On the invariant theory of finite unipotent groups generated by bireflections

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Abstract

Let k be a field of characteristic p and let V be a k-vector space. In Chapter 2 of this thesis we classify all unipotent groups $G \leq GL(V)$ consisting of bireflections for $p \neq 2$: we show that unipotent groups consisting of bireflections are either two–row groups, two–column groups, hook groups or one of two types of exceptional group.

The well known theorem of Chevalley-Shephard-Todd shows the importance of (pseudo-)reflection groups to invariant theory. Our interest in bireflection groups is motivated by the theorem of Kemper which tells us if $G \leq \mathrm{GL}(V)$ is a p-group and the invariant ring $k[V]^G$ is Cohen–Macaulay then G is generated by bireflections. We use our classification to investigate which groups consisting of bireflections have Cohen–Macaulay or complete intersection invariant rings.

In Chapter 3 we introduce techniques and notation which we use later to find invariant rings of groups by viewing them as subgroups of Nakajima groups. In Chapter 4 we show that for $k = \mathbb{F}_p$ there is a family of hook groups, including all non-abelian hook groups, which have complete intersection invariant rings.

It is well known that Cohen–Macaulay invariant rings of p-groups in characteristic p are Gorenstein. There has been speculation by experts in the area, that they might in fact be complete intersections. In Chapter 5 we settle this negatively by giving an example of a p-group which has Cohen–Macaulay but non complete intersection invariant ring. To the best of our knowledge this is the first example of that kind.

Finally in Chapter 6 we show that when $k = \mathbb{F}_p$ both types of exceptional group have complete intersection invariant rings.

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Chapter 1

Introduction

The results in this thesis can be divided into two sections. Chapter 2, although motivated by invariant theory, deals only with unipotent groups consisting of bireflections, their structure and classification. Later on we look at the invariant rings of some of these groups. This first chapter gives an introduction to invariant theory and why we are interested in unipotent bireflection groups. We start by defining the objects we are interested in.

Definition 1.0.1. Let k be a field and V an n-dimensional k-vector space. We denote the vectors fixed (pointwise) by a group $G \in \mathrm{GL}(V)$:

$$V^G = \{v \in V \mid g(v) = v, \text{ for all } g \in G\},$$

and for any $g \in \mathrm{GL}(V)$ we define:

$$V^g = \{v \in V \mid g(v) = v\}.$$

An element $g \in \mathrm{GL}(V)$ is called a **reflection** (sometimes a pseudoreflection) if

$$\dim_k(V^g) = n - 1.$$

If

$$\dim_k(V^g) \geq n - 2$$

then g is called a **bireflection**. A subgroup $G \leq \mathrm{GL}(V)$ is called a **reflection group** if it is generated by reflections. Similarly G is called a **bireflection group** if it is generated by bireflections.

Let R be a commutative ring on which a group G acts. In invariant theory we are interested in the subset of a ring R which remains invariant under the group action

$$R^G = \{f \in R \mid g(f) = f, \text{ for all } g \in G\}.$$

Let k be a field, later we shall always consider a finite field k with positive characteristic p, so $k = \mathbb{F}_q$ where $q = p^r$ for some $r \in \mathbb{N}\backslash 0$. Let V be an n-dimensional k-vector space with basis $\{e_1, \ldots, e_n\}$. We can choose a respective dual basis for $W = V^*$ by choosing $\{x_1, \ldots, x_n\}$ such that

$$x_i(e_j) = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ otherwise.} \end{cases}$$

For any finite group G we look at a fixed representation $\rho : G \to \mathrm{GL}(V)$ giving a left action of G on V. We will use the same convention as in Campbell and Wehlau's book ([10]) that G also acts on the left on W by the dual representation. The following lemma will be used to relate the two:

Lemma 1.0.2. *[10, 1.1.1] For a group G let $\rho : G \to \mathrm{GL}(V)$ be a fixed representation, $\rho^* : G \to \mathrm{GL}(W)$ the dual representation. Then for $g \in G$ the matrix representing $\rho(g) \in \mathrm{GL}(V)$ with respect to a fixed basis is the transpose inverse of the matrix representing $\rho^*(g)$ with respect to the dual basis.*

We will see that faithful representations of the same group can have very different invariant rings, so we will mainly view the groups we are interested in as subgroups of $\mathrm{GL}(V)$ with the natural representation. If we have a matrix M of $g \in \mathrm{GL}(V)$ with respect to a basis $e_1, \ldots, e_n$ for V, then we can read off the action of g on this basis by looking at the columns of M. To find the action of g^{-1} on the corresponding dual basis we can read across the rows of M.

From here we can extend the action of G on W to an action on the polynomial ring

$$k[V] = S(W) = k[x_1, \ldots, x_n].$$

We do this by setting:

$$g(f(x_1, \ldots, x_n)) = f(g(x_1), \ldots, g(x_n))$$

for all $f \in k[V]$, $g \in G$. In invariant theory we are interested in the fixed space of this action, the invariant ring:

$$k[V]^G = \{f \in S(V) \mid g(f) = f \text{ for all } g \in G\}.$$

***Example* 1.0.3.** *Let $k = \mathbb{R}$, $n = 2$ with $\{x, y\}$ a basis for W. Let $G = \langle g, h \rangle$ where*

$$g = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \ h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathrm{GL}(V).$$

Let $X = x^2$ and $Y = y^2$ then

$$\begin{aligned} g(X) = g(x^2) = (-x)^2 = X = h(X), \\ h(Y) = h(y^2) = (-y)^2 = Y = g(Y), \end{aligned}$$

so $X, Y \in k[V]^G$, furthermore we will see later that X, Y generate the whole ring of invariants, so $k[V]^G = k[X, Y]$.

Invariant theory of finite groups can be split into the modular case (where the characteristic of the field k divides the order of the group) and non-modular case. We will mainly be interested in the modular case where many of the questions that are answered in the non-modular case are still open.

Definition 1.0.4. Let A be the k-algebra generated by $f_1, \dots, f_n$ so $A = k[f_1, \dots, f_n]$. If the f_i are algebraically independent then we say that A is a **regular** or **polynomial** ring.

In the Example 1.0.3 we see that the invariant ring is polynomial, and so $k[V]^G \cong k[V]$. In invariant theory we are interested in when this happens, when is the ring of invariants a polynomial ring? We will use the following hierarchy to describe how far away a ring is from being a polynomial ring:

$$\text{Regular} \Rightarrow \text{Complete Intersection} \Rightarrow \text{Gorenstien} \Rightarrow \text{Cohen–Macaulay}.$$

These terms will be defined in the next section.

The groups that we will be interested in will be p-groups with k a finite field of characteristic p. Motivated by Theorems 1.3.6 and 1.3.7 we will look at groups which consist of bireflections, and the structure of their invariant rings. Our main groups of interest will be:

- two–column groups which are subsets of

$$\left\{ \begin{pmatrix} 1 & 0 & \mathbf{0} \\ c & 1 & \mathbf{0} \\ \mathbf{a} & \mathbf{b} & \mathrm{I}_{n-2} \end{pmatrix} \mid \mathbf{a}, \mathbf{b} \in k^{n-2},\, c \in k \right\}$$

  (see Section 2.3).

- two–row groups which are subsets of

$$\left\{ \begin{pmatrix} \mathrm{I}_{n-2} & \mathbf{0} & \mathbf{0} \\ \mathbf{a} & 1 & 0 \\ \mathbf{b} & c & 1 \end{pmatrix} \mid \mathbf{a}, \mathbf{b} \in (k^{n-2})^T,\, c \in k \right\}$$

  (see Section 2.3).

- hook groups which are subsets of

$$\left\{\begin{pmatrix} 1 & \mathbf{0} & 0 \\ \mathbf{b} & \mathrm{I}_{n-2} & \mathbf{0} \\ c & \mathbf{a} & 1 \end{pmatrix} \mid \mathbf{a} \in (k^{n-2})^T, \mathbf{b} \in k^{n-2},\, c \in k \right\}$$

  (see Section 2.4).

- exceptional groups of type one. These are distinct from the above groups. They are subgroups of a group consisting of bireflections which is isomorphic to

$$\mathrm{Syl}_p(\mathrm{SL}_3(q)) = \left\{\begin{pmatrix} 1 & m & n \\ 0 & 1 & l \\ 0 & 0 & 1 \end{pmatrix} \mid m, n, l \in k \right\}$$

  (see Section 2.5).

- exceptional groups of type two. These are subgroups of an elementary abelian group of order q^3 consisting of bireflections. These are distinct from hook groups but each pair of elements generates a hook group (see Section 2.6).

In section 2.2 we give a full classification of finite unipotent groups consisting of bireflections in characteristic $p > 2$.

Theorem 1.0.5. *Let $G \leq \mathrm{GL}(V)$ be a p-group consisting of bireflections with $p \neq 2$ then one of the following must hold:*

- *G is a two–row group.*
- *G is a two–column group.*
- *G is a hook group.*
- *G is an exceptional group of type one.*

- *G is an exceptional group of type two.*

The remainder of Chapter 2 is spent looking at the properties of these groups. This allows us to draw some more general conclusions in Section 2.7, for example showing that all unipotent groups conisisting of bireflections have class less than or equal to two for $p \neq 2$, $n \geq 3$ (see Corollary 2.7.3).

For $k = \mathbb{F}_p$ it turns out that a lot of these representations do have Cohen–Macaulay rings of invariants, many of which are also complete intersection rings.

Theorem 1.0.6. *Let $G \leq \mathrm{GL}(V)$ be a unipotent group consisting of bireflections, as above let $W = V^*$. If $k = \mathbb{F}_p$ and $k[V]^G$ is not a complete intersection ring then one of the following must hold*

- *G is a non-abelian two–column group.*
- *G is a two–column group which cannot be generated by reflections.*
- *G is an abelian hook group with $[G, [G, W]] \neq \{0\}$.*
- *G is a two–row group.*

If $k = \mathbb{F}_p$ and $k[V]^G$ is not a Cohen–Macaulay ring then one of the following must hold:

- *G is an abelian hook group with $[G, [G, W]] \neq \{0\}$.*
- *G is a two–row group.*

In Chapter 2 we formally define these groups and prove Theorem 1.0.5. In Chapter 4 we show that certain hook groups have complete intersection invariant rings for $k = \mathbb{F}_p$ (see Theorem 4.2.8). In Chapter 6 we show that the exceptional groups have complete intersection invariant rings, again for $k = \mathbb{F}_p$. Combining these with existing results about two–column groups we will prove Theorem 1.0.6.

In Chapter 5 we will find the invariant ring of a two–column group which has Cohen–Macaulay but non complete intersection ring of invariants. This is a counter example to speculation by experts that if k is a field of characteristic p, G a p-group, then $k[V]^G$ Cohen–Macaulay implies that $k[V]^G$ is a complete intersection ring.

Firstly though we need more of an introduction to invariant theory.

1.1 The invariant ring $k[V]^G$

The ring of invariants $k[V]^G$ is a subring of $k[V]$ and has some of the same nice properties, for example being finitely generated. This is a classical result of David Hilbert from 1890 ([19]) in the non-modular case and was proved later in 1915 by Emmy Noether in the modular case ([29]). One consequence of this is that the invariant ring is Noetherian which means that any chain of ascending ideals eventually terminates. Another nice property of $k[V]^G$ is that it is graded.

Definition 1.1.1. A ring R is called (positively) **graded** if we can find additive groups $R_i \leq R$ for $i \in \mathbb{N}$ such that

$$R = \bigoplus_{i \in \mathbb{N}} R_i$$

and if $r_i \in R_i$, $r_j \in R_j$ then $r_i r_j \in R_{i+j}$. We call $r \in R$ **homogeneous** if $r \in R_i$ for some $i \in \mathbb{N}$. A graded algebra R is called **connected** if $R_0 = k$. An R-module M is called a **graded module** if we can find additive groups $M_i \leq M$ for $i \in \mathbb{N}$ such that

$$M = \bigoplus_{i \in \mathbb{N}} M_i$$

and if $r_i \in R_i$, $m_j \in M_j$ then $r_i(m_j) \in M_{i+j}$.

We say that a monomial $x_1^{a_1} x_2^{a_2} \dots x_n^{a_n}$ has degree $d = a_1 + a_2 + \dots + a_n$. For a polynomial ring $R = k[x_1, \dots, x_n]$ we let R_d be the subspace spanned by monomials of degree d. In this way we see that polynomial rings have a natural positive grading given by degree, as the degree zero part is the field itself they are also connected. By the way we have defined the group action we can see that it respects this grading, and all elements in degree zero (elements of the field k) are fixed. If $f, g \in k[V]^G$, so are fg and $f+g$ so $k[V]^G$ is a graded connected k-algebra. An ideal in a graded ring R is called homogeneous if it can be generated by homogeneous elements. In the case of a graded connected ring there is a unique maximal homogeneous ideal R_+, generated by all elements of positive degree. In some situations we can use this to treat the ring similarly to a local ring and refer to it as *-local.

Let R be a ring with subring S. If for some element $a \in R$ we can find a monic polynomial f with coefficients in S such that $f(a) = 0$ then we say that a is integral over S. If every $a \in R$ is integral we say that R is integral over S or that R is an integral extension of S.

Theorem 1.1.2. *[10, 3.0.4] Let $G \leq \mathrm{GL}(V)$ be a finite group, then $k[V]$ is an integral extension of $k[V]^G$.*

Proof. For any $h \in k[V]$ we can construct the monic polynomial

$$F(t) = \prod_{g \in G} (t - g(h)) = t^{|G|} + f_{|G|-1} t^{|G|-1} + \ldots + f_0.$$

As at least the identity in G fixes h we must have $F(h) = 0$. We can extend the action of G on $R = k[V]$ to an action on $R[t]$ by letting $g(t) = t$ for all $g \in G$. As all elements of the group simply permute the factors of $F(t)$ we find that $F(t) \in R[t]^G$. This means that for any $\sigma \in G$:

$$t^{|G|} + \sigma(f_{|G|-1}) t^{|G|-1} + \ldots + \sigma(f_0) = t^{|G|} + f_{|G|-1} t^{|G|-1} + \ldots + f_1 t + f_0.$$

As all the f_i's are of different degrees this means that $f_i \in k[V]^G$ for $0 \leq i \leq |G| - 1$, and so $k[V]$ is integral over $k[V]^G$. $\square$

Definition 1.1.3. Let R be a Noetherian ring and let p be a prime ideal of R. If we can find a chain of prime ideals

$$p_0 \subsetneq p_1 \subsetneq \ldots \subsetneq p_i = p$$

which is of maximal length, i, then we call i the height of p. The **Krull dimension** of R is the maximum height of proper prime ideals in R.

For a polynomial ring $R = k[x_1, \ldots, x_n]$ it's Krull dimension is n: we can find the chain of prime ideals:

$$(0) \subsetneq (x_1) \subsetneq (x_1, x_2) \subsetneq \ldots \subsetneq (x_1, \ldots, x_n)$$

and this is the maximal possible length of a chain of prime ideals in R.

As $k[V]$ is integral over $k[V]^G$ we can use the Lying Over, Going-Up and Going-Down Theorems to relate prime ideals in $k[V]^G$ with prime ideals in $k[V]$ (see [10] Theorem 2.5.2). This tells us that $k[V]$ and $k[V]^G$ have the same Krull dimension. From here on for any ring R we shall mean the Krull dimension of R when we refer to its dimension or write $\dim(R)$ and will specifically state in any instances when we mean the dimension of R as a k-vector space (and write $\dim_k(R)$).

We know that $k[V]^G$ is finitely generated, suppose for some $m \in \mathbb{N}$ it is generated by homogeneous elements $f_1, \ldots, f_m \in k[V]^G_+$. There is a canonical surjective homomorphism of k-algebras ϕ from the polynomial ring $k[y_1, \ldots, y_m]$ onto $k[V]^G$, mapping y_i to f_i for $1 \leq i \leq m$. For a graded connected ring R the minimal number of generators m for R_+ is called the **embedding dimension** of R ($\operatorname{Emb}\dim(R)$).

Definition 1.1.4. Let R be a commutative graded k-algebra of Krull dimension n with homogeneous elements $f_1, \ldots, f_n$ of R and let $A = k[f_1, \ldots, f_n]$. We say that the f_i's form a **homogeneous system of parameters** (or **HSOP**) for R if R is finitely generated as an A-module: there exists $h_1, \ldots, h_m$ for some $m \in \mathbb{N}$ such that

$$R = \sum_{i=1}^{m} Ah_i.$$

In the case that $R = k[V]^G$ a HSOP is often referred to as a set of **primary invariants** and the module generators (the h_i's) are the respective **secondary invariants**.

As $k[V]^G$ is a graded connected k-algebra Noether's Normalisation Lemma (see [14, Theorem A1]) tells us that we can always find such a system for the invariant ring $k[V]^G$. As $k[V]$ is integral over $k[V]^G$ any HSOP for $k[V]^G$ also forms a HSOP for $k[V]$. In example 1.0.3 $\{X, Y\}$ can be shown to be a homogeneous system of parameters for $k[V]^G$ and in general a HSOP is not too difficult to find.

Lemma 1.1.5. *[10, 2.6.3] Let $k[V]^G$ have Krull dimension n and $f_1, \ldots, f_n \in k[V]^G$. Let $\overline{k}$ be the algebraic closure of k, and let*

$$\overline{V} = V \otimes_k \overline{k}.$$

Then $\{f_1, \ldots, f_n\}$ forms a HSOP for $k[V]^G$ if and only if $\mathcal{V}_{\overline{V}}(f_1, \ldots, f_n) = \{0\}$, where

$$\mathcal{V}_{\overline{V}}(f_1, \ldots, f_n) = \{\mathbf{x} \in \overline{V} : 0 = f_1(\mathbf{x}) = f_2(\mathbf{x}) = \ldots = f_n(\mathbf{x})\}.$$

***Example* 1.1.6.** *Let $k = \mathbb{F}_q$, $R = k[V]$ and let $G = \mathrm{GL}(V)$. We can form the following homogeneous polynomial over $R[t]$:*

$$F^W(t) = \prod_{w \in W} (t - w) = \sum_{i=0}^{n} d_{i,n} t^{q^i}.$$

We can see that $F^W(t) \in R^G[t]$, and as the $d_{i,n}$ all have different degrees we must have $d_{i,n} \in R^G$ for $1 \leq i \leq n$. These are known as the Dickson Invariants.

The smallest non-trivial case is $k = \mathbb{F}_2$ and $W = \langle x, y \rangle_k$. Here we have

$$F^W(t) = t(t+x)(t+y)(t+x+y) = t^4 + (x^2 + xy + y^2)t^2 + (x^2y + y^2x)$$

so $d_{1,2} = x^2 + xy + y^2$, $d_{0,2} = x^2y + y^2x$. *We see that*

$$d_{0,2} = xy(x+y)$$

so if $(c_1, c_2) \in \mathcal{V}_{\overline{V}}(d_{0,2})$ *either* $c_1 = 0$, $c_2 = 0$ *or* $c_1 = -c_2$. *Substituting these into* $d_{1,2}$ *shows that*

$$\mathcal{V}_{\overline{V}}(d_{1,2}, d_{0,2}) = \{0\}$$

and so we have found a HSOP for $k[V]^G$.

As $k[V]$ is integral over $k[V]^G$ if $H \leq G$ then $k[V]^H$ is integral over $k[V]^G$, so a HSOP for $k[V]^G$ is also a HSOP for $k[V]^H$. The Dickson Invariants can always be shown to form a HSOP for $\mathrm{GL}(V)$. As we are only interested in groups G which are subgroups of $\mathrm{GL}(V)$, this means that the Dickson Invariants always form a HSOP for $k[V]^G$ (though not usually the most convenient one to work with). They also have some other nice properties which we will make use of later.

Lemma 1.1.7. *[10, 3.3.1] Let* $x_1, \ldots, x_n$ *be a basis for* W. *For* $1 \leq i \leq n$ *define subspaces of* W *by*

$$W_i = \langle x_1, \ldots, x_i \rangle$$

and as above let

$$F_i(t) = F^{W_i}(t) = \prod_{w \in W_i} (t - w).$$

Then

1. $F_i(t) = F_{i-1}(t)^q - F_{i-1}(x_i)^{q-1} F_{i-1}(t)$;
2. $d_{j,i} = d_{j,i-1}^q - d_{j-1,i-1} F_{n-1}^{q-1}(x_i)$.

The following can be used to check if a HSOP generates the whole invariant ring.

Theorem 1.1.8. *[22, 16] If* $\{f_1, \ldots, f_n\}$ *are a HSOP for* $k[V]^G$ *then*

$$\prod_{i=1}^{n} \deg(f_i) = |G|$$

if and only if $k[V]^G = k[f_1, \ldots, f_n]$ *so* $k[V]^G$ *is a polynomial ring.*

In the case of the Dickson Invariants we can see that $\deg(d_{i,n}) = q^n - q^i$ for $1 \leq i \leq n$ and then can check that

$$\prod_{i=1}^{n} \deg(d_{i,n}) = \prod_{i=0}^{n-1} q^n - q^i = |\mathrm{GL}(V)|.$$

So $k[V]^G = k[d_{1,n}, \ldots, d_{n,n}]$. In Example 1.0.3 we also find that

$$\deg(X)\deg(Y) = 4 = |G|,$$

however this is not always the case.

***Example* 1.1.9.** *Let* $k = \mathbb{R}$*,* $n = 2$ *and* x, y *be a basis for* W*. Let* G *be as in Example 1.0.3 and* $H = \langle t \rangle \leq \mathrm{GL}(V)$ *where*

$$t = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We see that H *is a subgroup of* G *and so* X, Y *form a HSOP for* $k[V]^H$*, however they don't generate the whole ring:* $xy \in k[V]^H$ *but* $xy \notin k[X, Y]$*. It can be shown that* $k[V]^H = k[V]^G[xy]$ *and we cannot find a HSOP,* f_1, f_2 *such that* $k[V]^G = k[f_1, f_2]$ *and so* $k[V]^G$ *is not regular.*

When we can find a HSOP $f_1, \ldots, f_n$ such that $k[V]^G = k[f_1, \ldots, f_n]$ then $k[V]^G \cong k[V]$ and we say $k[V]^G$ is a polynomial or regular ring. In the non-modular case the well known theorem of Chevalley ([11]), Shephard and Todd([30]) tells us which groups have polynomial invariant rings.

Theorem 1.1.10 (Chevalley, Shephard-Todd)**.** *If* $|G| \in k^*$ *then* $k[V]^G$ *is polynomial if and only if* G *is generated by reflections*

In the modular case Serre's theorem (see [10, Corollary 12.2.5]) tells us that for $k[V]^G$ to be polynomial, G must be generated by reflections. However the converse is false.

1.2 How far away is $k[V]^G$ from being polynomial?

When a ring fails to be a polynomial ring we can ask how far away it is from being regular. In Example 1.1.9 we said that $k[V]^H = k[V]^G[xy]$ where $k[V]^G$ is a polynomial ring. This is an example where the invariant ring $R = k[V]^H$ is a hypersurface- this means that

$$\operatorname{Emb}\dim(R) \leq \dim(R) + 1.$$

In this section we will define other ring classifications that we will use later on.

1.2.1 Cohen–Macaulay rings and depth

Definition 1.2.1. Let R be a ring, M an R-module, and

$$\mathbf{x} = x_1, x_2, ..., x_n$$

a sequence of elements in R. The sequence $\mathbf{x}$ is called an **M-regular sequence** or **M-sequence** of length n if the following are satisfied:

- $M/\mathbf{x}M \neq 0$;
- x_i is not a zero divisor of $M/(x_1, ..., x_{i-1})M$ for $1 \leq i \leq n$.

When R is taken to be a module over itself this is simply called a **regular sequence**.

A regular sequence is maximal if it cannot be extended to a longer regular sequence. If R is a Noetherian ring, M an R-module and I an ideal in R with $IM \neq M$, then all maximal regular M-sequences in I have the same length (see [14] Chapter 17). For a local (or *-local) ring with maximal ideal m (or maximal homogeneous ideal R_+) we call the depth of M the length of a maximal regular

M-sequence in m (or R_+). We shall denote the depth of M (as an R-module) by $\text{depth}_R(M)$, or just $\text{depth}(M)$ where the ring is clear. Most often we will be interested in the depth of R as an R-module (this is the maximal length of homogeneous regular sequences in R). A ring is called Cohen–Macaulay if depth is equal to Krull dimension, in which case any HSOP is a regular sequence (see [10, 2.8.1]).

Theorem 1.2.2. *[3, 4.3.5] If R is a graded connected Noetherian k-algebra then the following are equivalent:*

1. *R is Cohen–Macaulay of dimension n;*
2. *R has a HSOP $f_1, ..., f_n$ such that if $A = k[f_1, ..., f_n]$ then R is free as an A-module;*
3. *for any HSOP $f_1, ..., f_n$ for R, $A = k[f_1, \dots, f_n]$, R is free as an A-module.*

This means that if R is Cohen–Macaulay with HSOP $f_1, \dots, f_n$ then we can find a set of secondary invariants $h_1, \dots, h_m \in R$ for some $m \in \mathbb{N}$ such that if $A = k[f_1, \dots, f_n]$ then

$$R = \oplus_{i=1}^m A h_i.$$

If we have a set of generators for $R = k[V]^G$ then we have a simple way to check if the ring is Cohen–Macaulay.

Theorem 1.2.3. *[13, 3.7.1] Let $G \leq \mathrm{GL}(V)$, and let $k[V]^G$ have a set of primary invariants $f_1, \dots, f_n$, and a minimal set of secondary invariants $h_1, \dots, h_m$. Then*

$$\prod_{i=1}^n \deg(f_i) \leq m|G|$$

with equality if and only if $k[V]^G$ is Cohen–Macaulay.

In the non-modular case $k[V]^G$ is always Cohen–Macaulay ([20]), however this is not true in the modular case. We can use the Cohen–Macaulay defect to give a measure of how far away a ring R is from being Cohen–Macaulay:

$$\mathrm{CMdef}(R) = \mathrm{Krull\ dim}(R) - \mathrm{depth}(R).$$

We introduce a little homological algebra which will allow us to put a lower bound on the depth of an invariant ring.

Let R be a ring. An R-module P is called projective if it is a direct summand of a free module or equivalently if for all maps $f : P \to N$, and all surjections $g : N \to M$ with M, N both R-modules there exists h such that the following diagram commutes:

$$\begin{tikzcd} & P \arrow[dl, dotted, "h"'] \arrow[d, "f"] \\ M \arrow[r, two heads, "g"'] & N. \end{tikzcd}$$

For a graded module M over a graded connected ring R, free and projective are equivalent (see [14] Theorem A3.2). Suppose M is a finitely generated R-module, with generators $m_1, \ldots, m_r$. We can always find a surjection from the free R-module

$$M_0 = \oplus_{i=1}^r R r_i$$

onto M by mapping r_i to m_i. Using this surjection we obtain the following exact sequence:

$$0 \longrightarrow I \longrightarrow M_0 \xrightarrow{p_0} M \longrightarrow 0$$

where I is the kernel of the map. The kernel I may not be free, but as above we can find an M_1 which is free and surjects onto I, and continue in this manner to find the exact sequence of free modules (and M):

$$\ldots \xrightarrow{p_3} M_2 \xrightarrow{p_2} M_1 \xrightarrow{p_1} M_0 \xrightarrow{p_0} M \longrightarrow 0\ .$$

The above sequence is called a free resolution for M. If we have an exact sequence as above, but with the M_i merely projective (not necessarily free) then this is called a projective resolution for M. If we can find some l such that $M_l \neq 0$ but $M_i = 0$ for $i > l$ then we say that the resolution terminates and has length l. A projective resolution of M is minimal if it has minimal length l, in this case we call l the projective dimension of M ($\mathrm{projdim}_R(M)$). If it is not possible to find a projective resolution which terminates we say that M has infinite projective dimension.

Let R be a ring with R-modules M and N. Suppose that we have a projective resolution $\mathbf{P}$ for M:

$$\ldots \xrightarrow{p_3} M_2 \xrightarrow{p_2} M_1 \xrightarrow{p_1} M_0 \xrightarrow{p_0} M \longrightarrow 0\,.$$

For $i \geq 1$ we can define maps

$$f_i : \mathrm{Hom}_R(M_{i-1}, N) \to \mathrm{Hom}(M_i, N)$$

such that for $m \in \mathrm{Hom}(M_{i-1}, N)$, $m' \in M_i$

$$f_i(m)(m') = m(\rho_i(m')).$$

We can form the following complex $\mathrm{Hom}(\mathbf{P}, N)$:

$$0 \longrightarrow \mathrm{Hom}(M_0, N) \xrightarrow{f_1} \mathrm{Hom}(M_1, N) \xrightarrow{f_2} \mathrm{Hom}(M_2, N) \xrightarrow{f_3} \ldots\,.$$

Unlike the projective resolution this is not necessarily an exact sequence. We measure how far away this sequence is from being exact by defining

$$\mathrm{Ext}^i_R(M, N) = H^i(\mathbf{P}, N) = \ker(f_i)/\mathrm{Im}(f_{i+1}).$$

In the case that we have a finite group G with kG-module A we define

$$H^i(G, A) = \mathrm{Ext}^i_{\mathbb{Z}G}(\mathbb{Z}, A).$$

We call the smallest $i > 0$ such that $H^i(G, A) \neq 0$ the cohomological connectivity of A and denote it by $cc_G(A)$.

Theorem 1.2.4. *[16, 1.2] Let $R = k[V]$, $G \leq \mathrm{GL}(V)$ then*

$$\mathrm{depth}(R^G) \geq \min\{\dim_k(V^p) + cc_G(R) + 1, \dim_k(V)\}.$$

The representation V of a group G is called flat if

$$\mathrm{depth}(R^G) = \dim_k(V^G) + cc_G(R) + 1.$$

For p-groups and k of characteristic p, where $cc_G(R) = 1$, this is a particularly useful theorem.

Theorem 1.2.5. *[15, 4] Let G be a group with Sylow-p-subgroup P. Let V be a finite kG-module, $R = k[V]$, and $m = cc_G(R)$. Suppose that $0 \neq \tau \in H^m(G, R)$ is a cohomology class such that*

$$\mathrm{res}^P_N(\tau) = 0$$

for each maximal subgroup $N < P$. Then V is flat.

Let G be a p-group. For a maximal subgroup M of G let

$$\mathcal{X}_M := (1 - g)W^M$$

where $g \in G \backslash M$. For some $u_N \in N \backslash M$ for $N \lhd_{\max} G$ let

$$\mathcal{Y}_M := \bigcap_{\substack{N \lhd_{\max} G \\ N \neq M}} (1 - u_N)W^{N \cap M}.$$

Theorem 1.2.6. *[15, 6] For a non-cyclic p-group G the following are equivalent:*

1. *$\bigcap_{M \lhd_{\max} P} \ker(\mathrm{res}^G_M|_{H^1(G,W)}) \neq 0$;*
2. *for some $M \lhd G$ maximal $\mathcal{X}_M < \mathcal{Y}_M \cap W^M$;*

3. for all $M \lhd G$ maximal $\mathcal{X}_M < \mathcal{Y}_M \cap W^M$.

From the above we see that homology is useful when looking at the depth of invariant rings. Another useful tool when looking at regular sequences is the Koszul complex. Let R be a commutative Noetherian ring with M an R-module. Let

$$\bigotimes^i M = \underbrace{M \otimes M \otimes \ldots \otimes M}_{i\text{-times}}$$

so $\bigotimes^0 M = R$, $\bigotimes^1 M = M$ and $\bigotimes^2 M = M \otimes M$. The tensor algebra of M, $\bigotimes M$, is defined to be

$$\bigotimes M = \bigoplus_{i \geq 0} \bigotimes^i M.$$

where the multiplication of $x_1 \otimes \ldots \otimes x_m \in \bigotimes^m M$ and $y_1 \otimes \ldots \otimes y_l \in \bigotimes^l M$ is given by

$$x_1 \otimes \ldots \otimes x_m \times y_1 \otimes \ldots \otimes y_l = x_1 \otimes \ldots \otimes x_m \otimes y_1 \otimes \ldots \otimes y_l \in \bigotimes^{l+m} M.$$

From here we can define the exterior algebra

$$\wedge M = \bigotimes M/J$$

where J is the ideal of $\bigotimes M$ generated by the elements $x \otimes x$ and $x \otimes y - y \otimes x$ for all $x, y \in M$.

The tensor algebra is graded by the $\bigotimes^i M$'s, and this naturally leads to a grading on $\wedge M$ by letting

$$\wedge^i M = \bigotimes^i \overline{M}$$

for $i \geq 0$ where $\bigotimes^i \overline{M}$ is the image of $\bigotimes^i M$ in $\wedge M$.

Let $\mathbf{x} = x_1, \ldots, x_m$ be a sequence in R and let N be the free R-module of rank m with $e_1, \ldots, e_m$ a basis for N. We can see that

$$\wedge^i N \cong R^{\binom{m}{i}}$$

as R-modules for $1 \leq i \leq m$ and $\wedge^i N = 0$ otherwise. Let $f : N \mapsto R$ be the map defined by $f(e_i) = x_i$ for $1 \leq i \leq m$. The Koszul complex of $\mathbf{x}$, $K(\mathbf{x})$, is defined to be:

$$0 \longrightarrow \wedge^m N \xrightarrow{d_m} \wedge^{m-1} N \xrightarrow{d_{m-1}} \ldots \xrightarrow{d_2} \wedge^1 N \xrightarrow{d_1} R \longrightarrow 0$$

where for $1 \leq i \leq m$ the map $d_i : \wedge^i N \mapsto \wedge^{i-1} N$ maps $a_1 \wedge ... \wedge a_i$ to

$$d_i(a_1 \wedge .. \wedge a_i) = \sum_{j=1}^{i} (-1)^j f(a_j) a_1 \wedge \ldots \wedge \widehat{a}_j \wedge \ldots \wedge a_i$$

where $\widehat{a}_j$ signifies that this term has been omitted.

Theorem 1.2.7. *[14, 17.4] Let M be a finitely generated module over the ring R and let $\mathbf{x} = x_1, \ldots, x_n$ be a sequence in R. If*

$$H^j(M \otimes K(\mathbf{x})) = 0 \text{ for } j < m$$

while

$$H^m(M \otimes K(\mathbf{x})) \neq 0$$

then every maximal M-sequence in $I = (\mathbf{x}) \subset R$ has length m.

1.2.2 Gorenstein rings and free resolutions

We are always interested in R a graded connected finitely generated k-algebra, and in this case the Hilbert Syzygy Theorem tells us that any R-module M has finite projective dimension. The following relates projective dimension and depth.

Theorem 1.2.8. *Let R be a graded connected Noetherian k-algebra generated by $f_1, \ldots, f_s$ and let $B = k[y_1, \ldots, y_s]$ be a polynomial ring. Then*

$$\operatorname{depth}(R) + \operatorname{projdim}_B(R) = \dim(B).$$

Proof. This is contained in [3] but not stated specifically: the ring R meets the criterion of Hypothesis 4.3.2 (of [3]) so by Theorem 4.4.3 (of [3]) $\operatorname{depth}(R) = \operatorname{hcodim}_B(R)$. By Theorem 4.4.4 (of [3])

$$\operatorname{hcodim}_B(R) + \operatorname{projdim}_B(R) = \dim(B)$$

and so

$$\operatorname{depth}(R) + \operatorname{projdim}_B(R) = \dim(B).$$

$\square$

This is a graded connected version of the Auslander-Buchsbaum Theorem which holds in the local case. Let R be a graded connected Cohen–Macaulay k-algebra, with Krull dimension n, which can be generated by $f_1, \ldots, f_s$ and let $B = k[y_1, \ldots, y_s]$ be a polynomial ring with canonical surjection ϕ onto R.

If l is the projective dimension of R as a B-module then using the above $l = s - n$. Let

$$0 \longrightarrow M_l \xrightarrow{\rho_l} \ldots \xrightarrow{\rho_1} M_0 \longrightarrow R \longrightarrow 0$$

be a projective resolution for R. For $1 \leq i \leq l$ let $M_i^* = \operatorname{Hom}_B(M_i, B)$ and $\rho_i^* : M_{i-1}^* \to M_i^*$ such that for $m \in M_{i-1}^*$, $n \in M_i$

$$\rho_i^*(m)(n) = m(\rho_i(n)).$$

Using [3] Corollary 4.5.2

$$\operatorname{Ext}_B^i(M, B) = 0$$

for $i \neq s - n = l$ so if we let $\Omega_B(M) = M_l^*/\operatorname{Im}(\rho_l)$ then

$$0 \longrightarrow M_0^* \xrightarrow{\rho_1^*} \ldots \xrightarrow{\rho_l^*} M_l^* \longrightarrow \Omega_B(R) \longrightarrow 0$$

is an exact sequence.

It can be shown that $\Omega_B(R)$ doesn't depend on the choice of generators $f_1, \ldots, f_s$: for any $f'_1, \ldots, f'_{s'}$ and regular ring $B' = k[y'_1, \ldots, y'_{s'}]$ we find that $\Omega_B(R) = \Omega_{B'}(R)$ is a free R-module (see Section 4.5 of [3]). We call $\Omega(R) = \Omega_B(R)$ the canonical module of R and say that the type of R is the rank of $\Omega(R)$ as a free R-module.

Definition 1.2.9. Let R be a graded connected k-algebra. Then R is a **Gorenstein** ring if it is a Cohen–Macaulay ring of type one (or equivalently Cohen–Macaulay such that $\Omega(R) \cong R$ as R-modules).

If $k[V]^G$ is a Gorenstein ring with HSOP $f_1, \ldots, f_n$, $I = (f_1, \ldots, f_n)$ then we know a that $k[V]^G/I$ has a particularly nice form, described by the definition below.

Definition 1.2.10. Let R be a zero dimensional graded connected k-algebra with top degree d, then R is called a **Poincaré duality algebra** if $\dim_k(R_d) = 1$ and for all $i \leq d/2$ there exists a bilinear form

$$\begin{aligned} R_i \otimes_k R_{d-i} &\to R_d, \\ a \otimes_k b &\mapsto ab, \end{aligned}$$

which is non-singular: if $a \in R_i$ then $a = 0$ if and only if $a \otimes_k b \mapsto 0$ for all $b \in R_{d-i}$.

If R is a Gorenstein ring with a HSOP $f_1, \ldots, f_n$ then $R/(f_1, \ldots, f_n)$ is a Poincaré duality algebra (see [28] Corollary 5.7.4).

1.2.3 Complete intersection rings

Definition 1.2.11. Let R be a finitely generated k-algebra such that there is a polynomial ring $A = k[y_1, \ldots, y_{n+s}]$ and some homogeneous ideal I with

$$R = A/I = k[\bar{y}_1, ..., \bar{y}_{n+s}].$$

We call R a **complete intersection ring** if it has dimension n and we can find a homogeneous regular sequence $a_1, \ldots, a_s$ which generates I.

This can be shown to be a ring property for R independent on the choice of A and I. We can also show that if $R = A/I$ is a complete intersection it must be Cohen–Macaulay: as A is a Cohen–Macaulay ring $a_1, \ldots, a_s$ can be extended to a regular sequence $a_1, \ldots, a_{n+s}$ which is a HSOP for A. From the definition of a regular sequence we see $a_{s+1}, \ldots, a_{n+s}$ must be a regular sequence of length n in A/I. This means that the depth of R is at least n, however as depth is bounded above by the Krull dimension, the depth of R must be equal to n and so R is a Cohen–Macaulay ring.

Proposition 1.2.12. *[32, 9.4] Let $S = k[y_1, \ldots, y_{n+s}]$ be a polynomial ring, $I = (a_1, \ldots, a_m)A$ an ideal of A and $R = A/I$. Then R is a complete intersection ring if and only if the Koszul complex, $K(r_1, \ldots, r_m)$, is a free resolution for R.*

The Koszul complex is self dual ([6, Proposition 1.6.10]) so from this we can see that if R is a complete intersection ring then R is Gorenstein. The next result gives us a practical way to check if a ring is a complete intersection.

Proposition 1.2.13. *[25] Let $G \leq \mathrm{GL}(V)$ and let $f_1, \ldots, f_n$ be a HSOP for $k[V]^G$. Let $A = k[f_1, \ldots, f_n]$ and $h_1, \ldots, h_s$ be a set of module generators for $k[V]^G$ over A. Let J be the kernel of the map*

$$A[y_1, \ldots, y_s] \to A[h_1, \ldots, h_s] = k[V]^G, \quad y_i \mapsto h_i$$

where the degree of the y_i are shifted such that $\deg(h_i) = \deg(y_i)$. If $S \subset J$ is the set containing

- *generators for the A linear relations between the h_i, elements of $J \cap (\oplus_{i=1}^m Ay_i)$,*
- *for each $1 \leq i \leq j \leq s$ a relation of the form $y_iy_j - f_{i,j}$ with $f_{i,j} \in \oplus_{i=1}^m Ay_i$,*

then the elements of S form a generating set for the ideal J in $A[y_1, \ldots, y_s]$.

Note that in the above as the HSOP is algebraically independent we do not need to worry about relations between them. Even if we have found a set of relations, showing that they are minimal can be difficult, so it can be useful to move the question to a ring with smaller Krull dimension.

Proposition 1.2.14. *Let $R = k[f_1, \ldots, f_{n+s}] \subseteq k[x_1, \ldots, x_n]$ be a complete intersection ring of dimension n. If $\mathbf{t} = t_1, \ldots, t_m$ is a regular sequence in R then $R/\mathbf{t}R$ is a complete intersection ring.*

Proof. We will first show that if $t_1 \in R$ is a not a zero divisor then $R/(t_1)$ is a complete intersection ring. Let $B = k[y_1, \ldots, y_{n+s}]$, then

$$\begin{aligned} &\phi : B \to R, \\ &\phi(y_i) \mapsto f_i \text{ for } 1 \leq i \leq n+s. \end{aligned}$$

is a surjection. As R is a complete intersection ring we can find a regular sequence $h_1, \ldots, h_s$ which generate the ideal $J = (h_1, \ldots, h_s)$ such that $\ker(\phi) = J$.

Let $t' \in B$ such that $\phi(t') = t_1$. As t_1 is not a zero divisor in R, $\bar{t}'$, the image of t' in B/J, is not a zero divisor. This means that $h_1, \ldots, h_s, t'$ form a regular sequence, and so

$$B/(h_1, \ldots, h_s, t') \cong R/(t_1)$$

is a complete intersection ring.

Our result is true for a regular sequence of length one, we can easily extend to a regular sequence of length m by induction. If $t_1, \ldots, t_m$ are a regular sequence, by the induction hypothesis we assume that $S = R/(t_1, \ldots, t_{m-1})$ is a complete intersection ring. By the property of regular sequences $\bar{t}_m$, the image of t_m in S, is not a zero divisor in S, and so

$$S/\bar{t}_m \cong R/(t_1, \ldots, t_m),$$

is a complete intersection ring. $\square$

1.3 Unipotent groups

Let k be a field of characteristic p, V a k-vector space and $G \leq \mathrm{GL}(V)$ a finite group. We will mainly be interested in finite p-groups, so $|G| = p^s$ for some $s \in \mathbb{N}$.

Definition 1.3.1. The **lower central series** of a group G, is a series of subgroups, $L_1, L_2, \ldots$, of G, defined by $L_1(G) = G$, and

$$L_i(G) = [L_{i-1}(G), G] \text{ for } i \geq 2$$

so $L_2(G) = [G, G] = G'$. If $L_m(G) = 1$ for some m, then G is called **nilpotent**. If $m = n + 1$ is the smallest integer such that $L_m(G) = 1$ then n is known as the **class** of G.

Let now G be a finite p-group. We can always find a normal subgroup N of G such that G/N is elementary abelian, the smallest such subgroup is known at the Frattini subgroup $\Phi(G)$, which can also be characterised by

$$\Phi(G) = \bigcap_{M <_{\max} G} M.$$

By the definition of $[G, G]$ if $g, h \in G$ then $[g, h] \in [G, G]$ so $G/[G, G]$ is abelian (this also means any subgroup of G containing $[G, G]$ is normal). To find an elementary abelian group we need to eliminate any non identity $g^p \in G$, so we see that

$$\Phi(G) = G^p[G, G].$$

A p-group is called special if it is either i)elementary abelian or ii)the Frattini subgroup is given by

$$\Phi(G) = Z(G) = [G, G]$$

in which case it is elementary abelian. A non-abelian special group such that $\Phi(G)$ is cyclic is called extraspecial. The extraspecial groups of order p^3 for $p = 2$

are the dihedral group, D_8, and the quaternions, Q_8, where

$$
\begin{aligned}
Q_8 &= \langle x, y \mid x^4 = y^4 = 1, [x,y] = x^2 = y^2 \rangle, \\
D_8 &= \langle x, y \mid y^4 = x^2 = 1, xyx^{-1} = y^{-1} \rangle.
\end{aligned}
$$

For $p \neq 2$ the extraspecial groups of order p^3 are $M(p)$ and $N(p)$, where

$$
\begin{aligned}
M(p) &= \langle x, y, z \mid x^p = y^p = z^p = 1, [x,z] = [y,z] = 1, [x,y] = z \rangle, \\
N(p) &= \langle x, y \mid x^{p^2} = y^p = 1, yxy^{-1} = x^{p+1} \rangle,
\end{aligned}
$$

(see [17] Theorem 5.1). All extraspecial groups can be written as a central product of copies of extraspecial groups of order p^3 (see [17] Theorem 5.2). Later we shall see some representations of these groups consisting of bireflections.

Let $B = \{x_1, x_2, \ldots, x_n\}$ be an ordered basis for W, and

$$
U_B = \{g \in \mathrm{GL}(V) \mid g(x_i) - x_i \in \langle x_1, \ldots, x_{i-1} \rangle\}.
$$

We can see that U_B (and hence subgroups) are p-groups, furthermore for any p-group $G \leq \mathrm{GL}(V)$ we can find a basis B' such that $G \leq U_{B'}$ (see [10, Lemma 4.0.2]). So we will be looking at groups which are generated by triangular matrices with 1's along the diagonal, we call these unipotent groups.

Definition 1.3.2. For $g \in \mathrm{GL}(V)$, we write $\delta_g \in \mathrm{End}_k(V)$ for the map which takes $v \in V$ to $(g-1)(v)$.

For a unipotent element $g \in \mathrm{GL}(V)$ the **index of** g, $\mathrm{ind}(g)$, is the nilpotence-index of δ_g, that is $c \in \mathbb{N}$ such that $\delta_g^c = 0$, $\delta_g^{c-1} \neq 0$. The **index of a group** $G \leq \mathrm{GL}(V)$ is defined to be $\mathrm{ind}(G) := \max\{\mathrm{ind}(g) | g \in G\}$.

Let $G \leq \mathrm{GL}(V)$, $w \in W$. We define the **stabilizer (or isotropy) subgroup** of w to be

$$
G_w = \{g \in G \mid g(w) = w\}.
$$

The following invariants appear frequently in future sections and are especially important for unipotent groups:

Definition 1.3.3. The **orbit product** of w is defined to be

$$O(w) = \prod_{g \in G/G_{x_i}} g(x_i).$$

For a given basis $B = \{x_1, \ldots, x_n\}$ for W we denote:

$$\mathbf{N}_i^G = \mathbf{N}_i = O(x_i)$$

As applying any element $g \in G$ only permutes it's factors, for any $w \in W$ we see that $O_G(w) \in k[V]^G$. If $G \leq U_B$ then $\mathbf{N}_1^G, \ldots, mathbf{N}_n^G$ form a homogeneous system of parameters for $k[V]^G$ ([10, Proposition 4.0.3]). We can ask for which groups does this HSOP generate the whole invariant ring? These groups are known as Nakajima groups, and we shall see more about them in Chapter 3.

In the non-modular case Nakajima characterised the groups with hypersurface invariant rings as subgroups of reflection groups (with polynomial rings of invariants), see [27]. There are several papers which investigate when the invariant ring is a hypersurface in the modular case (including [5], [21], [8]). The following result can be used to show that all maximal subgroups of Nakajima groups have hypersurface invariant rings:

Proposition 1.3.4. *[10, 11.0.1] Let R be an integral domain of characteristic p and suppose the finite group G acts faithfully on R. Suppose $H \leq G$ is a maximal subgroup of index less than or equal to p. Let $\sigma \in G \backslash H$. If there exists $f \in R^H$ such that if $x := (\sigma - 1)f \in R^G$ then $(\sigma - 1)(R^H) \subseteq Rx$, then $R^H = R^G[f]$.*

In their thesis Yinglin Wu proved the following which is useful in showing when invariant rings are complete intersections.

Proposition 1.3.5. *[33, 3.1.1] Let $G \leq \mathrm{GL}(V)$ be a p-group such that $k[V]^G$ is a complete intersection, and let H be a maximal proper subgroup of G. Then if $k[V]^H = k[V]^G[a]$ for some homogeneous element $a \in k[V]^H$, then $k[V]^H$ is a complete intersection.*

Invariant rings of p-groups are always unique factorisation domains (see [10, Theorem 3.8.1]), so if they are Cohen–Macaulay they are also Gorenstein ([6, Corollary 3.3.19]). This has led to the speculation that maybe Cohen–Macaulay invariant rings of p-groups (characteristic p) are complete intersection rings (we shall see a counter example to this in Chapter 5).

A result of Gordeev, Kac and Watanabe (see [28, Proposition 5.7.7]) says that if $k[V]^G$ is a complete intersection then G is generated by bireflections. The following Theorem by Kemper is stronger in our case where we restrict to unipotent groups.

Theorem 1.3.6. *[23, 3.7] If G is a p-group and $k[V]^G$ is Cohen–Macaulay then G is generated by bireflections.*

The next Theorem from the same paper can be used even for some groups generated by reflections to show their invariant rings are not Cohen–Macaulay.

Theorem 1.3.7. *[23, 3.9] Let $G \leq \mathrm{GL}(V)$ and $N \trianglelefteq G$ such that G/N is an elementary abelian p-group. Suppose there exists $\sigma_0 \in G\backslash N$ not a bireflection, such that for all bireflections $\sigma \in G\backslash N$ we have*

$$V^{\sigma_0} \nsubseteq V^{\sigma}.$$

Then $k[V]^G$ is not Cohen–Macaulay.

The theorems above motivate our interest in groups consisting of bireflections. Before we start to look at the classification of these groups we briefly review some tools which will will want to use later to find their invariant rings.

1.3.1 Monomial orders and SAGBI basis

A monomial in a polynomial ring $R = k[x_1, \ldots, x_n]$ is a term of the form $x_1^{a_1} \ldots x_n^{a_n}$ where $a_i \in \mathbb{N}$ for $1 \leq i \leq n$. A monomial order is a total ordering of monomials satisfying the following additional hypothesis: for monomials m_1, m_2, m, $m_1 > m_2$ implies $mm_1 > mm_2$. There are many types of monomial ordering, but throughout this thesis we will only use graded reverse lexicographical ordering with $x_i < x_j$ if $i < j$.

Definition 1.3.8. Let $R = k[x_1, \ldots, x_n]$ and order the x_i: $x_1 < x_2 < \ldots < x_n$. Let $m_1, m_2 \in R$ be monomials with

$$m_1 = x_1^{a_1} \ldots x_n^{a_n},$$
$$m_2 = x_1^{b_1} \ldots x_n^{b_n}.$$

In **graded reverse lexicographical ordering** $m_1 <_{\text{GRevLex}} m_2$ if and only if one of the following holds

- $a_1 + a_2 + \ldots + a_n < b_1 + \ldots + b_n$.
- $a_1 + a_2 + \ldots + a_n = b_1 + \ldots + b_n$ and we can find some $1 \leq j \leq n$ such that $a_i = b_i$ for $1 \leq i < j$ and $a_j > b_j$.

For any non-zero polynomial $f \in R$ we can write f uniquely in the form

$$f = d_1 m_1 + d_2 m_2 + \ldots + d_s m_s$$

where the m_i's are monomials in R with $m_i >_{\text{GRevLex}} m_{i+1}$ for $1 \leq i \leq s-1$ and for $1 \leq i \leq s$ we have coefficients $d_i \in k \backslash \{0\}$. We call $d_1 m_1$ the lead term of f, denoted $\text{LT}(f)$, and m_1 the lead monomial of f, denoted $\text{LM}(f)$.

Lemma 1.3.9. *Let $f \in k[x_1, \ldots, x_n]$ be homogeneous, then x_1 divides $\text{LT}(f)$ if and only if x_1 divides f.*

Proof. If x_1 divides f then it divides each non-zero term including $\text{LT}(f)$.

Let

$$f = d_1 m_1 + d_2 m_2 + \ldots + d_s m_s$$

and assume x_1 divides $\text{LT}(f)$ so

$$m_1 = x_1^{a_{1,1}} x_2^{a_{2,1}} \ldots x_n^{a_{n,1}}$$

with $a_{1,1} \geq 1$. For all

$$m_l = x_1^{a_{1,l}} x_2^{a_{2,l}} \dots x_n^{a_{n,l}}$$

with $l > 1$ we can find some $1 \leq j \leq n$ such that $a_{i,1} = a_{i,l}$ for $1 \leq i < j$ and $a_{j,1} < a_{j,l}$. Either $j = 1$ and

$$a_{1,l} > a_{1,1} \geq 1$$

or $j > 1$ and

$$a_{i,l} = a_{i,1} \geq 1$$

for all $i < j$, in particular $i = 1$. This means that x_1 divides each monomial m_l, and so x_1 divides f. □

A Gröbner basis for an ideal I is a generating set $f_1, \dots, f_m$ for I such that for any $h \in I$ the lead monomial of h can be written as a multiple of the lead monomial of some f_i. It can be shown that we can find such a generating set for any ideal $I \in R$ using the Buchberger algorithm (see [2, 5.3]), and that this can be used to answer the ideal membership problem.

The acronym SAGBI stands for Subalgebra Analogue of Gröbner Basis for Ideals: we want to find a generating set $f_1, \dots, f_m$ for a subalgebra $A \subseteq R$ such that for any $h \in A$

$$\mathrm{LM}(h) = \mathrm{LM}(f_1)^{a_i} \dots \mathrm{LM}(f_m)^{a_m}$$

with $a_i \in \mathbb{N}$ for $1 \leq i \leq m$. For any $h \in R$ we can perform a subduction of h using a set $\mathbf{f} = \{f_1, \dots, f_m\}$ and the following algorithm:

Algorithm 1 Subduction

function $\textsc{Subdt}(h, \mathbf{f})$
    $H := h$;
    $S := 0$;
    TEST:= true
    while TEST = true **do**
        if $\exists\, a_1, \dots, a_m \in \mathbb{N}$ such that $\mathrm{LM}(H) = \mathrm{LM}(f_1)^{a_1} \dots \mathrm{LM}(f_m)^{a_m}$ **then**
            $c := \mathrm{LT}(H)/\mathrm{LT}(f_1^{a_1} \dots f_m^{a_m})$;
            $H := H - c f_1^{a_1} \dots f_m^{a_m}$;
            $S := S + c f_1^{a_1} \dots f_m^{a_m}$;
            if $H = 0$ **then**
                TEST:= false;
        else
            TEST:=false;
    return H;

In this algorithm we find $S \in k[f_1, \dots, f_m]$ such that

$$h = S + \textsc{Subdt}(h, \mathbf{f}),$$

so if $\textsc{Subdt}(h, \mathbf{f}) = 0$ then $h \in k[f_1, \dots, f_m]$.

Lemma 1.3.10. *Let* $\mathbf{f} = \{f_1, \dots, f_r\}$ *with* $f_1, \dots, f_r \in R$. *Let*

$$h = \sum_{i=1}^{m} c_i h_i$$

where $c_i \in k \backslash 0$ *and* $h_i = f_1^{a_{1,i}} \dots f_r^{a_{r,i}}$ *for* $a_{1,i} \in \mathbb{N}$. *If*

$$\mathrm{LM}(h_i) \neq \mathrm{LM}(h_j)$$

for $i \neq j$ *then* $\textsc{Subdt}(h, \mathbf{f}) = 0$.

Proof. If

$$\mathrm{LM}(h_i) \neq \mathrm{LM}(h_j)$$

for $i \neq j$, then we can assume that the h_i are ordered so that

$$\mathrm{LM}(h_i) <_{\textsc{GRevLex}} \mathrm{LM}(h_j)$$

for $i < j$.

We prove by induction on m: if $m = 1$ then

$$h = c_1 f_1^{a_{1,q}} \dots f_r^{a_{r,q}}$$

so clearly

$$\mathrm{LM}(h) = \mathrm{LM}(f_1^{a_{1,q}} \dots f_r^{a_{r,q}}).$$

The first stage in the subduction process then is to find

$$h - c_1 f_1^{a_{1,q}} \dots f_r^{a_{r,q}} = 0$$

so after the first iteration we find $\textsc{Subdt}(h, \mathbf{f}) = 0$.

Suppose $m > 1$ and let

$$h' = \sum_{i=1}^{m-1} c_i h_i.$$

We see that

$$\mathrm{LM}(h) = \mathrm{LM}(h_m)$$

so the first step of the subduction algorithm is to find

$$h - c_m h_m = h'.$$

The next step of the algorithm is to reiterate the process with h', however by induction we can assume that $\textsc{Subdt}(h', \mathbf{f}) = 0$, and so we find that $\textsc{Subdt}(h, \mathbf{f}) = 0$. $\square$

Let $\mathbf{f} = \{f_1, \ldots, f_m\}$ with $f_i \in R$ as above. A tête–á–tête in $\mathbf{f}$ is a pair $\{F_1, F_2\}$ where

$$F_1 = f_1^{a_1} f_2^{a_2} \ldots f_m^{a_m},$$
$$F_2 = f_1^{b_1} f_2^{a_2} \ldots f_m^{b_m},$$

for some $a_i, b_i \in \mathbb{N}$ for $1 \leq i \leq m$, such that $\mathrm{LM}(F_1) = \mathrm{LM}(F_2)$. A tête–á–tête is called trivial if F_1 and F_2 share a common factor greater than one. If $B = \{f_1, \ldots, f_m\}$ generates an algebra $A \subseteq k[V]$, then the SAGBI algorithm can be used to find a SAGBI basis for A.

Algorithm 2 SAGBI

function SAGBI(B)
    $\mathcal{B} = B$;
    $\mathcal{B}' = B$;
    while $\mathcal{B}' \neq \emptyset$ **do**
        $\mathcal{B}' = \emptyset$;
        for $\{F_1, F_2\}$ a non-trivial tête–á–tête in $\mathcal{B}$ **do**
            $c := \mathrm{LT}(F_1)/\mathrm{LT}(F_2)$;
            $H := \textsc{Subdt}(F_1 - cF_2)$;
            if $H \neq 0$ **then**
                $\mathcal{B}' = \mathcal{B}' \cup \{H\}$;
                $\mathcal{B} = \mathcal{B} \cup \{H\}$;
    return $\mathcal{B}$;

This algorithm may not terminate: unlike a Gröbner basis for an ideal it is not always possible to find a finite SAGBI basis for a subalgebra. Fortunately in the case we are interested in, where the subalgebra A is the invariant ring of some unipotent group, we can always choose a basis and monomial order such that the algorithm terminates.

Theorem 1.3.11. *[10, 5.2.3] If $G \leq \mathrm{GL}(V)$ is triangular, then $k[V]^G$ has a finite SAGBI basis.*

1.3.2 The invariant field and SAGBI/divide-by-x algorithm

For $G \leq \mathrm{GL}(V)$, it is often easier to find the field of fractions, $\mathrm{Quot}(k[V]^G)$, than to find the ring $k[V]^G$. We will write

$$k(V)^G = \mathrm{Quot}(k[V]^G)$$

and refer to $k(V)^G$ as the invariant field.

Let $G \leq \mathrm{GL}(V)$ be a p-group which is triangular with respect to a basis $x_1, \ldots, x_n$ for W. As in the paper by Campbell and Chuai ([7]) we define $R[m] = k[x_1, \ldots, x_m]$ for $m = 1, \ldots, n$, and let $R[0] = k$.

Theorem 1.3.12. *[7, 2.4] Let $G \leq \mathrm{GL}(V)$ be a p-group. For $1 \leq i \leq n$ let $\phi_i \in R[i]^G$ be homogeneous of smallest positive degree in x_i, then*

$$k(V)^G = k(\phi_1, \ldots, \phi_n)$$

and further more we can find $f \in k[V]^G$ such that

$$k[V]^G_f = k[\phi_1, \ldots, \phi_n]_f.$$

Once we have found a set $\{\phi_1, \ldots, \phi_n\}$ we view each ϕ_i as a polynomial in x_i with coefficients in $R[i-1]$ and let c_i be the coefficient of the highest power of x_i in ϕ_i. We can use the c_i to find f in the above Theorem.

Lemma 1.3.13. *[7, 2.1] For any $h \in R[m]^G$, there exists an integer r such that $c_m^r h \in R[m-1]^G[\phi_m]$.*

When we have the invariant field or localised invariant ring, the next Theorem helps us to find the invariant ring, which is not generally an easy task.

Proposition 1.3.14. *[10, 10.0.8] Let $A := k[f_1, \ldots, f_m] \subseteq k[V]^G$. Suppose that*

1. *$k[V]$ is integral over A,*

2. $\mathrm{Quot}(A) = k(V)^G$,

3. *there exists* $h \in A$ *such that* hA *is a prime ideal of* A *and* A_h *is a unique factorisation domain.*

Then $A = k[V]^G$.

Proof. If hA is a prime ideal and A_h is a unique factorisation domain then A is a unique factorisation domain by [3, 6.3.1]. If A is integral over A and a unique factorisation domain then A is integrally closed by [10, Proposition 3.0.2]. If $f \in k[V]^G$ then $f \in \mathrm{Quot}(A)$ as $\mathrm{Quot}(A) = k(V)^G$, as A is integrally closed this means that $f \in A$. $\square$

Finding a ring A that meets the first two conditions is not usually too difficult: if A contains a HSOP for $k[V]^G$ then we know it is integral over A and we can use Theorem 1.3.12 to find a ring such that $\mathrm{Quot}(A) = k(V)^G$. Checking that A fulfils the third condition is usually more difficult and here it helps to have a SAGBI basis using the graded reverse lexicographical ordering. The following theorem is proved by combining Lemma 1.3.9 and Proposition 1.3.14.

Proposition 1.3.15. *[9, 1.1] Let* $G \leq \mathrm{GL}(V)$, $x_1, \ldots, x_n$ *a basis for* W *and let* $f_1, \ldots, f_r \in k[V]^G$ *be homogeneous. Let* $A = k[x_1, f_1, \ldots, f_r]$ *such that* $A_{x_1} = k[V]^G_{x_1}$. *Suppose that the following hold*

- A *is integral over* $k[V]^G$*;*
- $x_1, f_1, \ldots, f_r$ *are a SAGBI basis for* A*;*
- x_1 *doesn't divide* $\mathrm{LM}(f_i)$ *for* $1 \leq i \leq r$.

Then $A = k[V]^G$.

***Example* 1.3.16.** *Let* $G = \langle \sigma \rangle$ *where*

$$\sigma = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

with respect to the natural basis $B = \{x_1, x_2, x_3\}$ *for* W*. As* $kx_1 = W^G$*, we can choose* $\phi_1 = x_1$*.* G *acts as a Nakajima group on* $\langle x_1, x_2\rangle$*, so we can choose* $\phi_2 = \mathbf{N}_2$*. Let*

$$d = x_2^2 - x_1x_3$$

then as $\deg_{x_3}(d) = 1$*, we can choose* $\phi_3 = d$*. Using Theorem 1.3.12 if* $A' = k[\phi_1, \phi_2, \phi_3]$ *then*

$$\mathrm{Quot}(A') = k(V)^G.$$

If we view the ϕ_i *as polynomials in* x_i *and let* c_i *be the coefficient of the highest degree term of* x_i*, then*

$$c_1 = 1, \quad c_2 = 1, \quad c_3 = -x_1.$$

Using Lemma 1.3.13 this means that

$$A'_{x_1} = k[V]_{x_1}.$$

Let $A = A'[\mathbf{N}_3^G]$*, then* A *contains a HSOP for* $k[V]^G$ *and so is integral over* $k[V]^G$*. As* $\mathbf{N}_3^G \in k[V]^G$

$$A_{x_1} = A'_{x_1} = k[V]_{x_1}.$$

G *is a specific example of a symmetric square representation of* $(k, +)$ *as in Section 3 of [9]. They show that all tête–á–têtes in* $\mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3, d$ *subduct to zero and so using Theorem 1.3.14*

$$A = k[V]^G.$$

If we find A with generating set B such that A contains a HSOP for $k[V]^G$ and $A_{x_1} = k[V]^G_{x_1}$ then we can perform the SAGBI/divide-by-x algorithm as introduced in Section 1 of [9] in order to get to a new ring A' with generating set B' which meet the criteria of Proposition 1.3.15.

Algorithm 3 SAGBI/divide-by-x

function SAGBI DIVIDE(B,x)
  $\mathcal{B} = B$;
  while $\exists\, f \in B$ such that x divides $LT(f)$ **do**
    $f' = f/x$
    $\mathcal{B} = (\mathcal{B} \cup \{f'\}) \setminus \{f\}$
  $\mathcal{B}' = B$;
  while $\mathcal{B}' \neq \emptyset$ **do**
    $\mathcal{B}' = \emptyset$;
    for $\{F_1, F_2\}$ a non-trivial tête–á–tête in $\mathcal{B}$ **do**
      $c := \mathrm{LT}(F_1)/\mathrm{LT}(F_2)$;
      $H := \mathrm{SUBDT}(F_1 - cF_2)$;
      if $H \neq 0$ **then**
        if x divides $\mathrm{LM}(x)$ **then** $H := H/x$;
          $\mathcal{B}' = \mathcal{B}' \cup \{H\}$;
          $\mathcal{B} = \mathcal{B} \cup \{H\}$;
        else
          $\mathcal{B}' = \mathcal{B}' \cup \{H\}$;
          $\mathcal{B} = \mathcal{B} \cup \{H\}$;
  return $\mathcal{B}$;

1.4 Reflection and bireflection groups

As we have seen reflection and bireflection groups are important in invariant theory, however they are also interesting in their own right. For a reflection $g \in \mathrm{GL}(V)$ we call the fixed space V^g the hyperplane of g and the one-dimensional vector space $\mathrm{Im}(1-g)$ the direction of g. There are two types of reflections: diagonalisable reflections of order coprime to the characteristic of the field and transvections. Transvections only have finite order when the field k has positive characteristic p in which case they have order p.

A diagonalisable reflection g has one as an eigenvalue with multiplicity $n-1$ and another eigenvalue λ_g which is a root of unity. The second eigenvalue, λ_g is called the root of g. The linear transformation g is called a real reflection, in the case that $k = \mathbb{R}$, and $\lambda_g = -1$. The finite groups generated by real reflections are called Coxeter groups and were classified by Coxeter in [12] using ideas from the theory of Lie algebras (e.g. root systems) and the theory of hyperplane arrangements. Later similar techniques were extended to $k = \mathbb{C}$ by G.C.Shephard and J.A.Todd, who showed that these also had polynomial invariant rings ([30]).

The study of real reflection groups is important to Lie Theory and is very well developed. Whilst we don't have the same descriptions with roots for modular reflection groups a classification for the irreducible representations can be found in [24] where it is then used to prove the following.

Theorem 1.4.1. *Suppose V is an irreducible representation of the modular group G. Then $k[V]^G$ is a polynomial ring if and only if G is generated by reflections and if W is any non-trivial subspace of V, then $k[V]^{G_W}$ has a polynomial ring of invariants.*

A classification for the irreducible bireflection groups can be found in [18] by Guralnick and Saxl, however in the modular case we are interested in reducible bireflection groups.

In [31, 8.2] Smith looks at the modular groups consisting entirely of reflections.

Proposition 1.4.2. *[31, 8.2.18] Let k be a field of characteristic $p \neq 0$, $G \leq \mathrm{GL}(V)$ such that every non identity element of G is a reflection. Either V^G has dimension $n-1$ or $(V^*)^G$ has dimension $n-1$.*

Using this it can be shown that all groups consisting of reflections have polynomial rings of invariants. In the next chapter we will see that finding even just the unipotent groups consisting of bireflections becomes more complicated.

From Theorems 1.3.6 and 1.3.7 we may hope that these would all have Cohen–Macaulay rings of invariants, however this is not true in general. Let $G = \langle g, h\rangle$

where

$$g := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}, \quad h := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

with respect to a basis $\{e_1, \ldots, e_n\}$ for V. In [15] they show that $k[V]^G$ is flat for $p = 2$ and so

$$\operatorname{depth}(k[V]^G) = 4 < \dim(k[V]^G).$$

Therefore G is a group consisting of bireflections (an example of a two–row group) which has non Cohen–Macaulay invariant ring. There is a class of unipotent groups consisting of bireflections where the invariant ring is always Cohen–Macaulay:

Theorem 1.4.3. *[10, 3.9.1] Let* $G \leq \mathrm{GL}(V)$, $\dim_k(V) = n$ *then:*

1. *if* $\dim_k(V^G) = n - 1$, *then* $k[V]^G$ *is a polynomial ring.*

2. *if* $\dim_k(V^G) = n - 2$, *then* $k[V]^G$ *is Cohen–Macaulay.*

Due to the form of their matrix representations on V we shall call a group G with $\dim_k(V^G) \geq n - 2$ a two–column group. In his thesis ([33]) Yinglin Wu investigates the invariant rings of these groups, however considering the matrices on $W = V^*$ rather than V he calls them two–row groups. Wu studies the conjecture that these groups always have complete intersection invariant rings and in the modular case where $k = \mathbb{F}_p$ he shows that if G is an abelian p-group generated by reflections with $\dim_k(V^G) \geq n-2$, then $k[V]^G$ is a complete intersection ring. In the non-modular case he finds the conjecture to be false, by finding a two–column group G such that $k[V]^G$ is not Gorenstein, and hence not complete intersection. We will see in Chapter 5 that it is not true in the modular case either. Our example G is a p-group, so $k[V]^G$ is Gorenstein. This example also shows that Cohen–Macaulay does not imply complete intersection for invariant rings of unipotent groups.

Chapter 2

Bireflection Groups

From here onwards we need a field of characteristic p so we set $k = \mathbb{F}_q$ where $q = p^r$. We will let V be a k-vector space, and later, when we start to look for invariant rings, we will need $W = V^*$. In this chapter we will classify all unipotent groups consisting of bireflections for $p \neq 2$. We will look at the different types of unipotent bireflections and their properties, but first we want to look at the correspondence between the properties of the group action on V and its action on W. We start with a definition linking subspaces of V with subspaces of W.

Definition 2.0.4. Let V be a vector space, $U \subseteq V$. We define

$$U^{\perp} = \{\lambda \in V^* \mid \lambda(u) = 0 \text{ for all } u \in U\} \subseteq V^*.$$

The next lemma will be used many times in this thesis to move between groups and their dual representations.

Lemma 2.0.5. *Let V be a finite-dimensional k-vector space and $G \leq \mathrm{GL}(V)$. Then the following hold:*

1. $[G, V^*]^{\perp} = V^G$ *and* $[G, V]^{\perp} = (V^*)^G$.

2. $\dim_k([G, V^*]) = \dim_k(V) - \dim_k(V^G) = \mathrm{codim}(V^G)$.

3. $V^G \leq [G, V]$ *if and only if* $(V^*)^G \leq [G, V^*]$.

4. *the canonical map $V \to V/V^G$ induces an isomorphism*

$$[G,V]/[G,V]^G \cong [G, V/V^G].$$

Proof. 1. If $v \in [G,V^*]^\perp$ then for all $\lambda \in V^*$, $g \in G$

$$(g(\lambda)-\lambda)(v) = 0.$$

If $v \in V^G$ then $g(v) - v = 0$ for all $g \in G$. For $\lambda \in V^*$, $v \in V$ and $g \in G$ we have

$$(g(\lambda)-\lambda)(v) = \lambda(gv - v)$$

hence the claim.

2. As $[G,V^*]^\perp = V^G$,

$$\dim_k(V) = \dim_k(V^G) + \dim_k([G,V^*]).$$

Rearranging gives the result.

3. $V^G \leq [G,V] \iff (V^G)^\perp \geq [G,V]^\perp \iff [G,V^*] \geq (V^*)^G$.

4. $[V/V^G, G] = ([V,G] + V^G \cong [V,G]/([V,G]\cap V^G) = [V,G]/[V,G]^G$.

□

2.1 Bireflections

Definition 2.1.1. For $u \in V$ and $0 \neq \gamma \in u^\perp$ we set $t_u^\gamma \in \mathrm{GL}(V)$ to be the transvection mapping $s \in V$ to $s + \gamma(s)u$.

In a field of characteristic p, transvections are reflections of order p. It isn't hard to see that $V^{t_u^\gamma} = \ker(\gamma)$ and $[V, t_u^\gamma] = \langle u \rangle$. We start by proving some other general results for transvections which will be useful later on.

Lemma 2.1.2. *For $u_1, u_2 \in V$, $\gamma_1 \in u_1^\perp$, $\gamma_2 \in u_2^\perp$:*

1. *if* $u_2 \in \ker(\gamma_1)$ *then*

$$|t_{u_1}^{\gamma_1} t_{u_2}^{\gamma_2}| = \begin{cases} p^2 & \text{if } p = 2 \text{ and } u_1 \notin \ker(\gamma_2), \\ p & \text{otherwise.} \end{cases}$$

2. *if* $u_1 \in \ker(\gamma_2)$, *then* $t_{u_1}^{\gamma_1} t_{u_2}^{\gamma_2} = t_{u_2'}^{\gamma_2} t_{u_1}^{\gamma_1}$ *where* $u_2' = t_{u_1}^{\gamma_1}(u_2)$.

3. $|t_{u_1}^{\gamma_1} t_{u_2}^{\gamma_2}|$ *is a power of* p *if and only if either* $\gamma_1(u_2) = 0$ *or* $\gamma_2(u_1) = 0$.

4. *if* $u_1 \in \ker(\gamma_2)$ *then* $t_{u_1}^{\gamma_1} t_{u_1}^{\gamma_2} = t_{u_1}^{\gamma_1+\gamma_2}$.

5. *if* $u_2 \in \ker(\gamma_1)$ *then* $t_{u_1}^{\gamma_1} t_{u_2}^{\gamma_1} = t_{u_1+u_2}^{\gamma_1}$.

6. $t_{u_1}^{c\gamma_1} = t_{cu_1}^{\gamma_1}$ *for all* $c \in k$.

Proof. 1. Let $t = t_{u_1}^{\gamma_1} t_{u_2}^{\gamma_2}$. For $p \neq 2$ we will show that if $u_2 \in \ker(\gamma_1)$ then for $a \in \mathbb{N}$, $w \in V$:

$$t^a(w) = w + a\gamma_1(w)u_1 + a\gamma_2(w)u_2 + \frac{a(a-1)}{2}\gamma_1(w)\gamma_2(u_1)u_2$$

We do this by induction, it is clear for $a = 1$, and then:

$$\begin{aligned} t^a(w) &= tt^{a-1}(w) \\ &= t(w + (a-1)\gamma_1(w)u_1 + (a-1)\gamma_2(w)u_2 + \frac{(a-1)(a-2)}{2}\gamma_1(w)\gamma_2(u_1)u_2) \\ &= w + a\gamma_1(w)u_1 + a\gamma_2(w)u_2 + \frac{a(a-1)}{2}\gamma_1(w)\gamma_2(u_1)u_2 \end{aligned}$$

so $|t| = p$. If $p = 2$ we can see that:

$$t^2(w) = w + \gamma_1(w)\gamma_2(u_1)u_2$$

so either $t^2 = 1$ or t^2 is a transvection with order 2, and so $(t^2)^2 = t^4 = 1$ and t has order p^2.

2. Let $w \in V$ so:

$$\begin{aligned} t_{u_1}^{\gamma_1} t_{u_2}^{\gamma_2}(w) &= t_{u_1}^{\gamma_1}(w + \gamma_2(w)u_2) \\ &= w + \gamma_1(w)u_1 + \gamma_2(w)u_2 + \gamma_2(w)\gamma_1(u_2)u_1 \\ &= (w + \gamma_1(w)u_1) + \gamma_2(w)(u_2 + \gamma_1(u_2)u_1) \\ &= t_{u_2'}^{\gamma_2} t_{u_1}^{\gamma_1}(w) \end{aligned}$$

where u_2' is as given above.

3. We can see using the first two parts that if $\gamma_1(u_2) = 0$ or $\gamma_2(u_1) = 0$ then $|t_{u_1}^{\gamma_1} t_{u_2}^{\gamma_2}|$ is a power of p. Let $t = t_{u_1}^{\gamma_1} t_{u_2}^{\gamma_2}$, if $|t|$ is a power of p, then $[t, V]^t \neq \{0\}$. If $u_2 \in ku_1$ then we already know that:

$$\gamma_1(u_2) = \gamma_2(u_1) = 0$$

so assume u_1, u_2 linearly independent. We can see that $[t, V] \leq \langle u_1, u_2 \rangle$, so we can find $a_1, a_2 \in k$ not both zero such that $a_1u_1 + a_2u_2 \in [t, V]^t$:

$$\begin{aligned} a_1u_1 + a_2u_2 &= t(a_1u_1 + a_2u_2) \\ &= (a_1 + a_2\gamma_1(u_2) + a_1\gamma_2(u_1)\gamma_1(u_2))u_1 + (a_2 + a_1\gamma_2(u_1))u_2 \end{aligned}$$

comparing u_2 terms we see that $a_1\gamma_2(u_1) = 0$, so either $\gamma_2(u_1) = 0$ or $a_1 = 0$. If $a_1 = 0$ then $a_2 \neq 0$ and comparing u_2 terms $a_2\gamma_1(u_2) = 0$ so $\gamma_1(u_2) = 0$.

4. For all $v \in V$

$$\begin{aligned} t_{u_1}^{\gamma_1} t_{u_1}^{\gamma_2}(v) &= t_{u_1}^{\gamma_1}(v + \gamma_2(v)u_1) \\ &= v + (\gamma_1(v) + \gamma_2(v))u_1 \\ &= v + (\gamma_1 + \gamma_2)(v)u_1 \\ &= t_{u_1}^{\gamma_1+\gamma_2}(v) \end{aligned}$$

so $t_{u_1}^{\gamma_1} t_{u_1}^{\gamma_2} = t_{u_1}^{\gamma_1+\gamma_2}$.

5. Similarly for all $v \in V$

$$\begin{aligned} t_{u_1}^{\gamma_1} t_{u_2}^{\gamma_1}(v) &= t_{u_1}^{\gamma_1}(v + \gamma_1(v)u_1) \\ &= v + \gamma_1(v)u_1 + \gamma_1(v)u_2 \\ &= v + \gamma_1(v)(u_1 + u_2) \\ &= t_{u_1+u_2}^{\gamma_1} \end{aligned}$$

so $t_{u_1}^{\gamma_1} t_{u_2}^{\gamma_1} = t_{u_1+u_2}^{\gamma_1}$.

6. For any $c \in k$ and $v \in V$

$$\begin{aligned} t_{u_1}^{c\gamma_1}(v) &= v + c\gamma_1(v)(u_1) \\ &= v + \gamma_1(v)(cu_1) \\ &= t_{cu_1}^{\gamma_1} \end{aligned}$$ □

Later we will want to write bireflections as products of transvections. The next lemma will be useful when rewriting and comparing them.

Lemma 2.1.3. *Let $m \in \mathbb{N}$, $\gamma_1, \ldots, \gamma_m \in V^*$ and $u_1, \ldots, u_m$ such that $\gamma_i(u_j) = 0$ for $1 \le i \le j \le m$. Let*

$$g = t_{u_1}^{\gamma_1} t_{u_2}^{\gamma_2} \ldots t_{u_m}^{\gamma_m}$$

then

1. *if $\gamma_1, \ldots, \gamma_m$ are linearly independent and*

$$h = t_{u'_1}^{\gamma_1} t_{u'_2}^{\gamma_2} \ldots t_{u'_m}^{\gamma_m}$$

such that $\gamma_i(u'_j) = 0$ for $1 \le i \le j \le m$. Then $g = h$ if and only if $u'_i = u_i$ for $1 \le i \le m$.

2. if $u_1, \ldots, u_m$ are linearly independent and

$$h = t_{u_1}^{\gamma_1'} t_{u_2}^{\gamma_2'} \ldots t_{u_m}^{\gamma_m'}$$

such that $\gamma_i'(u_j) = 0$ for $1 \leq i \leq j \leq m$. Then $g = h$ if and only if $\gamma_i' = \gamma$ for $1 \leq i \leq m$.

Proof. Part 1) is equivalent to Lemma 3.0.3. Part 2) is dual to part 1), which we can see from Lemma 2.0.5. □

The following can be used to check the commutator and fixed spaces of elements of $\mathrm{GL}(V)$ to see if they are bireflections.

Lemma 2.1.4. *Let $g, h \in \mathrm{GL}(V)$ be unipotent, $w \in V$. Then:*

$$\begin{aligned}
\delta_{gh}(w) &= \delta_g(w) + \delta_h(w) + \delta_g\delta_h(w),\\
\delta_{g^i}(w) &= \sum_{j=1}^{i} \binom{i}{j} \delta_g^j(w).
\end{aligned}$$

If g is a bireflection then

$$\delta_{g^i}(w) = i\delta_g(w) + \tfrac{i(i-1)}{2}\delta_g^2(w).$$

Proof. For $g, h \in G$, $w \in V$:

$$\begin{aligned}
gh(w) &= g(w + \delta_h(w))\\
&= w + \delta_g(w) + \delta_h(w) + \delta_g\delta_h(w)
\end{aligned}$$

so the first result holds

$$\delta_{gh}(w) = \delta_g(w) + \delta_h(w) + \delta_g\delta_h(w).$$

For the second result

$$\begin{aligned}\delta_{g^i}(w) &= (g^i - 1)(w) = (((g-1)+1)^i - 1)(w)\\ &= \left(\sum_{j=0}^{i}\binom{i}{j}(g-1)^j(w)\right) - w = \sum_{j=1}^{i}\binom{i}{j}(g-1)^j(w)\\ &= \sum_{j=1}^{i}\binom{i}{j}\delta_g^j(w).\end{aligned}$$

If g is a bireflection then $\delta_g^j(v) = 0$ for any $j > 2$ and $v \in V$ and so we get:

$$\delta_{g^ih}(w) = i\delta_g + \tfrac{i(i-1)}{2}\delta_g^2(w). \qquad \square$$

From this we find the following:

Lemma 2.1.5. *Let $G \leq \mathrm{GL}(V)$. If $[G,V] \leq V^G$ then $[G,G] = 1$ and G is elementary abelian. If G is a unipotent transvection group then $[G,G] = 1$ implies $[G,V] \leq V^G$.*

Proof. By the above, for all $g, h \in G$, $w \in V$

$$\delta_{gh}(w) = \delta_g\delta_h(w) + \delta_g(w) + \delta_h(w).$$

If $[G,V] \leq V^G$ then $\delta_g\delta_h(w) = 0$ and so

$$\delta_{gh}(w) = \delta_g(w) + \delta_h(w) = \delta_{hg}(w),$$

which means that $[G,G] = 1$.

Since $(g-1)^2 = 0$, for all $g \in G$, we see that $0 = (g-1)^p = g^p - 1$ so all elements of G have order p and G is elementary abelian.

Let $G = \langle t_i \mid i = 1, \cdots, \ell\rangle$ where $t_i = t_{v_i}^{\gamma_i}$ for $1 \leq i \leq \ell$. If G is unipotent then for $1 \leq i, j \leq \ell$ either $\gamma_i(v_j) = 0$ or $\gamma_j(v_i) = 0$ (otherwise $|t_it_j|$ is not a power of p by Lemma 2.1.2(3)). Suppose that $\gamma_i(v_j) = 0$ then by Lemma 2.1.2(2)

$$t_{v_i}^{\gamma_i}t_{v_j}^{\gamma_j} = t_{v_j'}^{\gamma_j}t_{v_i}^{\gamma_i}$$

where $v'_j = t_i(v_j)$. If G is abelian then $v_j = t_i(v_j)$. This means that

$$\langle v_1, \ldots, v_l \rangle = [G, V] \leq V^{t_1} \cap \ldots \cap V^{t_\ell} = V^G. \qquad \square$$

Definition 2.1.6. All reflections are bireflections (and so also all reflection groups are bireflection groups), so unipotent bireflections include **transvections**:

$$\begin{pmatrix} \mathrm{J}_2 & \mathbf{0} \\ \mathbf{0} & \mathrm{I}_{n-2} \end{pmatrix}$$

as well as those elements of $\mathrm{GL}(V)$ conjugate to one of

$$\begin{pmatrix} \mathrm{J}_3 & \mathbf{0} \\ \mathbf{0} & \mathrm{I}_{n-3} \end{pmatrix} \qquad \textbf{index 3 bireflection,}$$

$$\begin{pmatrix} \mathrm{J}_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathrm{J}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathrm{I}_{n-4} \end{pmatrix} \qquad \textbf{double transvection,}$$

where J_2, J_3 are Jordan 2 and 3 blocks respectively. If g is a unipotent bireflection it can be written as either t_u^γ for some $u \in V$ with $\gamma \in u^\perp$ in the case of a transvection, or as $t_{u_1}^{\gamma_1} t_{u_2}^{\gamma_2}$ for some $u_1, u_2 \in V$, $\gamma_1 \in u_1^\perp$, $\gamma_2 \in u_2^\perp$ with $\gamma_1(u_2) = 0$. If $\gamma_2(u_1) \neq 0$ it is an index 3 bireflection, if $\gamma_2(u_1) = 0$ then it is a double transvection.

2.2 Groups consisting of bireflections

We now look for the p-groups G which are not only generated by bireflections, but g is a bireflection for all elements $g \in G$, we shall call these *pure bireflection groups*. We define certain classes of group with this property.

Definition 2.2.1. Let $G = \langle g_1, \ldots, g_l \rangle \leq \mathrm{GL}(V)$. Then:

- If $\dim_k(\cap_{i=1}^{l} V^{g_i}) \geq n-2$ or equivalently $\dim_k(V^G) \geq n-2$ then G is a **two–column group** on V (if $\dim_k(V^G) \geq n-1$ then G is a **one–column group**).
- If $\dim_k(\sum_{i=1}^{l}[g_i, V]) \leq 2$ or equivalently $\dim_k([G, V]) \leq 2$ then G is a **two–row group** on V (if $\dim_k([G, V]) \leq 1$ then G is a **one–row group**).
- If there exists $U \subset V$ such that $\dim_k(U) = n-1$ and $[G, U] \leq kv$ for some $v \in U^G$, then G is a **hook group** on V with hyperplane U and line kv.

Looking at the unipotent groups consisting of reflections (see Proposition 1.4.2) we might expect these to be the only types of unipotent pure bireflection group, however we shall see that there are some exceptional types.

Definition 2.2.2. Let $n \geq 5$ and $G \leq \mathrm{GL}(V)$ a unipotent group. Let $g, h \in G$ be bireflections and $U = V^g + V^h$. We call g, h a **special pair** if we can find $r_1, r_2, v \in V$ linearly independent such that the following hold:

$$\begin{aligned} &\dim_k(U) = n-1, \quad \dim_k(V^g \cap V^h) = n-3, \\ &v \notin U, \quad r_1, r_2 \in V^g \cap V^h \end{aligned}$$

and:

$$\begin{aligned} \delta_g(U) &= kv, & \delta_g(v) &= r_2, \\ \delta_h(U) &= k(v + r_1), & \delta_h(v) &= 2r_1 + r_2. \end{aligned}$$

If $g, h \in G$ are a special pair and G is a pure bireflection group then we call G an **exceptional pure bireflection group** (or exceptional group) **of type one**, and g, h an **exceptional pair** (for a matrix example see Chapter 6.1).

Lemma 2.2.3. *For $g, h \in \mathrm{GL}(V)$ the following are equivalent:*

1. *g, h are a special pair;*

2. $g = t_{\hat{v}}^{\zeta_1} t_{\hat{r}_2}^{\hat{v}^*}$ *for* $\zeta_1, \hat{v}^* \in V^*$ *linearly independent,* $\hat{r}_2, \hat{v} \in \ker(\zeta_1)$ *such that*

$$\hat{v}^*(\hat{r}_2) = 0, \quad \hat{v}^*(\hat{v}) = 1$$

and we can find $a, b \in k$*,* $\hat{r}_1 \in \ker(\zeta_1) \cap \ker(\hat{v}^*)$ *and some* $\zeta_2 \in V^*$ *linearly independent to* ζ_1 *and* $\hat{v}^*$ *such that*

$$\zeta_2(\hat{r}_2) = \zeta_2(\hat{r}_1) = \zeta_2(\hat{v}) = \zeta_1(\hat{r}_1) = 0$$

and $h = t_{\beta_1}^{\zeta_1} t_{\beta_2}^{\zeta_2} t_{\beta_3}^{\hat{v}^*}$ *where*

$$\begin{aligned}
\beta_1 &= b\hat{v} + (a - ab)\hat{r}_2 + (2a + b)\hat{r}_1, \\
\beta_2 &= v - a\hat{r}_2 + \hat{r}_1, \\
\beta_3 &= 2\hat{r}_1 + \hat{r}_2;
\end{aligned}$$

3. *we can find some* $\gamma_1, \gamma_2, v^* \in V^*$ *linearly independent, and*

$$r_1, r_2, v \in \ker(\gamma_1) \cap \ker(\gamma_2)$$

linearly independent with

$$\begin{aligned}
v^*(r_1) &= v^*(r_2) = 0, \\
v^*(v) &= 1
\end{aligned}$$

such that $g = t_v^{\gamma_1} t_{r_2}^{v^*}$ *and* $h = t_{v+r_1}^{\gamma_2} t_{2r_1+r_2}^{v^*}$.

Proof. 1) $\Rightarrow$ 2) Suppose g, h are a special pair. This means g is an index 3 bireflection so we can find $\zeta_1, \hat{v}^* \in V^*$ linearly independent, $\hat{r}_2, \hat{v} \in \ker(\gamma_1)$ such that $\hat{v}^*(\hat{r}_2) = 0$, $\hat{v}^*(\hat{v}) = 1$ and $g = t_{\hat{v}}^{\zeta_1} t_{\hat{r}_2}^{\hat{v}^*}$. Let $u_1 \in \ker(\hat{v}^*)$ such that $\zeta_1(u_1) = 1$.

As g, h are a special pair if $U = V^g + V^h$ then $\dim_k(U) = n - 1$. As V^g is $n-2$ dimensional and $\hat{v}, u_1 \notin V^g$ are linearly independent we can see that

$$V = ku_1 + k\hat{v} + V^g.$$

This means we can find some $a', a \in k$ such that if $u = a'u_1 - a\hat{v}$ then $U = ku + V^g$. As $\delta_g(U) \not\leq V^g$ and $\delta_g(\hat{v}) \in V^g$, we see $a' \neq 0$ and so we can assume $a' = 1$.

We can now find $v = \delta_g(U)$, let

$$v = \delta_g(u) = \delta_g(u_1 - a\hat{v}) = \hat{v} - a\hat{r}_2.$$

By the definition of a special pair we can find $\hat{r}_1 \in V^g \cap V^h$ such that

$$\delta_h(U) = k(v + \hat{r}_1).$$

Since

$$\delta_g(v) = \delta_g(\hat{v} - a\hat{r}_2) = \delta_g(\hat{v})$$

we know that $\hat{r}_2 \in V^g \cap V^h$ and

$$\delta_h(v) = \delta_h(\hat{v} - a\hat{r}_2) = \delta_h(\hat{v}) = 2\hat{r}_1 + \hat{r}_2.$$

Let $u_2 \in \ker(\zeta_1) \cap \ker(\hat{v}^*)$ such that $\delta_h(u_2) = v + \hat{r}_1$. We know

$$u_2 \in U \backslash V^h$$

so

$$U = ku_2 + V^h.$$

We know that

$$u = u_1 - a\hat{v} \in U$$

so we can find some $b \in k$ such that $u_1 - a\hat{v} - bu_2 \in V^h$ and

$$\delta_h(u_1 - a\hat{v} - bu_2) = 0.$$

This means that

$$\delta_h(u_1) = bv + (a - ab)r_2 + (2a + b)r_1.$$

Let $\zeta_2 \in V^*$ such that $\ker(\zeta_2) = V^g \cap V^h + ku_1 + k\hat{v}$ and $\zeta_2(u_2) = 1$. Let

$$\begin{aligned}
\beta_1 &= b\hat{v} + (a - ab)\hat{r}_2 + (2a + b)\hat{r}_1,\\
\beta_2 &= v - a\hat{r}_2 + \hat{r}_1,\\
\beta_3 &= 2\hat{r}_1 + \hat{r}_2,
\end{aligned}$$

and $\tilde{h} = t_{\beta_1}^{\zeta_1} t_{\beta_2}^{\zeta_2} t_{\beta_3}^{\hat{v}^*}$. We find

$$\begin{aligned}
\delta_{\tilde{h}}(u_1) &= u_1 + b\hat{v} + (a - ab)\hat{r}_2 + (2a + b)\hat{r}_1 = \delta_h(u_1),\\
\delta_{\tilde{h}}(u_2) &= u_2 + \hat{v} - a\hat{r}_2 + \hat{r}_1 = \delta_h(u_2),\\
\delta_{\tilde{h}}(\hat{v}) &= \hat{v} + 2\hat{r}_1 + \hat{r}_2 = \delta_h(\hat{v}),
\end{aligned}$$

and

$$\delta_{\tilde{h}}(V^g \cap V^h) = 0 = \delta_h(V^g \cap V^h)$$

so $h = \tilde{h}$ as required.

2) $\Rightarrow$ 3) Let g, h be as described in part 2). Then we see that $\beta_1 = b\beta_2 + a\beta_3$ and:

$$\begin{aligned}
g &= t_{\hat{v}}^{\zeta_1} t_{\hat{r}_2}^{\hat{v}^*} = t_{\hat{v}}^{\zeta_1} t_{-a\hat{r}_2}^{\zeta_1} t_{a\hat{r}_2}^{\zeta_1} t_{\hat{r}_2}^{\hat{v}^*} = t_{\hat{v}-\hat{r}_2}^{\zeta_1} t_{\hat{r}_2}^{\hat{v}^*+a\zeta_1},\\
h &= t_{b\beta_2+a\beta_3}^{\zeta_1} t_{\beta_2}^{\zeta_2} t_{\beta_3}^{\hat{v}*} = t_{\beta_2}^{b\zeta_1} t_{\beta_3}^{a\zeta_1} t_{\beta_2}^{\zeta_2} t_{\beta_3}^{\hat{v}*} = t_{\beta_2}^{\zeta_2+b\zeta_1} t_{\beta_3}^{\hat{v}^*+a\zeta_1}
\end{aligned}$$

(using Lemma 2.1.2).

Let

$$\gamma_1 = \zeta_1, \quad \gamma_2 = \zeta_2 + b\zeta_1, \quad v^* = \hat{v}^* + a\zeta_1$$
$$r_1 = \hat{r}_1, \quad r_2 = \hat{r}_2, \quad v = \hat{v} - a\hat{r}_2$$

then we can write

$$g = t_v^{\gamma_1} t_{r_2}^{v^*},$$
$$h = t_{v+r_1}^{\gamma_2} t_{2r_1+r_2}^{v^*}$$

so they are in the form required.

3) $\Rightarrow$ 1) Let g, h be as described in part 3). As v, r_2 and $v + r_1, 2r_1 + r_2$ are linearly independent, $V^g = \ker(\gamma_1) \cap \ker(v^*)$ and $V^h = \ker(\gamma_2) \cap \ker(v^*)$ so

$$\begin{aligned} V^g + V^h &= \ker(v^*) \cap \ker(\gamma_1) + \ker(v^*) \cap \ker(\gamma_2) \\ &= \ker(v^*) \cap (\ker(\gamma_1) + \ker(\gamma_2)) \\ &= \ker(v^*) \cap V = \ker(v^*) \end{aligned}$$

which has dimension $n - 1$. We can also find

$$V^g \cap V^h = \ker(\gamma_1) \cap \ker(\gamma_2) \cap \ker(v^*)$$

and check it has the correct dimension

$$\begin{aligned} \dim_k(V^g \cap V^h) &= \dim_k(V^g) + \dim_k(V^h) - \dim_k(V^g + V^h) \\ &= n - 2 + n - 2 - n + 1 = n - 3. \end{aligned}$$

We can see that

$$\delta_g(U) = kv, \quad \delta_g(v) = r_2,$$
$$\delta_h(U) = k(v + r_1), \quad \delta_h(v) = 2r_1 + r_2,$$

so g, h are a special pair. $\square$

We now check that exceptional groups of type one exist.

Lemma 2.2.4. *If G is generated by a special pair then G is an exceptional group of type one. Moreover, for $p \neq 2$, $G \cong M(p)$ is an extraspecial group of order p^3.*

Proof. Let $g, h \in \mathrm{GL}(V)$ be a special pair and $G = \langle g, h\rangle$. By Lemma 2.2.3 we can find some $\gamma_1, \gamma_2, v^* \in V^*$ linearly independent, and $r_1, r_2, v \in \ker(\gamma_1) \cap \ker(\gamma_2)$ linearly independent with

$$\begin{aligned} v^*(r_1) &= v^*(r_2) = 0, \\ v^*(v) &= 1, \end{aligned}$$

such that we can write:

$$\begin{aligned} g &= t_v^{\gamma_1} t_{r_2}^{v^*}, \\ h &= t_{v+r_1}^{\gamma_2} t_{2r_1+r_2}^{v^*}. \end{aligned}$$

Using this and Lemma 2.1.2 we can find the commutator $z = ghg^{-1}h^{-1}$:

$$\begin{aligned} ghg^{-1}h^{-1} &= t_v^{\gamma_1} t_{r_2}^{v^*} t_{v+r_1}^{\gamma_2} t_{2r_1+r_2}^{v^*} t_{-r_2}^{v^*} t_{-v}^{\gamma_1} t_{-2r_1-r_2}^{v^*} t_{-v-r_1}^{\gamma_2} \\ &= t_v^{\gamma_1} t_{-v-2r_1-r_2}^{\gamma_1} t_{v+r_1+r_2}^{\gamma_2} t_{-v-r_1}^{\gamma_2} t_{r_2}^{v^*} t_{2r_1}^{v^*} t_{-2r_1-r_2}^{v^*} \\ &= t_{-2r_1-r_2}^{\gamma_1} t_{r_2}^{\gamma_2} \end{aligned}$$

and see that z commutes with g and h. As G is a p-group

$$\Phi(G) = G^p[G, G].$$

Suppose $p \neq 2$. As g and h are index 3 bireflections $g^p = h^p = 1$ and so

$$\langle z\rangle = Z(G) = \Phi(G).$$

Knowing this and using Lemma 2.1.2 we see that any $\sigma \in G$ can be written as:

$$\begin{aligned}\sigma &= g^l h^m z^n \\ &= (t_v^{\gamma_1} t_{r_2}^{v^*})^l (t_{v+r_1}^{\gamma_2} t_{2r_1+r_2}^{v^*})^m (t_{-2r_1-r_2}^{\gamma_1} t_{r_2}^{\gamma_2})^n \\ &= t_{\alpha_1}^{\gamma_1} t_{\alpha_2}^{\gamma_2} t_{\alpha_3}^{v^*}\end{aligned}$$

for some $0 \leq l, m, n \leq p-1$, where:

$$\begin{aligned}\alpha_1 &= lv - 2nr_1 + \tfrac{l(l-1)-2n}{2} r_2, \\ \alpha_2 &= mv + m^2 r_1 + \tfrac{m(m-1+2l)+2n}{2} r_2, \\ \alpha_3 &= 2mr_1 + (m+l) r_2.\end{aligned}$$

We find that:

$$0 = 2m\alpha_1 - 2l\alpha_2 + (2n - lm)\alpha_3$$

and so G is an extra special group consisting of bireflections with $|G| = p^3$. As all $\sigma \in G$ have order p, we see that $G \cong M(p)$.

For $p = 2$ we find that

$$\begin{aligned}g &= t_v^{\gamma_1} t_{r_2}^{v^*}, \\ h &= t_{v+r_1}^{\gamma_2} t_{r_2}^{v^*}\end{aligned}$$

are still index 3 bireflections and so

$$ghg^{-1}h^{-1} = g^2 h^2 = t_{r_2}^{\gamma_1} t_{r_2}^{\gamma_2} \in Z(G).$$

Let $z_1 = g^2$ and $z_2 = h^2$ then $z_1, z_2 \in Z(G)$. This means that for any $\sigma \in G$ can be written as

$$\sigma = g^{a_1} h^{a_2} z_1^{a_3} z_2^{a_4} = t_{\alpha_1}^{\gamma_1} t_{\alpha_2}^{\gamma_2} t_{\alpha_3}^{v^*}$$

where $a_1, \dots, a_4 \in \mathbb{F}_2$ and

$$\alpha_1 = a_1 v + a_3 r_2,$$

$$\alpha_2 = a_2 v + a_2 r_1 + a_4 r_2,$$

$$\alpha_3 = (a_1 + a_2) r_2.$$

We see that if $a_1 = a_2$ then $\alpha_3 = 0$, otherwise we must have $a_i = 0$, for either $i = 1$ or $i = 2$, in which case

$$\alpha_i \in k\alpha_3.$$

In any of these cases we see that σ is a bireflection, and G is a pure bireflection group. □

We see from the above that exceptional groups of type one look quite different when $p = 2$: they are abelian groups generated by elements of order p^2 unlike in the case that p is odd, where they are non abelian groups and all elements have order p. This isn't the case for our next type of exceptional pure bireflection group.

Definition 2.2.5. Let $G \leq \mathrm{GL}(V)$ be a unipotent group with $g_1, g_2, g_3 \in G$. We call g_1, g_2, g_3 a **special triple** if there exists $r_1, r_2, r_3 \in V$, $\gamma_1, \gamma_2, \gamma_3 \in r_1^\perp \cap r_2^\perp \cap r_3^\perp$ with

$$\dim_k(r_1, r_2, r_3) = \dim_k(\gamma_1, \gamma_2, \gamma_3) = 3$$

and we can find $f \in k$ such that

$$g_1 = t_{r_1}^{\gamma_1} t_{r_2}^{\gamma_2}, \quad g_2 = t_{r_3}^{\gamma_1} t_{r_2}^{\gamma_3}, \quad g_3 = t_{fr_3}^{\gamma_2} t_{-fr_1}^{\gamma_3}.$$

If G is a pure bireflection group then we call G an **exceptional pure bireflection group** (or exceptional group) **of type two**, and g_1, g_2, g_3 an **exceptional triple**. For a matrix example see Chapter 6.2.

We see that special triples g_1, g_2, g_3 have the nice property that the group generated by any pair $\langle g_i, g_j \rangle$ with $1 \leq i < j \leq 3$ is a hook group, so they are not an extension on exceptional groups of type one. Again with exceptional groups of type two we need to check that these groups exist.

Proposition 2.2.6. *If G is generated by a special triple then G is an exceptional group of type 2, moreover G is elementary abelian of order p^3.*

Proof. Let g_1, g_2, g_3 be a special triple, so for some $r_1, r_2, r_3 \in V$, $\gamma_1, \gamma_2, \gamma_3 \in r_1^\perp \cap r_2^\perp \cap r_3^\perp$, $f \in k$:

$$g_1 = t_{r_1}^{\gamma_1} t_{r_2}^{\gamma_2}, \quad g_2 = t_{r_3}^{\gamma_1} t_{r_2}^{\gamma_3}, \quad g_3 = t_{fr_3}^{\gamma_2} t_{-fr_1}^{\gamma_3}.$$

From their definitions we can see g_1, g_2, g_3 commute, so for any $\sigma \in G$:

$$\sigma = g_1^a g_2^b g_3^c = t_{\alpha_1}^{\gamma_1} t_{\alpha_2}^{\gamma_2} t_{\alpha_3}^{\gamma_3}$$

with

$$\alpha_1 = ar_1 + br_3, \quad \alpha_2 = ar_2 + cfr_3, \quad \alpha_3 = br_2 - cfr_1$$

So

$$cf\alpha_1 = b\alpha_2 - a\alpha_3$$

and for all $\sigma \in G$, σ is a bireflection. This means G is an exceptional group of type two, which is an abelian group of order p^3. $\square$

If $G \leq \mathrm{GL}(V)$ is a pure bireflection group then the dual representation is also a pure bireflection group. Using Lemma 1.0.2 the dual representation of a hook group is also a hook group, and similarly for exceptional groups of types 1 and 2, however the dual of a two–row group is a two–column group (and visa versa).

We will show that the above are the only types of pure unipotent bireflection groups for $p \neq 2$, to do this we will make regular use of Proposition 2.1.4. First we show that an index 3 bireflection defines a unique hyperplane and line for any hook group containing it.

Lemma 2.2.7. *Let $G \leq \mathrm{GL}(V)$, and $g \in G$ an index three bireflection so we can find $\gamma_1, \gamma_2 \in V^*$ and $u_1, u_2 \in V$ linearly independent such that*

$$\gamma_1(u_1) = \gamma_1(u_2) = \gamma_2(u_2) = 0,$$

$\gamma_2(u_1) \neq 0$ and $g = t_{u_1}^{\gamma_1} t_{u_2}^{\gamma_2}$. If G is a hook group then it has hyperplane $U = \ker(\gamma_1)$ and line $ku_2 = [g, [g, V]]$.

Proof. Let G be a hook group with hyperplane U. If $v \notin \ker(\gamma_1)$ then $\delta_g(v) \notin V^g$ so $v \notin U$, so $U \leq \ker(\gamma_1)$. As

$$\dim_k(U) = n - 1 = \dim_k \ker(\gamma_1)$$

we see that $U = \ker(\gamma_1)$. As $\delta_g(U) = ku_2$ we see that the line of G must be ku_2. As $[g, V] = \langle u_1, u_2 \rangle$ we see that $[g, [g, V]] = ku_2$. □

We now begin to look at pure bireflection groups generated by two elements.

Lemma 2.2.8. *Let $G = \langle g, h \rangle$ be a pure unipotent bireflection group which is not a two–column or two–row group. Then $U = V^g + V^h < V$ is a hyperplane with codimension one, and $\dim_k(\delta_g(U)) = \dim_k(\delta_h(U)) = 1$. Furthermore G is a hook group if and only if*

$$\delta_g(U) = \delta_h(U) \leq U.$$

Proof. As h is a bireflection $\dim_k(V^h) \geq n - 2$, however as G isn't a two–column group

$$\dim_k(V^g \cap V^h) < n - 2.$$

This means that $V^h \neq V^g \cap V^h$ and so $V^h \not\leq V^g$. As g is also a bireflection

$$1 \leq \dim_k(\delta_g(V^h)) \leq 2.$$

Suppose $\dim_k(\delta_g(V^h)) = 2$ then $\delta_g(V^h) = [g, V] \leq [gh, V]$. For any $w \in V$

$$\delta_{gh}(w) = \delta_g(w) + \delta_h(w) + \delta_g\delta_h(w) \in [gh, V].$$

As $\delta_g(w), \delta_g\delta_h(w) \in [g, V]$ this means that $\delta_h(w) \in [gh, V]$ and so $[h, V] \leq [gh, V]$. However as G is not a two–row group this would mean that

$$\dim_k([gh, V]) = \dim_k([g, V] + [h, V]) > 2$$

and gh is not a bireflection. So $\dim_k(\delta_g(V^h)) = \dim_k(\delta_h(V^g)) = 1$.

Let $U = V^g + V^h$, then $(\delta_g(U) + \delta_h(U)) \leq 2$ and so $U \neq V$. We can also see that:

$$\begin{aligned}\dim_k(V^g + V^h) &= \dim_k(V^g) + \dim_k(V^h) - \dim_k(V^g \cap V^h)\\ &> (n-2) + (n-2) - (n-2) = n-2\end{aligned}$$

so $\dim_k(U) = n - 1$.

If $\delta_g(U) = \delta_h(U) \leq U$ then G is a hook group with hyperplane U and line $\delta_g(U)$. Suppose G is a hook group with hyperplane U' and line $kv \leq U'$. If $V^g \not\leq U'$ we can find some $u \in V^g \backslash U'$ such that $V = U' + ku$. Then

$$\begin{aligned}&[g, V] = [g, U'] = kv,\\ &[h, V] = [h, ku] + [h, U'] = k\delta_h(u) + kv.\end{aligned}$$

This would mean that

$$[G, V] = [g, V] + [h, V] = k\delta_h(u) + kv$$

but G is not a two–row group and so we must have $V^g \leq U'$. Similarly $V^h \leq U'$ and so

$$V^g + V^h = U \leq U'.$$

As $\dim_k(U) = \dim_k(U')$ this means that $U = U'$ and

$$\delta_g(U) = \delta_h(U) = kv \leq U. \qquad \square$$

The next two lemmas look at conditions under which a group generated by two elements is either a two–row, two–column or hook group. It is here we start restricting to characteristic $p \neq 2$.

Lemma 2.2.9. *Let $G = \langle g, h \rangle$ be a pure unipotent bireflection group, $p \neq 2$, $U = V^g + V^h$. If $\delta_h(U) \leq U$ then G is either a hook, two–row or two–column group.*

Proof. Assume G is not a two–row or two–column group. We have shown in Lemma 2.2.8 that $U = V^g + V^h < V$ is a hyperplane with codimension 1, and $\dim_k(\delta_g(U)) = \dim_k(\delta_h(U)) = 1$. Let $u_1, u_2 \in V$ such that:

$$\delta_g(U) = ku_1, \quad \delta_h(U) = ku_2$$

and choose some $v \in V \backslash U$ so $V = U + kv$. Assume $\delta_h(U) \subseteq U$, $u_2 \in U$. This means that $\delta_h(u_2) = a_1 u_2$ for some $a_1 \in k$. Since δ_h is nilpotent $a_1 = 0$ and $u_2 \in V^h$. Similarly if $\delta_h(v) = a_2 v + r$ with $a_2 \in k$ and $r \in U$ then $a_2 = 0$, $\delta_h(v) \in U$ so $[h, V] \leq U$.

We look at $gh \in G$. Let $u \in V^h \backslash V^g$ and $u' \in V^g \backslash V^h$, then:

$$\begin{aligned}
\delta_{gh}(u) &= \delta_g(u), \\
\delta_{gh}(u') &= \delta_g \delta_h(u') + \delta_h(u').
\end{aligned}$$

We can see that $\delta_g(u), \delta_g\delta_h(u') \in ku_1$ and $\delta_h(u') \in ku_2$. Since $\delta_g(u)$ and $\delta_h(u')$ are non-zero

$$ku_1 + ku_2 \subseteq [gh, V].$$

Suppose that $\dim_k(ku_1 + ku_2) = 1$. Then $ku_1 = ku_2 \leq V^g \cap V^h$ and G is a hook group with hyperplane U and line ku_1. Assume $\dim_k(ku_1 + ku_2) = 2$. Then:

$$[gh, V] = ku_1 + ku_2.$$

From this we know:

$$\delta_{gh}(v) = \delta_g(v) + \delta_h(v) + \delta_g\delta_h(v) \in ku_1 + ku_2.$$

As $[h, V] \leq U$, we must have $\delta_g\delta_h(v) \leq ku_1$ and so for some $c_1, c_2 \in k$:

$$\begin{aligned}\delta_g(v) + \delta_h(v) &= c_1u_1 + c_2u_2,\\ \delta_h(v) &= c_1u_1 + c_2u_2 - \delta_g(v).\end{aligned}$$

As $[G, V] = [g, V] + [h, V] = \langle u_1, u_2, \delta_g(v), \delta_h(v)\rangle$ has dimension greater than two $\{u_1, u_2, \delta_g(v)\}$ must be linearly independent.

Looking at the action of gh^i on U for $2 \leq i \leq p-1$ we find that:

$$\begin{aligned}\delta_{gh^i}(u) &= \delta_g(u),\\ \delta_{gh^i}(u') &= i\delta_h(u') + i\delta_g\delta_h(u').\end{aligned}$$

We see that $\delta_g(u), \delta_g\delta_h(u') \in ku_1$ and $\delta_h(u') \in ku_2$ so:

$$[g^ih, V] = ku_1 + ku_2$$

Using Lemma 2.1.4 we find

$$\begin{aligned}\delta_{gh^i}(v) &= \delta_g(v) + i\delta_h(v) + \frac{i(i-1)}{2}\delta_h^2(v) + i\delta_g\delta_h(v) + \frac{i(i-1)}{2}\delta_g\delta_h^2(v)\\ &\in ku_1 + ku_2.\end{aligned}$$

As $[h, V] \leq U$ we can see that:

$$\delta_h^2(v) \in ku_2, \quad \delta_g\delta_h(v), \delta_g\delta_h^2(v) \in ku_1.$$

so for some $b_1, b_2 \in k$

$$\delta_g(v) + i\delta_h(v) = b_1u_1 + b_2u_2.$$

Substituting in $\delta_h(v) = c_1u_1 + c_2u_2 - \delta_g(v)$,

$$(i-1)\delta_g(v) = (b_1 - c_1)u_1 + (b_2 - c_2)u_2$$

but then $\delta_g(v), u_1, u_2$ are not linearly independent and we have a contradiction. □

Now we note what happens if our group generated by two elements is not a two–row, two–column or hook group.

Lemma 2.2.10. *Let $G = \langle g, h\rangle$ be a p-group consisting of bireflections which is not a two–row, two–column or hook group, $p \neq 2$. Then $U = V^g + V^h$ has codimension one, $\delta_h(U), \delta_g(U) \not\subseteq U$ and $v \in V \backslash U$, $r \in U$ such that $\delta_g(U) = kv$, $\delta_h(U) = k(v + r)$. We can find $c \in k$ such that either:*

$$\delta_g(v) = -cr + (c-1)\delta_h(v + r)$$

or

$$\delta_h(v + r) = cr + (c-1)\delta_g(v).$$

Proof. Using Lemma 2.2.8 we know that if G is not a two–row or two–column group then U has codimension one. By Lemma 2.2.9 if G is not a hook group then $\delta_h(U), \delta_g(U) \not\subseteq U$.

Let $v \in V$ such that $kv = \delta_g(U)$. As $\delta_g(U) \nsubseteq U$ we can write $V = U + kv$. As $\delta_h(U) \nsubseteq U$ we can find some $r \in U$ such that $\delta_h(U) = k(v + r)$.

We look at $gh \in G$. Let $u \in V^h \backslash V^g$, $u' \in V^g \backslash V^h$:

$$\begin{aligned}
\delta_{gh}(u) &= \delta_g(u) \in kv, \\
\delta_{gh}(u') &= \delta_h(u') + \delta_g \delta_h(u') \in k(v + r + \delta_g(v + r)).
\end{aligned}$$

As $r \in U$, $\delta_g(r) \in kv$ so:

$$kv + k(v + r + \delta_g(v)) \subseteq [gh, V].$$

Suppose

$$\dim_k(kv + k(v + r + \delta_g(v)) = 1$$

then:

$$k(v + r + \delta_g(v)) \leq kv$$

which would mean that $r + \delta_g(v) \in kv$. As g is a bireflection and $v \in [g, V]$ we know that $\delta_g(v) \in \delta_g^2(V) \leq V^g$. Since $r, \delta_g(v) \in U$ this tells us $\delta_g(v) = -r$, so

$$\delta_g(v) = -cr + (c - 1)\delta_h(v + r)$$

for $c = 1$.

Suppose

$$\dim_k(kv + k(v + r + \delta_g(v)) = 2$$

then as gh is a bireflection

$$kv + k(v + r + \delta_g(v)) = [gh, V]$$

and

$$\begin{aligned}\delta_{gh}(v + r) &= \delta_g(v + r) + \delta_h(v + r) + \delta_g\delta_h(v + r)\\ &\in kv + k(r + \delta_g(v)).\end{aligned}$$

As h is a bireflection $\delta_h(v + r) \in V^h \subseteq U$ and so $\delta_g\delta_h(v + r), \delta_g(r) \in kv$. We can find $c_1, c_2 \in k$ such that:

$$\delta_h(v + r) = c_1(r + \delta_g(v)) + c_2v - \delta_g(v).$$

As v is the only term not in U, we can see $c_2 = 0$ and so if $c = c_1$ we have:

$$\delta_h(v + r) = cr + (c - 1)\delta_g(v).$$

$\square$

Lemma 2.2.11. *Let $p \neq 2$ and let $G = \langle g, h\rangle \leq \mathrm{GL}(V)$ be a p-group. Then G is a pure bireflection group if and only if one of the following holds:*

- *G is a hook group.*
- *G is a two–row group.*
- *G is a two–column group.*
- *G is an exceptional group of type one, and g and h are a special pair.*

Proof. If G is a two–column, two–row or hook group then we can easily check it consists of bireflections (see Lemmas 2.5.2, 2.4.2 and 2.3.3) and exceptional groups consist of bireflections by definition. Suppose $G = \langle g, h\rangle$ isn't a two–row, two–column, hook or exceptional group. Using Lemma 2.2.10 $U = V^g + V^h$ has codimension one, $\delta_h(U), \delta_g(U) \not\subseteq U$ and $v \in V \backslash U$, $r \in U$ such that $\delta_g(U) = kv$, $\delta_h(U) = k(v+r)$. We can choose g, h such that:

$$\delta_h(v+r) = cr + (c-1)\delta_g(v).$$

As we are assuming our group is not a two–row group this means that $v, r, \delta_g(v)$ are linearly independent.

As $r \in U$ we can find $s, t \in k$ such that:

$$\delta_h(r) = s(v+r), \quad \delta_g(r) = tv.$$

We will show that either:

- $c \neq 0$ and $s = t = 0$,
- $c = 0$ and $t = 0$, or
- $c = 0$ and $s = 0$.

We do this by looking at the descending commutator series. Firstly we find that

$$[G, V] = \langle v, r, \delta_g(v) \rangle.$$

We want to find $[G, [G, V]]$, so we look at

$$\begin{aligned}
&\delta_g(v) = \delta_g(v),\\
&\delta_g(r) = tv,\\
&\delta_g^2(v) = 0,\\
&\delta_h(v) = (c-1)\delta_g(v) + cr - s(v+r),\\
&\delta_h(r) = s(v+r).
\end{aligned}$$

This gives us

$$[G,[G,V]] \geq \langle \delta_g(v), tv, cr, s(v+r)\rangle.$$

As G is a p-group we know that $\dim_k([G,V]) > \dim_k([G,[G,V]]$, so two of c, s, t must equal 0.

First assume $c \neq 0$, $s = t = 0$ so $r \in V^G$. We look at $g^i h \in G$ for $1 < i \leq p-1$. Let $u \in U \backslash V^g$, $u' \in U \backslash V^h$ then using Lemma 2.1.4:

$$\begin{aligned}
\delta_{g^i h}(u) &= i\delta_g(u) + \tfrac{i(i-1)}{2}\delta_g^2(u),\\
\delta_{g^i h}(u') &= \delta_h(u') + i\delta_g\delta_h(u') + \tfrac{i(i-1)}{2}\delta_g^2\delta_h(u').
\end{aligned}$$

We have already found $\delta_g(v), v, r$ to be linearly independent, and

$$\begin{aligned}
&i\delta_g(u) + \tfrac{i(i-1)}{2}\delta_g^2(u) \in k(2v + (i-1)\delta_g(v)),\\
&\delta_h(u') \in k(v+r),\\
&i\delta_g(v+r) + \tfrac{i(i-1)}{2}\delta_g^2(v+r) = i\delta_g(v)
\end{aligned}$$

so we can see that:

$$[g^i h, V] = k(2v + (i-1)\delta_g(v)) + k(v + r + i\delta_g(v)).$$

This means that

$$\begin{aligned}
\delta_{g^i h}(v) &= i\delta_g(v) + \delta_h(v) + i\delta_g\delta_h(v) + \tfrac{i(i-1)}{2}\delta_g^2(v) + \tfrac{i(i-1)}{2}\delta_g^2\delta_h(v)\\
&\in k(2v + (i-1)\delta_g(v)) + k(v + r + i\delta_g(v)).
\end{aligned}$$

We know that:

$$\delta_h(v) = cr + (c-1)\delta_g(v), \quad \delta_g^2(v) = \delta_g(r) = 0$$

so for some $\alpha_1, \alpha_2 \in k$:

$$(i + c - 1)\delta_g(v) + cr = \alpha_1(2v + (i-1)\delta_g(v)) + \alpha_2(v + r + i\delta_g(v)).$$

Comparing r terms $\alpha_2 = c$, then comparing v terms $\alpha_1 = -\frac{c}{2}$. Looking at the $\delta_g(v)$ terms:

$$\begin{aligned} i + c - 1 &= -\tfrac{c}{2}(i-1) + ci, \\ c(i-1) &= 2(i-1), \\ c &= 2. \end{aligned}$$

Now we can see that:

$$\dim_k(U) = n - 1, \quad \dim_k(V^g \cap V^h) = n - 3,$$

and if we let $r_1 = r$, $\delta_g(v) = r_2$ then:

$$\begin{aligned} \delta_g(U) &= kv, & \delta_g(v) &= r_2, \\ \delta_h(U) &= k(v + r_1), & \delta_h(v) &= 2r_1 + r_2, \end{aligned}$$

and so g, h are a special pair and G is as described in Lemma 2.2.4 and is an exceptional group of type one.

Now suppose $c = 0$. If $t = 0$ we have:

$$\delta_h(v + r) = -\delta_g(v + r).$$

As above let $u \in U \backslash V^g$, $u' \in U \backslash V^h$ then:

$$\begin{aligned}\delta_{g^ih}(u) &= i\delta_g(u) + \tfrac{i(i-1)}{2}\delta_g^2(u),\\ \delta_{g^ih}(u') &= \delta_h(u') + i\delta_g\delta_h(u') + \tfrac{i(i-1)}{2}\delta_g^2\delta_h(u').\end{aligned}$$

We know that

$$\begin{aligned}&i\delta_g(u) + \tfrac{i(i-1)}{2}\delta_g^2(u) \in k(2v + (i-1)\delta_g(v)),\\ &\delta_h(u') \in k(v + r),\\ &\delta_g(r) = \delta_g^2(v) = 0\end{aligned}$$

so we find:

$$[g^ih, V] = k(2v + (i-1)\delta_g(v)) + k(v + r + i\delta_g(v)).$$

This means

$$\begin{aligned}\delta_{g^ih}(v) &= i\delta_g(v) + \delta_h(v) + i\delta_g\delta_h(v) + \tfrac{i(i-1)}{2}\delta_g^2(v) + \tfrac{i(i-1)}{2}\delta_g^2(v)\\ &\in k(2v + (i-1)\delta_g(v)) + k(v + r + i\delta_g(v))\end{aligned}$$

We know

$$\begin{aligned}&\delta_h(v) = -\delta_g(v) - sv - sr,\\ &\delta_g^2(v) = 0,\\ &\delta_g(r) = 0\end{aligned}$$

so for some $\alpha_1, \alpha_2 \in k$:

$$(i - 1 - is)\delta_g(v) - sv - sr = \alpha_1(2v + (i-1)\delta_g(v)) + \alpha_2(v + r + i\delta_g(v)).$$

Comparing V and r terms $\alpha_2 = -s$, $\alpha_1 = 0$, but comparing $\delta_g(v)$ terms

$$i - 1 - is = -is$$

which only holds for $i = 1$, so we have a contradiction.

If $s = 0$ then we have:

$$\delta_h(v) = -\delta_g(v)$$

which can be dealt with using the symmetric argument to the one above where $t = 0$. □

We need to exclude $p = 2$ in the above proposition as we can find additional groups, which don't exist in the odd p case.

***Example* 2.2.12.** *Let $H := \langle g_1, g_2 \rangle$ where:*

$$g_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \qquad g_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

We find that $H \cong C_2 \times C_2$ so it is an abelian group of order four. This just leaves one non-identity element not given explicitly. As

$$g_1 g_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

we see that H consists of bireflections but isn't a two–row, two–column or hook group.

Let

$$h = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

The group H is a maximal subgroup of the Nakajima group $G := \langle g_1, g_2, h\rangle$ (see Chapter 3). Using Theorem 1.3.4 H has hypersurface invariant ring.

We now want to see what happens if our group has more than two generators. First we want to find non-transvection bireflections.

Lemma 2.2.13. *Let $g_1, \ldots, g_l$ be bireflections such that $G = \langle g_1, \ldots, g_l\rangle$ is a p-group which isn't a one–row or one–column group. Then we can find $g \in G$ such that V^g has codimension two.*

Proof. By Proposition 8.2.12 of [31] G consists of transvections if and only if it is either a one–row or one–column group. This means that either g_i is not a transvection for some $1 \leq i \leq l$ or there exists $g \in G$ not a transvection which is a product of two transvections. Suppose for some $\gamma_1, \gamma_2 \in V^*$, $u_1, u_2 \in V$

$$g = t_{u_1}^{\gamma_1} t_{u_2}^{\gamma_2}.$$

If either $\gamma_1 \in k\gamma_2$ or $u_1 \in ku_2$ we see that g is a transvection. Otherwise V^g has codimension 2. □

We want to be able to use Lemma 2.2.11 to help us with pure bireflection groups with more than two generators. The next lemma allows us to find a useful subgroup with two generators for groups which are not two–row or two–column groups.

Lemma 2.2.14. *Let G be a unipotent bireflection group which isn't a two–row or two–column group. Then we can find $g_1, g_2 \in G$ such that $H = \langle g_1, g_2 \rangle$ isn't a two–row or two–column group.*

Proof. By the previous lemma we can pick $g \in G$ such that V^g has codimension 2.

As G is not a 2–column group we can find $\sigma_1 \in G$ such that $V^g \not\leq V^{\sigma_1}$. If also $[\sigma_1, V] \not\leq [g, V]$ then choose $g_1 = g$, $g_2 = \sigma_1$ and we are done. Otherwise, as G is not a 2–row group, we can find $\sigma_2 \in G$ such that:

$$[\sigma_2, V] \not\leq [g, V].$$

Either:

- $V^g \not\leq V^{\sigma_2}$, then pick $g_1 = g$, $g_2 = \sigma_2$,
- $V^g \leq V^{\sigma_2}$ and $\dim_k(V^{\sigma_1}) = \dim_k(V^{\sigma_2}) = n - 2$, then $V^g = V^{\sigma_2}$ so $V^{\sigma_2} \not\leq V^{\sigma_1}$ so pick $g_1 = \sigma_1$, $g_2 = \sigma_2$, or
- $V^g \leq V^{\sigma_2}$ and either $\dim_k(V^{\sigma_1}) > n - 2$ or $\dim_k(V^{\sigma_2}) > n - 2$.

In the third case, as $V^g \not\leq V^{\sigma_1}$, $V^g \leq V^{\sigma_2}$ we can find $u \in V^g \backslash V^{\sigma_1}$ so:

$$\sigma_1\sigma_2(u) = \sigma_1(u)$$

so $u \notin V^{\sigma_1\sigma_2}$.

As $[\sigma_1, V] \leq [g, V]$ and $[\sigma_2, V] \not\leq [g, V]$ we can find some $v, r \in V$ such that $r \notin [g, V]$:

$$\begin{aligned} \sigma_2(v) &= v + r, \\ \sigma_1\sigma_2(v) &= v + r + \delta_{\sigma_1}(v + r). \end{aligned}$$

We know that $\delta_{\sigma_1}(v+r) \in [\sigma_1, V] \leq [g, V]$ and so $r + \delta_{\sigma_1}(v+r) \notin [g, V]$ therefore $V^g \not\leq V^{\sigma_1\sigma_2}$ and $[\sigma_1\sigma_2, V] \not\leq [g, V]$ so choose $g_1 = g$, $g_2 = \sigma_1\sigma_2$. $\square$

Now we are able to move up to looking at groups with three generators.

Lemma 2.2.15. *Suppose p is odd. Suppose $G = \langle g_1, g_2, h\rangle$ is a pure bireflection group such that $H = \langle g_1, g_2\rangle$ is a hook group with hyperplane U and line kv which isn't a two–row or two–column group. Then either:*

- *G is a hook group with hyperplane U and line kv,*
- *G is an exceptional group of type one and either g_1, h or g_2, h are a special pair,*
- *G is an exceptional group of type two and g_1, g_2, h are double transvections.*

Proof. Let $G_1 = \langle g_1, h\rangle$ and $G_2 = \langle g_2, h\rangle$. Suppose that neither g_1, h or g_2, h are special pairs (in which case G is an exceptional group of type one). As both G_1 and G_2 must consist of bireflections up to duality we only need to consider the following four cases:

1. G_1 and G_2 two–column groups,
2. G_1 a two–row group, G_2 a two–column group,
3. G_1 a two–column group, G_2 a hook group but not a two–column or two–row group,
4. G_1 and G_2 hook groups which aren't two–column or two–row groups.

We will use that as H is not a two–column or two–row group we can see by Lemma 2.2.8 that $U = V^{g_1} + V^{g_2}$, and we can find

$$u_1 \in V^{g_1}\backslash V^{g_2}, \quad u_2 \in V^{g_2}\backslash V^{g_1}$$

such that

$$\delta_{g_1}(u_2) = \delta_{g_2}(u_1) = v.$$

As U is of codimension one there exists some $w \notin U$ such that

$$\dim_k(k\delta_{g_1}(w) + k\delta_{g_2}(w)) = 2.$$

Case 1 If G_1 and G_2 are two–column groups, then $V^{g_1} \leq V^h$ and $V^{g_2} \leq V^h$ and so

$$V^{g_1} + V^{g_2} = U \leq V^h.$$

This means that $\delta_h(U) = \{0\} < kv$ and G is a hook group with hyperplane U and line kv.

Case 2 If G_1 is a two–row group and G_2 is a two–column group then

$$[h, V] \leq [g_1, V] \leq U$$

and $V^{g_2} \leq V^h$. We see that

$$\begin{aligned}
g_1g_2h(w) &= w + \delta_{g_1}(w) + \delta_{g_2}(w) + \delta_h(w) + c_1v, \\
g_1g_2h(u_1) &= u + \delta_h(u_1) + c_2v, \\
g_1g_2h(u_2) &= u' + v
\end{aligned}$$

for some $c_1, c_2 \in k$. As G_1 is a two–row group:

$$\delta_h(u_1) \in \langle v, \delta_{g_1}(w)\rangle.$$

Suppose that G not a hook group. Then $v, \delta_h(u_1)$ are linearly independent, so in order for g_1g_2h to be a bireflection:

$$[g_1g_2h, V] = \langle v, \delta_h(u_1)\rangle = \langle v, \delta_{g_1}(w)\rangle.$$

This would mean that

$$\delta_{g_1}(w) + \delta_{g_2}(w) + \delta_h(w) + c_1 v \in \langle v, \delta_{g_1}(w) \rangle$$
$$\delta_{g_2}(w) \in \langle v, \delta_{g_1}(w) \rangle$$

and G (and therefore H) is a two–row group, which is a contradiction.

Case 3 If G_2 is a hook group but not a two–row or two–column group then by Lemma 2.2.8 it has hyperplane $U' = V^{g_2} + V^h$. Suppose the line of G_2 is kv'. If G_1 is a two–column group $V^{g_1} \leq V^h$ and so

$$U = V^{g_1} + V^{g_2} \leq V^h + V^{g_2} = U'.$$

As $\dim_k(U) = \dim_k(U')$ this means that $U = U'$. As $\delta_{g_2}(U) = kv$ we see that $kv' = kv$ and so G is a hook group with hyperplane U and line kv.

Case 4 Suppose G_1 and G_2 are both hook groups which are not two–row or two–column groups. Let $U_1 = V^{g_1} + V^h$, $U_2 = V^{g_2} + V^h$ be the hyperplanes of G_1 and G_2 with lines kv_1 and kv_2 respectively. If there exists $u \in (V^h \cap U) \backslash V^{g_1}$ then $u \in U_1$ so $U_1 = V^g + ku = U$, $kv_1 = kv$ and $\langle g_1, g_2, h \rangle$ is a hook group. Similarly if there exists $u \in (V^h \cap U) \backslash V^{g_2}$.

Assume this is not the case. If we take u_1, u_2 as defined above then $u_1 \in U_1 \backslash U_2$ and $u_2 \in U_2 \backslash U_1$. We can see

$$U_1 + U_2 = U + U_1 = U + U_2 = V$$

and by definition $\dim_k(U_1) = \dim_k(U_2) = \dim_k(U) = n - 1$. From this we see that

$$\dim_k(U_1 \cap U_2) = n - 2, \quad \dim_k(U \cap U_1 \cap U_2) = n - 3.$$

As $V^h \leq U_1$, $V^h \leq U_2$ and $\dim_k(V^h) \geq n-2$ we see that $U_1 \cap U_2 = V^h$. Similarly

$$V^{g_1} = U \cap U_1, \quad V^{g_2} = U \cap U_2.$$

We can assume $w \in V^h \backslash U$, and as H not a two–row or two–column group, $\delta_{g_1}(w), \delta_{g_2}(w), v$ are linearly independent. Since $w \in U_1 \cap U_2$ and we can see that $kv_1 = k\delta_{g_1}(w) \in V^h$, $kv_2 = k\delta_{g_2}(w) \in V^h$. Let $a_1, a_2 \in k$ such that

$$\delta_h(u_1) = a_1\delta_{g_1}(w), \quad \delta_h(u_2) = a_2\delta_{g_2}(w).$$

We now look at $G_3 = \langle g_1g_2, h\rangle$ and see that

$$\begin{aligned} g_1g_2(w) &= w + \delta_{g_1}(w) + \delta_{g_2}(w) + \delta_{g_1}\delta_{g_2}(w), \\ g_1g_2(u_1) &= u_1 + v, \\ g_1g_2(u_2) &= u_2 + v. \end{aligned}$$

As

$$[g_1g_2, V] = \langle \delta_{g_1}(w) + \delta_{g_2}(w), v\rangle \neq \langle \delta_{g_1}(w), \delta_{g_2}(w)\rangle = [h, V]$$

we know that G_3 is not a two–row group. As

$$\dim_k(V^{g_1g_2}) = \dim_k(V^h) = n-2$$

and $w \in V^{g_1g_2}\backslash V^h$, we see that G_3 is not a two–column group either. By Lemma 2.2.8 this means that $U_3 = V^h + V^{g_1g_2}$ has codimension one. As

$$u_1 - u_2 \in V^{g_1g_2} \leq U_3$$

and

$$\delta_h(u_1 - u_2) = \delta_h(u_1) - \delta_h(u_2) = a_1\delta_{g_1}(w) - a_2\delta_{g_2}(w) \in V^h.$$

by Lemma 2.2.9 we see that G_3 must be a hook group with line

$$k(a_1\delta_{g_1}(w) - a_2\delta_{g_2}(w)).$$

As $w \in V^h \leq U$ we find:

$$k(a_1\delta_{g_1}(w) - a_2\delta_{g_2}(w)) = k(\delta_{g_1}(w) + \delta_{g_2}(w) + \delta_{g_1}\delta_{g_2}(w)).$$

As $\delta_{g_1}(w), \delta_{g_2}(w), v$ are linearly independent and $\delta_{g_1}\delta_{g_2}(w) \in kv$ we see that

$$\delta_{g_1}\delta_{g_2}(w) = \delta_{g_2}\delta_{g_1}(w) = 0$$

and $a_1 = -a_2$.

Let $\gamma_0, \gamma_1, \gamma_2 \in V^*$ such that:

$$\begin{aligned}
\gamma_0(w) = 1, &\quad \ker(\gamma_0) = U, \\
\gamma_1(u_2) = 1, &\quad \ker(\gamma_1) = U_1, \\
\gamma_2(u_1) = 1, &\quad \ker(\gamma_2) = U_2
\end{aligned}$$

Let $G' = \langle \tilde{g}_1, \tilde{g}_2, \tilde{h} \rangle$ where

$$\tilde{g}_1 = t^{\gamma_0}_{\delta_{g_1}(w)} t^{\gamma_1}_v, \quad \tilde{g}_2 = t^{\gamma_0}_{\delta_{g_2}(w)} t^{\gamma_2}_v, \quad \tilde{h} = t^{\gamma_1}_{a_1\delta_{g_2}(w)} t^{\gamma_2}_{-a_1\delta_{g_1}(w)},$$

then G' is an exceptional group of type two. We can see that for $i = 1, 2$

$$V^{\tilde{g}_i} = U \cap U_i = V^{g_i}$$

and

$$V^{\tilde{h}} = U_1 \cap U_2 = V^h.$$

We know that

$$V = V^{g_1} \oplus ku_2 \oplus kw = V^{g_2} \oplus ku_1 \oplus kw = V^h \oplus ku_1 \oplus ku_2.$$

From the definition of $\tilde{g_1}$

$$\begin{aligned}\tilde{g_1}(u_2) &= u_2 + v = g_1(u_2)\\ \tilde{g_1}(w) &= w + \delta_{g_1}(w) = g_1(w)\end{aligned}$$

and so $\tilde{g_1} = g_1$, similarly $\tilde{g_2} = g_2$ and $\tilde{h} = h$. This means that $G = G'$ is an exceptional group of type two, and g_1, g_2, h are double transvections.

$\square$

We can finally now prove our main result of this section, the full classification of unipotent pure bireflection groups for $p \neq 2$.

Proof. (of Theorem 1.0.5) Suppose G is not a two–row, two–column group or an exceptional group. By Lemma 2.2.14 we can find $g_1, g_2 \in G$ such that $\dim_k(V^{g_1} \cap V^{g_2}) < n - 2$ and $\dim_k([g_1, V] + [g_2, V]) > 2$. Let $N := \langle g_1, g_2 \rangle$.

As G (and therefore N) consists of bireflections by Lemma 2.2.11 N must be a hook group with hyperplane U for some $U \subset V$, and line kv for some $v \in U^N$. As G is not an exceptional group by Lemma 2.2.15 for any $g \in G$, $\langle g, N \rangle$ is a hook group with hyperplane U and line kv. This means that

$$[G, U] \leq kv \leq V^G$$

and so G is a hook group with hyperplane U and line kv. $\square$

2.3 Two–column and two–row groups

Now that we know the groups consisting of bireflections for $p \neq 2$ we can start to look at them in more detail. Although we don't have the same classification of pure bireflection groups, two–row, two–column and hook groups are still of interest for $p = 2$, so we don't restrict to $p \neq 2$ for these sections. We start by looking at two–row groups.

Definition 2.3.1. Let $r_1, r_2 \in V$ be linearly independent, $\zeta \in V$ be such that $\zeta(r_1) = 1$, $\zeta(r_2) = 0$. Then for all $\gamma_1, \gamma_2 \in r_1^\perp \cap r_2^\perp$, $c \in k$ define $\kappa^{r_1,r_2,\zeta}_{\gamma_1,\gamma_2,c} = t^{\gamma_1}_{r_1} t^{\gamma_2}_{r_2} t^{\zeta}_{cr_2}$. Let

$$\begin{aligned}
K^{r_1,r_2,\zeta} &= \{\kappa^{r_1,r_2,\zeta}_{\gamma_1,\gamma_2,c} \mid \gamma_1, \gamma_2 \in r_1^\perp \cap r_2^\perp, c \in k\}, \\
L^{r_1,r_2} &= \{\kappa^{r_1,r_2,\zeta}_{0,\gamma,0} \mid \gamma \in r_1^\perp \cap r_2^\perp\}.
\end{aligned}$$

Where r_1, r_2, ζ are fixed in context we will write $\kappa_{\gamma_1,\gamma_2,c}$ for $\kappa^{r_1,r_2,\zeta}_{\gamma_1,\gamma_2,c}$.

We will see that for any two–row group G we can choose r_1, r_2 and ζ such that $G \leq K^{r_1,r_2,\zeta}$, and that $Z(K^{r_1,r_2,\zeta}) = L^{r_1,r_2}$.

Lemma 2.3.2. *Let $r_1, r_2 \in V$ and $\zeta_1, \zeta_2 \in V^*$ such that*

$$\begin{aligned}
\zeta_1(r_1) &= \zeta_2(r_1) = 1, \\
\zeta_1(r_2) &= \zeta_2(r_2) = 0.
\end{aligned}$$

Then

$$K^{r_1,r_2,\zeta_1} = K^{r_1,r_2,\zeta_2}.$$

Proof. Let

$$g = \kappa^{r_1,r_2,\zeta_1}_{\gamma_1,\gamma_2,c} \in K^{r_1,r_2,\zeta_1}.$$

As ζ_1 and ζ_2 agree on r_1 and r_2 we can find $\gamma_3 \in r_1^\perp \cap r_2^\perp$ such that

$$\zeta_1 = \zeta_2 + \gamma_3.$$

This means

$$g = \kappa^{r_1,r_2,\zeta_1}_{\gamma_1,\gamma_2,c} = \kappa^{r_1,r_2,\zeta_2}_{\gamma_1,\gamma_2+c\gamma_3,c} \in K^{r_1,r_2,\zeta_2}.$$

So $K^{r_1,r_2,\zeta_1} \leq K^{r_1,r_2,\zeta_2}$. A symmetric argument tells us that $K^{r_1,r_2,\zeta_1} \leq K^{r_1,r_2,\zeta_1}$ and so

$$K^{r_1,r_2,\zeta_1} = K^{r_1,r_2,\zeta_2}. \qquad \square$$

From here on we shall write $K^{r_1,r_2} = K^{r_1,r_2,\zeta}$. we look at multiplication between the elements of this set.

Lemma 2.3.3. *If we fix r_1, r_2, ζ then*

1. $\kappa_{\gamma_1,\gamma_2,c} = \kappa_{\gamma_1',\gamma_2',c'} \Leftrightarrow \gamma_1 = \gamma_1'$, $\gamma_2 = \gamma_2'$ *and* $c = c'$.
2. $\kappa_{\gamma_1,\gamma_2,c}\kappa_{\gamma_1',\gamma_2',c'} = \kappa_{\hat{\gamma}_1,\hat{\gamma}_2,\hat{c}}$ *where:*
   - $\hat{\gamma}_1 = \gamma_1 + \gamma_1'$,
   - $\hat{\gamma}_2 = \gamma_2 + \gamma_2' + c\gamma_1'$,
   - $\hat{c} = c + c'$.
3. $\kappa_{\gamma_1,\gamma_2,c}$ *and* $\kappa_{\gamma_1',\gamma_2',c'}$ *commute iff* $c\gamma_1' = c'\gamma_1$.
4. $\kappa^{-1}_{\gamma_1,\gamma_2,c} = \kappa_{-\gamma_1,c\gamma_1-\gamma_2,-c}$.
5. $\kappa_{\gamma_1,\gamma_2,c}\kappa_{\gamma_1',\gamma_2',c'}\kappa^{-1}_{\gamma_1,\gamma_2,c}\kappa^{-1}_{\gamma_1',\gamma_2',c'} = \kappa_{c\gamma_2'-c'\gamma_2,0,0} \in L^{r_1,r_2}$.
6. *For any* $\kappa_{\gamma_1,\gamma_2,c} \in \mathrm{GL}(V)$*:*

$$|\kappa_{\gamma_1,\gamma_2,c}| = \begin{cases} p^2, & \text{if } p = 2 \text{ and } c \neq 0, \\ p, & \text{otherwise.} \end{cases}$$

Proof. 1. If $\kappa_{\gamma_1,\gamma_2,c} = \kappa_{\gamma_1',\gamma_2',c'}$ then:

$$t_{r_1}^{\gamma_1} t_{r_2}^{\gamma_2} t_{cr_2}^{\zeta} = t_{r_1}^{\gamma_1'} t_{r_2}^{\gamma_2'} t_{c'}^{\zeta}.$$

As r_1, r_2 are linearly independent by Lemma 2.1.3

$$\gamma_1 = \gamma_1', \quad \gamma_2 + c\zeta = \gamma_2' + c'\zeta'$$

so $\gamma_2 - \gamma_2' = (c - c')\zeta$. As $\gamma_2 - \gamma_2' \in r_2^{\perp}$, $\zeta \notin r_2^{\perp}$ we see $c = c'$ and $\gamma_2 = \gamma_2'$.

2. If $g = \kappa_{\gamma_1,\gamma_2,c}$ and $h = \kappa_{\gamma'_1,\gamma'_2,c'}$ then

$$gh = t_{r_1}^{\gamma_1} t_{r_2}^{\gamma_2} t_{cr_2}^{\zeta} t_{r_1}^{\gamma'_1} t_{r_2}^{\gamma'_2} = t_{r_1}^{\gamma_1} t_{r_1+cr_2}^{\gamma'_1} t_{r_2}^{\gamma_2+\gamma'_2} t_{(c+c')r_2}^{\zeta}$$
$$= t_{r_1}^{\gamma_1+\gamma'_1} t_{r_2}^{\gamma_2+\gamma'_2+c\gamma'_1} t_{(c+c')r_2}^{\zeta}.$$

3.,4.,5. and 6. follow from 2..

□

We now move from looking at a set to looking at a group and its properties.

Proposition 2.3.4. *Let $G = \langle \kappa_{\gamma_1,\gamma_2,c} | \gamma_1, \gamma_2 \in r_1^{\perp} \cap r_2^{\perp}, c \in k\rangle$. Then $G = K^{r_1,r_2}$ and $|G| = q^{2n-3}$.*

Proof. We know that $K^{r_1,r_2} \subseteq G$. By Proposition 2.3.3(2) all elements of the group can be written as $\kappa_{\gamma_1,\gamma_2,c}$ for some γ_1, γ_2, c, so $G = K^{r_1,r_2}$.

By Proposition 2.3.3(1) $\kappa_{\gamma_1,\gamma_2,c} = \kappa_{\gamma'_1,\gamma'_2,c'}$ if and only if $\gamma_1 = \gamma'_1$, $\gamma_2 = \gamma'_2$ and $c = c'$. This means that

$$|K^{r_1,r_2}| = |r_1^{\perp} \cap r_2^{\perp}|^2 \cdot |k| = (q^{n-2})^2 q = q^{2n-3}.$$ □

We want to see when different choices of r_1, r_2 determine different groups.

Lemma 2.3.5. *Let $r_1, r_2, u_1, u_2 \in V$, $G = K^{r_1,r_2}$ and $H = K^{u_1,u_2}$. Then $G = H$ if and only if $kr_2 = ku_2$ and*

$$\langle r_1, r_2\rangle = \langle u_1, u_2\rangle.$$

Proof. Let $\zeta_1, \zeta_2 \in V^*$ such that $\zeta_1(r_2) = \zeta_2(u_2) = 0$ and $\zeta_1(r_1) = \zeta_2(u_1) = 1$.

Suppose to start with that $kr_2 = ku_2$ and

$$\langle r_1, r_2\rangle = \langle u_1, u_2\rangle.$$

Firstly we note that this means $r_1 = a_1u_1 + a_2u_2$ and $r_2 = a_3u_2$ with $a_1, a_3 \neq 0$. Also

$$r_1^{\perp} \cap r_2^{\perp} = u_1^{\perp} \cap u_2^{\perp}.$$

For any $g \in G$ we can write $g = \kappa^{r_1,r_2,\zeta_1}_{\gamma_1,\gamma_2,c} \in K^{r_1,r_2}$ for some $\gamma_1, \gamma_2 \in r_1^\perp \cap r_2^\perp$, $c \in k$. Using Lemma 2.1.2 this means that

$$g = t^{\gamma_1}_{r_1} t^{\gamma_2}_{r_2} t^{\zeta_1}_{cr_2} = t^{\gamma_1}_{a_1u_1+a_2u_2} t^{\gamma_2}_{a_3r_2} t^{\zeta_1}_{cr_2} = t^{a_1\gamma_1}_{u_1} t^{a_3\gamma_2+a_2\gamma_1}_{u_2} t^{\zeta_1}_{ca_3u_2}.$$

As $kr_2 = ku_2$ we can see that $\zeta_1(u_2) = 0$. Then

$$\zeta_1(a_1u_1 + a_2u_2) = a_1\zeta_1(u_1) = 1$$

and $\zeta_1(u_1) = 1/a_1$. Let $b = 1/a_1$. As ζ_1 and $b\zeta_2$ agree on r_1, r_2 we can find some $\gamma_3 \in r_1^\perp \cap r_2^\perp$ such that $\zeta_1 = \gamma_3 + b\zeta_2$. Using Lemma 2.1.2

$$\begin{aligned} g &= t^{a_1\gamma_1}_{u_1} t^{a_3\gamma_2+a_2\gamma_1}_{u_2} t^{\zeta_1}_{cr_2} \\ &= t^{a_1\gamma_1}_{u_1} t^{a_3\gamma_2+a_2\gamma_1}_{u_2} t^{\gamma_3+b\zeta_2}_{ca_3u_2} \\ &= t^{a_1\gamma_1}_{u_1} t^{a_3\gamma_2+a_1\gamma_1}_{u_2} t^{\gamma_3}_{ca_3u_2} t^{\zeta_2}_{bca_3u_2} \\ &= t^{a_1\gamma_1}_{u_1} t^{a_3\gamma_2+a_1\gamma_1+ca_3\gamma_3}_{u_2} t^{\zeta_2}_{bca_3u_2} \\ &= \kappa^{u_1,u_2,\zeta_2}_{\gamma_1',\gamma_2',c'} \in H, \end{aligned}$$

where

$$\begin{aligned} \gamma_1' &= a_1\gamma_1, \\ \gamma_2' &= a_3\gamma_2 + a_1\gamma_1 + ca_3\gamma_3, \\ c' &= bca_3. \end{aligned}$$

This means for any $g \in G$, $g \in H$. We can use the symmetric argument to show that for any $h \in H$, $h \in G$ and so $G = H$.

Suppose that

$$\langle r_1, r_2 \rangle \neq \langle u_1, u_2 \rangle.$$

This means that

$$[G, V] \neq [H, V]$$

and so $G \neq H$. Suppose that

$$\langle r_1, r_2 \rangle = \langle u_1, u_2 \rangle$$

but $kr_2 \neq ku_2$. This means that

$$kr_2 = [G, V]^G \neq [H, V]^H = ku_2$$

so $G \neq H$. □

Clearly K^{r_1,r_2} is a two–row group for any $r_1, r_2 \in V$. We check that any two–row group can be written as a subgroup of K^{r_1,r_2} for some $r_1, r_2 \in V$.

Lemma 2.3.6. *Let H be a two–row group. If*

$$[H, V]^H = [H, V]$$

then $H \leq K^{r_1,r_2}$ for any $r_1, r_2 \in V$ such that

$$[H, V] = \langle r_1, r_2 \rangle.$$

If

$$kv = [H, V]^H < [H, V]$$

then $H \leq K^{r_1,r_2}$ for any $r_2 \in kv$ and $r_1 \in V$ such that

$$[H, V] = \langle r_1, r_2 \rangle.$$

Proof. Suppose H is a two–row group with

$$[H, V]^H = [H, V].$$

If we choose any $r_1, r_2 \in V$ such that

$$[H, V] \leq \langle r_1, r_2 \rangle$$

then any $h \in H$ can be written as

$$h = t_{r_1}^{\gamma_1} t_{r_2}^{\gamma_2}$$

for some $\gamma_1, \gamma_2 \in r_1^{\perp} \cap r_2^{\perp}$. Then for any $\zeta \in V^*$ such that $\zeta(r_1) = 1$ and $\zeta(r_2) = 0$

$$h = t_{r_1}^{\gamma_1} t_{r_2}^{\gamma_2} = t_{r_1}^{\gamma_1} t_{r_2}^{\gamma_2} t_0^{\zeta} = \kappa_{\gamma_1,\gamma_2,0}^{r_1,r_2,\zeta} \in K^{r_1,r_2}.$$

This means that $H \leq K^{r_1,r_2}$.

Suppose that H is a two–row group with

$$kv = [H, V]^H < [H, V].$$

If we choose any $r_2 \in kv$ and $r_1 \in V$ such that

$$[H, V] = \langle r_1, r_2 \rangle$$

then we can write and $h \in H$ as

$$h = t_{r_1}^{\gamma_1} t_{r_2}^{\gamma_2}$$

for some $\gamma_1 \in r_1^{\perp} \cap r_2^{\perp}$, $\gamma_2 \in V^*$. If $\gamma_2 \in r_1^{\perp} \cap r_2^{\perp}$ then $h \in K^{r_1,r_2}$ by the above argument. If $\gamma_2(r_1) = c \neq 0$ then let $\zeta = \frac{1}{c}\gamma_2$ and write

$$h = t_{r_1}^{\gamma_1} t_{r_2}^{0} t_{cr_2}^{\zeta} = \kappa_{\gamma_1,0,c}^{r_1,r_2,\zeta}$$

so $h \in K^{r_1,r_2}$ and $H \leq K^{r_1,r_2}$. $\square$

Proposition 2.3.7. *For $n \geq 3$ if $G = K^{r_1,r_2}$ then it is a special group with:*

$$Z(G) = \Phi(G) = [G, G] = L^{r_1,r_2}$$

Proof. As G is a p-group we know that $\Phi(G) = G^p[G,G]$. We have shown in Proposition 2.3.3 that $[G,G] \leq L^{r_1,r_2}$, and that $G^p = \{e\}$ for p odd and $G^p \leq L^{r_1,r_2}$ for p even. Putting this together we find that $\Phi(G) \leq L^{r_1,r_2}$.

For $\gamma \in r_1^\perp \cap r_2^\perp$ take $g_1 = \kappa_{\gamma,0,0}$ and $g_2 = \kappa_{0,0,1}$ then $g_1 g_2 g_1^{-1} g_2^{-1} = \kappa_{0,\gamma,0}$ so

$$[G,G] = L^{r_1,r_2} = \Phi(G).$$

If $g \in L^{r_1,r_2}$ then it commutes with all elements $\kappa_{\gamma_1,\gamma_2,c}$ so $L^{r_1,r_2} \leq Z(G)$. If we choose $\kappa_{\gamma_1,\gamma_2,c} \in Z(G)$ then for any γ_1', c' we have that $c\gamma_1' = c'\gamma_1$ so $\gamma = 0$ and $c = 0$, so $\kappa_{\gamma_1,\gamma_2,c} \in L^{r_1,r_2}$ and $Z(G) = L^{r_1,r_2} = \Phi(G) = [G,G]$. G is a special p-group. $\square$

We can see that for any $\gamma_1, \gamma_2 \in V^*$, $G = (K^{\gamma_1,\gamma_2})^*$ is a two–row group. Results for two–column groups can be obtain by dualising the results of this section using Lemma 2.0.5.

2.4 Hook groups

We now move on to look at properties of hook groups. First we establish some notation.

Definition 2.4.1. Let $U < V^*$ be a subspace of codimension 1 and fix $0 \neq v \in U$ and $w \in V^* \backslash U$. For every $\lambda \in v^\perp \cap w^\perp$ and $u \in U$ define $b_{u,\lambda}^{w,U,v} \in \mathrm{GL}(V)$ by:

$$b_{u,\lambda}^{w,U,v}(w) = w + u,$$
$$b_{u,\lambda}^{w,U,v}|_U = t_v^\lambda,$$

so that $b_{u,\lambda}^{w,U,v}(u') = u' + u'(\lambda)v$ for any $u' \in U$.
Choose $w^* \in V^*$ such that $w^*(w) = 1$ and $U = \ker(w^*)$. For $c \in k$ we can then

define:

$$\mathcal{B}_c^{w,U,v} := \{b_{u,\lambda}^{w,U,v} | \lambda \in v^\perp \cap w^\perp, \lambda(u) = c\},$$
$$B^{U,v} := \{b_{u,\lambda}^{w,U,v} | \lambda \in v^\perp \cap w^\perp, u \in U\},$$
$$R_{\hat{v},U} := \{t_{av}^{w^*} | a \in k\} = \{b_{av,0}^{w,U,v} | a \in k\}.$$

Note that we have chosen hyperplane U and line kv to be in the dual space $W = V^*$. This is to ease in calculations of invariant rings later on. As hook groups are self dual we would achieve the same results by specifying a hyperplane and line in V.

For $c \neq 0$ the elements of $\mathcal{B}_c^{w,U,v}$ are index 3 bireflections. If w, U, v are fixed in context we will write $b_{u,\lambda}$ and $\mathcal{B}_c$ instead of $b_{u,\lambda}^{w,U,v}$ and $\mathcal{B}_c^{w,U,v}$.

We look at multiplication of the elements of $B^{U,v}$.

Lemma 2.4.2. *If we fix w, U, v then*

1. $b_{u,\lambda} = b_{u',\lambda'} \Leftrightarrow u = u'$ *and* $\lambda = \lambda'$,
2. $b_{u,\lambda} b_{u',\lambda'} = b_{\hat{u},\hat{\lambda}}$ *where* $\hat{\lambda} = \lambda + \lambda'$ *and* $\hat{u} = u + u' + \lambda(u')v$,
3. $b_{u,\lambda}$ *and* $b_{u',\lambda'}$ *commute iff* $\lambda(u') = \lambda'(u)$,
4. $b_{u,\lambda}^{-1} = b_{-u+\lambda(u)v,\lambda}$,
5. $b_{u',\lambda'} b_{u,\lambda} b_{u',\lambda'}^{-1} b_{h,\lambda}^{-1} = b_{(\lambda'(u)-\lambda(u'))v,0} \in R_{\hat{v},U}$,
6. *For* $b_{u,\lambda} \in \mathcal{B}_c$

$$|b_{u,\lambda}| = \begin{cases} p^2, & \textit{if } p = 2 \textit{ and } c \neq 0, \\ p, & \textit{otherwise.} \end{cases}$$

Proof. 1. If $b_{u,\lambda} = b_{u',\lambda'}$ then:

$$w + u = w + u'$$
$$u = u'.$$

For any $s \in U$ we find

$$\begin{aligned} s + \lambda(s)v &= s + \lambda'(s)v \\ \lambda(s)v &= \lambda'(s)v, \end{aligned}$$

so $\lambda = \lambda'$.

2. Let $\hat{\lambda} = \lambda + \lambda'$ and $\hat{u} = u + u' + \lambda(u')v$. We can look at the action of $b_{u,\lambda}b_{u',\lambda'}$ on w and on U. We start with w:

$$\begin{aligned} b_{u,\lambda}b_{u',\lambda'}(w) &= b_{u,\lambda}(w + u') \\ &= w + (u + u' + \lambda(u')v) \\ &= b_{\hat{u},\hat{\lambda}}(w). \end{aligned}$$

Let $s \in U$ then

$$\begin{aligned} b_{u,\lambda}b_{u',\lambda'}(s) &= b_{u,\lambda}(s + \lambda(s)v) \\ &= s + (\lambda(s) + \lambda'(s))v \\ &= b_{\hat{u},\hat{\lambda}}(s) \end{aligned}$$

so $b_{u,\lambda}b_{u',\lambda'} = b_{\hat{u},\hat{\lambda}}$.

3.,4.,5.,6. follow from 2.

□

We see that $B^{U,v}$ is closed under multiplication, the next few propositions look at it's group structure.

Proposition 2.4.3. *Let $G = \langle b_{u,\lambda} | u \in U, \lambda \in v^{\perp} \cap w^{\perp} \rangle$. Then $G = B^{U,v}$ and $|G| = q^{2n-1}$.*

Proof. From the definition of G,

$$\{b_{u,\lambda} | u \in U, \lambda \in v^{\perp} \cap w^{\perp}\} \subseteq G.$$

By Proposition 2.4.2(2) all elements of the group can be written as $b_{u,\lambda}$ for some u, λ, so

$$G = \{b_{u,\lambda} | u \in U, \lambda \in v^{\perp} \cap w^{\perp}\}.$$

By Proposition 2.4.2(1) $b_{u,\lambda} = b_{u',\lambda'}$ if and only if $u = u'$ and $\lambda = \lambda'$ so

$$\begin{aligned} |\{b_{u,\lambda} | u \in U, \lambda \in v^{\perp} \cap w^{\perp}\}| &= |\{u \in U\}| \cdot |\{\lambda \in v^{\perp} \cap w^{\perp}\}| \\ &= q^{n-1} q^{n-2} = q^{2n-3}. \end{aligned}$$

□

Proposition 2.4.4. *For $n \geq 3$, the group $G = B^{U,v}$ is a special group with:*

$$Z(G) = \Phi(G) = [G, G] = R_{\hat{v},U}$$

Proof. As G is a p-group we know that $\Phi(G) = G^p[G, G]$. We have shown in Proposition 2.4.2 that $[G, G] \leq R_{\hat{v},U}$, and that $G^p = \{e\}$ for p odd and $G^p \leq R_{\hat{v},U}$ for p even. So we have that $\Phi(G) \leq R_{\hat{v},U}$.

Let $u \in V$, for any $d \in k$ we can choose $\lambda \in V^*$ such that $\lambda(u) = -d$. Then $b_{u,0}, b_{0,\lambda} \in G$ and:

$$b_{u,0} b_{0,\lambda} b_{u,0}^{-1} b_{0,\lambda}^{-1} = b_{dv,0} \in R_{\hat{v},U}.$$

so $[G, G] = R_{\hat{v},U} = \Phi(G)$.

If $t \in R_{\hat{v},U}$ then it commutes with all elements $b_{u,\lambda}$ so $R_{\hat{v},U} \leq Z(G)$. If we choose $b_{u,\lambda} \in Z(G)$ then for any u', λ' we know $\lambda(u') = \lambda'(u)$. This can only happen if $u = cv$ and $\lambda = 0$, so $b_{u,\lambda} \in R_{\hat{v},U}$ and

$$Z(G) = R_{\hat{v},U} = \Phi(G) = [G, G].$$

This means that G is a special p-group. □

We know that $B^{U,v}$ is special, for $k = \mathbb{F}_p$ it is extra special so we know we can write it as a central product of copies of extraspecial groups of order p^3.

Lemma 2.4.5. *Let $k = \mathbf{F}_p$, $n \geq 3$ and $G = B^{U,v}$. If $p = 2$ then*

$$G \cong \underbrace{D_8 * D_8 * \ldots * D_8}_{n-2 \ copies}.$$

Otherwise

$$G \cong \underbrace{M(p) * M(p) * \ldots * M(p)}_{n-2 \ copies}.$$

Proof. Let $\{e_1, \ldots, e_n\}$ be a basis for V such that $e_1 = v$ and $\langle e_1, \ldots, e_{n-1}\rangle = U$. For $1 \leq i \leq n-2$ let

$$H_i = \langle b_{e_{i+1},0}, b_{0,e^*_{i+1}}\rangle.$$

The H_i are groups of order p^3 and we can check using Lemma 2.4.2 that

$$[H_i, H_i] = Z(H_i) = R_{\hat{v},U}.$$

This means that H_i is extraspecial for $1 \leq i \leq n-2$. If $p = 2$ then as $|b_{u,0}| = 2$ and $b_{u,0} \notin \Phi(H_i)$ we see that $H_i \cong D_8$. If p is odd then as all elements have order p, $H_i \cong M(p)$.

Let $H = H_1 H_2 \ldots H_{n-2}$. For any $2 \leq i, j \leq n-1$, $i \neq j$ we see $Z(H_i) = Z(H_j)$, and H_i centralises H_j so for p even

$$H \cong \underbrace{D_8 * D_8 * \ldots * D_8}_{n-2 \text{ copies}},$$

and for p odd

$$H \cong \underbrace{M(p) * M(p) * \ldots * M(p)}_{n-2 \text{ copies}}.$$

Clearly $H \leq G$ and $|H| = p^{2n-1} = |G|$ so $H = G$. □

The next Proposition relates $\mathcal{B}_c$ and $B^{U,v}$. It is useful when looking for generators of $B^{U,v}$.

Proposition 2.4.6. *Let $n > 3$. For $c \in k$ let $G_c = \langle \mathcal{B}_c \rangle$. Then $G_c = B^{U,v}$.*

Proof. We know $G_c \leq B^{U,v}$. We will show that for any element $b_{u,\lambda} \in B^{U,v}$, $b_{u,\lambda} \in G_c$ and so $B^{U,v} \leq G_c$.

Since $\dim_k(V) = n > 3$ we can choose u', λ' such that:

$$\lambda'(u') = c,$$
$$\lambda'(u) = 0,$$
$$\lambda(u') = 0,$$

so $b_{u',\lambda'}, b_{u'+u,\lambda'}, b_{u',\lambda+\lambda'} \in \mathcal{B}_c$. Then:

$$b_{u'+u,\lambda'} b^{-1}_{u',\lambda'} = b_{u,0} \in G_c,$$
$$b_{u',\lambda'+\lambda} b^{-1}_{u',\lambda'} = b_{0,\lambda} \in G_c,$$
$$b_{u,0} b_{0,\lambda} = b_{u,\lambda} \in G_c. \qquad \square$$

We now look at some subgroups of $B^{U,v}$.

Proposition 2.4.7. *Let $G = \langle b_1, \ldots, b_l \rangle$, where $b_i = b_{u_i,\lambda_i} \in B^{U,v}$ for $1 \leq i \leq l$ minimally generate G. Then $p^l \leq |G| \leq p^{l+r}$*

Proof. As G is a p-group $\Phi(G) = G^p[G,G]$. We know that $G^p \leq R_{\hat{v},U}$ and $[G,G] \leq R_{\hat{v},U}$ so $\Phi(G) \leq R_{\hat{v},U}$ and $1 \leq |\Phi(G)| \leq q$.

By [1, Theorem 23.1] $\langle X \rangle = G$ if and only if $\langle X, \Phi(G) \rangle = G$. As $G/\Phi(G)$ is elementary abelian this means if l is the minimal number of generators then $|G/\Phi(G)| = p^l$, and so $p^l \leq |G| \leq qp^l = p^{l+r}$. $\square$

Proposition 2.4.8.

1. *Let $G = \langle b_1, \ldots, b_l, R_{\hat{v},U} \rangle$ where the set of $b_i = b_{h_i,\lambda_i} \in B^{U,v}$ for $1 \leq i \leq l$, and the b_i's and $R_{\hat{v},U}$ minimally generate G. Then $|G| = p^{l+r}$.*

2. *Let $G = \langle b_1, \ldots, b_l \rangle$ where the set of $b_i = b_{h_i,\lambda_i} \in B^{U,v}$ for $1 \leq i \leq l$, and the b_i's minimally generate G. Suppose $[G,[G,V]] = 0$, then $|G| = p^l$.*

Proof. 1. Since $\Phi(G) \leq R_{\hat{v},U} \leq Z(G)$, $G/R_{\hat{v},U}$ is elementary abelian of order p^l and $|R_{\hat{v},U}| = q = p^r$, so $|G| = p^{l+r}$.

2. If $[G,[G,V]] = 0$ it can be seen from Proposition 2.4.2 that G is elementary abelian and if it is minimally generated by l elements it has order p^l.

□

2.5 Exceptional groups of type one

We now look at the exceptional groups of type one. In Lemma 2.2.4 we see that a group generated by a exceptional pair $G = \langle g, h \rangle$ for $p = 2$ is quite different to a group generated by a special pair for odd p. To start with we note that g and h have order p^2 and not order p. The centre of G also has order p^2 rather than p, and G is not an extra-special group. The types of bireflection we find are also quite different:

$$ghg^{-1}h^{-1} = t_{r_2}^{\gamma_1} t_{r_2}^{\gamma_2}$$

is a transvection and not a double transvection for $p = 2$. We will see in the odd case that exceptional groups do not contain any transvections (Lemma 2.5.9). For even p the exceptional groups of type one are part of a larger family of pure bireflection groups containing a pair of elements

$$g = t_{u_1}^{\zeta_1} t_{u_3}^{\zeta_3}, \quad h = t_{u_2}^{\zeta_2} t_{u_3}^{\zeta_3}$$

for $\zeta_1, \zeta_2, \zeta_3 \in V^*$, $u_1, u_2, u_3 \in V$. We have already seen another one of these groups in Example 2.2.12 but we will not look at them in any detail.

We will restrict to $p \neq 2$ for this section, we also need $n \geq 5$ for our definition of an exceptional group of type one to make sense. We start by defining some groups containing a special pair, and then show that these are the only possible exceptional groups of type one.

Definition 2.5.1. Define linearly independent sets

$$\mathbf{r} = \{r_1, r_2, v\} \text{ and } \gamma = \{\gamma_1, \gamma_2, v^*\}$$

with $r_1, r_2, v \in V$, $\gamma_1, \gamma_2 \in r_1^\perp \cap r_2^\perp \cap v^\perp$ such that $v^* \in r_1^\perp \cap r_2^\perp$ and $v^*(v) = 1$. For all $l, m, n \in k$ define $\chi^{\mathbf{r},\gamma}_{l,m,n}$ by:

$$\chi^{\mathbf{r},\gamma}_{l,m,n} = t^{\gamma_1}_{\alpha_1} t^{\gamma_2}_{\alpha_3} t^{v^*}_{\alpha_2}$$

where

$$\begin{aligned}
\alpha_1 &= lv - 2nr_1 + \tfrac{l(l-1)-2n}{2} r_2, \\
\alpha_2 &= mv + m^2 r_1 + \tfrac{m(m-1+2l)+2n}{2} r_2, \\
\alpha_3 &= 2mr_1 + (m+l) r_2.
\end{aligned}$$

Define:

$$\begin{aligned}
X^{\mathbf{r},\gamma} &= \{\chi^{\mathbf{r},\gamma}_{l,m,n} \mid l, m, n \in k\}, \\
J_{\mathbf{r},\gamma} &= \{\chi^{\mathbf{r},\gamma}_{0,0,n} \mid n \in k\}.
\end{aligned}$$

If $\mathbf{r}, \gamma$ are fixed in context we will write $\chi_{l,m,n}$.

Note that for all $l, m, n \in k$

$$2m\alpha_1 - 2l\alpha_2 + (2n + ml)\alpha_3 = 0$$

and so $\chi_{l,m,n}$ is a bireflection.

Lemma 2.5.2. *For fixed* $\mathbf{r}, \gamma$ *we have:*

1. $\chi_{l,m,n} = \chi_{l',m',n'} \Leftrightarrow l = l', m = m', n = n'$,

2. $\chi_{l,m,n}\chi_{l',m',n'} = \chi_{l+l',m+m',n+n'-ml'}$,

3. $\chi_{l,m,n}$ *and* $\chi_{l',m',n'}$ *commute iff* $ml' = m'l$,

4. $\chi^{-1}_{l,m,n} = \chi_{-l,-m,-n-ml}$,

5. $\chi_{l,m,n}\chi_{l',m',n'}\chi^{-1}_{l,m,n}\chi^{-1}_{l',m',n'} = \chi_{0,0,lm'-l'm}$.

Proof. 1. This is a direct application of Lemma 2.1.3.

2. Let $l, m, n, l', m', n' \in k$ then

$$\begin{aligned}\chi_{l,m,n}\chi_{l',m',n'} &= t^{\gamma_1}_{\alpha_1} t^{\gamma_2}_{\alpha_2} t^{v^*}_{\alpha_3} t^{\gamma_1}_{\alpha'_1} t^{\gamma_2}_{\alpha'_2} t^{v^*}_{\alpha'_3} \\ &= t^{\gamma_1}_{\alpha_1+\alpha'_1+l'\alpha_3} t^{\gamma_2}_{\alpha_2+\alpha'_2+m'\alpha_3} t^{v^*}_{\alpha_3+\alpha'_3}\end{aligned}$$

where:

$$\begin{aligned}\alpha_1 &= lv - 2nr_1 + \tfrac{l(l-1)-2n}{2} r_2, \\ \alpha_2 &= mv + m^2 r_1 + \tfrac{m(m-1+2l)+2n}{2} r_2, \\ \alpha_3 &= 2mr_1 + (m+l) r_2, \\ \alpha'_1 &= l'v - 2n'r_1 + \tfrac{l(l-1)-2n}{2} r_2, \\ \alpha'_2 &= m'v + (m')^2 r_1 + \tfrac{m'(m'-1+2l')+2'n}{2} r_2. \\ \alpha'_3 &= 2m'r_1 + (m'+l') r_2.\end{aligned}$$

We find that

$$\begin{aligned}\alpha_1 + \alpha'_1 + l'\alpha_3 =& (l+l')v - 2(n+n'-ml')r_1 \\ &+ \tfrac{(l+l')(l+l'-1)-2(n+n'-ml')}{2} r_2, \\ \alpha_2 + \alpha'_2 + m'\alpha_3 =& (m+m')v + (m+m')^2 r_1 \\ &+ \tfrac{(m+m')(m+m'-1+2(l+l'))+2(n+n'-ml')}{2} r_2, \\ \alpha_3 + \alpha'_3 =& 2(m+m')r_1 + (m+m'+l+l')r_2,\end{aligned}$$

and so $\chi_{l,m,n}\chi_{l',m',n'} = \chi_{l+l',m+m',n+n'-ml'}$.

3.,4.,5. and 6. follow from 2..

∎

We know that $X^{\mathbf{x},\gamma}$ is closed under multiplication, we can now start to look at it as a group.

Proposition 2.5.3. *Let $G = \langle \chi_{l,m,n} \mid l, m, n \in k\rangle$. Then $G = X^{\mathbf{r},\gamma}$ and $|G| = q^3$.*

Proof. By Proposition 2.5.2(2) all elements of the group can be written as $\chi_{l,m,n}$ for some $l, m, n \in k$, so $G = \{\chi_{l,m,n} | l, m, n \in k\} = X^{\mathbf{r},\gamma}$.

By Proposition 2.5.2(1) $\chi_{l,m,n} = \chi_{l',m',n'}$ if and only if $l = l', m = m', n = n'$ so $|X^{\mathbf{r},\gamma}| = |k|^3 = q^3$. $\square$

Proposition 2.5.4. *Let $G = X^{\mathbf{r},\gamma}$. Then G is a special group with:*

$$Z(G) = \Phi(G) = [G, G] = J_{\mathbf{r},\gamma}.$$

Proof. As G is a p-group we know that $\Phi(G) = G^p[G, G]$. We have shown in Proposition 2.5.2 that $[G, G] \leq J_{\mathbf{r},\gamma}$. As G is a pure bireflection group with $p \neq 2$, $G^p = \{e\}$. So we see that $\Phi(G) \leq J_{\mathbf{r},\gamma}$.

For any $l \in k$ if we let $b_1 = \chi_{l,0,0}, b_2 = \chi_{0,1,0}$ then $b_1 b_2 b_1^{-1} b_2^{-1} = \chi_{0,0,l}$, and so

$$[G, G] = J_{\mathbf{r},\gamma} = \Phi(G).$$

If $t \in J_{\mathbf{r},\gamma}$ then it commutes with all elements $\chi_{l,m,n}$ so $J_{\mathbf{r},\gamma} \leq Z(G)$. If we choose $\chi_{l,m,n} \in Z(G)$ then for any l', m' we have that $ml' = m'l$ so $m = l = 0$ and so $\chi_{l,m,n} \in J_{\mathbf{r},\gamma}$. This means that

$$Z(G) = \Phi(G) = [G, G]$$

and G is a special p-group. $\square$

Using the above we see that $X^{\mathbf{r},\gamma}$ is isomorphic to a group we recognise.

Proposition 2.5.5. *Let $G = X^{\mathbf{r},\gamma}$ and*

$$P = \left\langle \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in k \right\rangle.$$

Then $G \cong P$ for any $\mathbf{r}, \gamma$.

Proof. We can define a map $\phi : G \to P$ such that

$$\phi(\chi_{l,m,n}) = \begin{pmatrix} 1 & m & -n \\ 0 & 1 & l \\ 0 & 0 & 1 \end{pmatrix}.$$

Clearly ϕ is surjective, we check that the map is a group homomorphism. Let $\chi_{l,m,n}, \chi_{l',m',n'} \in G$ then

$$\begin{aligned} \phi(\chi_{l,m,n}\chi_{l',m',n'}) &= \phi(\chi_{l+l',m+m',n+n'-ml'}) \\ &= \begin{pmatrix} 1 & m+m' & -n-n'+ml' \\ 0 & 1 & l+l' \\ 0 & 0 & 1 \end{pmatrix}, \end{aligned}$$

and

$$\begin{aligned} \phi(\chi_{l,m,n})\phi(\chi_{l',m',n'}) &= \begin{pmatrix} 1 & m & -n \\ 0 & 1 & l \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & m' & -n' \\ 0 & 1 & l' \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & m+m' & -n-n'+ml' \\ 0 & 1 & l+l' \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

If $\phi(\chi_{l,m,n}) = \mathbf{1}$ then $l = m = n = 0$ and $\chi_{0,0,0} = 1$, so ϕ is an isomorphism and $G \cong P$. □

In the above P is the Sylow p-subgroup of $\mathrm{SL}_3(q)$. The next couple of lemmas will help us towards our goal of showing that all exceptional groups of type one are isomorphic to P.

Lemma 2.5.6. *Let $G_1, G_2 \leq \mathrm{GL}(V)$ be hook groups with hyperplanes U_1, U_2 and lines v_1, v_2 respectively. Let $\gamma_1, \gamma_2 \in V^*$ such that* $\ker(\gamma_1) = U_1$ *and* $\ker(\gamma_2) = U_2$.

If $U_1 \neq U_2$, $kv_1 \neq kv_2$ then for any $t \in G_1 \cap G_2$ we can find $a, b \in k$ such that

$$t = t^{\gamma_1}_{av_2} t^{\gamma_2}_{bv_1}.$$

Let $\gamma_3 \in V^*$ and $v_3 \in V$ such that

$$\dim_k \langle v_1, v_2, v_3 \rangle = \dim_k \langle \gamma_1, \gamma_2, \gamma_3 \rangle = 3.$$

If $G_3 \leq \mathrm{GL}(V)$ is also a hook group with hyperplane $U_3 = \ker(\gamma_3)$ and line v_3 and $t \in G_1 \cap G_2 \cap G_3$ then $t = 1$.

Proof. For any $u \in \ker(\gamma_1) \cap \ker(\gamma_2)$ we see that $\delta_t(u) \in kr_2 \cap k(2r_1 + r_2) = \{0\}$ so we can find $r_3, r_4 \in V$ such that

$$t = t^{\gamma_1}_{r_3} t^{\gamma_2}_{r_4}.$$

As $\ker(\gamma_2) \not\leq \ker(\gamma_1)$ we see that $r_3 \in kv_2$ and similarly $r_4 \in kv_1$, so we can find some $a, b \in k$ such that:

$$t = t^{\gamma_1}_{av_2} t^{\gamma_2}_{bv_1}.$$

If $t \in G_1 \cap G_2 \cap G_3$ as above then we see that for some $c_1, c_2, c_3, c_4 \in k$

$$\begin{aligned} t &= t^{\gamma_1}_{c_1 v_2} t^{\gamma_2}_{c_2 v_1} = t^{\gamma_1}_{c_3 v_3} t^{\gamma_3}_{c_4 v_1} \\ &= t^{\gamma_1}_{c_1 v_2} t^{\gamma_2}_{c_2 v_1} t^{\gamma_3}_{0} = t^{\gamma_1}_{c_3 v_3} t^{\gamma_2}_{0} t^{\gamma_3}_{c_4 v_1} \end{aligned}$$

so using Lemma 2.1.3 $c_1 = c_2 = c_3 = c_4 = 0$. □

Lemma 2.5.7. *Let $g_1 = \chi_{1,0,0}$, $g_2 = \chi_{0,1,0}$ and $\sigma \in \mathrm{GL}(V)$. If $G = \langle g_1, g_2, \sigma \rangle$ is a pure bireflection group then either σ is a double transvection and for some $a \in k$, $\sigma = \chi_{0,0,a}$ or σ is an index 3 bireflection and g_1, σ or g_2, σ are a special pair.*

Proof. Let $z = \chi_{0,0,1}$, $G_1 = \langle g_1, z, \sigma \rangle$ and $G_2 = \langle g_2, z, \sigma \rangle$. As G_1, G_2 are not two–row or two–column groups, by Lemma 2.2.15 each could be a hook group, exceptional group of type one or exceptional group of type two. As g_1 is an index 3 bireflection G_1 is not an exceptional group of type two. As z is a double transvection if G_1 is an exceptional group of type one then g_1, σ are an exceptional pair. Similarly either G_2 is a hook group or g_2, σ are an exceptional pair. Suppose both G_1 and G_2 are hook groups.

As g_1 is an index 3 bireflection we see by Lemma 2.2.7 that G_1 has hyperplane $\ker(\gamma_1)$ and line kr_2. Similarly G_2 has hyperplane $\ker(\gamma_2)$ and line $k(2r_1 + r_2)$. Using Lemma 2.5.6 we can find some $a, b \in k$ such that:

$$\sigma = t^{\gamma_1}_{a(2r_1+r_2)} t^{\gamma_2}_{br_2}.$$

As $r_1, r_2 \in \ker(\gamma_1) \cap \ker(\gamma_2)$ we see σ is a double transvection.

Let $G_3 = \langle g_1 g_2, z, \sigma \rangle$. Using Lemma 2.2.15 again, G_3 is either a hook or an exceptional group. As $g_1 g_2$ is an index 3 bireflection it isn't an exceptional group of type two, and as σ, z are double transvections G_3 isn't a exceptional group of type one. This means that G_3 is a hook group. We know that

$$\begin{aligned} g_1 g_2 &= t^{\gamma_1}_{v+2r_1+r_2} t^{\gamma_2}_{v+r_1} t^{v^*}_{2r_1+2r_2} \\ &= t^{\gamma_1+\gamma_2}_{v+r_1} t^{2v^*+\gamma_1}_{r_1+r_2} \end{aligned}$$

and

$$[g_1 g_2, [g_1 g_2, V]] = k(r_1 + r_2).$$

By Lemma 2.2.7 $k(r_1+r_2)$ is the line of G_3, and $U = \ker(\gamma_1+\gamma_2)$ is the hyperplane. If $u_1, u_2 \in \ker(v^*)$ such that for $i, j \in \{1, 2\}$

$$\gamma_i(u_j) = \begin{cases} 1 \text{ if } i = j, \\ 0 \text{ otherwise.} \end{cases}$$

We can see that $u_1 - u_2 \in \ker(\gamma_1 + \gamma_2) = U$, so

$$\delta_\sigma(u_1 - u_2) \in k(r_1 + r_2)$$

$$2ar_1 + (a-b)r_2 \in k(r_1 + r_2)$$

For this to happen we must have $b = -a$ and then $\sigma = \chi_{0,0,a}$. □

We can now prove that all exceptional groups of type one are as described above.

Proposition 2.5.8. *If $G \leq \mathrm{GL}(V)$ is an exceptional group of type one then we can find $\mathbf{r}, \gamma$ such that $G \leq X^{\mathbf{r},\gamma}$.*

Proof. If G is an exceptional group of type one then we can find $\mathbf{r} = \{r_1, r_2, v\}$ and $\gamma = \{\gamma_1, \gamma_2, v^*\}$ such that

$$\chi^{\mathbf{r},\gamma}_{1,0,0}, \chi^{\mathbf{r},\gamma}_{0,1,0} \in G.$$

Let $g_1 = \chi^{\mathbf{r},\gamma}_{1,0,0}$, $g_2 = \chi^{\mathbf{r},\gamma}_{0,1,0}$. If $G \not\leq X^{\mathbf{r},\gamma}$ then we can find $\sigma \in G \backslash X^{\mathbf{r},\gamma}$. If G consists of bireflections then $\langle g_1, g_2, \sigma\rangle$ consists of bireflections so by Lemma 2.5.7 if $\sigma \notin X^{\mathbf{r},\gamma}$ then either g_1, σ or g_2, σ are an exceptional pair.

Without loss of generality we can assume g_1, σ are an exceptional pair. By Lemma 2.2.3 we can find $a, b \in k$, $r_3 \in \ker(\gamma_1) \cap \ker(v^*)$ and $\gamma_3 \in V^*$ linearly independent to γ_1 and v^* such that

$$\gamma_3(r_2) = \gamma_2(r_3) = \gamma_3(v) = \gamma_1(r_3) = 0$$

and $\sigma = t^{\gamma_1}_{\beta_1} t^{\gamma_3}_{\beta_2} t^{v^*}_{\beta_3}$ where

$$\begin{aligned}
\beta_1 &= bv + (a-ab)r_2 + (2a+b)r_3,\\
\beta_2 &= v - ar_2' + r_3,\\
\beta_3 &= 2r_3 + r_2.
\end{aligned}$$

Using Lemma 2.1.2 we can find

$$z' := \sigma g_1 \sigma^{-1} g_1^{-1} = t^{\gamma_1}_{2r_3+(1-b)r_2} t^{\gamma_3}_{-r_2}.$$

As z' is not an index 3 bireflection it can't be part of an exceptional pair so because $\langle g_1, g_2, z'\rangle$ must be a pure bireflection group by Lemma 2.5.7

$$z' = \chi_{0,0,c}$$

for some $c \in k$. This means that

$$z' = t^{\gamma_1}_{c(2r_1+r_2)} t^{\gamma_2}_{-cr_2} = t^{\gamma_1}_{2r_3+(1-b)r_2} t^{\gamma_3}_{-r_2}.$$

As γ_1 is linearly independent to γ_2 and γ_3 we can find some $u_1 \in V$ such that $\gamma_1(u_1) = 1$ and $\gamma_2(u_1) = \gamma_3(u_1) = 0$. We find

$$\delta_{z'}(u_1) = 2r_3 + (1-b)r_2 = 2cr_1 + cr_2$$

so $r_3 = cr_1 + \frac{(c-1+b)r_2}{2}$. By multiplying z' on the right by $(t^{\gamma_1}_{2cr_1+cr_2})^{-1}$ and using Lemma 2.1.2 we get:

$$\begin{aligned}
t^{\gamma_1}_{c(2r_1+r_2)} t^{\gamma_3}_{-r_2} &= t^{\gamma_1}_{c(2r_1+r_2)} t^{\gamma_2}_{-cr_2},\\
t^{\gamma_3}_{-r_2} &= t^{\gamma_2}_{-cr_2},\\
t^{\gamma_3}_{-r_2} &= t^{c\gamma_2}_{-r_2}.
\end{aligned}$$

Using Lemma 2.1.3 we see that $\gamma_3 = c\gamma_2$. Now we see that $\sigma = t^{\gamma_1}_{\beta_1} t^{\gamma_2}_{c\beta_2} t^{v^*}_{\beta_3}$ for

$$\begin{aligned}
\beta_1 &= bv + (a-ab)r_2 + (2a+b)(cr_1 + \frac{(c-1+b)}{2}r_2),\\
&= bv + (2ac+bc)r_1 + \frac{b(b-1)+2ac+bc}{2}r_2,\\
c\beta_2 &= cv - car_2 + c^2 r_1 + \frac{c(c-1+b)r_2}{2},\\
&= cv + c^2 r_1 + \frac{c(c-1+2b)-2ac-bc}{2},\\
\beta_3 &= 2(cr_1 + \frac{(c-1+b)r_2}{2}) + r_2,\\
&= 2cr_1 + (c+b)r_2.
\end{aligned}$$

If $L = b$, $M = c$ and $L = -\frac{2ac+bc}{2}$ then $\sigma = \chi^{\mathbf{r},\gamma}_{L,M,N}$, so $\sigma \in X^{\mathbf{r},\gamma}$. $\square$

This allows us to say some more about exceptional groups of type one.

Corollary 2.5.9. *If G is an exceptional group of type one then it contains no transvections and any double transvections in G are contained within $J_{\mathbf{r},\gamma}$, which is a two–row and two–column group.*

Corollary 2.5.10. *If $k = \mathbb{F}_p$, for fixed $\mathbf{r}, \gamma$, there is only one exceptional group of type one which is an extra special group of order p^3 which is isomorphic to $M(p)$.*

Proof. If G is an exceptional group of type one then by the above proposition $G \leq X^{\mathbf{r},\gamma}$, for some $\mathbf{r}, \gamma$, however G has no non-trivial subgroups which contain a special pair, so $G = X^{\mathbf{r},\gamma}$. We can see that

$$\Phi(G) = [G, G] = Z(G) = J_{\mathbf{r},\gamma}$$

so G is extraspecial, and the order of G is p^3. As G has no elements of order greater than p, $G \cong M(p)$. $\square$

2.6 Exceptional groups of type two

In this section we will treat exceptional groups of type two, as we have with exceptional groups of type one above. Unlike exceptional groups of type one, many of our results for exceptional groups of type two still hold for $p = 2$, so we do not restrict odd characteristic when we define some groups containing a special triple. We cannot, however, use our earlier classification results for even characteristic and so we restrict to $p \neq 2$ when we show that these are all possible exceptional groups of type two in Proposition 2.6.4.

To be able to find $G \leq \mathrm{GL}(V)$ an exceptional group of type two we need $n \geq 6$.

Definition 2.6.1. Let $\mathbf{r} = \{r_1, r_2, r_3\}$ with $r_1, r_2, r_3 \in V$, $\gamma = \{\gamma_1, \gamma_2, \gamma_3\}$ with $\gamma_1, \gamma_2, \gamma_3 \in r_1^\perp \cap r_2^\perp \cap r_3^\perp$ and

$$\dim_k\langle r_1, r_2, r_3\rangle = \dim_k\langle \gamma_1, \gamma_2, \gamma_3\rangle = 3.$$

For all $a, b, c \in k$ define $w^{\mathbf{r},\gamma}_{a,b,c} = t^{\gamma_1}_{\alpha_1} t^{\gamma_2}_{\alpha_2} t^{\gamma_3}_{\alpha_3}$ where:

$$\alpha_1 = ar_1 + br_3,$$
$$\alpha_2 = ar_2 + cr_3,$$
$$\alpha_3 = br_2 - cr_1,$$

and $W^{\mathbf{r},\gamma} = \{w^{\mathbf{r},\gamma}_{a,b,c} | a, b, c \in k\}$. Where $\mathbf{r}, \gamma$ are fixed in context we shall write $w^{\mathbf{r},\gamma}_{a,b,c} = w_{a,b,c}$.

Lemma 2.6.2. *For fixed* $\mathbf{r}, \gamma$ *we have:*

1. $w_{a,b,c} = w_{a',b',b'} \Leftrightarrow l = l', m = m', n = n'$.
2. $w_{a,b,c} w_{a',b',c'} = w_{a+a',b+b',c+c'}$.
3. $w_{a,b,c}$ *and* $w_{a',b',c'}$ *commute for all* $a, b, c, a', b', c' \in k$.
4. $w^{-1}_{a,b,c} = w_{-a,-b,-c}$.

Proof. 1. We can see by using Lemma 2.1.3..

2. For $a, b, c, a', b', c' \in k$ we see that

$$\begin{aligned} w_{a,b,c} w_{a'b',c'} &= t^{\gamma_1}_{ar_1+br_3} t^{\gamma_2}_{ar_2+cr_3} t^{\gamma_3}_{br_2-cr_3} t^{\gamma_1}_{a'r_1+b'r_3} t^{\gamma_2}_{a'r_2+c'r_3} t^{\gamma_3}_{b'r_2-c'r_3}, \\ &= t^{\gamma_1}_{(a+a')r_1+(b+b')r_3} t^{\gamma_2}_{(a+a')r_2+(c+c')r_3} t^{\gamma_3}_{(b+b')r_2-(c+c')r_3}. \end{aligned}$$

3.,4.,5.,6. follow from 2..

□

Proposition 2.6.3. *Let* $G = \langle w^{\mathbf{r},\gamma}_{a,b,c} | a, b, c \in k\rangle$*. Then* $G = W^{\mathbf{r},\gamma}$ *is an elementary abelian group with* $|G| = q^3$.

Proof. By Proposition 2.6.2(2) all elements of the group can be written as $w_{a,b,c}$ for some $a, b, c \in k$, so

$$G = \{w_{a,b,c} | a, b, c \in k\} = W^{\mathbf{r},\gamma}.$$

As all elements commute and have order p we see that G is elementary abelian.

By Proposition 2.6.2(1) $w_{a,b,c} = w_{a',b',c'}$ if and only if $a = a', b = b', c = c'$ so $|W^{\mathbf{r},\gamma}| = q^3$. □

Proposition 2.6.4. *Let $p \neq 2$. If $G \in \mathrm{GL}(V)$ is an exceptional group of type two then there exists some $\mathbf{r}, \gamma$ such that for all $h \in G$, $h = w^{\mathbf{r},\gamma}_{a,b,c}$ for some $a, b, c \in k$.*

Proof. As G is an exceptional group of type two we can find a subgroup $H = \langle g_1, g_2, g_3 \rangle$ such that g_1, g_2, g_3 are a special triple. This means that for some $\mathbf{r} = \{r_1, r_2, r_3\}$, $\gamma = \{\gamma_1, \gamma_2, \gamma_3\}$ and $s \in k$:

$$g_1 = w^{\mathbf{r},\gamma}_{1,0,0}, \quad g_2 = w^{\mathbf{r},\gamma}_{0,1,0}, \quad g_3 = w^{\mathbf{r},\gamma}_{0,0,s}.$$

We will show that for all $h \in G$ we can find some $a, b, c \in k$ such that $h = w^{\mathbf{r},\gamma}_{a,b,c}$.

From Proposition 2.5.9 we can see that G is not an exceptional group of type one: for any exceptional group of type one all elements which are not index three bireflections are contained within the centre which is a two–column (and two–row group). The special triple g_1, g_2, g_3 are all double transvections which are not contained in any single two–row or two–column group.

For all $h \in G$ the subgroups

$$\langle g_i, g_j, h \rangle \text{ for } 1 \leq i < j \leq 3$$

consist of bireflections so by Lemma 2.2.15 they are either hook groups or exceptional groups of type two. Suppose $\langle g_1, g_2, h \rangle$ is a hook group then it has hyperplane $\ker(\gamma_1)$ and line kr_2. Similarly if $\langle g_1, g_3, h \rangle$ is a hook group then it has hyperplane $\ker(\gamma_2)$ and line kr_1, and if $\langle g_2, g_3, h \rangle$ is a hook group then it has hyperplane γ_3 and line r_3. As $\gamma_1, \gamma_2, \gamma_3$ and r_1, r_2, r_3 are linearly independent this means if all three groups are hook groups by Lemma 2.5.6, $h = 1$.

For h not the identity we know that for some $1 \leq i < j \leq 3$ that $\langle g_i, g_j, h \rangle$ is not a hook group, we can assume $i = 1$ and $j = 2$ without loss of generality.

We can find $\mathbf{r}' = \{r_1', r_2', r_3'\}$, $\gamma' = \{\gamma_1', \gamma_2', \gamma_3'\}$ such that for some $n \in k$:

$$g_1 = w_{1,0,0}^{\mathbf{r}',\gamma'}, \quad g_2 = w_{0,1,0}^{\mathbf{r}',\gamma'}, \quad h = w_{0,0,n}^{\mathbf{r}',\gamma'}.$$

Then:

$$\begin{aligned} t_{r_1}^{\gamma_1} t_{r_2}^{\gamma_2} &= t_{r_1'}^{\gamma_1'} t_{r_2'}^{\gamma_2'}, \\ t_{r_3}^{\gamma_1} t_{r_2}^{\gamma_3} &= t_{r_3'}^{\gamma_1'} t_{r_2'}^{\gamma_3'}. \end{aligned}$$

As

$$(k\gamma_1 + k\gamma_2) \cap (k\gamma_1 + k\gamma_3) = k\gamma_1, \quad (kr_1 + kr_2) \cap (kr_2 + kr_3) = kr_2$$

for some $l, m \in k$

$$\begin{aligned} &\gamma_1 = \gamma_1', \quad \gamma_2' = \gamma_2 + l\gamma_1, \quad \gamma_3' = \gamma_3 + m\gamma_1 \\ &r_1' = r_1 - lr_2, \quad r_2 = r_2', \quad r_3' = r_3 - mr_2. \end{aligned}$$

Using this we find that

$$\begin{aligned} h &= t_{nr_3'}^{\gamma_2'} t_{-nr_1'}^{\gamma_3'}, \\ &= t_{n(r_3 - mr_2)}^{l\gamma_1} t_{-n(r_1 - lr_2)}^{m\gamma_1} t_{n(r_3 - mr_2)}^{\gamma_2} t_{-n(r_1 - lr_2)}^{\gamma_3}, \\ &= t_{-mnr_1 + lnr_2}^{\gamma_1} t_{nr_3 - mnr_2}^{\gamma_2} t_{-nr_1 + lnr_2}^{\gamma_3}, \\ &= w_{-mn,ln,n}^{\mathbf{r},\gamma} \end{aligned}$$

as required. $\square$

Corollary 2.6.5. *If G is an exceptional group of type two then it contains no transvections or index 3 bireflections.*

Corollary 2.6.6. *If $k = \mathbb{F}_p$ then for fixed $\mathbf{r}, \gamma$ there is only one exceptional group of type two which is an elementary abelian group of order p^3.*

2.7 Pure bireflection groups

We now know a little about each type of bireflection group for $p \neq 2$ so we can say a bit more about them than at the end of Section 2.2.

Definition 2.7.1. A subgroup $G \leq \mathrm{GL}(V)$ is called a **maximal pure bireflection group** if it is a pure bireflection group, and for all $G \leq H \leq \mathrm{GL}(V)$ either $H = G$ or H is not a pure bireflection group.

Lemma 2.7.2. *Let $p \neq 2$, $n > 3$. If G is a maximal pure unipotent bireflection group then it is a special group and one of the following holds:*

- *$G = B^{U,v}$ for some $U < V$ of codimension 1, $v \in U$. $|G| = q^{2n-3}$.*
- *$G = K^{r_1,r_2}$ or $G = (K^{\gamma_1,\gamma_2})^*$ for some $r_1, r_2 \in V$ or $\gamma_1, \gamma_2 \in V^*$. Then $|G| = q^{2n-3}$.*
- *$G = X^{\mathbf{r},\gamma}$ for some $\mathbf{r} = \{r_1, r_2, v\}$, $\gamma = \{\gamma_1, \gamma_2, v^*\}$. Then $|G| = q^3$.*
- *$G = W^{\mathbf{r},\gamma}$ for some $\mathbf{r} = \{r_1, r_2, r_3\}$, $\gamma = \{\gamma_1, \gamma_2, \gamma_3\}$. Then $|G| = q^3$.*

If $k = \mathbb{F}_p$, then G is extra special or abelian if and only if it is self-dual.

Proof. We show in Proposition 1.0.5 that if G is a pure bireflection group then it is either a hook, two–row, two–column or exceptional group. Suppose it is a hook group. Then we can find some U, v such that $[G, V] \leq kv$, so $G \leq B^{U,v}$, as G is maximal $G = B^{U,v}$, similarly for G a two–row, two–column and exceptional group.

Let $k = \mathbb{F}_p$. By Proposition 2.3.7 if $G = K^{r_1,r_2}$ then

$$|\Phi(G)| = |L^{r_1,r_2}| = p^{n-2} > p$$

for $n \geq 3$, so G is not extra special if it is a two–row or two–column group. If G is not a two–row or two–column group then it is either a hook group or an exceptional group and is self dual. If G is a hook group then

$$|\Phi(G)| = |R_{v,U}| = p,$$

so G is extraspecial. If G is exceptional of type one then

$$|\Phi(G)| = |J_{\mathbf{r},\gamma}| = p,$$

so it is extra special. If it is exceptional of type two then it is abelian. □

Corollary 2.7.3. *If G is a pure unipotent bireflection group, $p \neq 2$, $n > 3$, then it is a subgroup of one of the groups in Lemma 2.7.2 and it has class less than or equal to two.*

Proof. If G is a pure unipotent bireflection group then it must be either a maximal pure bireflection group or contained in a maximal pure bireflection group. Above gives the list of all possible pure bireflection groups which are all special, so each of their subgroups must have class less than or equal to two. □

The following Proposition summarises the results of this Chapter.

Proposition 2.7.4. *Let $p > 2$, $n \geq 3$. Let G be a unipotent group consisting of bireflections with $g \in G$.*

1. *If $g = t_u^\zeta$ is a transvection then G is one of the following*

   - *A subgroup of K^{r_1,r_2} with $u \in \langle r_1, r_2\rangle$,*
   - *A subgroup of $(K^{\gamma_1,\gamma_2})^*$ with $\zeta \in \langle \gamma_1, \gamma_2\rangle$,*
   - *A subgroup of $B^{U,v}$ with either $U = \ker(\zeta)$ or $u \in kv$.*

2. *If $g = t_{u_1}^{\zeta_1} t_{u_2}^{\zeta_2}$ is a double transvection so $u_1, u_2 \in \ker(\zeta_1) \cap \ker(\zeta_2)$ then G is one of the following*

   - *A subgroup of K^{r_1,r_2} with $\langle r_1, r_2\rangle = \langle u_1, u_2\rangle$,*

- *A subgroup of* $(K^{\gamma_1,\gamma_2})^*$ *with* $\langle \gamma_1, \gamma_2 \rangle = \langle \zeta_1, \zeta_2 \rangle$,
- *A subgroup of* $B^{U,v}$ *such that* $v \in \langle u_1, u_2 \rangle$
- *A subgroup of* $G \leq X^{\mathbf{r},\gamma}$ *where*

$$\langle r_1, r_2 \rangle = \langle u_1, u_2 \rangle,$$

  and

$$\langle \gamma_1, \gamma_2 \rangle = \langle \zeta_1, \zeta_2 \rangle,$$

- *A subgroup of* $G \leq W^{\mathbf{r},\gamma}$ *where*

$$\langle r_1, r_2, r_3 \rangle > \langle u_1, u_2 \rangle,$$

  and

$$\langle \gamma_1, \gamma_2, \gamma_3 \rangle > \langle \zeta_1, \zeta_2 \rangle.$$

3. *If* $g = t_{u_1}^{\zeta_1} t_{u_2}^{\zeta_2}$ *is an index 3 bireflection so* $u_1 \notin \ker(\zeta_2)$ *and* $u_2 \in \ker(\zeta_1)$ *then* G *is one of the following*

- *A subgroup of* K^{u_1,u_2},
- *A subgroup of* $(K^{\zeta_1,\zeta_2})^*$,
- *A subgroup of* $B^{U,v}$ *where* $U = \ker(\zeta_2)$ *and* $v \in ku_2$,
- *A subgroup of* $G \leq X^{\mathbf{r},\gamma}$ *where*

$$\langle r_1, r_2, v \rangle > \langle u_1, u_2 \rangle,$$

  and

$$\langle \gamma_1, \gamma_2, v^* \rangle > \langle \zeta_1, \zeta_2 \rangle.$$

Proof. 1. Suppose $g = t_u^{\zeta}$ is a transvection. By the above corollary we know that G must be a subgroup of one of the groups in Lemma 2.7.2. If G is a two–row group then by Lemma 2.3.6 we can find $r_1, r_2 \in V$ such that

$$ku = [g, V] \leq [G, V] \leq \langle r_1, r_2 \rangle$$

and $G \leq K^{r_1,r_2}$.

If G is a two–column group then G^* is a two–row group so we can find $\gamma_1, \gamma_2 \in V^*$ such that

$$k\zeta \leq [G, V^*] \leq \langle \gamma_1, \gamma_2 \rangle$$

and $G \leq (K^{\gamma_1,\gamma_2})^*$.

Suppose G is a hook group with line kv and hyperplane U. Either $u \in kv$ or $U = \ker(\zeta)$.

By Corollaries 2.5.9 and 2.6.5 we know that G is not contained in an exception group of type one or type two.

2. Suppose $g = t_{u_1}^{\zeta_1} t_{u_2}^{\zeta_2}$ is a double transvection. If G is a two–row group then by Lemma 2.3.6 we can find $r_1, r_2 \in V$ with

$$\langle u_1, u_2 \rangle = [G, V] = \langle r_1, r_2 \rangle$$

such that $G \leq K^{r_1,r_2}$. If G is two–column group then G^* is a two–row group and by Lemma 2.3.6 we can find γ_1, γ_2 with

$$\langle \zeta_1, \zeta_2 \rangle = [G, V^*] = \langle \zeta_1, \zeta_2 \rangle$$

such that $G \leq (K^{\gamma_1,\gamma_2})^*$.

If $G \leq B^{U,v}$ is a hook then as V^g has codimension two $U \neq V^g$. This means that $kv \leq \langle u_1, u_2 \rangle$. If G is an exceptional group of type one then by Corollary 2.5.9

$$g \in J_{\mathbf{r},\gamma} = \{\chi_{0,0,n}^{\mathbf{r},\gamma} \mid n \in k\}.$$

This means that

$$\langle u_1, u_2\rangle = [J_{\mathbf{r},\gamma}, V] = \langle r_1, r_2\rangle$$

and

$$\langle \gamma_1, \gamma_2\rangle = \langle \zeta_1, \zeta_2\rangle.$$

Let $H = W^{\mathbf{r},\gamma}$, if $G \leq H$ then

$$[g, V] = \langle u_1, u_2\rangle \leq [H, V] = \langle r_1, r_2, r_3\rangle$$

and similarly $V^H \leq V^g$ so

$$\langle \zeta_1, \zeta_2\rangle < \langle \gamma_1, \gamma_2, \gamma_3\rangle.$$

3. Suppose $g = t_{u_1}^{\zeta_1} t_{u_2}^{\zeta_2}$ is an index 3 bireflection. If G is a two–row group then we can again use Lemma 2.3.6 to see that $G = K^{u_1,u_2}$. Similarly by looking at the dual space we see that if G is a two–column group then $G \leq (K^{\zeta_1,\zeta_2})^*$.

   If G is a hook group we just apply Lemma 2.2.7. If $G \leq X^{\mathbf{r},\gamma}$ then

   $$[g, V] \leq [X^{\mathbf{r},\gamma}, V] = \langle r_1, r_2, v\rangle$$

   so

   $$\langle r_1, r_2, v\rangle > \langle u_1, u_2\rangle.$$

   By looking at the fixed space (or by looking at the duals of both groups) we see that

   $$\langle \gamma_1, \gamma_2, v^*\rangle > \langle \zeta_1, \zeta_2\rangle.$$

   By Corollary 2.6.5 we know that G is not contained in an exceptional group of type two.

$\square$

Chapter 3

Nakajima Groups and their Subgroups

We now move on from the classification of pure bireflection groups and start looking at their invariant rings. We introduce Nakajima groups which are an important class of unipotent groups in invariant theory. By viewing other unipotent groups as subgroups of Nakajima groups, we hope to be able to find their invariant rings. Here we introduce methods and notation which will help us in later chapters where we put it to use finding invariant rings of pure bireflection groups.

Definition 3.0.1. Let $B = \{x_1, x_2, \ldots, x_n\}$ an ordered basis for W. Let $G \leq \mathrm{GL}(V)$ be a p-group, for $1 \leq i \leq n$ define

$$G_i = \{g \in G \mid g(x_j) = x_j \text{ for all } i \neq j\}.$$

We say that G is a **Nakajima group** (with respect to B) if:

1. $G \leq U_B$ and

2. $G = G_n G_{n-1} \ldots G_1 = \{g_n \ldots g_1 \mid g_i \in G_i \text{ for } 1 \leq i \leq n\}$.

We define B-Nak $= \{G \leq U_B \mid G = G_n \ldots G_1\}$, the set of all groups which are Nakajima groups with respect to B.

In the above definition we can see that the subgroups G_i are one-row groups consisting of bireflections. As seen in the example below, the triangular group U_B is itself a Nakajima group meaning that all unipotent groups are a subgroup of a Nakajima group.

***Example* 3.0.2.** *Let $B = \{x_1, x_2, x_3, x_4\}$ be a basis for W. With respect to this basis, let*

$$G = \left\{ \begin{pmatrix} 1 & a & b & c \\ 0 & 1 & d & e \\ 0 & 0 & 1 & f \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid a, b, c, d, e, f \in k \right\}.$$

Then for any

$$g = \begin{pmatrix} 1 & a & b & c \\ 0 & 1 & d & e \\ 0 & 0 & 1 & f \\ 0 & 0 & 0 & 1 \end{pmatrix} \in G,$$

with $a, b, c, d, e, f \in k$, with can find

$$\sigma_1 = \begin{pmatrix} 1 & a & b & c \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & d & e \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & f \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and $\sigma_4 = \mathrm{I}_4$ so that $g = \sigma_4\sigma_3\sigma_2\sigma_1$. Hence G is a Nakajima group.

Lemma 3.0.3. *[10, Lemma 8.0.6.] Let $B = \{x_1, \ldots, x_n\}$ be an ordered basis for W, $G \leq \mathrm{GL}(V)$ a p-group. Then every element $g \in G_n \ldots G_1$ has a unique expression of the form $g = \sigma_n \ldots \sigma_1$ with $\sigma_i \in G_i$ for $1 \leq i \leq n$.*

In the above Lemma we can see that $g(x_i) = \sigma_i(x_i)$ for $1 \leq i \leq n$: G is upper triangular so for some $a_j \in k$ for $1 \leq j \leq i-1$:

$$g(x_i) = \sigma_n \dots \sigma_1(x_i) = \sigma_n \dots \sigma_i(x_i) = \sigma_n \dots \sigma_{i+1}(x_i + \sum_{j=1}^{i-1} a_j x_j),$$
$$= x_i + \sum_{j=0}^{i-1} a_j x_j = \sigma_i(x_i).$$

The following can be seen to follow from Lemma 2.1.2.

Lemma 3.0.4. *[10, Lemma 8.0.5] Let $G \leq \mathrm{GL}(V)$, $B = \{x_1, \dots, x_n\}$ a basis for W. For $i < j$, G_i normalizes G_j; in particular $G_iG_j = G_jG_i$*

Lemma 3.0.5. *Let $B = \{x_1, \dots, x_n\}$ be a basis for W, $G = \langle N_1, \dots, N_r \rangle$ for some $r \in \mathbb{N}$ such that $N_i \in B$-*Nak *for $1 \leq i \leq r$, then $G \in B$-*Nak.

Proof. Let $g \in G$ then

$$g = \sigma'_m \dots \sigma'_1$$

where $\sigma'_i \in (N_{j_i})_{s_i}$ for $1 \leq i \leq m$ for some $1 \leq j_i \leq r$, $1 \leq s_i \leq n$. For any $h_i \in G_i$, $h_j \in G_j$ with $i < j$ we can use Lemma 3.0.4 to see that $h_ih_j = h'_jh'_i$ for some $h'_i \in G_i$ and $h'_j \in G_j$. This means we can rearrange the σ'_i to write g as $g = \sigma_n \dots \sigma_1$ where $\sigma_i \in G_i$ for $1 \leq i \leq n$ and so G is a Nakajima group with respect to B. □

Theorem 3.0.6. *[10, Theorem 8.0.7] Let $B = \{x_1, \dots, x_n\}$ be an ordered basis for W, and $G \leq U_B$ be a p-group. Then G is a Nakajima group with respect to B if and only if $k[V]^G = k[\mathbf{N}_1, \dots, \mathbf{N}_n]$.*

So the invariant rings of Nakajima groups are always polynomial. In the case that $k = \mathbb{F}_p$ it can be shown that G is a p-group with a polynomial invariant ring if and only if it is a Nakajima group with respect to some basis ([26, Theorem 1.4]). Theorems 1.3.4 and 1.3.5 can be used to gain more information about the subgroups of Nakajima groups. In his thesis Yinglin Wu shows that if $G \leq \mathrm{GL}(V)$

is an abelian reflection group which is a two–column group on V then $k[V]^G$ is a complete intersection ring (he looks at $S(V)$ rather then $k[V]$ and so refers to these as two–row groups). We will use a similar approach to Yinglin Wu: finding the invariant ring of a group G by working down through maximal subgroups from a Nakajima group containing G. This has lead to the following definition.

Definition 3.0.7. Let $B = \{x_1, \dots, x_n\}$ be a basis for W, $G < U_B$ a p-group. For $1 \leq i \leq n$ let:

$$S_i(G) = \{h \in \mathrm{GL}(V) \mid h(x_j) = x_j \\ \text{for } j \neq i, \text{ there exists } g \in G \text{ such that } h(x_i) = g(x_i)\}$$

Then define

$$\mathrm{Nak}_B^+(G) = \langle g \mid g \in S_i(G) \text{ for some } 1 \leq i \leq n \rangle$$
$$\mathrm{Nak}_B^-(G) = G_n \dots G_1$$

We call G **nice with respect to** B if $[\mathrm{Nak}_B^+(G), \mathrm{Nak}_B^+(G)] \leq G$.

Lemma 3.0.8. *[21, Lemma 3] Let B be an ordered basis for W. Let $G, N \leq \mathrm{GL}(V)$ with $G \leq N \leq U_B(V)$. If $N \in B$-*Nak *then*

- $G \triangleleft N$ *implies* $\mathrm{Nak}_B^-(G) \triangleleft N$*,*
- $[N, N] \leq G$ *implies* $[N, N] \leq \mathrm{Nak}_B^-(G)$*.*

Proposition 3.0.9. *Let $B = \{x_1, \dots, x_n\}$ an ordered basis for W, $G \leq U_B$, then:*

1. $\mathrm{Nak}_B^-(G) \leq G$ *and if $N \leq G$ with N a Nakajima group with respect to B then $N \leq \mathrm{Nak}_B^-(G)$;*
2. $G \leq \mathrm{Nak}_B^+(G)$ *and if $G \leq N$ with N a Nakajima group with respect to B then $\mathrm{Nak}_B^+(G) \leq N$;*

3. *If* G *is nice with respect to* B *then*

$$\mathrm{Nak}_B^-(G) \trianglelefteq G \trianglelefteq \mathrm{Nak}_B^+(G);$$

4. $G^p \leq \mathrm{Nak}_B^+(G)^p \leq [\mathrm{Nak}_B^+(G), \mathrm{Nak}_B^+(G)]$ *and so*

$$\Phi(\mathrm{Nak}_B^+(G)) = [\mathrm{Nak}_B^+(G), \mathrm{Nak}_B^+(G)].$$

Proof. 1. For the first part if $g \in N$ then by Lemma 3.0.3 we can find a unique expression $g = \sigma_n \dots \sigma_1$ with $\sigma_i \in N_i$ for $1 \leq i \leq n$. As $N \leq G$ we see that $\sigma_i \leq G_i$ for $1 \leq i \leq n$, so $g \in \mathrm{Nak}_B^-(G)$ and $N \leq \mathrm{Nak}_B^-(G)$.

2. For the second part, first note that G and N are upper triangular unipotent with respect to B so

$$G \leq N \leq U_B \in B\text{-Nak}.$$

If $h \in S_i(G)$ for some $1 \leq i \leq n$ then there exists $g \in G$ such that $h(x_i) = g(x_i)$. We can find a unique expression of the form $g = \sigma_1 \dots \sigma_n$ (again by Lemma 3.0.3), where $\sigma_j \in (U_B)_j$ for $1 \leq j \leq n$. If N is a Nakajima group containing g then $\sigma_1, \dots, \sigma_n \in N$. By their definitions it's clear that $\sigma_i(x_j) = h(x_j) = x_j$ for $j \neq i$, and we can see that

$$g(x_i) = \sigma_1 \dots \sigma_n(x_i) = \sigma_i(x_i) = h(x_i)$$

and so $h = \sigma_i \in N$, so $g \in \mathrm{Nak}_B^+(G) \leq N$, and $G \leq \mathrm{Nak}_B^+(G)$.

3. For the third part, using Lemma 3.0.5 we see that both $\mathrm{Nak}_B^+(G)$ and $\mathrm{Nak}_B^-(G)$ are Nakajima groups and from there we can use Lemma 3.0.8.

4. For the fourth part, let $\mathrm{Nak}_B^+(G) = N$ and $\overline{N} = N/[N, N]$. Let

$$g = \sigma_n \dots \sigma_1 \in N$$

where $\sigma_i \in N_i$ for $1 \leq i \leq n$ and let $\overline{g}$ denote the image of g in $\overline{N}$. Then as $\overline{N}$ is abelian and σ_i is a transvection of order p for $1 \leq i \leq n$

$$\overline{(g^p)} = \overline{g}^p = \overline{\sigma}_n^p \ldots \overline{\sigma}_1^p = \overline{e}.$$

So for all $g \in N$, $g^p \in [N, N]$. This means that $G^p \leq N^p \leq [N, N]$. As N is a p-group $\Phi(N) = N^p[N, N] = [N, N]$.

$\square$

Corollary 3.0.10. *Let $G, H_1, H_2 \leq \mathrm{GL}(V)$ such that $G = \langle H_1, H_2 \rangle$ and let B be a basis with respect to which G is triangular, then*

$$\mathrm{Nak}_B^+(G) = \langle \mathrm{Nak}_B^+(H_1), \mathrm{Nak}_B^+(H_2) \rangle.$$

Proof. As $H_i \leq G \leq \mathrm{Nak}_B^+(G)$ we see that $\mathrm{Nak}_B^+(H_i) \leq \mathrm{Nak}_B^+(G)$ for $i = 1, 2$ so

$$\langle \mathrm{Nak}_B^+(H_1), \mathrm{Nak}_B^+(H_2) \rangle \leq \mathrm{Nak}_B^+(G).$$

By Proposition 3.0.5 $\langle \mathrm{Nak}_B^+(H_1), \mathrm{Nak}_B^+(H_2) \rangle$ is a Nakajima group, and so by the above Proposition 3.0.9

$$\mathrm{Nak}_B^+(G) = \langle \mathrm{Nak}_B^+(H_1), \mathrm{Nak}_B^+(H_2) \rangle. \qquad \square$$

We can see that given a basis $B = \{x_1, \ldots, x_n\}$ for W and a group $G \leq U_B$ then $\mathrm{Nak}_B^+(G)$ is the smallest Nakajima group with respect to B containing G, and $\mathrm{Nak}_B^-(G)$ is the largest Nakajima group with respect to B contained in G. If G is nice with respect to B then the quotient groups

$$\mathrm{Nak}_B^+(G)/\mathrm{Nak}_B^-(G), \quad G/\mathrm{Nak}_B^-(G), \quad \mathrm{Nak}_B^+(G)/G$$

are elementary abelian.

Proposition 3.0.11. *Let B be an ordered basis for W, $G \leq U_B$ a p-group such that G is nice with respect to B and:*

$$N^+ := \mathrm{Nak}_B^+(G), \quad N^- := \mathrm{Nak}_B^-(G).$$

For some $m, l \in \mathbb{N}$ we can find $\mathbf{h} = \{h_1, \ldots, h_m\}$ with $h_i \in G$ and $\mathbf{g} = \{g_1, \ldots, g_l\}$ with $g_i \in \mathrm{Nak}_B^+(\langle h_{j_i} \rangle)$ for some $1 \leq j_i \leq m$ such that for any ordering of $\mathbf{g}$ and $\mathbf{h}$ if:

$$\begin{aligned} H_0 &= N^-, & H_i &= \langle H_{i-1}, h_i \rangle \text{ for } 1 \leq i \leq m, \\ N_0 &= G, & N_i &= \langle N_{i-1}, g_i \rangle \text{ for } 1 \leq i \leq l, \end{aligned}$$

then:

$$\begin{aligned} &N^- = H_0 \lhd_{\max} H_1 \lhd_{\max} \ldots \lhd_{\max} H_m = G, \\ &G = N_0 \lhd_{\max} N_1 \lhd_{\max} N_2 \lhd_{\max} \ldots \lhd_{\max} N_l = N^+. \end{aligned}$$

Proof. Let $|G/N^-| = p^m$ then we can find $h_1, \ldots, h_m$ such that

$$G/N^- = \langle \overline{h}_1, \ldots, \overline{h}_m \rangle$$

where $\overline{h}_i$ is the image of h_i in G/N^- for $1 \leq i \leq m$. Then

$$G = \langle N^-, h_1, \ldots, h_m \rangle.$$

If $G = \langle N^-, h_1, \ldots, h_{i-1}, h_{i+1}, \ldots, h_m \rangle$ for any $1 \leq i \leq m$ then

$$G/N^- = \langle \overline{h}_1, \ldots, \overline{h}_{i-1}, \overline{h}_{i+1}, \overline{h}_m \rangle$$

which doesn't have order p^m, so

$$h_i \notin \langle N^-, h_1, \ldots, h_{i-1} \rangle$$

for $1 \leq i \leq m$. Let

$$H_0 = N^-, \qquad\qquad H_i = \langle H_{i-1}, h_i\rangle \text{ for } 1 \leq i \leq m$$

then

$$N^- = H_0 \lhd_{\max} H_1 \lhd_{\max} \ldots \lhd_{\max} H_m = G.$$

If N is a Nakajima group containing G then $\mathrm{Nak}_B^+(\langle h_i\rangle) \leq N$ and $N^+ \leq N$ so

$$\widetilde{N} := \langle N^-, \mathrm{Nak}_B^+(\langle h_1\rangle), \ldots, \mathrm{Nak}_B^+(\langle h_n\rangle)\rangle \leq N.$$

By Lemma 3.0.5 $\widetilde{N}$ is a Nakajima group and so $\widetilde{N} = N^+$ using Lemma 3.0.9. Let $l = |N^+/G|$ then we can find $\mathbf{g} = \{g_1, \ldots, g_l\}$ with $g_i \in \mathrm{Nak}_B^+(\langle h_{j_i}\rangle)$ for $1 \leq i \leq l$ and some $1 \leq j_i \leq m$ such that:

$$N^+ = \langle G, g_1, \ldots, g_l\rangle.$$

Similarly to above $\mathbf{g}$ must be minimal and if

$$G_0 = G, \qquad\qquad G_i = \langle G_{i-1}, g_i\rangle \text{ for } 1 \leq i \leq l$$

then

$$G = G_0 \lhd_{\max} G_1 \lhd_{\max} G_2 \lhd_{\max} \ldots \lhd_{\max} G_l = N^+. \qquad \square$$

Proposition 3.0.12. *If $G \leq \mathrm{GL}(V)$ is nice with respect to B then for $1 \leq i \leq n$*

$$\{g(x_i) - x_i \mid g \in G\} = \{g(x_i) - x_i \mid g \in \mathrm{Nak}_B^+(G)\}$$

and consequently:

$$\mathbf{N}_i^G = \mathbf{N}_i^{\mathrm{Nak}_B^+(G)}$$

where as defined above $\mathbf{N}_i^G$, $\mathbf{N}_i^{\mathrm{Nak}_B^+(G)}$ *are the orbit products of* x_i *for* G *and* $\mathrm{Nak}_B^+(G)$ *respectively.*

Proof. Let $g \in \mathrm{Nak}_B^+(G)$ such that:

$$g(x_i) = x_i + r.$$

We need to show that there exists $h \in G$ such that:

$$h(x_i) = x_i + r.$$

By the definition of $\mathrm{Nak}_B^+(G)$

$$g = g_m g_{m-1} \cdots g_1$$

where $m \in \mathbb{N}$ and $g_j \in S_{l_j}(G)$ for $1 \leq j \leq m$ for some $1 \leq l_j \leq n$. We proceed by induction on m.

If $m = 1$ result is trivially true. Let

$$g = g_m g'.$$

By the induction hypothesis we can find some $h' \in G$ such that:

$$h'(x_i) = g'(x_i) = x_i + r'$$

for some $r' \in \langle x_1, \ldots, x_{i-1} \rangle$. If $g_m \in S_i(G)$ then we can find some $h_m \in G$ such that

$$h_m(x_i) = x_i + r - r'$$

so $h_m h' \in G$ and

$$h_m h'(x_i) = x_i + r + \delta_{h_m}(r').$$

As $h_m, h' \in \mathrm{Nak}_B^+(G)$ which is a Nakajima group we can find $\sigma'_j, \sigma_{m,j} \in \mathrm{Nak}_B^+(G)_j$ for $1 \leq j \leq n$ such that:

$$h' = \sigma'_n \dots \sigma'_1,$$
$$h_m = \sigma_{m,n} \dots \sigma_{m,1}.$$

Let $t = \sigma_{m,i-1} \dots \sigma_{m,0}$, then

$$\sigma'_i(x_i) = x_i + r', \qquad \sigma'_i(r') = r',$$
$$t(x_i) = x_i, \qquad t(r') = r' + \delta_{h_m}(r').$$

Let $\theta = \sigma'_i t \sigma_i'^{-1} t^{-1}$, then

$$\theta(x_i) = x_i - \delta_{h_m}(r')$$
$$\theta(r) = r.$$

As $\theta \in [\mathrm{Nak}_B^+(G), \mathrm{Nak}_B^+(G)]$, $\theta \in G$. Let $h = \theta h_m h'$ then $h \in G$ and

$$h(x_i) = x_i + r$$

as required.

Suppose $g_m \notin S_i(G)$ then $g_m(r') = r' + r'' = r$, with $r'' \in W^{g_m}$. Again we can find $\sigma_i \in \mathrm{Nak}^+(G)$ such that

$$\sigma_i(x_i) = x_i + r', \quad \sigma_i(r') = r'.$$

If $\theta = g_m^{-1}\sigma_i^{-1}g_m\sigma_i$, then

$$\begin{aligned}\theta(x_i) &= x_i + r'',\\ \theta(r') &= r'.\end{aligned}$$

As $\theta \in [\mathrm{Nak}_B^+(G), \mathrm{Nak}_B^+(G)]$, $\theta \in G$. Let $h = \theta h'$ then $h \in G$ and

$$h(x_i) = x_i + r.$$

$\square$

The choice of basis is important. Let $G = \langle g, h\rangle$ where:

$$g = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & -1\\ 0 & 0 & 0 & 1\end{pmatrix}, \quad h = \begin{pmatrix} 1 & 1 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

with respect to the basis $B = \{x_1, x_2, x_3, x_4\}$ for W. We can see that if

$$t = \begin{pmatrix} 1 & 1 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

then $N := \mathrm{Nak}_B^+(G) = \langle g, h, t\rangle$. The commutator subgroup $[N, N]$ is then generated by

$$\sigma = \begin{pmatrix} 1 & 0 & 0 & 1\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

and we see that $\sigma \notin G$ and so G is not nice with respect to B. We also see that $\mathbf{N}_4^G \neq \mathbf{N}_4^N$. Let $B' = \{y_1, y_2, y_3, y_4\}$ with:

$$y_1 = x_1, \quad y_2 = x_2 - x_3, \quad y_3 = x_3, \quad y_4 = x_4.$$

With respect to this basis:

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and so G is a Nakajima group with respect to B'.

Lemma 3.0.13. *Let $G \leq \mathrm{GL}(V)$ a p-group. Then for any basis B for W such that $G \leq U_B$, $[\mathrm{Nak}_B^+(G), W] = [G, W]$ and $V^G = V^{\mathrm{Nak}_B^+(G)}$.*

Proof. Let B be a basis such that $G \leq U_B$, $N = \mathrm{Nak}_B^+(G)$. By Lemma 3.0.9 we know that $G \leq N$ and so $[G, W] \leq [N, W]$.

For any group $H = \langle g_1, \ldots, g_l \rangle \leq \mathrm{GL}(V)$

$$[H, W] = \sum_{i=1}^{l} [g_i, W].$$

If $g \in S_i(G)$ for some $1 \leq i \leq n$ then $[g, W] \leq [G, W]$ and so:

$$[G, W] \geq \sum_{i=1}^{n} \sum_{g \in S_i(G)} [g, W] = [N, W],$$

so $[G, W] = [N, W]$.

Let $\{e_1, \ldots, e_n\}$ be the corresponding basis for V, then by Lemma 1.0.2 we see that for $g \in N$ the matrix representing g with respect to this basis is given by the transpose inverse and so $V^G = V^N$. $\square$

Lemma 3.0.14. *For $G \leq \mathrm{GL}(V)$ a p-group we can find a basis B for W such that $G \leq U_B$, $W^G = W^{\mathrm{Nak}_B^+(G)}$ and $[G, V] = [\mathrm{Nak}_B^+(G), V]$.*

Proof. Let $m = \dim_k(W^G)$, then we can choose a basis $B = \{x_1, \ldots, x_n\}$ for W such that $G \leq U_B$ and $\langle x_1, \ldots, x_m \rangle = W^G$.

Let $N = \mathrm{Nak}_B^+(G)$. As

$$S_i(G) = \{e\} \text{ for } 1 \leq i \leq m$$

$x_i \in W^N$ for $1 \leq i \leq m$ and so $W^G \leq W^N$. As $G \leq N$ we know that $W^N \leq W^G$ and so $W^G = W^N$.

Similarly to above using Lemma 1.0.2 we see that $W^G = W^N$ means $[G, V] = [N, V]$. $\square$

Lemma 3.0.15. *Let $U_1 \leq U_2 \leq \cdots U_\ell \leq W$ be G-stable subspaces. Let $B = \cup_i B_i$ be a basis for W such that $\langle B_i \rangle = U_i$ and $G \leq U_B$. Then the U_i are also $\mathrm{Nak}_B^+(G)$-stable with $[G, U_i] = [\mathrm{Nak}_B^+(G), U_i]$ for all i.*

Proof. Let $N := \mathrm{Nak}_B^+(G)$, then

$$[N, U_i] = \sum_{x \in B_i} \langle n(x) - x \mid n \in N \rangle.$$

Let $n \in S_j(G)$ and $x \in B_i$, then $n(x) = g(x)$ for some $g \in G$ if $x = x_j$ or $n(x) = x$ otherwise. Hence $n(x) - x \in [g, U_i]$, so $[n, U_i] \subseteq [G, U_i]$. We have

$$nn'x - x = nn'x - n'x + n'x - x \in [n, n'U_i] + [n', U_i].$$

Since N is generated by the $S_j(G)$'s, we see inductively that

$$[N, U_i] \leq [G, U_i] \leq [N, U_i] \text{ for } 1 \leq i \leq \ell. \qquad \square$$

Lemma 3.0.16. *A Nakajima group G is abelian if and only if $[G, W] \leq W^G$.*

Proof. This is just a special case of Lemma 2.1.5. $\square$

Lemma 3.0.17. *Let $G \leq \mathrm{GL}(V)$ be a p-group with $[G, W] \leq W^G$. Then G is abelian and we can find a basis B of W such that $\mathrm{Nak}_B^+(G)$ is also abelian and therefore G is nice.*

Proof. By Lemma 3.0.14 we can find a basis B for W such that $G \leq U_B$ and $W^G = W^N$ where $N := \mathrm{Nak}_B^+(G)$. By Lemma 3.0.13 $[G, W] = [N, W]$ and so $[N, W] \leq W^N$. By Proposition 3.0.16 N is an abelian group so $[N, N] = \{e\} \leq G$ and G is nice with respect to B. As $G \leq N$, G is also abelian. □

3.1 Maximal Pure Bireflection Groups

We now apply the above to the maximal pure bireflection groups as defined in Chapter 2.

Proposition 3.1.1. *For $G \leq \mathrm{GL}(V)$:*

- *if G is a two–row group (on V) then $\mathrm{Nak}_B^+(G)$ is a two–row group (on V) for any basis B for W with respect to which G is triangular;*
- *if G is a two–column group (on V) then $\mathrm{Nak}_B^+(G)$ is a two–column group (on V) for any basis B for W with respect to which G is triangular and $[G, V] = [\mathrm{Nak}_B^+(G), V]$;*
- *if G is a hook group on W with hyperplane U and line kv then for any basis $B = \{x_1, \dots, x_n\}$ with respect to which G is triangular and*

$$U = \langle x_1, \dots, x_{n-1} \rangle.$$

then $\mathrm{Nak}_B^+(G)$ is a hook group on W with hyperplane U and line kv.

If $G \leq \mathrm{GL}(V)$ is a maximal bireflection group which is either a hook, two–row or two–column group then we can find some basis B with respect to which G is a Nakajima group. Consequentially G is nice with respect to B.

Proof. If G is a two–row group on V, then

$$\dim_k([G, V]) \leq 2$$

and for any basis B with respect to which G is triangular. $[G, V] = [\mathrm{Nak}_B^+(G), V]$. This means that

$$\dim_k([\mathrm{Nak}_B^+, V]) \leq 2$$

and $\mathrm{Nak}_B^+(G)$ is a two–row group.

If G is two–column group on V, then by Lemma 3.0.13 for any basis B for W with respect to which G is triangular

$$V^G = V^{\mathrm{Nak}_B^+(G)},$$

so $\mathrm{Nak}_B^+(G)$ is also a two–column group.

Suppose that G is a hook group with hyperplane U and line kv. Let $B = \{x_1, \ldots, x_n\}$ be a basis for W with respect to which G is triangular and $U = \{x_1, \ldots, x_{n-1}\}$. By Proposition 3.0.15 U is a $\mathrm{Nak}_B^+(G)$ stable subspace and

$$[\mathrm{Nak}_B^+(G), U] = [G, U] \leq kv.$$

This means that $\mathrm{Nak}_B^+(G)$ is a hook group with hyperplane U and line kv.

So if G any two–row, two–column or hook group using the above and Propositions 3.0.14, 3.0.13 and 3.0.15 we can find some basis B for W such that $\mathrm{Nak}_B^+(G)$ is a pure bireflection group containing G. If G is maximal then $\mathrm{Nak}_B^+(G) = G$. This means that

$$[\mathrm{Nak}_B^+(G), \mathrm{Nak}_B^+(G)] \leq G$$

and so G is nice with respect to B. □

Proposition 3.1.2. *Let $p \neq 2$, and let $G \leq \mathrm{GL}(V)$ be a unipotent pure bireflection group then we can find a basis B with respect to which $\mathrm{Nak}_B^+(G)$ is a pure bireflection group if and only if G is not an exceptional group (of type one or two).*

Let G be a maximal unipotent pure bireflection group. Then G is not a Nakajima group with respect to any basis if and only if G is an exceptional group *of type 1 or 2. We can find some basis with respect to which G is nice if and only if G is not an exceptional group of type 1.*

Proof. Let G be a pure bireflection group. If G is not an exceptional group of type one or two then G is a two–row, two–column or hook group by Theorem 1.0.5. By Proposition 3.1.1 we can find a basis B such that $\mathrm{Nak}_B^+(G)$ is a pure bireflection group.

If G is an exceptional group of type one or two it is not a two–row, two–column or hook group. This means that $\mathrm{Nak}_B^+(G)$ is not a two–row, two–column or hook group for any choice of basis B. As exceptional groups do not contain any reflections (Lemmas 2.5.9 and 2.6.5), $\mathrm{Nak}_B^+(G)$ is not an exceptional group either, and so by Theorem 1.0.5 $\mathrm{Nak}_B^+(G)$ is not a pure bireflection group.

Now suppose G is a maximal pure bireflection group. If G is an exceptional group (of type 1 or 2) then G is not generated by reflections (Lemmas 2.5.9 and 2.6.5) and so it is not a Nakajima group with respect to any basis. If G is not an exceptional group then it is either a two–row, two–column or hook group so by Proposition 3.1.1 it is a Nakajima group, and hence also nice, with respect to some basis.

Suppose G is an exceptional group of type one which is nice with respect to some basis B and let $N = \mathrm{Nak}_B^+(G)$. By Lemma 2.5.9 there are no reflections in G and so $\mathrm{Nak}_B^-(G)$ contains only the identity. By Lemma 3.0.8 as G is nice

$$[N, N] \leq \mathrm{Nak}_B^-(G).$$

This would mean that N is abelian, however as $G \leq N$ is not abelian we have a contradiction.

If G is a maximal exceptional group of type two then we can find $r_1, r_2, r_3 \in W$ and $\gamma_1, \gamma_2, \gamma_3 \in W^*$ such that if

$$\mathbf{r} = \{r_1, r_2, r_3\}, \quad \gamma = \gamma_1, \gamma_2, \gamma_3$$

then $G = W^{\mathbf{r},\gamma}$. We see that

$$\langle r_1, r_2, r_3 \rangle = [G, W] \leq W^G$$

(see Definition 2.2.5). By Lemma 3.0.17 we can find some basis with respect to which G is nice. $\square$

Corollary 3.1.3. *For $p \neq 2$, if $G \leq \mathrm{GL}(V)$ is a maximal unipotent pure bireflection group then $k[V]^G$ is not a polynomial ring if and only if G is an exceptional group of type 1 or 2.*

Proof. By the above Proposition if G is not an exceptional group then it is a Nakajima group with respect to some basis and so $k[V]^G$ is polynomial. If G is an exceptional group of type one or two then G contains no reflections, so $k[V]^G$ is not polynomial. $\square$

Proposition 3.1.4. *Let $p \neq 2$ and $G \leq \mathrm{GL}(V)$ be a unipotent pure bireflection group. Let $H = [G, G]$, then $k[V]^H$ is Cohen Macaulay. Furthermore the invariant ring $k[V]^H$ is polynomial if and only if G is not an exceptional group of type one.*

Proof. If G is a hook group or a two–column group $\dim_k(V^H) \geq n - 1$ and so by Theorem 1.4.3 $k[V]^H$ is polynomial. If G is a two–row group then W^H has codimension one and so H is a Nakajima group with polynomial ring of invariants. If G is an exceptional group of type one then by Corollary 2.5.9 it does not contain any reflections. This means that H cannot be generated by reflections and so $k[V]^H$ is not a polynomial ring, however since V^H has codimension two (see Corollary 2.5.9) by Theorem 1.4.3 $k[V]^G$ is Cohen–Macaulay. If G is an exceptional group of type two then H is the trivial group as G is abelian. $\square$

Chapter 4

Invariant rings of hook groups

In this chapter we look at invariant rings of hook groups. We fix $w, v \in W$ and U a subspace of W with codimension one, and look at subgroups of $B^{U,v}$. These groups are generated by elements of the form $b_{u,\lambda}$ where $u \in U$ and $\lambda \in V = W^*$. We will look at transvection subgroups G of $B^{U,v}$ to see when the invariant ring $k[V]^G$ is polynomial. For $k = \mathbb{F}_p$ we find necessary and sufficient conditions for $G \leq B^{U,v}$ to be nice with respect to some basis. We will then find generators for $k[V]^G$ in these cases using 1.3.4 and results from the previous chapter. Using Theorem 1.3.5 we will show that, for $k = \mathbb{F}_p$, if a hook group is nice with respect to some basis then it's ring of invariants is a complete intersection ring. We start more generally though, with $k = \mathbb{F}_q$ where $q = p^r$, and look at some definitions and general properties of hook groups which we will want to make use of in the next few sections.

The following connects hook groups with the results of the previous chapter.

Lemma 4.0.1. *If $G \leq B^{U,v}$ with $R_{\hat{v},U} \leq G$ then we can find some basis for W with respect to which G is nice.*

Proof. By Proposition 3.1.1 we can choose $\mathcal{B}$ such that $B^{U,v}$ is a Nakajima group with respect to $\mathcal{B}$, then

$$[\mathrm{Nak}^+_{\mathcal{B}}(G), \mathrm{Nak}^+_{\mathcal{B}}(G)] \leq [B^{U,v}, B^{U,v}] = R_{\hat{v},U} \leq G$$

so G is nice with respect to $\mathcal{B}$. $\square$

We define two $\mathbb{F}_p$-vector spaces associated to each hook group.

Definition 4.0.2. Let $G = \langle b_1, \ldots, b_l \rangle$ where $b_i = b_{u_i,\lambda_i} \in B^{U,v}$ for $1 \leq i \leq l$. Then define:

$$\begin{aligned}
\hat{U}(G) &= \langle u_1, \ldots, u_l \rangle_{\mathbb{F}_p} + kv, \\
\Lambda(G) &= \langle \lambda_1, \ldots, \lambda_l \rangle_{\mathbb{F}_p}, \\
D_{\hat{U}}(G) &= \begin{cases} \dim_{\mathbb{F}_p}(\hat{U}(G)) \text{ for } R_{\hat{v},U} \leq G, \\ \dim_{\mathbb{F}_p}(\hat{U}(G)) - r \text{ for } R_{\hat{v},U} \not\leq G, \end{cases} \\
D_\Lambda(G) &= \dim_{\mathbb{F}_p}(\Lambda(G)).
\end{aligned}$$

Lemma 4.0.3. *Let $G \leq B^{U,v}$ then:*

1. *if $b_{u,\lambda} \in G$ then $u \in \hat{U}(G)$ and $\lambda \in \Lambda(G)$;*
2. *if $\lambda \in \Lambda(G)$ then $b_{u,\lambda} \in G$ for some $u \in \hat{U}(G)$;*
3. *$\Lambda(G)$ is independent of choice of generators for G and*

$$\Lambda(G) = \{\lambda \in W^* \mid b_{u,\lambda} \in G \textit{ for some } u \in W\};$$

4. *if $u \in \hat{U}$ then $b_{u+cv,\lambda} \in G$ for some $\lambda \in \Lambda(G)$ and some $c \in k$;*
5. *$\hat{U}(G) = \langle g(w) - w \mid g \in G \rangle_{\mathbb{F}_p} + kv$;*
6. *if $k = \mathbb{F}_p$ and $v \in [G, W]$ then $\hat{U}(G) = [G, W]$;*
7. *if $R_{\hat{v},U} \leq G$ and $u \in \hat{U}(G)$ then $b_{u,\lambda} \in G$ for some $\lambda \in w^\perp \bigcap v^\perp$.*

Proof. 1) Let $G = \langle b_1, \ldots, b_l \rangle$ where $b_i = b_{u_i,\lambda_i} \in B^{U,v}$. For any element $b_{u,\lambda} \in G$, we can write

$$b_{u,\lambda} = b^{a_1}_{u_1,\lambda_1} \ldots b^{a_l}_{u_l,\lambda_l} t$$

for some $t \in \Phi(G) \leq R_{\hat{v},U} \leq Z(G)$ and $0 \leq a_1, \ldots, a_l \leq p-1$. Using Proposition 2.4.2(2)

$$
\begin{aligned}
u &= a_1 u_1 + \ldots a_l u_l + cv, \\
\lambda &= a_1 \lambda_1 + \ldots + a_l \lambda_l,
\end{aligned}
$$

for some $c \in k$. This means that $u \in \hat{U}(G)$, and $\lambda \in \Lambda(G)$.

2) For any $\lambda \in \Lambda(G)$ we can write

$$\lambda = a_1 \lambda_1 + \ldots + a_l \lambda_l$$

for some $0 \leq a_1, \ldots, a_l \leq p-1$. We can take

$$b_{u,\lambda} = b_{u_1,\lambda_1}^{a_1} \ldots b_{u_l,\lambda_l}^{a_l}$$

so then $b_{u,\lambda} \in G$.

3) Combining parts 1) and 2) gives us

$$\Lambda(G) = \{\lambda \in W^* \mid b_{u,\lambda} \in G \text{ for some } u \in W\}$$

which must be independent on the choice of generators for G.

4) For any $u \in \hat{U}(G)$ we can write:

$$u = a_1 u_1 + \ldots + a_l u_l + bv$$

for some $0 \leq a_1, \ldots, a_l \leq p-1$ and $b \in k$. We can then find $\lambda \in \Lambda(G)$ and $c \in k$ such that $b_{u+cv,\lambda} \in G$: take

$$b_{u',\lambda} = b_{u_1,\lambda_1}^{a_1} \ldots b_{u_l,\lambda_l}^{a_l},$$

then

$$u' = a_1 u_1 + \ldots + a_l u_l + c' v$$

for some $c' \in k$. If we let $c = c' - b$ then $u' = u + cv$.

5) Using part 1) and the definition of $\hat{U}(G)$ we see that

$$\langle g(w) - w \mid g \in G \rangle_{\mathbb{F}_p} + kv \leq \hat{U}(G).$$

Using part 4) we know

$$\hat{U}(G) \leq \langle g(w) - w \mid g \in G \ \rangle_{\mathbb{F}_p} + kv,$$

and so

$$\hat{U}(G) = \langle g(w) - w \mid g \in G \rangle_{\mathbb{F}_p} + kv.$$

Parts 6) and 7) are direct consequences of part 5). □

The next lemma allows us to choose a useful generating set for later results.

Lemma 4.0.4. *Let $k = \mathbb{F}_p$ and $G \leq B^{U,v}$ with $\dim_k(\hat{U}(G)) = m+1$ then we can find a set of generators $\{b_{u_1,\lambda_1}, \ldots, b_{u_l,\lambda_l}\}$ for G such that $\{u_1, \ldots, u_m, v\}$ form a basis for $\hat{U}(G)$ and for $m < i \leq l$, $u_i \in kv$.*

Proof. If $\hat{U}(G) = kv$ then the result is trivial. Otherwise let $\dim_k(\hat{U}(G)) > 1$ and

$$G = \langle b_{\hat{u}_1,\hat{\lambda}_1}, b_{\hat{u}_2,\hat{\lambda}_2}, \ldots, b_{\hat{u}_l,\hat{\lambda}_l} \rangle.$$

We can assume that $\hat{U}(G) = span(\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_m, v)$, so for $1 \leq i \leq m$ let $b_{u_i,\lambda_i} = b_{\hat{u}_i,\hat{\lambda}_i}$.

For $j > m$ as $\hat{u}_j \in \hat{U}(G)$ for $1 \leq i \leq m$ we can find some $a_i, b \in k$ such that:

$$\hat{u}_j = bv + \sum_{i=1}^{m} a_i u_i.$$

Let:

$$b_{u_j,\lambda_j} = b_{\hat{u}_j,\hat{\lambda_j}} \prod_{i=1}^{m} b_{u_i,\lambda_i}^{-a_i}.$$

For some $c \in k$

$$u_j = \hat{u}_j - \sum_{i=1}^{m} a_i u_i + cv$$

so $u_j \in kv$. $\square$

As we have seen in previous sections, finding a good basis is useful for finding invariant rings.

Definition 4.0.5. Let $k = \mathbb{F}_p$ and $G \leq B^{U,v}$ with $L = \{\lambda_1, \dots, \lambda_m\}$ a basis for $\Lambda(G)$. We define a basis $\mathcal{C} = \{x_1, \dots, x_{n-1}\}$ for U to be a **Λ-basis with respect to L** if $x_1 = v$ and for $1 \leq i \leq m$, $1 \leq j \leq n-1$

$$\lambda_i(x_j) = \begin{cases} 1 \text{ if } j = n-i, \\ 0 \text{ otherwise.} \end{cases}$$

Lemma 4.0.6. *Let $k = \mathbb{F}_p$ and $G \leq B^{U,v}$ with $L = \{\lambda_1, \dots, \lambda_m\}$ a basis for $\Lambda(G)$. If $\mathcal{B} = \{x_1, \dots, x_n\}$ is a basis for W such that $\{x_1, \dots, x_{n-1}\}$ is a Λ-basis for U with respect to L, and $N = \mathrm{Nak}^+_{\mathcal{B}}(G)$ then*

1. *N is a hook group with hyperplane U and line kv;*

2. *$U^G = U^N$;*

3. *for $1 \leq i \leq n-1$:*

$$\deg(\mathbf{N}_i^G) = \begin{cases} 1 \textit{ for } 1 \leq i \leq n-m-1, \\ p \textit{ for } n-m \leq i \leq n-1; \end{cases}$$

4. *if $b_{u,\lambda_i} \in G$ for some $1 \leq i \leq m$ and $u \in \hat{U}(G)$ with $b_{u,\lambda_i}(x_n) = x_n + u$ then*

$$\mathrm{Nak}^+_{\mathcal{B}}(\langle b_{u,\lambda_i} \rangle) = \langle b_{u,\lambda_i}, b_{0,\lambda_i} \rangle.$$

Proof. 1) The first part can be seen from Proposition 3.1.1.

2) For the next part we first note that as G is a subgroup of N

$$U^N \leq U^G.$$

For $1 \leq i \leq n-m-1$ we see that $g(x_i) = x_i$ for any $g \in G$, therefore $S_i = \{1\}$ and so $x_i \in U^N$. For $u \in U^G$ we must have that $\lambda(u) = 0$ for all $\lambda \in \Lambda(G)$, therefore $u \in \langle x_1, \ldots, x_{n-m-1} \rangle$, so

$$U^G = U^N.$$

3) By definition $[G, U] \leq kv$, and so

$$1 \leq \deg(\mathbf{N}_i^G) \leq p$$

for $1 \leq i \leq n-1$. From the above $x_i \in W^G$, for $1 \leq i \leq n-m-1$, and so $\deg(\mathbf{N}_i) = 1$. By Proposition 4.0.3 for all $c \in k$ there exists $u \in \hat{U}(G)$ such that $b_{u,c\lambda_i} \in G$ for $1 \leq i \leq m$ so

$$b_{u,c\lambda_i}(x_{n-i}) = x_{n-i} + cv$$

and this means that $\deg(\mathbf{N}_i^G) = p$ for $n-m \leq i \leq n-1$.

4) If $b_{u,\lambda_i} \in G$ for some $1 \leq i \leq m$ and $u \in \hat{U}(G)$ with $b_{u,\lambda_i}(x_n) = x_n + u$ then for $1 \leq j \leq n$

$$b_{u,\lambda_i}(x_j) = \begin{cases} x_j + u \text{ for } j = n, \\ x_j + v \text{ for } j = n-i, \\ x_j \text{ otherwise.} \end{cases}$$

We can see that for $1 \leq j \leq n$

$$b_{u,0}(x_j) = \begin{cases} x_j + u = b_{u,\lambda_i}(x_j) \text{ if } j = n, \\ x_j \text{ otherwise,} \end{cases}$$

$$b_{0,\lambda_i}(x_j) = \begin{cases} x_j + v = b_{u,\lambda_i}(x_j) \text{ if } j = n - i, \\ x_j \text{ otherwise,} \end{cases}$$

so $b_{u,0} \in S_n$ and $b_{0,\lambda} \in S_i$. This means that $H = \langle b_{u,\lambda_i}, b_{0,\lambda_i} \rangle$ is a Nakajima group by Proposition 3.0.5 and

$$H \leq \mathrm{Nak}^+_{\mathcal{B}}(\langle b_{u,\lambda_i} \rangle).$$

As $b_{u,\lambda} = b_{u,0} b_{0,\lambda}$ we see $G \leq H$ and so by Proposition 3.0.9

$$H = \mathrm{Nak}^+_{\mathcal{B}}(\langle b_{u,\lambda_i} \rangle).$$ □

4.1 Transvection subgroups

If $k[V]^G$ is a polynomial ring then it is generated by reflections. In this section we look at which hook groups generated by reflections have polynomial rings of invariants.

Lemma 4.1.1. *Let $R_{\hat{v},U} \leq G \leq B^{U,v}$ then the subgroup of G generated by all reflections in G has a polynomial ring of invariants (and so for $k = \mathbb{F}_p$ it is a Nakajima group).*

Proof. The element $b_{u,\lambda} \in G$ is a reflection if $\lambda = 0$ or if $u \in kv$ so we are interested in the subgroup of G generated by elements of the form $b_{cv,\lambda}$ and $b_{u,0}$. Since $R_{\hat{v},U} \leq G$ we know that if $b_{cv,\lambda} \in G$ then $b_{0,\lambda} \in G$.

Let

$$T = \langle g \in G \mid \dim_k(W^g) = n - 1 \rangle.$$

Now define subgroups of T:

$$T_U = \langle b_{u,0} \in G \mid u \in \hat{U}(G) \rangle, \quad T_\Lambda = \langle b_{0,\lambda} \mid \lambda \in \Lambda(G) \rangle.$$

As $R_{\hat{v},U} \leq T_U$ the subgroup T_U is normal, we can also see that

$$T = T_U T_\Lambda = \{t_u t_\lambda \mid t_u \in T_U, t_\lambda \in T_\Lambda\}.$$

Let $x_1, x_2, \ldots, x_n$ be a basis for W such that $U = \langle x_1, \ldots, x_{n-1} \rangle$, and $x_n = w$ then T_U is a Nakajima group with respect to this basis as it fixes all basis elements except x_n. So:

$$k[V]^{T_U} = k[x_1, \ldots, x_{n-1}, \mathbf{N}_n^{T_U}].$$

For all $t \in T$ we have that $t(x_n) = t_u(x_n)$ for some $t_u \in T_U$ so $\mathbf{N}_n^T = \mathbf{N}_n^{T_U}$ and $\mathbf{N}_n^{T_U} \in k[V]^T$.

Let A be the k vector space spanned by $\{x_1, \ldots, x_{n-1}, \mathbf{N}_n^T\}$ and let $H = T/T_U$ then $k[V]^T = S(A)^H$. As $H \cong T_\lambda$ and the action of H on A is linear with $\dim_k((A^*)^H) = n-1$ by Theorem 1.4.3 $(k[V]^{T_u})^H = k[V]^T$ is polynomial.

If $k = \mathbb{F}_p$ this means that T is a Nakajima group (as in this case these are the only p-groups with polynomial rings of invariants). $\square$

Proposition 4.1.2. *Let $G = \langle b_1, \ldots, b_l, R_{\hat{v},U} \rangle$ where $b_i = b_{u_i,\lambda_i} \in B^{U,v}$ are a minimal set of generators, then the following are equivalent:*

1. *$k[V]^G$ is a polynomial ring,*
2. *G is generated by reflections,*
3. *$|G| = p^{a+b}$, $l = a + b - r$ where $a = D_{\hat{U}}(G)$ and $b = D_\Lambda(G)$.*

For $k = \mathbb{F}_p$ this is equivalent to G being a Nakajima group.

Proof. 1) $\Rightarrow$ 2) If $k[V]^G$ is a polynomial ring then we know G is generated by reflections.

2) $\Rightarrow$ 3) As G is minimally generated by $\{b_1, \ldots, b_l, R_{\hat{v},U}\}$ by Proposition 2.4.8 $|G| = p^{l+r}$.

If G is generated by reflections then as shown in proof of Lemma 4.1 then $G = T = T_U T_\Lambda$. From this we see $|G| = |T_U||T_\Lambda| = p^{a+b}$ for $a = D_{\hat{U}}(G)$ and $b = D_\Lambda(G)$, and so $l + r = a + b$, $l = a + b - r$.

3) $\Rightarrow$ 1) Let $\{u_1, \ldots, u_b, v\}$ be a basis for $\hat{U}(G)$, $\{\lambda_1, \ldots, \lambda_a\}$ a basis for $\Lambda(G)$ and let

$$H = \langle b_{u_1,0}, \ldots, b_{u_b,0}, b_{0,\lambda_1}, \ldots, b_{0,\lambda_a}, R_{\hat{v},U} \rangle.$$

If $b_{u,\lambda} \in G$ then by Lemma 4.0.3 $u \in \hat{U}(G)$ and $\lambda \in \Lambda(G)$ and so we can find

$$1 \leq \alpha_1, \ldots, \alpha_a, \beta_1, \ldots, \beta_b \leq p - 1$$

and $t \in R_{\hat{v},U}$ such that

$$b_{u,\lambda} = b_{u_1,0}^{\beta_1} \ldots b_{u_b,0}^{\beta_b} b_{0,\lambda_1}^{\alpha_1} \ldots b_{0,\lambda_a}^{\alpha_a} t \in H$$

so $G \leq H$. We can see that $|H| = p^{a+b}$ and $|G| = p^{l+r}$ so if $l = a + b - r$ then $G = H$.

As $G = H$ is clearly a reflection group by Proposition 4.1.1 G has polynomial ring of invariants.

For $k = \mathbb{F}_p$, G is a p-group so $k[V]^G$ is polynomial if and only if G is a Nakajima group [26, Proposition 4.1]. $\square$

The previous Proposition tells us that if $G \leq B^{U,v}$ with $R_{\hat{v},U} \leq G$ then $k[V]^G$ is polynomial and in the case that $k = \mathbb{F}_p$ it is a Nakajima group. The following lemma provides some information about bases with respect to which these reflection groups are Nakajima groups, and we will build on it later to find invariant rings of other nice hook groups.

Lemma 4.1.3. *Let $G \leq B^{U,v}$ be generated by reflections such that either $R_{\hat{v},U} \leq G$ or $[G, W] \leq W^G$, let $k = \mathbb{F}_p$. Then $|G| = p^{a+b}$ where $a = D_{\hat{U}}(G)$ and $b = D_\Lambda(G)$.*

Let $L = \{\lambda_1, \ldots, \lambda_b\}$ be a basis for $\Lambda(G)$. If $\mathcal{C} = \{x_1, \ldots, x_{n-1}\}$ is a Λ-basis for U with respect to L then we can find some $x_n \in W \backslash U$ such that $\mathcal{B} = \{x_1, \ldots, x_n\}$ is a basis for W with respect to which $k[V]^G$ is a Nakajima group.

Proof. First suppose that $R_{\hat{v},U} \leq G$. The first part is Lemma 4.1.2(3). In this case if $b_{cv,\lambda} \in G$ then $b_{0,\lambda} \in G$ and we can choose a minimal set of generators

$$G = \langle b_{u_1,0}, \ldots, b_{u_{a-1},0}, b_{0,\lambda'_1}, \ldots, b_{0,\lambda'_b}, R_{\hat{v},U} \rangle.$$

By Lemma 4.0.3

$$\{u \in U \mid b_{u,\lambda} \in G \text{ for some } \lambda \in \Lambda\} = \hat{U}(G)$$

so if we let $x_n = w$ then $\deg(\mathbf{N}_n^G) = |\hat{U}(G)| = a$. We know from Lemma 4.0.6 that

$$\deg(\mathbf{N}_i^G) = \begin{cases} 1 \text{ for } 1 \leq i \leq n-b-1, \\ p \text{ for } n-b \leq i \leq n-1. \end{cases}$$

This means that

$$\deg(\mathbf{N}_1^G) \deg(\mathbf{N}_2^G) \ldots \deg(\mathbf{N}_n^G) = p^{a+b} = |G|.$$

Since $\mathbf{N}_1^G, \ldots, \mathbf{N}_n^G$ form a HSOP by Theorem 1.1.8 this means that

$$k[V]^G = k[\mathbf{N}_1, \ldots, \mathbf{N}_n]$$

is a polynomial ring and by Theorem 3.0.6 G is a Nakajima group with respect to this basis.

Now assume $R_{\hat{v},U} \not\leq G$, $[G, W] \leq W^G$. Let:

$$G = \langle b_{u_1,0}, \ldots, b_{u_l,0}, b_{c_1 v,\lambda'_1}, \ldots, b_{c_m v,\lambda'_m} \rangle$$

so that G is minimally generated by these elements. This means that

$$\{u_1, \ldots, u_l, v\}$$

are linearly independent (by Lemma 4.0.4). Suppose that $\{\lambda'_1, \ldots, \lambda'_m\}$ not linearly independent, then there exists some i such that:

$$\lambda_i = a_1\lambda'_1 + \ldots + a_{i-1}\lambda'_{i-1} + a_{i+1}\lambda'_{i+1} + \ldots + a_m\lambda'_m$$

for some $a_1, \ldots, a_m \in k$. Without loss of generality suppose $i = m$, so:

$$b_{c_m v, \lambda'_m} = b_{dv,0}(b^{a_1}_{c_1 v, \lambda'_1} \ldots b^{a_{m-1}}_{c_{m-1}v, \lambda'_{m-1}}).$$

Where $d = c_m - a_1c_1 - \ldots - a_{m-1}c_{m-1}$. If $d = 0$ then G was not minimally generated, however if $d \neq 0$ then $R_{\hat{v},U} \leq G$ so we can assume that $\{u_1, \ldots, u_l\}$ and $\{\lambda'_1, \ldots, \lambda'_m\}$ are both linearly independent sets. This means that

$$\begin{aligned} \hat{U}(G) &= \langle u_1, \ldots, u_l, v\rangle, \\ \Lambda(G) &= \langle \lambda'_1, \ldots, \lambda'_m\rangle, \end{aligned}$$

and so $\dim_k(\hat{U}(G)) - 1 = l = a$ and $\dim_k(\Lambda(G)) = m = b$. By 2.4.8 $|G| = p^{l+m} = p^{a+b}$.

By the additivity of the λ we can pick generators such that $\lambda'_i = \lambda_i$ for $1 \leq i \leq m = b$.

Let $\mathcal{B} = \{x_1, \ldots, x_n\}$ where

$$x_n = w - \sum_{i=1}^{b} c_i x_{n-i}.$$

Then for $1 \leq i \leq a$:

$$b_{u_i,0}(x_n) = x_n + u_i + d_i v$$

for some $d_i \in k$. Let $u_i' = u_i + d_i v$ for $1 \leq i \leq l$, then as $u_1, \ldots, u_a, v$ are linearly independent, so are $u_1', \ldots, u_a', v$. For $1 \leq j \leq b$

$$\begin{aligned} b_{c_j v, \lambda_j}(x_n) &= b_{c_j v, \lambda_j}(w - \sum_{i=1}^{b} c_i x_{n-i}) \\ &= w + c_j v - \sum_{i=1}^{b} c_i x_{n-i} - c_j v \\ &= x_n. \end{aligned}$$

As $[G, W] \leq W^G$ we know that G is commutative so for any $g \in G$ we can find some $0 \leq \alpha_1, \ldots, \alpha_l, \beta_1, \ldots, \beta_m \leq p-1$ such that

$$g = b_{u_1,0}^{\alpha_1} \ldots b_{u_l,0}^{\alpha_l} b_{c_1 v, \lambda_1}^{\beta_1} \ldots b_{c_m v, \lambda_m}^{\beta_m}.$$

This means that (using Proposition 2.4.2)

$$g(x_n) = x_n + \alpha_1 u_1' + \ldots + \alpha_a u_a'.$$

If $\mathbf{N}_i$ is the orbit product of x_i as previously defined, then using the above and Lemma 4.0.6:

$$\deg(\mathbf{N}_i) = \begin{cases} 1 \text{ for } 1 \leq i < n-b, \\ p \text{ for } n-b \leq i < n, \\ p^a \text{ for } i = n, \end{cases}.$$

We know that $\mathbf{N}_1, \ldots, \mathbf{N}_n$ is a HSOP (by [10, Proposition 4.0.3]) for $k[V]^G$, and $\prod_{i=1}^{n} \deg(\mathbf{N}_i) = p^{a+b} = |G|$ so by Proposition 1.1.8 $k[V]^G$ is a polynomial ring and G is a Nakajima group with respect to $\mathcal{B} = \{x_1, \ldots, x_n\}$. $\square$

Proposition 4.1.4. *Let $k = \mathbb{F}_p$ and $G \leq B^{U,v}$ with $R_{\hat{v},U} \not\leq G$ and $[G, W] \leq W^G$ then the following are equivalent:*

1. *G is generated by reflections,*

2. *$|G| = p^{a+b}$ where $a = D_{\hat{U}}(G)$ and $b = D_{\Lambda}(G)$,*

3. $k[V]^G$ *is a polynomial ring,*

4. G *is a Nakajima Group.*

Proof. Using Lemma 4.1.3 3) $\Leftrightarrow$ 4) $\Leftrightarrow$ 1) $\Leftarrow$ 2), so we just need to prove 2) $\Rightarrow$ 3).

By Lemma 4.0.4 we can find a set of generators

$$\{b_{u_1,\lambda_1}, \ldots, b_{u_a,\lambda_a}, b_{c_{a+1}v,\lambda_{a+1}}, \ldots, b_{c_l,\lambda_l}\}$$

for G such that $\{u_1, \ldots, u_a, v\}$ form a basis for $\hat{U}(G)$.

Suppose that

$$\lambda_i \in \langle \lambda_{a+1}, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_l \rangle$$

for some $a + 1 \leq i \leq l$. Without loss of generality we assume that $i = l$, then

$$\lambda_l = \alpha_{a+1}\lambda_{a+1} + \ldots + \alpha_{l-1}\lambda_{l-1}$$

for some $\alpha_{a+1}, \ldots, \alpha_{l-1} \in k$, and for some $d \in k$

$$b_{c_l v,\lambda_l} = b^{\alpha_{a+1}}_{c_{a+1}v,\lambda_{a+1}} \ldots b^{\alpha_{l-1}}_{c_{l-1}v,\lambda_{l-1}} b_{dv,0}.$$

If $d = 0$ then G wasn't minimally generated however if $d \neq 0$ then $R_{\hat{v},U} \in G$, so we can assume $L = \{\lambda_{a+1}, \ldots, \lambda_l\}$ to be linearly independent.

By Proposition 2.4.8 we know that $|G| = p^l = p^{a+b}$ so $b = l - a$, and

$$\Lambda(G) = \langle \lambda_{a+1}, \ldots, \lambda_l \rangle.$$

Choose a basis $\mathcal{B} = \{x_1, \ldots, x_n\}$ such that $\{x_1, \ldots, x_{n-1}\}$ is a Λ-basis with respect to L and

$$x_n = w - \sum_{i=1}^{b} c_i x_{n-i}.$$

By Lemma 4.0.6

$$\deg(\mathbf{N}_i) = \begin{cases} 1 \text{ for } 1 \leq i < n-b, \\ p \text{ for } n-b \leq i \leq n-1. \end{cases}$$

For $a+1 \leq i \leq l$

$$b_{c_i v, \lambda_i}(x_n) = x_n.$$

As G is abelian, for all $g \in G$ we can find $\alpha_1, \ldots, \alpha_l \in k$ such that

$$\begin{aligned} g(x_n) &= b_{u_1,\lambda_1}^{\alpha_1} \ldots b_{c_l v,\lambda_l}^{\alpha_l}(x_n) \\ &= b_{u_1,\lambda_1}^{\alpha_1} \ldots b_{u_a,\lambda_a}^{\alpha_a}(x_n) \\ &= x_n + \alpha_1 u_1 + \ldots + \alpha_a u_a + cv \end{aligned}$$

for some $c \in k$. As $u_1, \ldots, u_a$ are linearly independent (and $R_{\hat{v},U} \not\leq G$), this means that $\deg(\mathbf{N}_n) = p^a$ so

$$\deg(\mathbf{N}_i) = \begin{cases} 1 \text{ for } 1 \leq i < n-b, \\ p \text{ for } n-b < i \leq n-1, \\ p^a \text{ for } i = n. \end{cases}$$

We know that $\mathbf{N}_1, \ldots, \mathbf{N}_n$ is a HSOP for $k[V]^G$ and since

$$\prod_{i=1}^{n} \deg(\mathbf{N}_i) = p^{a+b} = |G|$$

this means that $k[V]^G$ is a polynomial ring. $\square$

Corollary 4.1.5. *Let $k = \mathbb{F}_p$, $G \leq B^{U,v}$ is generated by reflections if and only if $k[V]^G$ is a polynomial ring.*

Proof. We will show that either $[G, W] \leq W^G$ or $R_{\hat{v},U} \leq G$, so we can use Lemma 4.1.3. If G is a reflection group then:

$$G = \langle b_{u_1,0}, \ldots, b_{u_l,0}, b_{c_1 v,\lambda_1}, \ldots, b_{c_m v,\lambda_m} \rangle$$

for some $u_i \in U$, $c_i \in k$ and $\lambda_i \in W^*$. If $[G, W] \not\leq W^G$ then we can find some u_i and λ_j such that $\lambda_j(u_i) \neq 0$ then using Proposition 2.4.2 (5):

$$b_{u_i,0} b_{c_j v,\lambda_j} b_{u_i,0}^{-1} b_{c_j v,\lambda_j}^{-1} = b_{\lambda_j(u_i)v,0}$$

so $R_{\hat{v},U} = \langle b_{\lambda_j(u_i)v,0} \rangle \leq G$. $\square$

If $k = \mathbb{F}_p$ then $k[V]^G$ being a polynomial ring means that G is a Nakajima group with respect to some basis however if $k = \mathbb{F}_{p^r}$ for $r \neq 1$ then this isn't necessarily the case. For larger fields subgroups similar to Stong's example appear (see [10, Section 8.1]).

***Example* 4.1.6.** *Let $k = \mathbb{F}_4$ with $\{1, \alpha\}$ a vector space basis for $\mathbb{F}_4$ over $\mathbb{F}_2$ and $\alpha^2 = \beta$. Let:*

$$h_1 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad h_2 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad h_3 = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$h_4 = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad h_5 = \begin{pmatrix} 1 & 0 & \alpha & \alpha & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & \beta \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad h_6 = \begin{pmatrix} 1 & 1 & \alpha & \alpha & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$h_7 = \begin{pmatrix} 1 & 1 & \alpha & \alpha & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & \beta \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad h_8 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

with respect to the basis $\mathcal{B} = \{x_1, x_2, x_3, x_4, x_5\}$ for W.

Let $G = \langle h_1, \ldots, h_8, R_{\hat{v},U} \rangle$. Then we can see that if we let $a = D_{\hat{U}}(G)$, $b = D_\Lambda(G)$ then

$$a + b - r = 6 + 4 - 2 = 8.$$

By Proposition 4.1.2 we know G is generated by reflections and $k[V]^G$ is polynomial, however $\prod_{i=1}^n \deg(\mathbf{N}_i) = 2^{11} > 2^{10} = |G|$ so G is not a Nakajima group with respect to this basis (more needs to be done to check it is not Nakajima with respect to any basis). If:

$$Y = x_3^2 + x_3 x_1 + x_4^2 + x_4 x_1$$

then $Y \in k[V]^G$. We find that $\mathcal{V}_{\overline{V}}(\mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3, Y, \mathbf{N}_5) = \{0\}$ and so, using 1.1.5, $\mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3, Y, \mathbf{N}_5$ form a homogeneous system of parameters for $k[V]^G$. Since

$$\deg(\mathbf{N}_1)\deg(\mathbf{N}_2)\deg(\mathbf{N}_3)\deg(Y)\deg(\mathbf{N}_5) = 2^{10} = |G|$$

by Theorem 1.1.8 this means that $k[V]^G = k[\mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3, Y, \mathbf{N}_5]$.

4.2 Nice groups with $k = \mathbb{F}_p$

We now look back to Chapter 3. We want to see which hook groups G are nice with respect to some basis $\mathcal{B}$, so that if $N = \mathrm{Nak}^+_{\mathcal{B}}(G)$ we can try to find a sequence of maximal subgroups from N to G as described in Proposition 3.0.11.

Lemma 4.2.1. *Let $k = \mathbb{F}_p$, G a hook group with hyperplane U and line kv. Let $L = \{\lambda_1, \ldots, \lambda_a\}$ be a basis for $\Lambda(G)$ and let $\mathcal{B} = \{x_1, \ldots, x_n\}$ be a basis for W such that $\{x_1, \ldots, x_{n-1}\}$ is a Λ-basis for U with respect to L. Then $\mathrm{Nak}^+_{\mathcal{B}}(G)$ is a hook group with hyperplane U and line kv, and if $R_{\hat{v},U} \leq G$ or $[G, W] \leq W^G$ then G is nice with respect to this basis.*

Proof. Let $N = \mathrm{Nak}^+_{\mathcal{B}}(G)$. Firstly we note that by Proposition 3.1.1 N is a hook group with hyperplane U and line kv. This means that

$$[N, N] \leq R_{\hat{v},U}$$

so if $R_{\hat{v},U} \leq G$ then $[N, N] \leq G$ and so G is nice with respect to B.

By Lemma 4.0.6 $U^G = U^N$. If $[G, W] \leq W^G$ then $[G, W] \leq U^G$. By Proposition 3.0.13 $[G, W] = [N, W]$ and so

$$[N, W] = [G, W] \leq U^G = U^N \leq W^N.$$

This means by Proposition 3.0.16 that N is abelian and so G is nice with respect to $\mathcal{B}$. $\square$

Proposition 4.2.2. *Let $k = \mathbb{F}_p$, G a hook group with hyperplane U and line kv, then we can find a basis with respect to which G is nice if and only if either $R_{\hat{v},U} \leq G$ or $[G, W] \leq W^G$.*

Proof. We see by Lemma 4.2.1 that if $R_{\hat{v},U} \leq G$ or $[G, W] \leq W^G$ then we can find some basis with respect to which G is nice.

Let G be a hook group such that $R_{\hat{v},U} \not\leq G$ and $[G, W] \not\leq W^G$. Suppose we can find a basis $\mathcal{B} = \{x_1, \ldots, x_n\}$ for W with respect to which G is nice and let $N = \mathrm{Nak}^+_{\mathcal{B}}(G)$.

Let $u \in [G, W] \backslash W^G$, this means we can find some $\lambda \in \Lambda(G)$ and $c \in k$ such that $b_{u+cv,\lambda} \in G$.

Let

$$I = \{1 \leq i \leq n \mid x_i \notin U\}.$$

For $i \notin I$ let $u_i = x_i \in U$. For all $i \in I$ we can assume that $x_i = w + u_i$ for some $u_i \in U$ so

$$b_{u+cv,\lambda}(x_i) = x_i + u + dv$$

for some $d \in k$. This means that

$$u \in \langle x_1, \ldots, x_{i-1} \rangle$$

and there exists $s_i \in S_i$ such that

$$s_i(x_j) = \begin{cases} x_j + u + dv \text{ for } i = j, \\ x_j \text{ otherwise.} \end{cases}$$

As $u \notin W^G$ and $v \in W^G$ we can find some $g \in G$ such that $g(u + dv) = u + (d + 1)v$ and $g(v) = v$. Note that this means that $v \in \langle x_1, \ldots, x_{i-1} \rangle$. As $G \leq \mathrm{Nak}^+_{\mathcal{B}}(G)$, which is a Nakajima group, by Lemma 3.0.3 we can find $g_j \in N_j$ for $1 \leq j \leq n$ such that

$$g = g_n \ldots g_1.$$

Let

$$h = g_{i-1} \dots g_1.$$

then

$$h(u) = u + v, \quad h(v) = v$$

Now let $\theta_i = h^{-1}s_i^{-1}hs_i$ then

$$\begin{aligned} h^{-1}s_i^{-1}hs_i(x_i) &= h^{-1}s_i^{-1}h(x_i + u + dv) \\ &= h^{-1}s_i^{-1}(x_i + u + (d+1)v) \\ &= h^{-1}(x_i + v) \\ &= x_i + v. \end{aligned}$$

For $j \neq i$ we can find some $u' \in \langle x_1, \dots, x_{i-1} \rangle$ such that

$$\begin{aligned} \theta_i(x_j) &= h^{-1}s_i^{-1}hs_i(x_j) \\ &= h^{-1}s_i^{-1}h(x_j) \\ &= h^{-1}s_i^{-1}(x_j + u') \\ &= h^{-1}(x_j + u') \\ &= x_j. \end{aligned}$$

For $1 \leq i \leq n$ such that $i \notin I$ let $\theta_i = 1$. Let $\Theta = \theta_n \dots \theta_1 \in [N, N]$. Let l be the smallest number such that $\theta_i \neq 1$, then as $\theta_l(x_l) = x_l + v$ this means that $v \in \langle x_1, \dots, x_{l-1} \rangle$ and $\theta_i(v) = v$ for $1 \leq i \leq n$. This means that

$$\Theta(x_i) = \begin{cases} x_i + v \text{ for } i \in I, \\ x_i \text{ for } i \notin I. \end{cases}$$

For $u' \in U$ we find $a_1, \ldots, a_n \in k$ such that

$$u' = \sum_{i=1}^{n} a_i x_i.$$

Let

$$b = \sum_{i \in I} a_i$$

then

$$u' = bw + \sum_{i=1}^{n} u_i$$

so $b = 0$. Since

$$\Theta(u') = u' + bv$$

this means that $\Theta(u') = u'$ for all $u' \in U$. We can see that $\Theta(w) = w + v$ so

$$\langle \Theta \rangle = R_{\hat{v},U}.$$

The above means that $R_{\hat{v},U} \leq [N, N]$, however we have assumed that $R_{\hat{v},U} \not\leq G$ and $[N, N] \leq G$, so we have a contradiction. $\square$

We shall call a hook group nice if we can find a basis B with respect to which G is nice. For the rest of this section we will restrict to nice groups with $k = \mathbb{F}_p$. This restriction still leaves us with a lot of interesting cases: as any non-zero commutator generates $R_{\hat{v},U}$ we include all non abelian hook groups. When $p = 2$, if g is an index 3 bireflection then $\langle g^2 \rangle = R_{\hat{v},U}$ so any group containing at least one index three bireflection is nice with respect to some basis for W.

Restricting to $k = \mathbb{F}_p$ also means that now both $\hat{U}(G)$ and $\Lambda(G)$ are vector spaces over k, and $\dim_k(\Lambda(G)) = \dim_k(U) - \dim_k(W^G)$.

Lemma 4.2.3. *Let $G \leq B^{U,v}$ with $H_0 \leq G$ generated by all reflections in G. Either $G = H_0$ is a reflection group, or we can find $b_{u,\lambda} \in G$ and some maximal subgroup H of G, with $H_0 \leq H$ such that $G = \langle H, b_{u,\lambda} \rangle$, $u \notin \hat{U}(H)$ and $\lambda \notin \Lambda(H)$.*

Proof. If $\hat{U}(G) = kv$ then G is a reflection group. Let $a = \dim_k(\hat{U}(G)) > 1$ by Proposition 4.0.4 we can find generators for G such that:

$$G = \langle b_{u_1,0}, \ldots, b_{u_t,0}, b_{u_{t+1},\lambda_{t+1}}, \ldots, b_{u_a,\lambda_a}, b_{c_{a+1}v,\lambda_{a+1}}, \ldots, b_{c_l v,\lambda_l} \rangle$$

where $\hat{U}(G) = \langle u_1, \ldots, u_a, v \rangle$. Suppose that t is the maximal number such that we can choose

$$\lambda_1 = \ldots = \lambda_t = 0$$

for given $b_{c_{a+1}v,\lambda_{a+1}}, \ldots, b_{c_l v,\lambda_l}$. If G is not a reflection group then $t \neq a$. We can assume that

$$\lambda_{t+1} \notin \langle \lambda_{t+2}, \ldots, \lambda_l \rangle,$$

otherwise

$$\lambda_{t+1} = c_{t+2}\lambda_{t+2} + \ldots + c_l \lambda_l$$

and we can replace $b_{u_{t+1},\lambda_{t+1}}$ with

$$b_{u'_{t+1},0} = b_{u_{t+1},\lambda_{t+1}} b^{-c_{t+2}}_{u_{t+2},\lambda_{t+2}} \ldots b^{-c_a}_{u_a,\lambda_a} b^{-c_{a+1}}_{c_{a+1}v,\lambda_{a+1}} \ldots b^{-c_l}_{u_l,\lambda_l}$$

so

$$G = \langle b_{u_1,0}, \ldots, b_{u_t,0}, b_{u'_{t+1},0}, b_{u_{t+2},\lambda_{t+2}}, \ldots, b_{u_a,\lambda_a}, b_{c_{a+1}v,\lambda_{a+1}}, \ldots, b_{u_l,\lambda_l} \rangle.$$

and t wasn't maximal.

Let

$$H = \langle b_{u_1,0}, \ldots, b_{u_t,0}, b_{u_{t+2},\lambda_{t+2}}, \ldots, b_{u_a,\lambda_a}, b_{c_{a+1}v,\lambda_{a+1}}, \ldots, b_{u_l,\lambda_l}, \Phi(G) \rangle,$$

then H is maximal in G and $G = \langle H, b_{u_{t+1},\lambda_{t+1}} \rangle$ with $u_{t+1} \notin \hat{U}(H)$ and $\lambda_{t+1} \notin \Lambda(H)$.

Suppose there exists a reflection $g \in G \backslash H$, then

$$g = b^{b}_{u_{t+1},\lambda_{t+1}} b^{d_1}_{u_1,0} \ldots b^{d_t}_{u_t,0} b^{d_{t+2}}_{u_{t+2},\lambda_{t+2}} \ldots b^{d_a}_{u_a,\lambda_a} b^{d_{a+1}}_{c_{a+1}v,\lambda_{a+1}} \ldots b^{d_l}_{u_l,\lambda_l} t$$

for some $t \in \Phi(G)$, $0 \leq b, d_1, ..., d_l \leq p-1$, with $b \neq 0$ (otherwise g would be in H). If $g = b_{cv,\lambda}$ for some $c \in k$, $\lambda \in \Lambda(G)$ then

$$cv = bu + d_1 u_1 + ... + d_a u_a + dv$$

for some $d \in k$, so

$$-bu = a_1 u_1 + ... + a_m u_m + (d-c)v$$

and $u \in \hat{U}(H)$ so we have a contradiction. Alternatively for some $u \in \hat{U}(G)$ we could have $g = b_{u,0}$ so

$$0 = b\lambda_{t+1} + d_{t+2}\lambda_{t+2} + ... + d_l \lambda_l$$

and

$$-b\lambda_{t+1} = d_{t+2}\lambda_{t+2} + ... + d_l \lambda_l$$

however this would mean that $\lambda \in \Lambda(H)$. We know H contains all reflections in G so $H_0 \leq H$. $\square$

From the last section we know that the reflection subgroups of hook groups are Nakajima with respect to some basis. We want to use the previous lemma to split up our nice hook groups and make it easier to find Nakajima groups containing them. The next result means that we can do this without losing the niceness of the group.

Lemma 4.2.4. *Let G be a hook group with H a subgroup of G containing all reflections in G. If $R_{\hat{v},U} \leq G$ then $R_{\hat{v},U} \leq H$, if $[G, W] \leq W^G$ then $[H, W] \leq W^H$.*

Proof. We know $R_{\hat{v},U}$ is a reflection group so if H contains all reflections in G and $R_{\hat{v},U} \leq G$ then $R_{\hat{v},U} \leq H$.

As H is a subgroup of G we know that $[H, W] \leq [G, W]$ and $W^G \leq W^H$, this means if $[G, W] \leq W^G$ then $[H, W] \leq W^H$. $\square$

Lemma 4.2.5. *Let $H \leq B^{U,v}$ with $R_{\hat{v},U} \leq H$ or $[H,W] \leq W^H$. Let $a = D_{\hat{U}}(H)$, $b = D_\Lambda(H)$ and $|H| = p^{a+b-m}$ for some $m \in \mathbb{N}$. Let H_0 be the group generated by all reflections in H. Then we can find a set $\{b_{u_1,\lambda_1}, \ldots, b_{u_m,\lambda_m}\}$, where $b_{u_i,\lambda_i} \in H$ for $1 \leq i \leq m$, such that:*

$$H = \langle H_0, b_{u_1,\lambda_1}, \ldots, b_{u_m,\lambda_m} \rangle$$

and

$$u_1, \ldots, u_m \in \hat{U}(H) \backslash \hat{U}(H_0),$$
$$\lambda_1, \ldots, \lambda_m \in \Lambda(H) \backslash \Lambda(H_0)$$

are linearly independent sets.

Proof. In Propositions 4.1.2 and 4.1.4 we proved this for $m = 0$, we proceed by induction on m.

For $m > 0$ by the previous Lemma 4.2.3 we can find some $b_{u_m,\lambda_m} \in H$ and maximal subgroup H' of H containing all reflections in H such that

$$H = \langle H', b_{u_m,\lambda_m} \rangle,$$

$u_m \notin \hat{U}(H')$ and $\lambda_m \notin \Lambda(H')$.

Let $a' = D_{\hat{U}}(H') = D_{\hat{U}}(H) - 1$, $b' = D_\Lambda(H') = D_\Lambda(H) - 1$ then:

$$\begin{aligned} |H'| &= \frac{|H|}{p} \\ &= p^{a+b-m-1} \\ &= p^{a'+1+b'+1-m-1} \\ &= p^{a'+b'-(m-1)}. \end{aligned}$$

Now by Lemma 4.2.4 we can use the induction hypothesis to find some

$$b_{u_1,\lambda_1}, \ldots, b_{u_{m-1}\lambda_{m-1}}$$

with $u_1, \ldots, u_{m-1} \in \hat{U}(H') \backslash \hat{U}(H'_0)$ and $\lambda_1, \ldots, \lambda_{m-1} \in \Lambda(H') \backslash \Lambda(H'_0))$ linearly independent sets, such that $H' = \langle H'_0, b_{u_1,\lambda_1}, \ldots, b_{u_{m-1}\lambda_{m-1}} \rangle$ where H'_0 is the group generated by all reflections in H'. As H' contains all reflections in H we see that $H'_0 = H_0$. As $u_m \notin \hat{U}(H')$ and $\lambda_m \notin \Lambda(H')$ we can see that $u_1, \ldots, u_m \in \hat{U}(H) \backslash \hat{U}(H_0)$ and $\lambda_1, \ldots, \lambda_m \in \Lambda(H) \backslash \Lambda(H_0))$ are linearly independent sets. As $H = \langle H', b_{u_m,\lambda_m} \rangle$ we find that

$$H = \langle H_0, b_{u_1,\lambda_1}, \ldots, b_{u_m,\lambda_m} \rangle.$$

□

Lemma 4.2.6. *Let $H \leq B^{U,v}$ with $R_{\hat{v},U} \leq H$ or $[H, W] \leq W^H$ and*

$$H = \langle H_0, b_{u_1,\lambda_1}, \ldots, b_{u_m,\lambda_m} \rangle,$$

where H_0 is the subgroup generated by all reflections in H. Let

$$\lambda_{m+1}, \ldots, \lambda_{D_{\Lambda(H)}} \in \Lambda(H_0)$$

such that

$$L = \{\lambda_1, \ldots, \lambda_m, \lambda_{m+1}, \ldots, \lambda_{D_{\Lambda(H)}}\}$$

is a basis for $\Lambda(H)$. For any $\mathcal{C} = \{x_1, \ldots, x_{n-1}\}$ which a Λ-basis for U with respect to L we can find $x_n \in W \backslash U$ such that

- *$\mathcal{B} = \{x_1, \ldots, x_n\}$ is a basis for W,*
- *$H_0 = \mathrm{Nak}^-_{\mathcal{B}}(H)$,*
- *$b_{u_i,\lambda_i}(x_n) = x_n + u_i$ for $1 \leq i \leq m$,*
- *$\mathrm{Nak}^+_{\mathcal{B}}(H) = \langle H, b_{0,\lambda_1}, \ldots, b_{0,\lambda_m} \rangle$.*

Proof. The case $m = 0$ is covered in Lemma 4.1.3 so we proceed by induction on m.

Let

$$H' = \langle H_0, b_{u_1,\lambda_1}, \ldots, b_{u_{m-1},\lambda_{m-1}} \rangle.$$

If $\mathcal{C}$ is a Λ-basis for U with respect to $\{\lambda_1, \ldots, \lambda_m\}$ then $\mathcal{C}$ is a Λ-basis for U with respect to $\{\lambda_1, \ldots, \lambda_{m-1}.\}$ By Lemma 4.2.4 we can apply the induction hypothesis to H': we can find x'_n such that $\mathcal{C} = \{x_1, \ldots, x_{n-1}, x'_n\}$ is a basis for W and

$$\begin{aligned}
\mathrm{Nak}^{-}_{\mathcal{B}'}(H') &= H_0, \\
b_{u_i,\lambda_i}(x_n) &= x_n + u_i \text{ for } 1 \leq i \leq m-1, \\
\mathrm{Nak}^{+}_{\mathcal{B}'}(H') &= \langle H', b_{0,\lambda_1}, \ldots, b_{0,\lambda_{m-1}} \rangle.
\end{aligned}$$

As $x'_n \notin U$, $x'_n = w + u$ for some $u \in U$, so:

$$b_{u_m,\lambda_m}(x'_n) = x'_n + u_m + cv$$

for some $c \in k$. Let $x_n = x'_n - cx_{n-m}$ then

$$b_{u_m,\lambda_m}(x_n) = x_n + u_m.$$

For all $h \in H'$

$$\delta_h(x_n) = \delta_h(x'_n),$$

so if $h(x'_n) = x'_n$ then $h(x_n) = x_n$. Let $\mathcal{B} = \{x_1, \ldots, x_n\}$. For $1 \leq i \leq n-1$ let

$$\begin{aligned}
H'_i &= \{h \in H \mid h(x'_n) = x_n \text{ and } h(x_j) = x_j \text{ for } 1 \leq j \leq n-1, j \neq i\} \\
H_i &= \{h \in H \mid h(x_j) = x_j \text{ for } 1 \leq j \leq n, j \neq i\}
\end{aligned}$$

and let

$$\begin{aligned}
H'_n &= \{h \in H \mid h(x_j) = x_j \text{ for } 1 \leq j \leq n-1\} \\
H_n &= \{h \in H \mid h(x_j) = x_j \text{ for } 1 \leq j \leq n-1\}.
\end{aligned}$$

Then for $1 \leq i \leq n$

$$H_i' = H_i$$

and so

$$\mathrm{Nak}^-_{\mathcal{B}'}(H') = \mathrm{Nak}^-_{\mathcal{B}}(H') = H_0.$$

As H_0 contains all reflections in H, we know that $\mathrm{Nak}^-_{\mathcal{B}}(H) \leq H_0$. By the above H_0 is a Nakajima group with respect to $\mathcal{B}$ so

$$\mathrm{Nak}^-_{\mathcal{B}}(H) = H_0.$$

Let $H[i] = \langle b_{u_i,\lambda_i} \rangle$ for $1 \leq i \leq m$ then we see that

$$\mathrm{Nak}^+_{\mathcal{B}}(H[i]) = \langle H[i], b_{0,\lambda_i} \rangle \text{ for } 1 \leq i \leq m.$$

Using Corollary 3.0.10

$$\begin{aligned} \mathrm{Nak}^+_{\mathcal{B}}(H) &= \langle \mathrm{Nak}^+_{\mathcal{B}}(H_0), \mathrm{Nak}^+_{\mathcal{B}}(H[1]), \dots, \mathrm{Nak}^+_{\mathcal{B}}(H[m]) \rangle, \\ &= \langle H_0, b_{u_1,\lambda_1}, b_{0,\lambda_1}, \dots, b_{u_m,\lambda_m}, b_{0,\lambda_m} \rangle, \\ &= \langle H, b_{0,\lambda_1}, \dots, b_{0,\lambda_m}. \rangle \qquad \square \end{aligned}$$

Now we have broken our groups up and can use Theorems 1.3.4 and 1.3.5 to show that their invariant rings are complete intersections. For this we will use Properties of the Dickson Invariants from Lemma 1.1.7.

Lemma 4.2.7. *Let*

$$H = \langle H_0, b_{u_1,\lambda_1}, \dots, b_{u_m,\lambda_m} \rangle \leq B^{U,v}$$

where H_0 is the subgroup of H generated by reflections, with $R_{\hat{v},U} \leq H$ or $[H, W] \leq W^H$. Let $\lambda_{m+1}, \dots, \lambda_{D_{\Lambda(H)}} \in \Lambda(H_0)$ such that

$$L = \{\lambda_1, \dots, \lambda_m, \lambda_{m+1}, \dots, \lambda_{D_{\Lambda(H)}}\}$$

is a basis for $\Lambda(H)$. *Let* $\mathcal{B} = \{x_1, \ldots, x_n\}$ *is a basis for* W *such that* $\{x_1, \ldots, x_{n-1}\}$ *is a* Λ*-basis with respect to* L *and*

- $H_0 = \mathrm{Nak}^-_{\mathcal{B}}(H)$,
- $b_{u_i,\lambda_i}(x_n) = x_n + u_i$ *for* $1 \leq i \leq m$,
- $\mathrm{Nak}^+_{\mathcal{B}}(H) = \langle H, b_{0,\lambda_1}, \ldots, b_{0,\lambda_m}\rangle$.

Let $W_0 = \langle \delta_h(x_n) \mid h \in H_0\rangle$ *and* $W_i = W_{i-1} + ku_{m-i-1}$ *for* $1 \leq i \leq m$. *Let*

$$f_i = x_1 F^{W_{m-i}}(x_n) - \sum_{j=1}^{i} x_{n-j} F^{W_{m-i}}(u_j).$$

for $1 \leq i \leq m$, *then*

$$k[V]^G = k[\mathbf{N}_1^H, \mathbf{N}_2^H, \ldots, \mathbf{N}_n^H, f_1, \ldots, f_m]$$

is a complete intersection ring.

Proof. For $m = 0$, $H = H_0$ is a Nakajima group with respect to $\mathcal{B}$ so proceed by induction on m. Assume the result holds for $m-1$. Let:

$$H^+ = \langle H, b_{0,\lambda_m}\rangle,$$
$$H^- = \langle H_0, b_{u_1,\lambda_1}, \ldots, b_{u_{m-1},\lambda_{m-1}}\rangle.$$

Suppose that $g \in H^+$ is a reflection with $g \notin \langle H_0, b_{u_m,0}, b_{0,\lambda_m}\rangle$ then there exists $b_{u,\lambda} \in H^-$ such that

$$g = b_{u,\lambda} b^a_{u_m,0} b^b_{0,\lambda_m}$$

with one of $a, b \neq 0$. Either $u + au_m \in kv$ or $\lambda + b\lambda_m = 0$. In the first instance as $u_m \notin \hat{U}(H^-)$ this means that $u = 0$ and $a = 0$, and $b_{u,\lambda} \in H_0$. In the second instance as $\lambda_m \notin \Lambda(H^-)$ this means that $\lambda = 0$ and $b = 0$ so again $b_{u,\lambda} \in H_0$. From this we see that the group generated by all reflections in H^+ is

$$H_0^+ = \langle H_0, b_{u_m,\lambda_m}, b_{0,\lambda_m}\rangle$$

From this we see that

$$H^+ = \langle H_0^+, b_{u_1,\lambda_1}, \ldots, b_{u_{m-1},\lambda_{m-1}} \rangle.$$

If $R_{\hat{v},U} \leq H$ then clearly $R_{\hat{v},U} \leq H^+$. Let $N = \mathrm{Nak}^+_{\mathcal{B}}(G)$, then

$$H^- \leq H \leq H^+ \leq N$$

so $\mathrm{Nak}^+_{\mathcal{B}}(H^+) = N$ (using Proposition 3.0.9). Using Lemma 4.0.6 $W^N = W^H$ so $W^H = W^{H^+}$. This means that if $[H, W] \leq W^H$ then $[H^+, W] \leq W^{H^+}$.

As clearly $b_{u_i,\lambda_i}(x_n) = x_n + u_i$ for $1 \leq i \leq n$ this means that in order to use the induction hypothesis we just need to check that

$$H_0^+ = \mathrm{Nak}^-_{\mathcal{B}}(H^+).$$

As H_0^+ contains all reflections in H^+ we know that

$$\mathrm{Nak}^-_{\mathcal{B}}(H^+) \leq H_0^+.$$

As H_0 and

$$\langle b_{u_m,\lambda_m}, b_{0,\lambda_m} \rangle$$

are both Nakajima groups with respect to $\mathcal{B}$, by Lemma 3.0.5 H_0^+ is a Nakajima group and so

$$H_0^+ = \mathrm{Nak}^-_{\mathcal{B}}(H^+).$$

If we let

$$\begin{aligned}
W_0^+ &= \langle \delta_h(x_n) \mid h \in H_0^+ \rangle = W_0 + ku_m, & \\
W_i^+ &= W_{i-1}^+ \oplus ku_{m-i-1} & \text{for } 1 \leq i \leq m-1, \\
f_i^+ &= x_1 F^{W^+_{(m-1)-i}}(x_n) - \sum_{j=1}^{i} x_{n-j} F^{W^+_{(m-1)-i}}(u_j) & \text{for } 1 \leq i \leq m-1,
\end{aligned}$$

then we can find $k[V]^{H^+}$ by the induction hypothesis:

$$k[V]^{H^+} = k[\mathbf{N}_1^{H^+}, \ldots, \mathbf{N}_n^{H^+}, f_1^+, \ldots, f_{m-1}^+]$$

is a complete intersection ring.

As

$$\mathrm{Nak}_{\mathcal{B}}^+(H^+) = \mathrm{Nak}_{\mathcal{B}}^+(H)$$

and H^+ and H are both nice with respect to $\mathcal{B}$ by Proposition 3.0.12

$$\mathbf{N}_i^H = \mathbf{N}_i^{H^+} = \mathbf{N}_i^N$$

for $1 \le i \le n$. Looking at the W_i

$$\begin{aligned} W_0^+ &= W_1, \\ W_i^+ &= W_{i+1} && \text{for } 1 \le i \le m-1, \\ f_i^+ &= x_1 F^{W_{m-i}}(x_n) - \sum_{j=1}^{i} x_{n-j} F^{W_{m-i}}(u_j) = f_i && \text{for } 1 \le i \le m-1, \end{aligned}$$

so

$$k[V]^{H^+} = k[\mathbf{N}_1^H, \ldots, \mathbf{N}_n^H, f_1, \ldots, f_{m-1}].$$

This means what we want to show is that $k[V]^{H^+}[f_m] = k[V]^H$.

We can see that H is a maximal subgroup of H^+ with $\sigma := b_{u_m,0} \in H^+ \backslash H$, and clearly $f_m \notin k[V]^{H^+}$. Let:

$$\begin{aligned} x &:= (\sigma - 1) f_m \\ &= x_1 F^{W_0}(u_m) \end{aligned}$$

then in order to use Theorem 1.3.4 we just need to prove that $(\sigma - 1)(k[V]^H) \subseteq (x)$.

Using Lemma 4.2.4 we see we can also use the induction hypothesis to find $k[V]^{H^-}$. Let

$$\begin{aligned} W_0^- &= \langle \delta_h(x_n) \mid h \in H_0 \rangle = W_0, \\ W_i^- &= W_{i-1}^- \oplus k u_{m-1-i} && \text{for } 1 \le i \le m-1, \\ f_i^- &= x_1 F^{W^-_{(m-1)-i}}(x_n) - \sum_{j=1}^{i} x_{n-j} F^{W^-_{(m-1)-i}}(u_j) && \text{for } 1 \le i \le m-1, \end{aligned}$$

then

$$k[V]^{H^-} = k[\mathbf{N}_1^-, \ldots, \mathbf{N}_n^-, f_1^-, \ldots, f_{m-1}^-].$$

As $H^- \le H$ we know that $k[V]^H \subseteq k[V]^{H^-}$ so $(\sigma - 1)k[V]^H \subseteq (\sigma - 1)k[V]^{H^-}$. Let:

$$h = a_1 (\mathbf{N}_1^-)^{c_1} \ldots (\mathbf{N}_n^-)^{c_n} f_1^{d_1} \ldots f_{m-1}^{d_{m-1}} \in k[V]^{H^-}$$

be a monomial in $k[V]^{H^-}$. If $(\sigma - 1)(h) \neq 0$ then at least one of $c_n, d_1, \ldots, d_{m-1}$ is not zero. We can see that:

$$\begin{aligned} \mathbf{N}_n^- &= F^{W^-_{m-1}}(x_n) \\ \delta_\sigma(\mathbf{N}_n^-) &= F^{W^-_{m-1}}(u_m) \end{aligned}$$

and for $1 \le i \le m-1$:

$$\begin{aligned} f_i^- &= x_1 F^{W^-_{(m-1)-i}}(x_n) - \sum_{j=1}^{i} x_{n-j} F^{W^-_{(m-1)-i}}(u_j) \\ \delta_\sigma(f_i^-) &= x_1 F^{W^-_{(m-1)-i}}(u_m). \end{aligned}$$

From the nice properties of Dickson invariants we know that for $1 \le i \le m-1$:

$$F^{W_i^-}(u_m) = F^{W^-_{i-1}}(u_m)^p - F^{W^-_{i-1}}(u_{m-1-i})^{p-1} F^{W^-_{i-1}}(u_m).$$

This means $F^{W_0^-}(u_m) = F^{W_0}(u_m)$ divides $\delta_\sigma(h) \neq 0$ for any $h \in k[V]^{H^-}$ and hence for any $h \in k[V]^H$. If $h \in k[V]^H$ then:

$$\begin{aligned} b_{u_m,\lambda_m}(h) &= h, \\ b_{0,\lambda_m} b_{u_m,0}(h) &= h, \\ \sigma(h) &= b_{0,\lambda_m}^{-1}(h). \end{aligned}$$

From this we can see that x_1 divides $\delta_\sigma(h) \neq 0$ for any $h \in k[V]^H$. As H is a p-group $k[V]^H$ is a unique factorisation domain and so $(\sigma - 1)(k[V]^H) \subseteq xk[V]$ as required. Now

$$k[V]^H = k[V]^{H^+}[f_m] = k[\mathbf{N}_1^H, \ldots, \mathbf{N}_n^H, f_1, \ldots, f_m].$$

By Theorem 1.3.5 $k[V]^H$ is a complete intersection ring. $\square$

Theorem 4.2.8. *For $k = \mathbb{F}_p$, all hook groups G which are nice with respect to some basis $\mathcal{B}$ for W have complete intersection rings of invariants.*

Proof. If G is a hook group then it is a subgroup of $B^{U,v}$ for some hyperplane U and $v \in U$. By Lemma 4.2.2 if G is nice with respect to some basis $\mathcal{B}$ either $R_{\hat{v},U} \leq G$ or $[G, W] \leq W^G$, then we can apply Lemmas 4.2.5, 4.2.6 and 4.2.7 to show that it has complete intersection ring of invariants. $\square$

This gives us the following corollaries:

Corollary 4.2.9. *For $k = \mathbb{F}_p$, all non-abelian hook groups have complete intersection rings of invariants.*

Proof. As noted at the beginning of this section if $k = \mathbb{F}_p$ all non-abelian hook groups contain $R_{\hat{v},U}$ and so are nice with respect to some basis, hence by Theorem 4.2.8 they have complete intersection invariant rings. $\square$

Corollary 4.2.10. *Let $k = \mathbb{F}_2$, then all hook groups containing at least one index 3 bireflection have complete intersection rings of invariants.*

Proof. If $k = \mathbb{F}_2$ and G is a hook group which contains an index 3 bireflection g, then

$$\langle g^2 \rangle = R_{\hat{v},U}$$

so $R_{\hat{v},U} \leq G$ and so G is nice with respect to some basis. By Theorem 4.2.8 they have complete intersection invariant rings. □

4.2.1 Example: quaternions

We cannot hope to extend the result above to show that all hook groups have complete intersection invariant rings, or even that their invariant rings are all Cohen–Macaulay. Here we see two four dimensional representations of Q_8 which are both hook groups- one has complete intersection invariant ring and the other is not Cohen–Macaulay. This also shows Lemma 4.2.7 being used on an example.

Lemma 4.2.11. *Let $G \leq \mathrm{GL}(V)$. If $G \cong Q_8$ is a bireflection group then G is a hook group containing at least one index 3 bireflection.*

Proof. The group Q_8 is an extraspecial group with $[Q_8, Q_8] = Z(Q_8)$ a cyclic group of order two. Let $G \leq GL(V)$ such that $G \cong Q_8$, and let $t \in G$ such that $Z(G) = \langle t \rangle$.

If $g \in G \backslash \Phi(G)$ then $|g| = 4$ and $g^2 = t$. The only bireflections with order 4 are index 3 bireflections, and if g is an index three bireflection of order four then g^2 is a transvection (see Lemma 2.1.2). If $G = \langle g_1, g_2 \rangle$ then $g_1, g_2 \in G \backslash \Phi(G)$, so we can find $v \in V$ and $\gamma \in v^{\perp}$ such that:

$$g_1^2 = g_2^2 = t_v^{\gamma} = t.$$

Let $U = V^t$, then U is a hyperplane, and $\delta_{g_1}(U) = \delta_{g_2}(U) = kv$, so G is a hook group with hyperplane U and line kv. □

Proposition 4.2.12. *Let $k = \mathbb{F}_2$. If $G \leq \mathrm{GL}(V)$ with $G \cong Q_8$ then $k[V]^G$ is either complete intersection or not Cohen–Macaulay.*

Proof. If G has a Cohen–Macaulay ring of invariants then G is generated by bireflections (Theorem 1.3.6). By Lemma 4.2.11 G is a hook group containing at least one index 3 bireflection so by Corollary 4.2.10 $k[V]^G$ is a complete intersection ring. $\square$

***Example* 4.2.13.** *Let* $k = \mathbb{F}_2$*,* $H = \langle g_1, g_2 \rangle$ *where*

$$g_1 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with respect to basis $\mathcal{B} = \{x_1, x_2, x_3, x_4\}$ *for* W*. We see that* $H \cong Q_8$ *is a hook group with hyperplane* $U = \langle x_1, x_2, x_3 \rangle$ *and line* kx_1*. We find it's invariant ring (which is a complete intersection ring) by using Proposition 4.2.7.*

Let

$$\sigma_1 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad t = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

As in Proposition 3.0.9 we can find

$$\begin{aligned} G &= \langle g_1, g_2, \sigma_1, \sigma_2 \rangle = \mathrm{Nak}^+_{\mathcal{B}}(H), \\ H_0 &= \langle t \rangle = \mathrm{Nak}^-_{\mathcal{B}}(G), \end{aligned}$$

both Nakajima groups such that $H_0 \leq H \leq G$*. Let* $H_1 = \langle g_1, g_2, \sigma_2 \rangle$ *then*

$$k[V]^{H_1} = k[\mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3, \mathbf{N}_4, f_1]$$

where the $\mathbf{N}_i$ *are orbit products and*

$$\begin{aligned}f_1 = x_1((x_4^2 - x_1x_4)^2 - (x_2^2 - x_4x_2)(x_4^2 - x_1x_4)) - \\ x_3(((x_2 + x_3)^2 - x_1(x_2 + x_3))^2 - (x_2^2 - x_4x_2)((x_2 + x_3)^2 - x_1(x_2 + x_3))).\end{aligned}$$

From here we can find $k[V]^H = k[V]^{H_1}[f_2]$ *where*

$$f_2 = x_1(x_4^2 - x_1x_4) - x_3((x_2 + x_3)^2 - x_1(x_2 + x_3)) - x_2(x_2^2 - x_2x_1).$$

***Example* 4.2.14.** *Let* $k = \mathbb{F}_4$ *with* $a \neq 1$, $a^3 = 1$. *Let* $G = \langle g_1, g_2 \rangle$ *where*

$$g_1 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & 0 & a^2 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with respect to basis $\mathcal{B} = \{x_1, x_2, x_3, x_4\}$ *for* W. *Again* $G \cong Q_8$. *The maximal subgroups of* G *are* $L = \langle g_1g_2 \rangle$, $M = \langle g_1 \rangle$ *and* $N = \langle g_2 \rangle$. *Let* $t \in G$ *such that*

$$\langle t \rangle = Z(G) = [G, G],$$

then

$$L \cap M = L \cap N = M \cap N = \langle t \rangle.$$

We want to show that the invariant ring of G *is not Cohen–Macaulay, we do this using Theorem 1.2.6 by showing that:*

$$(g_1 - 1)W^N < (g_1 - 1)W^t \cap (g_1g_2 - 1)W^t \cap W^N.$$

As $W^N = W^M = W^L = \langle x_1, x_2 \rangle$ *we see that* $(g_1 - 1)W^N = \{0\}$. *We need to find some* $(g_1 - 1)W^t \cap (g_1g_2 - 1)W^t \cap W^N \neq \{0\}$. *As*

$$W^t = \langle x_1, x_2, x_3 \rangle = U$$

and G is a hook group with line kx_1 and hyperplane U, on which none of the generators act trivially

$$kx_1 = (g_1 - 1)W^t \cap (g_1 g_2 - 1)W^t \cap W^N.$$

From this we see V is linearly flat and so for $R = k[V]^G$

$$\mathrm{depth}(R^G) = V^G + cc_G(R) + 1 = 1 + 1 + 1 = 3 < 4$$

so R^G is not Cohen–Macaulay.

Chapter 5

Invariant rings of two–column groups

5.1 Non complete intersection example

By Proposition 1.4.3 all two–column groups have Cohen–Macaulay invariant rings, however it is not the case that they are necessarily complete intersection rings as we will see with this four dimensional representation of the extra-special group $M(3)$. This is a counter example to the conjecture that Cohen–Macaulay implies complete intersection for invariant rings of unipotent groups. It is also a counter example to the conjecture that all two–column groups have complete intersection invariant rings in the modular case (a counter example in the non-modular case was given by Wu in [33]).

Proposition 5.1.1. *Let $G = \langle g_1, g_2 \rangle$ where*

$$g_1 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with respect to basis $B = \{x_1, x_2, x_3, x_4\}$ for W. If $p = 3$ then $k[V]^G$ is Cohen–Macaulay but not complete intersection.

Firstly we look at a subgroup of G, let $H = \langle g_1, h\rangle$ with

$$h = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

As H is a nice hook group using Proposition 4.2.7,

$$k[V]^H = k[\mathbf{N}_1^H, \mathbf{N}_2^H, \mathbf{N}_3^H, \mathbf{N}_4^H, d]$$

where

$$d = (x_4^p - x_1^{p-1}x_4)x_1 - (x_2^p - x_1^{p-1}x_2)x_3.$$

We will use the notation

$$\mathbf{N}_i = \mathbf{N}_i^G, \quad n_i = \mathbf{N}_i^H.$$

To find the invariant ring of G we first find it's localisation at x_1.

Lemma 5.1.2. *For G as given above, $p \neq 2$,*

$$k[V]^G_{x_1} = k[\mathbf{N}_1^G, \mathbf{N}_2^G, \mathbf{N}_3^G, \mathbf{N}_4^G, h_1, h_2]_{x_1}$$

where

$$\begin{aligned} h_1 &= n_3x_1 - x_2\mathbf{N}_2 \\ h_2 &= 2x_1d + (x_2^2 - x_1x_2)\mathbf{N}_2. \end{aligned}$$

Proof. As in Theorem 1.3.12 we wish to find $\phi_1, \phi_2, \phi_3, \phi_4$ with $\phi_i \in R[i]$ such that $k(V) = k(\phi_1, \ldots, \phi_4)$. As always we can choose $\phi_1 = x_1$. If we look at the action of G restricted to $\langle x_1, x_2\rangle$ we see that it acts as a Nakajima group and we can choose

$$\phi_2 = x_2^p - x_1^{p-1}x_2 = \mathbf{N}_2.$$

As H is a subgroup of G we know that $R[i]^G \subseteq R[i]^H$ for $1 \leq i \leq n$. The minimal degree of polynomials in x_3 in $R[3]^H$ is p. As $h_1 \in R[3]^G$ with p the maximal degree of x_3 we can choose $\phi_3 = h_1$. The minimal degree of polynomials in x_4 in $R[4]^H$ is also p. Similarly as $h_2 \in R[4]^G$ with p the maximal degree of x_4 we can choose $\phi_4 = h_2$.

As in Lemma 1.3.13 we let c_i be the leading coefficient of ϕ_i viewed as a polynomial in x_i for $1 \leq i \leq 4$ so

$$c_1 = c_2 = 1, \quad c_3 = x_1, \quad c_4 = x_1^2.$$

We can therefore use Lemma 1.3.13 to see that

$$k[V]^G_{x_1} = k[\mathbf{N}_1, \mathbf{N}_2, h_1, h_2]_{x_1} = k[\mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3, \mathbf{N}_4, h_1, h_2]_{x_1}. \qquad \square$$

We have found a ring A which is integral over $k[V]^G$ and such that $A_{x_1} = k[V]_{x_1}$, so we can apply the SAGBI divide by x algorithm. First we find the leading terms of the invariants we have:

$$\begin{array}{lll} \mathrm{LT}(\mathbf{N}_1) = x_1, & \mathrm{LT}(\mathbf{N}_2) = x_2^p, & \mathrm{LT}(\mathbf{N}_3) = x_3^{p^2}, \\ \mathrm{LT}(\mathbf{N}_4) = x_4^{p^2}, & \mathrm{LT}(h_1) = x_2^{p+1}, & \mathrm{LT}(h_2) = x_2^{p+2}. \end{array}$$

For any $p \neq 2$ we have a non-trivial tete-a-tete $\{h_1^2, h_2\mathbf{N}_3^G\}$ and can find:

$$\begin{aligned} h_3 &= \left(h_2\mathbf{N}_3^G - h_1^2\right)/x_1 \\ &= 2d\mathbf{N}_2 - x_2\mathbf{N}_2^2 - x_1n_3^2 + 2x_2\mathbf{N}_2n_3 \end{aligned}$$

which has $\mathrm{LT}(h_3) = x_2^{2p}x_3$.

Lemma 5.1.3. *For $p = 3$ and $p = 5$, for all $0 \leq i, j \leq p - 1$ with*

$$p \leq i + 2j \leq 2p - 1$$

we can find $c_{i,j}$ such that if $f_{i,j}$ are defined recursively (in lexicographic order) to be:

$$f_{i,j} = \begin{cases} \left(h_2^{(p+1)/2} - c_{i,j}h_1\mathbf{N}_2^{(p+1)/2}\right)/x_1 & \text{for } i = 0, j = \frac{p+1}{2}, \\ \left(h_2 f_{0,j-1} - c_{i,j}h_1h_3^{(2j-p-1)/2}\mathbf{N}_2^{p+1-j}\right)/x_1 & \text{for } i = 0, j > \frac{p+1}{2}, \\ \left(h_1h_2^j - c_{i,j}\mathbf{N}_2^{(p+3)/2}\right)/x_1 & \text{for } i = 1, j = \frac{p-1}{2}, \\ \left(h_1h_2^jh_3^{(i-1)/2} - c_{i,j}\mathbf{N}_2 f_{i-2,j+1}\right)/x_1 & \text{for } i > 1 \text{ odd}, j = \frac{p-i}{2}, \\ \left(h_3^{i/2}h_2^j - c_{i,j}h_1 f_{i-1,j}\right)/x_1 & \text{for } i \text{ even}, j = \frac{p+1-i}{2}, \\ \left(h_1 f_{i-1,j} - c_{i,j}h_2 f_{i,j-1}\right)/x_1 & \text{otherwise} \end{cases}$$

then $f_{i,j} \in k[V]^G$ and $\mathrm{LT}(f_{i,j}) = x_2^{a_j}x_3^{b_{i,j}}$ where:

$$a_j = (p^2 - 1 + 2j)/2$$
$$b_{i,j} = p(2i + 2j - p + 1)/2$$

Proof. We start with $p = 3$, and find if we let

$$c_{0,2} = 2, \quad c_{1,1} = 2, \quad c_{1,2} = 2, \quad c_{2,1} = 1$$

and define $f_{i,j}$ as above then

$$
\begin{aligned}
h_4 = f_{1,1} &= \left(h_1h_2 + N_2^3\right)/x_1\\
&= 2x_1n_3d + (x_2^2 - x_1x_2)\mathbf{N}_2n_3 + x_2\mathbf{N}_2d + x_2^2\mathbf{N}_2^2 + x_1x_2\mathbf{N}_2^2,\\
h_5 = f_{0,2} &= \left(h_2^2 + N_2^2h_1\right)/x_1\\
&= x_1d^2 + (x_2^2 - x_1x_2)d\mathbf{N}_2 + \mathbf{N}_2^2n_3 + x_2^3\mathbf{N}_2^2 - x_1x_2^2\mathbf{N}_2^2,\\
h_6 = f_{2,1} &= (h_3h_2 - h_1f_{1,1}/\, x_1\\
&= d^2\mathbf{N}_2 + x_2d\mathbf{N}_2 - x_1dn_3^2 + x_2d\mathbf{N}_2^2 - x_2\mathbf{N}_2n_3d - x_2^2\mathbf{N}_2^3\\
&\quad - (x_2^2 - x_1x_2)\mathbf{N}_2n_3^2 - x_2^2\mathbf{N}_2^2n_3 - x_1x_2\mathbf{N}_2^2n_3,\\
h_7 = f_{1,2} &= (h_1f_{0,2} + h_2f_{1,1})/x_1\\
&= 2x_1n_3d^2 + x_2\mathbf{N}_2d^2 - x_2^2\mathbf{N}_2^2d - x_1x_2\mathbf{N}_2^2d - n_3d + \mathbf{N}_2^2n_3^2 - x_2^3\mathbf{N}^2n_3\\
&\quad + x_1x_2^2\mathbf{N}_2^2n_3 + x_2(x_2^2 - x_1x_2)\mathbf{N}_2^3.
\end{aligned}
$$

These have the following leading terms:

$$
\begin{aligned}
\mathrm{LT}(f_{0,2}) &= x_2^6x_3^3, & \mathrm{LT}(f_{1,1}) &= x_2^5x_3^3,\\
\mathrm{LT}(f_{1,2}) &= x_2^6x_3^6, & \mathrm{LT}(f_{2,1}) &= x_2^5x_3^6.
\end{aligned}
$$

For $p = 5$ we let

$$
\begin{aligned}
&c_{0,3} = 4, \quad c_{0,4} = 2, \quad c_{1,2} = 4, \quad c_{1,3} = 4,\\
&c_{1,4} = 1, \quad c_{2,2} = 3, \quad c_{2,3} = 4, \quad c_{3,1} = 3,\\
&c_{3,2} = 3, \quad c_{3,3} = 1, \quad c_{4,1} = 2, \quad c_{4,2} = 2.
\end{aligned}
$$

The terms are larger still than in the case of $p = 3$, but the following can be shown to be in $k[V]^G$:

$$
\begin{aligned}
f_{0,3} &= \left(h_2^3 + \mathbf{N}_2^3 h_1\right)/x_1, \\
f_{0,4} &= \left(h_2 f_{0,3} - 2h_1 h_3 \mathbf{N}_2^2\right)/x_1, \\
f_{1,2} &= \left(h_1 h_2^2 + \mathbf{N}_2^4\right)/x_1, \\
f_{1,3} &= \left(h_1 f_{0,3} + h_2 f_{1,2}\right)/x_1, \\
f_{1,4} &= \left(h_1 f_{0,4} - h_2 f_{1,3}\right)/x_1, \\
f_{2,2} &= \left(h_2^2 h_3 - 3h_1 f_{1,2}\right)/x_1, \\
f_{2,3} &= \left(h_1 f_{1,2} + h_2 f_{2,2}\right)/x_1, \\
f_{3,1} &= \left(h_1 h_2 h_3 - 3\mathbf{N}_2 f_{1,2}\right)/x_1, \\
f_{3,2} &= \left(h_1 f_{2,2} - 3h_2 f_{3,1}\right)/x_1, \\
f_{3,3} &= \left(h_1 f_{2,3} - h_2 f_{3,2}\right)/x_1, \\
f_{4,1} &= \left(h_2 h_3^2 - 2h_1 f_{3,1}\right)/x_1, \\
f_{4,2} &= \left(h_1 f_{3,2} - 2h_2 f_{4,1}\right)/x_1.
\end{aligned}
$$

The leading terms are:

$$
\begin{aligned}
\mathrm{LT}(f_{0,3}) &= x_2^{15} x_3^5, & \mathrm{LT}(f_{0,4}) &= x_2^{16} x_3^{10}, \\
\mathrm{LT}(f_{1,2}) &= x_2^{14} x_3^5, & \mathrm{LT}(f_{1,3}) &= x_2^{15} x_3^{10}, \\
\mathrm{LT}(f_{1,4}) &= x_2^{16} x_3^{15}, & \mathrm{LT}(f_{2,2}) &= x_2^{14} x_3^{10}, \\
\mathrm{LT}(f_{2,3}) &= x_2^{15} x_3^{15}, & \mathrm{LT}(f_{3,1}) &= x_2^{13} x_3^{10}, \\
\mathrm{LT}(f_{3,2}) &= x_2^{14} x_3^{15}, & \mathrm{LT}(f_{3,3}) &= x_2^{15} x_3^{20}, \\
\mathrm{LT}(f_{4,1}) &= x_2^{13} x_3^{15}, & \mathrm{LT}(f_{4,2}) &= x_2^{14} x_3^{20}.
\end{aligned}
$$

□

Proposition 5.1.4. *Let $p = 3$ and $h_1, \ldots, h_7$ be as previously defined, then*

$$k[V]^G = k[\mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3, \mathbf{N}_4, h_1, \ldots, h_7].$$

Proof. Let

$$h_8 = h_4^2 = \frac{2h_1^2h_2^2 + h_1h_2\mathbf{N}_2^3 - \mathbf{N}_2^6}{x_1}.$$

In Lemma 5.1.3 for $1 \leq i \leq 7$ we showed that $h_i \in k[V]^G$ and that x_1 doesn't divide $\mathrm{LT}(h_i)$. Using Lemma 5.1.3 and Lemma 5.1.2 we know that:

$$A_{x_1} = k[\mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3, \mathbf{N}_4, h_1, \ldots, h_7]_{(x_1)} = k[V]^G_{(x_1)}.$$

If we can show that

$$\mathbf{N}_1, \ldots, \mathbf{N}_4, h_1, \ldots, h_7$$

form a SAGBI basis then we can use Proposition 1.3.15 to prove that

$$k[\mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3, \mathbf{N}_4, h_1, \ldots, h_7] = k[V]^G.$$

We find that

$$\begin{aligned}
h_1^p &= x_1^p\mathbf{N}_3 - \mathbf{N}_2^{p+1} + x_1^{p-1}\mathbf{N}_2^{p-1}h_1,\\
h_2^p &= 2x_1^{2p}\mathbf{N}_4 - 2x_1^{p-1}\mathbf{N}_2^{p}h_1 + \mathbf{N}_2^{p+2} - x_1^p\mathbf{N}_2^{p+1} + x_1^{2p-2}\mathbf{N}_2^{p-1}h_2,
\end{aligned}$$

and using this obtain the following relations:

$$
\begin{aligned}
h_1^2 - h_2\mathbf{N}_2 &= -x_1h_3,\\
h_1h_2 + \mathbf{N}_2^3 &= x_1h_4,\\
h_1h_3 - \mathbf{N}_2h_4 &= -x_1^2\mathbf{N}_3 - x_1\mathbf{N}_2^2h_1,\\
h_1h_4 + \mathbf{N}_2h_5 &= x_1h_7,\\
h_1h_5 - \mathbf{N}_2^2h_3 &= -x_1h_7,\\
h_1h_6 + \mathbf{N}_2h_7 &= \mathbf{N}_2^2h_1h_2 + x_1h_2\mathbf{N}_3,\\
h_1h_7 + h_8 &= \mathbf{N}_2^4h_1 + x_1\mathbf{N}_2^2\mathbf{N}_3,\\
h_2^2 + \mathbf{N}_2^2h_1 &= x_1h_5,\\
h_2h_3 + \mathbf{N}_2h_5 &= 2x_1h_6,\\
h_2h_4 + \mathbf{N}_2^2h_3 &= 2x_1h_7,\\
h_2h_5 - \mathbf{N}_2^2h_4 &= x_1\mathbf{N}_2h_1 - x_1^2\mathbf{N}_2^4 + x_1^3\mathbf{N}^2h_2 + 2x_1^5\mathbf{N}_4,\\
h_2h_6 - h_8 &= \mathbf{N}_2^4h_1 + 2x_1^4\mathbf{N}_2\mathbf{N}_4 - x_1\mathbf{N}_2^4 + x_1^2\mathbf{N}_2^3h_2,\\
h_2h_7 - \mathbf{N}_2^2h_6 &= 2\mathbf{N}_2^3h_1^2 + x_1h_1\mathbf{N}_2^4 - x_1^2h_1\mathbf{N}_2^2h_2 + x_1^4h_1\mathbf{N}_4,\\
h_3^2 - \mathbf{N}_2h_6 &= \mathbf{N}_2^{p-1}h_1^2 + x_1\mathbf{N}_3h_1,\\
h_3h_4 - 2\mathbf{N}_2h_7 &= 2\mathbf{N}_2^2h_1h_2 + 2x_1h_2\mathbf{N}_3,\\
h_3h_5 + h_8 &= \mathbf{N}_2^4h_1 - x_1\mathbf{N}_2^2\mathbf{N}_3 - x_1\mathbf{N}_2^4 + x_1^2\mathbf{N}_2^3h_2 + 2x_1^4\mathbf{N}_2\mathbf{N}_4,\\
h_3h_6 + \mathbf{N}_2^3\mathbf{N}_3 &= \mathbf{N}_2^3h_5 + 2\mathbf{N}_2^6 + 2x_1\mathbf{N}_3h_4 + 2x_1\mathbf{N}_2^2h_7 + x_1\mathbf{N}_2^4h_2 + 2x_1^3\mathbf{N}_2^2\mathbf{N}_4,\\
h_3h_7 - \mathbf{N}_2^2\mathbf{N}_3h_1 &= \mathbf{N}_2^4h_3 + \mathbf{N}_2^5h_1 + x_1\mathbf{N}_3h_5 + 2x_1\mathbf{N}_2^2h_7 + x_1\mathbf{N}_2^6\\
&\quad + 2x_1^2\mathbf{N}_2^3h_4 + x_1^3\mathbf{N}_2\mathbf{N}_4h_1,
\end{aligned}
$$

$$\begin{aligned}
h_4h_5 - \mathbf{N}_2^2h_6 &= \mathbf{N}_2^4h_2 + 2x_1\mathbf{N}_2^3h_3 + 2x_1\mathbf{N}_2^4h_1 + 2x_1^2\mathbf{N}_2^5 \\
&\quad + x_1^3\mathbf{N}_2^2h_4 + 2x_1^4\mathbf{N}_4h_1, \\
h_4h_6 + \mathbf{N}_2^2\mathbf{N}_3h_1 &= \mathbf{N}_2^4h_3 + 2\mathbf{N}_2^5h_1 + x_1\mathbf{N}_3h_5 + 2x_1\mathbf{N}_2^2h_7 + 2x_1\mathbf{N}_2^6 \\
&\quad + x_1^2\mathbf{N}_2^3h_4 + 2x_1^2\mathbf{N}_2\mathbf{N}_4h_1, \\
h_4h_7 - \mathbf{N}_2^2\mathbf{N}_3h_2 &= \mathbf{N}_2^4h_4 + \mathbf{N}_2^5h_2 + 2x_1\mathbf{N}_2^4h_3 + 2x_1^2\mathbf{N}_2^3\mathbf{N}_3 + x_1^2\mathbf{N}_2^3h_5 \\
&\quad + x_1^3\mathbf{N}_2\mathbf{N}_4h_2 + 2x_1^3\mathbf{N}_2^2h_6 + 2x_1^4\mathbf{N}_4h_3, \\
h_5^2 - \mathbf{N}_2^2h_7 &= 2\mathbf{N}_2^6 + x_1\mathbf{N}_2^3h_4 + 2x_1\mathbf{N}_2^4h_2 + 2x_1^2\mathbf{N}_2^4h_1 \\
&\quad + x_1^3\mathbf{N}_2^2h_5 + 2x_1^4\mathbf{N}_4h_2, \\
h_5h_6 - \mathbf{N}_2^2\mathbf{N}_3h_2 &= 2\mathbf{N}_2^4h_4 + \mathbf{N}_2^5h_2 + x_1\mathbf{N}_2^4h_3 + 2x_1\mathbf{N}_2^5h_1 + x_1^2\mathbf{N}_2^3\mathbf{N}_3 \\
&\quad + x_1^3\mathbf{N}_2\mathbf{N}_4h_2 + x_1^3\mathbf{N}_2^2h_6 + x_1^4\mathbf{N}_4h_3, \\
h_5h_7 - \mathbf{N}_2^4\mathbf{N}_3 &= \mathbf{N}_2^4h_5 + \mathbf{N}_2^7 + 2x_1\mathbf{N}_2^3h_6 + x_1\mathbf{N}_2^4h_4 + 2x_1\mathbf{N}_2^5h_2 \\
&\quad + x_1^2\mathbf{N}_2^4h_3 + x_1^3\mathbf{N}_2^3\mathbf{N}_4 + x_1^3\mathbf{N}_2^2h_7 + x_1^4\mathbf{N}_4h_4, \\
h_6^2 + \mathbf{N}_2^2\mathbf{N}_3h_3 &= \mathbf{N}_2^2h_8 + 2\mathbf{N}_2^5h_3 + 2\mathbf{N}_2^6h_1 + 2x_1\mathbf{N}_3h_7 + 2x_1\mathbf{N}_2^4\mathbf{N}_3 \\
&\quad + 2x_1\mathbf{N}_2^4h_5 + 2x_1^2\mathbf{N}_2^3h_6 + 2x_1^3\mathbf{N}_2\mathbf{N}_4h_3, \\
h_6h_7 - \mathbf{N}_2^2\mathbf{N}_3h_4 &= \mathbf{N}_2^5h_4 + 2\mathbf{N}_2^6h_2 + 2x_1\mathbf{N}_2^3\mathbf{N}_3h_1 + 2x_1\mathbf{N}_2^5h_3 + x_1\mathbf{N}_2^6h_1 \\
&\quad + x_1^2\mathbf{N}_2^3h_7 + x_1^2\mathbf{N}_2^4\mathbf{N}_3 + x_1^2\mathbf{N}_2^7 + x_1^3\mathbf{N}_2\mathbf{N}_4h_4 + 2x_1^3\mathbf{N}_2^2\mathbf{N}_3h_2 \\
&\quad + 2x_1^3\mathbf{N}_2^4h_4 + x_1^4\mathbf{N}_2^2\mathbf{N}_4h_1 + x_1^5\mathbf{N}_3\mathbf{N}_4, \\
h_7^2 - \mathbf{N}_2^2\mathbf{N}_3h_5 &= 2\mathbf{N}_2^4h_7 + \mathbf{N}_2^5h_5 + 2\mathbf{N}_2^8 + x_1\mathbf{N}_2^3\mathbf{N}_3h_2 + 2x_1\mathbf{N}_2^4h_6 \\
&\quad + x_1^2\mathbf{N}_2^2h_8 + x_1^3\mathbf{N}_2\mathbf{N}_4h_5 + 2x_1^4\mathbf{N}_4h_6.
\end{aligned}$$

These, along with Lemma 1.3.10 can be used to show that all tête–á–têtes subduct to zero, and

$$\{\mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3, \mathbf{N}_4, h_1, \ldots, h_7\}$$

forms a SAGBI basis for A.

□

We have now found $k[V]^G$ for $p = 3$ and will go on to show that it is not a complete intersection ring. Using MAGMA, [4], it can be shown that $f_{i,j}$ as defined in Lemma 5.1.3 are contained in $k[V]^G$ for primes up to and including 19 (at this point the algorithm begins to run quite slowly finding the $\frac{p^2+5}{2}$ invariants). For $p = 5$ MAGMA can be used to show that the $f_{i,j}$'s, along with $\mathbf{N}_1, \ldots, \mathbf{N}_4, h_1, h_2, h_3$ generate $k[V]^G$.

Proof. (of Proposition 5.1.1) Let $I = (\mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3, \mathbf{N}_4)$ and

$$\overline{A} = k[V]^G/I.$$

As $k[V]^G$ is a Cohen–Macaulay ring any HSOP forms a regular sequence (see [10, Theorem 2.8.1]), so by Proposition 1.2.14 if $k[V]^G$ is a complete intersection ring then $\overline{A}$ is also a complete intersection ring.

For $1 \leq i \leq 7$ let $\overline{h}_i$ be the image of h_i in $\overline{A}$. $\overline{A}$ is a Poincaré duality algebra with each degree either a zero- or one-dimensional vector space. Let $\overline{A}_i$ denote the degree i part of $\overline{A}$. The non-empty degrees are:

$$0, 4, 5, 7, 8, 9, 11, 12, 16.$$

Each non empty degree part is a one-dimension k-vector space

$$\begin{array}{llll}
\overline{A}_0 = k, & \overline{A}_4 = k\overline{h}_1, & \overline{A}_5 = k\overline{h}_2, & \overline{A}_7 = k\overline{h}_3, \\
\overline{A}_8 = k\overline{h}_4, & \overline{A}_9 = k\overline{h}_5, & \overline{A}_{11} = k\overline{h}_6, & \overline{A}_{12} = k\overline{h}_7, \\
\overline{A}_{16} = k\overline{h}_4^2. & & &
\end{array}$$

Let $R = k[Y_1, \ldots, Y_7]$ with

$$\begin{array}{llll}
\deg(Y_1) = 4, & \deg(Y_2) = 5, & \deg(Y_3) = 7, & \deg(Y_4) = 8, \\
\deg(Y_5) = 9, & \deg(Y_6) = 11, & \deg(Y_7) = 12, &
\end{array}$$

and let J be the ideal in R generated by

$$S = \{Y_iY_j \mid 1 \leq i \leq j \leq 7, i + j \neq 8\} \cup \{Y_1Y_7 + Y_4^2, Y_3Y_5 + Y_4^2, Y_2Y_6 - Y_4^2\}.$$

We define the surjective map:

$$\phi : R \longrightarrow \overline{A}$$
$$\phi(Y_i) \longmapsto h_i.$$

The kernel of ϕ is J and so $R/J \cong \overline{A}$.

We claim that J cannot be generated by a regular sequence. Firstly we look at the degrees of the generators for J in ascending order:

$$\begin{aligned} \deg(Y_1^2) &= 8, \\ \deg(Y_1Y_2) &= 9, \\ \deg(Y_2^2) &= 10, \\ \deg(Y_1Y_3) &= 11, \\ &\vdots \\ \deg(Y_7^2) &= 24. \end{aligned}$$

If Y_1^2 is not needed as a generator for J then we can find some r and $X_i \in J$ with $\deg(X_i) \leq \deg(Y^2)$ such that

$$Y_1^2 = \sum_{i=1}^{r} a_i X_i$$

where $a_i \in R$ for $1 \leq i \leq r$. If $t_1 \in J$ with $\deg(t_1) \leq 8$ then $t_1 = c_1 Y_1^2$ for some $c \in k$, so Y_1^2 must be part of our generating set. Suppose $f_2 \in J$ with $\deg(f_2) = 9$, then as the lowest positive degree in R is 4 we see that $f_2 = c_2 Y_1 Y_2$ for some $c_2 \in k$, so Y_1Y_2 must be part of our generating set. This is already enough to show that J cannot be generated by a regular sequence, as Y_1^2, Y_1Y_2 do not form a regular sequence. If we continue working through degrees, then we see we cannot find a generating set with fewer elements than $|S| = 27$. This means that $\overline{A}$ is not a complete intersection ring, and so $k[V]^G$ is not a complete intersection ring. $\square$

Chapter 6

Invariant rings of exceptional pure bireflection groups

In this chapter we will see that both types of exceptional group defined in Chapter 2 have complete intersection invariant ring for $k = \mathbb{F}_p$, $p \neq 2$. At the end of the chapter we put this together with earlier results to see when pure bireflection groups have complete intersection or Cohen–Macaulay invariant rings.

6.1 Exceptional groups of type one

We start with exceptional groups of type one, and restrict to $p \neq 2$. Let

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

with respect to a basis $\{y_1, \dots, y_n\}$ for W. Let

$$r_1 = y_1, \qquad r_2 = y_2, \qquad v = y_3,$$
$$\gamma_1 = y_5^*, \qquad \gamma_2 = y_4^*, \qquad v^* = y_3^*$$

and fix

$$\mathbf{r} = \{r_1, r_2, v\},$$
$$\gamma = \{\gamma_1, \gamma_2, v^*\}.$$

Then using the above matrices

$$g = t_{y_3}^{y_5^*} t_{y_2}^{y_3^*} = t_v^{\gamma_1^*} t_{r_2}^{v^*} = \chi_{1,0,0}^{\mathbf{r},\gamma},$$
$$h = t_{y_3+y_1}^{y_4^*} t_{2y_1+y_2}^{y_3^*} = t_{v+r_1}^{\gamma_1^*} t_{2r_1+r_2}^{v^*} = \chi_{0,1,0}^{\mathbf{r},\gamma}.$$

Let $z = \chi_{0,0,1}^{\mathbf{r},\gamma}$ then $\langle z \rangle = [G, G]$ and with respect to this basis:

$$z = \begin{pmatrix} 1 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The group $G \leq X^{\mathbf{r},\gamma}$ is an exceptional group of type one. If $k = \mathbb{F}_p$ then $G = X^{\mathbf{r},\gamma}$.

We want to look at the invariant ring of this group, to do this we will make a change of basis for W in order to make later calculations simpler. Let

$$x_1 = \frac{1}{2} y_2,$$
$$x_2 = y_1 + \frac{1}{2} y_2,$$
$$x_3 = \frac{1}{2} y_3 - \frac{1}{4} y_2,$$
$$x_4 = y_4,$$
$$x_5 = y_5.$$

then with respect to this basis:

$$g = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and

$$z = \begin{pmatrix} 1 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

We will show that $k[V]^G$ is a complete intersection ring. We first we determine the invariant field $k(V)^G$ by finding $\phi_1, \ldots, \phi_5$ as described in Theorem 1.3.12. As G is triangular we can see that $U = \langle x_1, x_2, x_3, x_4 \rangle$ is a G-stable subspace of W, we want to find $R[4]^G$ by looking at how G acts on U.

For this section four-by-four matrices describe the action of $\mathrm{GL}(V)$ on U, so if g', h', z' are the restrictions to U of g, h, z respectively then

$$g' = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad h' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad z' = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Lemma 6.1.1. *Let G be as above then $R[4]^G = k[\mathbf{N}_1^G, \mathbf{N}_2^G, \mathbf{N}_3^G, \mathbf{N}_4^G, d_1]$ where*

$$d_1 = (x_3^p - x_1^{p-1}x_3)^2 - (x_2^p - x_1^{p-1}x_2)(x_4^p - x_1^{p-1}x_4)$$

Proof. To find the invariant ring of H, we will first find the invariant ring of $H = \langle G, \sigma \rangle$ where

$$\sigma = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We know from the previous section on Exceptional Groups of type 1 that $[G, G] = \langle z \rangle$. Let $a \in \mathbb{F}_p$ such that $2a = 1$. As σ commutes with g and z and

$$\sigma^{-1}(h)^{-1}\sigma h = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = gz^a$$

the subgroup $N = \langle \sigma, g', z' \rangle$ is a normal subgroup of H.

As N is a Nakajima group we can easily find it's invariant ring

$$S(U)^N = k[\mathbf{N}_1^N, \mathbf{N}_2^N, \mathbf{N}_3^N, \mathbf{N}_4^N],$$

where

$$\begin{aligned} \mathbf{N}_1^N &= x_1 = \mathbf{N}_1^H = \mathbf{N}_1^G \qquad , \\ \mathbf{N}_2^N &= x_2^p - x_1^{p-1}x_2 = \mathbf{N}_2^H, \\ \mathbf{N}_3^N &= x_3^p - x_1^{p-1}x_3, \\ \mathbf{N}_4^N &= x_4^p - x_1^{p-1}x_4. \end{aligned}$$

Letting H/N act on $k[V]^N$, we can see that:

$$\delta_{h_1}(\mathbf{N}_1^N) = \delta_{h_1}(\mathbf{N}_2^N) = 0,$$
$$\delta_{h_1}(\mathbf{N}_3^N) = \mathbf{N}_2^N,$$
$$\delta_{h_1}(\mathbf{N}_4^N) = 2\mathbf{N}_3^N + \mathbf{N}_2^N.$$

Let $\widetilde{H} = \langle h \rangle$. The action of $\widetilde{H}/N$ on $k[V]^N$ is isomorphic to the action of $\widetilde{H}$ on $S(U) = k[x_1, x_2, x_3, x_4]$. Identifying x_1, x_2, x_3, d of Example 1.3.16 to $\mathbf{N}_2, \mathbf{N}_3, \mathbf{N}_4, d_1$ respectively we obtain:

$$k[V]^{H'} = k[\mathbf{N}_1^{H'}, \mathbf{N}_2^{H'}, \mathbf{N}_3^{H'}, \mathbf{N}_4^{H'}, d_1].$$

As G is a maximal subgroup of H we can now we use Theorem 1.3.4 to find $S(U)^G$: we know that $x_2 \in S(U)^G \backslash S(U)^H$ and $\delta_\sigma(x_2) = x_1$. As $\delta_\sigma(x_i) = 0$ for $i \neq 2$, it isn't hard to see that $\delta_\sigma S(U)^H \leq x_1 S(U)^H$ so

$$S(U)^G = S(U)^H[x_2] = k[\mathbf{N}_1^G, \mathbf{N}_2^G, \mathbf{N}_3^G, \mathbf{N}_4^G, d_1]. \qquad \square$$

Lemma 6.1.2. *Let* $A = k[\mathbf{N}_1^G, \mathbf{N}_2^G, \mathbf{N}_3^G, \mathbf{N}_4^G, \mathbf{N}_5^G, h_1, h_2]$ *with*

$$h_1 = x_1x_5 + x_2x_4 - x_3^2,$$
$$h_2 = x_1(x_5^p - x_2^{p-1}x_5) + x_2(x_4^p - x_1^{p-1}x_4) - 2x_3^{p+1} + x_1^{p-1}x_3^2 + x_2^{p-1}x_3^2.$$

Then $A_{x_1} = k[V]^G_{x_1}$.

Proof. As in Theorem 1.3.12 we can find $\phi_1, \ldots, \phi_5$ such that $\phi_i \in R[i]$ is of minimal positive degree in x_i. As $\deg_{x_5}(h_1) = 1$ we cannot find any invariants of lesser positive degree in x_5 and so we can take $\phi_5 = h_1$.

Using Lemma 6.1.1 we know $R[4]^G = k[\mathbf{N}_1^G, \ldots, \mathbf{N}_4^G, d_1]$ so we can choose

$$\phi_i = \mathbf{N}_i^G \text{ for } 1 \leq i \leq 3,$$
$$\phi_4 = d_1.$$

Let $A' = k[\phi_1, \dots, \phi_5]$. Using Theorem 1.3.12

$$\mathrm{Quot}(A') = k(V)^G.$$

As

$$d_1 = -h_1^p + x_1^{p-1}x_2^{p-1}h_1 + x_1^{p-1}h_2$$

we see that $d_1 \in A$ and so

$$A'[\mathbf{N}_4, \mathbf{N}_5, h_2] = A.$$

As $\mathbf{N}_4, \mathbf{N}_5, h_2 \in k[V]^G$ we see that

$$\mathrm{Quot}(A) = \mathrm{Quot}(A') = k(V)^G$$

furthermore we know that $R[4]^G \subseteq A$.

By [7] Lemma 2.1 for any $f \in R[m]^G$ we can find some $r \in \mathbb{N}$ such that $c_m^r f \in R[m-1]^G[\phi_m]$ (where c_i as defined in [7] for $1 \leq i \leq n$). We know that $\phi_5 \in A$ with $c_5 = x_1$, and we know that $R[4] \subset A$ so for any $f \in k[V]^G = R[5]^G$ we can find some r such that $x_1^r f \in R[4]^G[h_1] = A$. This means that

$$A_{x_1} = k[V]^G_{x_1}. \qquad \square$$

We would like to use Theorem 1.3.15 and for this we need a SAGBI basis for A. We start by looking at relations in A. First of all we find that

$$\mathrm{LM}(h_2) = \mathrm{LM}(h_1^{(p+1)/2}) = x_3^{p+1}$$

so in our SAGBI basis for A we will use $h_2' = h_2 - 2h_1^{(p+1)/2}$ instead of h_2 (note that $\mathrm{LM}(h_2') = x_2 x_4^p$).

To make notation easier we let

$$\begin{aligned} f_1(t) &= t^p - x_1^{p-1}t, \\ f_2(t) &= t^p - x_2^{p-1}t. \end{aligned}$$

We previously defined

$$\begin{aligned} d_1 &= -h_1^p + x_1^{p-1}x_2^{p-1}h_1 + x_1^{p-1}h_2 \\ &= f_1(x_3)^2 - f_1(x_2)f_1(x_4) \end{aligned}$$

we similarly let

$$\begin{aligned} d_2 &= -h_1^p + x_1^{p-1}x_2^{p-1}h_1 + x_2^{p-1}h_2 \\ &= f_2(x_3)^2 - f_2(x_1)f_2(x_5). \end{aligned}$$

We can write $\mathbf{N}_3$ in terms of $f_1(x_3)$ and $f_1(x_2)$ using the properties of the Dickson invariants (see 1.1.7). Let $U = \langle x_1, x_2 \rangle$, then

$$\begin{aligned} \mathbf{N}_3 = \prod_{u \in U}(x_3 + u) &= f_1(x_3)^p - f_1(x_2)^{p-1}f_1(x_3) \\ = \prod_{u \in U}(x_3 + u) &= f_2(x_3)^p - f_2(x_1)^{p-1}f_2(x_3), \end{aligned}$$

and so

$$\mathbf{N}_3^2 = f_1(x_3)^{2p} - 2f_1(x_2)^{p-1}f_1(x_3)^{p+1} + f_1(x_2)^{2p-2}f_1(x_3)^2.$$

Let

$$\begin{aligned} H_1(d_1) &= -2f_1(x_2)^{p-1}d_1^{(p+1)/2} + f_1(x_2)^{2p-2}d_1, \\ H_2(d_2) &= -2f_2(x_1)^{p-1}d_2^{(p+1)/2} + f_2(x_1)^{2p-2}d_2. \end{aligned}$$

Lemma 6.1.3.

$$\mathbf{N}_3^2 - d_1^p - H_1(d_1) + f_1(x_2)^p\mathbf{N}_4 = \mathbf{N}_3^2 - d_2^p - H_2(d_2) + f_2(x_1)^p\mathbf{N}_5 = 0$$

Proof. Let

$$P_1 = \mathbf{N}_3^2 - d_1^p - H_1(d_1),$$
$$P_2 = \mathbf{N}_3^2 - d_2^p - H_2(d_2).$$

As we can write

$$d_1 = f_1(x_3)^2 - f_1(x_2)f_1(x_4),$$
$$d_2 = f_2(x_3)^2 - f_2(x_1)f_2(x_5),$$

we see that x_4 divides P_1 and x_5 divides P_2. As $P_1 \in k[V]^G$ this means that all elements in the orbit of x_4 under G divide P_1: for some $c_1 \in k[V]^G$ we know $P_1 = c_1\mathbf{N}_4$. As the highest power of x_4 in P is $x_4^{p^2}$ with coefficient $-f_1(x_2)^p$ we must have:

$$P = -f_1(x_2)^p\mathbf{N}_4$$

so

$$N_3^2 + d_1^p + H_1(d_1) + f_1(x_2)^pN_4 = 0.$$

Similarly for some $c_2 \in k[V]^G$ we know $P_2 = c_2\mathbf{N}_5$. As the highest power of x_5 in P is $x_5^{p^2}$ with coefficient $-f_2(x_1)^p$ we must have:

$$P_2 = -f_2(x_1)^p\mathbf{N}_5$$

so

$$N_3^2 + d_2^p + H_2(d_2) + f_2(x_1)^pN_5 = 0. \qquad \square$$

For any $f \in k[V]$ such that

$$f = \sum_{s=1}^{m} c_s x_1^{\alpha_{s,1}} \dots x_n^{\alpha_{s,n}}$$

with $c_s \in k\backslash 0$ we define

$$(f)_j^{\deg(x_i)} = \sum_{\{s|\alpha_{s,i}=j\}} c_s x_1^{\alpha_{s,1}} \dots x_n^{\alpha_{s,n}}$$

the part of f with x_i-degree j.

Lemma 6.1.4. *For $1 \leq r$ we can find*

$$D_r = h_1^{pr} - x_1^{p-1} x_2^{p-1} \sigma_r \in A$$

where $\sigma_r \in k[x_1, x_2, h_1, h_2]$ such that

$$(D_r)_0^{\deg(x_3)} = x_1^p f_2(x_1)^{r-1} f_2(x_5)^{r+1} + x_2^p f_1(x_2)^{r-1} f_1(x_4)^{r+1}$$

Proof. Firstly if we let

$$\begin{aligned} D_1 &= h_1^p - x_1^{p-1} x_2^{p-1} h_1 \\ &= x_1^p f_2(x_5) + x_2^p f_1(x_4) + x_3^{2p} - x_1^{p-1} x_2^{p-1} x_3^2, \end{aligned}$$

so that

$$\sigma_1 = h_1,$$

then we see that

$$(D_1)_0^{\deg(x_3)} = x_1^p f_2(x_5) + x_2^p f_1(x_4).$$

Now let

$$D_2 = D_1^2 - x_1^{p-1} x_2^{p-1} h_2^2,$$

so

$$\sigma_2 = 2h_1^{p+1} - x_1^{p-1} x_2^{p-1} h_1^2 + h_2^2.$$

As

$$(h_2)_0^{\deg(x_3)} = x_1 f_2(x_5) + x_2 f_1(x_4)$$

we find

$$\begin{aligned}(D_1^2)_0^{\deg(x_3)} &= x_1^{2p} f_2(x_5)^2 + x_2^{2p} f_1(x_4)^2 + 2x_1^p x_2^p f_1(x_4) f_2(x_5),\\ (h_2^2)_0^{\deg(x_3)} &= x_1^2 f_2(x_5)^2 + x_2^2 f_1(x_4)^2 + 2x_1 x_2 f_1(x_4) f_2(x_5),\end{aligned}$$

so

$$(D_2)_0^{\deg(x_3)} = x_1^p f_2(x_1) f_2(x_5)^2 + x_2^p f_1(x_2) f_1(x_4)^2$$

as required.

Now we can proceed by induction. Assume for some $r > 1$ we can find

$$D_{r-1} = h_1^{p(r-1)} - x_1^{p-1} x_2^{p-1} \sigma_{r-1}$$

and

$$D_r = h_1^{pr} - x_1^{p-1} x_2^{p-1} \sigma_r$$

with

$$\begin{aligned}(D_{r-1})_0^{\deg(x_3)} &= x_1^p f_2(x_1)^{r-2} f_2(x_5)^{r-1} + x_2^p f_1(x_2)^{r-2} f_1(x_4)^{r-1}\\ (D_r)_0^{\deg(x_3)} &= x_1^p f_2(x_1)^{r-1} f_2(x_5)^r + x_2^p f_1(x_2)^{r-1} f_1(x_4)^r.\end{aligned}$$

Let

$$\begin{aligned}D =& - D_r(d_1 + d_2) - D_{r-1}(d_1 d_2)\\ =&(h_1^{pr} - x_1^{p-1} x_2^{p-1} \sigma_r)(2h_1^p - 2x_1^{p-1} x_2^{p-1} h_1 - x_1^{p-1} h_2 - x_2^{p-1} h_2)\\ &- (h_1^{p(r-1)} - x_1^{p-1} x_2^{p-1} \sigma_{r-1})(h_1^p - x_1^{p-1} x_2^{p-1} h_1 - x_1^{p-1} h_2)\\ &(h_1^p - x_1^{p-1} x_2^{p-1} h_1 - x_2^{p-1} h_2)\end{aligned}$$

then

$$D = h_1^{p(r+1)} - x_1^{p-1}x_2^{p-1}\sigma$$

where

$$\begin{aligned}\sigma = x_1^{p-1}x_2^{p-1}h_1^{p(r-1)+2} + x_1^{p-1}h_1^{p(r-1)+1}h_2 + x_2^{p-1}h_1^{p(r-1)+1}h_2 - h_1^{p(r-1)}h_2^2 \\ + \sigma_r(d_1+d_2) + \sigma_{r-1}d_1d_2.\end{aligned}$$

Note that if $\sigma_r, \sigma_{r-1} \in k[x_1, x_2, h_1, h_2]$ then $\sigma \in k[x_1, x_2, h_1, h_2]$. We find the part of D with x_3 degree 0, firstly we note that

$$\begin{aligned}(d_1+d_2)_0^{\deg(x_3)} &= -f_1(x_2)f_1(x_4) - f_2(x_1)f_2(x_5),\\ (d_1d_2)_0^{\deg(x_3)} &= f_1(x_2)f_2(x_1)f_1(x_4)f_2(x_5),\end{aligned}$$

so

$$\begin{aligned}(-D_r(d_1+d_2))_0^{\deg(x_3)} =& x_1^p f_2(x_1)^r f_2(x_5)^{r+1} + x_2^p f_1(x_2)^r f_1(x_4)^{r+1}\\ &+ x_1^p f_1(x_2)f_2(x_1)^{r-1}f_1(x_4)f_2(x_5)^r\\ &+ x_2^p f_2(x_1)f_1(x_2)^{r-1}f_2(x_5)f_1(x_4)^r,\\ (D_{r-1}(d_1d_2))_0^{\deg(x_3)} =& x_1^p f_1(x_2)f_2(x_1)^{r-1}f_1(x_4)f_2(x_5)^r\\ &+ x_2^p f_2(x_1)f_1(x_2)^{r-1}f_2(x_5)f_1(x_4)^r.\end{aligned}$$

We find that

$$D_0^{\deg(x_3)} = x_1^p f_2(x_1)^r f_2(x_5)^{r+1} + x_2^p f_1(x_2)^r f_1(x_4)^{r+1}$$

and so we can let $D_{r+1} = D$, $\sigma_{r+1} = \sigma$. □

Lemma 6.1.5.

$$(h_2')^p - x_2^p\mathbf{N}_4 - x_1^p\mathbf{N}_5 - 2x_1^{p-1}x_2^{p-1}\sigma_{\frac{p+1}{2}} + f_1^{p-1}(x_2)d_1 + f_2^{p-1}(x_1)d_2 = 0.$$

Proof. From Lemma 6.1.3 we see that

$$\begin{aligned} f_1(x_2)^p\mathbf{N}_4 &= \mathbf{N}_3^2 + d_1^p + H(d_1), \\ (f_1(x_2)^p\mathbf{N}_4)_0^{\deg(x_3)} &= f_1(x_2)^p f_1(x_4)^p - 2f_1(x_2)^{\frac{3p-1}{2}} f_1(x_4)^{\frac{p+1}{2}} + f_1(x_2)^{2p-1} f_2(x_4), \end{aligned}$$

and so dividing by $f_1(x_2)^p$ gives

$$(\mathbf{N}_4)_0^{\deg(x_3)} = f_1(x_4)^p - 2f_1(x_2)^{\frac{p-1}{2}} f_1(x_4)^{\frac{p+1}{2}} + f_1(x_2)^{p-1} f_2(x_4).$$

Similarly to above, Lemma 6.1.3 gives

$$\begin{aligned} f_2(x_1)^p\mathbf{N}_5 &= \mathbf{N}_3^2 + d_2^p + H(d_2), \\ (f_2(x_1)^p\mathbf{N}_5)_0^{\deg(x_3)} &= f_2(x_1)^p f_2(x_5)^p - 2f_2(x_1)^{\frac{3p-1}{2}} f_2(x_5)^{\frac{p+1}{2}} + f_2(x_1)^{2p-1} f_1(x_5), \end{aligned}$$

and so

$$(\mathbf{N}_5)_0^{\deg(x_3)} = f_2(x_5)^p - 2f_2(x_1)^{\frac{p-1}{2}} f_2(x_5)^{\frac{p+1}{2}} + f_2(x_1)^{p-1} f_2(x_5).$$

Let

$$\begin{aligned} T &= (h')_2^p - x_2^p\mathbf{N}_4 - x_1^p\mathbf{N}_5 + 2x_1^{p-1}x_2^{p-1}\sigma_{\frac{p+1}{2}} + x_2^p f_1(x_2)^{p-2} d_1 + x_1^p f_2(x_1)^{p-2} d_2, \\ &= h_2^p - 2D_{(p+1)/2)} - x_2^p\mathbf{N}_4 - x_1^p\mathbf{N}_5 + f_1(x_2)^{p-1} d_1 + f_2(x_1)^{p-1} d_2, \end{aligned}$$

then using the above we see that

$$T_0^{\deg(x_3)} = 0.$$

If $T \neq 0$ then x_3 must divide all terms of T and so all elements in the orbit of x_3 must divide T.

We look at the highest degree of x_3 in each of the parts of T

$$\begin{aligned}
\deg_{x_3}((h')_2^p) &\leq p^2 - p, \\
\deg_{x_3}(N_4) = \deg_{x_3}(N_5) &\leq p^2 - 1, \\
\deg_{x_3}(\sigma_{\frac{p+1}{2}}) &\leq p^2 - p + 2, \\
\deg_{x_3}(d_1) = \deg_{x_3}(d_2) &= 2.
\end{aligned}$$

This shows us that

$$\deg_{x_3}(T) \leq p^2 - 1 < p^2 = \deg_{x_3}(\mathbf{N}_3)$$

so $\mathbf{N}_3$ cannot divide T which means $T = 0$. $\square$

Proposition 6.1.6. $k[V]^G = k[\mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3, \mathbf{N}_4, \mathbf{N}_5, h_1, h_2]$.

Proof. Let $B = \{\mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3, \mathbf{N}_4, \mathbf{N}_5, h_1, h_2'\}$. In order to show that B is a SAGBI basis for A we need to find all tête–á–têtes in B and show that they subduct to zero. We know

$$\begin{aligned}
\mathrm{LM}(\mathbf{N}_1^G) &= x_1, \\
\mathrm{LM}(\mathbf{N}_2^G) &= x_2, \\
\mathrm{LM}(\mathbf{N}_3^G) &= x_3^{p^2}, \\
\mathrm{LM}(\mathbf{N}_4^G) &= x_4^{p^2}, \\
\mathrm{LM}(\mathbf{N}_5^G) &= x_5^{p^2}, \\
\mathrm{LM}(h_1) &= x_3^2, \\
\mathrm{LM}(h_2') &= x_2 x_4^p.
\end{aligned}$$

If

$$\mathrm{LM}(\mathbf{N}_1^{a_1}\mathbf{N}_2^{a_2}\mathbf{N}_3^{a_3}\mathbf{N}_4^{a_4}\mathbf{N}_5^{a_5}h_1^{a_6}(h_2')^{a_7}) = \mathrm{LM}(\mathbf{N}_1^{b_1}\mathbf{N}_2^{b_2}\mathbf{N}_3^{b_3}\mathbf{N}_4^{b_4}\mathbf{N}_5^{b_5}h_1^{b_6}(h_2')^{b_7})$$

then

$$\begin{aligned}
&a_1 = b_1,\\
&a_2 + a_7 = b_1 + b_7,\\
&a_3p^2 + 2a_6 = b_3p^2 + 2b_6,\\
&a_4p^2 + a_7p = b_4p^2 + b_7p,\\
&a_5 = b_5.
\end{aligned}$$

We see that the only non trivial tête–á–têtes in B are $\{\mathbf{N}_3^2, h_1^{p^2}\}$ and $\{x_4^p\mathbf{N}_2, (h_2')^p\}$.

From Lemma 6.1.3 we know

$$\mathbf{N}_3^2 - d_2^p - H_2(d_2) + f_2(x_1)^p\mathbf{N}_5 = 0,$$

where

$$d_2 = -h_1^p + x_1^{p-1}x_2^{p-1}h_1 + x_2^{p-1}h_2.$$

This means that

$$\begin{aligned}
\mathbf{N}_3^2 + h_1^{p^2} &= x_1^{p^2-p}x_2^{p^2-p}h_1^p + x_2^{p^2-p}h_2^p + H_2(d_2) - f_2(x_1)^p\mathbf{N}_5\\
&\in k[x_1, x_2, h_1, h_2', \mathbf{N}_5].
\end{aligned}$$

As there are no non-trivial tête–á–têtes in $\{x_1, x_2, h_1, h_2', \mathbf{N}_5\}$ we see that we can use Lemma 1.3.10 to show that $\textsc{Subdt}(\mathbf{N}_3^2 + h_1^{p^2}) = 0$.

From Lemma 6.1.5

$$\begin{aligned}
(h_2')^p - x_2^p\mathbf{N}_4 &= x_1^p\mathbf{N}_5 + 2x_1^{p-1}x_2^{p-1}\sigma_{\frac{p+1}{2}} - f_1(x_2)^{p-1}d_1 - f_2(x_1)^{p-1}d_2\\
&\in k[x_1, x_2, h_1, h_2', \mathbf{N}_5]
\end{aligned}$$

so we can again use Lemma 1.3.10 to see that $\textsc{Subdt}((h_2')^p - x_2^p\mathbf{N}_4) = 0$.

As both tête–á–têtes subduct to zero, B forms a SAGBI basis for $k[V]^G$. The orbit products form a HSOP for $k[V]^G$ and so it is integral over A and by Lemma 6.1.2 $A_{x_1} = k[V]^G_{x_1}$. Using Theorem 1.3.14 we see that $k[V]^G = A$.

Viewing invariant ring $k[V]^G$ as a module of $B = k[\mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3, \mathbf{N}_4, \mathbf{N}_5]$ we find that it is generated by

$$\{h_1^i (h_2')^j \mid 0 \leq i \leq p^2 - 1, \quad 0 \leq j \leq p-1\},$$

so the number of secondary generators is p^3. We find that

$$\prod_{i=1}^{5} \deg(\mathbf{N}_i) = p^6 = p^3|G|$$

so by Theorem 1.2.3 $k[V]^G$ is a Cohen–Macaulay ring.

For $0 \leq i \leq p^2 - 1$, $0 \leq j \leq p - 1$ let

$$\phi(y_{i,j}) = h_1^i (h_2')^j.$$

Let

$$\mathbf{y} = \{y_{i,j} \mid 1 \leq i \leq p^2 - 1, \, 0 \leq j \leq p - 1\}$$

so $B[\mathbf{y}]$ is a polynomial ring. We define canonical surjective map ϕ from $B[\mathbf{y}]$ onto $k[V]^G$ by

$$\phi(y_{i,j}) = f_{i,j}$$

for $0 \leq i \leq p^2 - 1$, $0 \leq j \leq p - 1$.

For $i + i' < p^2$ and $j + j' < p$ we see that

$$\phi(y_{i,j})\phi(y_{i',j'}) = f_{i+i',j+j'}$$

so we can write any polynomial which has degree less than p^2 in h_1 and less than p in h_2 as a linear combination of the $f_{i,j}$. Let

$$F_2 = x_1^p \mathbf{N}_5 + 2x_1^{p-1} x_2^{p-1} \sigma_{\frac{p+1}{2}} - f_1(x_2)^{p-1} d_1 - f_2(x_1)^{p-1} d_2 + x_2^p \mathbf{N}_4$$

and then let

$$F_1 = x_1^{p^2-p} x_2^{p^2-p} h_1^p + x_2^{p^2-p} F_2 - 2x_2^{p^2-p} h_1^{p(p+1)/2} + H_2(d_2) - f_2(x_1)^p \mathbf{N}_5 - \mathbf{N}_3^2.$$

Using Lemmas 6.1.3 and 6.1.5 we find that

$$\phi(y_{i,j})\phi(y_{i',j'}) = \begin{cases} f_{i+i',j+j'} & \text{for } i+i' < p^2, \quad j+j' < p, \\ f_{i+i'-p^2,j+j'} F_1 & \text{for } i+i' \geq p^2, \quad j+j' < p, \\ f_{i+i',j+j'-p} F_2 & \text{for } i+i' < p^2, \quad j+j' \geq p, \\ f_{i+i'-p^2,j+j'-p} F_1 F_2 & \text{for } i+i' \geq p^2, \quad j+j' \geq p. \end{cases}$$

The terms on the right can be rewritten as linear expressions in the $f_{i,j}$ with coefficients in B, and we find a set of relations as described in Proposition 1.2.13. We can find a preimages of F_1, F_2 as linear expressions in the $y_{i,j}$ with coefficients in B, let these be denoted by F_1', F_2' respectively. The kernel I of the map above can therefore be generated by $(p^2-1)(p-1)$ generators:

$$\begin{aligned} & y_{i,j} - y_{i,0}y_{0,j} \quad \text{for } 1 < i+j, \\ & y_{1,0}y_{p^2-1,0} - F_1', \\ & y_{0,1}y_{0,p-1} - F_2'. \end{aligned}$$

As

$$(p^2-1)(p-1) = \dim(B) - \dim(k[V]^G)$$

we see that the generators for I must form a regular sequence and so $k[V]^G$ is a complete intersection ring. □

Corollary 6.1.7. *Let E be an exceptional group of type 1 generated by an exceptional pair. Then $k[V]^E$ is a complete intersection ring. For $k = \mathbb{F}_p$ all exceptional groups of type one have complete intersection rings of invariants.*

Proof. If E is generated by an exceptional pair then E is congruent to G and so $k[V]^E$ is a complete intersection ring. For $k = \mathbb{F}_p$ all exceptional groups are generated by special pairs, so all have complete intersection invariant rings. $\square$

6.2 Exceptional groups of type two

We now consider exceptional groups of type two. Let $G = \langle g_1, g_2, g_3 \rangle$ where

$$g_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$g_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

with respect to the basis $B = \{x_1, \ldots, x_6\}$ for W. Let

$$\begin{aligned} r_1 &= x_1, & r_2 &= x_2, & r_3 &= x_3, \\ \gamma_1 &= x_4^*, & \gamma_2 &= x_6^*, & \gamma_3 &= x_5^* \end{aligned}$$

$\mathbf{r} = \{r_1, r_2, r_3\}$ and $\gamma = \{\gamma_1, \gamma_2, \gamma_3\}$. Then

$$g_1 = w^{\mathbf{r},\gamma}_{0,0,1}, \quad g_2 = w^{\mathbf{r},\gamma}_{1,0,0}, \quad g_3 = w^{\mathbf{r},\gamma}_{0,1,0}$$

so $G \leq W^{\mathbf{r},\gamma}$ is an exceptional group of type 2. If $k = \mathbb{F}_p$ then $G = W^{\mathbf{r},\gamma}$.

We know from Proposition 3.1.2 that G is nice with respect to B, so to find the invariant ring of G we look at finding a chain for maximal subgroups from the abelian group

$$N = \mathrm{Nak}_B^+(G) = \langle G, h_1, h_2, h_3 \rangle$$

where

$$h_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad h_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$h_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let

$$\begin{aligned} N_0 &= \mathrm{Nak}^+(G), \\ N_1 &= \langle G, h_1, h_2 \rangle, \\ N_2 &= \langle G, h_1 \rangle, \\ N_3 &= G, \end{aligned}$$

then we have the inclusions of maximal subgroups:

$$G = N_3 <_{\max} N_2 <_{\max} N_1 <_{\max} N_0 = N$$

as described in 3.0.11. We shall work through these making repeated use of Theorem 1.3.4 to find the invariant ring of G.

Lemma 6.2.1. $k[V]^{N_1} = k[\mathbf{N}_1^N, \ldots, \mathbf{N}_6^N, f_1]$ *where*

$$f_1 = (x_5^p - x_1^{p-1}x_5)(x_3^p - x_1^{p-1}x_3) - (x_4^p - x_1^{p-1}x_4)(x_2^p - x_1^{p-1}x_2).$$

Proof. N_0 is a Nakajima group so $k[V]^{N_0} = k[\mathbf{N}_1^{N_0}, \ldots, \mathbf{N}_6^{N_0}]$ where

$$\begin{aligned}
\mathbf{N}_1^{N_0} &= x_1 \\
\mathbf{N}_2^{N_0} &= x_2 \\
\mathbf{N}_3^{N_0} &= x_3 \\
\mathbf{N}_4^{N_0} &= (x_4^p - x_3^{p-1}x_4)^p - (x_1^p - x_3^{p-1}x_1)^{p-1}(x_4^p - x_3^{p-1}x_4) \\
\mathbf{N}_5^{N_0} &= (x_5^p - x_2^{p-1}x_5)^p - (x_1^p - x_2^{p-1}x_1)^{p-1}(x_5^p - x_2^{p-1}x_5) \\
\mathbf{N}_6^{N_0} &= (x_6^p - x_3^{p-1}x_6)^p - (x_2^p - x_3^{p-1}x_2)^{p-1}(x_6^p - x_3^{p-1}x_6)
\end{aligned}$$

As G is nice with respect to the current basis by Proposition 3.0.12

$$\mathbf{N}_i^N = \mathbf{N}_i^{N_0} = \mathbf{N}_i^{N_1} = \mathbf{N}_i^{N_2} = \mathbf{N}_i^{N_3} = \mathbf{N}_i^G$$

for $1 \leq i \leq 6$. We shall therefore denote $\mathbf{N}_i^N$ by $\mathbf{N}_i$ for $1 \leq i \leq 6$.

We want to use Theorem 1.3.4 to show that

$$k[V]^{N_1} = k[V]^{N_0}[f_1].$$

Firstly it is easy to check that $f_1 \in k[V]^{N_1}$. Let

$$d_1 = \delta_{h_3}(f_1) = (x_2^p - x_1^{p-1}x_2)(x_3^p - x_1^{p-1}x_3)$$

we will show that $\delta_{h_3}k[V]^{N_1} \subseteq d_1k[V]^{N_1}$. To do this we look at the action of h_3 on a ring containing $k[V]^{N_1}$, let

$$H_1 = \mathrm{Nak}_B^-(N_1) = \langle g_1, g_2, h_1, h_2 \rangle$$

then

$$k[V]^{H_1} = k[\mathbf{N}_1^{H_1}, \mathbf{N}_2^{H_1}, \ldots, \mathbf{N}_6^{H_1}] \supseteq k[V]^{N_1},$$

where

$$\begin{aligned}
\mathbf{N}_1^{H_1} &= x_1 \\
\mathbf{N}_2^{H_1} &= x_2 \\
\mathbf{N}_3^{H_1} &= x_3 \\
\mathbf{N}_4^{H_1} &= x_4^p - x_1^{p-1}x_4 \\
\mathbf{N}_5^{H_1} &= x_5^p - x_1^{p-1}x_5 \\
\mathbf{N}_6^{H_1} &= (x_6^p - x_3^{p-1}x_6)^p - (x_2^p - x_3^{p-1}x_2)^{p-1}(x_6^p - x_3^{p-1}x_6).
\end{aligned}$$

This means that we can write any $f \in k[V]^{N_1}$ in terms of $\mathbf{N}_1^{H_1}, \mathbf{N}_2^{H_1}, \ldots, \mathbf{N}_6^{H_1}$. We look at what h_3 does to these orbit products:

$$\begin{aligned}
&\delta_{h_3}(\mathbf{N}_i^{H_1}) = \delta_{h_3}(\mathbf{N}_6^{H_1}) = 0 \text{ for } 1 \leq i \leq 4 \\
&\delta_{h_3}(\mathbf{N}_5^{H_1}) = x_2^p - x_1^{p-1}x_2.
\end{aligned}$$

So any $f \in k[V]^{N_1}$ can be written as a polynomial in N_5 with coefficients in $k[V]^{h_3}$, and so $x_2^p - x_1^{p-1}x_2$ must divide $\delta_{h_3}k[V]^{N_0}$.

Let

$$\sigma_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

so $\sigma_3 h_3 = g_3$. If $f \in k[V]^{N_1}$ then

$$\begin{aligned} g_3(f) = \sigma_3 h_3(f) &= f, \\ h_3(f) &= \sigma_3^{-1}(f), \end{aligned}$$

so $\delta_{h_3} k[V]^{N_1} = \delta_{\sigma_3} k[V]^{N_1}$. We find

$$\begin{aligned} &\delta_{\sigma_3}(\mathbf{N}_i^{H_1}) = \delta_{\sigma_3}(\mathbf{N}_5^{H_1}) = \delta_{\sigma_3}(\mathbf{N}_6^{H_1}) = 0 \text{ for } 1 \leq i \leq 3, \\ &\delta_{\sigma_3}(\mathbf{N}_4^{H_1}) = x_3^p - x_1^{p-1}x_3 \end{aligned}$$

and so $x_3^p - x_1^{p-1}x_3$ divides $\delta_{h_3} k[V]^{N_1}$. Since $x_2^p - x_1^{p-1}x_2$ and $x_3^p - x_1^{p-1}x_3$ have no common factors and $k[V]^{N_1}$ is a unique factorisation domain

$$d_1 = (x_2^p - x_1^{p-1}x_2)(x_3^p - x_1^{p-1}x_3)$$

must divide $\delta_{h_3}(f)$ for any $f \in k[V]^{N_1}$. This means that $\delta_{h_3} k[V]^{N_1} \subseteq d_1 k[V]^{N_1}$ and so

$$k[V]^{N_1} = k[V]^{N_0}[f_1]. \qquad \square$$

Lemma 6.2.2. *$k[V]^{N_2} = k[V]^{N_1}[f_2]$ where*

$$\begin{aligned} f_2 = (x_5^p - x_1^{p-1}x_5)x_3 &+ (x_6^p - x_3^{p-1}x_6)x_1 \\ &- x_4(x_2^p - x_3^{p-1}x_2) - x_4(x_2^p - x_1^{p-1}x_2) - x_2(x_4^p - x_2^{p-1}x_4). \end{aligned}$$

Proof. Similarly to above we want to use Theorem 1.3.4 to show that

$$k[V]^{N_2} = k[V]^{N_1}[f_2].$$

Again it is easy to check that $f_2 \in k[V]^{N_2}$. Let

$$d_2 = \delta_{h_2}(f_2) = (x_2^p - x_3^{p-1}x_2)x_1$$

we will show that $\delta_{h_3}k[V]^{N_2} \subseteq d_2k[V]^{N_2}$. To do this we look at the action of h_2 on a ring containing $k[V]^{N_2}$: let

$$H_2 = \mathrm{Nak}^-(N_2) = \langle g_1, h_1\rangle$$

then

$$k[V]^{H_2} = k[\mathbf{N}_1^{H_2}, \mathbf{N}_2^{H_2}, \ldots, \mathbf{N}_6^{H_2}] \supseteq k[V]^{N_2},$$

where

$$\begin{aligned}
\mathbf{N}_1^{H_2} &= x_1, & \mathbf{N}_2^{H_2} &= x_2, & \mathbf{N}_3^{H_2} &= x_3,\\
\mathbf{N}_4^{H_2} &= x_4, & \mathbf{N}_5^{H_2} &= x_5^p - x_1^{p-1}x_5, & \mathbf{N}_6^{H_2} &= x_6^p - x_3^{p-1}x_6.
\end{aligned}$$

We find that

$$\begin{aligned}
&\delta_{h_2}(\mathbf{N}_i^{H_2}) = 0 \text{ for } 1 \le i \le 5\\
&\delta_{h_2}(\mathbf{N}_6^{H_2}) = x_2^p - x_3^{p-1}x_2.
\end{aligned}$$

so $x_2^p - x_3^{p-1}x_2$ must divide $\delta_{h_2}k[V]^{N_2}$.

Let

$$\sigma_2 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

so $\sigma_2 h_2 = g_2$. If $f \in k[V]^{N_2}$ then

$$\begin{aligned} g_2(f) = \sigma_2 h_2(f) &= f, \\ h_2(f) &= \sigma_2^{-1}(f). \end{aligned}$$

so $\delta_{h_2} k[V]^{N_2} = \delta_{\sigma_2} k[V]^{N_2}$. We find

$$\begin{aligned} &\delta_{\sigma_2}(\mathbf{N}_i^{H_2}) = \delta_{\sigma_2}(\mathbf{N}_5^{H_2}) = \delta_{\sigma_2}(\mathbf{N}_6^{H_2}) = 0 \text{ for } 1 \leq i \leq 3 \\ &\delta_{\sigma_2}(\mathbf{N}_4^{H_2}) = x_1 \end{aligned}$$

and so x_1 divides $\delta_{h_2} k[V]^{N_2}$. Since $x_2^p - x_1^{p-1}x_2$ and x_1 have no common factors and $k[V]^{N_2}$ is a unique factorisation domain $d_2 = (x_2^p - x_1^{p-1}x_2)x_1$ must divide $\delta_{h_2}(f)$ for any $f \in k[V]^{N_2}$. This means that $\delta_{h_2} k[V]^{N_2} \subseteq d_2 k[V]^{N_2}$ and so

$$k[V]^{N_2} = k[V]^{N_1}[f_2]. \qquad \square$$

Proposition 6.2.3. $k[V]^G = k[\mathbf{N}_1, \ldots, \mathbf{N}_6, f_1, f_2, f_3]$ *where*

$$f_3 = x_1x_6 - x_2x_4 + x_3x_5.$$

Proof. As in the previous two Lemmas we want to show that

$$k[V]^G = k[V]^{N_2}[f_3].$$

We check $f_3 \in k[V]^G$. Let

$$d_3 = \delta_{h_1}(f_3) = x_1 x_3$$

we will show that $\delta_{h_1} k[V]^G \subseteq d_3 k[V]^G$.

$$\begin{aligned} &\delta_{h_1}(x_i) = 0 \text{ for } 1 \leq i \leq 5 \\ &\delta_{h_1}(x_6) = x_3. \end{aligned}$$

so x_3 must divide $\delta_{h_1} k[V]^G$.

Let

$$\sigma_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

so $\sigma_1 h_1 = g_1$. If $f \in k[V]^G$ then

$$\begin{aligned} g_1(f) =& \sigma_1 h_1(f) = f, \\ & h_1(f) = \sigma_1^{-1}(f). \end{aligned}$$

so $\delta_{h_1} k[V]^G = \delta_{\sigma_1} k[V]^G$. We find

$$\begin{aligned} &\delta_{\sigma_1}(x_i) = \delta_{\sigma_1}(x_6) = 0 \text{ for } 1 \leq i \leq 3, \\ &\delta_{\sigma_1}(x_5) = -x_1, \end{aligned}$$

and so x_1 divides $\delta_{h_1} k[V]^G$. Since x_3 and x_1 have no common factors and $k[V]^G$ is a unique factorisation domain $d_3 = x_1 x_3$ must divide $\delta_{h_1}(f)$ for any $f \in k[V]^{N_2}$.

This means that $\delta_{h_1} k[V]^G \subseteq d_3 k[V]^G$ and so

$$k[V]^G = k[V]^{N_2}[f_3].$$ □

We use that any exceptional group of type two generated by an exceptional triple is congruent to G to find the next result.

Proposition 6.2.4. *If E is an exceptional group of type two generated by an exceptional triple then $k[V]^E$ is a complete intersection ring. For $k = \mathbb{F}_p$ all exceptional groups of type two have complete intersection invariant rings.*

6.3 Proof of Theorem 1.0.6 and open problems

Combining the results of this Chapter with those of Chapter 4 gives us some information about when the invariant rings of pure unipotent bireflection groups are Cohen-Macaulay or complete intersection rings. We are finally able to prove Theorem 1.0.6.

Proof. (of Theorem 1.0.6). By Theorem 1.0.5 we know that if G is a unipotent group consisting of bireflections then it is either a two–row group, two–column group, hook group or an exceptional group (of type one or two). By [33] Theorem 3.2.1 if G is an abelian reflection two-column group then $k[V]^G$ is a complete intersection ring. By Theorem 4.2.8 if G is a nice hook group then it has complete intersection ring of invariants, and by Proposition 4.2.2 if G is not nice then G is abelian and $[G,[G,W] \neq \{0\}$. If G is an exceptional group then it has complete intersection invariant ring by Propositions 6.2.4 and 6.1.7. The only remaining groups are those listed above.

By Theorem 1.4.3 if G is a two–column group then $k[V]^G$ is Cohen–Macaulay and so the only groups which could have non-Cohen–Macaulay invariant rings are two–row groups or hook groups which are not nice with respect to any basis. □

We end with a brief discussion of problems left to solve, for example finding the invariant ring of the group in Chapter 5 for $p \neq 3$. Is this a special case or are there many more invariant rings which are Cohen–Macaulay and not complete intersection rings? Are there examples for $p = 2$? Suppose $f_1, \ldots, f_n$ are a HSOP for $k[V]^G$ a Cohen–Macaulay but non complete intersection ring, what does the Poincaré duality algebra $k[V]/(f_1, \ldots, f_n)$ look like in this case?

It would also be interesting to extend the results of Theorem 1.0.6 to other fields. As shown in Example 2.2.12 we would need to start by finding a new classification for k of even characteristic, here the structure of exceptional groups of type one are also quite different. Work on hook groups is still valid for $p = 2$, but more needs to be done to extend from $k = \mathbb{F}_p$ to larger fields.

When G is a p-group which is nice with respect to some basis B, we know that $[G, G]$ is contained within $N^- = \mathrm{Nak}_B^-(G)$. Can we find a set of hook groups $H_i < G$ such that

$$N^- \leq \langle N^-, H_1 \rangle \leq \ldots \leq \langle N^-, H_1, \ldots, H_m \rangle = G,$$

which would help us to find $k[V]^G$ by extending the results of Chapter 4? Firstly though we would need to look at when G is a nice bireflection group but $k[V]^G$ is not Cohen–Macaulay. Suppose that G is nice with respect to some basis B, $N^+ := \mathrm{Nak}_B^+(G)$ and we have a chain of maximal subgroups as in Proposition 3.0.11

$$G = N_0 \lhd_{\max} N_1 \lhd_{\max} N_2 \lhd_{\max} \ldots \lhd_{\max} N_l = N^+.$$

If $k[V]^G$ is Cohen–Macaulay can we always find a chain such that for some $f_i \in k[V]^{N_i}$

$$k[V]^{N_i} = k[V]^{N_{i+1}}[f_i]$$

for $0 \leq i < l$? If this is the case then $k[V]^G$ would either be complete intersection or not Cohen–Macaulay for all groups which are nice with respect to some basis.

References

[1] Aschbacher, M. (1986). Finite group theory.

[2] Becker, T. and Weispfenning, V. (1993). *Gröbner Bases: A Computational Approach to Commutative Algebra*. Springer Verlag.

[3] Benson, D. J. (1993). *Polynomial invariants of finite groups*, volume 190. Cambridge University Press.

[4] Bosma, W., Cannon, J., and Playoust, C. (1997). The Magma algebra system. I. The user language. *J. Symbolic Comput.*, 24(3-4):235–265. Computational algebra and number theory (London, 1993).

[5] Broer, A. (2006). Hypersurfaces in modular invariant theory. *Journal of Algebra*, 306(2):576–590.

[6] Bruns, W. and Herzog, H. J. (1998). *Cohen-Macaulay rings*. Cambridge University Press.

[7] Campbell, H. and Chuai, J. (2007). Invariant fields and localized invariant rings of p-groups. *Quarterly journal of mathematics*, 58(2).

[8] Campbell, H. and Hughes, I. (1999). Rings of invariants of certain p-groups over the field $\mathbb{F}_p$. *Journal of Algebra*, 211(2):549–561.

[9] Campbell, H., Shank, R. J., and Wehlau, D. L. (2011). Rings of invariants for modular representations of elementary abelian p-groups. *arXiv preprint arXiv:1112.0230*.

[10] Campbell, H. and Wehlau, D. L. (2011). Modular invariant theory, encyclopaedia of mathematical sciences, vol. 139.

[11] Chevalley, C. (1955). Invariants of finite groups generated by reflections. *American Journal of Mathematics*, 77(4):778–782.

[12] Coxeter, H. S. (1934). Discrete groups generated by reflections. *Annals of Mathematics*, pages 588–621.

[13] Derksen, H. and Kemper, G. (2002). Computational invariant theory, volume 130 of encyclopaedia of mathematical sciences.

[14] Eisenbud, D. (2013). *Commutative Algebra: with a view toward algebraic geometry*, volume 150. Springer Science & Business Media.

[15] Elmer, J. and Fleischmann, P. (2010). On the depth of modular invariant rings for the groups $C_p \times C_p$. In *Symmetry and spaces*, pages 45–61. Springer.

[16] Fleischmann, P., Kemper, G., and Shank, R. (2005). Depth and cohomological connectivity in modular invariant theory. *Transactions of the American Mathematical Society*, 357(9):3605–3621.

[17] Gorenstein, D. (2007). *Finite groups*, volume 301. American Mathematical Soc.

[18] Guralnick, R. M. and Saxl, J. (2003). Generation of finite almost simple groups by conjugates. *Journal of Algebra*, 268(2):519–571.

[19] Hilbert, D. (1890). Über die theorie der algebraischen formen. *Mathematische annalen*, 36(4):473–534.

[20] Hochster, M. and Eagon, J. A. (1971). Cohen-Macaulay rings, invariant theory, and the generic perfection of determinantal loci. *American Journal of Mathematics*, 93(4):1020–1058.

[21] Hughes, I. and Kechagias, N. (2005). Rings of invariants of modular p-groups which are hypersurfaces. *Journal of Algebra*, 291(1):72–89.

[22] Kemper, G. (1996). Calculating invariant rings of finite groups over arbitrary fields. *Journal of Symbolic Computation*, 21(3):351–366.

[23] Kemper, G. (1999). On the Cohen–Macaulay property of modular invariant rings. *Journal of Algebra*, 215(1):330–351.

[24] Kemper, G. and Malle, G. (1997). The finite irreducible linear groups with polynomial ring of invariants. *Transformation Groups*, 2(1):57–89.

[25] Kemper, G. and Steel, A. (1999). Some algorithms in invariant theory of finite groups. In *Computational methods for representations of groups and algebras*, pages 267–285. Springer.

[26] Nakajima, H. (1983a). Regular rings of invariants of unipotent groups. *Journal of Algebra*, 85(2):253–286.

[27] Nakajima, H. (1983b). Rings of invariants of finite groups which are hypersurfaces. *Journal of Algebra*, 80(2):279–294.

[28] Neusel, M. D. and Smith, L. (2010). *Invariant theory of finite groups*, volume 94. American Mathematical Soc.

[29] Noether, E. (1926). Der endlichkeitssatz der invarianten endlicher linearer gruppen der charakteristik p. *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse*, 1926:28–35.

[30] Shephard, G. C. and Todd, J. A. (1954). Finite unitary reflection groups. *Canad. J. Math*, 6(2):274–301.

[31] Smith, L. (1995). *Polynomial invariants of finite groups.* AK Peters/CRC Press.

[32] Stanley, R. P. (1979). Invariants of finite groups and their applications to combinatorics. *Bulletin of the American Mathematical Society*, 1(3):475–511.

[33] Wu, Y. (2009). *Invariants of Modular Two-Row Groups*. PhD thesis, Queen's University. https://qspace.library.queensu.ca/bitstream/1974/5258/1/Wu_Yinglin_200909_PhD.pdf.