ON LANDAU-GINZBURG MODELS FOR QUADRICS AND FLAT SECTIONS OF DUBROVIN CONNECTIONS

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Abstract. This paper proves a version of mirror symmetry expressing the (small) Dubrovin connection for even-dimensional quadrics in terms of a mirror-dual Landau-Ginzburg model $(\check{X}_{\mathrm{can}}, \mathcal{W}_q)$. Here $\check{X}_{\mathrm{can}}$ is the complement of an anticanonical divisor in a Langlands dual quadric. The superpotential $\mathcal{W}_q$ is a regular function on $\check{X}_{\mathrm{can}}$ and is written in terms of coordinates which are naturally identified with a cohomology basis of the original quadric. This superpotential is shown to extend the earlier Landau-Ginzburg model of Givental, and to be isomorphic to the Lie-theoretic mirror introduced in [Rie08]. We also introduce a Laurent polynomial superpotential which is the restriction of $\mathcal{W}_q$ to a particular torus in $\check{X}_{\mathrm{can}}$. Together with results from [PR13] for odd quadrics, we obtain a combinatorial model for the Laurent polynomial superpotential in terms of a quiver, in the vein of those introduced in the 1990's by Givental for type A full flag varieties. These Laurent polynomial superpotentials form a single series, despite the fact that our mirrors of even quadrics are defined on dual quadrics, while the mirror to an odd quadric is naturally defined on a projective space. Finally, we express flat sections of the (dual) Dubrovin connection in a natural way in terms of oscillating integrals associated to $(\check{X}_{\mathrm{can}}, \mathcal{W}_q)$ and compute explicitly a particular flat section.

Contents

1. Introduction

Suppose X is a smooth projective complex Fano variety of dimension N. Starting from X as the 'A-model', Dubrovin constructed a flat connection on a trivial bundle with fiber $H^*(X,\mathbb{C})$, using Gromov-Witten invariants of X, see Section 5. The 'B-models' of Fano varieties were first introduced in [Wit97] and [Giv95]. In our setting X will always have Picard rank 1. In this case the base of the trivial bundle on the A-side can be taken to be the two-dimensional complex torus $\mathbb{C}^*_q \times \mathbb{C}^*_\hbar$ with coordinates q and $\hbar$. The Dubrovin connection is flat and therefore defines a D-module M_A, where $D = \mathbb{C}[\hbar^{\pm 1}, q^{\pm 1}]\langle \partial_\hbar, \partial_q \rangle$.

In [Giv96], Givental computed the 'small J-function' and the 'quantum differential equation' of projective hypersurfaces, such as quadrics (see Section 6). He also proved the first mirror theorem in this setting, which states that the coefficients of the J-function (and hence the solutions to the quantum differential equation) can be expressed as oscillating integrals. When the cohomology of the hypersurface is generated in degree 2, e.g. for odd-dimensional quadrics, then the coefficients of the J-function generate the A-model D-module M_A. For even-dimensional quadrics this is no longer the case.

In this paper, we exploit the fact that quadrics are homogeneous spaces for the special orthogonal group and thus also have mirror LG models defined using Lie theory [Rie08]. We express these Lie theoretic mirrors in certain canonical coordinates and show how to reconstruct in a natural way the entire D-module M_A on the mirror side from a Gauss-Manin system M_B. In particular, we obtain formulas for flat sections of the Dubrovin connection where the coefficients are oscillating integrals. We also investigate the comparison between various choices of mirrors for quadrics including particularly Givental's mirror and our canonical LG model.

We begin describing our results by giving an overview of various LG models for quadrics, including the new ones introduced in this paper. We are then able to state our comparison results followed by our versions of the mirror theorem and some applications.

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1.1. **Overview of LG models for quadrics.**

Givental's mirror. Givental's mirror to the quadric $X = Q_N$ is defined by a smooth affine variety (the Givental mirror manifold)

$$\check{X}_{q,\mathrm{Giv}} = \left\{ (\nu_1, \dots, \nu_{N+2}) \in (\mathbb{C}^*)^{N+2} \;\middle|\; \prod_{i=1}^{N+2} \nu_i = q, \quad \nu_{N+1} + \nu_{N+2} = 1 \right\} \tag{1}$$

with superpotential

$$\mathcal{W}_{q,\mathrm{Giv}}(\nu_1, \dots, \nu_{N+2}) = \nu_1 + \ldots + \nu_N, \tag{2}$$

and volume form

$$\omega_{q,Giv} = \frac{\bigwedge_{i=1}^{N+2} d\log \nu_i}{d(\nu_{N+1} + \nu_{N+2}) \wedge d\log(\prod_{i=1}^{N+2} \nu_i)}. \tag{3}$$

Note that $\check{X}_{q,\mathrm{Giv}}$ is a hypersurface in an $(N+1)$-dimensional torus and $\omega_{q,Giv}$ is the residue of the standard holomorphic volume form on the torus (compare e.g. [Pha11]). Givental's mirror theorem expresses the coefficients of the J-function of Q_N as oscillating integrals involving $\mathcal{W}_{q,\mathrm{Giv}}$ and $\omega_{q,Giv}$ over some middle-dimensional cycles in $\check{X}_{q,\mathrm{Giv}}$.

A Laurent polynomial mirror. A Laurent polynomial LG model $(\check{X}_{\mathrm{Prz}}, \mathcal{W}_{q,\mathrm{Prz}})$ for the N-dimensional quadric $X = Q_N$,

$$\check{X}_{\mathrm{Prz}} = (\mathbb{C}^*)^N, \quad \mathcal{W}_{q,\mathrm{Prz}} = z_1 + z_2 + \ldots + z_{N-1} + \frac{(z_N+q)^2}{z_1 z_2 \cdots z_N}, \tag{4}$$

can be obtained from Givental's mirror by a change of variables which is essentially the one found in [Prz13, Remark 19], see also [GS13]. We recall the change of variables in Sections 2.2 and 3.7. This LG model is a partial compactification of Givental's mirror. The torus-invariant volume form on $\check{X}_{\mathrm{Prz}}$ restricts to Givental's volume form (3).

A Lie-theoretic mirror. The smooth quadric Q_N inside $\mathbb{P}^{N+1}$ is naturally a homogeneous space for the group $\mathrm{Spin}_{N+2}(\mathbb{C})$ associated to the defining quadratic form. The mirror construction from [Rie08] applies in this setting. It gives a regular function $\mathcal{W}_{q,\mathrm{Lie}}$ on an N-dimensional affine subvariety $\check{X}_{\mathrm{Lie}}$ inside the full flag variety for the Langlands dual group, namely the full flag variety for $\mathrm{PSp}_{N+1}(\mathbb{C})$ if N is odd, and for $\mathrm{PSO}_{N+2}(\mathbb{C})$ otherwise. The precise definition of $(\check{X}_{\mathrm{Lie}}, \mathcal{W}_{q,\mathrm{Lie}})$ is recalled in Section 3.4.

The affine variety $\check{X}_{\mathrm{Lie}}$ also has a holomorphic volume form ω_{can}, which is explicitly described in [Rie08]. Indeed $\check{X}_{\mathrm{Lie}}$ is an affine Richardson variety and it is also log Calabi-Yau as seen by combining [KLS14, Appendix A] and [KS14, Section 4.2].

By the main result of [Rie08] there is an isomorphism between the Jacobi ring of $\mathcal{W}_{q,\mathrm{Lie}}$ and the quantum cohomology ring of Q_N (with the quantum parameter inverted). This is not true for the mirrors $(\check{X}_{q,\mathrm{Giv}}, \mathcal{W}_{q,\mathrm{Giv}})$ and $(\check{X}_{\mathrm{Prz}}, \mathcal{W}_{q,\mathrm{Prz}})$.

A canonical mirror. The canonical mirror of an odd-dimensional quadric Q_{2m-1} was introduced in [PR13], and is defined on the complement $\check{X}_{\mathrm{can}}$ of an anticanonical divisor in the projective space $\check{\mathbb{X}} = \mathbb{P}(H^*(Q_{2m-1}, \mathbb{C})^*)$. Suppose $p_0, \ldots, p_{2m-1}$ are the homogeneous coordinates on $\check{\mathbb{X}}$ corresponding to the Schubert basis of $H^*(Q_{2m-1}, \mathbb{C})$. Then $\mathcal{W}_q : \check{X}_{\mathrm{can}} \to \mathbb{C}$ is given by

$$\mathcal{W}_q = \frac{p_1}{p_0} + \sum_{\ell=1}^{m-1} \frac{p_{\ell+1} p_{2m-1-\ell}}{\delta_\ell} + q \frac{p_1}{p_{2m-1}},$$

where

$$\delta_\ell = \sum_{k=0}^{\ell} (-1)^k p_{\ell-k} p_{N-\ell+k} \text{ for } 1 \le \ell \le m-1 \tag{5}$$

with $N = 2m-1$.

The canonical mirror of an even-dimensional quadric Q_{2m-2} introduced here is similar in appearance, however the mirror projective space is replaced by a 'mirror quadric' $\check{\mathbb{X}} = \check{Q}_{2m-2}$. Note first that $\mathbb{P}(H^*(Q_{2m-2}, \mathbb{C})^*)$ has dimension $2m-1$ and homogeneous coordinates $p_0, \ldots, p_{m-1}, p'_{m-1}, \ldots p_{2m-2}$ corresponding to the Schubert basis of $H^*(Q_{2m-2}, \mathbb{C})$. The mirror quadric $\check{Q}_{2m-2}$ is the quadratic hypersurface inside $\mathbb{P}(H^*(Q_{2m-2}, \mathbb{C})^*)$ defined by

$$p_{m-1} p'_{m-1} - p_m p_{m-2} + \cdots + (-1)^{m-1} p_{2m-2} p_0 = 0.$$

The superpotential $\mathcal{W}_q$ is defined by the formula

$$\mathcal{W}_q = \frac{p_1}{p_0} + \sum_{\ell=1}^{m-3} \frac{p_{\ell+1}p_{2m-2-\ell}}{\delta_\ell} + \frac{p_m}{p_{m-1}} + \frac{p_m}{p'_{m-1}} + q\frac{p_1}{p_{2m-2}},$$

which is regular on the the complement $\check{X}_{\mathrm{can}}$ of an anticanonical divisor in $\check{Q}_{2m-2}$. Here δ_ℓ is defined by the formula in equation (5), with $N = 2m-2$.

Laurent polynomial mirrors with a quiver description. For $X = Q_{2m-1}$ the Laurent polynomial mirror

$$\mathcal{W}_{q,\mathrm{Lus}} = a_1 + \cdots + a_{m-1} + c + b_{m-1} + \cdots + b_1 + q\frac{a_1 + b_1}{a_1 \ldots a_{m-1} c b_{m-1} \ldots b_1}. \tag{6}$$

was introduced in [PR13, Proposition 8]. It was obtained by restricting $\mathcal{W}_{q,\mathrm{Lie}}$ to a natural choice of torus $\check{X}_{\mathrm{Lus}}$ in $\check{X}_{\mathrm{Lie}}$, on which we consider coordinates like the ones used by Lusztig in [Lus94].

For the even quadric $X = Q_{2m-2}$ we define here an analogous Laurent polynomial mirror

$$\mathcal{W}_{q,\mathrm{Lus}} = a_1 + \cdots + a_{m-2} + c + d + b_{m-2} + \cdots + b_1 + q\frac{a_1 + b_1}{a_1 \ldots a_{m-2} c d b_{m-2} \ldots b_1}, \tag{7}$$

also obtained from a torus $\check{X}_{\mathrm{Lus}}$ in $\check{X}_{\mathrm{Lie}}$. Note that $(\check{X}_{\mathrm{Lus}}, \mathcal{W}_{q,\mathrm{Lus}})$ is not isomorphic to the other Laurent polynomial mirror $(\check{X}_{\mathrm{Prz}}, \mathcal{W}_{q,\mathrm{Prz}})$.

In Section 4, we interpret $(\check{X}_{\mathrm{Lus}}, \mathcal{W}_{q,\mathrm{Lus}})$ in terms of a quiver, in the spirit of [Giv97, BCFKvS98, BCFKvS00]. The quiver we associate to Q_N looks like an augmentation of a type D_N quiver (see Figure 3). Note that the mirrors for type A homogeneous spaces from [Giv97, EHX97, BCFKvS98, BCFKvS00] also relate to Lusztig coordinates, see [Rie06, Rie08].

1.2. Comparison of the canonical LG model with the other mirrors.

Isomorphism with the Lie-theoretic mirror. It was proved in [PR13] that for the odd-dimensional quadric Q_{2m-1} viewed as a homogeneous space for Spin_{2m+1}, there is an isomorphism between the domain $\check{X}_{\mathrm{Lie}}$ of $\mathcal{W}_{q,\mathrm{Lie}}$ and the domain $\check{X}_{\mathrm{can}}$ of the canonical mirror. This isomorphism identifies the superpotentials $\mathcal{W}_{q,\mathrm{Lie}}$ and $\mathcal{W}_q$.

Theorem 1.1 ([PR13, Theorem 1])**.** *If $X = Q_{2m-1}$ is an odd-dimensional quadric, there is an isomorphism of affine varieties $\check{X}_{\mathrm{Lie}} \to \check{X}_{\mathrm{can}}$ such that the following diagram commutes*

$$\begin{array}{ccc} \check{X}_{\mathrm{Lie}} & \xrightarrow{\sim} & \check{X}_{\mathrm{can}} \\ & {\scriptstyle \mathcal{W}_{q,\mathrm{Lie}}} \searrow \quad \swarrow {\scriptstyle \mathcal{W}_q} & \\ & \mathbb{C} & \end{array}$$

A key ingredient in the construction of the isomorphism is the geometric Satake correspondence of [Lus83, Gin95, MV07], which identifies the projective space $\check{\mathbb{X}} = \mathbb{P}(H^*(Q_{2m-1}, \mathbb{C})^*)$ containing $\check{X}_{\mathrm{can}}$ as the projectivisation of a representation of $\mathrm{PSp}_{2m}(\mathbb{C})$.

In this paper, we prove the same result in the case of even-dimensional quadrics Q_{2m-2} (see Theorem 3.2).

Comparison with the Givental mirror. In Sections 2.2 and 3.7, we relate $(\check{X}_{\mathrm{can}}, \mathcal{W}_q)$ to the Givental mirror $(\check{X}_{q,\mathrm{Giv}}, \mathcal{W}_{q,\mathrm{Giv}})$. In particular, we prove the following proposition.

Proposition 1.2. *There is an embedding, $\check{X}_{q,\mathrm{Giv}} \hookrightarrow \check{X}_{\mathrm{can}}$, of the Givental mirror manifold into the canonical mirror such that the volume form ω_{can} on $\check{X}_{\mathrm{can}}$ (suitably normalized) pulls back to $\omega_{q,Giv}$, and the superpotential $\mathcal{W}_q$ pulls back to $\mathcal{W}_{q,\mathrm{Giv}}$.*

An advantage of the mirror $\mathcal{W}_q$ over its predecessor $\mathcal{W}_{q,\mathrm{Giv}}$ is that the former has the expected number of critical points (at fixed generic value of q), namely $\dim(H^*(Q_N, \mathbb{C}))$.

Proposition 1.3. *The superpotential $\mathcal{W}_q : \check{X}_{\mathrm{can}} \to \mathbb{C}$ for the mirror of Q_N has $\dim H^*(Q_N, \mathbb{C})$ many non-degenerate critical points. Precisely two of these in the even N case, and one of these in the odd N case, are not contained in the image of the embedding, $\check{X}_{q,\mathrm{Giv}} \hookrightarrow \check{X}_{\mathrm{can}}$, of the Givental mirror manifold.*

In the special case of Q_4 this lack of critical points of the classical mirror was already observed in [EHX97]. It was suggested there to solve it using a partial compactification and this was carried out for the first time, albeit in an ad hoc fashion. This was also a motivation for introducing the Lie-theoretic mirrors $(\check{X}_{\mathrm{Lie}}, \mathcal{W}_{q,\mathrm{Lie}})$ in [Rie08]. In the odd quadrics case Proposition 1.2 is proved using a combination of results from [GS13] and [PR13]. In the even quadrics case we prove it in the present paper.

The first part of Proposition 1.3 is an immediate consequence of analogous result for $(\check{X}_{\mathrm{Lie}}, \mathcal{W}_{q,\mathrm{Lie}})$ from [Rie08] together with Theorem 2.1 and Theorem 3.2, respectively. The second part comes from a direct calculation, see Propositions 2.3 and 3.13.

Comparison with $(\check{X}_{\mathrm{Prz}}, \mathcal{W}_{q,\mathrm{Prz}})$. For odd quadrics Q_{2m-1} it was proved in [PR13] that after a change of variables, $\check{X}_{\mathrm{Prz}}$ gets identified with a particular torus inside $\check{X}_{\mathrm{can}}$. This embedding identifies the two superpotentials $\mathcal{W}_{q,\mathrm{Prz}}$ and $\mathcal{W}_q$. We recall this result in Section 2.2.

For even quadrics Q_{2m-2}, the situation is more complicated. We consider the complement of a particular hyperplane section in $\check{X}_{\mathrm{Prz}}$ for which we construct an embedding into $\check{X}_{\mathrm{can}}$ such that $\mathcal{W}_q$ pulls back to $\mathcal{W}_{q,\mathrm{Prz}}$ and show that this embedding cannot be extended. Moreover we observe that the image of the embedding is precisely the embedded Givental mirror manifold inside $\check{X}_{\mathrm{can}}$. Therefore the Givental mirror manifold is in a sense the intersection of the mirrors $\check{X}_{\mathrm{Prz}}$ and $\check{X}_{\mathrm{can}}$. These results are contained in Section 3.7.

Comparison with the quiver mirror. The quiver mirror $(\check{X}_{\mathrm{Lus}}, \mathcal{W}_{q,\mathrm{Lus}})$ is obtained from the Lie-theoretic mirror $(\check{X}_{\mathrm{Lie}}, \mathcal{W}_{q,\mathrm{Lie}})$, and hence from the canonical mirror $(\check{X}_{\mathrm{can}}, \mathcal{W}_q)$, by restricting it to a torus (see Propositions 2.2 and 3.11).

1.3. The mirror theorem for A-model and B-model D-modules.

. Recall that the Dubrovin connection for Q_N gives rise to a module M_A over the ring of differential operators $D = \mathbb{C}[\hbar^{\pm 1}, q^{\pm 1}]\langle \partial_\hbar, \partial_q \rangle$, see (48). On the B-side we obtain a D-module M_B by considering a Gauss-Manin system associated to the mirror $(\check{X}_{\mathrm{can}}, \mathcal{W}_q)$, see Definition 5.1. For odd-dimensional quadrics it is already known that there is an isomorphism between M_A and M_B. This follows from

[GS13, Section 4 & Appendix A] together with the comparison result in [PR13]. The isomorphism takes a particularly natural form in the canonical coordinates, as recalled in Theorem 5.2.

For even dimensional quadrics we construct in Section 5 an explicit isomorphism from the A-model D-module M_A to a natural submodule of the B-model D-module M_B, see Theorem 5.3. We conjecture that this submodule is in fact all of M_B, so that M_A and M_B are isomorphic. Here our canonical mirror $(\check{X}_{\mathrm{can}}, \mathcal{W}_q)$ takes place on a dual quadric. We note that there is a non-trivial cluster algebra structure on the coordinate ring of $\check{X}_{\mathrm{can}}$, which plays an important role in our proof of the isomorphism.

1.4. **Applications.**

. In Section 6, we turn to the problem of constructing flat sections $S : \mathbb{C}^*_\hbar \times \mathbb{C}^*_q \to H^*(X, \mathbb{C})$ for a dual version of the Dubrovin connection of the quadric Q_N, using the B-model. Namely we are interested in solutions to the partial differential equation

$$\begin{array}{rcl} q\frac{\partial S}{\partial q} & = & \frac{1}{\hbar}\sigma_1 \star_q S, \\ \hbar\frac{\partial S}{\partial \hbar} & = & -\frac{1}{\hbar}c_1(TX) \star_q S - \mathrm{Gr}(S). \end{array} \tag{8}$$

First we observe that one can write coefficients of flat sections from the B-model as oscillating integrals using $(\check{X}_{\mathrm{can}}, \mathcal{W}_q)$. This goes as follows. Consider any critical point p of $\mathcal{W}_q$. By a procedure outlined by Givental in the setting of full flag varieties in [Giv97, Section 2], there should be an associated non-compact, middle-dimensional cycle Γ_p in $\check{X}_{\mathrm{can}}$ for which $\Re(\frac{1}{\hbar}\mathcal{W}_q) \to -\infty$ rapidly in any unbounded direction of Γ_p (here we suppress the dependence on $\hbar$ and q in the notation for simplicity). Then, as in [MR13, Section 4.2], the integrals $\int_{\Gamma_p} e^{\frac{1}{\hbar}\mathcal{W}_q} p_i \omega$ locally determine coefficients of a section S_{Γ_p}. This section is given by the formula

$$S_{\Gamma_p} = \frac{1}{(2\pi i)^N} \sum_{i=0}^{N} \left(\int_{\Gamma_p} e^{\frac{1}{\hbar}\mathcal{W}_q} p_i \omega_{can} \right) \sigma_{N-i}$$

in the odd quadric case, and by a similar formula in the even quadric case. The local section S_{Γ_p} is a solution to (8) as a consequence of Theorems 5.2 and 5.3. With an appropriate partial compactification these cycles should have an interpretation in terms of Lefschetz thimbles, compare [Sei08].

If we replace the cycle Γ_p with a compact torus $(S^1)^N$ we obtain a global holomorphic flat section of the dual Dubrovin connection whose coefficients are given by residue integrals. In Section 6 we construct this solution explicitly by expanding it as a power series, using the quiver mirror $(\check{X}_{\mathrm{Lus}}, \mathcal{W}_{q,\mathrm{Lus}})$ to express the integrals in coordinates. Moreover we verify the resulting formula in a different way on the A-side.

2. Landau-Ginzburg models for odd quadrics

The quadrics are cominuscule homogeneous spaces (for the Spin groups). Therefore, in addition to the Givental approach [Giv98] for constructing LG models, there is another LG model for each quadric on an affine variety (generally larger than a torus), which was defined by the second-named author using a Lie-theoretic construction [Rie08]. Namely for any projective homogeneous space $X = G/P$ of a simple complex algebraic group, [Rie08] constructed a conjectural LG model, which is a regular function on an affine subvariety of the Langlands dual group. We call it the *Lie-theoretic LG model*. It was shown in [Rie08] that this LG model recovers the Peterson variety presentation [Pet97] of the quantum cohomology of $X = G/P$. It therefore defines an LG model whose Jacobi ring has the correct dimension. In this section we will rewrite the Lie-theoretic LG model in terms of natural projective coordinates on $\mathbb{P}(H^*(Q_N, \mathbb{C})^*)$. We call the resulting LG model the *canonical LG model* of Q_N.

Note that for odd-dimensional quadrics Q_{2m-1} a recent paper [GS13] of Gorbounov and Smirnov constructed directly a partial compactification of the Givental mirrors, without making use of [Rie08].

2.1. **The canonical LG model for Q_{2m-1}.** LG models for odd-dimensional quadrics with the expected number of critical points have been constructed in [Rie08] (where they appear as a special case), [GS13], and finally [PR13]. Here we recall the main results from the paper [PR13], which contains the formulation for the LG model which we will adopt.

In this section our A-model variety $X = X_N = X_{2m-1}$ is the quadric $Q_N = Q_{2m-1}$. Recall that an odd-dimensional quadric has one-dimensional cohomology groups in even degrees spanned by Schubert classes $\sigma_i \in H^{2i}(Q_{2m-1}, \mathbb{C})$ for $0 \leq i \leq 2m-1$, and no other cohomology. To construct its canonical mirror first consider the projective space $\check{\mathbb{X}} = \check{\mathbb{X}}_{2m-1} = \mathbb{P}^{2m-1}$ with homogeneous coordinates $(p_0 : p_1 : \cdots : p_{2m-1})$ in one-to-one correspondence with these Schubert classes σ_i. Inside $\check{\mathbb{X}}$ we have the open affine subvariety $\check{X}_{\mathrm{can}} \subset \mathbb{P}^{2m-1}$ defined by:

$$\check{X}_{\mathrm{can}} = \check{X}_{2m-1} := \check{\mathbb{X}} \setminus D, \tag{9}$$

where $D := D_0 + D_1 + \ldots + D_{m-1} + D_m$, the divisors D_i being given by

$$D_0 := \{p_0 = 0\},$$
$$D_\ell := \left\{ \sum_{k=0}^{\ell} (-1)^k p_{\ell-k} p_{2m-1-\ell+k} = 0 \right\} \text{ for } 1 \leq \ell \leq m-1,$$
$$D_m := \{p_{2m-1} = 0\}.$$

The divisor D is an anticanonical divisor. Indeed, the index of $\check{\mathbb{X}} = \mathbb{P}^{2m-1}$ is $2m$. As a result, there is a unique up to scalar $(2m-1)$-form ω_{can} which is regular on $\check{X}_{\mathrm{can}}$ and has logarithmic poles on D. For all $1 \leq j \leq m-1$, take $r_j \in \{p_j, p_{2m-1-j}\}$. Setting $p_0 = 1$, the restriction of ω_{can} to the torus $\{r_j \neq 0 \mid 1 \leq j \leq m-1\}$ inside $\check{X}_{\mathrm{can}}$ is given by

$$\omega_{can} = \frac{\bigwedge_{1\leq j\leq m-1} \mathrm{d}r_j \wedge \bigwedge_{1\leq \ell\leq m-1} \mathrm{d}\delta_\ell \wedge \mathrm{d}p_{2m-1}}{\delta_1 \ldots \delta_{m-1} p_{2m-1}}. \tag{10}$$

We have:

Theorem 2.1 ([PR13, Theorem 1])**.** *The Lie-theoretic LG model $\mathcal{W}_{q,\mathrm{Lie}} : \check{X}_{\mathrm{Lie}} \to \mathbb{C}$ from* [Rie08] *for $X = Q_{2m-1}$ is isomorphic to the canonical LG model $\mathcal{W}_q : \check{X}_{2m-1} \to \mathbb{C}$ defined by*

$$\mathcal{W}_q = \frac{p_1}{p_0} + \sum_{\ell=1}^{m-1} \frac{p_{\ell+1} p_{2m-1-\ell}}{\delta_\ell} + q \frac{p_1}{p_{2m-1}}, \tag{11}$$

where δ_ℓ is given by (5) *with $N = 2m-1$.*

We also have another expression for the superpotential:

Proposition 2.2 ([PR13, Proposition 8]). *For $X = Q_{2m-1}$ and $\mathcal{W}_q$ as above, there is a torus $\check{X}_{\mathrm{Lus}} := (\mathbb{C}^*)^{2m-1} \hookrightarrow \check{X}_{\mathrm{can}}$ to which $\mathcal{W}_q$ pulls back giving the Laurent polynomial expression*

$$\mathcal{W}_{q,\mathrm{Lus}} = a_1 + \cdots + a_{m-1} + c + b_{m-1} + \cdots + b_1 + q\frac{a_1 + b_1}{a_1 \ldots a_{m-1} c b_{m-1} \ldots b_1}.$$

We call the Laurent polynomial LG model $(\check{X}_{\mathrm{Lus}}, \mathcal{W}_q)$ from Proposition 2.2 the *quiver mirror*. The reason for this denomination will be made clear in Section 4.

2.2. **Comparison with the Givental and Laurent polynomial mirrors for odd quadrics.** Let us recall the Laurent polynomial LG model of Q_{2m-1} from Equation (4)

$$\mathcal{W}_{q,\mathrm{Prz}} = z_1 + \cdots + z_{2m-2} + \frac{(z_{2m-1} + q)^2}{z_1 z_2 \ldots z_{2m-1}},$$

defined over the torus

$$\check{X}_{\mathrm{Prz}} := \{(z_1, \ldots, z_{2m-1}) \mid z_i \neq 0 \quad \forall\, i\},$$

and the Givental LG model from Equation (2)

$$\mathcal{W}_{q,\mathrm{Giv}} = \nu_1 + \cdots + \nu_{2m-1},$$

defined over the affine variety

$$\check{X}_{q,\mathrm{Giv}} = \left\{ (\nu_1, \ldots, \nu_{2m+1}) \mid \nu_i \neq 0 \;\forall\, i,\; \prod_{i=1}^{2m+1} \nu_i = q, \quad \nu_{2m} + \nu_{2m+1} = 1 \right\}.$$

These two LG models are related by a birational change of coordinates analogous to that of [Prz13, Rmk. 19], namely

$$z_i = \begin{cases} \nu_{i+1} & \text{for } 1 \leq i \leq 2m-2; \\ q\frac{\nu_{2m}}{\nu_{2m+1}} & \text{for } i = 2m-1; \end{cases}$$

and conversely

$$\nu_i = \begin{cases} \frac{(z_{2m-1}+q)^2}{z_1 \ldots z_{2m-1}} & \text{for } i = 1; \\ z_{i-1} & \text{for } 2 \leq i \leq 2m-1; \\ \frac{z_{2m-1}}{z_{2m-1}+q} & \text{for } i = 2m; \\ \frac{q}{z_{2m-1}+q} & \text{for } i = 2m+1. \end{cases}$$

This change of variables defines an isomorphism

$$\check{X}_{\mathrm{Prz}} \setminus \{z_{2m-1} + q = 0\} \cong \check{X}_{q,\mathrm{Giv}}$$

which identifies the superpotentials $\mathcal{W}_{q,\mathrm{Prz}}$ and $\mathcal{W}_{q,\mathrm{Giv}}$.

Let us now compare these two LG models with ours. Consider the change of coordinates

$$z_i = \begin{cases} \frac{p_i}{p_{i-1}} & \text{for } 1 \leq i \leq m-1; \\ \frac{p_{2m-1-i}\delta_{2m-3-i}}{p_{2m-2-i}\delta_{2m-2-i}} & \text{for } m \leq i \leq 2m-3; \\ q\frac{p_1}{p_{2m-1}} & \text{for } i = 2m-2; \\ q\frac{\delta_{m-2}}{\delta_{m-1}} & \text{for } i = 2m-1. \end{cases}$$

It is well-defined on the cluster torus $\{p_i \neq 0 \mid \forall 1 \leq i \leq m-1\}$ inside $\check{X}_{\text{can}}$. Moreover, an easy calculation shows that it transforms the canonical LG model (11) into the Laurent polynomial LG model (4) for odd quadrics.

Indeed, using this change of variables we see that $z_1 \dots z_{2m-1}$ maps to

$$\frac{p_{m-1}}{p_0} \cdot \frac{p_{m-1}\delta_0}{\delta_{m-2}p_1} \cdot q\frac{p_1}{p_{2m-1}} \cdot q\frac{\delta_{m-2}}{\delta_{m-1}} = q^2\frac{(p_{m-1})^2}{\delta_{m-1}}.$$

Moreover $(z_{2m-1}+q)^2$ maps to $\left(q\frac{p_{m-1}p_m}{\delta_{m-1}}\right)^2$ since $\delta_{m-1}+\delta_{m-2} = p_{m-1}p_m$. It follows that $\frac{(z_{2m-1}+q)^2}{z_1 \dots z_{2m-1}}$ maps to $\frac{p_m^2}{\delta_{m-1}}$. We also see that for $2 \leq j \leq m-1$, $z_j + z_{2m-1-j}$ maps to $\frac{p_j p_{2m-j}}{\delta_{j-1}}$ since $\delta_{j-1} + \delta_{j-2} = p_{j-1}p_{2m-j}$. Hence via this change of variables, the Laurent polynomial superpotential $\mathcal{W}_{q,\text{Prz}}$ maps to

$$\frac{p_1}{p_0} + \sum_{j=2}^{m-1} \frac{p_j p_{2m-j}}{\delta_{j-1}} + q\frac{p_1}{p_{2m-1}} + \frac{p_m^2}{\delta_{m-1}},$$

which is precisely the expression of $\mathcal{W}_q$.

Note that this change of coordinates between $(\check{X}_{\text{Prz}}, \mathcal{W}_{q,\text{Prz}})$ and $(\check{X}_{\text{can}}, \mathcal{W}_q)$ may also be obtained by combining the isomorphism between (11) and the Gorbounov-Smirnov mirror from [PR13, Section 6], with the comparison between the Gorbounov-Smirnov mirror and the Laurent polynomial mirror (there called the Hori-Vafa mirror) in [GS13].

Combining both changes of coordinates, we obtain an embedding of the Givental mirror variety $\check{X}_{q,\text{Giv}} \hookrightarrow \check{X}_{\text{can}}$, corresponding to the change of coordinates

$$\nu_i = \begin{cases} \frac{p_m^2}{\delta_{m-1}} & \text{for } i = 1; \\ \frac{p_{i-1}}{p_{i-2}} & \text{for } 2 \leq i \leq m; \\ \frac{p_{2m-i}\delta_{2m-2-i}}{p_{2m-1-i}\delta_{2m-1-i}} & \text{for } m+1 \leq i \leq 2m-2; \\ q\frac{p_1}{p_{2m-1}} & \text{for } i = 2m-1; \\ \frac{\delta_{m-2}}{p_{m-1}p_m} & \text{for } i = 2m; \\ \frac{\delta_{m-1}}{p_{m-1}p_m} & \text{for } i = 2m+1. \end{cases}$$

The embedding identifies $\check{X}_{q,\text{Giv}}$ with the intersection of cluster tori $\{p_i \neq 0 \mid \forall 1 \leq i \leq m\}$ in $\check{X}$, the superpotential $\mathcal{W}_{q,\text{Giv}}$ with $\mathcal{W}_q$, and the form $\omega_{q,Giv}$ with ω_{can}. This proves Proposition 1.2 from the introduction in the case of odd quadrics.

2.3. **The critical points of the canonical mirror.** Since the canonical mirror $(\check{X}_{\text{can}}, \mathcal{W}_q)$ is isomorphic to the Lie-theoretic mirror $(\check{X}_{\text{Lie}}, \mathcal{W}_{q,\text{Lie}})$, it follows from [Rie08] that $\mathcal{W}_q$ has the 'correct' number of critical points on $\check{X}_{\text{can}}$, that is, $\dim H^*(Q_{2m-1}, \mathbb{C}) = 2m$. Here we give explicit expression for the critical points, and compare with the critical points of the classical mirrors $(\check{X}_{q,\text{Giv}}, \mathcal{W}_{q,\text{Giv}})$ and $(\check{X}_{\text{Prz}}, \mathcal{W}_{q,\text{Prz}})$.

Proposition 2.3. *The critical points of the superpotential $\mathcal{W}_q$ on $\check{X}_{\text{can}}$ are given by*

$$p_j = \begin{cases} \zeta^j & \text{if } 1 \leq j \leq m-1\ ; \\ \frac{1}{2}\zeta^j & \text{if } m \leq j \leq 2m-2\ ; \\ q & \text{if } j = 2m-1, \end{cases}$$

where ζ is a primitive $(2m-1)$-st root of $4q$. The associated critical value is $(2m-1)\zeta$. Moreover there is an extra critical point given by $p_1 = \cdots = p_{2m-2} = 0$, $p_{2m-1} = -q$ with corresponding critical value 0. This critical point does not belong to $\check{X}_{\mathrm{Prz}}$, $\check{X}_{q,\mathrm{Giv}}$ or $\check{X}_{\mathrm{Lus}}$.

Proof. Setting $p_0 = 1$ we get the following relations at a critical point of $\mathcal{W}_q$:

(12)
$$\frac{\partial \mathcal{W}_q}{\partial p_1} = 1 + \left(\sum_{\ell=1}^{m-1} (-1)^{\ell} \frac{p_{\ell+1}p_{2m-1-\ell}}{\delta_\ell^2}\right) p_{2m-2} + \frac{q}{p_{2m-1}} = 0$$

(13)
$$\frac{\partial \mathcal{W}_q}{\partial p_j} = \frac{p_{2m-j}}{\delta_{j-1}} + \left(\sum_{\ell=j}^{m-1} (-1)^{\ell+1-j} \frac{p_{\ell+1}p_{2m-1-\ell}}{\delta_\ell^2}\right) p_{2m-1-j} = 0 \ (2 \le j \le m-1)$$

(14)
$$\frac{\partial \mathcal{W}_q}{\partial p_m} = \frac{p_m(p_{m-1}p_m - 2\delta_{m-2})}{\delta_{m-1}^2} = 0$$

(15)
$$\frac{\partial \mathcal{W}_q}{\partial p_j} = -\frac{p_{2m-j}\delta_{2m-2-j}}{\delta_{2m-1-j}^2} + \left(\sum_{\ell=2m-j}^{m-1} (-1)^{\ell+j-2m} \frac{p_{\ell+1}p_{2m-1-\ell}}{\delta_\ell^2}\right) p_{2m-1-j} = 0 \ (m+1 \le j \le 2m-2)$$

(16)
$$\frac{\partial \mathcal{W}_q}{\partial p_{2m-1}} = \sum_{\ell=1}^{m-1} (-1)^{\ell-1} \frac{p_{\ell+1}p_{2m-1-\ell}}{\delta_\ell^2} - q\frac{p_1}{p_{2m-1}^2} = 0.$$

From Equation (14) it follows that we have two possibilities, i.e. $p_m = 0$, or $p_{m-1}p_m = 2\delta_{m-2}$. If $p_m = 0$, using (13) for $j = m-1, m-2, \ldots, 2$ shows that $p_m = p_{m+1} = p_{2m-2} = 0$. Then (12) implies $p_{2m-1} = -q$. Using (15) for $j = m+1, m+2, \ldots, 2m-2$ shows that $p_{m-1} = p_{m-2} = \cdots = p_2 = 0$. Finally (16) implies $p_1 = 0$. At the corresponding critical point $(0, \ldots, 0, -q)$, the value of $\mathcal{W}_q$ (the critical value) is clearly 0.

Let us now assume $p_m \neq 0$ and $p_{m-1}p_m = 2\delta_{m-2}$, so that $\delta_{m-1} = \delta_{m-2}$. Combining Equations (13) for $j = m-1$ and (15) for $j = m+1$, we obtain $p_{m-2}p_{m+1} = 2\delta_{m-3}$, hence $\delta_{m-2} = \delta_{m-3}$. Iteratively, we obtain

$$\delta_{m-1} = \delta_{m-2} = \cdots = \delta_0; \tag{17}$$
$$p_j p_{2m-1-j} = 2\delta_{j-1} \quad \forall\, 1 \le j \le m-1. \tag{18}$$

Combining Equations (12) and (16) with the identity (17), we get that

$$p_{2m-1} = q, \tag{19}$$

hence all the δ_j are equal to q.

Now Equations (13) for $j = m-1$ and (17) imply $p_{m-1}p_{m+1} = 2p_m^2$. Then Equation (13) for $j = m-2$ and (17) imply that $p_{m-2}p_{m+2} = 2(p_{m-1}p_{m+1} - p_m^2)$. Inductively for $j = m-3, \ldots, 2$ we obtain

$$\sum_{\ell=j}^{m} (-1)^{\ell-j} p_\ell p_{2m-\ell} = p_m^2 \tag{20}$$

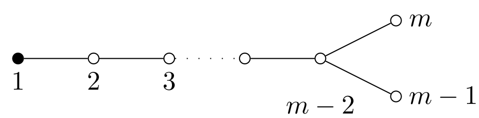
for all $2 \leq j \leq m$. Then (12) implies that $p_m^2 = qp_1$.

Finally, (15) and (20) imply $p_j = \frac{1}{2}p_1^j$ for $m \leq j \leq 2m-2$, while (13) and (20) imply $p_j = p_1^j$ for $1 \leq j \leq m-1$. Then $p_m = \frac{1}{2}p_1^m$ together with $p_m^2 = qp_1$ implies that $p_1^{2m-1} = 4q$, which concludes the proof. □

3. Landau-Ginzburg models for even quadrics

We view the quadric $X = X_{2m-2} := Q_{2m-2}$ of dimension $2m-2$ as a homogeneous space for the Spin group $\mathrm{Spin}_{2m}(\mathbb{C})$. In this section we will introduce a *canonical LG model* for X_{2m-2} which will be defined on an open subvariety of a dual quadric $\check{\mathbb{X}}_{2m-2} = P\backslash \mathrm{PSO}_{2m}(\mathbb{C})$, see Section 3.2. Note that the projective special orthogonal group $\mathrm{PSO}_{2m}(\mathbb{C})$ is the Langlands dual group to $\mathrm{Spin}_{2m}(\mathbb{C})$, and both groups have the same Dynkin diagram, namely the Dynkin diagram of type D_m. The main result of this section, Proposition 3.3, shows that the new LG-model is isomorphic to one defined earlier [Rie08] on a Richardson variety $\check{X}_{\mathrm{Lie}}$ inside the full flag variety of $\mathrm{PSO}_{2m}(\mathbb{C})$.

Note that in the following we will denote the group $\mathrm{PSO}_{2m}(\mathbb{C})$ by G, since this is the group we will primarily be working with. Then the A-model symmetry group is $G^\vee = \mathrm{Spin}_{2m}(\mathbb{C})$, and we have $X_{2m-2} = G^\vee/P^\vee$, where $P^\vee$ is the parabolic subgroup associated to the first node of the Dynkin diagram of type D_m.



3.1. **Notations and definitions.** Let $V = \mathbb{C}^{2m}$ with fixed quadratic form

$$Q = \begin{pmatrix} & & & & 1 \\ & & & -1 & \\ & & \iddots & & \\ & -1 & & & \\ 1 & & & & \end{pmatrix}.$$

In other words $Q(v_i, v_j) = (-1)^{\max(i,j)}\delta_{i+j,2m+1}$ where $\{v_i\}$ is the standard basis of $\mathbb{C}^{2m}$. For $G = \mathrm{PSO}(V,Q) = \mathrm{PSO}(V)$ we fix Chevalley generators $(e_i)_{1\leq i\leq m}$ and $(f_i)_{1\leq i\leq m}$. To be explicit we embed $\mathfrak{so}(V,Q)$ into $\mathfrak{gl}(V)$ and set

$$e_i = \begin{cases} E_{i,i+1} + E_{2m-i,2m-i+1} & \text{if } 1 \leq i \leq m-1, \\ E_{m-1,m+1} + E_{m,m+2} & \text{if } i = m, \end{cases}$$

and $f_i := e_i^T$, the transpose matrix, for every $i = 1, \dots, m$. Here $E_{i,j} = (\delta_{i,k}\delta_{l,j})_{k,l}$ is the standard basis of $\mathfrak{gl}(V)$. For elements of the group $\mathrm{PSO}(V)$, we will take matrices to represent their equivalence classes. We have Borel subgroups $B_+ = TU_+$ and $B_- = TU_-$ consisting of upper-triangular and lower-triangular matrices in $\mathrm{PSO}(V)$, respectively. Here U_+ and U_- are the unipotent radicals of B_+ and B_-, respectively, and T is the maximal torus of $\mathrm{PSO}(V)$, consisting of diagonal matrices (d_{ij}) with non-zero entries $d_{i,i} = d^{-1}_{2m-i+1,2m-i+1}$. We let $X(T) = \mathrm{Hom}(T, \mathbb{C}^*)$, $R \subset X(T)$ the set of roots, and R^+ the positive roots. We denote the set of simple roots by $\Pi = \{\alpha_i \mid 1 \leq i \leq m\} \subset R^+ \subset R \subset X(T)$, and the set of fundamental weights (which is the dual basis in $X(T)$) by $\{\omega_i \mid 1 \leq i \leq m\} \subset X(T) \otimes_{\mathbb{Z}} \mathbb{R}$.

The parabolic subgroup P of $\mathrm{PSO}(V)$ we are interested in is the one whose Lie algebra $\mathfrak{p}$ is generated by all of the e_i together with $f_2, \dots, f_m$, leaving out f_1. Let $x_i(a) := \exp(ae_i)$ and $y_i(a) := \exp(af_i)$. The Weyl group W of $\mathrm{PSO}(V)$ is generated by simple reflections s_i for which we choose representatives

$$\dot{s}_i = y_i(-1)x_i(1)y_i(-1). \tag{21}$$

We let W_P denote the parabolic subgroup of the Weyl group W, namely $W_P = \langle s_2, \dots, s_m\rangle$. The length of a Weyl group element w is denoted by $\ell(w)$. The longest element in W_P is denoted by w_P. We also let w_0 be the longest element in W. Next W^P is defined to be the set of minimal length coset representatives for W/W_P. The minimal length coset representative for w_0 is denoted by w^P.

We introduce the following notation for the elements of W^P. Namely, $W^P = \{e, w_1, \dots, w_{m-1}, w'_{m-1}, w_m, w_{m+1}, \dots w_{2m-2}\}$, where

$$w_k = \begin{cases} s_k s_{k-1} \dots s_1 & \text{if } 1 \le k \le m-2, \\ s_{m-1}s_{m-2}\dots s_1 & \text{if } k = m-1, \\ s_m s_{m-1} s_{m-2} \dots s_1 & \text{if } k = m, \\ s_{2m-1-k} \dots s_{m-2} s_m s_{m-1} s_{m-2} \dots s_1 & \text{if } m+1 \le k \le 2m-2. \end{cases}$$

and $w'_{m-1} = s_m s_{m-2} \dots s_1$.

For any $w \in W$ let $\dot{w}$ denote the representative of w in G obtained by setting $\dot{w} = \dot{s}_{i_1} \cdots \dot{s}_{i_r}$, where $w = s_{i_1} \cdots s_{i_r}$ is a reduced expression and $\dot{s}_i$ is as in (21). Each $\dot{w}_k \in \mathrm{PSO}(V)$ can be represented by a matrix $[w_k] \in \mathrm{SO}(V)$ such that

$$[w_k] \cdot v_{2m} = \begin{cases} v_{2m-k} & 1 \le k < m-1, \\ v_{2m-k-1} & m-1 < k \le 2m-2, \end{cases} \tag{22}$$

and $[w'_{m-1}] \cdot v_{2m} = v_m$ and $[w_{m-1}] \cdot v_{2m} = v_{m+1}$.

3.2. **The dual quadric and its Plücker coordinates.** Consider the homogeneous space $\check{\mathbb{X}}_{2m-2} = P\backslash\mathrm{PSO}(V)$. It is canonically identified with the isotropic Grassmannian of lines in V^*, when this Grassmannian is viewed as a homogeneous space via the action of $\mathrm{PSO}(V)$ from the right. Moreover the isotropic Grassmannian of lines is also a $(2m-2)$-dimensional quadric $\check{\mathbb{X}}_{2m-2} =: \check{Q}_{2m-2}$, now in $\mathbb{P}(V^*)$. So in this case, the varieties X and $\check{\mathbb{X}}$ are (non-canonically) isomorphic. The reason for this isomorphism of varieties is that the group $G^\vee$ is of simply-laced type. However Lie-theoretically we still think of X_{2m-2} and $\check{\mathbb{X}}_{2m-2}$ as being very different homogeneous spaces, with $X_{2m-2} = \mathrm{Spin}_{2m}(\mathbb{C})/P^\vee$ and $\check{\mathbb{X}}_{2m-2} = P\backslash\mathrm{PSO}_{2m}(\mathbb{C})$.

Definition 3.1 (Plücker coordinates)**.** The Plücker coordinates for $\check{\mathbb{X}}_{2m-2} = P\backslash\mathrm{PSO}(V)$ are the homogeneous coordinates coming from the embedding of $\check{\mathbb{X}}_{2m-2}$ into $\mathbb{P}(V^*)$ as the (right) G-orbit of the line $\mathbb{C}v^*_{2m}$:

$$\check{\mathbb{X}}_{2m-2} = P\backslash\mathrm{PSO}(V) \to \mathbb{P}(V^*) : Pg \mapsto (\mathbb{C}v^*_{2m}) \cdot g.$$

We think of the Plücker coordinates as corresponding to the elements of W^P. Let $v^-_{\omega_i}$ (respectively $v^+_{\omega_i}$) denote lowest and highest weight vectors in the highest weight representation V_{ω_i}. Then the Plücker coordinates may be defined by:

$$\begin{aligned} p_0(g) &= \langle v^*_{2m} \cdot [g], v_{2m}\rangle, \\ p_k(g) &= \langle v^*_{2m} \cdot [g], [w_k] \cdot v_{2m}\rangle \text{ for } 1 \le k \le 2m-2, \\ p'_{m-1}(g) &= \langle v^*_{2m} \cdot [g], [w'_{m-1}] \cdot v_{2m}\rangle, \end{aligned}$$

where $[g] \in \mathrm{SO}(V)$ is any fixed matrix representing $g \in \mathrm{PSO}(V)$. The homogeneous coordinates of Pg are then given by

$$(p_0(g) : \ldots : p_{m-2}(g) : p_{m-1}(g) : p'_{m-1}(g) : p_m(g) : \ldots \ : p_{2m-2}(g)).$$

These are simply the bottom row entries of $[g]$ read from right to left, keeping in mind (22).

We may now write down the equation of the quadric $\check{\mathbb{X}}_{2m-2}$ in terms of Plücker coordinates:

$$p_{m-1}p'_{m-1} - p_{m-2}p_m + p_{m-3}p_{m+1} - \cdots + (-1)^{m-1}p_0p_{2m-2} = 0. \tag{23}$$

We note that as in the case of the odd quadric these Plücker coordinates are to be thought of as B-model incarnations of the Schubert classes of Q_{2m-2}. Namely, recall that $H^*(Q_{2m-2}, \mathbb{C})$ has a Schubert basis $\{\sigma_w\}$ indexed by W^P. We will use the notation $\sigma_i = \sigma_{w_i}$, $\sigma'_{m-1} = \sigma_{w'_{m-1}}$, and $\sigma_0 = \sigma_e$, where the w_i are defined in Section 3.1. As a special case of the geometric Satake correspondence [Lus83, Gin95, MV07] we have that the (defining) projective representation V of $PSO_{2m}(V)$ is identified with the cohomology of Q_{2m-2},

$$V = H^*(Q_{2m-2}, \mathbb{C}),$$

and the standard basis v_i agrees with the Schubert basis via $v_{2m} = \sigma_0$ and

$$[w_i] \cdot v_{2m} = \sigma_i, \quad [w'_{m-1}] \cdot v_{2m} = \sigma'_{m-1}. \tag{24}$$

The Schubert classes σ_w are in this way naturally identified with the Plücker coordinates.

3.3. **The superpotential for Q_{2m-2} on a dual quadric.** In this section we state our theorem describing a superpotential for Q_{2m-2} in terms of Plücker coordinates on the dual quadric $\check{\mathbb{X}}_{2m-2} = \check{Q}_{2m-2}$. Consider

$$\check{X}_{\mathrm{can}} = \check{X}_{2m-2} := \check{\mathbb{X}} \setminus D, \tag{25}$$

where $D := D_0 + D_1 + \ldots + D_{m-2} + D_{m-1} + D'_{m-1}$, the D_i being given by

$$\begin{aligned}
D_0 &:= \{p_0 = 0\}, \\
D_\ell &:= \left\{ \sum_{k=0}^{\ell} (-1)^k p_{\ell-k} p_{2m-2-\ell+k} = 0 \right\} \text{ for } 1 \le \ell \le m-3, \\
D_{m-2} &:= \{p_{2m-2} = 0\}, \\
D_{m-1} &:= \{p_{m-1} = 0\}, \\
D'_{m-1} &:= \{p'_{m-1} = 0\}.
\end{aligned}$$

The divisor D is an anticanonical divisor in $\check{\mathbb{X}}$ (see [KLS14, Lemma 5.4]). For simplicity, we will define

$$\delta_\ell = \sum_{k=0}^{\ell} (-1)^k p_{\ell-k} p_{N-\ell+k} \text{ for } 1 \le \ell \le m-3. \tag{26}$$

(For even quadrics, $N = 2m-2$.)

As in the odd case, we have a unique up to scalar $(2m-2)$-form ω_{can} which is regular on $\check{X}_{\mathrm{can}}$ and has logarithmic poles along D. For all $1 \le j \le m-2$, take $r_j \in \{p_j, p_{2m-2-j}\}$. Setting $p_0 = 1$, the restriction of ω_{can} to the torus $\{r_j \neq 0 \mid 1 \leq j \leq m-2\}$ inside $\check{X}_{\mathrm{can}}$ is given by

$$\omega_{can} = \frac{\bigwedge_{1\leq j\leq m-2} r_j \wedge \bigwedge_{1\leq \ell\leq m-3} \delta_\ell \wedge p_{2m-2} \wedge p_{m-1} \wedge p'_{m-1}}{\delta_1 \dots \delta_{m-3} p_{2m-1} p_{m-1} p'_{m-1}}. \tag{27}$$

Our first result is the following theorem.

Theorem 3.2. *The Lie-theoretic LG model* $(\check{X}_{\mathrm{Lie}}, \mathcal{W}_{q,\mathrm{Lie}})$ *for* $Q_{2m-2} = \mathrm{Spin}_{2m}/P^\vee$ *from* [Rie08] *is isomorphic to the canonical LG model* $(\check{X}_{2m-2}, \mathcal{W}_q)$*, where* $\mathcal{W}_q : \check{X}_{2m-2} \to \mathbb{C}$ *is defined by*

$$\mathcal{W}_q = \frac{p_1}{p_0} + \sum_{\ell=1}^{m-3} \frac{p_{\ell+1} p_{2m-2-\ell}}{\delta_\ell} + \frac{p_m}{p_{m-1}} + \frac{p_m}{p'_{m-1}} + q\frac{p_1}{p_{2m-2}}. \tag{28}$$

This isomorphism is defined in Section 3.5. Before we begin the proof we need to recall the definition of the Lie-theoretic LG model from [Rie08].

3.4. **The Lie-theoretic LG model for Q_{2m-2}.** Following [Rie08] consider the (open) Richardson variety $\check{X}_{\mathrm{Lie}} := R_{w_P,w_0} \subset G/B_-$, namely

$$\check{X}_{\mathrm{Lie}} := R_{w_P,w_0} = (B_+\dot{w}_P B_- \cap B_-\dot{w}_0 B_-)/B_-.$$

This Richardson variety $\check{X}_{\mathrm{Lie}}$ is irreducible of dimension $2m-2$, and its closure is the Schubert variety $\overline{B_+\dot{w}_P B_-/B_-}$. Let T^{W_P} be the W_P-fixed part of the maximal torus T. Note that since we are in the setting of Section 3.1 we have that $T^{W_P} \cong \mathbb{C}^*$ with isomorphism given by α_1. The inverse isomorphism is $\omega_1^\vee : \mathbb{C}^* \to T^{W_P}$. We fix a $d \in T^{W_P}$. Then one can define

$$Z_d := B_-\dot{w}_0 \cap U_+ d\dot{w}_P U_- \subset G, \tag{29}$$

and the map

$$\pi_R : Z_d \to \check{X}_{\mathrm{Lie}} : g \mapsto gB_-. \tag{30}$$

is an isomorphism from Z_d to the open Richardson variety [Rie08, Section 4.1].

Let q be the non-vanishing coordinate on the 1-dimensional torus T^{W_P} given by $\alpha_1 : T^{W_P} \to \mathbb{C}^*$. The mirror LG model is a regular function on $\check{X}_{\mathrm{Lie}}$ depending also on q, and hence a regular function on $\check{X}_{\mathrm{Lie}} \times T^{W_P}$. It is defined as follows [Rie08]:

$$\mathcal{F} : (u_1\dot{w}_P B_-, d) \;\mapsto\; g = u_1 d\dot{w}_P \bar{u}_2 \in Z_d \;\mapsto\; \sum e_i^*(u_1) + \sum f_i^*(\bar{u}_2), \tag{31}$$

where $u_1 \in U_+, \bar{u}_2 \in U_-$, and where $\bar{u}_2$ is determined by u_1 and the property that $u_1 d\dot{w}_P \bar{u}_2 \in Z_d$.

The corresponding map from $\check{X}_{\mathrm{Lie}}$, when the coordinate q is fixed, is denoted

$$\mathcal{W}_{q,\mathrm{Lie}} : \check{X}_{\mathrm{Lie}} \to \mathbb{C} : u_1\dot{w}_P B_- \mapsto \mathcal{F}(u_1\dot{w}_P B_-, \omega_1^\vee(q)).$$

Remark 1. Note that if $g = u_1 d\dot{w}_P \bar{u}_2 \in Z_d$, then we have a simple identity concerning the Plücker coordinates:

$$(p_0(g) : \dots : p_{2m-2}(g)) = (p_0(\bar{u}_2) : \dots : p_{2m-2}(\bar{u}_2)).$$

The remainder of Section 3 will be devoted to proving Theorem 3.2, which now says that there is an isomorphism $\check{X}_{2m-2} \xrightarrow{\sim} \check{X}_{\mathrm{Lie}}$ under which $\mathcal{W}_q$ is identified with $\mathcal{W}_{q,\mathrm{Lie}}$.

3.5. **Isomorphism between $\check{X}_{\mathrm{can}}$ and $\check{X}_{\mathrm{Lie}}$.** To prove Theorem 3.2, the first step is to construct an isomorphism between $\check{X}_{2m-2}$ and the open Richardson variety $\check{X}_{\mathrm{Lie}}$. We define the following maps:

$$\begin{array}{rcccl} \check{\mathbb{X}} = P\backslash G & \xleftarrow{\pi_L} & Z_d = B_-\dot{w}_0 \cap U_+ d\dot{w}_P U_- & \xrightarrow{\pi_R} & \check{X}_{\mathrm{Lie}}, \\ Pg & \leftarrow\!\shortmid & g & \mapsto & gB_-, \end{array}$$

given by taking left and right cosets, respectively. Note that g is equal to $b_-\dot{w}_0$ in our previous notation and factorizes (a priori non-uniquely) as

$$g = u_1 d\dot{w}_P \bar{u}_2.$$

Moreover π_R is an isomorphism, so we have $\pi := \pi_L \circ \pi_R^{-1} : \check{X}_{\mathrm{Lie}} \to \check{\mathbb{X}}_{2m-2}$. Our next goal is to prove:

Proposition 3.3. *π_L defines an isomorphism from Z_d to $\check{X}_{2m-2}$. As a consequence, π defines an isomorphism from $\check{X}_{\mathrm{Lie}}$ to $\check{X}_{2m-2}$.*

Our proof uses a presentation of the coordinate ring of the unipotent cell

$$U_-^P := U_- \cap B_+(\dot{w}^P)^{-1}B_+ \tag{32}$$

due to [GLS11]. The strategy of the proof of Proposition 3.3 is as follows.

- The first step is to show that the natural map $\pi_L : Z_d \to \check{\mathbb{X}}$ factorizes as $\phi\circ\theta$ where $\phi : U_-^P \to \check{\mathbb{X}}$ with $\phi(\bar{u}) = P\bar{u}$ and $\theta : Z_d \to U_-^P$ is an isomorphism which will be constructed in Lemma 3.4.
- We then use the presentation of the coordinate ring of U_-^P to show that the image of the map ϕ lands in $\check{X}_{2m-2}$ and not just $\check{\mathbb{X}}_{2m-2}$. That is, the Plücker coordinates p_0, p_{2m-2}, p_{m-1}, p'_{m-1} and the functions δ_ℓ (defined in (5)) do not vanish. Finally, we show that ϕ is an isomorphism from U_-^P to $\check{X}_{2m-2}$. The main step is to find a pre-image for each of the functions generating $\mathbb{C}[U_-^P]$.

Lemma 3.4. *There exists an isomorphism $\theta : Z_d \to U_-^P$ such that for $b\dot{w}_0 \in Z_d$,*

$$Pb\dot{w}_0 = P\bar{u}_2, \tag{33}$$

where $\bar{u}_2 := \theta(b\dot{w}_0)$.

To prove Lemma 3.4 we use an isomorphism introduced by Berenstein and Zelevinsky in [BZ97] (and joint with Fomin in type A [BFZ96]) which is sometimes called the BZ twist (or BFZ twist).

Theorem 3.5. [BZ97, Theorem 1.2] *Let $y \in U_- \cap B_+\dot{w}^{-1}B_+$. There exists a unique $x \in U_+ \cap B_-\dot{w}B_-$ such that $U_+ \cap B_-\dot{w}y = \{x\}$. The resulting map $\tilde{\eta}_w : U_- \cap B_+\dot{w}^{-1}B_+ \to U_+ \cap B_-\dot{w}B_-$ sending y to x is an isomorphism. In particular we have an inverse isomorphism*

$$\varepsilon_w : U_+ \cap B_-\dot{w}B_- \to U_- \cap B_+\dot{w}^{-1}B_+.$$

Remark 2. We note that the original twist map of Berenstein and Zelevinsky is an automorphism $\eta_w : U_+ \cap B_-\dot{w}B_- \to U_+ \cap B_-\dot{w}B_-$. Our map $\tilde{\eta}_w$ is related to η_w by

$$\tilde{\eta}_w(y) = \eta_w(y^T),$$

where y^T denotes the transpose of y. We have

$$\tilde{\eta}_w(y) = x \quad \Longleftrightarrow \quad B_-wy = B_-x.$$

Here we may write B_-w for $B_-\dot{w}$, as the coset doesn't depend on the representative of w.

Proof of Lemma 3.4. The idea is to consider the two birational maps

$$\begin{array}{llllll} \Psi_1: & U_-^P & \to & P\backslash G, & \bar{u}_2 & \mapsto & P\bar{u}_2, \\ \pi_L: & Z_d & \to & P\backslash G, & b_-\dot{w}_0 = u_1 d\dot{w}_P\bar{u}_2 & \mapsto & Pb_-\dot{w}_0, \end{array}$$

and to show that the composition

$$\theta := \Psi_1^{-1} \circ \pi_L : Z_d \to U_-^P. \tag{34}$$

is an isomorphism. We construct a commutative triangle of maps as follows.

$$\begin{array}{ccc} & U_-\dot{w}_0 \cap B_+\dot{w}_P U_- & \\ {\scriptstyle\mu} \nearrow & & \searrow {\scriptstyle\xi} \\ Z_d & \xrightarrow[\theta]{} & U_-^P \end{array}$$

Here $\mu : Z_d \to U_-\dot{w}_0 \cap B_+\dot{w}_P U_-$ is an isomorphism defined by $b_-\dot{w}_0 \mapsto [b_-]_0^{-1} b_-\dot{w}_0$, where $[b_-]_0$ is the torus part of b_-. The inverse isomorphism μ^{-1} is given by $b_+\dot{w}_P u_- \mapsto d[b_+]_0^{-1} b_+\dot{w}_P u_-$. Note that clearly $Pz = P\mu(z)$ for all $z \in Z_d$.

We now define a composition ξ of isomorphisms as follows,

$$U_-\dot{w}_0 \cap B_+\dot{w}_P U_- \xrightarrow{\ell_{\dot{w}_0^{-1}}} U_+ \cap B_- w^P B_- \xrightarrow{\varepsilon_{w^P}} U_- \cap B_+(\dot{w}^P)^{-1} B_+,$$

where $\ell_{\dot{w}_0^{-1}}$ is the left multiplication by $\dot{w}_0^{-1}$ map. Hence we obtain an isomorphism

$$\xi : U_-\dot{w}_0 \cap B_+\dot{w}_P U_- \longrightarrow U_-^P.$$

Suppose $u_-\dot{w}_0 \in U_-\dot{w}_0 \cap B_+\dot{w}_P U_-$. To prove the identity (33) it remains to check that $Pu_-\dot{w}_0 = P\bar{u}_2$ where $\bar{u}_2 = \xi(u_-\dot{w}_0)$. This follows from the defining property of ε_{w^P}. Namely if $u_-\dot{w}_0 \in U_-^P\dot{w}_0$ then if $y = \varepsilon_{s^P}(\dot{w}_0^{-1} w\dot{w}_0)$, we have

$$B_-\dot{w}_0^{-1} u_-\dot{w}_0 = B_- w^P \bar{u}_2 = B_- w_0 w_P \bar{u}_2.$$

Therefore $B_+ u_-\dot{w}_0 = B_+ w_P \bar{u}_2$. □

For the second step of the proof of Proposition 3.3 we use a result of [GLS11] to describe the coordinate ring of the unipotent cell U_-^P. In Lemma 3.10 we then explicitly relate the coordinates on U_-^P to the coordinates on $\check{X}_{2m-2}$, which are the Plücker coordinates from Definition 3.1. In this way we show that the map

$$\phi : U_-^P \to \check{\mathbb{X}}, \quad \bar{u}_2 \mapsto P\bar{u}_2$$

restricts to an isomorphism onto its image, and that this image is $\check{X}_{2m-2}$.

We must first define the generalized minors involved in the presentation due to [GLS11]. Let G^{sc} be the simply-connected covering group of $G = \mathrm{PSO}(V)$, with Borel subgroup B_-^{sc} and unipotent radical U_-^{sc} projecting to B_- and U_- in G. Here $G^{sc} = \mathrm{Spin}(V)$. Since $U_-^{sc} \cong U_-$ via this projection, we may use representations of G^{sc} to define generalized minors of elements of U_-. For $u \in U_-$ we denote by u^{sc} its lift to U_-^{sc}, and similarly for elements of U_+.

Let $w \in W$ have reduced expression $w = s_{i_1} s_{i_2} \dots s_{i_r}$. Write

$$\bar{s}_j = y_j^{sc}(1) x_j^{sc}(-1) y_j^{sc}(1)$$

and $\bar{w} = \bar{s}_{i_1}\bar{s}_{i_2}\dots\bar{s}_{i_r}$.

Definition 3.6. Let $w \in W$ and ω_j be a fundamental weight of G^{sc}. Let V_{ω_j} be the irreducible representation of G^{sc} with highest weight ω_j and $v^+_{\omega_j}$ be a fixed highest weight vector. Define for any $u \in U_-$:

$$\Delta_{\omega_j, w\cdot\omega_j}(u) = \langle u^{sc}\cdot v^+_{\omega_j}, \bar{w}\cdot v^+_{\omega_j}\rangle.$$

Here $\langle u^{sc}\cdot v^+_{\omega_j}, \bar{w}\cdot v^+_{\omega_j}\rangle = \langle \bar{w}^{-1}u^{sc}\cdot v^+_{\omega_j}, \cdot v^+_{\omega_j}\rangle$ denotes the highest weight vector coefficient of $\bar{w}^{-1}u^{sc}\cdot v^+_{\omega_j}$ in terms of the weight space decomposition.

Note that the smallest representative w^P in W of $[w_0] \in W/W_P$ has the following reduced expression:

$$w^P = s_1\dots s_{m-2}s_{m-1}s_m s_{m-2}\dots s_1. \tag{35}$$

Here we state the result from [GLS11] applied to our particular setting.

Theorem 3.7 ([GLS11, Section 8])**.** *Consider the reduced expression* $s_{i_1}\dots s_{i_{2m-2}} = s_1\dots s_{m-2}s_m s_{m-1}s_{m-2}\dots s_1$ *for* $(\dot{w}^P)^{-1}$ *coming from* (35)*. The coordinate ring of the unipotent cell* $U^P_- := U_- \cap B_+(\dot{w}^P)^{-1}B_+$ *inside* PSO_{2m} *is*

$$\mathbb{C}\left[U^P_-\right] = \mathbb{C}\left[\Delta_{\omega_{i_r},(\dot{w}^P)^{-1}_{\le r}\cdot\omega_{i_r}}, \Delta^{-1}_{\omega_{2m-2-s},(\dot{w}^P)^{-1}_{\le s}\cdot\omega_{2m-2-s}}\right]$$

where

- $1 \le r \le 2m-2$*;* $m-1 \le s \le 2m-2$ *;*
- $(\dot{w}^P)^{-1}_{\le r} := s_{i_1}\dots s_{i_r}$*.*

If $j < m$ then $\Delta_{\omega_j,w\cdot\omega_j}(u)$ is a minor in the usual sense for the unique matrix $u^{\mathrm{SO}_{2m}}$ in $U^{\mathrm{SO}_{2m}}_-$ representing u. We denote the minor of $u^{\mathrm{SO}_{2m}}$ with row set $\{i_1,\dots,i_p\}$ and column set $\{j_1,\dots,j_p\}$ by $D^{i_1,\dots,i_p}_{j_1,\dots,j_p}(u)$. We now reformulate Theorem 3.7 as follows.

Corollary 3.8. *The coordinate ring* $\mathbb{C}\left[U^P_-\right]$ *is generated by the minors*

$$D^{2,\dots,r,r+1}_{1,2,\dots,r}, \quad 1 \le r \le m-2;$$

$$D^{2,\dots,2m-1-s,m+1}_{1,2,\dots,2m-1-s}, \quad m+1 \le s \le 2m-3, \ \textit{and}\ D^{2m}_1;$$

the functions

$$\Delta_{\omega_m,\frac{1}{2}[-\epsilon_1+\epsilon_2+\dots+\epsilon_{m-1}-\epsilon_m]} \ \textit{and}\ \Delta_{\omega_{m-1},\frac{1}{2}[-\epsilon_1+\epsilon_2+\dots+\epsilon_m]},$$

which are Pfaffians; the inverses of minors

$$\left(D^{2,\dots,2m-1-s,m+1}_{1,2,\dots,2m-1-s}\right)^{-1}, \quad m+1 \le s \le 2m-3, \ \textit{and}\ \left(D^{2m}_1\right)^{-1};$$

and the inverses of Pfaffians

$$\Delta^{-1}_{\omega_m,\frac{1}{2}[-\epsilon_1+\epsilon_2+\dots+\epsilon_{m-1}-\epsilon_m]} \ \textit{and}\ \Delta^{-1}_{\omega_{m-1},\frac{1}{2}[-\epsilon_1+\epsilon_2+\dots+\epsilon_m]}.$$

To relate the minors and Pfaffians of Corollary 3.8 to the Plücker coordinates we will need to use a specific factorisation of generic elements of U^P_-. By an application of Bruhat's lemma [Lus94], a generic element in U^P_- can be assumed to have a particular factorisation:

$$\bar{u}_2 = y_1(a_1)\dots y_{m-2}(a_{m-2})y_m(d)y_{m-1}(c)y_{m-2}(b_{m-2})\dots y_1(b_1), \tag{36}$$

where $a_i, c, d, b_j \neq 0$.

We have the following standard expression for the p_k on factorized elements, which is a simple consequence of their definition.

Lemma 3.9. *Fix $0 \leq k \leq 2m-2$ an integer. Then if $\bar{u}_2$ is of the form (36) we have*

$$p_k(\bar{u}_2) = \begin{cases} 1 & \text{if } k = 0, \\ a_1 \dots a_{k-1}(a_k + b_k) & \text{if } 1 \leq k \leq m-2, \\ a_1 \dots a_{m-2}c & \text{if } k = m-1, \\ a_1 \dots a_{m-2}cd & \text{if } k = m, \\ a_1 \dots a_{m-2}cdb_{m-2} \dots b_{2m-1-k} & \text{otherwise.} \end{cases}$$

and

$$p'_{m-1}(\bar{u}_2) = a_1 \dots a_{m-2}d. \quad \square$$

We can now prove the lemma we need.

Lemma 3.10. *We have the following equalities of generalised minors and Plücker coordinates evaluated on $\bar{u}_2 \in U_-^P$:*

$$D_1^{2m}(\bar{u}_2) = p_{2m-2}(\bar{u}_2), \tag{37}$$

$$\Delta_{\omega_{m-1}, \frac{1}{2}[-\epsilon_1+\epsilon_2+\dots+\epsilon_m]}(\bar{u}_2) = p_{m-1}(\bar{u}_2), \tag{38}$$

$$\Delta_{\omega_m, \frac{1}{2}[-\epsilon_1+\epsilon_2+\dots+\epsilon_{m-1}-\epsilon_m]}(\bar{u}_2) = p'_{m-1}(\bar{u}_2), \tag{39}$$

$$D_{1,2,\dots,2m-1-s}^{2,\dots,2m-1-s,m+1}(\bar{u}_2) = \delta_{s-m}(\bar{u}_2), \ \text{for } m+1 \leq s \leq 2m-3, \tag{40}$$

where we recall that $\delta_{s-m} = \sum_{k=s}^{m}(-1)^{s-k}p_{k-m}p_{3m-2-k}$.

Proof. The identity (37) follows immediately from the definition of the Plücker coordinates. For the identity (38), write

$$\Delta_{\omega_{m-1}, \frac{1}{2}[-\epsilon_1+\epsilon_2+\dots+\epsilon_m]}(\bar{u}_2) = (D_{1,\dots,m-1,m+1}^{2,\dots,m,2m}(\bar{u}_2))^{\frac{1}{2}}.$$

Note that in the definition of $\Delta_{\omega_j, w\cdot\omega_j}$ we have chosen the representative $\bar{w}$ in such a way that evaluated on a factorized $\bar{u}_2$ the generalized minors will be nonnegative for any positive choice of the coordinates a_i, b_i, c, d (i.e. on 'totally positive' $\bar{u}_2$). This determines the choice of square root. Then developing $D_{1,\dots,m-1,m+1}^{2,\dots,m,2m}(\bar{u}_2)$ with respect to the last column, we get

$$D_{1,\dots,m-1,m+1}^{2,\dots,m,2m}(\bar{u}_2) = D_{1,\dots,m-1}^{2,\dots,m}(\bar{u}_2)D_{m+1}^{2m}(\bar{u}_2) = p_{m-1}(\bar{u}_2)D_{1,\dots,m-1}^{2,\dots,m}(\bar{u}_2)$$

using the definition of $p_{m-1}(\bar{u}_2)$. Finally, since the matrix is $\bar{u}_2$ orthogonal:

$$D_{1,\dots,m-1}^{2,\dots,m}(\bar{u}_2) = D_{1,\dots,m+1}^{1,\dots,m,2m}(\bar{u}_2).$$

Developing again with respect to the last column, we obtain

$$D_{1,\dots,m+1}^{1,\dots,m,2m}(\bar{u}_2) = D_{1,\dots,m}^{1,\dots,m}(\bar{u}_2)D_{m+1}^{2m}(\bar{u}_2) = p_{m-1}(\bar{u}_2),$$

using the definition of $p_{m-1}(\bar{u}_2)$ and the fact that $\bar{u}_2$ is lower unipotent. The identity (38) then follows. The proof of the identity (39) is similar.

Let us now prove the identity (40). Developing $D_{1,2,\dots,2m-1-s}^{2,\dots,2m-1-s,m+1}(\bar{u}_2)$ with respect to the $(2m-1-s)$-th column, we see that it is equal to

$$D_{2m-1-s}^{m+1}(\bar{u}_2)D_{1,2,\dots,2m-2-s}^{2,\dots,2m-1-s}(\bar{u}_2) - D_{1,\dots,2m-2-s}^{2,\dots,2m-2-s,m+1}(\bar{u}_2).$$

Since $\bar{u}_2$ is orthogonal for Q, we have

$$D_{1,2,\dots,2m-2-s}^{2,\dots,2m-1-s}(\bar{u}_2) = D_{1,\dots,s+2}^{1,\dots,s+1,2m}(\bar{u}_2),$$

and since $\bar{u}_2$ is in U_-,

$$D^{1,\dots,s+1,2m}_{1,\dots,s+2}(\bar{u}_2) = D^{2m}_{s+2}(\bar{u}_2) = p_{2m-2-s}(\bar{u}_2).$$

Finally

$$D^{2,\dots,2m-1-s,m+1}_{1,2,\dots,2m-1-s}(\bar{u}_2) = D^{m+1}_{2m-1-s}(\bar{u}_2)p_{2m-2-s}(\bar{u}_2) - D^{2,\dots,2m-2-s,m+1}_{1,\dots,2m-2-s}(\bar{u}_2),$$

hence

$$D^{2,\dots,2m-1-s,m+1}_{1,2,\dots,2m-1-s}(\bar{u}_2) = \sum_{k=s}^{2m-2}(-1)^{s-k}D^{m+1}_{2m-1-s}(\bar{u}_2)p_{2m-2-s}(\bar{u}_2).$$

We also have $D^{m+1}_{2m-1-s}(\bar{u}_2) = db_{2m-2}\dots b_{2m-1-s}$ for $m+1 \le s \le 2m-2$. Indeed, by definition

$$D^{m+1}_{2m-1-s}(\bar{u}_2) = \langle v^*_{m+1}\cdot\bar{u}_2, v_{2m-1-s}\rangle = db_{2m-2}\dots b_{2m-1-s}.$$

Hence

$$\begin{aligned} D^{2,\dots,2m-1-s,m+1}_{1,2,\dots,2m-1-s}(\bar{u}_2) &= \sum_{k=s}^{2m-2}(-1)^{s-k}db_{2m-2}\dots b_{2m-1-s}p_{2m-2-s} \\ &= \sum_{k=s}^{m}(-1)^{s-k}p_{k-m}(\bar{u}_2)p_{3m-2-k}(\bar{u}_2). \end{aligned}$$ □

Proof of Proposition 3.3. Recall that $\pi_L = \phi\circ\theta$ where θ is the isomorphism constructed in Lemma 3.4 and $\phi : U^P_- \to \check{\mathbb{X}}$ is the natural map $\bar{u}_2 \mapsto P\bar{u}_2$. It remains to prove that ϕ is an isomorphism onto $\check{X}_{2m-2}$. We start by proving that the image of ϕ is contained in $\check{X}_{2m-2}$.

Indeed, if $\bar{u}_2 \in U^P_-$, then by Corollary 3.8 the minors $D^{2,\dots,2m-1-s,m+1}_{1,2,\dots,2m-1-s}(\bar{u}_2)$ and $D^{2m}_1(\bar{u}_2)$ and the Pfaffians $\Delta_{\omega_m,\frac{1}{2}[-\epsilon_1+\epsilon_2+\dots+\epsilon_{m-1}-\epsilon_m)]}(\bar{u}_2)$ and $\Delta_{\omega_{m-1},\frac{1}{2}[-\epsilon_1+\epsilon_2+\dots+\epsilon_m)]}(\bar{u}_2)$ do not vanish. Since we have proved in Lemma 3.10 that those correspond precisely to the divisors involved in defining $\check{X}_{2m-2}$, it follows that $P\bar{u}_2 \in \check{X}_{2m-2}$. We may now prove that ϕ is an isomorphism between U^P_- and $\check{X}_{2m-2}$.

Injectivity of the pullback map $\phi^* : \mathbb{C}[\check{X}_{2m-2}] \to \mathbb{C}[U^P_-]$ is a simple consequence of the fact that the map $U^P_- \to \check{\mathbb{X}}_{2m-2}$ is dominant. We now prove that ϕ^* is surjective by observing that each of the functions generating $\mathbb{C}[U^P_-]$ (as in Corollary 3.8) has a preimage.

We have already seen that the inverses of minors and Pfaffians correspond to the inverses of denominators of $\mathcal{W}_q$. Let us now consider the minors $D^{2,\dots,r,r+1}_{1,2,\dots,r}$ for $1 \le r \le m-2$ and $D^{2,\dots,2m-1-s,m+1}_{1,2,\dots,2m-1-s}$ for $m+1 \le s \le 2m-3$. In Lemma 3.10, we proved that

$$D^{2,\dots,2m-1-s,m+1}_{1,2,\dots,2m-1-s} = \phi^*(\delta_{s-m})$$

and

$$D^{2,\dots,r+1}_{1,\dots,r} = D^{1,\dots,2m-1-r}_{1,\dots,2m-r} = D^{2m}_{2m-r} = \phi^*(p_r).$$

Finally, $D^{2m}_1 = \phi^*(p_{2m-2})$, and the Pfaffians

$$\Delta_{\omega_m,\frac{1}{2}[-\epsilon_1+\epsilon_2+\dots+\epsilon_{m-1}-\epsilon_m)]} \text{ and } \Delta_{\omega_{m-1},\frac{1}{2}[-\epsilon_1+\epsilon_2+\dots+\epsilon_m)]}.$$

are pullbacks of the Plücker coordinates p'_{m-1} and p_{m-1}, by Lemma 3.10. This concludes the proof. □

3.6. **Comparison of the superpotentials.** In this section we will prove Theorem 3.2. We saw in the previous section that $\pi = \pi_L \circ \pi_R^{-1} : \check{X}_{\mathrm{Lie}} \to \check{X}_{2m-2}$ is an isomorphism. Note that we have a commutative diagram

$$\begin{array}{ccc} Z_d & \xrightarrow[\sim]{\pi_R} & \check{X}_{\mathrm{Lie}} \\ & {\scriptstyle F_q}\searrow \quad \swarrow{\scriptstyle \mathcal{W}_{q,\mathrm{Lie}}} & \\ & \mathbb{C} & \end{array}$$

Therefore

$$(\pi^{-1})^*(\mathcal{W}_{q,\mathrm{Lie}}) = (\pi_L^{-1})^*(F_q).$$

This gives a regular function on $\check{X}_{2m-2}$ which we denote by $\widetilde{\mathcal{W}_q}$. The statement of Theorem 3.2 says that $\widetilde{\mathcal{W}_q}$ and $\mathcal{W}_q$ agree. We will prove this by expressing both functions in terms of coordinates introduced earlier. Namely we consider the set of factorized elements $P\bar{u}_2$ with $\bar{u}_2$ as in (36) with nonzero coordinates a_i, b_i, c, d as defining an open dense subvariety inside $\check{X}_{2m-2}$ which is isomorphic to a torus. We call this subvariety $\check{X}_{2m-2}$. To finish the proof we will show that the restrictions of $\widetilde{\mathcal{W}_q}$ and of $\mathcal{W}_q$ to $\check{X}_{2m-2}$ agree. This will additionally give an interesting Laurent polynomial formula for the superpotential, which we will use in Section 6 to describe a flat section of the Dubrovin connection.

Proposition 3.11. *$\widetilde{\mathcal{W}_q}$ and $\mathcal{W}_q$ restricted to a particular torus $\check{X}_{\mathrm{Lus}}$ inside $\check{X}_{2m-2}$ have the following Laurent polynomial expression*

$$\mathcal{W}_{q,\mathrm{Lus}} = a_1 + \cdots + a_{m-2} + c + d + b_{m-2} + \cdots + b_1 + q\frac{a_1 + b_1}{a_1 \ldots a_{m-2} c d b_{m-2} \ldots b_1}.$$

We call $(\check{X}_{\mathrm{Lus}}, \mathcal{W}_{q,\mathrm{Lus}})$ the *quiver mirror*. To prove Proposition 3.11 we will need the following:

Lemma 3.12. *If $u_1 \in U_+$, $\bar{u}_2 \in U_-$, $u_1 d\dot{w}_P \bar{u}_2 \in Z_d$, and $\bar{u}_2$ can be written as in* (36)*, then we have the following identities:*

$$f_i^*(\bar{u}_2) = \begin{cases} a_i + b_i & \text{if } 1 \le i \le m-2, \\ c & \text{if } i = m-1, \\ d & \text{if } i = m. \end{cases} \tag{41}$$

$$e_i^*(u_1) = \begin{cases} 0 & \text{if } 2 \le i \le m, \\ q\frac{a_1 + b_1}{a_1 \ldots a_{m-1} c d b_{m-1} \ldots b_1} & \text{if } i = 1. \end{cases} \tag{42}$$

Proof. Equation (41) is obtained immediately from the definition of $\bar{u}_2$. For Equation (42), notice that

$$\begin{aligned} e_i^*(u_1) &= \frac{\langle u_1^{-1} \cdot v_{\omega_i}^-, e_i \cdot v_{\omega_i}^- \rangle}{\langle u_1^{-1} \cdot v_{\omega_i}^-, v_{\omega_i}^- \rangle} \\ &= \frac{\langle d\dot{w}_P \bar{u}_2 \cdot v_{\omega_i}^+, e_i \cdot v_{\omega_i}^- \rangle}{\langle d\dot{w}_P \bar{u}_2 \cdot v_{\omega_i}^+, v_{\omega_i}^- \rangle}. \end{aligned}$$

Assume $2 \le i \le m$. Then $e_i^*(u_1) = 0$ if and only if $\langle \bar{u}_2 \cdot v_{\omega_i}^+, \dot{w}_P^{-1} e_i \cdot v_{\omega_i}^- \rangle = 0$. Now the vector $w_P^{-1} e_i \cdot v_{\omega_i}^-$ is in the μ-weight space of the i-th fundamental representation, where $\mu = w_P^{-1}s_i(-\omega_i)$. Moreover, $\bar{u}_2 \in B_+(\dot{w}^P)^{-1}B_+$, hence $\bar{u}_2 \cdot v_{\omega_i}^+$ can have non-zero components only down to the weight space of weight $(w^P)^{-1}(\omega_i) = w_P^{-1}(-\omega_i)$. Since $l(w_P^{-1}s_i) > l(w_P^{-1})$ for $2 \le i \le m$, this is higher than μ, which proves that $e_i^*(u_1) = 0$.

Now assume $i = 1$. We have

$$\begin{aligned} e_1^*(u_1) &= \frac{\langle d\dot{w}_P\bar{u}_2 \cdot v_{\omega_1}^+, e_1 \cdot v_{\omega_1}^-\rangle}{\langle d\dot{w}_P\bar{u}_2 \cdot v_{\omega_1}^+, v_{\omega_1}^-\rangle} \\ &= (\omega_1 + \alpha_1 - \omega_1)(d)\frac{\langle \bar{u}_2 \cdot v_{\omega_1}^+, \dot{w}_P^{-1}e_1 \cdot v_{\omega_1}^-\rangle}{\langle \bar{u}_2 \cdot v_{\omega_1}^+, \dot{w}_P v_{\omega_1}^-\rangle} \\ &= q\frac{\langle \bar{u}_2 \cdot v_{\omega_1}^+, \dot{w}_P^{-1}e_1 \cdot v_{\omega_1}^-\rangle}{\langle \bar{u}_2 \cdot v_{\omega_1}^+, v_{\omega_1}^-\rangle}. \end{aligned}$$

First look at the denominator. The only way to go from the highest weight vector $v_{\omega_1}^+$ of the first fundamental representation to the lowest weight vector $v_{\omega_1}^-$ is to apply $g \in B_+wB_+$ for $w \ge (w^P)^{-1}$. Since $\bar{u}_2 \in B_+(\dot{w}^P)^{-1}B_+$, it follows that we need to take all factors of $\bar{u}_2$, and normalising $v_{\omega_1}^-$ appropriately, we get

$$\langle \bar{u}_2 \cdot v_{\omega_1}^+, v_{\omega_1}^-\rangle = a_1 \dots a_{m-1}cdb_{m-1}\dots b_1.$$

Finally, we look at the numerator $\langle \bar{u}_2 \cdot v_{\omega_1}^+, \dot{w}_P^{-1}e_1 \cdot v_{\omega_1}^-\rangle$. The vector $\dot{w}_P^{-1}e_1 \cdot v_{\omega_1}^-$ has weight

$$\mu' = \dot{w}_P^{-1}s_1(-\omega_1) = \dot{w}_P^{-1}(-\epsilon_2) = \epsilon_2.$$

Write $w_P^{-1}s_1$ as a prefix $w' = s_1s_2\dots s_{m-2}s_ms_{m-1}s_{m-2}\dots s_2$ of $(w^P)^{-1}$. We have $w's_1 = (w^P)^{-1}$, hence the way from $v_{\omega_1}^+$ to $w' \cdot v_{\omega_1}^-$ is through s_1. From the factorization of $\bar{u}_2$ in (36), it follows that $\langle \bar{u}_2 \cdot v_{\omega_1}^+, \dot{w}_P^{-1}e_1 \cdot v_{\omega_1}^-\rangle = a_1 + b_1$. □

Proof of Proposition 3.11. Using the expression (31) of the superpotential from [Rie08], we immediately deduce expression for $\widetilde{\mathcal{W}}_q$ as a Laurent polynomial from Lemma 3.12. □

Next, using Lemma 3.9 and Proposition 3.11, we express $\widetilde{\mathcal{W}}_q$ in terms of Plücker coordinates and deduce the theorem.

Proof of Theorem 3.2. From Lemma 3.9, it follows that for $\bar{u}_2$ as in (36)

$$p_{\ell+1}(\bar{u}_2)p_{2m-2-\ell}(\bar{u}_2) = (a_{\ell+1} + b_{\ell+1})(a_1\dots a_\ell)^2a_{\ell+1}\dots a_{m-2}cdb_{m-2}\dots b_{\ell+1}$$

for $0 \le \ell \le m-3$. We also get that $p_k(\bar{u}_2)p_{2m-2-k}(\bar{u}_2)$ is equal to

$$\begin{cases} a_1\dots a_{m-2}cdb_{m-2}\dots b_1 & \text{if } k = 0; \\ (a_1 + b_1)a_1\dots a_{m-2}cdb_{m-2}\dots b_2 & \text{if } k = 1; \\ (a_k + b_k)(a_1\dots a_{k-1})^2a_k\dots a_{m-2}cdb_{m-2}\dots b_{k+1} & \text{if } 2 \le k \le m-3. \end{cases} \tag{43}$$

Using (43), we find that most terms in $\delta_\ell(\bar{u}_2) = \sum_{k=0}^{\ell}(-1)^kp_{\ell-k}(\bar{u}_2)p_{2m-2+k-\ell}(\bar{u}_2)$ cancel, and

$$\delta_\ell(\bar{u}_2) = (a_1\dots a_\ell)^2a_{\ell+1}\dots a_{m-2}cdb_{m-2}\dots b_{\ell+1}.$$

This proves that

$$\frac{p_{\ell+1}p_{2m-2-\ell}}{\delta_\ell}(\bar{u}_2) = a_{\ell+1} + b_{\ell+1}$$

for $0 \leq \ell \leq m-3$. Moreover:

$$\frac{p_m}{p_{m-1}}(\bar{u}_2) = \frac{a_1 \dots a_{m-2}cd}{a_1 \dots a_{m-2}c} = d,$$

and

$$\frac{p_m}{p'_{m-1}}(\bar{u}_2) = \frac{a_1 \dots a_{m-2}cd}{a_1 \dots a_{m-2}d} = c.$$

For the first and last terms, we obtain

$$\frac{p_1}{p_0}(\bar{u}_2) = a_1 + b_1,$$

and

$$\frac{p_1}{p_{2m-2}}(\bar{u}_2) = \frac{a_1 + b_1}{a_1 \dots a_{m-1}cdb_{m-1} \dots b_1}$$

as easy consequences of Lemma 3.9. Using Proposition 3.11, this proves that $\widetilde{\mathcal{W}_q}$ coincides with the definition of $\mathcal{W}_q$ from Equation (28)

$$\mathcal{W}_q = \frac{p_1}{p_0} + \sum_{\ell=1}^{m-3} \frac{p_{\ell+1}p_{2m-2-\ell}}{\delta_\ell} + \frac{p_m}{p_{m-1}} + \frac{p_m}{p'_{m-1}} + q\frac{p_1}{p_{2m-2}}. \qquad \square$$

3.7. **Comparison with the Givental and Laurent polynomial mirrors for even quadrics.** Let us recall the Laurent polynomial LG model of Q_{2m-2} from Equation (4)

$$\mathcal{W}_{q,\mathrm{Prz}} = z_1 + \dots + z_{2m-3} + \frac{(z_{2m-2}+q)^2}{z_1 z_2 \dots z_{2m-2}},$$

defined over the torus

$$\check{X}_{\mathrm{Prz}} := \{(z_1, \dots, z_{2m-2}) \mid z_i \neq 0 \quad \forall\, i\},$$

and the Givental LG model from Equation (2)

$$\mathcal{W}_{q,\mathrm{Giv}} = \nu_1 + \dots + \nu_{2m-2},$$

defined over the affine variety

$$\check{X}_{q,\mathrm{Giv}} = \left\{ (\nu_1, \dots, \nu_{2m}) \mid \nu_i \neq 0 \;\forall\, i,\; \prod_{i=1}^{2m} \nu_i = q, \quad \nu_{2m-1} + \nu_{2m} = 1 \right\}.$$

These two LG models are related by a birational change of coordinates analogous to that of [Prz13, Rmk. 19], namely

$$z_i = \begin{cases} \nu_{i+1} & \text{for } 1 \leq i \leq 2m-3; \\ q\frac{\nu_{2m-1}}{\nu_{2m}} & \text{for } i = 2m-2; \end{cases}$$

and conversely

$$\nu_i = \begin{cases} \frac{(z_{2m-2}+q)^2}{z_1 \dots z_{2m-2}} & \text{for } i = 1; \\ z_{i-1} & \text{for } 2 \leq i \leq 2m-2; \\ \frac{z_{2m-2}}{z_{2m-2}+q} & \text{for } i = 2m-1; \\ \frac{q}{z_{2m-2}+q} & \text{for } i = 2m. \end{cases}$$

This change of variables defines an isomorphism

$$\check{X}_{\mathrm{Prz}} \setminus \{z_{2m-2} + q = 0\} \cong \check{X}_{q,\mathrm{Giv}}$$

which identifies the superpotentials $\mathcal{W}_{q,\mathrm{Prz}}$ and $\mathcal{W}_{q,\mathrm{Giv}}$.

Let us now compare these two LG models with ours. Consider the change of coordinates

$$z_i = \begin{cases} \frac{p_i}{p_{i-1}} & \text{for } 1 \le i \le m-2; \\ \frac{p_{2m-3-i}\delta_{2m-5-i}}{p_{2m-4-i}\delta_{2m-4-i}} & \text{for } m-1 \le i \le 2m-5; \\ \frac{p_m}{p_{m-1}} & \text{for } i = 2m-4; \\ \frac{p_m}{p'_{m-1}} & \text{for } i = 2m-3; \\ q\frac{\delta_{m-3}}{\delta_{m-2}} & \text{for } i = 2m-2. \end{cases}$$

It is well-defined on the following intersection $\tilde{T}$ of two cluster tori

$$\tilde{T} := \{x \in \check{X}_{\text{can}} \mid p_i(x) \neq 0 \text{ for all } 0 \le i \le m-2 \text{ and } p_m(x) \neq 0\}.$$

The inverse change of coordinates is given by

$$p_i = \begin{cases} z_1 \dots z_i & \text{for } 1 \le i \le m-2; \\ qz_1 \dots z_{m-2}\frac{z_{2m-3}}{z_{2m-2}} & \text{for } i = m-1; \\ qz_1 \dots z_{m-2}\frac{z_{2m-4}z_{2m-3}}{z_{2m-2}} & \text{for } i = m; \\ qz_1 \dots z_{i-2}\left(1 + \frac{z_{2m-1-i}}{z_{i-2}}\right)\frac{z_{2m-4}z_{2m-3}}{z_{2m-2}+q} & \text{for } m+1 \le i \le 2m-3; \\ qz_1 \dots z_{2m-3}\frac{z_1}{z_{2m-2}+q} & \text{for } i = 2m-2. \end{cases}$$

and $p'_{m-1} = qz_1 \dots z_{m-2}\frac{z_{2m-4}}{z_{2m-2}}$. Moreover, we have

$$\delta_j = \begin{cases} \frac{z_2 \dots z_{j+1}}{z_{2m-4-j} \dots z_{2m-5}} & \text{for } 1 \le j \le m-3; \\ \frac{q}{z_{2m-2}} \cdot \frac{z_2 \dots z_{m-2}}{z_{m-1} \dots z_{2m-5}} & \text{for } i = 2m-2. \end{cases}$$

We see that the inverse change of coordinates is well-defined over $\check{X}_{\text{Prz}} \setminus \{z_{2m-2}+q = 0\}$, which is isomorphic to the Givental mirror manifold $\check{X}_{q,\text{Giv}}$. Hence we obtain an isomorphism

$$\check{X}_{\text{can}} \supset \tilde{T} \cong \check{X}_{\text{Prz}} \setminus \{z_{2m-2}+q=0\} \cong \check{X}_{q,\text{Giv}} \subset \check{X}_{\text{Prz}}$$

which identifies the (restrictions of) the superpotentials $\mathcal{W}_q$ and $\mathcal{W}_{q,\text{Prz}}$. It also identifies the form $\omega_{q,Giv}$ with ω_{can}. This proves Proposition 1.2 from the introduction in the case of even quadrics.

3.8. **The critical points of the canonical mirror.** Since the canonical mirror $(\check{X}_{\text{can}}, \mathcal{W}_q)$ is isomorphic to the Lie-theoretic mirror $(\check{X}_{\text{Lie}}, \mathcal{W}_{q,\text{Lie}})$, it follows from [Rie08] that $\mathcal{W}_q$ has the 'correct' number of critical points on $\check{X}_{\text{can}}$, that is, $\dim H^*(Q_{2m-2}, \mathbb{C}) = 2m$. Here we give explicit expression for the critical points, and compare with the critical points of the classical mirrors $(\check{X}_{q,\text{Giv}}, \mathcal{W}_{q,\text{Giv}})$ and $(\check{X}_{\text{Prz}}, \mathcal{W}_{q,\text{Prz}})$.

Proposition 3.13. *The critical points of the superpotential $\mathcal{W}_q$ on $\check{X}_{\text{can}}$ are given by*

$$p_j = \begin{cases} \zeta^j & \text{if } 1 \le j \le m-2 \ ; \\ \frac{1}{2}\zeta^j & \text{if } m-1 \le j \le 2m-3 \ ; \\ q & \text{if } j = 2m-2, \end{cases}$$

and $p'_{m-1} = \frac{1}{2}\zeta^{m-1}$, where ζ is a primitive $(2m-2)$-st root of $4q$. The associated critical value is $(2m-2)\zeta$. Moreover there are two extra critical points given by

$p_1 = \cdots = p_{m-2} = p_m = p_{2m-3} = 0$, $p_{m-1} = -p'_{m-1} = \pm\sqrt{q}$, $p_{2m-2} = -q$ with corresponding critical value 0. These two critical points do not belong to $\check{X}_{\mathrm{Prz}}$, $\check{X}_{q,\mathrm{Giv}}$ or $\check{X}_{\mathrm{Lus}}$.

Proof. The proof is very similar to that of Proposition 2.3 and we don't repeat it here. □

4. The quiver mirrors $(\check{X}_{\mathrm{Lus}}, \mathcal{W}_{q,\mathrm{Lus}})$

In this section we will explain how our quiver superpotential $\mathcal{W}_{q,\mathrm{Lus}}$ for Q_N can be read off from a certain quiver, justifying its name. This is analogous to the type A complete flag variety case [Giv97] and partial flag variety case [BCFKvS98, BCFKvS00], where one can also read off Laurent polynomial superpotentials from quivers.

We begin by explaining the [BCFKvS98] formula for the Grassmannian $Gr_2(4)$. Note that since $Gr_2(4)$ is defined by a single (quadratic) Plücker relation, it is isomorphic to the quadric Q_4.

For $Gr_2(4)$ the quiver from [BCFKvS98] is shown in Figure 1. The Laurent polynomial superpotential can be read off easily. There are two versions. In the left hand picture the coordinates t_{ij} of the torus $(\mathbb{C}^*)^4$ are in bijection with vertices of the quiver. To each arrow we associate a Laurent monomial by taking the coordinate at the head of the arrow divided by the coordinate at the tail. The Laurent polynomial corresponding to the quiver is the sum of all of the Laurent monomials associated to the arrows.

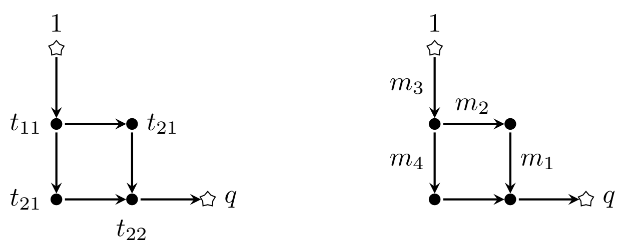


Figure 1. The quiver for $Gr_2(4)$ and two choices of coordinates.

The labels m_i of the arrows in the right hand version are another natural choice of coordinates on the same torus. Indeed these are coordinates related to factorizations into one-parameter subgroups of Lie-theoretic mirrors used in [Lus94], compare [MR13]. We suppose the remaining arrows are labelled in such a way that the square commutes and any path leading from 1 to q has labels whose product equals q. These are Laurent monomials in the variables m_i (depending on q). Then the Laurent polynomial superpotential is obtained in [BCFKvS98] as the sum of the labels of all of the arrows of the quiver. In the case of $Gr_2(4)$ it is

$$m_1 + m_2 + m_3 + m_4 + \frac{m_1 m_2}{m_4} + q\frac{1}{m_1 m_2 m_3}. \tag{44}$$

Since $Gr_2(4)$ is isomorphic to Q_4, this suggests it should be related to the superpotential $(\check{X}_{\mathrm{Lus}}, \mathcal{W}_{q,\mathrm{Lus}})$ from (7) for Q_4,

$$a_1 + c + d + b_1 + q\frac{a_1 + b_1}{a_1 b_1 c d}. \tag{45}$$

There is indeed a toric change of coordinates turning Equation (44) into Equation (45):

$$m_1 \mapsto \frac{q}{a_1cd};\ m_2 \mapsto a_1;\ m_3 \mapsto c;\ m_4 \mapsto b_1.$$

Note that the torus of the other Laurent polynomial mirror $(\check{X}_{\mathrm{Prz}}, \mathcal{W}_{q,\mathrm{Prz}})$ for Q_4 is a different one, as seen in Section 3.7.

The superpotential (45) also comes from a quiver, see Figure 2. This generalises

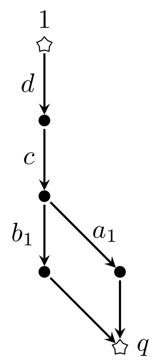


FIGURE 2. The quiver for Q_4.

to all quadrics Q_N. Indeed our Laurent polynomial superpotentials (6) and (7) for Q_N can be described using quivers as in Figure 3. The factorisation of $\bar{u}_2$ from (36) can also be naturally read off the quiver (compare with [MR13, Section 5.3]). This goes as follows.

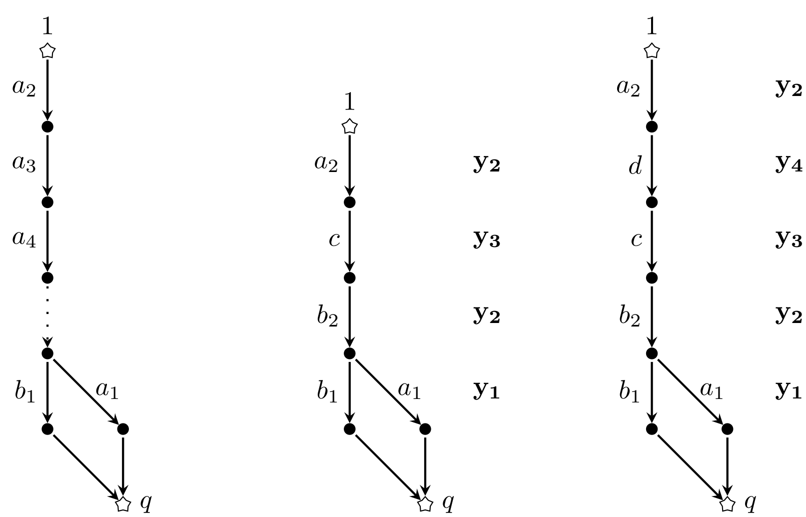


FIGURE 3. The quiver for Q_N, and the labelled quivers for Q_5 and Q_6.

Let the $N-2$ vertical arrows on the left-hand edge be labelled from top to bottom by $a_2, a_3, \ldots, a_{m-1}, c, b_{m-1}, \ldots, b_2$ for odd quadrics Q_{2m-1}, and by $a_2, a_3, \ldots, a_{m-2}, d, c, b_{m-2}, \ldots, b_2$ for even quadrics Q_{2m-2}. The diagonal arrow with the same tail as b_1 is labelled by a_1. The arrows below are not labelled. The labelled arrows can be organized into 'levels' starting with a_1, b_1 at the bottom level. The levels are associated to the one-parameter subgroups y_i (of PSO_{2m} for $X = Q_{2m-2}$, respectively of PSp_{2m} for $X = Q_{2m-1}$) as shown in the Q_5 and Q_6 examples. Reading off column by column from right to left and from top to bottom we recover the factorization (36).

Remark 3. It is interesting to note that our quivers (restricted to the vertices which are not labelled by q) are orientations of type D Dynkin diagrams with a special vertex added at either end. So we have three ways to associate a Dynkin diagram to a quadric: the type of its symmetry group, the type of the cluster algebra associated to the coordinate ring of its mirror, and the type of the quiver defining its superpotential. See Table 1.

Quadric	Symmetry group	Cluster type of mirror	Superpotential Quiver
Q_3	B_2	A_1	D_3
Q_4	D_3	A_1	D_4
Q_5	B_3	A_1^2	D_5
Q_6	D_4	A_1^2	D_6
Q_7	B_4	A_1^3	D_7
$\vdots$	$\vdots$	$\vdots$	$\vdots$

TABLE 1. Dynkin diagrams associated to quadrics

5. THE A-MODEL AND B-MODEL CONNECTIONS

Our expression for the canonical LG model $\mathcal{W}_q$ in terms of homogeneous coordinates coming from $\check{X}_{\mathrm{can}} \subset \mathbb{P}(H^*(X,\mathbb{C})^*)$ makes it possible to compare in a very natural way the (small) Dubrovin connection on the A side and the Gauss-Manin connection on the B side. We recall first the relevant definitions on the A side.

Let $X = Q_N$. Consider $H^*(X, \mathbb{C}[\hbar, q])$ as a space of sections on a trivial bundle with fiber $H^*(X,\mathbb{C})$, over the base $\mathbb{C}_\hbar \times \mathbb{C}_q$, where the $\hbar$ and q are the coordinates. Let Gr be the operator on sections defined on the fibres as the 'grading operator' $H^*(X,\mathbb{C}) \to H^*(X,\mathbb{C})$ which multiplies $\sigma \in H^{2k}(X,\mathbb{C})$ by k. We define the Dubrovin connection by

$$ {}^A\nabla_{q\partial_q} S := q\frac{\partial S}{\partial q} + \frac{1}{\hbar}\sigma_1 \star_q S, \tag{46} $$

$$ {}^A\nabla_{\hbar\partial_\hbar} S := \hbar\frac{\partial S}{\partial \hbar} - \frac{1}{\hbar}c_1(TX) \star_q S + \mathrm{Gr}(S), \tag{47} $$

following the conventions of Iritani [Iri09], where $\star_q$ denotes the quantum cup product in the quantum cohomology, and S may be any meromorphic or formal section of the above vector bundle. The above defines a meromorphic connection which is flat, see also [Dub96, Giv96, CK99]. It therefore turns $H^*(X, \mathbb{C}[\hbar^{\pm 1}, q^{\pm 1}])$ into a D-module for $\mathbb{C}[\hbar^{\pm 1}, q^{\pm 1}]\langle \partial_\hbar, \partial_q \rangle$, which we will call the *A-model D-module* and denote by M_A. Explicitly

$$M_A := H^*(X, \mathbb{C}[\hbar^{\pm 1}, q^{\pm 1}]), \text{ with } \partial_\hbar \sigma := {}^A\nabla_{\partial_\hbar}\sigma \text{ and } \partial_q \sigma := {}^A\nabla_{\partial_q}\sigma. \tag{48}$$

This is the D-module we consider on the A-model side.

We now define the D-module M_B. Let $\Omega^k(\check{X}_{\text{can}})$ denote the space of all algebraic k-forms on $\check{X}_{\text{can}}$.

Definition 5.1. Define the $\mathbb{C}[\hbar, q]$-module

$$G_0^{\mathcal{W}_q} := \Omega^n(\check{X}_{\text{can}})[\hbar, q]/(\hbar d + d\mathcal{W}_q \wedge -)\Omega^{n-1}(\check{X}_{\text{can}})[\hbar, q].$$

It has a meromorphic (Gauss-Manin) connection given by

$$ {}^B\nabla_{q\partial_q}[\alpha] = q\frac{\partial}{\partial q}[\alpha] + \frac{1}{\hbar}\left[q\frac{\partial \mathcal{W}_q}{\partial q}\alpha\right], \tag{49}$$

$$ {}^B\nabla_{\hbar\partial_\hbar}[\alpha] = \hbar\frac{\partial}{\partial \hbar}[\alpha] - \frac{1}{\hbar}[\mathcal{W}_q\alpha]. \tag{50}$$

Let $M_B = G_0^{\mathcal{W}_q} \otimes_{\mathbb{C}[\hbar,q]} \mathbb{C}[\hbar^{\pm 1}, q^{\pm 1}]$. We view M_B as a $\mathbb{C}[\hbar^{\pm 1}, q^{\pm 1}]\langle \partial_\hbar, \partial_q \rangle$-module with ∂_q acting by ${}^B\nabla_{\partial_q}$ and $\partial_\hbar$ acting by ${}^B\nabla_{\partial_\hbar}$.

On the A-model side a special role is played by the element $1 \in M_A$ corresponding to the identity in $H^*(X, \mathbb{C})$. For the B-model there is also a distinguished element. Recall that $\check{X}_{\text{can}}$ is the complement of an anticanonical divisor in $\check{\mathbb{X}}$. Therefore we saw that there is an up to scalar unique non-vanishing logarithmic N-form on $\check{X}_{\text{can}}$ which we called ω_{can} (see Equations (10) and (27)). This is the same form as the one appearing in [GHK11, Lemma 5.14], and it also agrees with the one from [Rie08] after the isomorphism of $\check{X}_{\text{can}}$ with $\check{X}_{\text{Lie}}$. It determines an element $[\omega_{can}]$ in M_B.

5.1. **The case of odd-dimensional quadrics.** For odd-dimensional quadrics we recall the isomorphism between the D-modules on the two sides, proved using results from [GS13].

Theorem 5.2 ([PR13, Corollary 13])**.** *For $X = Q_{2m-1}$ with its mirror LG-model $(\check{X}_{2m-1}, \mathcal{W}_q)$ from Theorem 2.1, the map*

$$\begin{array}{ccc} M_A & \to & M_B \\ \sigma_i & \mapsto & [p_i\omega_{can}] \end{array}$$

defines an isomorphism of D-modules.

5.2. **The case of even-dimensional quadrics.** For even quadrics Q_{2m-2} we prove the following.

Theorem 5.3. *For $X = Q_{2m-2}$ and the canonical mirror $(\check{X}_{2m-2}, \mathcal{W}_q)$, see* (28)*, the map*

$$\begin{array}{cccc} \Psi: & M_A & \to & M_B \\ & \sigma_i & \mapsto & [p_i\omega_{can}] \\ & \sigma'_{m-1} & \mapsto & [p'_{m-1}\omega_{can}] \end{array}$$

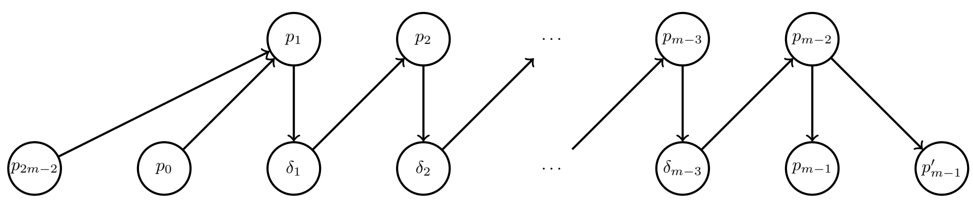
defines an injective homomorphism of D-modules. In particular, the $\mathbb{C}[\hbar^{\pm 1}, q^{\pm 1}]$-submodule of M_B generated by the classes $[p_i\omega_{can}]$ and $[p'_{m-1}\omega_{can}]$ is a submodule also for $D = \mathbb{C}[\hbar^{\pm 1}, q^{\pm 1}]\langle \partial_\hbar, \partial_q \rangle$.

Remark 4. In the odd quadrics case, [GS13] (with Némethi and Sabbah) prove an additional property, cohomological tameness, for the superpotential, which implies that the dimension of M_B agrees with the number of critical points of $\mathcal{W}_q$. It is an interesting question whether this proof could be adapted to give a proof of cohomological tameness in the even case. Since by Proposition 1.3 the number of critical points of $\mathcal{W}_q$ agrees with the dimension of $H^*(X,\mathbb{C})$ this would imply that the injective homomorphism in Theorem 5.3 is an isomorphism.

To prove Theorem 5.3 we consider a cluster algebra structure on our mirror $\check{X}_{2m-2}$. Cluster algebras were introduced by Fomin and Zelevinsky in the seminal paper [FZ02], which was the first of the series [FZ02, FZ03, BFZ05, FZ07].

The coordinate ring $\mathbb{C}[\check{X}_{2m-2}]$ has a cluster algebra structure of type A_1^{m-2} which is described in detail in [GLS08b, Section 2] and [GLS08a, Section 12], and which we review here. Note that the coordinates $\{y_1, y_2, \dots, y_{2m}\}$ in [GLS08b, GLS08a] correspond to our coordinates $\{p_0, p_1, \dots, p_{m-2}, p_{m-1}, p'_{m-1}, p_m, \dots, p_{2m-2}\}$ here, while the coordinates $\{p_\ell\}$ in [GLS08b, GLS08a] correspond to our coordinates $\{\delta_\ell\}$.

Consider the following initial quiver:



Here the initial cluster variables correspond to the vertices in the top row of the quiver, while the frozen variables (or coefficients) correspond to the vertices in the bottom row. Recall that the p_i are Plücker coordinates, and the δ_i are defined as in (26). We see from this description that the coordinate ring of $\check{X}_{2m-2}$ has a cluster structure of type A_1^{m-2}. In particular, it is of finite type, and there are 2^{m-2} different clusters, consisting of

- the cluster variables $r_1, \dots, r_{m-2}$, where $r_i \in \{p_i, p_{2m-2-i}\}$;
- the frozen variables (or coefficients) $\delta_1, \dots, \delta_{m-3}$, p_0, p_{m-1}, p'_{m-1}, and p_{2m-2}.

The exchange relations are

$$p_i p_{2m-2-i} = \begin{cases} p_0 p_{2m-2} + \delta_1 & \text{for } i = 1; \\ \delta_{i-1} + \delta_i & \text{for } 1 \le i \le m-3; \\ \delta_{m-3} + p_{m-1} p'_{m-1} & \text{for } i = m-2. \end{cases} \tag{51}$$

Note that the exchange relation for $i = m-2$ is a Plücker relation: it is the equation of the dual quadric (23).

Remark 5. In the case of $\check{\mathbb{X}}^\circ_{2m-3}$ the isomorphism with the Richardson variety combined with [GLS11] also gives a cluster algebra structure of type A_1^{m-2}, with a similar quiver to the one shown on page 28 but where the frozen vertices labelled p_{m-1} and p'_{m-1} are identified.

Proof of Theorem 5.3. For the injectivity of Ψ we refer to [MR13, Section 5]. It remains to prove that Ψ preserves the D-module structure. We use a change of coordinates to reduce the problem to checking only the action of $q\partial_q$. Namely, this follows by replacing $(p_i, q, \hbar)$ with $(\mathbf{p}_i, \mathbf{q}, \hbar)$, where

$$\mathbf{p}_i = \hbar^{-i} p_i, \qquad \mathbf{p}'_{m-1} = \hbar^{1-m} p'_{m-1}, \qquad \mathbf{q} = \hbar^{-N} q, \qquad \hbar = \hbar,$$

and observing that written in these coordinates the Gauss-Manin system for $\frac{1}{\hbar}\mathcal{W}_q$ no longer involves the $\hbar$.

Now we check that the map Ψ preserves the action of $q\partial_q$. We consider the following identities in $QH^*(Q_{2m-2}, \mathbb{C})$, which are a special case of results in [FW04]:

$$\sigma_1 \star_q \sigma_i = \begin{cases} \sigma_{i+1} & \text{for } 0 \le i \le m-3 \text{ or } m-1 \le i \le 2m-4; \\ \sigma_{m-1} + \sigma'_{m-1} & \text{for } i = m-2; \\ \sigma_{2m-2} + q\sigma_0 & \text{for } i = 2m-3; \\ q\sigma_1 & \text{for } i = 2m-2, \end{cases} \tag{52}$$

$$\sigma_1 \star_q \sigma'_{m-1} = \sigma_m. \tag{53}$$

We need to prove that there are similar identities on the B side:

$$\left[q\frac{\partial \mathcal{W}_q}{\partial q} p_i \omega_{can}\right] = \begin{cases} [p_{i+1}\omega_{can}] & \text{for } 0 \le i \le m-3 \text{ or } m-1 \le i \le 2m-4; \\ [(p_{m-1} + p'_{m-1})\omega_{can}] & \text{for } i = m-2; \\ [(p_{2m-2} + q)\omega_{can}] & \text{for } i = 2m-3; \\ [qp_1\omega_{can}] & \text{for } i = 2m-2, \end{cases} \tag{54}$$

$$\left[q\frac{\partial \mathcal{W}_q}{\partial q} p'_{m-1} \omega_{can}\right] = [p_m \omega_{can}], \tag{55}$$

where ω_{can} is the canonical $(2m-2)$-form on $\check{X}_{\text{can}}$.

The proof of these identities in M_B proceeds by constructing closed $(2m-3)$-forms ν_i and ν'_{m-1} such that the relation corresponding to p_i will follow from the fact that

$$[d\mathcal{W}_q \wedge \nu_i] = [(\hbar d + d\mathcal{W}_q \wedge -)\nu_i] = 0$$

and similarly for p'_{m-1}. (The first equality above comes from the fact that ν_i is closed, and the second comes from the definition of M_B.)

Concretely, we will pick a cluster $\mathcal{C}$ containing a particular Plücker coordinate, say p_i, and use the following Ansatz for constructing ν_i. We define a vector field

$$\xi_i = p_i \left(\sum_{c \in \mathcal{C} \setminus \{p_i\}} m_c c \partial_c \right)$$

and define an associated $(2m-3)$-form by insertion $\nu_i = \iota_{\xi_i}\omega_{can}$, and analogously for $\nu'_{m-1} = \iota_{\xi'_{m-1}}\omega_{can}$. Here the m_c's are constants and ι is the interior product.

To see that these $(2m-3)$-forms are closed, write $\omega_{can} = \bigwedge_{p\in\mathcal{C}} \frac{dp}{p}$. For $c \in \mathcal{C}$, we have $\iota_{c\partial_c}\omega_{can} = \bigwedge_{p\in\mathcal{C}\setminus\{c\}} \frac{dp}{p}$, and so ν_i is a $\mathbb{C}$-linear combination of terms of the form $p_i \bigwedge_{p\in\mathcal{C}\setminus\{c\}} \frac{dp}{p}$ for $c \ne p_i$. Such a term is closed, because p_i lies in $\mathcal{C} \setminus \{c\}$.

Using the fact that $d\mathcal{W}_q \wedge \omega_{can} = 0$, we get $d\mathcal{W}_q \wedge \nu_i = \pm d\mathcal{W}_q(\xi_i)\omega_{can}$. It follows that

$$d\mathcal{W}_q \wedge \nu_i = p_i \left(\sum_{c \in \mathcal{C} \setminus \{p_i\}} m_c c \frac{\partial \mathcal{W}_q}{\partial c} \right) \omega_{can}.$$

Therefore e.g. in order to prove that $\left[q\frac{\partial \mathcal{W}_q}{\partial q} p_i \omega_{can} \right] - [p_{i+1}\omega_{can}] = 0$, we will show that $q\frac{\partial \mathcal{W}_q}{\partial q} p_i - p_{i+1}$ has the form $p_i \left(\sum_{c \in \mathcal{C}\setminus\{p_i\}} m_c c \frac{\partial \mathcal{W}_q}{\partial c} \right)$, for some choice of coefficients m_c.

To prove these identities, we will work with two clusters:

- the initial cluster $\mathcal{C}_1 = \{p_1, \dots, p_{m-2}, \delta_1, \dots, \delta_{m-3}, p_0, p_{m-1}, p'_{m-1}, p_{2m-2}\}$;
- the cluster $\mathcal{C}_2 = \{p_{2m-3}, \dots, p_m, \delta_1, \dots, \delta_{m-3}, p_0, p_{m-1}, p'_{m-1}, p_{2m-2}\}$.

Let us first start with $\mathcal{C}_1$ and express $\mathcal{W}_q$ in terms of it using the exchange relations (51). To simply our calculations, we set $p_0 = 1$, and let δ_0 denote $p_0 p_{2m-2} = p_{2m-2}$.

$$\begin{aligned}\mathcal{W}_q = p_1 + \sum_{\ell=1}^{m-3} \left(\frac{p_{\ell+1}\delta_{\ell-1}}{p_\ell \delta_\ell} + \frac{p_{\ell+1}}{p_\ell} \right) + \frac{\delta_{m-3}}{p_{m-2}p_{m-1}} + \frac{\delta_{m-3}}{p_{m-2}p'_{m-1}} \\ + \frac{p_{m-1}}{p_{m-2}} + \frac{p'_{m-1}}{p_{m-2}} + q\frac{p_1}{\delta_0}.\end{aligned}$$

The partial derivatives of $\mathcal{W}_q$ are:

$$\begin{aligned}
q\frac{\partial \mathcal{W}_q}{\partial q} &= q\frac{p_1}{\delta_0}, \\
p_1 \frac{\partial \mathcal{W}_q}{\partial p_1} &= p_1 - \frac{p_2\delta_0}{p_1\delta_1} - \frac{p_2}{p_1} + q\frac{p_1}{\delta_0}, \\
p_i \frac{\partial \mathcal{W}_q}{\partial p_i} &= \frac{p_i \delta_{i-2}}{p_{i-1}\delta_{i-1}} + \frac{p_i}{p_{i-1}} - \frac{p_{i+1}\delta_{i-1}}{p_i\delta_i} - \frac{p_{i+1}}{p_i} \text{ for } 2 \le i \le m-3, \\
p_{m-2}\frac{\partial \mathcal{W}_q}{\partial p_{m-2}} &= \frac{p_{m-2}\delta_{m-4}}{p_{m-3}\delta_{m-3}} + \frac{p_{m-2}}{p_{m-3}} - \frac{\delta_{m-3}}{p_{m-2}p_{m-1}} - \frac{\delta_{m-3}}{p_{m-2}p'_{m-1}} - \frac{p_{m-1}}{p_{m-2}} - \frac{p'_{m-1}}{p_{m-2}}, \\
\delta_0 \frac{\partial \mathcal{W}_q}{\partial \delta_0} &= \frac{p_2\delta_0}{p_1\delta_1} - q\frac{p_1}{\delta_0}, \\
\delta_i \frac{\partial \mathcal{W}_q}{\partial \delta_i} &= -\frac{p_{i+1}\delta_{i-1}}{p_i\delta_i} + \frac{p_{i+2}\delta_i}{p_{i+1}\delta_{i+1}} \text{ for } 1 \le i \le m-4, \\
\delta_{m-3}\frac{\partial \mathcal{W}_q}{\partial \delta_{m-3}} &= -\frac{p_{m-2}\delta_{m-4}}{p_{m-3}\delta_{m-3}} + \frac{\delta_{m-3}}{p_{m-2}p_{m-1}} + \frac{\delta_{m-3}}{p_{m-2}p'_{m-1}}, \\
p_{m-1}\frac{\partial \mathcal{W}_q}{\partial p_{m-1}} &= -\frac{\delta_{m-3}}{p_{m-2}p_{m-1}} - \frac{\delta_{m-3}}{p_{m-2}p'_{m-1}} + \frac{p_{m-1}}{p_{m-2}}, \text{ and} \\
p'_{m-1}\frac{\partial \mathcal{W}_q}{\partial p'_{m-1}} &= -\frac{\delta_{m-3}}{p_{m-2}p_{m-1}} - \frac{\delta_{m-3}}{p_{m-2}p'_{m-1}} + \frac{p'_{m-1}}{p_{m-2}}.
\end{aligned}$$

Hence

$$q\frac{\partial \mathcal{W}_q}{\partial q}p_i - p_{i+1} = -p_i\left(\sum_{j=i+1}^{m-1} p_j\frac{\partial \mathcal{W}_q}{\partial p_j} + p'_{m-1}\frac{\partial \mathcal{W}_q}{\partial p'_{m-1}} + \sum_{j=0}^{m-3}\delta_j\frac{\partial \mathcal{W}_q}{\partial \delta_j} + \sum_{j=i}^{m-3}\delta_j\frac{\partial \mathcal{W}_q}{\partial \delta_j}\right)$$

for $0 \le i \le m-3$, and

$$q\frac{\partial \mathcal{W}_q}{\partial q}p_{m-2} - (p_{m-1} + p'_{m-1}) = -p_{m-2}\left(p_{m-1}\frac{\partial \mathcal{W}_q}{\partial p_{m-1}} + p'_{m-1}\frac{\partial \mathcal{W}_q}{\partial p'_{m-1}} + \sum_{j=0}^{m-3}\delta_j\frac{\partial \mathcal{W}_q}{\partial \delta_j}\right).$$

Since the right-hand sides of the above equations have the form $p_i\left(\sum_{c\in\mathcal{C}\setminus\{p_i\}} m_c c\partial_c\mathcal{W}_q\right)$, this proves identity (54) for $0 \le i \le m-2$.

To prove the remaining identities, we use the cluster $\mathcal{C}_2$. In this cluster chart, $\mathcal{W}_q$ takes the following form:

$$\begin{aligned}\mathcal{W}_q = &\frac{\delta_0}{p_{2m-3}} + \frac{\delta_1}{p_{2m-3}} + \sum_{\ell=1}^{m-4}\left(\frac{p_{2m-2-\ell}}{p_{2m-3-\ell}} + \frac{p_{2m-2-\ell}\delta_{\ell+1}}{p_{2m-3-\ell}\delta_\ell}\right) + \frac{p_m}{p_{m-1}}\\ &+ \frac{p_m}{p'_{m-1}} + \frac{p_{m+1}}{p_m} + \frac{p_{m-1}p'_{m-1}p_{m+1}}{p_m\delta_{m-3}} + \frac{q}{p_{2m-3}} + q\frac{\delta_1}{p_{2m-3}\delta_0}.\end{aligned}$$

Working out the partial derivatives of $\mathcal{W}_q$ as before, we get

$$q\frac{\partial \mathcal{W}_q}{\partial q}p_{m-1} - p_m = p_{m-1}\left(p'_{m-1}\frac{\partial \mathcal{W}_q}{\partial p'_{m-1}} + \sum_{j=m}^{2m-3}p_j\frac{\partial \mathcal{W}_q}{\partial p_j} + \sum_{j=0}^{m-3}\delta_j\frac{\partial \mathcal{W}_q}{\partial \delta_j}\right) \tag{56}$$

$$q\frac{\partial \mathcal{W}_q}{\partial q}p'_{m-1} - p_m = p'_{m-1}\left(p_{m-1}\frac{\partial \mathcal{W}_q}{\partial p_{m-1}} + \sum_{j=m}^{2m-3}p_j\frac{\partial \mathcal{W}_q}{\partial p_j} + \sum_{j=0}^{m-3}\delta_j\frac{\partial \mathcal{W}_q}{\partial \delta_j}\right) \tag{57}$$

$$q\frac{\partial \mathcal{W}_q}{\partial q}p_i - p_{i+1} = p_i\left(-\sum_{j=i+1}^{2m-3}p_j\frac{\partial \mathcal{W}_q}{\partial p_j} - \sum_{j=0}^{2m-3-i}\delta_j\frac{\partial \mathcal{W}_q}{\partial \delta_j}\right) \tag{58}$$

for $m \le i \le 2m-4$,

Recall that δ_0 is p_{2m-2}. The final two relations are

$$q\frac{\partial \mathcal{W}_q}{\partial q}p_{2m-3} - (p_{2m-2} + q) = -p_{2m-3}\delta_0\frac{\partial \mathcal{W}_q}{\partial \delta_0} \quad \text{and} \tag{59}$$

$$q\frac{\partial \mathcal{W}_q}{\partial q}p_{2m-2} - qp_1 = 0 \tag{60}$$

This gives us the identities (54) for $m-1 \le i \le 2m-2$, as well as (55). □

6. The hypergeometric flat section of a quadric

Givental in [Giv96] constructed flat sections of a dual version of the Dubrovin connection (see Equations (61) and (62) below) in terms of Gromov-Witten invariants. In this section we directly and explicitly compute all the components of a distinguished flat section and the resulting invariants, in two different ways. The first component we consider is also a particular component of Givental's J-function.

6.1. **The dual Dubrovin connection and the J-function.** We begin by defining Givental's J-function and what we call the 'quantum differential operators'. Consider the dual connection to ${}^A\nabla$ with respect to the pairing

$$\langle \sigma, \tau \rangle = (2\pi i\hbar)^N \int_X \sigma \cup \tau.$$

Here $\sigma \cup \tau$ is the usual cup product of σ and τ, which we will subsequently also denote by $\sigma\tau$. Explicitly, the dual connection is given by the formulas:

$$ {}^A\nabla^\vee_{q\partial_q} S \quad := \quad q\frac{\partial S}{\partial q} - \frac{1}{\hbar}\sigma_1 \star_q S, \tag{61}$$

$$ {}^A\nabla^\vee_{\hbar\partial_\hbar} S \quad := \quad \hbar\frac{\partial S}{\partial \hbar} + \frac{1}{\hbar}c_1(TX) \star_q S + \mathrm{Gr}(S), \tag{62}$$

compare [Iri09, Definition 3.1]. For the purposes of the J-function we ignore the ${}^A\nabla^\vee_{\hbar\partial_\hbar}$ part of the covariant derivative and consider ${}^A\nabla^\vee_{q\partial_q}$ as a family of connections (in the parameter $\hbar$). Formal flat sections indexed by the cohomology basis were written down by Givental [Giv96] in terms of descendent Gromov-Witten invariants. We denote these sections by $S_0, \dots, S_{2m-1}$ in the case of Q_{2m-1}, and by $S_0, \dots, S_{m-1}, S'_{m-1}, S_m, \cdots, S_{2m-2}$ for Q_{2m-2}, in keeping with the notation from (24) for Schubert classes. See [CK99, (10.14)] for a precise definition of the sections S_i.

Definition 6.1. We define Givental's J-function in our setting as

$$J = (2\pi i\hbar)^N \sum \langle S_j, \sigma_0 \rangle \sigma_{PD(j)},$$

where the sum is over all the Schubert classes, including σ'_{m-1} in the even case, and where $\sigma_{PD(j)}$ stands for the Poincaré dual cohomology class to σ_j.

In the case of a quadric (or, indeed, of any projective Fano complete intersection), the J-function is computed explicitly in [Giv96, Theorem 9.1] from the J-function of projective space. Namely

$$J^{Q_N} = e^{\frac{\ln(q)\sigma_1}{\hbar}} \sum_{d\geq 0} \frac{\prod_{j=1}^{2d}(2\sigma_1 + j\hbar)}{\prod_{j=1}^{d}(\sigma_1 + j\hbar)^N} q^d. \tag{63}$$

We consider a family of differential operators which annihilate the J-function:

Definition 6.2 ([CK99, Definition 10.3.2])**.** The differential operators P which are formal power series in $\hbar q\partial_q, q, \hbar$ and which annihilate the coefficients of Givental's J-function are called *quantum differential operators.*

6.2. **The hypergeometric term of the J-function.** Among Givental's flat sections S_i, the flat section S_N corresponding to the class of a point has the property that all its coefficients are power series in $\mathbf{q} = \hbar^{-N} q$. Moreover, a special role is played by the coefficient $(2\pi i\hbar)^N\langle S_N, \sigma_0\rangle$, also appearing as the coefficient of the fundamental class in the definition of J-function. We define it as in [BCFKvS98, Definition 5.1.1]:

Definition 6.3. The *hypergeometric series* A_X of X is the unique power series of the form $A_X(q) = 1 + \sum_{k=1}^\infty a_k q^k$, for which $P(q\partial_q, q, 1)A_X = 0$ for all quantum differential operators $P(\hbar q\partial_q, q, \hbar)$ specialized to $\hbar = 1$. We denote the hypergeometric series A_{Q_N} of the quadric Q_N by A_N.

The hypergeometric series A_N of the quadric Q_N may be obtained by setting $\hbar$ to 1 in $(2\pi i\hbar)^N\langle S_N,\sigma_0\rangle$. Alternatively we have $\langle S_N,\sigma_0\rangle = A_N(\hbar^{-N}q)$.

We recall the geometric interpretation of the coefficients of A_X below. The flat sections S_i and in particular the J-function encode certain descendent Gromov-Witten invariants. Let

$$I_k(\psi_1^{a_1}\gamma_1,\dots,\psi_r^{a_r}\gamma_r) \tag{64}$$

denote the degree k descendent Gromov-Witten invariant associated to the cohomology classes $\gamma_1,\dots,\gamma_r$, where the ψ-class ψ_i denotes the first Chern class of the ith cotangent bundle of the moduli space of degree k genus 0 stable maps with r marked points, see [CK99, Section 10.1]. Let ψ stand for ψ_1. If we write

$$J^{Q_N} = (2\pi i\hbar)^N \sum J_i^{Q_N}\sigma_{\mathrm{PD}(i)},$$

then in fact $A_N(\hbar^{-N}q) = \langle S_N,\sigma_0\rangle = J_N^{Q_N}$ and we have

$$\begin{aligned} A_N(\hbar^{-N}q) = J_N^{Q_N} &= 1+\sum_{k=1}^{\infty} q^k I_k\left(\frac{\sigma_N e^{\frac{\ln(q)\sigma_1}{\hbar}}}{\hbar-\psi},\sigma_0\right)\\ &= 1+\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}\frac{q^k}{\hbar}I_k\left(\sigma_N\left(\frac{\ln(q)\sigma_1}{\hbar}\right)^j\frac{1}{j!}\left(\frac{\psi}{\hbar}\right)^i,\sigma_0\right).\end{aligned}$$

The cup-product $\sigma_N\cup\left(\frac{\ln(q)\sigma_1}{\hbar}\right)^j$ is nonzero if and only if $j=0$. Therefore we have

$$J_N^{Q_N} = 1+\sum_{k=1}^{\infty}\sum_{i=0}^{\infty}\frac{q^k}{\hbar}I_k\left(\sigma_N\left(\frac{\psi}{\hbar}\right)^i,\sigma_0\right).$$

Now the dimension of the moduli space of stable maps $\overline{\mathcal{M}}_{0,2}(Q_N,k)$ is equal to $(k+1)N-1$, hence

$$J_N^{Q_N} = 1+\sum_{k=1}^{\infty}\frac{q^k}{\hbar}I_k\Big(\sigma_N\left(\frac{\psi}{\hbar}\right)^{kN-1},\sigma_0\Big).$$

Next we use the fundamental class axiom to get

$$J_N^{Q_N} = 1+\sum_{k=1}^{\infty}\left(\frac{q}{\hbar^N}\right)^k I_k\left(\sigma_N\psi^{kN-2}\right).$$

If we set $\hbar=1$ in $J_N^{Q_N}$, this gives exactly the hypergeometric series of the quadric, since $J_N^{Q_N} = A_N(\hbar^{-N}q)$. Hence we obtain the following geometric interpretation of the coefficient a_k of q^k in $A_N(q)$:

$$a_k = I_k\left(\sigma_N\psi^{kN-2}\right). \tag{65}$$

6.3. **The hypergeometric flat section of the dual Dubrovin connection.** In this Section, as an illustration of the mirror theorem, we compute explicitly the coefficients of the hypergeometric flat section S_N of the Dubrovin connection for Q_N, once using the A-model and once using the B-model. The main result of the computations is the following.

Theorem 6.4. *The hypergeometric flat section S_N of the dual Dubrovin connection for Q_N is given by the expansion*

$$S_N = \frac{1}{(2\pi i\hbar)^N} {\sum_{\ell=0}^{N}}' \langle S_N, \sigma_\ell\rangle \sigma_{PD(\ell)},$$

where $\sum'$ means that we add an extra summand $\langle S_N, \sigma'_{m-1}\rangle\sigma_{m-1}$ when $N = 2m-2$. The coefficients are given by the following formulas:

$$\langle S_N, \sigma_\ell\rangle = \begin{cases} \sum_{k\geq 0} \frac{k^\ell}{\hbar^{kN-\ell}(k!)^N} \cdot \binom{2k}{k} \cdot q^k & \text{if } 0 \leq \ell \leq \lfloor \frac{N-1}{2} \rfloor, \\ \sum_{k\geq 0} \frac{k^\ell}{2\hbar^{kN-\ell}(k!)^N} \cdot \binom{2k}{k} \cdot q^k & \text{if } \lfloor \frac{N+1}{2} \rfloor \leq \ell \leq N-1, \\ \sum_{k\geq 0} \frac{1}{\hbar^{(k-1)N}(k-1)!^N} \cdot \frac{k-1}{k} \cdot \binom{2k-2}{k-1} \cdot q^k & \text{if } \ell = N. \end{cases}$$

Moreover, when $N = 2m-2$ is even, we have

$$\langle S_N, \sigma'_{m-1}\rangle = \sum_{k\geq 0} \frac{k^{m-1}}{2\hbar^{kN+1-m}(k!)^N} \cdot \binom{2k}{k} \cdot q^k.$$

The $\ell = 0$ special case of Theorem 6.4 gives the following.

Corollary 6.5. *The hypergeometric series of the quadric Q_N is*

$$A_N(q) = 1 + \sum_{k\geq 1} \frac{1}{(k!)^N}\binom{2k}{k} q^k. \tag{66}$$

The Gromov-Witten invariant $I_k(\sigma_N\psi^{Nk-2})$ is given by

$$I_k(\sigma_N\psi^{Nk-2}) = \frac{1}{(k!)^N}\binom{2k}{k}. \tag{67}$$

This corollary is easily verified in the A-model. The formula (67) follows from equations (66) and (65), while the second formula, (66), follows easily from the formula (63) for the J-function of Q_N. In the odd quadrics case the D-module is cyclic and hence the constant term determines all of the other terms of the flat section. However for even quadrics this is not the case. We now give a direct A-model proof of Theorem 6.4 which works in the even and odd case alike.

A-model proof. Our A-model proof works by recovering Theorem 6.4 from the recurrence relations of Kontsevich-Manin for Gromov-Witten invariants [KM98]. Define

$$\beta_{\ell,k} = I_k(\psi^{Nk-1-\ell}\sigma_N, \sigma_\ell). \tag{68}$$

Let us first assume that $N = 2m-1$. Using the divisor axiom and topological recursion, we get:

$$k\beta_{\ell,k} = I_k(\psi^{Nk-1-\ell}\sigma_N, \sigma_\ell, \sigma_1) = \begin{cases} \beta_{\ell+1,k} & \text{if } \ell \notin \{m-1, N-1, N\}, \\ 2\beta_{m,k} & \text{if } \ell = m-1, \\ \beta_{N,k} + \beta_{0,k-1} & \text{if } \ell = N-1, \\ \beta_{1,k-1} & \text{if } \ell = N. \end{cases}$$

A straightforward computation then gives

$$\frac{\beta_{\ell,k+1}}{\beta_{\ell,k}} = \begin{cases} \frac{2(2k+1)}{k^\ell(k+1)^{N+1-\ell}} & \text{if } 0 \leq \ell \leq N-1, \\ \frac{2(2k-1)}{(k-1)k^{N-1}(k+1)} & \text{if } \ell = N, \end{cases}$$

and $\beta_{1,1} = 2$, which yields Theorem 6.4. □

Similarly, in the case where $N = 2m - 2$:

$$k\beta_{\ell,k} = I_k(\psi^{Nk-1-\ell}\sigma_N, \sigma_\ell, \sigma_1) = \begin{cases} \beta_{\ell+1,k} & \text{if } \ell \notin \{m-2, N-1, N\}, \\ \beta_{m-1,k} + \beta'_{m-1,k} & \text{if } \ell = m-2, \\ \beta_{N,k} + \beta_{0,k-1} & \text{if } \ell = N-1, \\ \beta_{1,k-1} & \text{if } \ell = N, \end{cases}$$

and

$$k\beta'_{m-1,k} = \beta_{m,k}.$$

Theorem 6.4 is then easily checked. □

B-model proof. We consider the distinguished flat section of the Dubrovin connection whose coefficients are expressed in terms of the B-model as residue integrals, see Section 1.4 and compare with [MR13, Theorem 4.2]. Explicitly, we let $\Gamma_0 \cong (S^1)^N$ be a compact cycle inside $\check{X}_{\mathrm{can}}$ such that $\int_{\Gamma_0} \omega_{can} = 1$. Then the integral formula

$$S_{\Gamma_0}(\hbar, q) := \frac{1}{(2\pi i\hbar)^N} \sum \left(\int_{\Gamma_0} e^{\frac{1}{\hbar}\mathcal{W}_q} p_i \omega_{can} \right) \sigma_{N-i} \tag{69}$$

defines a flat section of the Dubrovin connection in the $N = 2m-1$ case, and with $(\int_{\Gamma_0} e^{\frac{1}{\hbar}\mathcal{W}_q} p_{m-1}\omega_{can})\sigma_{m-1}$ replaced by $(\int_{\Gamma_0} e^{\frac{1}{\hbar}\mathcal{W}_q} p'_{m-1}\omega_{can})\sigma_{m-1} + (\int_{\Gamma_0} e^{\frac{1}{\hbar}\mathcal{W}_q} p_{m-1}\omega_{can})\sigma'_{m-1}$ in the $N = 2m-2$ case.

We will prove the formula in Theorem 6.4 in one representative case, but omit the other cases, which are extremely similar.

Let us consider the case that $N = 2m-2$, and $m \leq \ell \leq 2m-3$. In this case recall that $p_\ell = a_1 \dots a_{m-2} c d b_{m-2} \dots b_{2m-1-\ell}$, and recall from (7) that the superpotential $\mathcal{W}_q$ equals

$$a_1 + \cdots + a_{m-2} + c + d + b_{m-2} + \cdots + b_1 + \frac{q}{a_2 \dots a_{m-2} c d b_{m-2} \dots b_1} + \frac{q}{a_1 \dots a_{m-2} c d b_{m-2} \dots b_2}$$

in terms of the usual coordinates on $\check{X}_{\mathrm{Lus}}$ viewed as a torus chart in $\check{X}_{\mathrm{can}}$.

To compute the constant term of $p_\ell \exp(\frac{1}{\hbar}\mathcal{W}_q)$, we consider

$$p_\ell \left(1 + \frac{1}{\hbar}\mathcal{W}_q + \frac{1}{\hbar^2}\frac{\mathcal{W}_q^2}{2!} + \frac{1}{\hbar^3}\frac{\mathcal{W}_q^3}{3!} + \dots \right),$$

and we pick out from each $p_\ell \frac{\mathcal{W}_q^i}{\hbar^i i!}$ every term which has the form λq^j where $\lambda \in \mathbb{Q}[\frac{1}{\hbar}]$. Here we just need to look at each $\frac{\mathcal{W}_q^{kN-\ell}}{\hbar^{kN-\ell}(kN-\ell)!}$ for $k = 1, 2, \dots$, because the expansion of $p_\ell \frac{\mathcal{W}_q^i}{\hbar^i i!}$ for i not of the form $kN - \ell$ will contain no terms of the form λq^j for $\lambda \in \mathbb{Q}[\frac{1}{\hbar}]$.

Now let us analyze $p_\ell \frac{\mathcal{W}_q^{kN-\ell}}{\hbar^{kN-\ell}(kN-\ell)!}$ for $N = 2m-2$. A (Laurent) monomial in the expansion of $p_\ell \mathcal{W}_q^{k(2m-2)-\ell}$ is obtained by choosing one term in each of the $k(2m-2)-\ell$ factors. Some of the monomials in the expansion will be pure in the variable q alone – in which case they will equal q^k. We need to show that the number of such monomials divided by $(k(2m-2)-\ell)!$ equals $\frac{1}{2}\binom{2k}{k} k^\ell/(k!)^{k(2m-2)}$. To count the number of such monomials, we need to pick one term in each of the $k(2m-2)-\ell$ factors so that we:

- choose i terms which are $\frac{q}{a_2 \dots a_{m-2} c d b_{m-2} \dots b_1}$ for some $0 \leq i \leq k$;

- choose $k-i$ terms which are $\frac{q}{a_1\dots a_{m-2}cdb_{m-2}\dots b_2}$;
- choose $k-1$ terms which are c;
- choose $k-1$ terms which are d;
- choose i terms which are b_1;
- choose $k-i-1$ terms which are a_1;
- for each j such that $2 \le j \le m-2$, choose $k-1$ terms which are a_j;
- for each j such that $2 \le j \le 2m-2-\ell$, choose k terms which are b_j.
- for each j such that $2m-2-\ell < j \le m-2$, choose $k-1$ terms which are b_j.

The number of ways to do this is the sum of multinomial coefficients

$$\sum_{i=0}^{k} \binom{k(2m-2)-\ell}{i, i, k-i, k-i-1, k\dots k, k-1\dots k-1}, \tag{70}$$

where the number of k's in the string $k\dots k$ above is $2m-2-\ell-1$, and the number of $k-1$'s in the string $k-1\dots k-1$ above is $\ell-1$. When we simplify (70) and divide by $(k(2m-2)-\ell)!$, we obtain $\frac{1}{2}\binom{2k}{k}k^\ell/(k!)^{k(2m-2)}$, as desired. □

References

[BCFKvS98] Victor V. Batyrev, Ionuţ Ciocan-Fontanine, Bumsig Kim, and Duco van Straten, *Conifold transitions and mirror symmetry for Calabi-Yau complete intersections in Grassmannians*, Nuclear Phys. B **514** (1998), no. 3, 640–666. MR 1619529 (99m:14074)

[BCFKvS00] ———, *Mirror symmetry and toric degenerations of partial flag manifolds*, Acta Math. **184** (2000), no. 1, 1–39. MR 1756568 (2001f:14077)

[BFZ96] Arkady Berenstein, Sergey Fomin, and Andrei Zelevinsky, *Parametrizations of canonical bases and totally positive matrices*, Adv. Math. **122** (1996), no. 1, 49–149. MR 1405449 (98j:17008)

[BFZ05] ———, *Cluster algebras. III. Upper bounds and double Bruhat cells*, Duke Math. J. **126** (2005), no. 1, 1–52. MR 2110627 (2005i:16065)

[BZ97] Arkady Berenstein and Andrei Zelevinsky, *Total positivity in Schubert varieties*, Comment. Math. Helv. **72** (1997), no. 1, 128–166. MR 1456321 (99g:14064)

[CK99] David A. Cox and Sheldon Katz, *Mirror symmetry and algebraic geometry*, Mathematical Surveys and Monographs, vol. 68, American Mathematical Society, Providence, RI, 1999. MR 1677117 (2000d:14048)

[Dub96] Boris Dubrovin, *Geometry of 2D topological field theories*, Integrable Systems and Quantum Groups **1620** (1996), 120 – 348.

[EHX97] Tohru Eguchi, Kentaro Hori, and Chuan-Sheng Xiong, *Gravitational quantum cohomology*, Internat. J. Modern Phys. A **12** (1997), no. 9, 1743–1782.

[FW04] W. Fulton and C. Woodward, *On the quantum product of Schubert classes*, J. Algebraic Geom. **13** (2004), no. 4, 641–661.

[FZ02] Sergey Fomin and Andrei Zelevinsky, *Cluster algebras. I. Foundations*, J. Amer. Math. Soc. **15** (2002), no. 2, 497–529 (electronic). MR 1887642 (2003f:16050)

[FZ03] ———, *Cluster algebras. II. Finite type classification*, Invent. Math. **154** (2003), no. 1, 63–121. MR 2004457 (2004m:17011)

[FZ07] ———, *Cluster algebras. IV. Coefficients*, Compos. Math. **143** (2007), no. 1, 112–164. MR 2295199 (2008d:16049)

[GHK11] Mark Gross, Paul Hacking, and Sean Keel, *Mirror symmetry for log Calabi-Yau surfaces I*, 2011.

[Gin95] V. Ginzburg, *Perverse sheaves on a Loop group and Langlands' duality*, arXiv:9511007, 1995.

[Giv95] Alexander B. Givental, *Homological geometry and mirror symmetry*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994), Birkhäuser, Basel, 1995, pp. 472–480. MR 1403947 (97j:58013)

[Giv96] ———, *Equivariant Gromov-Witten invariants*, IMRN **13** (1996), 613–663.

[Giv97] ———, *Stationary phase integrals, quantum Toda lattices, flag manifolds and the mirror conjecture*, Topics in singularity theory, Amer. Math. Soc. Transl. Ser. 2, vol. 180, Amer. Math. Soc., Providence, RI, 1997, pp. 103–115. MR 1767115 (2001d:14063)

[Giv98] ———, *A mirror theorem for toric complete intersections*, Topological field theory, primitive forms and related topics (Kyoto, 1996), Progr. Math., vol. 160, Birkhäuser Boston, Boston, MA, 1998, pp. 141–175. MR 1653024 (2000a:14063)

[GLS08a] Christof Geiß, Bernard Leclerc, and Jan Schröer, *Partial flag varieties and preprojective algebras*, Ann. Inst. Fourier (Grenoble) **58** (2008), no. 3, 825–876. MR 2427512 (2009f:14104)

[GLS08b] ———, *Preprojective algebras and cluster algebras*, Trends in representation theory of algebras and related topics, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2008, pp. 253–283. MR 2484728 (2009m:16024)

[GLS11] ———, *Kac-Moody groups and cluster algebras*, Adv. Math. **228** (2011), no. 1, 329–433.

[GS13] Vassily Gorbounov and Maxim Smirnov, *Some remarks on Landau-Ginzburg potentials for odd-dimensional quadrics*, 2013.

[Iri09] Hiroshi Iritani, *An integral structure in quantum cohomology and mirror symmetry for toric orbifolds*, Adv. Math. **222** (2009), no. 3, 1016–1079.

[KLS14] Allen Knutson, Thomas Lam, and David E Speyer, *Projections of Richardson varieties*, Journal für die reine und angewandte Mathematik (Crelles Journal) **2014** (2014), no. 687, 133–157.

[KM98] M. Kontsevich and Yu. Manin, *Relations between the correlators of the topological sigma-model coupled to gravity*, Comm. Math. Phys. **196** (1998), no. 2, 385–398. MR 1645019 (99k:14040)

[KS14] Shrawan Kumar and Karl Schwede, *Richardson varieties have Kawamata log terminal singularities*, Int. Math. Res. Not. IMRN (2014), no. 3, 842–864. MR 3163569

[Lus83] George Lusztig, *Singularities, character formulas, and a q-analog of weight multiplicities*, Analysis and topology on singular spaces, II, III (Luminy, 1981), Astérisque, vol. 101, Soc. Math. France, Paris, 1983, pp. 208–229.

[Lus94] G. Lusztig, *Total positivity in reductive groups*, Lie theory and geometry, Progr. Math., vol. 123, Birkhäuser Boston, Boston, MA, 1994, pp. 531–568. MR 1327548 (96m:20071)

[MR13] R. Marsh and K. Rietsch, *The B-model connection and T-equivariant mirror symmetry for Grassmannians*, arXiv:1307.1085, 2013.

[MV07] I. Mirković and K. Vilonen, *Geometric Langlands duality and representations of algebraic groups over commutative rings*, Ann. of Math. (2) **166** (2007), no. 1, 95–143.

[Pet97] D. Peterson, *Quantum cohomology of G/P*, Lecture Course, MIT, Spring Term, 1997.

[Pha11] Frédéric Pham, *Singularities of integrals*, Universitext, Springer, London; EDP Sciences, Les Ulis, 2011, Homology, hyperfunctions and microlocal analysis, With a foreword by Jacques Bros, Translated from the 2005 French original, With supplementary references by Claude Sabbah. MR 2798679 (2012b:58031)

[PR13] C. Pech and K. Rietsch, *A comparison of Landau-Ginzburg models for odd-dimensional Quadrics*, arXiv:1306.4016, 2013.

[Prz13] V. V. Przyjalkowski, *Weak Landau-Ginzburg models of smooth Fano threefolds*, Izv. Ross. Akad. Nauk Ser. Mat. **77** (2013), no. 4, 135–160. MR 3135701

[Rie06] Konstanze Rietsch, *A mirror construction for the totally nonnegative part of the Peterson variety*, Nagoya Math. J. **183** (2006), 105–142. MR 2253887 (2007i:14055)

[Rie08] ———, *A mirror symmetric construction of $qH^*_T(G/P)_{(q)}$*, Adv. Math. **217** (2008), no. 6, 2401–2442.

[Sei08] Paul Seidel, *Fukaya categories and Picard-Lefschetz theory*, Zurich Lectures in Advanced Mathematics, European Mathematical Society (EMS), Zürich, 2008. MR 2441780 (2009f:53143)

[Wit97] Edward Witten, *Phases of $N = 2$ theories in two dimensions*, Mirror symmetry, II, AMS/IP Stud. Adv. Math., vol. 1, Amer. Math. Soc., Providence, RI, 1997, pp. 143–211. MR 1416338