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Some integrable maps and their Hirota bilinear forms

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Abstract
We introduce a two-parameter family of birational maps, which reduces to a family previously found by Demskoi, Tran, van der Kamp and Quispel (DTKQ) when one of the parameters is set to zero. The study of the singularity confinement pattern for these maps leads to the introduction of a tau function satisfying a homogeneous recurrence which has the Laurent property, and the tropical (or ultradiscrete) analogue of this homogeneous recurrence confirms the quadratic degree growth found empirically by Demskoi *et al*. We prove that the tau function also satisfies two different bilinear equations, each of which is a reduction of the Hirota–Miwa equation (also known as the discrete KP equation, or the octahedron recurrence). Furthermore, these bilinear equations are related to reductions of particular two-dimensional integrable lattice equations, of discrete KdV or discrete Toda type. These connections, as well as the cluster algebra structure of the bilinear equations, allow a direct construction of Poisson brackets, Lax pairs and first integrals for the birational maps. As a consequence of the latter results, we show how each member of the family can be lifted to a system that is integrable in the Liouville sense, clarifying observations made previously in the original DTKQ case.

Keywords: Hirota bilinear form, Liouville integrability, discrete KdV, discrete Toda lattice, tropical

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1. Introduction

In recent work [4], Demskoi, Tran, van der Kamp and Quispel (DTKQ) introduced a one-parameter family of birational maps, given by the Nth-order difference equation

$$\Big(u_n + u_{n+1} + \ldots + u_{n+N}\Big)\, u_{n+1}u_{n+2}\cdots u_{n+N-1} = \alpha, \tag{1}$$

for each integer $N \geqslant 2$. It was shown that the equation (1) admits $\left\lfloor \frac{N+1}{2} \right\rfloor$ independent first integrals, explicitly derived in terms of multi-sums of products, and from a conjectured formula for the degrees d_n of the iterates (quadratic in the index n) it was inferred that $\lim_{n\to\infty} n^{-1}\log d_n = 0$ for each N, so that the corresponding map should have vanishing algebraic entropy in the sense of [15]. These results suggested that (1) should correspond to a finite-dimensional system that is integrable in the Liouville sense [27, 41].

For all N it was noted in [4] that, up to orientation, the map

$$\sigma: \quad (u_0, \ldots, u_{N-1}) \mapsto (u_1, \ldots, u_N)$$

defined by (1) preserves the canonical volume form

$$\Omega = \mathrm{d}u_0 \wedge \cdots \wedge \mathrm{d}u_{N-1},$$

so that $\sigma^*\Omega = (-1)^N\Omega$ for each N. This means that for $N = 2$ the map σ is symplectic, while for $N = 3$ it can be reduced to an anti-symplectic (orientation-reversing) map by restricting to a level set of one of the first integrals; so this is enough to imply Liouville integrability for $N = 2, 3$. However, in the absence of a suitable symplectic or Poisson structure, it is not possible to assert that the maps are Liouville integrable for $N \geqslant 4$.

In this paper we consider a generalization of (1) with two parameters, given by

$$\left(\sum_{j=0}^{N} u_{n+j} + \beta\right) \prod_{k=1}^{N-1} u_{n+k} = \alpha, \tag{2}$$

and show that this slight extension allows a natural interpretation of the observations made in [4], in terms of a Liouville integrable system with $\left\lfloor \frac{N}{2} \right\rfloor$ degrees of freedom. In fact, for even N we shall show that the solutions of (2) are related to the $(N, -1)$ periodic reduction of Hirota's discrete KdV equation, for which Liouville integrability was proved in [20], while for odd N they are connected to reductions of a discrete Toda lattice, considered recently in [22]. Furthermore, for all N these maps are linked to reductions of the Hirota–Miwa equation (also known as the discrete KP equation, or the octahedron recurrence), which connects them to certain cluster algebras [8] and leads to some associated symplectic maps, referred to as U-systems [21].

The original derivation of the equation (1) was based on the fact that it is dual to a linear difference equation of order N, in the sense introduced in [34]: the pair of dual equations have a first integral in common, and each equation appears as an integrating factor for the other one in the total difference of this first integral. The same observation applies to the more general version (2), by introducing

$$\zeta = \left(\sum_{j=0}^{N-1} u_{n+j} + \beta\right)\left(\alpha - \prod_{k=0}^{N-1} u_{n+k}\right). \tag{3}$$

The latter quantity is seen to be a first integral of (2) by noting the identity

$$\Delta\zeta = (u_{n+N} - u_n)\left(\alpha - \left(\sum_{j=0}^{N} u_{n+j} + \beta\right)\prod_{k=1}^{N-1} u_{n+k}\right),$$

in terms of the total difference $\Delta = \mathcal{S} - 1$, with $\mathcal{S}$ being the shift operator such that $\mathcal{S}F_n = F_{n+1}$ (where F_n is any function of n). The linear factor above shows that (2) is dual to the linear equation $u_{n+N} - u_n = 0$, in the same sense that (1) is, and for $\beta = 0$ the quantity ζ reduces to the first integral that is denoted by the same letter in [4].

The key to our results is the use of the singularity confinement pattern of (2) to obtain its 'Laurentification' [12], i.e. a lift to a higher-dimensional system which has the Laurent property in the sense of [5]. (A formal definition of Laurentification will be given in the next section.) The solution of the higher-dimensional system can then be shown to satisfy a Hirota bilinear equation (in fact, two bilinear equations for each N). Our main result, from which all the rest can be derived, is the following.

Theorem 1.1. *Suppose that*

$$u_n = \frac{\tau_{n+3}\tau_n}{\tau_{n+2}\tau_{n+1}} \tag{4}$$

is a solution of (2). Then τ_n satisfies the bilinear equation

$$\tau_{n+N+2}\tau_n = \gamma_n\,\tau_{n+N+1}\tau_{n+1} + \alpha\,\tau_{n+N}\tau_{n+2}, \tag{5}$$

where the quantity γ_n is 2-periodic, that is

$$\gamma_{n+2} = \gamma_n \qquad \forall n;$$

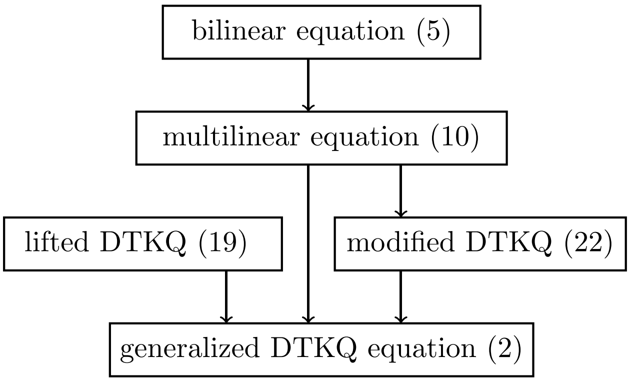
and conversely, the equation (5) for τ_n, with the 2-periodic coefficient γ_n, has a first integral β such that u_n given by (4) satisfies (2). Moreover, if u_n is given by (4), then when N is even *(2) has a first integral K such that τ_n satisfies*

$$\tau_{n+2N+1}\tau_n = -\alpha\,\tau_{n+2N}\tau_{n+1} + K\,\tau_{n+N+1}\tau_{n+N}, \tag{6}$$

while for N odd *(2) has a first integral $\bar{K}$ such that*

$$\tau_{n+2N+2}\tau_n = \alpha^2\,\tau_{n+2N}\tau_{n+2} + \bar{K}\,\tau_{n+N+1}^2. \tag{7}$$

An outline of the paper is as follows. In the next section, we explain how we originally obtained the above result, using the singularity confinement method (or an arithmetical analogue of it) to find a Laurentification of (2), given by a multilinear equation for a tau function (equation (10) in section 2). We also present a tropical (ultradiscrete) version of the multilinear equation, as well as a corresponding tropical version of (2), and show how this can be used to obtain an explicit formula for degree growth (quadratic in n). For any fixed N, it is then possible to derive the bilinear equation (5), as well as (6) or (7), either numerically (with specific initial data) or symbolically (working with rational functions of initial data). The complete proof of theorem 1.1, for arbitrary N, is reserved until section 3, where we begin by deriving (5) via a modified version of (2) (see equation (22) below), before treating the rest of the result and its detailed consequences for even/odd N separately. The interpretation in terms of Liouville integrability is naturally achieved by considering a lift of (2) to dimension $N+1$, obtained by eliminating the parameter β: this yields equation (19) in section 3. A schematic picture of the connections between the main equations involved, valid for arbitrary N, is provided by the following diagram:



The vertical arrows above denote maps between solutions of an equation and the one below it. The other results in section 3 are based on the connection between bilinear equations and cluster algebras, as explained in [8], which leads to a Poisson structure for the lifted DTKQ equation (19). For N even, both bilinear equations (5) and (6) reveal the connection with reductions of Hirota's discrete KdV equation; while for N odd, the second bilinear equation (7) leads to a link with reductions of a discrete time Toda equation, as well as an associated Bäcklund transformation (or BT, in the sense of [25]). In order to illustrate these results, we provide full details for the particular even case $N = 4$ in section 4, and for the odd case $N = 5$ in section 5, before finishing with some conclusions.

Shortly after completing this work, we were made aware of recent results by Svinin [39, 40], who has constructed a large class of difference equations admitting a Lax pair, and described the first integrals for some of them. In particular, the generalized DTKQ equation (2) arises as the case $k = 1$ of a first integral for one of the systems presented by Svinin; see the unnumbered equation appearing before (66) in [40]. We make a further comment on the connection with Svinin's work in our conclusions.

2. Singularity confinement and Laurentification

In this section we describe the experimental approach which led us to theorem 1.1.

The first relevant tool here is the singularity confinement test, which was introduced in [9] as a heuristic method for identifying discrete systems that may be integrable. In its original form, this method has the drawback that many systems with confined singularities have positive algebraic entropy [15], but recently singularity confinement has been refined to include information about deautonomization, which renders it a more effective tool [10, 31, 35]. If we apply the basic singularity confinement test to (2) for a few small values of N, then in all cases we find that the singularity pattern is

$$\ldots, \epsilon, \epsilon^{-1}, \epsilon^{-1}, \epsilon, \ldots, \tag{8}$$

where the latter is the leading power of ϵ when the singularity is approached as $\epsilon \to 0$.

In fact, in order to see the singularity pattern, we do not really need to apply the singularity confinement test per se, but rather an arithmetical version of it, by considering orbits of (2) defined over $\mathbb{Q}$, for rational values of the parameters α, β. This can be turned into a semi-numerical method for measuring the growth of complexity [1], with the rate of growth of the logarithmic heights of the iterates being taken as a measure of entropy [11]. Furthermore, if the map is defined over $\mathbb{Q}$, then one can consider reduction modulo a prime p, in which case the appearance of a singularity at some iterate $u_n \in \mathbb{Q}$ means that the p-adic norm $|u_n|_p > 1$, and the p-adic expansion of the iterates (expanding in powers of p) is analogous to the expansion in powers of ϵ in the usual singularity confinement test (see [23, 24] for an application of this idea).

To see this method in practice, consider (2) for $N = 5$ with $\alpha = 3$, $\beta = -10$, which gives the recurrence

$$u_{n+5} = 10 - (u_n + u_{n+1} + u_{n+2} + u_{n+3} + u_{n+4}) + \frac{3}{u_{n+1}u_{n+2}u_{n+3}u_{n+4}},$$

defined over $\mathbb{Q}$, and choose the five rational initial data $u_0 = u_1 = u_2 = u_3 = 1$, $u_4 = 4$. The sequence continues as

$$\frac{11}{4}, \frac{23}{44}, \frac{316}{253}, \frac{1628}{1817}, \frac{7153}{2923}, \frac{194735}{46028}, \frac{2800493}{3066460}, \frac{115286767}{186573385}, \ldots, \tag{9}$$

and if we factorize each of the above terms then we find

$$\frac{11}{2^2}, \frac{23}{2^2{\cdot}11}, \frac{2^2{\cdot}79}{11{\cdot}23}, \frac{2^2{\cdot}11{\cdot}37}{23{\cdot}79}, \frac{23{\cdot}311}{37{\cdot}79}, \frac{5{\cdot}17{\cdot}29{\cdot}79}{2^2{\cdot}37{\cdot}311}, \frac{37{\cdot}75689}{2^2{\cdot}5{\cdot}17{\cdot}29{\cdot}311}, \frac{59{\cdot}61{\cdot}103{\cdot}311}{5{\cdot}17{\cdot}29{\cdot}75689}, \ldots,$$

which can be taken $\bmod p$ for $p = 11, 23, 37, 79, 311$ etc to reveal the singularity pattern p^{-1}, p, p, p^{-1} at leading order, in accordance with (8) (while the choice $p = 2$ is special here because the coefficient β vanishes $\bmod 2$ in this example).

There is a close link between singularity confinement for discrete systems and the Laurent property [18, 29]. In the context of integrability, the Laurent property appears at the level of Hirota bilinear equations: the Hirota–Miwa (discrete KP) equation can be derived from mutations in a cluster algebra [32], which means that it has the Laurent property, and this property is inherited by its reductions to recurrences of Somos (or Gale-Robinson) type [5, 30]. Furthermore, it seems likely that any birational map with confined singularities can be lifted to a higher-dimensional 'Laurentified' system, i.e. one that has the Laurent property. In specific examples, Laurentification in this sense has been obtained by passing to homogeneous coordinates [42], or by using recursive factorization [12, 13].

For completeness, here we provide a general definition of what it means to 'Laurentify' a birational map, which includes the case of recurrences or difference equations considered in the sequel.

Definition 2.1. Let ψ be a birational map of M-dimensional affine space $\mathbb{A}^M$, defined by rational functions with coefficients belonging to an integral domain $\mathbb{K}$, where $\mathbb{A} = \mathbb{A}(\mathbb{F})$ for some field $\mathbb{F}$ containing the field of fractions of $\mathbb{K}$. Suppose that a fixed set of coordinates $\boldsymbol{\tau} = (\tau_0, \tau_1, \ldots, \tau_{M-1})$ is chosen for $\mathbb{A}^M$; these coordinates are regarded as initial data for ψ. Then, in terms of these coordinates, ψ is said to have the Laurent property if for all $n \in \mathbb{Z}$, each component of the iterates $\psi^n\boldsymbol{\tau}$ belongs to the ring $\mathcal{R}$ of Laurent polynomials in the initial data, that is

$$\mathcal{R} := \mathbb{K}[\tau_0^{\pm 1}, \ldots, \tau_{M-1}^{\pm 1}].$$

Definition 2.2. Let φ be a birational map of N-dimensional affine space. A birational map ψ in dimension $M \geqslant N$ is said to be a Laurentification of φ whenever there is a rational map π such that the diagram

$$
\begin{array}{ccc}
\mathbb{A}^M & \xrightarrow{\psi} & \mathbb{A}^M \\
\Big\downarrow \pi & & \Big\downarrow \pi \\
\mathbb{A}^N & \xrightarrow{\varphi} & \mathbb{A}^N
\end{array}
$$

is commutative, and ψ has the Laurent property.

For the equation (2), regardless of the means by which we obtain the singularity pattern, the form of (8) immediately suggests that we should try the substitution (4) in order to obtain the Laurentification.

Proposition 2.3. *Given the substitution (4), u_n is a solution of (2) whenever the tau function τ_n satisfies the multilinear relation*

$$
\begin{aligned}
\tau_{n+N+3}\tau_{n+N}^2 \prod_{j=1}^{N-1} \tau_{n+j} = \alpha\tau_{n+3}\tau_{n+N} \prod_{j=2}^{N+1} \tau_{n+j} - \beta \prod_{j=1}^{N+2} \tau_{n+j} \\
-\tau_n\tau_{n+3} \prod_{j=3}^{N+2} \tau_{n+j} - \sum_{k=1}^{N-1} \tau_{n+k}\tau_{n+k+3} \prod_{\substack{j=1 \\ j\neq k+1,k+2}}^{N+2} \tau_{n+j},
\end{aligned}
\tag{10}
$$

which is of order $N+3$ and homogeneous of degree $N+2$. For each $N \geqslant 2$ the recurrence (10) has the Laurent property, i.e. the iterates are Laurent polynomials in the initial data $\boldsymbol{\tau} = (\tau_0, \tau_1, \ldots, \tau_{N+2})$. More precisely,

$$
\tau_n \in \mathbb{Z}[\alpha, \beta, \tau_0^{\pm 1}, \ldots, \tau_{N+2}^{\pm 1}] \qquad \forall n \in \mathbb{Z}.
$$

Proof. To prove the Laurent property for the equation (10), we make use of the bilinear relation (5) in theorem 1.1, of which an independent proof is given in the next section. Let $\mathcal{R} := \mathbb{Z}[\alpha, \beta, \tau_0^{\pm 1}, \ldots, \tau_{N+2}^{\pm 1}]$. The period 2 coefficient appearing in (5) takes two distinct values, given by

$$
\gamma_0 = (\tau_{N+1}\tau_1)^{-1}\Big(\tau_{N+2}\tau_0 - \alpha\tau_N\tau_2\Big), \quad \gamma_1 = (\tau_{N+2}\tau_2)^{-1}\Big(\tau_{N+3}\tau_1 - \alpha\tau_{N+1}\tau_3\Big) \in \mathcal{R},
$$

using the fact that $\tau_{N+3} \in \mathcal{R}$, which follows directly from (10). So the iterates of (10) coincide with those of (5), subject to fixing γ_0, γ_1 as above. Now we can make use of proposition 5.4 in [30], which implies that the nonautonomous Somos recurrence (5) has the Laurent property, meaning that

$$
\tau_n \in \mathbb{Z}[\alpha, \gamma_0, \gamma_1, \tau_0^{\pm 1}, \ldots, \tau_{N+1}^{\pm 1}]
$$

for all n. Upon substituting $\gamma_0, \gamma_1 \in \mathcal{R}$ into the Laurent polynomials obtained from (5), the result follows. $\square$

For $\beta = 0$, corresponding to (1), particular cases of the preceding result have been proved before. The case $N = 2$ of the recurrence (10) with $\beta = 0$ was found previously by the recursive factorization method: this is theorem 8 in [12], while theorem 10 in the same paper corresponds to the case $N = 3$, but with the inclusion of certain periodic coefficients; and some results for general N are found in [14]. For any particular N it can be verified directly with computer algebra that the Laurent property holds, by applying a method attributed to Hickerson, that is described in [36]; but a direct proof along these lines for all N is not so straightforward.

Example 2.4. When $N = 5$, the recurrence (10) becomes

$$\begin{aligned} \tau_8\tau_5^2\tau_4\tau_3\tau_2\tau_1 \quad = \quad & \alpha\tau_6\tau_5^2\tau_4\tau_3^2\tau_2 - \beta\tau_7\tau_6\tau_5\tau_4\tau_3\tau_2\tau_1 - \tau_7\tau_6\tau_5\tau_4\tau_3^2\tau_0 - \tau_7\tau_6\tau_5\tau_4^2\tau_1^2 \\ & -\tau_7\tau_6\tau_5^2\tau_2^2\tau_1 - \tau_7\tau_6^2\tau_3^2\tau_2\tau_1 - \tau_7^2\tau_4^2\tau_3\tau_2\tau_1, \end{aligned} \tag{11}$$

where we have set the index $n \to 0$ for brevity.

The Laurent property means that the iterates of (10) can be written as

$$\tau_n = \frac{\mathrm{N}_n(\boldsymbol{\tau})}{\boldsymbol{\tau}^{\mathbf{d}_n}}, \tag{12}$$

where the numerator N_n is a polynomial in the initial data that is not divisible by any of the variables $\tau_0, \tau_1, \ldots, \tau_{N+2}$, while $\boldsymbol{\tau}^{\mathbf{d}_n}$ denotes the Laurent monomial in these variables specified by the denominator vector $\mathbf{d}_n$, an integer vector whose components give the exponents for each variable. Due to the homogeneity of (10), the degree growth of the iterates can be determined from that of the denominators. If we further assume that there are no cancellations between numerators and denominators on the right-hand side, then (as is well known in the context of cluster algebras [6, 7]), the denominator vector $\mathbf{d}_n$ satisfies the max-plus tropical (or ultradiscrete) analogue of (10), which takes the form

$$\mathbf{d}_{n+N+3} + 2\mathbf{d}_{n+N} + \sum_{j=1}^{N-1} \mathbf{d}_{n+j} = \max\left(\mathbf{d}_{n+3} + \mathbf{d}_{n+N} + \sum_{j=2}^{N+1} \mathbf{d}_{n+j}, \sum_{j=1}^{N+2} \mathbf{d}_{n+j}, \ldots\right), \tag{13}$$

where each of the omitted terms in the max corresponds to one of the terms on the right-hand side of (10). The vector form of (13) means that each component of $\mathbf{d}_n$ satisfies the same tropical equation.

Example 2.5. When $N = 5$, the tropical version of (11) can be written as

$$\mathbf{d}_8 + 2\mathbf{d}_5 + \mathbf{d}_4 + \mathbf{d}_3 + \mathbf{d}_2 + \mathbf{d}_1 = \max\left(\mathbf{d}_6 + 2\mathbf{d}_5 + \mathbf{d}_4 + 2\mathbf{d}_3 + \mathbf{d}_2, \sum_{j=1}^{7} \mathbf{d}_j, \sum_{j=4}^{7} \mathbf{d}_j + 2\mathbf{d}_3 + \mathbf{d}_0, \hat{M}\right),$$

where

$$\hat{M} = \mathbf{d}_7 + \mathbf{d}_1 + \max\left(\mathbf{d}_6 + \mathbf{d}_5 + 2\mathbf{d}_4 + \mathbf{d}_1, \mathbf{d}_6 + 2\mathbf{d}_5 + 2\mathbf{d}_2, 2\mathbf{d}_6 + 2\mathbf{d}_3 + \mathbf{d}_2, \mathbf{d}_7 + 2\mathbf{d}_4 + \mathbf{d}_3 + \mathbf{d}_2\right);$$

once again we have set $n \to 0$ for brevity.

From the explicit form of the above equation, the task of solving (13) for general N looks like it might be a formidable one, but in fact there is an enormous simplification that can be made. The point is that the substitution (4) that lifts (2) to (10) also has a tropical analogue, which allows (13) to be reduced to a max-plus version of (2), and the latter turns out to have a very simple behaviour: all solutions reach a fixed point after finitely many steps.

Proposition 2.6. *Given the substitution*

$$U_n = d_{n+3} - d_{n+2} - d_{n+1} + d_n, \tag{14}$$

the quantity U_n is a solution of a tropical version of (2), given by

$$U_{n+N} = \Big[\max(-S_n, U_n, U_{n+1}, \dots, U_{n+N-1})\Big]_+, \qquad S_n = \sum_{j=1}^{N-1} U_{n+j} \tag{15}$$

(with $[x]_+$ denoting $\max(x,0)$ for $x \in \mathbb{R}$), whenever d_n is a scalar solution of (13). Moreover, for any choice of real initial data for (15) there exists $C \geqslant 0$ and an integer $m \geqslant 0$ such that

$$U_n = C \qquad \forall n \geqslant m. \tag{16}$$

Proof. The max-plus equation (15) is obtained from (2) by solving for the highest iterate u_{n+N} and replacing $(+, \times)$ with $(\max, +)$ in the usual way (setting all coefficients to 1), while a direct calculation verifies that if d_n satisfies the scalar version of (13) then U_n given by (14) is a solution of (15). Now given an N-tuple of real initial values $(U_0, \dots, U_{N-1})$, we consider the iteration of the equivalent map in $\mathbb{R}^N$ given by

$$(U_0, \dots, U_{N-1}) \mapsto (U_0', \dots, U_{N-1}') \tag{17}$$

where

$$U_j' = U_{j+1} \text{ for } 0 \leqslant j \leqslant N-2, \qquad U_{N-1}' = [\max(-S, U_0, \dots, U_{N-1})]_+, \quad S = \sum_{j=1}^{N-1} U_j.$$

The initial data can be divided into four disjoint subsets of $\mathbb{R}^N$, defined by

(i) $S \geqslant 0, \quad \max_{j\in\{1,\dots,N-1\}}(U_j) \geqslant U_0$;
(ii) $S \geqslant 0, \quad U_j < U_0 \quad \forall j \in \{1, \dots, N-1\}$;
(iii) $S < 0, \quad U_j < -S \quad \forall j \in \{0, \dots, N-1\}$;
(iv) $S < 0, \quad \max_{j\in\{0,\dots,N-1\}}(U_j) \geqslant -S$.

By examining each of these regions, it follows that the quantity S is non-decreasing under the action of the map (17), and after finitely many iterations it attains a maximum value $S = NC$ at a fixed point $U_j = C \geqslant 0$ for $0 \leqslant j \leqslant N-1$. To see this, begin by considering initial data lying in region (i). In that case, the maximum of the U_j is attained at some $k \in \{1\ \dots, N-1\}$, and $U_{N-1}' = U_k = C \geqslant 0$, with $S' = \sum_{j=1}^{N-1} U_j' = S + U_{N-1}' - U_1 \implies S' - S = U_k - U_1 \geqslant 0$. All subsequent iterations stay in this region, C remains the maximum value, and all components take this same value after a finite number of steps. Next, take initial data in region (ii), to find $U_{N-1}' = U_0$ and $S' - S = U_0 - U_1 > 0$. In that case, the maximum value is $C = U_{N-1}' > U_0' = U_1$, and so region (i) is reached after a single step. For case (iii), one step of (17) gives $U_{N-1}' = -S > 0$, so $S' - S = -S - U_1 > 0$, and hence $\max_{j\in\{0,\dots,N-1\}}(U_j') = U_{N-1}' > -S'$ which means that either region (iv) is attained when $S' < 0$, or otherwise $S' \geqslant 0$ and region (i) has been reached instead. Finally, starting off in region (iv) gives $U_{N-1}' = U_k = C \geqslant -S > 0$ for some k, and $S' - S = U_k - U_1 \geqslant 0$. If the sum $S' < 0$, a finite number of subsequent steps remain in region (iv), with C as the maximum value, until this sum changes sign, so that eventually region (i) is reached, and the proof is complete. □

Upon comparing the substitution (14) with (16), the explicit form of the scalar solution of (13) is obtained immediately, for sufficiently large n.

Corollary 2.7. *For any real $(N+3)$ − tuple of initial values $(d_0, d_1, \ldots, d_{N+2})$, there exist real parameters $A, \bar{A}, B, C$ with $C \geqslant 0$ and an integer $m \geqslant 0$ such that the solution of the scalar version of (13) is given by*

$$d_n = \frac{1}{4}Cn^2 + Bn + A + \bar{A}(-1)^n \qquad \forall n \geqslant m. \tag{18}$$

Remark 2.8. The iterates of (10) are given by (12), where the vector $\mathbf{d}_n$ has components $d_n^{(j)}$, giving the degree of the exponent of τ_j, for $0 \leqslant j \leqslant N+2$. For each j in this range, this gives the initial data $d_k^{(j)} = -\delta_{jk}$, $0 \leqslant k \leqslant N+2$, for a scalar solution of (13). Hence the degrees of the denominators of the iterates of (10) can be calculated exactly, while (by homogeneity) the degree of each numerator is one more than the degree of the denominator. In particular, $C = 0$ in the solution for $j = 0$, and in fact $d_n^{(0)} = 0$ $\forall n \geqslant 1$, since there is no division by τ_0 when (10) is iterated forwards, and $C = 1$ in the solution for $d_n^{(j)}$, $1 \leqslant j \leqslant N-1$. With more detailed analysis, the precise degree growth of (2) can also be derived, but it follows from (18) that it must be quadratic, which is consistent with the empirical results found in [4] for the case $\beta = 0$. More detailed results for symmetric QRT maps, which include the case $N = 2$ of (2), are given in [13].

The Laurent property, combined with the quadratic degree growth of the iterates of (10), suggests that there should also be bilinear relations satisfied by the terms. Thus we can apply the method described in [19], starting with particular numerical values of the initial conditions and looking for the smallest integer q such that a matrix M of size $\lfloor \frac{q}{2} \rfloor + 1$, with entries $\mathrm{M}_{ij} = \tau_{q+i-j}\tau_{i+j-2}$, has vanishing determinant: this corresponds to a bilinear relation of minimal order, with constant coefficients. By considering matrix entries of the form $\mathrm{M}_{ij} = \tau_{q+\ell i-j+k}\tau_{\ell i+j+k-2}$, for different choices of offset k, one can also obtain minimal order relations whose coefficients have period $\ell > 1$.

To illustrate the method, we pick $N = 5$ with $\alpha = 3$, $\beta = -10$, and use (11) to generate the particular sequence that starts from $\tau_0 = \tau_1 = \ldots = \tau_6 = 1$, $\tau_7 = 4$, beginning

$$1, 1, 1, 1, 1, 1, 1, 4, 11, 23, 79, 148, 1244, 9860, 75\,689, 370\,697, \ldots,$$

whose ratios $\tau_{n+3}\tau_n/(\tau_{n+2}\tau_{n+1})$ produce (9). By considering bilinear relations with constant coefficients, the minimal relation is found to be of order $q = 12$, corresponding to the matrix M with entries $\mathrm{M}_{ij} = \tau_{12+i-j}\tau_{i+j-2}$. However, all but three of the entries in a vector spanning the one-dimensional kernel of M are zero, and it is sufficient to take the 3×3 minor

$$\mathrm{M}' = \begin{pmatrix} \tau_{12}\tau_0 & \tau_{10}\tau_2 & \tau_6^2 \\ \tau_{13}\tau_1 & \tau_{11}\tau_3 & \tau_7^2 \\ \tau_{14}\tau_2 & \tau_{12}\tau_4 & \tau_8^2 \end{pmatrix} = \begin{pmatrix} 1244 & 79 & 1 \\ 9860 & 148 & 16 \\ 75\,689 & 1244 & 121 \end{pmatrix}, \qquad \det \mathrm{M}' = 0,$$

whose kernel is spanned by the vector $(1, -9, -533)^T$, corresponding to the bilinear relation (7) with $N = 5$, $\alpha = 3$ and the particular value $\bar{K} = 533$ for the first integral. With this same numerical sequence, one can also obtain relations with periodic coefficients, starting from period $\ell = 2$, by taking the matrix with entries of the form $\mathrm{M}_{ij} = \tau_{q+2i-j+k}\tau_{2i+j+k-2}$, so that the minimum order relation has $q = 7$, and for $k = -1, 0$ one has 3×3 minors with

$$\det\begin{pmatrix} 4 & 1 & 1 \\ 23 & 11 & 4 \\ 148 & 79 & 23 \end{pmatrix} = 0 = \det\begin{pmatrix} 11 & 4 & 7 \\ 79 & 23 & 11 \\ 1244 & 148 & 316 \end{pmatrix}.$$

Each of the two matrices above has a one-dimensional kernel, spanned by $(1,-1,-3)^T$, $(1,-2,-3)^T$ respectively, corresponding to the three-term bilinear relation (5) with $N=5$, $\alpha=3$, $\gamma_0=1$, $\gamma_1=2$.

For fixed N, once bilinear relations have been obtained for one or more particular numerical sequences of values of τ_n, these relations can then be checked for arbitrary initial data and coefficients by symbolic computations with a computer algebra package. Such computations provide a computer-assisted proof of theorem 1.1, for any specific choice of N, but to prove it for all N requires some general arguments, presented in the next section.

Sequences generated by certain bilinear recurrences of Somos type also admit further bilinear relations of higher order (see [33], for example), with the coefficients being first integrals, and this has been used to obtain first integrals for four-term Somos-6 and Somos-7 recurrences, in [19] and [8], respectively. Another approach to finding first integrals, based on reduction of conservation laws for the discrete KP or BKP equations, was used in [28]. However, in what follows we will apply the method in [22], obtaining first integrals from Lax pairs arising by reduction of lattice equations (discrete KdV, discrete Toda, and/or discrete KP).

3. Proof and consequences of the main theorem

In order to understand how the solutions of (2) are related to certain Liouville integrable systems, it is helpful to consider the equation of order $N+1$ obtained by eliminating β:

$$E_n[u] := u_{n+N+1} - u_n + \alpha\left(\frac{1}{\prod_{j=1}^{N-1} u_{n+j}} - \frac{1}{\prod_{j=2}^{N} u_{n+j}}\right) = 0. \tag{19}$$

To see how this arises, one can solve (2) for β, which gives

$$\beta = \frac{\alpha}{\prod_{j=1}^{N-1} u_{n+j}} - \sum_{j=0}^{N} u_{n+j},$$

and then apply the total difference operator Δ to both sides; from this it follows that β defined as above is a first integral for (19). For all N, it can also be checked that (19) has the first integral

$$\hat{\zeta} = \prod_{j=0}^{N} u_{n+j} + \alpha \sum_{k=1}^{N-1} u_{n+k}, \tag{20}$$

which is related to the first integral (3) of (2) by $\hat{\zeta} = \zeta - \alpha\beta$.

For the proof of the first part of theorem 1.1, it is convenient to introduce another lift of (2) to dimension $N+1$, defined via (4) by setting

$$\pi_1: \quad u_n = w_n w_{n+1} \qquad \text{where} \quad w_n = \frac{\tau_n \tau_{n+2}}{\tau_{n+1}^2}. \tag{21}$$

This can be regarded as an intermediate step between (2) and (10). Upon making the substitution (21), we find that (2) produces the equation

$$\mathcal{E}_n[w] := \sum_{j=0}^{N} w_{n+j}w_{n+j+1} + \beta - \frac{\alpha}{\prod_{j=1}^{N-1} w_{n+j}w_{n+j+1}} = 0. \tag{22}$$

If the substitution (21) is interpreted as a Miura map, then (22) can be regarded as a modified version of the generalized DTKQ equation (2)

Lemma 3.1. *The quantity given in terms of w_n and shifts by*

$$\gamma_n[w] = \prod_{j=0}^{N} w_{n+j} - \frac{\alpha}{\prod_{j=1}^{N-1} w_{n+j}} \tag{23}$$

provides a 2-integral of the modified generalized DTKQ equation (22).

Proof. A direct calculation shows that

$$(\mathcal{S}^2 - 1)\, \gamma_n[w] = \prod_{j=2}^{N} w_{n+j}\, \Delta\mathcal{E}_n[w] = E_n[w] \prod_{j=2}^{N} w_{n+j},$$

where

$$E_n[w] := w_{n+N+2}w_{n+N+1} - w_{n+1}w_n + \frac{\alpha}{\prod_{j=2}^{N} w_{n+j}} \left(\frac{1}{\prod_{k=1}^{N-1} w_{n+k}} - \frac{1}{\prod_{k=3}^{N+1} w_{n+k}} \right) \tag{24}$$

denotes the lift of (19) to $N+2$ dimensions obtained from the first substitution in (21). Hence the quantity $\gamma_n[w]$ gives a 2-integral of both (22) and (24), where in the first case (23) can be rewritten as a function of $w_n, \ldots, w_{n+N-1}$ using (22). □

If we substitute for w_n with the ratio of tau functions given in (21), then we see that (23) is equivalent to the bilinear equation (5). Thus the following result is an immediate consequence of the preceding lemma, and proves the first part of theorem 1.1.

Corollary 3.2. *The quantity given in terms of τ_n and shifts by*

$$\gamma_n = \frac{\tau_{n+N+2}\tau_n - \alpha\tau_{n+N}\tau_{n+2}}{\tau_{n+N+1}\tau_{n+1}} \tag{25}$$

is a 2-integral of the multilinear equation (10).

Remark 3.3. In general, γ_n can be considered as a 2-integral for (22), (24) or (10), but it can only be considered as a 2-integral for (2) or (19) when N is odd, because only then can it be written purely in terms of u_n and shifts.

As the above remark indicates, there are considerable differences between the cases of even/odd N, so henceforth we consider these two cases separately.

3.1. The case of even N

In the case of even N, we introduce another dependent variable v_n, which is defined by

$$\pi_2: \quad v_n = \prod_{j=0}^{\frac{N-2}{2}} u_{n+2j} = \prod_{k=0}^{N-1} w_{n+k} = \frac{\tau_n \tau_{n+N+1}}{\tau_{n+1}\tau_{n+N}}. \tag{26}$$

It turns out that v_n satisfies a travelling wave reduction of Hirota's lattice KdV equation,

$$V_{k+1,l} - V_{k,l+1} = \alpha \left(\frac{1}{V_{k,l}} - \frac{1}{V_{k+1,l+1}} \right), \tag{27}$$

obtained by imposing the periodicity condition

$$V_{k+N,l+1} = V_{k,l} \Longrightarrow V_{k,l} = v_n, \qquad n = lN - k,$$

which is called the $(N,1)$-reduction of (27). By taking the reciprocal of the dependent variable and rescaling, this is equivalent to the $(N,-1)$-reduction considered in [20].

Proposition 3.4. *If N is even and u_n is a solution of (2), then $v_n = u_n u_{n+2} \cdots u_{n+N-2}$ is a solution of*

$$v_{n+N+1} - v_n = \alpha \left(\frac{1}{v_{n+N}} - \frac{1}{v_{n+1}} \right), \tag{28}$$

which is the $(N,1)$ periodic reduction of the lattice KdV equation (27).

Proof. For N even, (5) is a special case of the second bilinear equation in the statement of proposition 4.2 in [22]. Upon solving (5) for γ_n and noting that (since γ_n has period 2) $(\mathcal{S}^N - 1)\gamma_n = 0$, the equation (28) follows immediately by taking v_n to be the ratio of tau functions given in (26). □

The result of proposition 4.2 in [22] shows that the (L,M) periodic reduction of (27) is actually associated with two different bilinear equations, so applying this result to the case $(L,M) = (N,1)$ considered here, for even N we immediately obtain the second relation (6) in theorem 1.1, in the following form.

Corollary 3.5. *For even N, the quantity given in terms of τ_n and shifts by*

$$K = \frac{\tau_n \tau_{n+2N+1} + \alpha \tau_{n+1}\tau_{n+2N}}{\tau_{n+N}\tau_{n+N+1}} \tag{29}$$

is a first integral of (10), which via (4) produces a first integral of (2) or (19) defined by

$$K[u] := \left(\prod_{j=0}^{N-1} u_{n+2j} + \alpha \right) u_{n+N-1}^{\frac{N}{2}} \prod_{k=1}^{N-2} (u_{n+k} u_{n+2N-2-k})^{\lfloor \frac{k+1}{2} \rfloor}. \tag{30}$$

Remark 3.6. Observe that, as it is written, the expression (30) is a polynomial in u_{n+j} for $j = 0, \ldots, 2N-2$, which can be rewritten as a rational function of any N adjacent iterates by using the recurrence (2) to eliminate higher shifts. The corresponding first integral (29) in terms of tau functions satisfying equation (10) is regarded in a similar way. By setting $u_n = w_n w_{n+1}$, $K[u]$ also provides a first integral for (22) or (24).

3.1.1. U-systems and other Liouville integrable maps. We can now discuss Liouville integrability of various maps associated with (2) for N even.

Using the presymplectic form which comes from the cluster algebra associated with the bilinear recurrence (5) (see [8] and references), the variables w_n defined in terms of tau functions by (21) provide a set of symplectic coordinates. The corresponding symplectic map in dimension N is defined by

$$w_{n+N}w_n \prod_{j=1}^{N-1} w_{n+j}^2 = \gamma_n \prod_{j=1}^{N-1} w_{n+j} + \alpha, \tag{31}$$

which is an example of a U-system [21], and (up to overall scaling) the nondegenerate Poisson bracket preserved by (31) is the one given by equation (3.21) in lemma 3.13 of [20], that is

$$\{w_m, w_n\} = (-1)^{m-n+1} w_m w_n \qquad \text{for} \qquad 0 \leqslant m < n \leqslant N-1. \tag{32}$$

(For a specific example of this bracket, see equation (81) in section 4 below.)

As is shown in [20], theorem 3.14, the nondegenerate bracket for (31) lifts to a bracket for (28) in dimension $N+1$, which has rank N and one Casimir. There is another Poisson bracket for (28), coming from a discrete Lagrangian formulation, and the two different brackets are compatible with one another.

There is another cluster algebra that arises, namely the one associated with the bilinear recurrence (6). The variables v_n defined by (26) provide symplectic coordinates for the corresponding U-system, which is the map in dimension N given by

$$\prod_{j=0}^{N} v_{n+j} = -\alpha \prod_{k=1}^{N-1} v_{n+k} + K, \tag{33}$$

preserving a nondegenerate bracket that has the same form in these coordinates as the one for w_n above, i.e.

$$\{v_m, v_n\} = (-1)^{m-n+1} v_m v_n \qquad \text{for} \qquad 0 \leqslant m < n \leqslant N-1. \tag{34}$$

This lifts to another Poisson bracket for the reduced KdV map (28) in dimension $N+1$, which also has rank N and one Casimir; in fact, it is a linear combination of the two compatible brackets found in [20], so they all belong to the same Poisson pencil, consisting of the brackets

$$\lambda_1\{\,,\,\}_1 + \lambda_2\{\,,\,\}_2, \tag{35}$$

for arbitrary $(\lambda_1 : \lambda_2)$, where $\{\,,\,\}_{1,2}$ are any two fixed independent brackets in this family.

According to corollary 2.2 in [22], each of the bilinear equations (5) and (6) has a matrix Lax representation (2×2 and $N \times N$, respectively), and this yields Lax pairs for the corresponding U-systems (31) and (33). However, it is more convenient to make use of the 2×2 Lax pair for the discrete KdV reduction, as obtained in [20]. This produces a complete set of first integrals for the map (28), which Poisson commute with respect to any bracket in the pencil (35). Hence the Liouville integrability of the map (19) follows from that of (28), as described by the following result.

Theorem 3.7. *For N even, let φ and χ denote the birational maps in dimension $N+1$ defined by (19) and (28) respectively, and let ψ denote the lift of (31) to $N+2$ dimensions given by*

$$\psi: \qquad (w_0, w_1, \ldots, w_{N-1}, \gamma_0, \gamma_1) \mapsto (w_1, w_2, \ldots, w_N, \gamma_1, \gamma_0).$$

Then with π_j for $j=1,2$ defined by (21) and (26), each of the birational maps ψ,φ,χ preserves a Poisson bracket such that the diagram

$$
\begin{array}{ccc}
\mathbb{C}^{N+2} & \xrightarrow{\psi} & \mathbb{C}^{N+2} \\
\downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_1} \\
\mathbb{C}^{N+1} & \xrightarrow{\varphi} & \mathbb{C}^{N+1} \\
\downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_2} \\
\mathbb{C}^{N+1} & \xrightarrow{\chi} & \mathbb{C}^{N+1}
\end{array}
\tag{36}
$$

of rational Poisson maps is commutative. In particular, the bracket preserved by φ is of rank N, being specified by

$$\{u_0,u_1\}=u_0u_1,\quad \{u_0,u_{N-1}\}=-\frac{\alpha}{\prod_{j=1}^{N-2}u_j},\quad \{u_0,u_N\}=-u_0u_N+\frac{\alpha^2}{\left(\prod_{j=1}^{N-1}u_j\right)^2},\tag{37}$$

with $\{u_0,u_j\}=0$ for $j=2,\ldots,N-2$. Moreover, each of the three horizontal maps is integrable in the Liouville sense.

Proof. As already mentioned, theorem 3.14 in [20] says that the bracket (32) lifts to a bracket for (28). This can be made more explicit by first extending (31) to the map ψ in dimension $N+2$, which preserves a Poisson bracket of rank N, defined by extending (32) to include the extra coordinates γ_0,γ_1 as a pair of Casimirs. Then the formula $v_n=w_nw_{n+1}\cdots w_{n+N-1}$ from (26) defines a Poisson map $\pi:\mathbb{C}^{N+2}\to\mathbb{C}^{N+1}$, with a corresponding Poisson bracket for the variables v_n, $n=0,\ldots,N$, denoted by $\{\,,\,\}_2$ say (as in [20]), so that $\pi^*\{v_m,v_n\}_2=\{\pi^*v_m,\pi^*v_n\}$. This rational Poisson map factors as $\pi=\pi_2\circ\pi_1$, where π_1 is defined by $u_n=w_nw_{n+1}$, as in (21), and π_2 by $v_n=u_nu_{n+2}\cdots u_{n+N-2}$. A direct calculation shows that the pushforward of the bracket (32) by π_1 yields (37), and the product $\gamma_0\gamma_1$ pushes forward to a Casimir of the latter bracket, which by construction is preserved by (19). By theorem 4.1 in [20], the map χ is Liouville integrable: it has $D+1$ independent first integrals $I_0,I_1,\ldots,I_D$ with $D=\frac{N}{2}$, coming from the trace of a monodromy matrix, which commute with respect to the bracket $\{\,,\,\}_2$, with I_0 being a Casimir. The pullbacks of these integrals, $\pi_2^*I_j$, provide $D+1$ commuting integrals for the map φ; and pulling back once more gives the same number of integrals π^*I_j for ψ, including only one Casimir $\pi^*I_0=-(\gamma_0\gamma_1)^{N/2}$ (see remark 3.15 in [20]), so taking another independent Casimir, i.e. $\gamma_0+\gamma_1$, gives a full set of commuting first integrals. Hence the Liouville integrability of ψ and φ is proved. □

Remark 3.8. For a fixed value of K, the second U-system (33) can also be regarded as being Liouville integrable with respect to the nondegenerate bracket (34). As already mentioned, this lifts to another independent bracket in the pencil (35), $\{\,,\,\}_1$ say, for the map χ defined by (28).

3.2. The case of odd N

As is mentioned in remark 3.3, in the case that N is odd, γ_n can be considered as a 2-integral for (2), and via the substitution (4) the first bilinear equation (5) yields

$$\prod_{j=0}^{N-1} u_{n+j} = \gamma_n \prod_{k=0}^{\frac{N-3}{2}} u_{n+2k+1} + \alpha, \tag{38}$$

which is the U-system associated with this bilinear recurrence. The latter defines a symplectic map in dimension $N-1$, whose corresponding nondegenerate Poisson bracket is specified by equation (3.22) in lemma 3.13 of [20], namely

$$\{u_n, u_{n+1}\} = u_n u_{n+1}, \qquad \{u_n, u_{n+j}\} = 0 \quad \text{for} \quad 2 \leqslant j \leqslant N-2. \tag{39}$$

(For a particular example, see (91) below.) The U-system (38) can naturally be viewed as a reduction of the Hirota–Miwa equation, and a Lax pair and first integrals can be obtained immediately by applying corollary 2.2 in [22]. The case $N = 5$ is presented explicitly in section 5.

At this stage, for the odd case we can already state a partial analogue of theorem 3.7.

Theorem 3.9. *For N odd, let φ denote the birational map in dimension $N+1$ defined by (19) and let ψ denote the birational lift of (38) to $N+1$ dimensions given by*

$$\psi: \qquad (u_0, u_1, \ldots, u_{N-2}, \gamma_0, \gamma_1) \mapsto (u_1, u_2, \ldots, u_{N-1}, \gamma_1, \gamma_0).$$

Then there is a birational map $\hat{\pi}_1$ such that the diagram

$$\begin{array}{ccc} \mathbb{C}^{N+1} & \xrightarrow{\psi} & \mathbb{C}^{N+1} \\ \Big\downarrow \hat{\pi}_1 & & \Big\downarrow \hat{\pi}_1 \\ \mathbb{C}^{N+1} & \xrightarrow{\varphi} & \mathbb{C}^{N+1} \end{array} \tag{40}$$

of birational Poisson maps is commutative. Moreover, the Poisson bracket preserved by φ is of rank $N-1$, with non-zero brackets given by (37).

Proof. The bracket (39) for the U-system (38) in dimension $N-1$ extends to a bracket for ψ by including the additional coordinates γ_0, γ_1 as two Casimirs. Taking $n = 0, 1$ in (38) defines u_{N-1} and u_N as rational functions of $u_0, u_1, \ldots, u_{N-2}, \gamma_0, \gamma_1$, and conversely gives γ_0 and γ_1 as rational functions of $u_0, u_1, \ldots, u_N$, so this specifies a birational transformation $\hat{\pi}_1$ between these two sets of coordinates in dimension $N+1$. A direct calculation shows that the bracket preserved by φ takes the same form (37) as for N even, but in this case there are two independent Casimirs given by γ_0, γ_1. □

When N is odd, the fact that the coefficient γ_n in (38) is 2-periodic means that neither this U-system, nor the corresponding bilinear equation (5), can be related to a reduction of discrete KdV, which would require the period of γ_n to divide N (see proposition 3.7 in [20] and proposition 4.2 in [22]). However, it turns out that there is a connection with reductions of another integrable two-dimensional lattice equation, namely a discrete form of the Toda lattice. This connection arises from the fact that τ_n satisfies the other bilinear equation (7), which is the content of the following statement.

Proposition 3.10. *For odd N, the quantity given in terms of τ_n and shifts by*

$$\bar{K} = \frac{\tau_n \tau_{n+2N+2} - \alpha^2 \tau_{n+2} \tau_{n+2N}}{(\tau_{n+N+1})^2} \tag{41}$$

is a first integral of (10), which via (4) produces a first integral of (2) or (19) defined by

$$\bar{K}[u] := \left(\prod_{j=0}^{N-1} s_{n+2j} - \alpha^2\right) \left(s_{n+N-1}\right)^{\frac{N-1}{2}} \prod_{k=1}^{\frac{N-3}{2}} \left(s_{n+2k}s_{n+2N-2k-2}\right)^k, \tag{42}$$

where

$$s_n = u_n u_{n+1}. \tag{43}$$

Proof. Taking γ_n as given by (25) and applying the total difference operator to $\bar{K}$ yields the identity

$$\Delta\bar{K} = \frac{\tau_{n+2N+2}\tau_{n+1}}{\tau_{n+N+2}\tau_{n+N+1}} \left(\mathcal{S}^{N+1} - 1\right)\gamma_n - \alpha \frac{\tau_{n+2N+1}\tau_{n+2}}{\tau_{n+N+2}\tau_{n+N+1}} \left(\mathcal{S}^{N-1} - 1\right)\gamma_{n+1},$$

so that the right-hand side vanishes because $(\mathcal{S}^2 - 1)\gamma_n = 0$ and N is odd. This completes the proof of the statement, and also the proof of theorem 1.1. □

Remark 3.11. The preceding result means that, for each odd N, the Somos-$(N + 2)$ recurrence (5) is related to (7), which corresponds to two copies of a Somos-$(N + 1)$ recurrence with the iterates interlaced, since the iterates with even/odd indices decouple from each other. For the particular case $N = 3$, the relation between Somos-5 (with autonomous coefficients) and two copies of Somos-4 was shown in proposition 2.8 of [17], and interpreted as a Bäcklund transformation in [2].

3.2.1. Lax pair associated with a discrete Toda equation. The five-point lattice equation

$$\frac{V_{k,l}}{V_{k+1,l}} - \frac{V_{k-1,l}}{V_{k,l}} + \alpha^2 \left(\frac{V_{k+1,l-1}}{V_{k,l}} - \frac{V_{k,l}}{V_{k-1,l+1}}\right) = 0 \tag{44}$$

is a discrete time Toda equation [3, 16]. The $(1, -P)$ periodic reduction of (44) corresponds to imposing the condition

$$V_{k+1,l-P} = V_{k,l} \qquad \Longrightarrow \qquad V_{k,l} = v_n, \qquad n = kP + l, \tag{45}$$

which leads to the ordinary difference equation

$$\frac{v_n}{v_{n+P}} - \frac{v_{n-P}}{v_n} + \alpha^2 \left(\frac{v_{n+P-1}}{v_n} - \frac{v_n}{v_{n+1-P}}\right) = 0. \tag{46}$$

Upon introducing a tau function T_n such that

$$v_n = \frac{T_n}{T_{n+1}}, \tag{47}$$

we can immediately apply proposition 3.1 in [22], where the $(Q, -P)$ periodic reduction of (44) was considered; here we are only concerned with the case $Q = 1$, which gives the following result.

Proposition 3.12. *If v_n given by (47) satisfies (46), then there is a first integral $\bar{K}$ such that T_n satisfies the bilinear equation*

$$T_{n+2P}T_n = \alpha^2\, T_{n+2P-1}T_{n+1} + \bar{K}\, T_{n+P}^2, \tag{48}$$

and conversely every solution of (48) provides a solution of (46).

We now present a Lax representation for (44), which originates from a map associated with a discretization of the Toda lattice in [37, 38], and subsequently provides a Lax representation for (46).

Proposition 3.13. *The discrete Toda equation (44) is equivalent to the the discrete zero curvature equation*

$$L(\Pi_{k,l}, V_{k,l}, \eta)M(V_{k+1,l}, V_{k,l+1}, \eta) = M(V_{k+1,l-1}, V_{k,l}, \eta)L(\Pi_{k+1,l}, V_{k+1,l}, \eta), \tag{49}$$

where η is a spectral parameter, and

$$L(p, v, \eta) = \begin{pmatrix} p+\eta & v \\ -v^{-1} & 0 \end{pmatrix}, \; M(u, v, \eta) = \begin{pmatrix} 1-\alpha^2 uv^{-1} - \alpha\eta & -\alpha u \\ \alpha v^{-1} & 1 \end{pmatrix}. \tag{50}$$

Proof. The equation (49) implies that both

$$\Pi_{k+1,l} = \frac{\alpha V_{k+1,l}}{V_{k,l+1}} + \frac{1}{\alpha}\left(\frac{V_{k,l}}{V_{k+1,l}} - 1\right) \tag{51}$$

and

$$\Pi_{k,l} = \frac{\alpha V_{k+1,l-1}}{V_{k,l}} + \frac{1}{\alpha}\left(\frac{V_{k,l}}{V_{k+1,l}} - 1\right), \tag{52}$$

and these two relations together imply the discrete Toda equation (44). □

By imposing the periodicity condition (45), the Lax matrices in (49) reduce to

$$L_n := L(p_n, v_n, \eta), \qquad M_n := M(v_n, v_{n-P+1}, \eta),$$

where from (51) we have

$$p_n = \frac{\alpha v_n}{v_{n-P+1}} + \frac{1}{\alpha}\left(\frac{v_{n-P}}{v_n} - 1\right), \tag{53}$$

and the zero curvature equation reduces to

$$L_n M_{n+P} = M_{n+P-1} L_{n+P}. \tag{54}$$

With this notation, we can introduce the monodromy matrix as

$$\mathcal{M}_n := (1-\alpha\eta)\, M_n^{-1} L_{n-P+1} \ldots L_{n-1} L_n. \tag{55}$$

This satisfies a discrete Lax equation, which follows from the identity (54).

Corollary 3.14. *The $(1, -P)$ periodic reduction (46) obtained from the discrete Toda equation is equivalent to the discrete Lax equation*

$$\mathcal{M}_n L_{n+1} = L_{n+1} \mathcal{M}_{n+1}. \tag{56}$$

The equation (56) means that the shift $n \to n+1$ is an isospectral evolution for the monodromy matrix (55). From (50), the determinant is $\det \mathcal{M}_n = 1 - \alpha\eta$, while $\operatorname{tr} \mathcal{M}_n$ is a monic polynomial of degree P in η whose coefficients provide first integrals of (46).

Upon comparing (41) with (48), we see that for odd N the solutions of (2) correspond to two interlaced sets of tau functions,

$$T_n^{\rm even} = \tau_{2n}, \qquad T_n^{\rm odd} = \tau_{2n+1}, \tag{57}$$

such that for $P = \frac{N+1}{2}$ there are two sets of solutions of (46) given by

$$v_n^{\rm even} = \frac{T_n^{\rm even}}{T_{n+1}^{\rm even}}, \qquad v_n^{\rm odd} = \frac{T_n^{\rm odd}}{T_{n+1}^{\rm odd}}. \tag{58}$$

Then from (43) we may write the even/odd index quantities s_j as

$$s_{2n} = \hat{s}_n^{\rm even} := \frac{v_n^{\rm even}}{v_{n+1}^{\rm even}}, \qquad s_{2n+1} = \hat{s}_n^{\rm odd} := \frac{v_n^{\rm odd}}{v_{n+1}^{\rm odd}}.$$

The equation (46) for the reduced Toda map is invariant under the scaling $v_n \to \lambda v_n$, for any non-zero λ, as are the quantities $\hat{s}_n^{\rm even/odd}$. Hence in this case (46) becomes an equation of order $2P-1=N$ for each of the latter quantities, that is

$$\hat{s}_n \cdots \hat{s}_{n+P-1} - \hat{s}_{n-P} \cdots \hat{s}_{n-1} + \alpha^2 \left(\frac{1}{\hat{s}_n \cdots \hat{s}_{n+P-2}} - \frac{1}{\hat{s}_{n-P+1} \cdots \hat{s}_{n-1}} \right) = 0. \tag{59}$$

Each iteration of (19) intertwines two sets of solutions of the above equation.

Proposition 3.15. *For even/odd n taken separately, the formula (42) defines a U-system in dimension $N-1=2P-2$ with coordinates $\hat{s}_0, \hat{s}_1, \ldots, \hat{s}_{N-2}$, preserving a nondegenerate Poisson bracket given by*

$$\{\hat{s}_n, \hat{s}_{n+1}\} = \hat{s}_n \hat{s}_{n+1}, \quad \{\hat{s}_n, \hat{s}_{n+P-1}\} = -2\hat{s}_n \hat{s}_{n+P-1}, \quad \{\hat{s}_n, \hat{s}_{n+P}\} = 2\hat{s}_n \hat{s}_{n+P}, \tag{60}$$

with all other brackets $\{\hat{s}_n, \hat{s}_{n+j}\}$ for $0 \leqslant j \leqslant N-2$ being zero. This lifts to a bracket of rank $N-1$ in dimension N that is preserved by (59), with $\bar{K}$ being a Casimir, where the extra bracket is

$$\{\hat{s}_n, \hat{s}_{n+N-1}\} = -\frac{\alpha^2}{\hat{s}_{n+1} \cdots \hat{s}_{n+N-2}}. \tag{61}$$

Proof. According to theorem 4.6 in [8], the bilinear equation (48) preserves a log-canonical presymplectic form, which reduces to a symplectic structure for the U-system

$$\hat{s}_{n+P-1}^P \prod_{j=0}^{P-2} (\hat{s}_{n+j} \hat{s}_{n+N-1-j})^{j+1} = \alpha^2 \hat{s}_{n+P-1}^{P-1} \prod_{k=1}^{P-2} (\hat{s}_{n+k} \hat{s}_{n+N-1-k})^k + \bar{K} \tag{62}$$

in dimension $N-1$. The symplectic structure is equivalent to a nondegenerate log-canonical Poisson bracket, of the form $\{\hat{s}_m, \hat{s}_n\} = c_{mn} \hat{s}_m \hat{s}_n$, where (c_{mn}) is a constant skew-symmetric Toeplitz matrix. Taking the Poisson bracket of $\hat{s}_{n+j}$ with both sides of (62) for $j = 1, \ldots, P-2$ produces a system of $2P-4$ homogeneous linear equations for the entries of the first row of this matrix, which is readily solved to yield (60), up to overall scaling by an arbitrary non-zero constant. Upon lifting this bracket to dimension N and requiring that $\bar{K}$ be a Casimir, taking the bracket of $\hat{s}_n$ with both sides of (62) leads to the above expression for $\{\hat{s}_n, \hat{s}_{n+N-1}\}$. □

In the case of odd N, a partial analogue of the bottom part of the diagram (36) arises by taking two iterations of the map φ in (40), that is

$$
\begin{array}{ccc}
\mathbb{C}^{N+1} & \xrightarrow{\varphi^2} & \mathbb{C}^{N+1} \\
\Big\downarrow \hat{\pi}_2 & & \Big\downarrow \hat{\pi}_2 \\
\mathbb{C}^{N} & \xrightarrow{\hat{\chi}} & \mathbb{C}^{N}
\end{array}
\tag{63}
$$

where the vertical map $\hat{\pi}_2$ is defined by using (43) either for even values, or for odd values of n only, and

$$\hat{\chi}: \qquad (\hat{s}_0, \hat{s}_1, \ldots, \hat{s}_{N-1}) \mapsto (\hat{s}_1, \hat{s}_2, \ldots, \hat{s}_N)$$

is defined by (59) with $P = (N+1)/2$. The diagonal entries of the monodromy matrix $\mathcal{M}_n$ in (55) are functions of the ratios $\hat{s}_j = v_j/v_{j+1}$ (although the off-diagonal entries are not), so that $\operatorname{tr}\mathcal{M}_n$ directly provides first integrals for (59). Moreover, from the above diagram, the integrals provided by $\operatorname{tr}\mathcal{M}_n$ can be pulled back by $\hat{\pi}_2$ to give integrals for φ^2.

In fact, we can say rather more: the two sets of integrals obtained by taking even/odd n in (43) coincide, so they pull back to integrals for φ. The reason is that the map φ corresponds to a Bäcklund transformation for the discrete Toda reduction, in the sense of [25]. In order to show this, it is necessary to consider the two sets of reduced Lax matrices

$$L_n^{\mathrm{even/odd}} := L(p_n^{\mathrm{even/odd}}, v_n^{\mathrm{even/odd}}, \eta), \qquad M_n^{\mathrm{even/odd}} := M(v_n^{\mathrm{even/odd}}, v_{n-P+1}^{\mathrm{even/odd}}, \eta)$$

for even/odd n separately, and introduce a gauge transformation matrix defined by

$$G_n := \begin{pmatrix} \eta + u_{2n+1} - \kappa & v_n^{\mathrm{odd}} \\ -(v_{n+1}^{\mathrm{even}})^{-1} & -1 \end{pmatrix}, \qquad \kappa = \alpha^{-1}(1 - \gamma_0\gamma_1). \tag{64}$$

Lemma 3.16. *The gauge matrix (64) intertwines the even/odd reduced Lax matrices as follows:*

$$L_n^{\mathrm{even}} G_n = G_{n-1} L_n^{\mathrm{odd}}, \tag{65}$$

$$M_n^{\mathrm{even}} G_n = G_{n-P} M_n^{\mathrm{odd}}. \tag{66}$$

Proof. In order to prove these intertwining relations, it is helpful to note that $\det L_n^{\mathrm{even/odd}} = 1$ and $\det M_n^{\mathrm{even/odd}} = 1 - \alpha\eta$, while $\det G_n = \kappa - \eta$ follows from the fact that $u_{2n+1} = v_n^{\mathrm{odd}}/v_{n+1}^{\mathrm{even}}$, which is a consequence of the tau function formulae (57) and (58). Thus, in both (65) and (66), the determinants of the left/right-hand sides agree, and henceforth it is sufficient to check only three out of four matrix entries in each equation. The $(2, 2)$ entries on each side of (65) are identical, and the same is true for (66), so we need only consider the $(1, 2)$ and $(2, 1)$ entries. Taking the difference of the $(2, 1)$ entries on each side of (65) requires that

$$-(v_n^{\mathrm{even}})^{-1}\Big(u_{2n+1} - \kappa - p_n^{\mathrm{odd}}\Big) - (v_n^{\mathrm{odd}})^{-1} = 0 \tag{67}$$

should hold. By using tau functions it is clear that $u_{2n} = v_n^{\mathrm{even}}/v_n^{\mathrm{odd}}$, so that the equality (67) boils down to the identity

$$p_n^{\mathrm{odd}} = u_{2n} + u_{2n+1} - \kappa. \tag{68}$$

To prove the latter, we successively use ratios of tau functions to go between different variables. On the one hand, we have

$$\gamma_0\gamma_1 = \left(v^{\mathrm{even}}_{n-P+1} - \alpha v^{\mathrm{odd}}_n\right)\left(\frac{1}{v^{\mathrm{even}}_{n+1}} - \frac{\alpha}{v^{\mathrm{odd}}_{n-P+1}}\right), \tag{69}$$

which follows from (5), while on the other hand

$$\frac{v^{\mathrm{odd}}_{n-P}}{v^{\mathrm{odd}}_n} - \frac{v^{\mathrm{even}}_{n-P+1}}{v^{\mathrm{even}}_{n+1}} = u_{2n-N+1}\cdots u_{2n}(u_{2n-N} - u_{2n+1}) = \alpha(u_{2n} - u_{2n-N+1})$$

by (19). Combining the above with the fact that $u_{2n-N+1} = v^{\mathrm{even}}_{n-P+1}/v^{\mathrm{odd}}_{n-P+1}$, and then comparing (51) with (69), leads to the alternative formula (68) for p^{odd}_n. Similarly, the equality of the $(1,2)$ entries on each side of (65) boils down to an equivalent formula for p^{even}_n, that is

$$p^{\mathrm{even}}_n = u_{2n-1} + u_{2n} - \kappa. \tag{70}$$

The equality of the $(2,1)$ entries on each side of (66) is a direct consequence of the identity (69), while to verify the $(1,2)$ entries it is sufficient to note that

$$\gamma_0\gamma_1 = \left(v^{\mathrm{odd}}_{n-P} - \alpha v^{\mathrm{even}}_n\right)\left(\frac{1}{v^{\mathrm{odd}}_n} - \frac{\alpha}{v^{\mathrm{even}}_{n-P+1}}\right). \tag{71}$$

Both (69) and (71) are proved in the same way, by expressing the terms on their right-hand sides in terms of tau functions, and using (5) together with the 2-periodicity of γ_n. □

For the case of odd N, we can now state a closer analogue of the bottom part of the diagram (36).

Theorem 3.17. *For odd $N = 2P - 1$, the map φ defined by (19) corresponds to a Bäcklund transformation (BT) for the reduced Toda equation (46). The BT is a 2-valued Poisson correspondence between solutions of (59), which preserves all the first integrals obtained from the trace of the monodromy matrix (55), and there is a commutative diagram*

$$\begin{array}{ccc} \mathbb{C}^{N+1} & \xrightarrow{\varphi} & \mathbb{C}^{N+1} \\ \big\downarrow \hat{\pi}_2 & & \big\downarrow \hat{\pi}_2 \\ \mathbb{C}^N & \xrightarrow{\chi_{BT}} & \mathbb{C}^N \end{array} \tag{72}$$

where χ_{BT} denotes one of the branches of the correspondence.

Proof. To begin with, suppose that u_n is a solution of (19), with τ_n being a corresponding tau function, so that $u_n = \tau_{n+3}\tau_n/(\tau_{n+2}\tau_{n+1})$. Then for the gauge matrix G_n given by (64), repeated application of (65) shows that

$$\begin{aligned} (M^{\mathrm{even}}_n)^{-1}L^{\mathrm{even}}_{n-P+1}\ldots L^{\mathrm{even}}_{n-1}L^{\mathrm{even}}_n G_n &= (M^{\mathrm{even}}_n)^{-1}L^{\mathrm{even}}_{n-P+1}\ldots L^{\mathrm{even}}_{n-1}G_{n-1}L^{\mathrm{odd}}_n \\ &= \cdots \\ &= (M^{\mathrm{even}}_n)^{-1}G_{n-P}L^{\mathrm{odd}}_{n-P+1}\ldots L^{\mathrm{odd}}_{n-1}L^{\mathrm{odd}}_n, \end{aligned}$$

and then by applying (66) it follows that the monodromy matrices for the even/odd index solutions of (46) are related by

$$\mathcal{M}_n^{\mathrm{even}} G_n = G_n \mathcal{M}_n^{\mathrm{odd}}, \tag{73}$$

proving the claim that the first integrals for these two sets of solutions coincide.

Now suppose instead that an adjacent set of $2P$ variables v_j^{even} is given, corresponding to a set of initial data for (46), and define a transformation to another set v_j^{odd}, say with $j = 0, \ldots, 2P-1$ in each case, by the gauge transformation of monodromy matrices (73) for $n = 2P-1$. With this choice of indices, v_{2P}^{even} appearing in G_{2P-1} should be specified in terms of v_j^{even} for $0 \leqslant j \leqslant 2P-1$ according to (46), and the quantities u_k with even/odd indices can be defined as the ratios

$$u_{2j} = \frac{v_j^{\mathrm{even}}}{v_j^{\mathrm{odd}}}, \qquad u_{2j+1} = \frac{v_j^{\mathrm{odd}}}{v_{j+1}^{\mathrm{even}}}, \qquad j = 0, \ldots, 2P-1.$$

Imposing the condition (65) for $n = P, \ldots, 2P-1$ implies that v_j^{odd} for $j = 0, \ldots, 2P-1$ are determined completely by the initial v_j^{even}, together with u_{4P-1} and the Bäcklund parameter κ appearing in each of the G_n. The requirement that (66) should also hold then ensures that (73) is satisfied for $n = 2P-1$, that is

$$\mathcal{M}_{2P-1}^{\mathrm{even}}(\eta) G_{2P-1}(\eta) = G_{2P-1}(\eta) \mathcal{M}_{2P-1}^{\mathrm{odd}}(\eta) \tag{74}$$

for all η, so the monodromy matrices $\mathcal{M}_{2P-1}^{\mathrm{even/odd}}$ have the same spectrum, and this requirement imposes an additional relation on u_{4P-1}. In that case, the correspondence between $v_j^{\mathrm{even/odd}}$ is fixed up to a choice of square root, which can be seen directly by applying the notion of spectrality from [25]: upon noting that, for $\eta = \kappa$, a vector in the kernel of the transposed gauge matrix is given by

$$\mathbf{v} = \begin{pmatrix} (v_{2P}^{\mathrm{even}})^{-1} \\ u_{4P-1} \end{pmatrix} \Longrightarrow G_{2P-1}^T(\kappa)\mathbf{v} = \mathbf{0},$$

it follows that

$$\mathbf{v}^T \mathcal{M}_{2P-1}^{\mathrm{even}}(\kappa) G_{2P-1}(\kappa) = \mathbf{0}^T \Longrightarrow \mathbf{v}^T \mathcal{M}_{2P-1}^{\mathrm{even}}(\kappa) = \mu\, \mathbf{v}^T$$

for some μ, so $\mathbf{v}^T$ is a left eigenvector of the monodromy matrix for this value of η. Thus (κ, μ) is a point on the spectral curve

$$\mu^2 - \mathrm{tr}\, \mathcal{M}_{2P-1}^{\mathrm{even}}(\kappa)\, \mu + 1 - \alpha\kappa = 0.$$

So, for a fixed choice of the initial data and the parameter κ, there are two possible values of μ, and by writing

$$\mathcal{M}_{2P-1}^{\mathrm{even}}(\kappa) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

this leads to two possible values for

$$u_{4P-1} = \frac{\mu - a}{v_{2P}^{\rm even} c} = \frac{b}{v_{2P}^{\rm even}(\mu - d)}.$$

Hence the BT defined in this way is a 2-valued correspondence between $v_j^{\rm even/odd}$, and also provides a 2-valued correspondence between the quantities $\hat{s}_j^{\rm even/odd}$ for $j = 0, \ldots, N-1$. The iteration of the map φ defined by (19) corresponds to one particular branch of the correspondence, $\chi_{\rm BT}$ say, with the other branch corresponding to φ^{-1}.

It remains to verify that this is a Poisson correspondence, preserving the Poisson structure for the coordinates $\hat{s}_j$ in proposition 3.15. To see this, note that the branch $\chi_{\rm BT}$ is associated with the bilinear equation (7), which arises from a cluster algebra, and takes the form of two copies of (48) for even/odd indices. By applying theorem 4.6 in [8], the corresponding presymplectic form reduces to a symplectic form $\hat{\omega}$ in dimension $2N-2$ for the combined U-system coordinates $\hat{s}_j^{\rm even/odd}$ for $j = 0, \ldots, N-2$, being a sum of two identical symplectic forms which are switched under the action of the map φ, that is

$$\hat{\omega} = \hat{\omega}^{\rm even} + \hat{\omega}^{\rm odd}, \qquad \varphi^* \hat{\omega}^{\rm even/odd} = \hat{\omega}^{\rm odd/even}.$$

Now $\chi_{\rm BT}$ preserves all the first integrals of (59), including $\bar{K}$, hence it must also preserve the lifted bracket in dimension N given by (60) and (61). □

The Liouville integrability of the map $\hat{\chi}$ in (63), defined by (59), is worthy of a more detailed treatment elsewhere, as is the connection of the BT with that for the even Mumford systems in [26]. In section 5 below we merely present the details for the particular case $N = 5$.

4. An even example: $N = 4$

For $N = 4$ the equation (2) becomes

$$(u_n + u_{n+1} + u_{n+2} + u_{n+3} + u_{n+4} + \beta)u_{n+1}u_{n+2}u_{n+3} = \alpha, \tag{75}$$

and its lift (19) is the map φ in dimension 5 defined by

$$\varphi: \qquad u_{n+5} - u_n + \frac{\alpha}{u_{n+2}u_{n+3}}\left(\frac{1}{u_{n+1}} - \frac{1}{u_{n+4}}\right) = 0. \tag{76}$$

If we set $u_n = \frac{\tau_{n+3}\tau_n}{\tau_{n+2}\tau_{n+1}}$, then the tau function τ_n satisfies (10), which in this case is of degree 6, being given by

$$\begin{aligned}\tau_7\tau_4^2\tau_3\tau_2\tau_1 = {} & \alpha\tau_5\tau_4^2\tau_3^2\tau_2 - \beta\tau_6\tau_5\tau_4\tau_3\tau_2\tau_1 - \tau_6\tau_5\tau_4\tau_3^2\tau_0 - \tau_6\tau_5\tau_4^2\tau_1^2 \\ & - \tau_6\tau_5^2\tau_2^2\tau_1 - \tau_6^2\tau_3^2\tau_2\tau_1\end{aligned} \tag{77}$$

(with $n \to 0$ for brevity). The iterates of (77) satisfy a Somos-6 relation, namely the first bilinear equation (5), which takes the form

$$\tau_{n+6}\tau_n = \gamma_n\tau_{n+5}\tau_{n+1} + \alpha\tau_{n+4}\tau_{n+2}, \qquad \gamma_{n+2} = \gamma_n, \tag{78}$$

while the second bilinear equation (6) is

$$\tau_{n+9}\tau_n = K\tau_{n+5}\tau_{n+4} - \alpha\tau_{n+8}\tau_{n+1}, \tag{79}$$

in this case.

For $N = 4$, taking $w_n = \frac{\tau_n \tau_{n+2}}{\tau_{n+1}^2}$ yields the U-system (31) for (78). Each iteration of the U-system is symplectic, and it lifts to the map

$$\psi: \quad (w_0, w_1, w_2, w_3, \gamma_0, \gamma_1) \mapsto \left(w_1, w_2, w_3, \frac{\gamma_0 w_1 w_2 w_3 + \alpha}{w_0 w_1^2 w_2^2 w_3^2}, \gamma_1, \gamma_0 \right) \tag{80}$$

in six dimensions, preserving the log-canonical Poisson bracket given by

$$\{w_m, w_n\} = c_{mn} w_m w_n, \quad (c_{mn})_{0 \leqslant m,n \leqslant 3} = \begin{pmatrix} 0 & 1 & -1 & 1 \\ -1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 1 \\ -1 & 1 & -1 & 0 \end{pmatrix}, \quad \{\gamma_m, w_n\} = 0. \tag{81}$$

Under the map defined by setting $u_n = w_n w_{n+1}$, that is

$$\pi_1: \quad (w_0, w_1, w_2, w_3, \gamma_0, \gamma_1) \mapsto (w_0 w_1, w_1 w_2, w_2 w_3, w_3 w_4, w_4 w_5)$$

where

$$w_4 = \psi^* w_3 = \frac{\gamma_0 w_1 w_2 w_3 + \alpha}{w_0 w_1^2 w_2^2 w_3^2}, \qquad w_5 = \psi^* w_4 = \frac{\gamma_1 w_2 w_3 w_4 + \alpha}{w_1 w_2^2 w_3^2 w_4^2},$$

the bracket (81) yields the bracket (37) preserved by (76).

The second U-system (33), associated with (79), is obtained by taking $v_n = \frac{\tau_n \tau_{n+5}}{\tau_{n+1}\tau_{n+4}} = u_n u_{n+2}$, producing the birational map

$$\hat{\psi}: \quad (v_0, v_1, v_2, v_3) \mapsto \left(v_1, v_2, v_3, \frac{K - \alpha v_1 v_2 v_3}{v_0 v_1 v_2 v_3} \right), \tag{82}$$

which is symplectic with respect to the 2-form

$$\omega = \sum_{0 \leqslant i < j \leqslant 3} \frac{1}{v_i v_j} \mathrm{d}v_i \wedge \mathrm{d}v_j.$$

Up to overall scale, this symplectic form corresponds to the nondegenerate log-canonical Poisson bracket given by $\{v_m, v_n\} = c_{mn} v_m v_n$, with the same coefficients c_{mn} as in (81).

The $(4, 1)$ periodic reduction of the lattice KdV equation, given by setting $N = 4$ in (28), is equivalent to the 5-dimensional birational map

$$\chi: \quad (v_0, v_1, v_2, v_3, v_4) \mapsto \left(v_1, v_2, v_3, v_4, v_0 + \alpha \left(\frac{1}{v_4} - \frac{1}{v_1} \right) \right). \tag{83}$$

This arises either by lifting (82) to one dimension higher and eliminating K, which becomes a first integral for (83) in the form

$$K = v_0 v_1 v_2 v_3 v_4 + \alpha v_1 v_2 v_3, \tag{84}$$

or by using $v_n = u_n u_{n+2}$ to obtain the transformation

$$\pi_2: \quad (u_0, u_1, u_2, u_3, u_4) \mapsto (u_0 u_2, u_1 u_3, u_2 u_4, u_3 u_5, u_4 u_6).$$

In the first case, the nondegenerate bracket for (82) lifts to the bracket $\{\, , \}_1$ defined by

$$\{v_0, v_1\}_1 = v_0 v_1, \ \{v_0, v_2\}_1 = -v_0 v_2, \ \{v_0, v_3\}_1 = v_0 v_3, \ \{v_0, v_4\}_1 = -v_0 v_4 - \alpha, \tag{85}$$

while the bracket (37) is pushed forward by π_2 to the bracket $\{\, , \}_2$ specified by

$$\{v_0, v_1\}_2 = v_0 v_1 - \alpha, \qquad \{v_0, v_2\}_2 = -v_0 v_2 + \alpha^2 v_1^{-2},$$
$$\{v_0, v_3\}_2 = v_0 v_3 - \alpha^3 (v_1 v_2)^{-2}, \quad \{v_0, v_4\}_2 = -v_0 v_4 + \alpha^4 (v_1 v_2 v_3)^{-2}. \tag{86}$$

The Poisson brackets $\{\,,\}_{1,2}$ are compatible with each other, and both are preserved by (83).

From the Lax representation of the KdV equation we derive the corresponding monodromy matrix for the $(4,1)$-reduction, as in [20], that is

$$\mathcal{M}(v_0, v_1, v_2, v_3, v_4, \lambda) = M(v_4, \lambda) L(v_3, v_4, \lambda) L(v_2, v_3, \lambda) L(v_1, v_2, \lambda) L(v_0, v_1, \lambda),$$

where λ is a spectral parameter, and

$$L(V, W, \lambda) = \begin{pmatrix} V - \frac{\alpha}{W} & \lambda \\ 1 & 0 \end{pmatrix}, \qquad M(V, \lambda) = \begin{pmatrix} V & \lambda \\ 1 & \frac{\alpha}{V} \end{pmatrix}. \tag{87}$$

The associated discrete Lax equation for the map (83) is

$$L(v_0, v_1, \lambda)\mathcal{M}(v_0, v_1, v_2, v_3, v_4, \lambda) = \mathcal{M}(v_1, v_2, v_3, v_4, v_5, \lambda) L(v_0, v_1, \lambda),$$

and the trace of the monodromy matrix is given by

$$\operatorname{tr} \mathcal{M}(v_0, v_1, v_2, v_3, v_4, \lambda) = I_2 \lambda^2 + I_1 \lambda + I_0,$$

where the coefficients are three functionally independent integrals, namely

$$I_0 = g_0 g_1 g_2 g_3 g_4,$$
$$I_1 = g_0 g_1 g_2 + g_1 g_2 g_3 + g_0 g_1 g_4 + g_0 g_3 g_4 + g_2 g_3 g_4 + \frac{\alpha g_2 g_3}{g_0},$$
$$I_2 = g_0 + g_1 + g_2 + g_3 + g_4 + \frac{\alpha}{g_0},$$

conveniently expressed in terms of the quantities $g_0 = v_0$ and $g_i = v_i - \alpha / v_{i-1}$ for $i = 1, 2, 3, 4$. Comparison with (84) reveals that K, a Casimir for the bracket $\{\,,\,\}_1$, is expressed as

$$K = I_2 \alpha^2 + I_1 \alpha + I_0,$$

while I_0 is a Casimir for $\{\,,\,\}_2$, and all of these integrals are in involution with respect to both brackets.

By setting $v_n = u_n u_{n+2}$, the I_j pull back to three integrals for the map (76), which commute with respect to the bracket defined by (37) with $N = 4$. A further pullback provides three independent commuting integrals for (80), with a fourth one being the Casimir $\gamma_0 + \gamma_1$.

5. An odd example: $N = 5$

For $N = 5$ the equation (2) is

$$(u_n + u_{n+1} + u_{n+2} + u_{n+3} + u_{n+4} + u_{n+5} + \beta) u_{n+1} u_{n+2} u_{n+3} u_{n+4} = \alpha, \tag{88}$$

which via (4) corresponds to the degree 7 equation (11), whose iterates also satisfy a Somos-7 recurrence with a period 2 coefficient, given by

$$\tau_{n+7} \tau_n = \gamma_n \, \tau_{n+6} \tau_{n+1} + \alpha \tau_{n+5} \tau_{n+2}. \tag{89}$$

The U-system associated with (89) is

$$u_n u_{n+1} u_{n+2} u_{n+3} u_{n+4} = \gamma_n u_{n+1} u_{n+3} + \alpha, \qquad \gamma_n = \gamma_{n+2}, \tag{90}$$

and the nondegenerate log-canonical Poisson bracket in 4 dimensions for (90) is given by

$$\{u_n, u_{n+1}\} = u_n u_{n+1}, \qquad \{u_n, u_{n+2}\} = 0 = \{u_n, u_{n+3}\}. \tag{91}$$

By eliminating β from (88), or eliminating γ_n from (90), we obtain a lift to the same equation in 6 dimensions, namely the $N = 5$ case of (19), which is equivalent to

$$u_{n+6} - u_n = \frac{\alpha}{u_{n+2}u_{n+3}u_{n+4}} \left(\frac{1}{u_{n+5}} - \frac{1}{u_{n+1}} \right). \tag{92}$$

Upon taking the bracket of both sides of (90) with u_0 for $n = 0, 1$, we see that (91) lifts to a Poisson bracket of rank 4 in 6 dimensions, with the additional brackets being

$$\{u_n, u_{n+4}\} = -\frac{\alpha}{u_{n+1}u_{n+2}u_{n+3}}, \qquad \{u_n, u_{n+5}\} = -u_n u_{n+5} + \frac{\alpha^2}{u_{n+1}^2 u_{n+2}^2 u_{n+3}^2 u_{n+4}^2}. \tag{93}$$

This 6-dimensional bracket is preserved by (92).

From proposition 2.1 and corollary 2.2 in [22], the bilinear equation (89) is the compatibility condition of the scalar Lax pair

$$Y_n \phi_{n+6} + \alpha\nu\phi_{n+4} = \xi\phi_n, \qquad \phi_{n+2} = \frac{1}{u_{n+1}}\Big(-\nu\phi_n + \phi_{n+1} \Big),$$

where u_n is given in terms of the tau function by (4), ν and ξ are spectral parameters, and

$$Y_n = \frac{\tau_{n+8}\tau_n}{\tau_{n+6}\tau_{n+2}} = u_n u_{n+1} u_{n+2} u_{n+3} u_{n+4} u_{n+5}.$$

For $n = 0$ the scalar Lax pair can be rewritten as a 2×2 matrix system in terms of $u_0, u_1, \ldots, u_5$, leading directly to a Lax pair for the map

$$\varphi: \qquad (u_0, \ldots, u_5) \mapsto (u_1, \ldots, u_6)$$

corresponding to (92), given by

$$\mathbf{L}(\nu)\Phi = \xi\Phi, \qquad \tilde{\Phi} = \mathbf{M}(\nu)\Phi, \tag{94}$$

with the tilde denoting the index shift $n \to n + 1$, and

$$\mathbf{L}(\nu) = \sum_{j=0}^{3} \mathbf{L}^{(j)}\nu^j, \qquad \mathbf{M}(\nu) = \begin{pmatrix} 0 & 1 \\ -\frac{\nu}{u_1} & \frac{1}{u_1} \end{pmatrix},$$

where

$$\mathbf{L}^{(0)} = \begin{pmatrix} 0 & u_0 \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{L}^{(1)} = \begin{pmatrix} -u_0 & -\frac{\gamma_0}{u_2} - u_0(u_1 + u_2 + u_3) \\ -1 & \beta + u_0 \end{pmatrix},$$

$$\mathbf{L}^{(2)} = \begin{pmatrix} \frac{\gamma_0}{u_2} + u_0(u_2 + u_3) & \gamma_0 + u_0 u_1(u_3 + u_4) - \frac{\alpha}{u_2 u_3} \\ -\beta - u_0 - u_1 & \gamma_1\left(\frac{1}{u_1} + \frac{1}{u_3}\right) + u_1(u_3 + u_4) + u_2(u_4 + u_5) - \frac{\alpha}{u_1 u_3 u_4} \end{pmatrix},$$

$$\mathbf{L}^{(3)} = \begin{pmatrix} -\gamma_0 & 0 \\ -\frac{\gamma_1}{u_1} - u_2(u_4 + u_5) + \frac{\alpha}{u_1 u_3 u_4} & -\gamma_1 \end{pmatrix}.$$

In the above formulae, $\beta, \gamma_0, \gamma_1$ stand for the functions of u_j defined by (88) for $n = 0$, and by (90) for $n = 0, 1$, respectively.

The compatibility condition for the system (94) is the discrete Lax equation

$$\tilde{\mathbf{L}}(\nu)\mathbf{M}(\nu) = \mathbf{M}(\nu)\mathbf{L}(\nu).$$

The spectral curve corresponding to the Lax matrix $\mathbf{L}(\nu)$ is a curve of genus 2 in the (ν, ξ) plane,

$$\det(\mathbf{L}(\nu) - \xi\mathbf{1}) \equiv \xi^2 + (K_3\nu^3 - K_2\nu^2 + K_1\nu - 1)\xi + K_0\nu^6 + \alpha\nu^5 = 0, \tag{95}$$

whose coefficients K_j provide 4 functionally independent first integrals for (92), namely

$$K_0 = u_0u_1u_2u_3u_4u_5 - \alpha(u_0 + u_5) + \frac{\alpha^2}{u_1u_2u_3u_4}, \; K_1 = u_0 + u_1 + u_2 + u_3 + u_4 + u_5 - \frac{\alpha}{u_1u_2u_3u_4},$$

$$K_2 = \sum_{j=0}^{5} u_ju_{j+2} + \sum_{j=0}^{2} u_ju_{j+3} - \alpha\left(\frac{1}{u_1u_2u_3} + \frac{1}{u_1u_2u_4} + \frac{1}{u_1u_3u_4} + \frac{1}{u_2u_3u_4}\right),$$

$$K_3 = u_0u_2u_4 + u_1u_3u_5 - \alpha\left(\frac{1}{u_1u_3} + \frac{1}{u_2u_4}\right),$$

with indices read mod 6 in the first sum above. The first integral in (20) is

$$\hat{\zeta} = u_0u_1u_2u_3u_4u_5 + \alpha(u_1 + u_2 + u_3 + u_4) = K_0 + \alpha K_1.$$

From (88) and (90) we can identify

$$K_0 = \gamma_0\gamma_1, \qquad K_3 = \gamma_0 + \gamma_1, \qquad K_1 = -\beta = u_0 + u_1 + u_2 + u_3 + \frac{\gamma_0}{u_0u_2} + \frac{\gamma_1}{u_1u_3} + \frac{\alpha}{u_0u_1u_2u_3},$$

$$K_2 = u_0u_2 + u_0u_3 + u_1u_3 + \frac{\gamma_0(u_0 + u_1 + u_2)}{u_0u_2} + \frac{\gamma_1(u_1 + u_2 + u_3)}{u_1u_3} + \alpha\left(\frac{1}{u_0u_1u_3} + \frac{1}{u_0u_2u_3}\right).$$

By construction, if we consider γ_0, γ_1 as functions of u_j defined by (90) for $n = 0, 1$, then these are Casimirs of the bracket given by (91) and (93). Hence K_0, K_3 are also Casimirs of this bracket, and one can verify directly that $\{K_1, K_2\} = 0$, which shows that (92) is a Liouville integrable map in 6 dimensions.

For $N = 5$, the other bilinear equation in theorem 1.1 is (7), which in this case becomes

$$\tau_{n+12}\tau_n = \alpha^2\tau_{n+10}\tau_{n+2} + \bar{K}\tau_{n+6}^2. \tag{96}$$

From the substitution (4), the conserved quantity $\bar{K}$ in (96) can be written in terms of u_j for $0 \leqslant j \leqslant 9$, as defined in (42), which gives

$$\bar{K} = u_0u_1u_2^2u_3^2u_4^3u_5^3u_6^2u_7^2u_8u_9 - \alpha^2u_2u_3u_4^2u_5^2u_6u_7,$$

and then using (90) the resulting expression can be further rewritten as a function of γ_j and only four adjacent u_j, which reveals that it is a polynomial in the quantities K_j obtained from the Lax pair above, that is

$$\bar{K} = K_0^3 + \alpha K_0^2K_1 + \alpha^2K_0K_2 + \alpha^3K_3. \tag{97}$$

For $P=3$, the recurrence (46) corresponds to the six-dimensional map

$$(v_0, v_1, \ldots, v_5) \mapsto \left(v_1, v_2, \ldots, \frac{v_1 v_3^2}{v_0 v_1 + \alpha^2 (v_3^2 - v_1 v_5)} \right), \tag{98}$$

and in this case the monodromy matrix (55) is

$$\mathcal{M}_5(\eta) = (1-\alpha\eta) M(v_5, v_3, \eta)^{-1} L(p_3, v_3, \eta) L(p_4, v_4, \eta) L(p_5, v_5, \eta),$$

where, from (53), there is dependence on v_0, v_1, v_2 via $p_3 = \alpha v_3/v_1 + \alpha^{-1}(v_0/v_3 - 1)$, and similarly for p_4, p_5. By taking the ratios $\hat{s}_n = v_n/v_{n+1}$, the iterates of (98) can be reduced to those of (59), which in this case yields the map

$$\hat{\chi}: \qquad (\hat{s}_0, \hat{s}_1, \hat{s}_2, \hat{s}_3, \hat{s}_4) \mapsto \left(\hat{s}_1, \hat{s}_2, \hat{s}_3, \hat{s}_4, \frac{\hat{s}_0 \hat{s}_1^2 \hat{s}_2^2 \hat{s}_3 \hat{s}_4 + \alpha^2 (\hat{s}_3 \hat{s}_4 - \hat{s}_1 \hat{s}_2)}{\hat{s}_1 \hat{s}_2 \hat{s}_3^2 \hat{s}_4^2} \right), \tag{99}$$

and by proposition 3.15 this preserves the Poisson bracket in 5 dimensions given by

$$\begin{aligned} \{\hat{s}_n, \hat{s}_{n+1}\} &= \hat{s}_n \hat{s}_{n+1}, & \{\hat{s}_n, \hat{s}_{n+2}\} &= -2\hat{s}_n \hat{s}_{n+2}, \\ \{\hat{s}_n, \hat{s}_{n+3}\} &= 2\hat{s}_n \hat{s}_{n+3}, & \{\hat{s}_n, \hat{s}_{n+4}\} &= -\alpha^2 (\hat{s}_{n+1} \hat{s}_{n+2} \hat{s}_{n+3})^{-1}. \end{aligned} \tag{100}$$

By construction, the above bracket has

$$\bar{K} = \hat{s}_1 \hat{s}_2^2 \hat{s}_3 (\hat{s}_0 \hat{s}_1 \hat{s}_2 \hat{s}_3 \hat{s}_4 - \alpha^2)$$

as a Casimir. The trace of the monodromy matrix is a monic cubic polynomial in η,

$$\operatorname{tr} \mathcal{M}_5(\eta) = \eta^3 + H_2 \eta^2 + H_1 \eta + H_0,$$

where H_0, H_1, H_2 provide three functionally independent first integrals for the map (98), but since they depend only on the ratios v_n/v_{n+1} they are also first integrals for (99), with the explicit expressions

$$\begin{aligned} H_2 &= \frac{1}{\alpha}(\hat{s}_0 \hat{s}_1 \hat{s}_2 + \hat{s}_1 \hat{s}_2 \hat{s}_3 + \hat{s}_2 \hat{s}_3 \hat{s}_4 - 3) + \alpha \left(\frac{1}{\hat{s}_1 \hat{s}_2} + \frac{1}{\hat{s}_2 \hat{s}_3} \right), \\ H_1 &= \frac{1}{\alpha^2}(\hat{s}_0 \hat{s}_1^2 \hat{s}_2^2 \hat{s}_3 + \hat{s}_0 \hat{s}_1 \hat{s}_2^2 \hat{s}_3 \hat{s}_4 + \hat{s}_1 \hat{s}_2^2 \hat{s}_3^2 \hat{s}_4 - 3) + \frac{\hat{s}_3 \hat{s}_4}{\hat{s}_1} + \frac{\hat{s}_0 \hat{s}_1}{\hat{s}_3} - \hat{s}_2 + \frac{\alpha^2}{\hat{s}_1 \hat{s}_2^2 \hat{s}_3} - \frac{2H_2}{\alpha}. \end{aligned}$$

The formula for H_0 has been omitted, since it is related to H_1, H_2 and the Casimir $\bar{K}$ by

$$\bar{K} = \alpha^3 H_0 + \alpha^2 H_1 + \alpha H_2 + 1. \tag{101}$$

Then a direct computation of the bracket $\{H_1, H_2\} = 0$ using (100) shows that the map (99) is Liouville integrable.

By setting $\hat{s}_n = u_{2n} u_{2n+1}$, it follows from theorem 3.17 that the quantities H_j coming from the monodromy matrix pull back to first integrals for (92). By a slight abuse of notation, we use the same symbols to denote the pullbacks of these integrals, and explain how they can be rewritten as functions of the quantities K_j found previously. The key point is that, for fixed K_0, the spectral curve in the (η, μ) plane coming from the monodromy matrix, that is

$$\det(\mathcal{M}_5(\eta) - \mu \mathbf{1}) \equiv \mu^2 - (\eta^3 + H_2 \eta^2 + H_1 \eta + H_0)\mu + 1 - \alpha\eta = 0,$$

is isomorphic to (95) via the change of coordinates

$$\eta = \kappa - \nu^{-1}, \quad \mu = -\xi \nu^{-3}, \qquad \text{with} \qquad K_0 = 1 - \alpha\kappa.$$

This leads to the relations

$$H_0 = K_3 - \kappa K_2 + \kappa^2 K_3 - \kappa^3, \qquad H_1 = K_2 - 2\kappa K_1 + 3\kappa^2, \qquad H_2 = K_1 - 3\kappa,$$

so that the identity (101) for $\bar{K}$ follows immediately from (97).

6. Conclusions

We have shown that the key to understanding the integrability of the family of maps considered in [4] is to introduce an additional parameter β, as in (2), and then lift to one dimension higher, eliminating this parameter to obtain (19). Although the properties of the map differ according to the parity of the dimension N, the Poisson bracket preserved by (19), in $N+1$ dimensions, is given by the same formulae (37) for both even and odd N. For the case of even N, we have found that the Liouville integrability of (19) follows from the corresponding results for reductions of Hirota's lattice KdV equation, considered in previous work. For odd N, the situation is more complicated: the connection with a reduction of the bilinear discrete KP (Hirota–Miwa) equation provides a Poisson bracket, a Lax pair, and a set of first integrals, but showing that these are in involution requires more work, and a general proof is lacking. On the other hand, for $N = 2P - 1$, there is an intriguing connection with a Bäcklund transformation (BT) for the $(1, -P)$ reduction of the discrete time Toda equation (44). For the general $(Q, -P)$ Toda reductions, considered briefly in [22], it would be interesting to construct a BT and see if there is a natural analogue of (19) for $Q > 1$.

In fact, a different lift of (2) has already appeared in the work of Svinin: it is the equation obtained by eliminating α, namely

$$u_{i+N}\Big(u_{i+1} + \cdots + u_{i+N+1} + \beta\Big) = u_{i+1}\Big(u_i + \cdots + u_{i+N} + \beta\Big),$$

which corresponds to replacing $T(i) \to u_i$, $s \to N$, $H_0 \to 1$, $H_1 \to -\beta$ in the case case $k = 1$ of equation (54) in [40], where a large class of difference equations with Lax representations are presented. For all systems obtained from the choice of integer parameters $(h, n) = (1, 1)$ in [40], Svinin obtained hyperelliptic spectral curves and corresponding sets of first integrals, including the equation (2) as a special case. In the future, it would be instructive to derive Hirota bilinear forms and Poisson structures for other equations found in [39, 40], and understand them from the viewpoint of Liouville integrability.

The starting point for all of the results in section 3 was the derivation of the Hirota bilinear equations associated with (2). This was achieved in two ways: first of all, in section 2, via an experimental approach involving the singularity confinement test (or an arithmetical version of it), followed by the lift to a Laurentification of (2), whose tropical analogue yields an exact calculation of degree growth; and secondly, once numerical and symbolic calculations produced bilinear equations for particular (small enough) values of N, by proving suitable algebraic identities in the general case. This combination of analytical, numerical and algebraic methods appears to be very effective, and we propose to apply it to other families of difference equations or maps in the future.

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