A remark on the space of metrics having non-trivial harmonic spinors

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Abstract

Let M be a closed spin manifold of dimension $n \equiv 3 \mod 4$. We give a simple proof of the fact that the space of metrics on M with invertible Dirac operator is either empty or it has infinitely many path components.

1 Introduction

Let M be an n-dimensional closed spin manifold and let $R(M)$ be the space of all Riemannian metrics on M. For any choice of a metric $g \in R(M)$, we can build the associated spinor bundle $\Sigma_g M$ and obtain a natural first order operator D_g acting on sections of $\Sigma_g M$ and which we call the *Dirac operator*. Elements of $\ker D_g$ are called *harmonic spinors* and their existence has been studied for a long time. While one can show that on S^2 no non-trivial harmonic spinors exist for any choice of g (cf. [Ba92]), it is conjectured that on every closed spin manifold of dimension $n \geq 3$ there exists a Riemannian metric g such that $\ker D_g \neq 0$. The conjecture has been proved by N. Hitchin in [Hi74] if $n \equiv 0, \pm 1 \mod 8$ and by C. Bär in [Ba96] if $n \equiv 3 \mod 4$.
As a more general question, one may ask how many metrics exist on M such that the corresponding Dirac operator has non-trivial kernel. A possible way to study this question is to consider the complementary set of metrics $R^{\text{inv}}(M)$ consisting of all metrics $g \in R(M)$ such that $\ker D_g = 0$. M. Dahl showed in [Da08] that elements of $R^{\text{inv}}(M)$ can be extended to $R^{\text{inv}}(W)$ if W is the trace of a surgery of codimension at least 3 on M. By using the Atiyah-Singer index theorem and special metrics on the spheres originating from the study of positive scalar curvature, he concluded from this result that $R^{\text{inv}}(M)$ is in all dimensions $n \geq 5$ which were considered by Hitchin and Bär either empty or disconnected. Moreover, in the case $n \equiv 3 \mod 4$, $n \geq 7$, he even obtained that, if non-empty, $R^{\text{inv}}(M)$ has infinitely many path components. Recently he improved this conclusion in collaboration with N. Grosse to dimension 3 by studying extensions of metrics to attached handles (cf. [DG12]).
The aim of this article is to show that the existence of infinitely many connected components of $R^{\text{inv}}(M)$ in all dimensions $n \equiv 3 \mod 4$ can be derived easily from Bär's results in [Ba96] by using spectral flow and rather elementary homotopy arguments.
Finally, we want to mention that Bär improved his theorem in [Ba97] to twisted Dirac operators. Note that for any fixed pair (F, ∇) of a bundle F over M and a connection ∇ on F, we obtain a family $D^{(F,\nabla)}$ of twisted Dirac operators which is again parametrised by the space of Riemannian metrics $R(M)$ on M. We believe that one can extend our argument here to this case by using the results from [Ba97] instead of [Ba96]. Accordingly, we conjecture that the corresponding space $R^{\text{inv}}_{(F,\nabla)}(M) = \{g \in R(M) : \ker D_g^{(F,\nabla)} = 0\}$ is either empty or it has infinitely many path components.

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2 Preliminaries: Dirac operators

In this section we recall briefly the definition of spinor bundles and their Dirac operators. Among the many references for these topics we want to mention [Hij01] and [Am01], on which we base our exposition. In order to simplify the presentation we assume throughout that M is an oriented closed manifold of odd dimension $n \geq 3$.
We denote by $\mathrm{GL}^+(M)$ the principal $\mathrm{GL}^+(n;\mathbb{R})$-bundle of oriented bases over M and recall that $\mathrm{GL}^+(n;\mathbb{R})$ has a unique connected 2-fold covering $\Theta : \widetilde{\mathrm{GL}^+}(n;\mathbb{R}) \to \mathrm{GL}^+(n;\mathbb{R})$ since the fundamental group of $\mathrm{GL}^+(n;\mathbb{R})$ is of order two. A *spin structure* on M is a pair $(\widetilde{\mathrm{GL}^+}(M), \vartheta)$, where $\widetilde{\mathrm{GL}^+}(M)$ is a principal $\widetilde{\mathrm{GL}^+}(n;\mathbb{R})$-bundle over M and $\vartheta : \widetilde{\mathrm{GL}^+}(M) \to \mathrm{GL}^+(M)$ is a 2-fold covering such that

$$\pi \circ \vartheta = \widetilde{\pi} \quad \text{and} \quad \vartheta(u \cdot v) = \vartheta(u) \cdot \Theta(v), \quad \text{for all } v \in \widetilde{\mathrm{GL}^+}(n;\mathbb{R}),\, u \in \widetilde{\mathrm{GL}^+}(M),$$

where π and $\widetilde{\pi}$ denote the corresponding projections of the bundles. Henceforth we assume that M is a *spin manifold*, that is, M is oriented and a spin structure on M is given. Note that so far we have not required that M is endowed with a Riemannian metric.
Let now g be a Riemannian metric on M and denote by $\mathrm{SO}(M,g)$ the associated principal $\mathrm{SO}(n)$-bundle of positively oriented orthonormal bases. Then $\mathrm{Spin}(M,g) := \vartheta^{-1}(\mathrm{SO}(M,g))$ is a principal $\mathrm{Spin}(n)$-bundle over M, where $\mathrm{Spin}(n) := \Theta^{-1}(\mathrm{SO}(n))$ is the unique connected 2-fold covering of $\mathrm{SO}(n)$. Let $\rho : \mathbb{C}l_n \to \mathrm{End}(\Sigma_n)$ denote the usual irreducible representation of the complex Clifford algebra, where Σ_n is the space of complex spinors. We fix an inner product $\langle\cdot,\cdot\rangle$ on Σ_n such that $\langle\rho(x)\sigma_1, \rho(x)\sigma_2\rangle = \langle\sigma_1,\sigma_2\rangle$ for all $x \in \mathbb{R}^n$, $\|x\| = 1$, and $\sigma_1, \sigma_2 \in \Sigma_n$. If now $\rho' : \mathrm{Spin}(n) \to \mathrm{Aut}(\Sigma_n)$ denotes the complex spinor representation of $\mathrm{Spin}(n)$, which is obtained by restricting ρ to $\mathrm{Spin}(n) \subset \mathbb{C}l_n$, then the *spinor bundle* $\Sigma_g M$ of M with respect to g is defined as the associated vector bundle $\mathrm{Spin}(M,g) \times_{\rho'} \Sigma_n$.
The representation ρ induces a *Clifford multiplication* on $\Sigma_g M$, that is, a complex linear vector bundle homomorphism

$$m : T^*M \otimes \Sigma_g M \to \Sigma_g M, \quad X^\flat \otimes \varphi \mapsto X \cdot \varphi$$

such that $X \cdot (Y \cdot \varphi) + Y \cdot (X \cdot \varphi) = -2g(X,Y)\varphi$ for all $X, Y \in TM$ and $\varphi \in \Sigma_g M$. Moreover, the inner product on Σ_n gives rise to an Hermitian structure on the bundle $\Sigma_g M$ such that $\langle X \cdot \varphi, \psi\rangle = -\langle\varphi, X \cdot \psi\rangle$ for all $X \in TM$ and $\varphi, \psi \in \Sigma_g M$. Finally, the Levi-Civita connection on TM induces a connection on $\mathrm{SO}(M,g)$ and this connection lifts in a canonical way to a connection on $\mathrm{Spin}(M,g)$. The associated covariant derivative $\nabla : C^\infty(M, \Sigma_g M) \to C^\infty(M, T^*M \otimes \Sigma_g M)$ on the spinor bundle has the properties

$$X\langle\varphi,\psi\rangle = \langle\nabla_X\varphi,\psi\rangle + \langle\varphi,\nabla_X\psi\rangle \quad \text{and} \quad \nabla_X(Y \cdot \varphi) = (\nabla^{TM}_X Y) \cdot \varphi + Y \cdot (\nabla_X \varphi)$$

for vector fields X, Y and a spinor field φ.
Now the Dirac operator with respect to the metric g is defined by

$$D_g = m \circ \nabla : C^\infty(M, \Sigma_g M) \to C^\infty(M, \Sigma_g M)$$

and is an elliptic, essentially selfadjoint differential operator of first order.

3 The Proof

We assume from now on that M is a closed spin manifold of dimension $n \equiv 3 \mod 4$. We denote by $R(M)$ the space of all Riemannian metrics on M with the C^1-topology and note that it is obviously contractible. Moreover, we define

$$R^{\mathrm{inv}}(M) = \{g \in R(M) : \ker D_g = 0\} \subset R(M)$$

and recall that our aim is to show that this set has infinitely many path components if it is not empty. Accordingly, we assume henceforth that $R^{\mathrm{inv}}(M) \neq \emptyset$ and now we conclude in three steps the announced disconnectedness of this space.

Step 1: The spectral flow

Since our operators D_g, $g \in R(M)$, are essentially selfadjoint, they have real spectra. Moreover, by ellipticity their spectra are discrete and consist entirely of eigenvalues of finite multiplicity. We define for any compact interval $[a, b] \subset \mathbb{R}$ a non-negative integer by

$$m(g, [a, b]) = \sum_{\lambda \in [a,b]} \dim \ker(D_g - \lambda \cdot id).$$

Next we quote the following stability result for the spectra of the operators D_g that can be found in [Ba96, Prop. 7.1].

Theorem 3.1. *Let (M, g) be a closed Riemannian spin manifold with Dirac operator D_g. Let $\varepsilon > 0$ and let $\Lambda > 0$ such that $-\Lambda, \Lambda \notin \sigma(D_g)$. Write*

$$\sigma(D_g) \cap (-\Lambda, \Lambda) = \{\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k\}.$$

Then there exists a neighbourhood of g in the C^1-topology such that for any metric $\tilde{g}$ in this neighbourhood with Dirac operator $D_{\tilde{g}}$ the following holds:

- $\sigma(D_{\tilde{g}}) \cap (-\Lambda, \Lambda) = \{\mu_1 \leq \mu_2 \leq \ldots \leq \mu_k\}$,
- $|\lambda_i - \mu_i| < \varepsilon$, $i = 1, \ldots, k$.

The eigenvalues λ_i and μ_i are repeated according to their multiplicities.

We obtain immediately the following corollary.

Corollary 3.2. *For all $g_0 \in R(M)$ and $\Lambda > 0$ such that $\pm\Lambda \notin \sigma(D_{g_0})$ there exists an open neighbourhood $N(g_0, \Lambda) \subset R(M)$ such that $\pm\Lambda \notin \sigma(D_g)$ and $m(g, [-\Lambda, \Lambda]) = m(g_0, [-\Lambda, \Lambda])$ for all $g \in N(g_0, \Lambda)$.*

Let now $\gamma : I \to R(M)$ be a path of metrics. Because of corollary 3.2 we can find a decomposition $0 = t_0 < t_1 < \ldots < t_N = 1$ and positive numbers $a_1, \ldots, a_N$ such that the functions $[t_{i-1}, t_i] \ni t \mapsto m(\gamma(t), [-a_i, a_i])$ are constant. We define

$$\Gamma(\gamma) = \sum_{i=1}^{N} m(\gamma(t_i), [0, a_i]) - m(\gamma(t_{i-1}), [0, a_i]) \in \mathbb{Z} \tag{1}$$

and note that, roughly speaking, $\Gamma(\gamma)$ counts the number of negative eigenvalues of $D_{\gamma(0)}$ that become positive as the parameter t travels from 0 to 1 minus the number of positive eigenvalues of $D_{\gamma(0)}$ that become negative; i.e., the net number of eigenvalues which cross zero. The formula (1) corresponds precisely to the definition of the spectral flow for paths of selfadjoint Fredholm operators acting on a fixed Hilbert space which can be found for example in [Phi96] and [BLP05]. Accordingly, one can show verbatim as in [Phi96] that $\Gamma(\gamma)$ indeed does only depend on the path γ and not on the choices of the t_i, a_i, $i = 1, \ldots N$. Moreover, if $\gamma, \tilde{\gamma} : I \to R(M)$ are two paths of metrics, then the following properties hold:

i) $\Gamma(\gamma) = 0$ if $\gamma(t) \in R^{\text{inv}}(M)$ for all $t \in [0, 1]$,

ii) $\Gamma(\gamma * \tilde{\gamma}) = \Gamma(\gamma) + \Gamma(\tilde{\gamma})$, whenever the concatenation $\gamma * \tilde{\gamma}$ exists,

iii) $\Gamma(\gamma^{-1}) = -\Gamma(\gamma)$, where $\gamma^{-1}(t) = \gamma(1 - t)$, $t \in I$,

iv) $\Gamma(\gamma) = \Gamma(\tilde{\gamma})$ if $\gamma \simeq \tilde{\gamma}$ through a homotopy having ends in $R^{\text{inv}}(M)$.

Note that the first three properties are immediate consequences of the definition. The homotopy invariance can be obtained again verbatim as in [Phi96].

Step 2: The range of Γ

Our argument in this section is based on results from [Ba96] which we introduce before we proceed with the proof. At first, we need the existence of the following metrics on the sphere S^n, that were constructed in [Ba96, §3].

Proposition 3.3. *For $n \equiv 3 \mod 4$ and any integer $m > 0$, there exists a path of metrics g_t^m, $t \in [0, 1]$, on S^n such that the following holds for the associated Dirac operators $\not{D}_t^m$:*

- *there is $\lambda(t) \in \sigma(\not{D}_t^m)$ such that $\lambda(0) = -1$ and $\lambda(1) = 1$,*
- *$\lambda(t)$ depends linearly on t,*
- *the multiplicity of $\lambda(t)$ is constant in t and greater than m,*
- *$\lambda(t)$ is the only eigenvalue of $\not{D}_t^m$ in the interval $[-2, 2]$.*

Bär combined in [Ba96] proposition 3.3 and a general gluing theorem for Dirac operators [Ba96, theorem B] to conclude the existence of non-trivial harmonic spinors in dimensions $n \equiv 3 \mod 4$. Actually, in order to find the spinors he just needed a special case of his gluing theorem which reads as follows.

Theorem 3.4. *Let (M,g) be a closed Riemannian spin manifold of odd dimension $n \geq 3$. Let D_g be the corresponding Dirac operator and let $\not{D}$ denote the Dirac operator on S^n with respect to some Riemannian metric. Finally, let $\Lambda > 0$ be such that $\pm\Lambda \notin \sigma(D_g) \cup \sigma(\not{D})$. Write*

$$(\sigma(D_g) \cup \sigma(\not{D})) \cap (-\Lambda, \Lambda) = \{\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k\}.$$

Then for any $\varepsilon > 0$ there exists a Riemannian metric $\tilde{g}$ on M such that the corresponding Dirac operator $D_{\tilde{g}}$ has the following properties:

i) $\pm\Lambda \notin \sigma(D_{\tilde{g}})$,

ii) $\sigma(D_{\tilde{g}}) \cap (-\Lambda, \Lambda) = \{\mu_1 \leq \mu_2 \leq \ldots \leq \mu_k\}$

iii) $|\lambda_j - \mu_j| < \varepsilon$ *for* $j = 1, \ldots, k$.

The eigenvalues λ_i and μ_i are repeated according to their multiplicities.

We now take some metric $g_0 \in R^{\mathrm{inv}}(M)$. Because of the conformal covariance of the Dirac operator (cf. [Hij01, Prop. 5.13]), we can assume that $[-2,2] \cap \sigma(D_{g_0}) = \emptyset$ simply by rescaling the metric if necessary.
Let $m > 0$ be an integer and consider the operators $\not{D}_t^m$ on S^n from proposition 3.3. Recall that we denote by $\lambda(t)$ the unique eigenvalue of $\not{D}_t^m$ in the interval $[-2,2]$ and that $\lambda(t)$ depends linearly on t with $\lambda(0) = -1$, $\lambda(1) = 1$.
We now apply theorem 3.4 for $\Lambda = 2$ and $\varepsilon = \frac{1}{2}$ to D_{g_0} and the operators $\not{D}_t^m$, $t \in [0,1]$, on S^n. We obtain for any $t \in [0,1]$ a metric $\tilde{g}_t$ on M such that each eigenvalue of $D_{\tilde{g}_t}$ in the interval $[-2,2]$ is of distance less then $\frac{1}{2}$ to $\lambda(t)$. In particular, $D_{\tilde{g}_0}$ and $D_{\tilde{g}_1}$ are invertible and hence $\{\tilde{g}_t\}_{t\in[0,1]}$ defines a path $\gamma : (I, \partial I) \to (R(M), R^{\mathrm{inv}}(M))$. Moreover, the function $t \mapsto m(\gamma(t), [-2,2])$ is constant on the whole interval $[0,1]$. Hence we finally obtain from the definition of Γ

$$\Gamma(\gamma) = m(\tilde{g}_1, [0,2]) - m(\tilde{g}_0, [0,2]) = m(\tilde{g}_1, [0,2]) = \dim\ker(\not{D}_1^m - id) > m.$$

To sum up, we have shown that the set

$$\{\Gamma(\gamma) : \ \gamma : (I, \partial I) \to (R(M), R^{\mathrm{inv}}(M)) \,\text{continuous}\} \subset \mathbb{Z}$$

is not bounded from above.

Step 3: The final argument

We fix some $g_0 \in R^{\mathrm{inv}}(M)$. Our first aim of this final step is to construct inductively a sequence of paths $\gamma_k : (I, \partial I) \to (R(M), R^{\mathrm{inv}}(M))$, $k \in \mathbb{N}$, such that $\gamma_k(0) = g_0$ for all $k \in \mathbb{N}$ and $\Gamma(\gamma_i) \neq \Gamma(\gamma_j)$ for all $i \neq j$.
Let γ_1 be the constant path $\gamma_1 \equiv g_0 \in R^{\mathrm{inv}}(M)$. Assume that we have already constructed $\gamma_1, \ldots, \gamma_k : (I, \partial I) \to (R(M), R^{\mathrm{inv}}(M))$ such that $\gamma_i(0) = g_0$, $i = 1, \ldots k$, and $\Gamma(\gamma_i) \neq \Gamma(\gamma_j)$ for all $i \neq j$.
According to the second step of our proof we can find a path $\tilde{\gamma} : (I, \partial I) \to (R(M), R^{\mathrm{inv}}(M))$ such that

$$\Gamma(\tilde{\gamma}) > \max_{1\le i,j\le k} |\Gamma(\gamma_i) - \Gamma(\gamma_j)|. \tag{2}$$

Moreover, we choose a path $\hat{\gamma} : (I, \partial I) \to (R(M), R^{\text{inv}}(M))$ such that $\hat{\gamma}(0) = g_0$ and $\hat{\gamma}(1) = \tilde{\gamma}(0)$. Then $\hat{\gamma} * \tilde{\gamma} : (I, \partial I) \to (R(M), R^{\text{inv}}(M))$ and we set $\gamma_{k+1} = \hat{\gamma} * \tilde{\gamma}$ if $\Gamma(\hat{\gamma} * \tilde{\gamma}) \neq \Gamma(\gamma_j)$ for all $j = 1, \ldots, k$.
If, on the other hand, $\Gamma(\hat{\gamma} * \tilde{\gamma}) = \Gamma(\gamma_j)$ for some $j = 1, \ldots, k$, then we set $\gamma_{k+1} = \hat{\gamma}$. In order to justify this choice, assume that also $\Gamma(\hat{\gamma}) = \Gamma(\gamma_i)$ for some $1 \le i \le k$. Then we obtain

$$\Gamma(\gamma_j) = \Gamma(\hat{\gamma} * \tilde{\gamma}) = \Gamma(\hat{\gamma}) + \Gamma(\tilde{\gamma}) = \Gamma(\gamma_i) + \Gamma(\tilde{\gamma}),$$

which contradicts (2). Hence we indeed obtain a sequence $\{\gamma_k\}_{k\in\mathbb{N}}$ with the required properties.
We now finish our proof by claiming that the metrics $\gamma_k(1)$, $k \in \mathbb{N}$, all lie in different path components of $R^{\text{inv}}(M)$. Assume on the contrary that we can find $i, j \in \mathbb{N}$, $i \neq j$, and a path $\tilde{\gamma} : I \to R^{\text{inv}}(M)$ such that $\tilde{\gamma}(0) = \gamma_i(1)$ and $\tilde{\gamma}(1) = \gamma_j(1)$. Then $\gamma_i * \tilde{\gamma} * \gamma_j^{-1}$ is a closed path with initial point $g_0 \in R^{\text{inv}}(M)$. Since $R(M)$ is contractible, $\gamma_i * \tilde{\gamma} * \gamma_j^{-1}$ is homotopic to the constant path $\gamma_1 \equiv g_0$ through a g_0-preserving homotopy. We obtain from the properties of Γ

$$0 = \Gamma(\gamma_1) = \Gamma(\gamma_i * \tilde{\gamma} * \gamma_j^{-1}) = \Gamma(\gamma_i) + \Gamma(\tilde{\gamma}) + \Gamma(\gamma_j^{-1}) = \Gamma(\gamma_i) + \Gamma(\gamma_j^{-1})$$

and hence $\Gamma(\gamma_i) = \Gamma(\gamma_j)$ contradicting the construction of the sequence $\{\gamma_k\}_{k\in\mathbb{N}}$.

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