt}\xi\right) dt - \int_0^1 g(R(\gamma', \xi)\gamma', \xi)\, dt$$

on the space $H_0^1(\gamma)$, where $\frac{\nabla}{dt}$ denotes the covariant derivative along γ and R is the curvature of the Levi-Civita connection induced by g. If g is a Riemannian metric, then the Morse index of γ is by definition the dimension of the maximal subspace of $H_0^1(\gamma)$ on which h_γ is negative definite. For a non-Riemannian metric, there always exists an infinite dimensional subspace on which h_γ is negative definite, but it is shown in [MPP05] how one can define an index for geodesics in general semi-Riemannian manifolds by using the spectral flow. The construction goes as follows: the geodesic γ induces canonically a path Γ in Ω by setting $\Gamma(s)[t] = \gamma(s \cdot t)$, $s, t \in I$. Each $\Gamma(s)$ is a critical point of the restriction of f_g to $\Omega_{\gamma(0),\gamma(s)}$ and the associated second variations $h_{\Gamma(s)}$, $s \in I$, can be regarded as a map on the pullback bundle $\Gamma^*(TF(\pi))$, which is a Hilbert bundle over the unit interval I having as typical fibre the Sobolev space $H_0^1(I, \mathbb{R}^n)$. Now choose a global trivialisation $\psi : I \times H_0^1(I, \mathbb{R}^n) \to \Gamma^*(TF(\pi))$ and obtain a path of quadratic forms

$h_{\Gamma(s)}(\psi(s,u))$, $(s,u) \in I \times H_0^1(I,\mathbb{R}^n)$, whose Riesz representations L_s, $s \in I$, are shown to be Fredholm in [MPP05, Prop. 3.1]. Moreover, it is easy to see that L_0 is invertible. If $\ker h_\gamma = 0$, then L_1 is invertible as well and we call the geodesic γ *non-degenerate*. In this case we obtain a path $L : I \to \Phi_S(H_0^1(I,\mathbb{R}^n))$ having invertible endpoints and we define the *spectral index* of the non-degenerate geodesic γ by

$$\mu(\gamma) = -\operatorname{sf}(L) \in \mathbb{Z}.$$

If g is a Riemannian metric, then $\mu(\gamma)$ coincides with the Morse index of γ, which can be computed as the total number of conjugate points along γ according to the classical Morse index theorem. Moreover, let us mention that $\mu(\gamma)$ can also be obtained in general by counting algebraically the number of conjugate points along γ, which is the content of the semi-Riemannian Morse index theorem [MPP05].

Now let X be a smooth manifold of dimension n and let $p : \mathcal{S}^2_\nu(M) \to M$ be the bundle of non-degenerate symmetric two-forms of index ν on M. A *family of semi-Riemannian metrics* of index ν on M is a smooth map $g : X \times M \to \mathcal{S}^2_\nu(M)$ such that each map $g(\lambda,\cdot) : M \to \mathcal{S}^2_\nu(M)$, $\lambda \in X$, is a section of p. We assume that there exists a smooth map $\sigma : X \times I \to M$ such that each $\sigma(\lambda) = \sigma(\lambda,\cdot) \in \Omega$, $\lambda \in X$, is a geodesic in M with respect to the metric g_λ. We refer to σ as the trivial branch of geodesics.

We call $\lambda^* \in X$ a *bifurcation point for geodesics* from the trivial branch σ if there exists a sequence $(\lambda_n, \gamma_n) \to (\lambda^*, \sigma(\lambda^*))$ in $X \times \Omega$ such that each γ_n is a geodesic with respect to g_{λ_n},

$$\gamma_n(0) = \sigma(\lambda_n, 0), \quad \gamma_n(1) = \sigma(\lambda_n, 1)$$

and $\gamma_n \neq \sigma(\lambda_n)$, $n \in \mathbb{N}$.[1] We denote the set of all bifurcation points by $\mathcal{B}_\sigma \subset X$.

The main result of this section reads as follows:

Theorem 5.1. *Let X be a smooth connected manifold of finite dimension n.*

i) Assume that there exist $\lambda_0, \lambda_1 \in X$ such that $\sigma(\lambda_0)$, $\sigma(\lambda_1)$ are non-degenerate and

$$\mu(\sigma(\lambda_0)) \neq \mu(\sigma(\lambda_1)).$$

Then:

a) Either $\mathcal{B}_\sigma$ has interior points or it disconnects some open connected subset of X.

b) $\dim \mathcal{B}_\sigma \geq n-1$.

c) If X is simply connected, then $\mathcal{B}_\sigma$ disconnects X.

ii) If X is simply connected and there exists a sequence $\{\lambda_k\}_{k\in\mathbb{N}} \subset X$ such that $\sigma(\lambda_k)$ is non-degenerate for all $k \in \mathbb{N}$ and

$$\lim_{k\to+\infty} |\mu(\sigma(\lambda_k))| = +\infty,$$

then $X \setminus \mathcal{B}_\sigma$ has infinitely many path components.

[1] We want to point out that the definition of a bifurcation point of geodesics is stated in an incorrect way in [MPP07] and that all results proved in that paper actually concern bifurcation with respect to the definition given here.

Proof. We consider the endpoint map

$$e : X \to M \times M, \quad e(\lambda) = (\sigma(\lambda, 0), \sigma(\lambda, 1)).$$

The pullback $e^*(\pi)$ of the submersion π can be defined in the usual way and its total space is given by

$$E := \{(\lambda, \gamma) \in X \times \Omega : e(\lambda) = \pi(\gamma)\} \subset X \times \Omega.$$

Note that $E_\lambda = (e^*(\pi))^{-1}(\lambda) = \Omega_{\sigma(\lambda,0),\sigma(\lambda,1)}$, $\lambda \in X$. By standard transversality arguments (cf. [La95, II,§2]), $e^*(\pi)$ is a submersion as well and we obtain a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{\iota} & \Omega \\ {\scriptstyle e^*(\pi)}\downarrow & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{e} & M \times M \end{array} \tag{2}$$

Because of the commutativity of (2), the map $\sigma : X \to \Omega$ induces a section $\tilde{\sigma} : X \to E$ of E. We now consider the smooth function

$$f : E \to \mathbb{R}, \quad f(\lambda, \gamma) = \frac{1}{2} \int_0^1 g_\lambda(\gamma', \gamma')\, dt$$

and note that $\gamma \in E_\lambda = \Omega_{\sigma(\lambda,0),\sigma(\lambda,1)}$ is a critical point of $f_\lambda := f\,|_{E_\lambda}$ if and only if γ is a geodesic for the metric g_λ, $\lambda \in X$. Hence we have reduced the study of $\mathcal{B}_\sigma$ to the bifurcation of critical points of the functional f from the branch $\tilde{\sigma}(X)$.
In order to apply theorems 3.1 and 3.3, we need the vector-bundle neighbourhood theorem, which is proved in [MPP07, App. A] for bundles over compact base spaces. However, it is easily seen by arguing as in [BJ73, Lemma 12.6] that its statement can be proved in the non-compact case along the same lines. Accordingly, there exists a trivial Hilbert bundle $X \times H$ over X and a smooth map $\psi : X \times H \to E$ such that ψ is a diffeomorphism onto an open neighbourhood of $\tilde{\sigma}(X)$ in E and $\psi(\lambda, 0) = \tilde{\sigma}(\lambda)$, $\lambda \in X$. Now consider $\tilde{f} : X \times H \to \mathbb{R}$, $\tilde{f} = f \circ \psi$. Since the restriction ψ_λ to the fibre is a diffeomorphism onto an open subset of E_λ, we infer that $u \in H$ is a critical point of $\tilde{f}_\lambda$ if and only if $\psi_\lambda(u)$ is a critical point of f_λ. In particular, $0 \in H$ is a critical point of all maps $\tilde{f}_\lambda$, $\lambda \in X$, and, moreover, $B(\tilde{f}) = \mathcal{B}_\sigma$.
The Hessian $\tilde{L}_\lambda$ of $\tilde{f}_\lambda$ at $0 \in H$ is the Riesz representation of the quadratic form

$$\tilde{h}_\lambda(u) = h_\lambda((T_0\psi_\lambda)u), \quad u \in H,\ \lambda \in X,$$

and $\tilde{L} : X \to \Phi_S(H)$ is a continuous family of selfadjoint Fredholm operators.
We now choose any path $\gamma : I \to X$ such that $\gamma(0) = \lambda_0$ and $\gamma(1) = \lambda_1$ and we consider the composite map $\tilde{L} \circ \gamma : I \to \Phi_S(H)$, which is a path of selfadjoint Fredholm operators having invertible endpoints. Arguing as in [MPP07, Prop. 8.4], we obtain

$$\mathrm{sf}(\gamma, \tilde{L}) = \mathrm{sf}(\tilde{L} \circ \gamma) = \mu(\sigma(\lambda_0)) - \mu(\sigma(\lambda_1)).$$

It is now clear that the theorems 3.1 and 3.3 apply to the family $\tilde{f} : X \times H \to \mathbb{R}$. Since $B(\tilde{f}) = \mathcal{B}_\sigma$, we finally obtain the claim. □

We finish this section by giving an example of theorem 5.1. Let g be a semi-Riemannian metric on a smooth connected manifold M of finite dimension n such that (M, g) is geodesically complete. We set $X := TM$ and define as trivial branch of geodesics

$$\sigma : TM \times I \to M, \quad \sigma(v, t) = \exp_{\pi_M(v)}(t \cdot v),$$

where $\pi_M : TM \to M$ denotes the projection of the tangent bundle. Here we use that by assumption the exponential map is defined on the whole tangent bundle TM. Note that the family σ consists of all geodesics in (M, g).
The constant paths $\sigma(0, t) \equiv p \in M$, $0 \in T_pM \subset TM$ are geodesics having vanishing spectral index $\mu(\sigma(0)) = 0$. Accordingly, if there exists $v \in TM$ such that $\sigma(v)$ is non-degenerate and $\mu(\sigma(v)) \neq 0$, we infer that $\dim \mathcal{B}_\sigma \geq 2n - 1$. Now let us assume in addition that $\pi_1(M) = 0$. Then TM is simply connected and we obtain that $\mathcal{B}_\sigma$ disconnects TM. Finally, if we can find a sequence $\{v_k\}_{k\in\mathbb{N}} \subset TM$ such that $\sigma(v_k)$, $k \in \mathbb{N}$, is non-degenerate and $|\mu(\sigma(v_k))| \to \infty$, $k \to \infty$, then $TM \setminus \mathcal{B}_\sigma$ has infinitely many path components. Examples of the latter case are provided by the spheres S^n, $n \geq 2$, with their usual Riemannian metrics.

References

[APS76] M.F. Atiyah, V.K. Patodi, I.M. Singer, **Spectral Asymmetry and Riemannian Geometry III**, Math. Proc. Cambridge Philos. Soc. **79** (1976), 71-99.

[BJ73] T. Bröcker, K. Jänich, **Introduction to differential topology**, Cambridge University Press, Cambridge-New York, 1982.

[BT03] B. Buffoni, J. Toland, **Analytic Theory of Global Bifurcation**, Princeton Ser. Appl. Math., Princeton University Press, 2003.

[Da13] E.N. Dancer, **Some bifurcation results for rapidly growing nonlinearities**, Discrete Contin. Dyn. Syst. **33** (2013), 153-161.

[Fed90] V.V. Fedorchuk, **The Fundamentals of Dimension Theory**, Encyclopaedia of Mathematical Sciences **17**, General Topology I, 1990, 91-202.

[FPR99] P.M. Fitzpatrick, J. Pejsachowicz, L. Recht, **Spectral Flow and Bifurcation of Critical Points of Strongly-Indefinite Functionals Part I: General Theory**, J. Funct. Anal. **162** (1999), 52-95.

[FPR00] P.M. Fitzpatrick, J. Pejsachowicz, L. Recht, **Spectral Flow and Bifurcation of Critical Points of Strongly-Indefinite Functionals Part II: Bifurcation of periodic orbits of Hamiltonian systems**, J. Differential Equations **163** (2000), 18-40.

[GGPS11] F. Gladiali, M. Grossi, F. Pacella, P.N. Srikanth, **Bifurcation and symmetry breaking for a class of semilinear elliptic equations in an annulus**, Calc. Var. Partial Differential Equations **40** (2011), 295-317.

[Kr64] M.A. Krasnosel'skii, **Topological Methods in the Theory of Nonlinear Integral Equations**, Pergamon, Oxford, 1964.

[La95] S. Lang, **Differential and Riemannian Manifolds**, Grad. Texts in Math. **160**, Springer, 1995.

[MSTT06] G. Molteni, E. Serra, M.Tarallo, S. Terracini, **Asymptotic Resonance, Interaction of Modes and Subharmonic Bifurcation**, Arch. Rational Mech. Anal. **182** (2006), 77-123.

[MPP05] M. Musso, J. Pejsachowicz, A. Portaluri, **A Morse Index Theorem for Perturbed Geodesics on Semi-Riemannian Manifolds**, Topol. Methods Nonlinear Anal. **25** (2005), 69-99.

[MPP07] M. Musso, J. Pejsachowicz, A. Portaluri, **Morse Index and Bifurcation for p-Geodesics on Semi-Riemannian Manifolds**, ESAIM Control Optim. Calc. Var **13** (2007), 598-621.

[Na90] S. Nayatani, **Lower bounds for the Morse index of complete minimal surfaces in Euclidean 3-space**, Osaka J. Math. **27** (1990), 453-464.

[PPT03] P. Piccione, A. Portaluri, D. V. Tausk, **Spectral flow, Maslov index and bifurcation of semi-Riemannian geodesics**, Ann. Global Anal. Geom. **25** (2004), no. 2, 121–149.

[PP05] P. Piccione, A. Portaluri, **A bifurcation result for semi-Riemannian trajectories of the Lorentz force equation**, J. Differential Equations **210** (2005), no. 2, 233–262.

[Po11] A. Portaluri, **A K-theoretical invariant and bifurcation for a parameterized family of functionals**, J. Math. Anal. Appl. **377** (2011), 762-770.

[Ro02] W. Rossman, **Lower bounds for Morse index of constant mean curvature tori**, Bull. London Math. Soc. **34** (2002), 599-609.

[Wa12] N. Waterstraat, **On the space of metrics having non-trivial harmonic spinors**, submitted, arXiv:1206.0499v1.

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