

Painlevé Equations and Orthogonal Polynomials

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Abstract

In this thesis we classify all of the special function solutions to Painlevé equations and all their associated equations produced using their Hamiltonian structures. We then use these special solutions to highlight the connection between the Painlevé equations and the coefficients of some three-term recurrence relations for some specific orthogonal polynomials. The key idea of this newly developed method is the recognition of certain orthogonal polynomial moments as a particular special function. This means we can compare the matrix of moments with the Wronskian solutions, which the Painlevé equations are famous for. Once this connection is found we can simply read off the all important recurrence coefficients in a closed form. In certain cases, we can even improve upon this as some of the weights allow a simplification of the recurrence coefficients to polynomials and with it, the new sequences orthogonal polynomials are simplified too.

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise.

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1 Introduction

1.1 General Introduction

In this thesis we will discuss the following orthogonal polynomials: various deformed Laguerre polynomials, Pollaczek-Jacobi polynomials, time-dependent Jacobi polynomials, some polynomials on the unit circle and some deformed Jacobi polynomials. Once the logarithmic derivative of Hankel and Wronskian determinants are taken it can be compared directly to one of the Painlevé equations special function solutions [17]. Our goal in this thesis is to explore a new method of computing the recurrence coefficients for specific orthogonal polynomial weights using the comparison we mentioned above.

Some of the solutions to the Painlevé equations comprise of classical special functions, such as the Bessel functions, the Airy function, the Legendre functions and the confluent hypergeometric function. Recently [18] there has been much interest in the relationship between semi-classical orthogonal polynomials and these integrable equations. This relationship dates back to work by Shohat [62] in 1939. It took until 1995, in a paper by Magnus [44], to establish that these integrable equations were actually Painlevé equations. These relationships extend to many of the Painlevé equations. For example, see [1, 2, 7, 11, 17, 18, 21, 23, 25, 69]. The purpose of this thesis is to explore and clarify this connection.

The thesis is organised into the following sections:

- Introduction to Painlevé and all the material we will be using, including the vital Hamiltonian structures.
- Introduction to the special function solutions and how they are related to

the Painlevé equations $P_{II} - P_{VI}$.

- Introduce the idea of rational function solutions. Some of these solutions can be shown to be special cases of the special function solutions which we will see in more detail later.
- The applications of the special function and rational function solutions and how they relate explicitly to certain orthogonal polynomials. This is chapters 5, 6, 7 and 8 of the thesis and contains most of the original work. We will be applying a new method to some previously known orthogonal polynomial weights.

1.2 Painlevé equations

The six Painlevé equations ($P_I - P_{VI}$) were first discovered approximately 100 years ago by Painlevé and his colleagues whilst investigating ordinary differential equations of the form

$$\frac{d^2w}{dz^2} = F\left(z; w, \frac{dw}{dz}\right),$$

where F is rational in $\frac{dw}{dz}$ and w is analytic in z . They possess the property that their solutions have no movable essential singularities. Alternatively, the locations of multi-valued singularities of any of the solutions are independent of the particular solutions chosen and so are dependent only on the equation. This is now known to be the *Painlevé property*. Painlevé, Gambier and their colleagues managed to show that there are 50 canonical equations with this property up to a Möbius (bilinear rational) transformation

$$W(\zeta) = \frac{a(z)w + b(z)}{c(z)w + d(z)}, \quad \zeta = \phi(z),$$

where $a(z)$, $b(z)$, $c(z)$, $d(z)$ and $\phi(z)$ are locally analytic functions. Contained inside these 50 equations are the six Painlevé equations. The remaining 44 equations can either be reduced to linear equations and solved in terms of elliptic functions, or can be reduced to ordinary differential equations satisfied by the transcendental solutions. The solutions of $(P_I - P_{VI})$ are called the Painlevé transcendents and the general solutions of the Painlevé equations are transcendental. This means they are irreducible; they cannot be expressed in terms of previously known functions, such as rational functions, elliptic functions or special functions. The Painlevé equations have a plethora of interesting properties, some of which will be investigated in this thesis. Some of these properties include:

- Bäcklund transformations. These transformations relate one solution (from within a hierarchy) to another solution.
- Special function solutions (which are also known as one parameter solutions). Painlevé equations can be thought of as nonlinear analogues of the classical special function solutions and these solutions play a vital role in this thesis.
- Rational function solutions are found for $P_{II} - P_{VI}$ and are sometimes formed as a subset of the special function solutions. However, this is not the case for all Painlevé equations. For example, P_{IV} (1.1d) has its rational function solutions formed as a subset of the special function solutions of P_{VI} (1.1d). However, P_{III} (1.1c) has some rational solutions that cannot be produced from the special function solutions of P_{III} (1.1c).
- Painlevé asymptotics. These leading order asymptotics are a useful way for determining (from an applied point of view) which equation a suspected

exact Painlevé solution belongs to and where exactly it appears in the hierarchy.

The Painlevé equations arise in a large number of applications, for example; random matrix theory, the asymptotic theory of orthogonal polynomials, self-similar solutions of integrable equations, tiling problems as well as many more [20]. The six Painlevé equations ($P_I - P_{VI}$) are the nonlinear ordinary differential equations defined below.

$$\frac{d^2w}{dz^2} = 6w^2 + z, \quad (1.1a)$$

$$\frac{d^2w}{dz^2} = 2w^3 + zw + A, \quad (1.1b)$$

$$\frac{d^2w}{dz^2} = \frac{1}{w} \left(\frac{dw}{dz} \right)^2 - \frac{1}{z} \frac{dw}{dz} + \frac{Aw^2 + B}{z} + Cw^3 + \frac{D}{w}, \quad (1.1c)$$

$$\frac{d^2w}{dz^2} = \frac{1}{2w} \left(\frac{dw}{dz} \right)^2 + \frac{3}{2}w^3 + 4zw^2 + 2(z^2 - A)w + \frac{B}{w}, \quad (1.1d)$$

$$\begin{aligned} \frac{d^2w}{dz^2} = & \left(\frac{1}{2w} + \frac{1}{w-1} \right) \left(\frac{dw}{dz} \right)^2 - \frac{1}{z} \frac{dw}{dz} + \frac{(w-1)^2}{z^2} \left(Aw + \frac{B}{w} \right) \\ & + \frac{Cw}{z} + \frac{Dw(w+1)}{w-1}, \end{aligned} \quad (1.1e)$$

$$\begin{aligned} \frac{d^2w}{dz^2} = & \frac{1}{2} \left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-z} \right) \left(\frac{dw}{dz} \right)^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{w-z} \right) \frac{dw}{dz} \\ & + \frac{w(w-1)(w-z)}{z^2(z-1)^2} \left\{ A + \frac{Bz}{w^2} + \frac{C(z-1)}{(w-1)^2} + \frac{Dz(z-1)}{(w-z)^2} \right\}. \end{aligned} \quad (1.1f)$$

where A, B, C and D are arbitrary constants.

1.3 Hamiltonian structure

Each Painlevé equation has its own Hamiltonian structure and system to explore. The Painlevé system associated with P_J is, by definition, the Hamiltonian system

$$\delta_J q = \frac{\partial \mathcal{H}_J}{\partial p}, \quad \delta_J p = -\frac{\partial \mathcal{H}_J}{\partial q}, \quad (1.2)$$

where δ is the operator associated with that Painlevé equation for a unique Hamiltonian function \mathcal{H}_J [57] and is given for each case by

$$\delta = \frac{d}{dz} \quad \text{for } J = \text{I, II, IV}, \quad (1.3)$$

$$\delta = z \frac{d}{dz} \quad \text{for } J = \text{III, V}, \quad (1.4)$$

$$\delta = z(z-1) \frac{d}{dz} \quad \text{for } J = \text{VI}, \quad (1.5)$$

where the Hamiltonian functions \mathcal{H}_J are given by

$$\mathcal{H}_I(q, p, z) = \frac{1}{2}p^2 - 2q^3 - zq, \quad (1.6a)$$

$$\mathcal{H}_{II}(q, p, z) = \frac{1}{2}p^2 - (q^2 + \frac{1}{2}z)p - (\alpha + \frac{1}{2})q, \quad (1.6b)$$

$$\mathcal{H}_{III}(q, p, z) = q^2 p^2 - zp q^2 - (\lambda_0 - 1)pq + zp + \frac{1}{2}(\lambda_0 - 2 - \lambda_\infty)zq, \quad (1.6c)$$

$$\mathcal{H}_{IV}(q, p, z) = 2qp^2 - (q^2 + 2zq + 2\kappa_0)p + \kappa_\infty q, \quad (1.6d)$$

$$\begin{aligned} \mathcal{H}_V(q, p, z) &= q(q-1)^2 p^2 - \{(\beta + \vartheta)q^2 - (2\beta + \vartheta - z)q + \beta\}p \\ &\quad - \frac{1}{4}\{\alpha^2 - (\beta + \vartheta)^2\}q, \end{aligned} \quad (1.6e)$$

$$\begin{aligned} \mathcal{H}_{VI}(q, p, z) &= q(q-1)(q-z)p^2 - \{\vartheta_4(q-1)(q-z) + \vartheta_3 q(q-z) \\ &\quad + (\vartheta_0 - 1)q(q-1)\}p + \vartheta_2(\vartheta_1 + \vartheta_2)(q-z), \end{aligned} \quad (1.6f)$$

where $\alpha, \beta, \vartheta, \vartheta_0, \vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_\infty, \lambda_0, \lambda_\infty, \kappa_0$ and κ_∞ are arbitrary constants. To elaborate, the function $\sigma_n = \mathcal{H}_J + L(z)$ where $L(z)$ is a linear correction term

will satisfy a non-linear second-order, second-degree ordinary differential equation (ODE), often referred to as the Painlevé σ -equation.

It is also interesting to note that P_V (1.1e) has the option of using either the delta operator as $\delta = \frac{d}{dz}$ or $\delta = z\frac{d}{dz}$. We will discuss this unique feature of P_V (1.1e) in chapter 2.

Remark 1.1. *Each Hamiltonian function $\sigma = H_J$ satisfies a second-order second-degree ordinary differential equation whose solutions are in a correspondence with solutions of the associated Painlevé equation through (1.2) since*

$$q = F_J\left(\sigma, \frac{d\sigma}{dz}, \frac{d^2\sigma}{dz^2}, z\right), \quad p = G_J\left(\sigma, \frac{d\sigma}{dz}, \frac{d^2\sigma}{dz^2}, z\right),$$

for suitable functions $F_J(\sigma, \frac{d\sigma}{dz}, \frac{d^2\sigma}{dz^2}, z)$ and $G_J(\sigma, \frac{d\sigma}{dz}, \frac{d^2\sigma}{dz^2}, z)$. Thus, given q and p , one can determine σ and conversely, given σ , one can determine q and p . This will be shown in detail later.

The six Painlevé σ -equations ($S_I - S_{VI}$) are the nonlinear ordinary differential

equations defined below:

$$\left(\frac{d^2\sigma}{dz^2}\right)^2 + 4\left(\frac{d\sigma}{dz}\right)^3 + 2z\frac{d\sigma}{dz} - 2\sigma = 0, \quad (1.7a)$$

$$\left(\frac{d^2\sigma}{dz^2}\right)^2 + 4\left(\frac{d\sigma}{dz}\right)^3 + 2\frac{d\sigma}{dz}\left(z\frac{d\sigma}{dz} - \sigma\right) = \frac{1}{4}(\alpha + \frac{1}{2})^2, \quad (1.7b)$$

$$\left(z\frac{d^2\sigma}{dz^2} - \frac{d\sigma}{dz}\right)^2 + 4\left(\frac{d\sigma}{dz}\right)^2\left(z\frac{d\sigma}{dz} - 2\sigma\right) + 4z\vartheta_\infty\frac{d\sigma}{dz} = z^2\left(z\frac{d\sigma}{dz} - 2\sigma + 2\vartheta_0\right), \quad (1.7c)$$

$$\left(\frac{d^2\sigma}{dz^2}\right)^2 - 4\left(z\frac{d\sigma}{dz} - \sigma\right) + 4\left(\frac{d\sigma}{dz} + 2\vartheta_0\right)\left(\frac{d\sigma}{dz} + 2\vartheta_\infty\right) = 0, \quad (1.7d)$$

$$\left(z\frac{d^2\sigma}{dz^2}\right)^2 = \left[2\left(z\frac{d\sigma}{dz}\right)^2 - z\frac{d\sigma}{dz} + \sigma\right]^2 - 4\prod_{j=0}^3\left(\frac{d\sigma}{dz} + \kappa_j\right), \quad (1.7e)$$

$$\frac{d\sigma}{dz}\left(z(z-1)\frac{d^2\sigma}{dz^2}\right)^2 + \left(\frac{d\sigma}{dz}\left\{2\sigma - (2z-1)\frac{d\sigma}{dz}\right\} + \kappa_1\kappa_2\kappa_3\kappa_4\right)^2 = \prod_{j=1}^4\left(\frac{d\sigma}{dz} + \kappa_j^2\right), \quad (1.7f)$$

where $\beta, \vartheta_0, \vartheta_\infty$ and $\kappa_0, \dots, \kappa_4$ are arbitrary constants.

In the following sections we will derive all of the σ -equations ($S_I - S_{VI}$) that we will need in this thesis. The Hamiltonian functions $\sigma = H_J$ frequently arise in applications, e.g: random matrix theory and orthogonal polynomials. It is this connection with the orthogonal polynomial applications which make the derivations of each σ -equation fundamentally important. All of the calculation in the following section can be found on the USB flash drive in its appropriate folder.

1.3.1 Hamiltonian structure for the first Painlevé equation P_I

The Hamiltonian associated with P_I (1.1a) is

$$\mathcal{H}_I(q, p, z) = \frac{1}{2}p^2 - 2q^3 - zq, \quad (1.8)$$

where Hamilton's equations (1.2) yield the following system, which p and q satisfy:

$$\frac{dq}{dz} = p, \quad (1.9a)$$

$$\frac{dp}{dz} = 6q^2 + z. \quad (1.9b)$$

Eliminating p in (1.9a) then $q = w$ satisfies P_I (1.1a).

Theorem 1.1. *The Hamiltonian function*

$$\sigma(z) = \mathcal{H}_I(q, p, z), \quad (1.10)$$

with $\mathcal{H}_I(q, p, z)$ given by (1.8), satisfies the second-order, second-degree equation

$$\left(\frac{d^2\sigma}{dz^2}\right)^2 + 4\left(\frac{d\sigma}{dz}\right)^3 + 2z\frac{d\sigma}{dz} - 2\sigma = 0,$$

which is S_I (1.7a). Conversely, if $\sigma(z)$ satisfies S_I (1.7a) then the solutions of the Hamiltonian system (1.9) are given by

$$q(z) = -\sigma', \quad p(z) = -\sigma'', \quad ' = \frac{d}{dz}. \quad (1.11)$$

Proof. Substituting (1.8) into (1.10) and differentiating twice followed by substituting (1.9a) and (1.9b) where possible yields

$$\frac{d\sigma}{dz} = -q, \quad (1.12a)$$

$$\frac{d^2\sigma}{dz^2} = -p. \quad (1.12b)$$

Then, solving (1.12a) and (1.12b) simultaneously gives (1.11). Substituting (1.11) into H_I (1.8) we can generate S_I (1.7a) multiplied by some other expression. Also see Okamoto [54, 59] and Forrester and Witte [27]. \square

1.3.2 Hamiltonian structure for the second Painlevé equation P_{II}

The Hamiltonian associated with P_{II} (1.1b) is the following:

$$\mathcal{H}_{II}(q, p, z) = \frac{1}{2}p^2 - (q^2 + \frac{1}{2}z)p - (\alpha + \frac{1}{2})q, \quad (1.13)$$

where Hamilton's equations (1.2) yield the following system, which p and q satisfy:

$$\frac{dq}{dz} = p - q^2 - \frac{1}{2}z, \quad (1.14a)$$

$$\frac{dp}{dz} = 2qp + \alpha + \frac{1}{2}. \quad (1.14b)$$

Eliminating p in (1.14a) then $q = w$ satisfies P_{II} (1.1b). Whilst eliminating q yields

$$p \frac{d^2 p}{dz^2} = \frac{1}{2} \left(\frac{dp}{dz} \right)^2 + 2p^3 - 2p^2 - \frac{1}{2}(\alpha + \frac{1}{2})^2, \quad (1.15)$$

which is known as P_{34} .

Theorem 1.2. *The Hamiltonian function*

$$\sigma(z) = \mathcal{H}_{II}(q, p, z), \quad (1.16)$$

with $\mathcal{H}_{II}(q, p, z)$ given by (1.13), satisfies a second-order, second-degree equation

$$\left(\frac{d^2 \sigma}{dz^2} \right)^2 + 4 \left(\frac{d\sigma}{dz} \right)^3 + 2 \frac{d\sigma}{dz} \left(z \frac{d\sigma}{dz} - \sigma \right) = \frac{1}{4}(\alpha + \frac{1}{2})^2,$$

which is S_{II} (1.7b). Conversely, if $\sigma(z; \alpha)$ satisfies S_{II} (1.7b) then the solutions of the Hamiltonian system (1.14) are given by

$$q(z) = \frac{4\sigma'' + 2\alpha + 1}{8\sigma'}, \quad p(z) = -2\sigma', \quad ' = \frac{d}{dz}. \quad (1.17)$$

Proof. Substituting (1.13) into (1.16) and differentiating twice followed by substituting (1.14a) and (1.14b) where possible yields

$$\frac{d\sigma}{dz} = -\frac{1}{2}p, \quad (1.18a)$$

$$\frac{d^2\sigma}{dz^2} = -qp - \frac{1}{2}\alpha - \frac{1}{4}. \quad (1.18b)$$

Then, solving (1.18a) and (1.18b) simultaneously gives (1.17). Substituting (1.17) into H_{II} (1.13) we can generate S_{II} (1.7b) multiplied by some other expression. Also see Okamoto [54, 59] and Forrester and Witte [27]. \square

1.3.3 Hamiltonian structure for the third Painlevé equation P_{III}

The Hamiltonian associated with P_{III} (1.1c) is the following:

$$\mathcal{H}_{III}(q, p, z) = q^2p^2 - zpq^2 - (\lambda_0 - 1)qp + zp + \frac{1}{2}(\lambda_0 - 2 - \lambda_\infty)zq, \quad (1.19)$$

with λ_0 and λ_∞ parameters, where Hamilton's equations (1.2) yield the following system, which p and q satisfy:

$$z \frac{dq}{dz} = 2pq^2 - zq^2 - (\lambda_0 - 1)q + z, \quad (1.20a)$$

$$z \frac{dp}{dz} = -2qp^2 + 2zpq + (\lambda_0 - 1)p - \frac{1}{2}(\lambda_0 - 2 - \lambda_\infty)z. \quad (1.20b)$$

See Okamoto [53, 59]. Eliminating p in (1.20a) then $q = w$ satisfies P_{III} (1.1c).

Theorem 1.3. *The Hamiltonian function*

$$\sigma(z; \lambda_0, \lambda_\infty) = \frac{1}{2}\mathcal{H}_{III}(q, p, z) + \frac{1}{2}pq + \frac{1}{8}(\lambda_0 - 2)^2 - \frac{1}{4}z^2, \quad (1.21)$$

with $\mathcal{H}_{III}(q, p, z)$ given by (1.19), satisfies the second-order, second-degree equation

$$\left(z \frac{d^2\sigma}{dz^2} - \frac{d\sigma}{dz} \right)^2 + 4 \left(\frac{d\sigma}{dz} \right)^2 \left(z \frac{d\sigma}{dz} - 2\sigma \right) + 4z\vartheta_\infty \frac{d\sigma}{dz} = z^2 \left(z \frac{d\sigma}{dz} - 2\sigma + 2\vartheta_0 \right),$$

which is S_{III} (1.7c) with the parameters

$$\{\vartheta_0, \vartheta_\infty\} = \{-\frac{1}{4}\lambda_\infty(\lambda_0 - 2), \frac{1}{8}(\lambda_\infty^2 + (\lambda_0 - 2)^2)\}.$$

Conversely, if $\sigma(z; \lambda_0, \lambda_\infty)$ satisfies S_{III} (1.7c) then the solutions of the Hamiltonian system (1.20) are given by

$$q(z) = \frac{2z\sigma'' + 2(1 - \lambda_0)\sigma' - \lambda_\infty z}{z^2 - 4(\sigma')^2}, \quad p(z) = \sigma' + \frac{1}{2}z, \quad ' = \frac{d}{dz}. \quad (1.22)$$

Proof. Substituting (1.19) into (1.21) and differentiating twice followed by substituting (1.20a) and (1.20b) where possible yields

$$\frac{d\sigma}{dz} = p - \frac{1}{2}z, \quad (1.23a)$$

$$\frac{d^2\sigma}{dz^2} = \frac{1}{2z}(4qp(z - p) + 2p\lambda_0 - z(\lambda_0 - \lambda_\infty - 1) - 2p). \quad (1.23b)$$

Then, solving (1.23a) and (1.23b) simultaneously gives (1.22). Substituting (1.22) into H_{III} (1.19) we can generate S_{III} (1.7c) multiplied by some other expression. Also see Okamoto [55, 59] and Forrester and Witte [27]. \square

1.3.4 Hamiltonian structure for the third Painlevé equation $P_{III'}$

An alternative form of P_{III} (1.1c), due to Okamoto [54, 55, 59], is obtained by making the transformation $w(z) = u(t)/\sqrt{t}$, with $t = \frac{1}{4}z^2$ in P_{III} (1.1c) giving

$$\frac{d^2u}{dt^2} = \frac{1}{u} \left(\frac{du}{dt} \right)^2 - \frac{1}{t} \frac{du}{dt} + \frac{u^2}{2t^2}(A + 2u) + \frac{B}{2t} - \frac{1}{u}, \quad (1.24)$$

which is well known to be $P_{III'}$ (1.24). The Hamiltonian associated with $P_{III'}$ (1.24) is the following:

$$\mathcal{H}_{III'}(q, p, t) = q^2 p^2 - (q^2 + \vartheta_0 q - t)p + \frac{1}{2}(\vartheta_0 + \vartheta_\infty)q, \quad (1.25)$$

with ϑ_0 and ϑ_∞ parameters, where Hamilton's equations (1.2) yield the following system, which p and q satisfy:

$$t \frac{dq}{dt} = 2q^2 p - q^2 - \vartheta_0 q + t, \quad (1.26a)$$

$$t \frac{dp}{dt} = -2qp^2 + 2qp + \vartheta_0 p - \frac{1}{2}(\vartheta_0 + \vartheta_\infty). \quad (1.26b)$$

See Okamoto [54, 55, 59]. Eliminating p in (1.26a) then $q = w$ satisfies P_{III} (1.1c) with parameters $(A, B) = (-2\vartheta_\infty, 2(\vartheta_0 + 1))$. Eliminating q in (1.26b) then p satisfies

$$\begin{aligned} \frac{d^2 p}{dt^2} = & \frac{1}{2} \left(\frac{1}{p} + \frac{1}{p-1} \right) \left(\frac{dp}{dt} \right)^2 - \frac{1}{t} \frac{dp}{dt} - \frac{2p(p-1)}{t} \\ & + \frac{1}{8t^2} \left\{ 4\vartheta_0 \vartheta_\infty - \frac{(\vartheta_0 + \vartheta_\infty)^2}{p} - \frac{(\vartheta_0 - \vartheta_\infty)^2}{p-1} \right\}. \end{aligned} \quad (1.27)$$

Making the transformation $p(t) = 1/[1 - w(z)]$, with $z = t$ in (1.27) yields

$$\frac{d^2 w}{dz^2} = \left(\frac{1}{2w} + \frac{1}{w-1} \right) \left(\frac{dw}{dz} \right)^2 - \frac{1}{z} \frac{dw}{dz} + \frac{(w-1)^2}{z^2} \left\{ \frac{(\vartheta_0 + \vartheta_\infty)^2 w}{8} - \frac{(\vartheta_0 - \vartheta_\infty)^2}{8w} \right\} - \frac{2w}{z},$$

which is P_V (1.1e) with parameters

$$\{A, B, C, D\} = \left\{ \frac{1}{8}(\vartheta_0 + \vartheta_\infty)^2, -\frac{1}{8}(\vartheta_0 - \vartheta_\infty)^2, -2, 0 \right\}.$$

This is precisely the well known connection between P_{III} (1.24) and P_V (1.1e) when $D = 0$.

Theorem 1.4. *The Hamiltonian function*

$$\sigma(z; \vartheta_0, \vartheta_\infty) = t \mathcal{H}_{III'}(q, p, t) - \frac{1}{2}t + \frac{1}{4}\vartheta_0^2, \quad (1.28)$$

with $\mathcal{H}_{III'}(q, p, z)$ given by (1.25) satisfies the second-order, second-degree equation

$$\left(t \frac{d^2 \sigma}{dt^2} \right)^2 + \left\{ 4 \left(\frac{d\sigma}{dt} \right)^2 - 1 \right\} \left(t \frac{d\sigma}{dt} - \sigma \right) + \vartheta_0 \vartheta_\infty \frac{d\sigma}{dt} = \frac{1}{4}(\vartheta_0^2 + \vartheta_\infty^2). \quad (1.29)$$

Conversely, if $\sigma(z; \vartheta_0, \vartheta_\infty)$ satisfies $S_{III'}$ (1.29) then the solutions of the Hamiltonian system (1.26) are given by

$$q(t) = \frac{2t\sigma'' - 2\vartheta_0\sigma' + \vartheta_\infty}{1 - 4(\sigma')^2}, \quad p(t) = \sigma' + \frac{1}{2}, \quad ' = \frac{d}{dt}. \quad (1.30)$$

Proof. Substituting (1.25) into (1.28) and differentiating twice followed by substituting (1.26a) and (1.26b) where possible yields

$$\frac{d\sigma}{dt} = p - \frac{1}{2}, \quad (1.31a)$$

$$\frac{d^2\sigma}{dt^2} = \frac{1}{2t}(4qp(1-p) + 2\vartheta_0p - \vartheta_0 - \vartheta_\infty). \quad (1.31b)$$

Then, solving (1.31a) and (1.31b) simultaneously gives (1.30). Substituting (1.30) into $H_{III'}$ (1.25) we can generate $S_{III'}$ (1.29) multiplied by some other expression. Also see Okamoto [55, 59] and Forrester and Witte [26]. \square

1.3.5 Hamiltonian structure for the fourth Painlevé equation P_{IV}

The Hamiltonian associated with P_{IV} (1.1d) is the following:

$$\mathcal{H}_{IV}(q, p, z) = 2qp^2 - (q^2 + 2zq + 2\kappa_0)p + \kappa_\infty q, \quad (1.32)$$

with κ_0, κ_∞ parameters, where Hamilton's equations (1.2) yield the following system, which p and q satisfy:

$$\frac{dq}{dz} = 4qp - q^2 - 2zq - 2\kappa_0, \quad (1.33a)$$

$$\frac{dp}{dz} = -2p^2 + 2pq + 2zp - \kappa_\infty. \quad (1.33b)$$

Eliminating p then $q = w$ satisfies P_{IV} (1.1d) with the following parameters: $\{A, B\} = \{1 - \kappa_0 + 2\kappa_\infty, -2\kappa_0^2\}$. Whilst eliminating q , then $w = -2p$ satisfies P_{IV} (1.1d) with the following parameters: $\{A, B\} = \{2\kappa_0 - \kappa_\infty - 1, -2\kappa_\infty^2\}$. As

in the usual case for Painlevé equations, this Hamiltonian equation satisfies a second-order, second-degree equation.

Theorem 1.5. *The Hamiltonian function*

$$\sigma(z; \kappa_0, \kappa_\infty) = \mathcal{H}_{IV}(q, p, z), \quad (1.34)$$

with $\mathcal{H}_{IV}(q, p, z)$ given by (1.32) satisfies the second-order, second-degree equation

$$\left(\frac{d^2\sigma}{dz^2}\right)^2 - 4\left(z\frac{d\sigma}{dz} - \sigma\right) + 4\left(\frac{d\sigma}{dz} + 2\vartheta_0\right)\left(\frac{d\sigma}{dz} + 2\vartheta_\infty\right) = 0,$$

which is S_{IV} (1.7d) with the parameters

$$\{\vartheta_0, \vartheta_\infty\} = \{\kappa_0, \kappa_\infty\}.$$

Conversely, if $\sigma(z; \kappa_0, \kappa_\infty)$ satisfies S_{IV} (1.7d) then the solutions of the Hamiltonian system (1.33) are given by

$$q(z) = \frac{\sigma'' - 2z\sigma' + 2\sigma}{2(\sigma' + 2\kappa_\infty)}, \quad p(z) = \frac{\sigma'' + 2z\sigma' - 2\sigma}{4(\sigma' + 2\kappa_0)}, \quad ' = \frac{d}{dz}. \quad (1.35)$$

Proof. Substituting (1.32) into (1.34) and differentiating twice followed by substituting (1.33a) and (1.33b) where possible yields

$$\frac{d\sigma}{dz} = -2qp, \quad (1.36a)$$

$$\frac{d^2\sigma}{dz^2} = -4qp^2 + (4\kappa_0 - 2q^2)p + 2\kappa_\infty q. \quad (1.36b)$$

Then, solving (1.32), (1.36a) and (1.36b) simultaneously gives (1.35). Substituting (1.35) into H_{IV} (1.32) we can generate S_{IV} (1.7d) multiplied by some other expression. Also see Jimbo and Miwa [36] and Okamoto [58]. \square

1.3.6 Hamiltonian structure for the fifth Painlevé equation P_V

The Hamiltonian associated with P_V (1.1e) is the following:

$$\mathcal{H}_V(q, p, z) = q(q-1)^2 p^2 - \{(b+\vartheta)q^2 - (2b+\vartheta-z)q+b\}p - \frac{1}{4}\{a^2 - (b+\vartheta)^2\}q, \quad (1.37)$$

with a , b and ϑ as parameters and where Hamilton's equations (1.2) yield the following system, which p and q satisfy:

$$z \frac{dq}{dz} = 2q(q-1)^2 p - (b+\vartheta)q^2 + (2b+\vartheta-z)q - b, \quad (1.38a)$$

$$z \frac{dp}{dz} = -(3q-1)(q-1)p^2 + 2(b+\vartheta)qp - (2b+\vartheta-z)p + \frac{1}{4}\{a^2 - (b+\vartheta)^2\}. \quad (1.38b)$$

Proof. See Jimbo and Miwa [36] Okamoto [54, 55, 57]. \square

Eliminating p then $q = w$ satisfies (1.1e) with the following parameters: $\{A, B, C\} = \{\frac{1}{2}a^2, -\frac{1}{2}b^2, -\vartheta - 1\}$. As in the usual case for Painlevé equations, this Hamiltonian equation satisfies a second-order, second-degree equation.

Theorem 1.6. *The Hamiltonian function*

$$\sigma(z; a, b, \vartheta) = \mathcal{H}_V(q, p, z) + \frac{1}{4}(2b+\vartheta)z - \frac{1}{8}(2b+\vartheta)^2, \quad (1.39)$$

with $\mathcal{H}_V(q, p, z)$ given by (1.37), satisfies the second-order, second-degree equation

$$\left(z \frac{d^2\sigma}{dz^2}\right)^2 = \left[2\left(z \frac{d\sigma}{dz}\right)^2 - z \frac{d\sigma}{dz} + \sigma\right]^2 - 4 \prod_{j=0}^3 \left(\frac{d\sigma}{dz} + \kappa_j\right),$$

which is S_V (1.7e) with the parameters

$$\{\kappa_0, \kappa_1, \kappa_2, \kappa_3\} = \left\{\frac{1}{4}(\vartheta+2a), \frac{1}{4}(\vartheta-2a), -\frac{1}{4}(\vartheta+2b), \frac{1}{4}(2b-\vartheta)\right\}.$$

Conversely, if $\sigma(z; a, b, \vartheta)$ satisfies (1.7e) then the solutions of the Hamiltonian system (1.38) are given by

$$q(z) = \frac{z\sigma'' + 2(\sigma')^2 - z\sigma' + \sigma}{2(\sigma' + \frac{1}{4}\vartheta - \frac{1}{2}a)(\sigma' + \frac{1}{4}\vartheta + \frac{1}{2}a)}, \quad p(z) = \frac{z\sigma'' - 2(\sigma')^2 + z\sigma' - \sigma}{2(\sigma' - \frac{1}{4}\vartheta + \frac{1}{2}b)}, \quad ' = \frac{d}{dz}. \quad (1.40)$$

Proof. Substituting (1.37) into (1.39) and differentiating twice followed by substituting (1.38a) and (1.38b) where possible yields

$$\frac{d\sigma}{dz} = \frac{1}{4}(2b + \vartheta) - qp, \quad (1.41a)$$

$$\frac{d^2\sigma}{dz^2} = \frac{1}{z}(q^3p^2 - (b + \vartheta)q^2p - (\frac{1}{4}(a + b + \vartheta)(a - b - \vartheta) + p^2)q + bp). \quad (1.41b)$$

Solving (1.39), (1.41a) and (1.41b) simultaneously gives (1.40). Substituting (1.40) into H_V (1.37) we can generate S_V (1.7e) multiplied by some other expression. Also see Jimbo and Miwa [36] and Okamoto [55, 59]. \square

1.3.7 Hamiltonian structure for the sixth Painlevé equation P_{VI}

The Hamiltonian associated with P_{VI} (1.1f) is the following:

$$\begin{aligned} \mathcal{H}_{VI}(q, p, z) = & q(q-1)(q-z)p^2 - \{\vartheta_4(q-1)(q-z) + \vartheta_3q(q-z) \\ & + (\vartheta_0 - 1)q(q-1)\}p + \vartheta_2(\vartheta_1 + \vartheta_2)(q-z), \end{aligned} \quad (1.42)$$

where $\vartheta_0, \vartheta_1, \vartheta_2$ and ϑ_3 are parameters and are related in the following way:

$$\vartheta_0 + \vartheta_1 + 2\vartheta_2 + \vartheta_3 + \vartheta_4 = 1,$$

where Hamilton's equations (1.2) yield the following system, which p and q satisfy:

$$z(z-1)\frac{dq}{dz} = -3p^2q^2 + \{(2z+2)p^2 + (2\vartheta_0 + 2\vartheta_3 + 2\vartheta_4 - 2)p\}q - zp^2 - (+z\vartheta_3 + z\vartheta_4 + \vartheta_0 + \vartheta_4 - 1)p - \vartheta_2(\vartheta_1 + \vartheta_2), \quad (1.43a)$$

$$z(z-1)\frac{dp}{dz} = 2pq^3 - (2pz + 2p + \vartheta_0 + \vartheta_3 + \vartheta_4 - 1)q^2 + \{(2p + \vartheta_3 + \vartheta_4)z + \vartheta_0 + \vartheta_4 - 1\}q - z\vartheta_4. \quad (1.43b)$$

Proof. See Jimbo and Miwa [36] and Okamoto [54, 55, 57]. \square

Eliminating p then $q = w$ satisfies (1.1f) with the following parameters:

$$\{A, B, C, D\} = \{\frac{1}{2}\vartheta_1^2, -\frac{1}{2}\vartheta_4^2, \frac{1}{2}\vartheta_3^2, \frac{1}{2}(1 - \vartheta_0^2)\}.$$

As in the usual case for Painlevé equations, this Hamiltonian equation satisfies a second-order, second-degree equation.

Theorem 1.7. *The Hamiltonian function*

$$\sigma(z; \alpha, \beta, \vartheta) = H_{VI} + (\kappa_1\kappa_3 + \kappa_1\kappa_4 + \kappa_3\kappa_4)z - \frac{1}{2}\sum_{1 \leq i < j \leq 4} \kappa_i\kappa_j, \quad (1.44)$$

with $\mathcal{H}_{VI}(q, p, z)$ given by (1.42), satisfies the second-order, second-degree equation

$$\frac{d\sigma}{dz} \left(z(z-1)\frac{d^2\sigma}{dz^2} \right)^2 + \left(\frac{d\sigma}{dz} \left\{ 2\sigma - (2z-1)\frac{d\sigma}{dz} \right\} + \kappa_1\kappa_2\kappa_3\kappa_4 \right)^2 = \prod_{j=1}^4 \left(\frac{d\sigma}{dz} + k_j^2 \right),$$

which is S_{VI} (1.7f) with the parameters

$$\{\kappa_1, \kappa_2, \kappa_3, \kappa_4\} = \{-\frac{1}{2}(\vartheta_3 + \vartheta_4), \frac{1}{2}(\vartheta_4 - \vartheta_3), -\frac{1}{2}(\vartheta_0 + \vartheta_1 - 1), \frac{1}{2}(\vartheta_0 - \vartheta_1 - 1)\}.$$

Conversely, if $\sigma(z; \kappa_1, \kappa_2, \kappa_3, \kappa_4)$ satisfies (1.7e) then the solutions of the Hamiltonian system (1.43) are given by

$$\begin{aligned} q &= \frac{(\kappa_3 + \kappa_4)z(z-1)\sigma'' + 2z(\sigma')^2 - A_1\sigma' + 2\sigma\kappa_1\kappa_2\kappa_3 - \kappa_2A_2}{2\{(\sigma')^2 + (\kappa_3^2 + \kappa_4^2)\sigma' + \kappa_3^2\kappa_4^2\}}, \\ q(q-1)p &= \frac{(\sigma' + B_1)z(z-1)\sigma'' + (2B_2x - B_3)(\sigma')^2 - (B_2\sigma + B_4)\sigma' + B_5}{2\{(\sigma')^2 + (\kappa_3^2 + \kappa_4^2)\sigma' + \kappa_3^2\kappa_4^2\}}, \end{aligned} \quad (1.45)$$

where $' = \frac{d}{dz}$.

Proof. Substituting (1.42) into (1.44) and differentiating twice followed by substituting (1.43b) and (1.43a) where possible yields

$$\frac{d}{dz}\sigma = -q(q-1)p^2 + \{(2q-1)\nu_1 - \nu_2\}p - \nu_1^2. \quad (1.46)$$

If we then compute $\sigma - z\frac{d}{dz}\sigma$, substituting (1.46) where possible yields the first expression here. Differentiating again and substituting (1.43b), (1.43a) and (1.46) where possible yields the second

$$\sigma - z\frac{d\sigma}{dz} = \left(B_1 - \frac{d\sigma}{dz}\right)q - (\kappa_3 + \kappa_4)q(q-1)p - \frac{1}{2}\sum_{1 \leq i < j \leq 4} \kappa_i\kappa_j, \quad (1.47a)$$

$$z(z-1)\frac{d^2\sigma}{dz^2} = 2\left(B_2\frac{d\sigma}{dz} - \kappa_1\kappa_2\kappa_3\right)q + 2\left(\kappa_3\kappa_4 - \frac{d\sigma}{dz}\right)q(q-1)p - B_3\frac{d\sigma}{dz} + C_1. \quad (1.47b)$$

where

$$\begin{aligned}
A_1 &= 2\sigma + 2\kappa_3\kappa_4z + \kappa_1\kappa_2 - \kappa_3^2 - \kappa_3\kappa_4 - \kappa_4^2, \\
A_2 &= \kappa_1^2\kappa_3^2 + \kappa_1^2\kappa_3\kappa_4 + \kappa_1^2\kappa_4^2 + \kappa_1\kappa_3^2\kappa_4 + \kappa_1\kappa_3\kappa_4^2 + \kappa_3^2\kappa_4^2, \\
B_1 &= \kappa_1\kappa_3 + \kappa_1\kappa_4 + \kappa_3\kappa_4, \\
B_2 &= \kappa_1 + \kappa_3 + \kappa_4, \\
B_3 &= \kappa_1 + \kappa_2 + \kappa_3 + \kappa_4, \\
B_4 &= 2\kappa_1\kappa_2\kappa_3 + \kappa_1^2\kappa_2 + \kappa_1\kappa_3\kappa_4 - \kappa_1\kappa_3^2 - \kappa_2\kappa_4^2, \\
B_5 &= 2\sigma\kappa_1\kappa_3\kappa_4 - \kappa_2(\kappa_1^2\kappa_3^2 + \kappa_1^2\kappa_3\kappa_4 + \kappa_1^2\kappa_4^2 + \kappa_1\kappa_3^2\kappa_4 + \kappa_3^2\kappa_4^2), \\
C_1 &= \kappa_4(\kappa_1\kappa_2 + \kappa_1\kappa_3 + \kappa_2\kappa_3) + \kappa_1\kappa_2\kappa_3.
\end{aligned}$$

Solving (1.47a) and (1.47b) simultaneously gives (1.45). Then, solving (1.45) for q and p and substituting these into H_{VI} (1.42) we can generate S_{VI} (1.7f) multiplied by some other expression. Also see [56]. \square

1.3.8 Summary

Now that we have derived all the σ -equations we need to re-classify the special function and rational function solutions that solve both the Painlevé equations ($P_{II} - P_{VI}$) and the σ -equations ($S_{II} - S_{VI}$). This is so that we can compare these results with the Hankel determinants of orthogonal polynomial weights.

2 Special function solutions

2.1 Bounded solutions

The conditions that allow bounded special function solutions of the Painlevé equations are going to be of much interest in this thesis. In the following chapter, we will discuss the conditions that are necessary for bounded solutions with respect to all of the special function solutions to all the Painlevé equations and their associated Hamiltonian equation. The reason we are interested in the bounded special function solutions is because these are the types of solution that arise when discussing the connections to orthogonal polynomials. Generally speaking, the bounded solutions tend to be the solutions that have the physical relevance. Despite the obvious importance of the locations of the bounded special function solutions, to the best of my knowledge, it seems as if this work has not been completed before. There is in fact very little information in the literature regarding the classification of the bounded Painlevé type solutions.

When it comes to the rational solutions, the locations of the bounded solutions are very easy to write down. This is because all the rational solutions to the Painlevé equations are always logarithmic derivatives of polynomials. So assuming we know the roots of these polynomials it just remains to classify the conditions that allow no real roots. Then as long as the asymptotic behaviour around $\pm\infty$ is bounded as well, we will have bounded solutions. In summary, for both the rational and special function solutions, we are looking for the conditions that give no poles on the real line.

Consider the following polynomials:

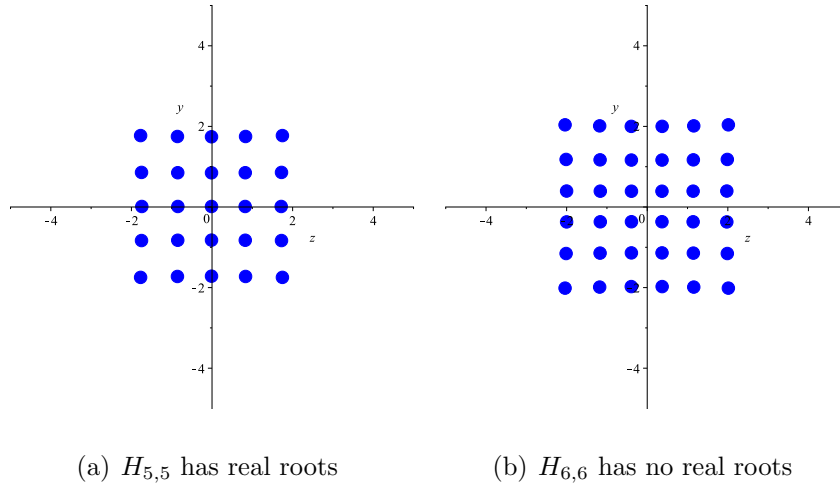


Figure 2.1: Roots of the generalised Hermite polynomials $H_{m,n}$, which we will formally define in section 3.1.3

As $H_{5,5}$ has real roots it is impossible for a solution containing the logarithmic derivative of $H_{5,5}$ to be bounded. As $H_{6,6}$ has no real roots it is possible for the logarithmic derivative of $H_{6,6}$ to be bounded.

Another interesting property of Painlevé equations is the fact that the form of the general solution of the Painlevé solutions are slightly different when compared with the form of the general solution of the associated Hamiltonian equations. The Painlevé type solutions always take the form of a logarithmic derivative of a ratio of functions. Regardless of the type of Painlevé solution we are discussing, be it polynomials or special functions. Whereas the Hamiltonian type solutions always take the form of logarithmic derivatives of a single function, not a ratio [56, 57, 58]. This highlights one of the main reasons that the Hamiltonian systems play such a vital role. They are easier to work with in this sense and all the applications in orthogonal polynomials involve only a logarithmic derivatives of a

single function. In most cases the solutions are just linear transformations away from the logarithmic derivatives of the matrix of moments. It's this fact alone that motivates this thesis.

2.2 Special functions

The Painlevé equations $P_{II} - P_{VI}$ possess hierarchies of solutions expressible in terms of classical special functions and for particular values of the parameters they satisfy an associated Riccati equation:

$$\frac{dw}{dz} = p_2(z)w^2 + p_1(z)w + p_0(z), \quad (2.1)$$

where $p_2(z)$, $p_1(z)$ and $p_0(z)$ are rational functions. Hierarchies of solutions, which are often referred to as one-parameter solutions, are generated from seed solutions derived from the Riccati equation using the associated Backlund transformation. The special function solutions of P_{II} (1.1b) are given in terms of Airy functions $Ai(z)$, $Bi(z)$; of P_{III} (1.1c) and $P_{III'}$ (1.24) are given in terms of Bessel functions $J_\nu(z)$ and $Y_\nu(z)$; of P_{IV} (1.1d) are given in terms of parabolic cylinder functions $D_\nu(z)$; of P_V (1.1e) are given in terms of Kummer functions $F(a, b; z)$ and $U(a, b; z)$; and of P_{VI} (1.1f) are given in terms of the general hypergeometric equation $F(a, b, c; z)$.

Table 2.1: Special function solutions of $P_{II} - P_{VI}$

	$p_2(z)$	$p_1(z)$	$p_0(z)$	Conditions on parameters
P_{II}	ε	0	$\frac{1}{2}\varepsilon$	$A = \frac{1}{2}\varepsilon$
P_{III}	ε_1	$\frac{A-\varepsilon_1}{\varepsilon_1 z}$	ε_2	$\varepsilon_1 A + \varepsilon_2 B = 4n + 2$
$P_{III'}$	ε_1	$\frac{\nu}{z}$	ε_2	$\varepsilon_1 A + \varepsilon_2 B = 4n + 2$
P_{IV}	ε_1	$2\varepsilon z$	2ν	$B = -2(2n + 1 + \varepsilon_A)^2$, or $-2n^2$
P_V	$\frac{\alpha}{z}$	$\varepsilon_3 - \frac{\alpha-\beta}{z}$	$-\frac{\beta}{z}$	$\alpha + \beta + \varepsilon_3 C = 2n + 1$, or $(\alpha - n)(\beta - n) = 0$
P_{VI}	$\frac{a}{z(z-1)}$	$\frac{(b+1-a)z-c}{z(z-1)}$	$\frac{c-b-1}{z-1}$	$\varepsilon_1\sqrt{2A} + \varepsilon_2\sqrt{-2B} + \varepsilon_3\sqrt{2C} + \varepsilon_4\sqrt{1-2D} = 2n + 1$

2.2.1 The Airy function

Definition 2.1. *The Airy function is the solution to Airy's equation*

$$\frac{d^2 w}{dz^2} = zw, \tag{2.2}$$

where all solutions are entire functions of z [60, §9.2(i)].

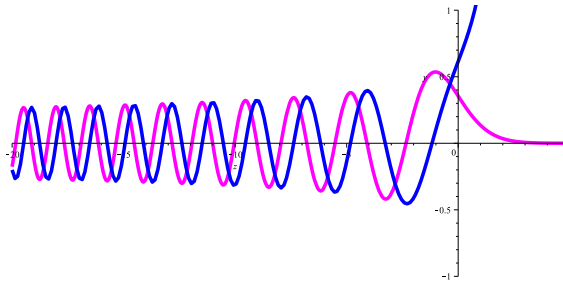


Figure 2.2: Plot of Airy functions $Ai(z)$ and $Bi(z)$ which are solutions to the Airy equation (2.2).

2.2.2 The Bessel function

Definition 2.2. *The Bessel function is the solution to Bessel's equation*

$$z^2 \frac{d^2 w}{dz^2} + z \frac{dw}{dz} + (z^2 - \nu^2)w = 0. \tag{2.3}$$

This differential equation has a regular singularity at $z = 0$ with indices $\pm\nu$ and an irregular singularity at $z = \infty$ of rank 1 [60, §10.2(i)].

Definition 2.3. *The Modified Bessel functions ($z \rightarrow \pm iz$) have the following relations to other functions [60, §10.39]:*

$$\mathcal{I}_{1/2}(z) = \left(\frac{2}{\pi z}\right)^{1/2} \sinh(z), \quad (2.4a)$$

$$\mathcal{I}_{-1/2}(z) = \left(\frac{2}{\pi z}\right)^{1/2} \cosh(z). \quad (2.4b)$$

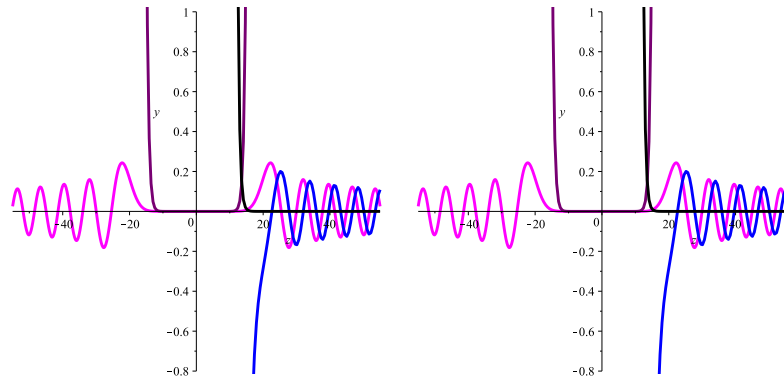
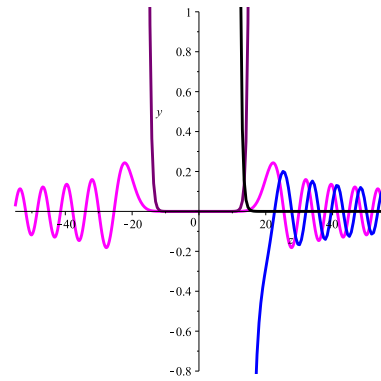
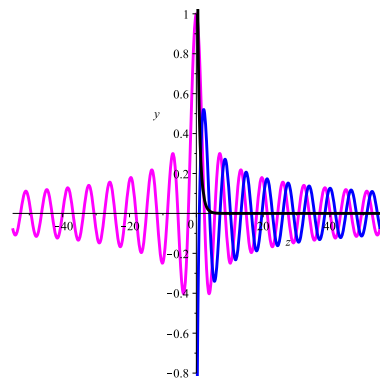
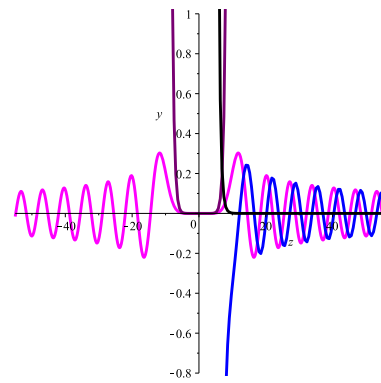
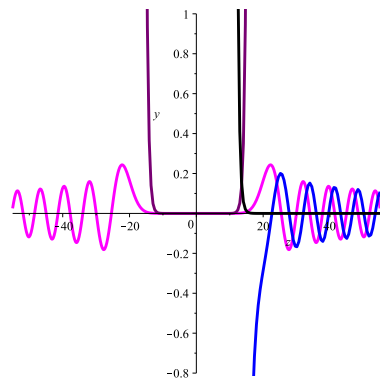
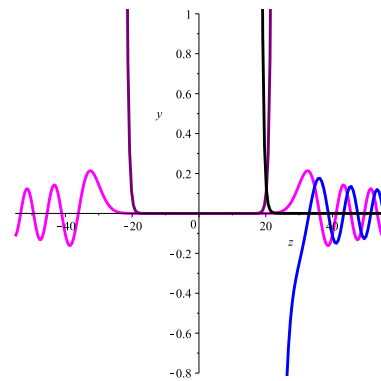
(a) $\nu = 20$,(b) $\nu = 10$,(c) $\nu = 0$,(d) $\nu = -10$,(e) $\nu = -20$,(f) $\nu = -30$,

Figure 2.3: Plot of Bessel functions of the first and second kind $J_\nu(z)$, $Y_\nu(z)$, $I_\nu(z)$ and $K_\nu(z)$ which are solutions to the Bessel equation (2.3).

2.2.3 The Parabolic cylinder functions

Definition 2.4. *The parabolic cylinder functions ψ are solutions of the differential equation*

$$\frac{d^2\psi}{dz^2} + \left(az^2 + bz + c\right)\psi = 0. \quad (2.5)$$

The parabolic cylinder functions ψ have the three distinct standard forms [60, §12.2(i)]

$$\frac{d^2U(-a)}{dz^2} - \left(\frac{1}{4}z^2 + a\right)U(-a) = 0, \quad (2.6a)$$

$$\frac{d^2W(a)}{dz^2} + \left(\frac{1}{4}z^2 - a\right)W(a) = 0, \quad (2.6b)$$

$$\frac{d^2D_\nu}{dz^2} - \left(\frac{1}{4}z^2 - \nu - \frac{1}{2}\right)D_\nu = 0. \quad (2.6c)$$

Each of these equations is, of course, transformable into the others. All solutions are entire functions of z and entire functions of a or ν . The form that we will be concerned with here is the D_ν type where

$$D_\nu = U\left(-\frac{1}{2} - \nu, z\right).$$

Definition 2.5. *The parabolic cylinder function D_ν has the following relations to Hermite polynomials [60, §12.7(i)]:*

$$U\left(-\frac{1}{2}, z\right) = D_0(z) = \exp\left(\frac{1}{4}z^2\right), \quad (2.7a)$$

$$U\left(n - \frac{1}{2}, z\right) = D_n(z) = \exp\left(-\frac{1}{4}z^2\right)He_n(z) = 2^{-n/2} \exp\left(-\frac{1}{4}z^2\right)H_n\left(\frac{1}{2}\sqrt{2}z\right), \quad (2.7b)$$

$$V\left(n + \frac{1}{2}, z\right) = \sqrt{\frac{2}{\pi}} \exp\left(\frac{1}{4}z^2\right)(-i)^n He_n(iz) = \sqrt{\frac{2}{\pi}} \exp\left(\frac{1}{4}z^2\right)(-i)^n 2^{-n/2} H_n\left(\frac{1}{2}\sqrt{2}iz\right), \quad (2.7c)$$

where $H_n(z)$, $He_n(z)$ are both Hermite polynomials, but $He_n(z)$ refers to the slightly unusual weight of $w(x) = e^{-\frac{1}{2}x^2}$.

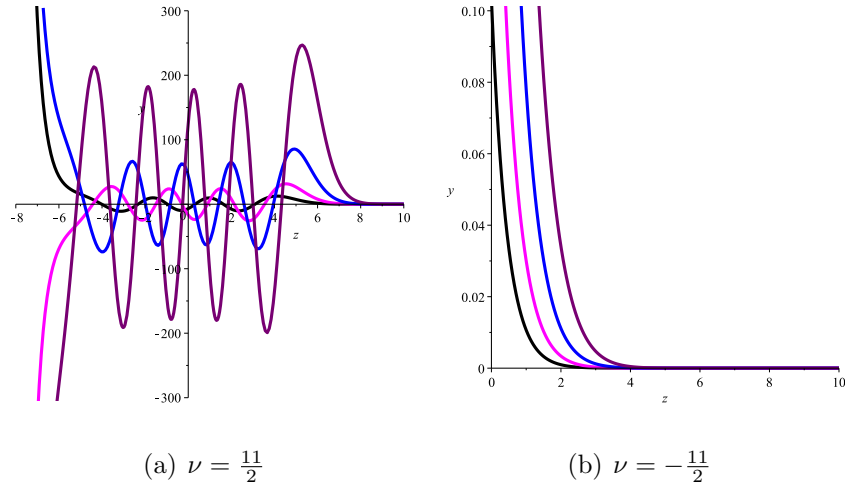


Figure 2.4: Plot of parabolic cylinder functions $D_\nu(z)$, $D_{\nu+1}(z)$, $D_{\nu+2}(z)$ and $D_{\nu+3}(z)$ which are solutions to the parabolic cylinder equation (2.5).

2.2.4 The confluent Hypergeometric function

Definition 2.6. A Kummer function is a solution to Kummer's equation [60, §13.2(i)]

$$z \frac{d^2 w}{dz^2} + (b - z) \frac{dw}{dz} - aw = 0.$$

This has two linearly independent solutions $M(a, b, z)$ and $U(a, b, z)$.

Both of these solutions will be of fundamental importance throughout this thesis.

Definition 2.7. The first two standard solutions are [60, §13.2(i)]

$$M(a, b, z) = \sum_{s=0}^{\infty} \frac{(a)_s}{(b)_s s!} z^s \quad \text{and} \quad \mathbf{M}(a, b, z) = \sum_{s=0}^{\infty} \frac{(a)_s}{\Gamma(b + s) s!} z^s. \quad (2.8)$$

where $\Gamma(a)$ is the Gamma function and $(a)_n$ is the Pochhammer symbol which is define by

$$(a)_n := \frac{\Gamma(a + n)}{\Gamma(a)}.$$

Definition 2.8. *The Kummer functions $M(a, b, z)$ and $U(a, b, z)$ have the following integral representations [60, §13.4(i)]:*

$$M(a, b, z) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)} \int_0^1 e^{uz} u^{a-1} (1-u)^{b-a-1} du, \quad (2.9a)$$

$$U(a, b, z) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-uz} u^{a-1} (1+u)^{b-a-1} du. \quad (2.9b)$$

Definition 2.9. *The Kummer functions $M(a, b, z)$ and $U(a, b, z)$ satisfy the following transformations [60, §13.2(vii)]:*

$$M(a, b, z) = e^z M(b-a, b, -z), \quad (2.10a)$$

$$U(a, b, z) = z^{1-b} U(a-b+1, 2-b, z). \quad (2.10b)$$

Definition 2.10. *The Kummer functions $M(a, b, z)$ and $U(a, b, z)$ satisfy the following differentiation formula [60, §13.3(ii)]:*

$$\frac{d^n}{dz^n} M(a, b, z) = \frac{(a)_n}{(b)_n} M(a+n, b+n, z), \quad (2.11a)$$

$$\frac{d^n}{dz^n} \left[e^{-z} U(a, b, z) \right] = (-1)^n e^{-z} U(a, b+n, z), \quad (2.11b)$$

$$\frac{d^n}{dz^n} \left[e^{-z} M(a, b, z) \right] = (-1)^n \frac{(b-a)_n}{(b)_n} e^{-z} M(a, b+n, z), \quad (2.11c)$$

$$\frac{d^n}{dz^n} \left[U(a, b, z) \right] = (-1)^n (a)_n U(a+n, b+n, z). \quad (2.11d)$$

Definition 2.11. *The Kummer functions $M(a, b, z)$ and $U(a, b, z)$ have the following relation to Laguerre polynomials when $n \in \mathbb{N}$ [60, §13.6(v)]:*

$$U(-n, a+1, z) = (-1)^n (a+1)_n M(-n, a+1, z) = (-1)^n n! L_n^{(a)}(z), \quad (2.12)$$

where $L_n^{(a)}$ is a associated Laguerre polynomial [60, §18.3].

2.2.5 The general Hypergeometric function

The hypergeometric function $F(a, b, c; z)$ is a solution of Euler's hypergeometric differential equation

$$z(1-z)\frac{d^2w}{dz^2} + [c - (a+b+1)z]\frac{dw}{dz} - abw = 0, \quad (2.13)$$

which has three regular singular points: 0, 1 and ∞ .

Definition 2.12. *The general hypergeometric function $F(a, b, c; z)$ is defined by the Gauss series [60, §15.2(i)]*

$$\begin{aligned} F(a, b, c; z) &= \sum_{s=0}^{\infty} \frac{(a)_s (b)_s}{(c)_s s!} z^s = 1 + \frac{ab}{c} z + \frac{a(a+1)b(b+1)}{c(c+1)2!} z^2 \\ &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{s=0}^{\infty} \frac{\Gamma(a+s)\Gamma(b+s)}{\Gamma(c+s)s!} z^s, \end{aligned}$$

on the disk $|z| < 1$.

Definition 2.13. *The general hypergeometric function $F(a, b, c; z)$ has the following integral representation [60, §15.6.1(i)]:*

$$F(a, b, c; z) = \frac{\Gamma(a)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-zt)^a} dt. \quad (2.14)$$

Definition 2.14. *The general hypergeometric function $F(a, b, c; z)$ satisfies the following differential formula [60, §15.5]:*

$$\frac{d^n}{dz^n} \left\{ z^{c-1} (1-z)^{a+b-c} F(a, b, c; z) \right\} = (c-n)_n z^{c-n-1} (1-z)^{a+b-c-n} F(a-n, b-n, c-n; z). \quad (2.15)$$

Definition 2.15. *The general hypergeometric function $F(a, b, c; z)$ has the following relation to Jacobi polynomials [60, §15.9]:*

$$\frac{(\alpha+1)_n}{n!} F\left(-n, n+\alpha+\beta+1, \alpha+1; \frac{1}{2}(1-z)\right) = P_n^{(\alpha, \beta)}(z), \quad (2.16)$$

where $P_n^{(\alpha,\beta)}(x)$ is the Jacobi polynomial.

2.3 Special function solutions to the Painlevé equations

2.3.1 The second Painlevé equation P_{II}

To obtain a special function solution of a P_{II} (1.1b) one supposes that $w(z)$ satisfies the Riccati equation (2.1) for some functions $p_2(z)$, $p_1(z)$ and $p_0(z)$. Differentiating (2.1) yields

$$\begin{aligned} \frac{d^2w}{dz^2} &= \frac{dp_2}{dz}w^2 + 2p_2w\frac{dw}{dz} + \frac{dp_1}{dz}w + p_1\frac{dw}{dz} + \frac{dp_0}{dz} \\ &= \frac{dp_2}{dz}w^2 + \frac{dp_1}{dz}w + \frac{dp_1}{dz} + (2p_2w + p_1)(p_2w^2 + p_1w + p_0) \\ &= 2p_2^2w^3 + \left(\frac{dp_2}{dz} + 3p_1p_2\right)w^2 + \left(\frac{dp_1}{dz} + 2p_0p_2 + p_1^2\right)w + \frac{dp_0}{dz} + p_1p_0. \end{aligned} \quad (2.17)$$

Substituting this into P_{II} (1.1b) gives

$$2(p_2-1)(p_2+1)w^3 + \left(3p_2p_1 + \frac{dp_2}{dz}\right)w^2 + \left(2p_2p_0 + p_1^2 + \frac{dp_1}{dz} - z\right)w + p_1p_0 + p_0 - A = 0.$$

Equating powers of w and solving gives

$$p_2(z) = \varepsilon, \quad p_1(z) = 0, \quad p_0(z) = \frac{1}{2}\varepsilon z, \quad \varepsilon^2 = 1,$$

with parameter $A = \frac{1}{2}\varepsilon$. P_{II} has solutions expressible in terms of solutions of the Riccati

$$\varepsilon \frac{dw}{dz} = w^2 + \frac{1}{2}z. \quad (2.18)$$

To solve (2.18) we have to make the following transformation:

$$w(z) = -\varepsilon \frac{d}{dz} \ln \psi(z),$$

then $\psi_\nu(z)$ satisfies the Airy equation, with $\psi(z) = C_1\text{Ai}(\zeta) + C_2\text{Bi}(\zeta)$, where $\text{Ai}(\zeta)$ and $\text{Bi}(\zeta)$ are the Airy functions. P_{II} (1.1b) has solutions expressible in terms of Airy functions if and only if $A = n + \frac{1}{2}$ for $n \in \mathbb{Z}$.

Theorem 2.1. Let $\tau_n(z, \varepsilon)$ be the bi-directional Wronskian determinant given by

$$\tau_n(z) := \begin{vmatrix} \psi & \delta(\psi) & \dots & \delta^{(n-1)}(\psi) \\ \delta(\psi) & \delta^{(2)}(\psi) & \dots & \delta^{(n)}(\psi) \\ \vdots & \vdots & \ddots & \vdots \\ \delta^{(n-1)}(\psi) & \delta^{(n)}(\psi) & \dots & \delta^{(2n-2)}(\psi) \end{vmatrix}, \quad \delta = \frac{d}{dz}.$$

Then for $n \geq 0$ the special function solutions for P_{II} (1.1b) in the form $w(z; A)$, are given by the following:

$$w(z; A) = \frac{d}{dz} \left(\frac{\tau_n(z)}{\tau_{n+1}(z)} \right),$$

for the parameters $A = n + \frac{1}{2}$, where $\zeta = -2^{-1/3}z$. Also note that $w(z; -n - \frac{1}{2}) = -w(z; n + \frac{1}{2})$.

Proof. See Okamoto [58]. □

2.3.2 The zeros of the Airy functions

It is interesting to note that these special functions will have no bounded solutions regardless of the choice of C_1 and C_2 . This is important from an application point of view because the non-linear ODE's that arise in the applications are almost always the bounded type. The bounded Painlevé solutions usually tend to be the solutions that have the physical relevancy. The lack of bounded solutions is easy to spot from the fact that the solutions can always be written in partial fractions with respect to the logarithmic derivative and this clearly shows singularities at $z = 0$ for all solutions in the hierarchy. This can also be easily seen by studying the zeros of the Airy type plots in figure 2.2.

2.3.3 The third Painlevé equation P_{III}

Without loss of generality we can set $C = -D = 1$ by rescaling w and z if necessary.

Theorem 2.2. P_{III} (1.1c) has solutions expressible in terms of Bessel functions if and only if

$$\varepsilon_1 A + \varepsilon_2 B = 4n + 2,$$

with $n \in \mathbb{Z}$, $\varepsilon_1 = \pm 1$, $\varepsilon_2 = \pm 1$ independently.

Proof. See Gromak [33], Mansfield and Webster [45] and Umemure H and Watanabe H [65]. □

To obtain a special function solution of a P_{III} (1.1c) we need to substitute (2.17) into P_{III} (1.1c). This yields

$$\begin{aligned} z(1 - p_2^2)w^4 + \left(A - p_2 - \frac{dp_2}{dz}z - p_1p_2z \right)w^3 - \left(\frac{dp_1}{dz}z + p_1 \right)w^2 \\ + \left(B + p_1p_0z - \frac{dp_0}{dz}z - p_0 \right)w + z(p_0^2 - 1) = 0. \end{aligned}$$

Equating powers of w and solving gives

$$p_2(z) = \varepsilon_1, \quad p_1(z) = \frac{A - \varepsilon_1}{\varepsilon_1 z}, \quad p_0(z) = \varepsilon_2, \quad \varepsilon_1^2 = \varepsilon_2^2 = 1,$$

with parameter $B = \varepsilon_1\varepsilon_2(2\varepsilon_1 - A)$. So, for P_{III} (1.1c) the associated Riccati equation is

$$\frac{dw}{dz} = \varepsilon_1 w^2 + \frac{A\varepsilon_1 - 1}{z}w + \varepsilon_2. \quad (2.19)$$

To solve (2.19) we have to make the following transformation:

$$w(z) = -\varepsilon_1 z \frac{d}{dz} \ln \psi_\nu(z),$$

then $\psi_\nu(z)$ satisfies

$$z \frac{d^2 \psi_\nu}{dz^2} + (1 - \nu) \frac{d\psi_\nu}{dz} + \varepsilon_1 \varepsilon_2 \psi_\nu = 0, \quad (2.20)$$

we have the following solution for the Riccati:

$$\psi_\nu(z) = \begin{cases} z^\nu \{C_1 J_\nu(z) + C_2 Y_\nu(z)\}, & \text{if } \varepsilon_1 = 1, \quad \varepsilon_2 = 1, \\ z^{-\nu} \{C_1 J_\nu(z) + C_2 Y_\nu(z)\}, & \text{if } \varepsilon_1 = -1, \quad \varepsilon_2 = -1, \\ z^\nu \{C_1 I_\nu(z) + C_2 K_\nu(z)\}, & \text{if } \varepsilon_1 = 1, \quad \varepsilon_2 = -1, \\ z^{-\nu} \{C_1 I_\nu(z) + C_2 K_\nu(z)\}, & \text{if } \varepsilon_1 = -1, \quad \varepsilon_2 = 1, \end{cases}$$

with C_1 and C_2 arbitrary constants and where $J_\nu(z)$, $Y_\nu(z)$, $I_\nu(z)$ and $K_\nu(z)$ are Bessel functions.

Theorem 2.3. *Let $\mathcal{F}_n(f)$ be the determinant given by*

$$\mathcal{F}_n(\psi) := \begin{vmatrix} \psi & \delta(\psi) & \dots & \delta^{(n-1)}(\psi) \\ \delta(\psi) & \delta^{(2)}(\psi) & \dots & \delta^{(n)}(\psi) \\ \vdots & \vdots & \ddots & \vdots \\ \delta^{(n-1)}(\psi) & \delta^{(n)}(\psi) & \dots & \delta^{(2n-2)}(\psi) \end{vmatrix}, \quad \delta = z \frac{d}{dz}$$

and $\mathcal{K}_n(\psi_\nu)$ be the determinant given by

$$\mathcal{K}_n(\psi_\nu) = \begin{vmatrix} \psi_\nu & \psi_{\nu-1} & \dots & \psi_{\nu-n+1} \\ \psi_{\nu+1} & \psi_\nu & \dots & \psi_{\nu-n+2} \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{\nu+n-1} & \psi_{\nu+n} & \dots & \psi_\nu \end{vmatrix}.$$

Then for $n \geq 0$ the special function solutions of P_{III} (1.1c) in the form

$$w_{\nu,n}^{[N]}(z; A^{[N]}, B^{[N]}, C^{[N]}, D^{[N]}, \varepsilon_1, \varepsilon_2),$$

for $N = 1, \dots, 4$, are given by the following:

$$\begin{aligned} w_{\nu,n}^{[1]}(A^{[1]}, B^{[1]}, C^{[1]}, D^{[1]}, 1, 1) &= \frac{n}{z} - \frac{d}{dz} \ln \frac{\mathcal{F}_{n+1}(\psi_\nu)}{\mathcal{F}_n(\psi')} = -\frac{d}{dz} \ln \frac{\mathcal{K}_{n+1}(\psi_\nu)}{\mathcal{K}_n(\psi_{\nu-1})}, \\ w_{\nu,n}^{[2]}(A^{[2]}, B^{[2]}, C^{[2]}, D^{[2]}, -1, 1) &= -\frac{n}{z} + \frac{d}{dz} \ln \frac{\mathcal{F}_{n+1}(\psi_\nu)}{\mathcal{F}_n(\psi')} = \frac{d}{dz} \ln \frac{\mathcal{K}_{n+1}(\psi_\nu)}{\mathcal{K}_n(\psi_{\nu+1})}, \\ w_{\nu,n}^{[3]}(A^{[3]}, B^{[3]}, C^{[3]}, D^{[3]}, 1, -1) &= \frac{n}{z} - \frac{d}{dz} \ln \frac{\mathcal{F}_{n+1}(\psi_\nu)}{\mathcal{F}_n(\psi')} = -\frac{d}{dz} \ln \frac{\mathcal{K}_{n+1}(\psi_\nu)}{\mathcal{K}_n(\psi_{\nu-1})}, \\ w_{\nu,n}^{[4]}(A^{[4]}, B^{[4]}, C^{[4]}, D^{[4]}, -1, -1) &= -\frac{n}{z} + \frac{d}{dz} \ln \frac{\mathcal{F}_{n+1}(\psi_\nu)}{\mathcal{F}_n(\psi')} = \frac{d}{dz} \ln \frac{\mathcal{K}_{n+1}(\psi_\nu)}{\mathcal{K}_n(\psi_{\nu+1})}, \end{aligned}$$

for the parameters

$$\begin{aligned} \{A^{[1]}, B^{[1]}, C^{[1]}, D^{[1]}\} &= \{2(\nu + n), 2(n - \nu + 1), 1, -1\}, \\ \{A^{[2]}, B^{[2]}, C^{[2]}, D^{[2]}\} &= \{2(\nu - n), 2(n + \nu + 1), 1, -1\}, \\ \{A^{[3]}, B^{[3]}, C^{[3]}, D^{[3]}\} &= \{2(\nu + n), -2(n - \nu + 1), 1, -1\}, \\ \{A^{[4]}, B^{[4]}, C^{[4]}, D^{[4]}\} &= \{2(\nu - n), -2(n + \nu + 1), 1, -1\}, \end{aligned}$$

with $' = \frac{d}{dt}$.

Proof. See Okamoto [58]; also Forrester and Witte [26]. \square

2.3.4 The zeros of the Bessel functions

The special function solutions of P_{III} (1.1c) are only bounded when $\varepsilon_1 = -1$, $\nu > 0$ and $C_1 C_2 > 0$, for all n , where n is the number of the solution in the hierarchy. The alternative case is when $\varepsilon_2 = 1$, $\nu < 0$ and $C_1 C_2 > 0$, for all n , where n is the number of the solution in the hierarchy.

2.3.5 The associated third Painlevé equation $P_{III'}$

Without loss of generality we can set $C = -D = 1$ by rescaling w and z , if necessary.

Theorem 2.4. $P_{III'}$ (1.24) has solutions expressible in terms of Bessel functions if and only if

$$\varepsilon_1 A + \varepsilon_2 B = 4n + 2,$$

with $n \in \mathbb{Z}$, $\varepsilon_1 = \pm 1$ and $\varepsilon_2 = \pm 1$ independently.

Proof. See Gromak [33], Mansfield and Webster [45] and Umemure H and Watanabe H [65]. \square

To obtain a special function solution of a $P_{III'}$ (1.24) we need to substitute (2.17) into $P_{III'}$ (1.24). This yields

$$\begin{aligned} t(1 - p_2^2)u^4 + \left(A - p_2 - \frac{dp_2}{dt}t - p_1 p_2 t \right) u^3 - \left(\frac{dp_1}{dt}t + p_1 \right) u^2 \\ + \left(B + p_1 p_0 t - \frac{dp_0}{dt}t - p_0 \right) u + t(p_0^2 - 1) = 0. \end{aligned}$$

Equating powers of w and solving gives

$$p_2(t) = \varepsilon_1, \quad p_1(t) = \frac{\nu}{t}, \quad p_0(t) = \varepsilon_2, \quad \varepsilon_1^2 = \varepsilon_2^2 = 1.$$

with parameters $A = \varepsilon_1(\nu + 1)$ and $B = \varepsilon_2(1 - \nu)$. So for P_{III} (1.1c) the associated Riccati equation is

$$\frac{du}{dt} = \varepsilon_1 u^2 + \frac{\nu u}{t} + \varepsilon_2. \quad (2.23)$$

To solve (2.23) we have to make the following transformation:

$$u(t) = -\varepsilon_1 t \frac{d}{dt} \ln \psi_\nu(t),$$

then $\psi_\nu(z)$ satisfies

$$t \frac{d^2 \psi_\nu}{dt^2} + (1 - \nu) \frac{d\psi_\nu}{dt} + \varepsilon_1 \varepsilon_2 \psi_\nu = 0. \quad (2.24)$$

We have the following solution for the Riccati:

$$\psi_\nu(t) = \begin{cases} t^{\nu/2} \{C_1 J_\nu(2\sqrt{t}) + C_2 Y_\nu(2\sqrt{t})\}, & \text{if } \varepsilon_1 = 1, \quad \varepsilon_2 = 1, \\ t^{-\nu/2} \{C_1 J_\nu(2\sqrt{t}) + C_2 Y_\nu(2\sqrt{t})\}, & \text{if } \varepsilon_1 = -1, \quad \varepsilon_2 = -1, \\ t^{\nu/2} \{C_1 I_\nu(2\sqrt{t}) + C_2 K_\nu(2\sqrt{t})\}, & \text{if } \varepsilon_1 = 1, \quad \varepsilon_2 = -1, \\ t^{-\nu/2} \{C_1 I_\nu(2\sqrt{t}) + C_2 K_\nu(2\sqrt{t})\}, & \text{if } \varepsilon_1 = -1, \quad \varepsilon_2 = 1, \end{cases}$$

with C_1 and C_2 arbitrary constants and where $J_\nu(2\sqrt{t})$, $Y_\nu(2\sqrt{t})$, $I_\nu(2\sqrt{t})$ and $K_\nu(2\sqrt{t})$ are Bessel functions.

Theorem 2.5. *Let $\mathcal{F}_n(f)$ be the determinant given by*

$$\mathcal{F}_n(\psi) := \begin{vmatrix} \psi & \delta(\psi) & \dots & \delta^{(n-1)}(\psi) \\ \delta(\psi) & \delta^{(2)}(\psi) & \dots & \delta^{(n)}(\psi) \\ \vdots & \vdots & \ddots & \vdots \\ \delta^{(n-1)}(\psi) & \delta^{(n)}(\psi) & \dots & \delta^{(2n-2)}(\psi) \end{vmatrix}, \quad \delta = z \frac{d}{dz}$$

and $\mathcal{K}_n(\psi_\nu)$ be the determinant given by

$$\mathcal{K}_n(\psi_\nu) = \begin{vmatrix} \psi_\nu & \psi_{\nu-1} & \dots & \psi_{\nu-n+1} \\ \psi_{\nu+1} & \psi_\nu & \dots & \psi_{\nu-n+2} \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{\nu+n-1} & \psi_{\nu+n} & \dots & \psi_\nu \end{vmatrix}.$$

Then for $n \geq 0$ the special function solutions of P_{III} (1.24) in the form

$$u_{\nu,n}^{[N]}(z; A^{[N]}, B^{[N]}, C^{[N]}, D^{[N]}, \varepsilon_1, \varepsilon_2),$$

for $N = 1, \dots, 4$, are given by the following:

$$\begin{aligned} u_{\nu,n}^{[1]}(A^{[1]}, B^{[1]}, C^{[1]}, D^{[1]}, 1, 1) &= \frac{n}{2} - t \frac{d}{dt} \ln \frac{\mathcal{F}_{n+1}(\psi_\nu)}{\mathcal{F}_n(\sqrt{t}\psi')} = -z \frac{d}{dt} \ln \frac{\mathcal{K}_{n+1}(\psi_\nu)}{\mathcal{K}_n(\psi_{\nu-1})}, \\ u_{\nu,n}^{[2]}(A^{[2]}, B^{[2]}, C^{[2]}, D^{[2]}, -1, 1) &= -\frac{n}{2} + t \frac{d}{dt} \ln \frac{\mathcal{F}_{n+1}(\psi_\nu)}{\mathcal{F}_n(\sqrt{t}\psi')} = z \frac{d}{dt} \ln \frac{\mathcal{K}_{n+1}(\psi_\nu)}{\mathcal{K}_n(\psi_{\nu+1})}, \\ u_{\nu,n}^{[3]}(A^{[3]}, B^{[3]}, C^{[3]}, D^{[3]}, 1, -1) &= \frac{n}{2} - t \frac{d}{dt} \ln \frac{\mathcal{F}_{n+1}(\psi_\nu)}{\mathcal{F}_n(\sqrt{t}\psi')} = -z \frac{d}{dt} \ln \frac{\mathcal{K}_{n+1}(\psi_\nu)}{\mathcal{K}_n(\psi_{\nu-1})}, \\ u_{\nu,n}^{[4]}(A^{[4]}, B^{[4]}, C^{[4]}, D^{[4]}, -1, -1) &= -\frac{n}{2} + t \frac{d}{dt} \ln \frac{\mathcal{F}_{n+1}(\psi_\nu)}{\mathcal{F}_n(\sqrt{t}\psi')} = z \frac{d}{dt} \ln \frac{\mathcal{K}_{n+1}(\psi_\nu)}{\mathcal{K}_n(\psi_{\nu+1})}, \end{aligned}$$

for the parameters

$$\begin{aligned} \{A^{[1]}, B^{[1]}, C^{[1]}, D^{[1]}\} &= \{2(\nu + n), 2(n - \nu + 1), 1, -1\}, \\ \{A^{[2]}, B^{[2]}, C^{[2]}, D^{[2]}\} &= \{2(\nu - n), 2(n + \nu + 1), 1, -1\}, \\ \{A^{[3]}, B^{[3]}, C^{[3]}, D^{[3]}\} &= \{2(\nu + n), -2(n - \nu + 1), 1, -1\}, \\ \{A^{[4]}, B^{[4]}, C^{[4]}, D^{[4]}\} &= \{2(\nu - n), -2(n + \nu + 1), 1, -1\}, \end{aligned}$$

with $' = \frac{d}{dt}$.

Proof. See Okamoto [58]; also Forrester and Witte [26]. Then, using the transformation from P_{III} (1.1c) to $P_{III'}$ (1.24), we can simply read off the special function solutions from the previous section. \square

The special function solutions of $P_{III'}$ (1.24) are only bounded when $\varepsilon_1 = -1$, $\nu > 0$ and $C_1 C_2 > 0$, for all n , where n is the number of the solution in the hierarchy. The alternative case is when $\varepsilon_2 = 1$, $\nu < 0$ and $C_1 C_2 > 0$, for all n , where n is the number of the solution in the hierarchy.

2.3.6 The fourth Painlevé equation P_{IV}

Theorem 2.6. P_{IV} has solutions expressible in terms of parabolic cylinder functions if and only if either:

$$B = -2(2n + 1 + \varepsilon A)^2,$$

or

$$B = -2n^2,$$

with $n \in \mathbb{Z}, \varepsilon = \pm 1$.

Proof. See [31, 32, 34, 35, 40, 42]. □

To obtain a special function solution of a P_{IV} (1.1d) we need to substitute (2.17) into P_{IV} (1.1d). This yields

$$\begin{aligned} & 3(p_2 - 1)(p_2 + 1)w^4 + \left(4p_2p_1 + 2\frac{dp_2}{dz} - 8z\right)w^3 \\ & + \left(2p_2p_0 + p_1^2 - 4z^2 + 2\frac{dp_1}{dz} + 4A\right)w^2 + 2\frac{dp_0}{dz}w - p_0^2 - 2B = 0. \end{aligned}$$

Equating powers of w and solving gives

$$p_2(z) = \varepsilon, \quad p_1(z) = 2\varepsilon z, \quad p_0(z) = 2\nu, \quad \varepsilon^2 = 1.$$

So for P_{IV} (1.1d) the associated Riccati equation is

$$\frac{dw}{dz} = \varepsilon(w^2 + 2zw) + 2\nu, \quad \varepsilon^2 = 1, \tag{2.27}$$

with parameters $A = -\varepsilon(\nu + 1)$ and $B = -2\nu^2$. To solve (2.27) we make the transformation $w(z) = \frac{d}{dz} \ln \psi(z)$ and this yields

$$\frac{d^2\psi_\nu}{dz^2} - 2\varepsilon z \frac{d\psi_\nu}{dz} + 2\varepsilon\nu\psi_\nu = 0. \tag{2.28}$$

The solution of (2.28) depends on whether $\nu \in \mathbb{Z}$ or $\nu \notin \mathbb{Z}$. The different solutions are characterised in the following [18]:

i) If $\nu \notin \mathbb{Z}$ then (2.28) has solutions

$$\psi_\nu(z; \varepsilon) = \begin{cases} \{C_1 D_\nu(\sqrt{2}z) + C_2 D_\nu(-\sqrt{2}z)\} \exp(\frac{1}{2}z^2), & \text{if } \varepsilon = 1, \\ \{C_1 D_{-\nu-1}(\sqrt{2}z) + C_2 D_{-\nu-1}(-\sqrt{2}z)\} \exp(-\frac{1}{2}z^2), & \text{if } \varepsilon = -1, \end{cases}$$

with C_1 and C_2 arbitrary constants.

ii) If $\nu = 0$ then (2.28) has solutions

$$\psi_0(z; \varepsilon) = \begin{cases} C_1 + C_2 \operatorname{erfi}(z), & \text{if } \varepsilon = 1, \\ C_1 + C_2 \operatorname{erfc}(z), & \text{if } \varepsilon = -1, \end{cases}$$

with C_1 and C_2 arbitrary constants, $\operatorname{erfc}(z)$ is the complementary error function and $\operatorname{erfi}(z)$ is the imaginary error function, respectively defined by

$$\operatorname{erfi} = \frac{2}{\sqrt{\pi}} \int_z^\infty \exp(-t^2) dt, \quad \operatorname{erfc} = \frac{2}{\sqrt{\pi}} \int_0^\infty \exp(t^2) dt. \quad (2.29)$$

iii) If $\nu = m$, for $m \geq 1$, then (2.28) has solutions

$$\psi_m(z; \varepsilon) = \begin{cases} \begin{aligned} &C_1 H_m(z) \\ &+ C_2 \exp(z^2) \frac{d^m}{dz^m} \{\operatorname{erfi}(z) \exp(-z^2)\}, \end{aligned} & \text{if } \varepsilon = 1, \\ \begin{aligned} &C_1 (-i)^m H_m(iz) \\ &+ C_2 \exp(-z^2) \frac{d^m}{dz^m} \{\operatorname{erfc}(z) \exp(z^2)\}, \end{aligned} & \text{if } \varepsilon = -1, \end{cases}$$

with C_1 and C_2 arbitrary constants and $H_n(z)$ is the Hermite polynomial defined by

$$H_m(z) = (-1)^m \exp(z^2) \frac{d^m}{dz^m} \exp(-z^2). \quad (2.30)$$

iv) If $\nu = -m$, for $m \geq 1$, then (2.28) has solutions

$$\psi_{-m}(z; \varepsilon) = \begin{cases} C_1(-i)^{m-1}H_{m-1}(iz)\exp(z^2) \\ \quad + C_2\frac{d^{m-1}}{dz^{m-1}}\{\operatorname{erfc}(z)\exp(z^2)\}, & \text{if } \varepsilon = 1, \\ C_1H_m(z)\exp(-z^2) \\ \quad + C_2\frac{d^{m-1}}{dz^{m-1}}\{\operatorname{erfi}(z)\exp(-z^2)\}, & \text{if } \varepsilon = -1, \end{cases}$$

with C_1 and C_2 arbitrary constants.

Theorem 2.7. Let $\tau_{n,\nu}(z, \varepsilon)$ be the bi-directional Wronskian determinant given by

$$\tau_{n,\nu}(z, \varepsilon) := \begin{vmatrix} \psi & \delta(\psi) & \dots & \delta^{(n-1)}(\psi) \\ \delta(\psi) & \delta^{(2)}(\psi) & \dots & \delta^{(n)}(\psi) \\ \vdots & \vdots & \ddots & \vdots \\ \delta^{(n-1)}(\psi) & \delta^{(n)}(\psi) & \dots & \delta^{(2n-2)}(\psi) \end{vmatrix}, \quad \delta = \frac{d}{dz}.$$

Then for $n \geq 0$ the special function solutions of P_{IV} (1.1d) in the form

$$w^{[N]}(z; A^{[N]}, B^{[N]}),$$

for $N = 1, 2, 3$, are given by the following:

$$\begin{aligned} w^{[1]}(z; A^{[1]}, B^{[1]}) &= -2z + \varepsilon \frac{d}{dz} \ln \frac{\tau_{n+1,\nu}(z; \varepsilon)}{\tau_{n,\nu}(z; \varepsilon)}, \\ w^{[2]}(z; A^{[2]}, B^{[2]}) &= \varepsilon \frac{d}{dz} \ln \frac{\tau_{n,\nu}(z; \varepsilon)}{\tau_{n,\nu-1}(z; \varepsilon)}, \\ w^{[3]}(z; A^{[3]}, B^{[3]}) &= \varepsilon \frac{d}{dz} \ln \frac{\tau_{n,\nu-1}(z; \varepsilon)}{\tau_{n+1,\nu}(z; \varepsilon)}, \end{aligned}$$

for the parameters

$$\begin{aligned} \{A^{[1]}, B^{[1]}\} &= \{\varepsilon(2n - \nu), -2(\nu + 1)^2\}, \\ \{A^{[2]}, B^{[2]}\} &= \{\varepsilon(2\nu - n + 1), -2n^2\}, \\ \{A^{[3]}, B^{[3]}\} &= \{\varepsilon(n + \nu + 1), -2(\nu - n)^2\}, \end{aligned}$$

where $\tau_{n,\nu-1} = \mathcal{W}(\frac{d}{dz}\psi, \frac{d^2}{dz^2}\psi, \dots, \frac{d^n}{dz^n}\psi)$ because $\frac{d}{dz}\psi_\nu = \psi_{\nu-1}$.

Proof. See Okamoto [58]; also Forrester and Witte [26]. □

2.3.7 The zeros of the parabolic cylinder function

The parabolic cylinder function $D_\nu(z)$ has no real zeros if $\nu < 0$, so $\psi_\nu(z)$ has no real zeros if $\nu < 0$ and $C_1C_2 > 0$ when $\varepsilon = 1$ or $\nu > -1$ and $C_1C_2 > 0$ when $\varepsilon = -1$ [60, §12.11]. The special function solutions of P_{IV} (1.1d) are only bounded solutions from the first hierarchy in the following two cases:

- $\varepsilon = 1, \nu < 0, C_1C_2 > 0,$
- $\varepsilon = -1, \nu > 1, C_1C_2 > 0.$

These are both the bounded situations for all n .

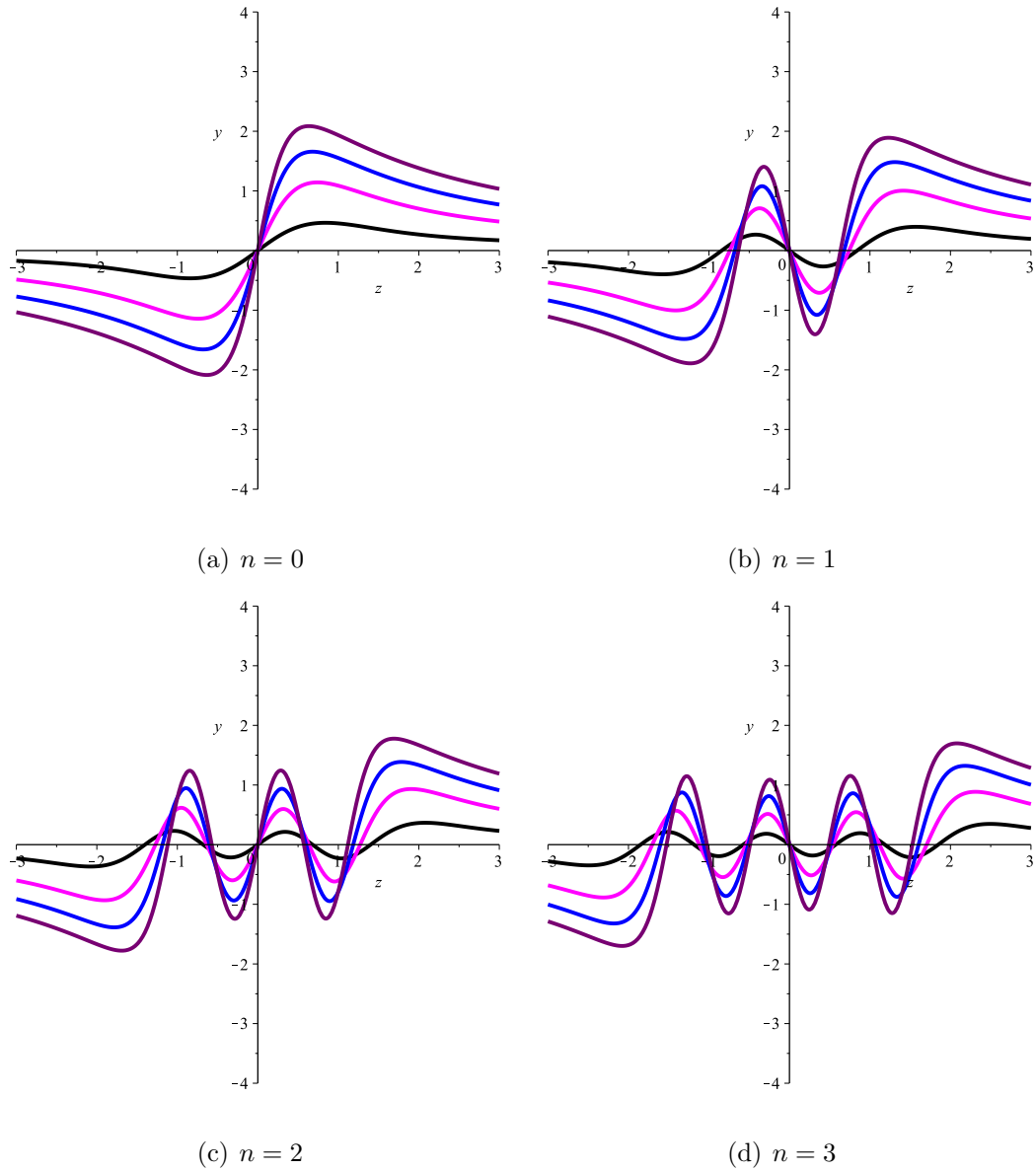
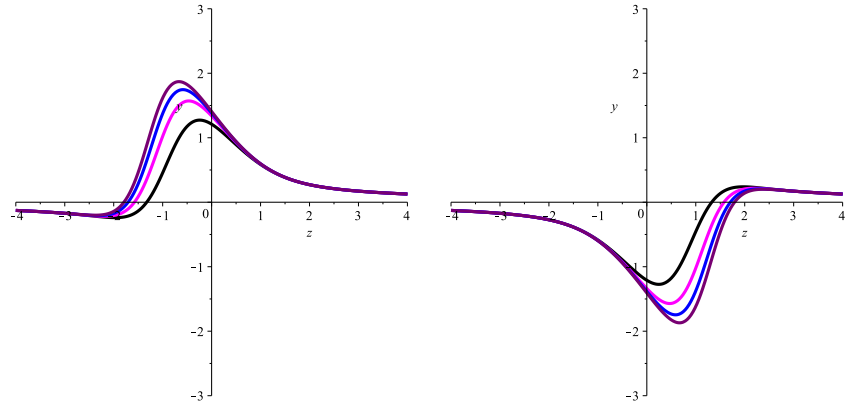
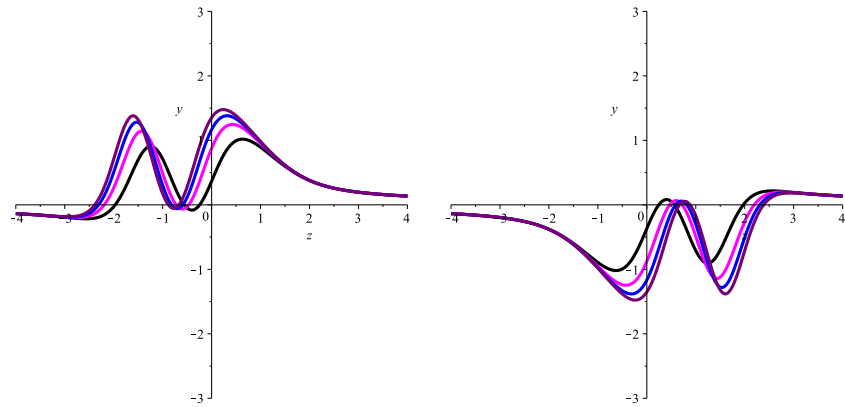


Figure 2.5: Parabolic cylinder function solution plots of $w^{[1]}(z; \nu = -\frac{3}{2})$, $w^{[1]}(z; \nu = -\frac{5}{2})$, $w^{[1]}(z; \nu = -\frac{7}{2})$, $w^{[1]}(z; \nu = -\frac{9}{2})$ with $C_1 = C_2 = 1$ and $\varepsilon = 1$.

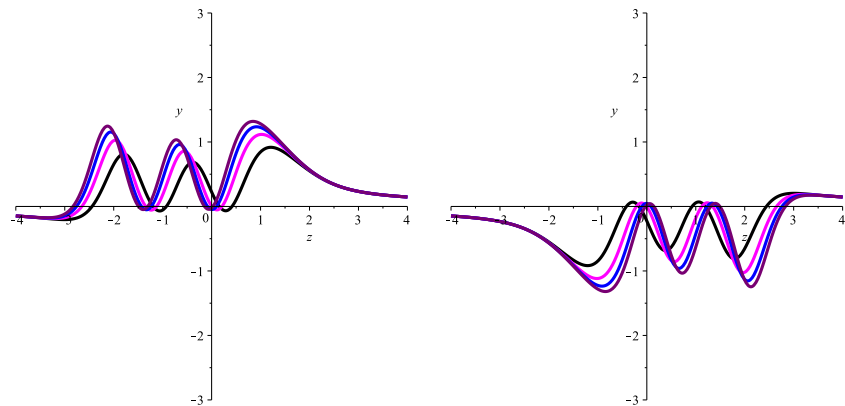
The number of turning points in these plots is dictated by $2(n + 1)$.



(a) $n=0, C_1=1, C_2=\{10,20,30,40\}$ (b) $n=0, C_2=1, C_1=\{10,20,30,40\}$



(c) $n=1, C_1=1, C_2=\{10,20,30,40\}$ (d) $n=1, C_2=1, C_1=\{10,20,30,40\}$



(e) $n=2, C_1=1, C_2=\{10,20,30,40\}$ (f) $n=2, C_2=1, C_1=\{10,20,30,40\}$

Figure 2.6: Parabolic cylinder function solution plots of $w^{[1]}(z; \nu = -\frac{3}{2}, \varepsilon = 1)$.

2.3.8 The fifth Painlevé equation P_V

Theorem 2.8. *Equation P_V (1.1e) has solutions expressible in terms of Kummer functions if and only if*

$$a + b + \varepsilon_3 C = 2n + 1, \text{ or } (a - n)(b - n) = 0,$$

where $n \in \mathbb{N}$, $a = \varepsilon_1 \sqrt{2A}$ and $b = \varepsilon_2 \sqrt{-2B}$, with $\varepsilon_1 = \pm 1$, $j = 1, 2, 3$ independently.

Proof. See Okamoto [57], Masuda [46] and Watanabe [68]; also [34, §40]. \square

To obtain a special function solution of a P_V (1.1e) we need to substitute (2.17) into P_V (1.1e) which yields a rather large expression. However, equating powers of w and solving gives

$$p_2(z) = \frac{a}{z}, \quad p_1(z) = \varepsilon_3 - \frac{a-b}{z}, \quad p_0(z) = -\frac{b}{z}, \quad \varepsilon_3^2 = 1,$$

with parameters $A = \frac{1}{2}a^2$, $B = -\frac{1}{2}b^2$, $C = \varepsilon_3(1 - a - b)$ and $D = -\frac{1}{2}$. So for P_V (1.1e) the associated Riccati equation is

$$z \frac{dw}{dz} = aw^2 + (b - a + \varepsilon_3 z)w - b. \quad (2.33)$$

To solve (2.33) we make the transformation $w(z) = -\frac{z}{a} \frac{d}{dz} \ln \phi(z)$ and this yields

$$z^2 \frac{d^2 \phi}{dz^2} + z(a - b - \varepsilon_3 z) \frac{d\phi}{dz} - ab\phi = 0, \quad (2.34)$$

which has solutions

$$\phi(z) = \begin{cases} z^b \{C_1 M(b, 1 + a + b, z) + C_2 U(b, 1 + a + b, z)\}, & \text{if } \varepsilon_3 = 1, \\ z^b \exp(-z) \{C_1 M(a + 1, a + b + 1, z) \\ \quad + C_2 U(a + 1, a + b + 1, z)\}, & \text{if } \varepsilon_3 = -1, \end{cases} \quad (2.35)$$

with C_1 and C_2 arbitrary constants and where $M(\alpha, \beta, z)$ and $U(\alpha, \beta, z)$ are Kummer functions. The special function solutions of (1.1e) and (1.7e) are given by the following theorem:

Theorem 2.9. *Let $\mathcal{F}_n(\psi)$ be the determinant given by*

$$\mathcal{F}_n(\psi) := \begin{vmatrix} \psi & \delta(\psi) & \dots & \delta^{(n-1)}(\psi) \\ \delta(\psi) & \delta^{(2)}(\psi) & \dots & \delta^{(n)}(\psi) \\ \vdots & \vdots & \ddots & \vdots \\ \delta^{(n-1)}(\psi) & \delta^{(n)}(\psi) & \dots & \delta^{(2n-2)}(\psi) \end{vmatrix}, \quad \delta = z \frac{d}{dz},$$

$\tau_n(f)$ be the bi-directional Wronskian determinant given by

$$\tau_n(\psi) := \begin{vmatrix} \psi & \delta(\psi) & \dots & \delta^{(n-1)}(\psi) \\ \delta(\psi) & \delta^{(2)}(\psi) & \dots & \delta^{(n)}(\psi) \\ \vdots & \vdots & \ddots & \vdots \\ \delta^{(n-1)}(\psi) & \delta^{(n)}(\psi) & \dots & \delta^{(2n-2)}(\psi) \end{vmatrix}, \quad \delta = \frac{d}{dz}$$

and

$$\psi_{a,b}(z; a, b) = C_1 M(a, b, z) + C_2 U(a, b, z), \quad (2.36)$$

with C_1 and C_2 arbitrary constants.

Then for $n \geq 0$ the special function solutions of P_V (1.1e) in the form

$$w_n^{[N]}(z; A^{[N]}, B^{[N]}, C^{[N]}, D^{[N]}, \varepsilon_3),$$

for $N = 1, 2$, are given by the following:

$$\begin{aligned}
w_n^{[1]}(z; A^{[1]}, B^{[1]}, C^{[1]}, D^{[1]}, 1) &= 1 + \frac{1}{\alpha - \beta - n} \left\{ \beta + z + z \frac{d}{dz} \ln \frac{\mathcal{F}_{n+1}(e^{-z}\psi_{\alpha, \beta+1})}{\mathcal{F}_n(e^{-z}\psi_{\alpha, \beta})} \right\} \\
&= 1 + \frac{1}{\alpha - \beta - n} \left\{ \beta + n + z \frac{d}{dz} \ln \frac{\tau_{n+1}(\psi_{\alpha-n+1, \beta-n+2})}{\tau_n(\psi_{\alpha-n+1, \beta-n+1})} \right\}, \\
w_n^{[2]}(z; A^{[2]}, B^{[2]}, C^{[2]}, D^{[2]}, -1) &= 1 + \frac{1}{\alpha + n} \left\{ z - \beta - z \frac{d}{dz} \ln \frac{\mathcal{F}_{n+1}(\psi_{\alpha+1, \beta+1})}{\mathcal{F}_n(\psi_{\alpha, \beta})} \right\} \\
&= 1 + \frac{1}{\alpha + n} \left\{ z - \beta - n - z \frac{d}{dz} \ln \frac{\tau_{n+1}(\psi_{\alpha+1, \beta-n+1})}{\tau_n(\psi_{\alpha, \beta-n+1})} \right\},
\end{aligned}$$

for the parameters

$$\{A^{[1]}, B^{[1]}, C^{[1]}, D^{[1]}\} = \left\{ \frac{1}{2}(-\alpha + \beta + n)^2, -\frac{1}{2}\alpha^2, 1 + n - \beta, -\frac{1}{2} \right\}, \quad (2.37a)$$

$$\{A^{[2]}, B^{[2]}, C^{[2]}, D^{[2]}\} = \left\{ \frac{1}{2}(\alpha + n)^2, -\frac{1}{2}(\beta - \alpha)^2, \beta - n - 1, -\frac{1}{2} \right\}. \quad (2.37b)$$

Proof. These results can be inferred from previous work done by Forrester and Witte [27] and Okamoto [59]. \square

2.3.9 The zeros of the Kummer functions

If $a - b \neq 0, -1, -2, \dots$ then ϕ has infinitely many zeros in \mathbb{C} . When $a, b \in \mathbb{R}$ the number of real zeros is finite [60, §13.9(i)]. The special function solutions of P_V (1.1e) are only bounded when $a = 0, -1, -2, \dots$ and n is even; this applies to both of the hierarchies here. This is no coincidence either as this is precisely the condition in the rational solutions when the polynomial roots that comprise the solutions are not sitting on the real line.

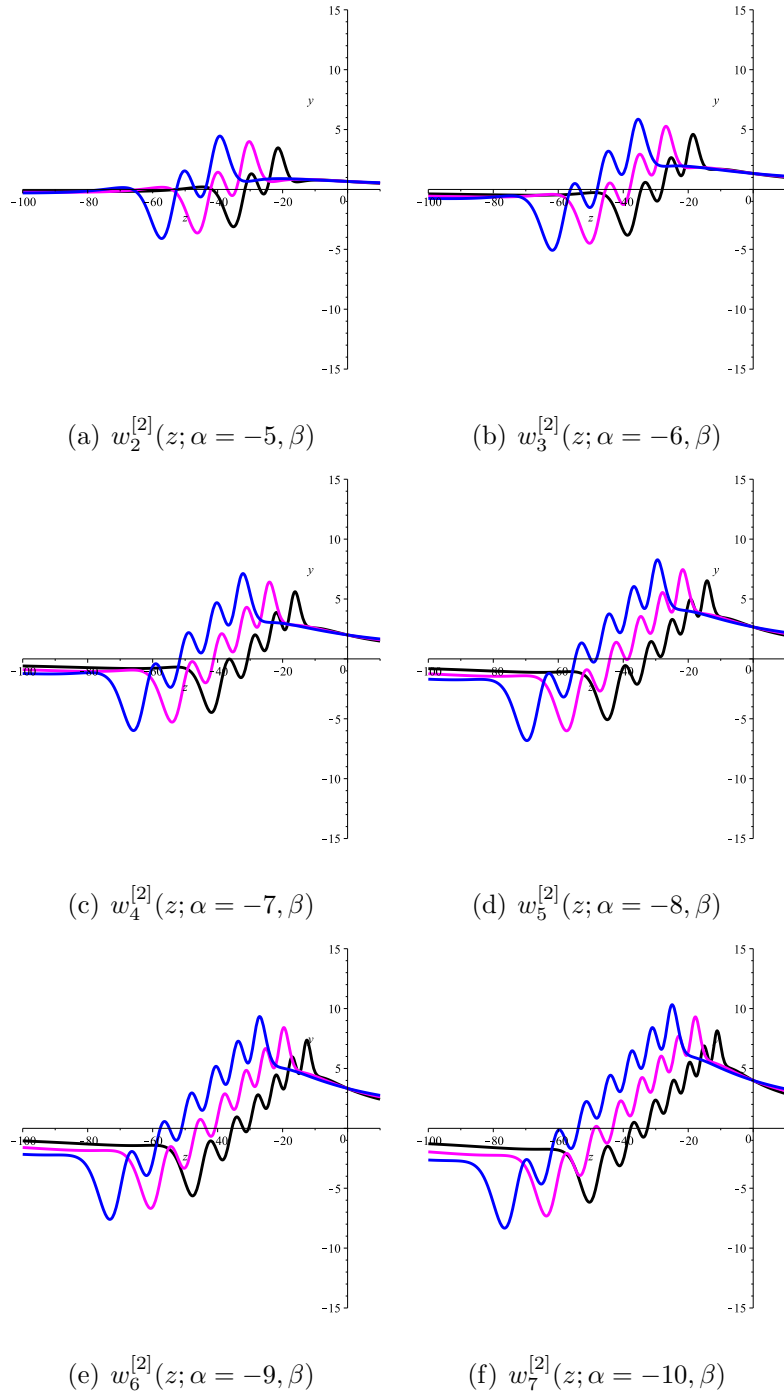


Figure 2.7: Kummer function solution plots to P_V (1.1e) with $\beta = \{-30, -40, -50\}$.

2.3.10 The sixth Painlevé equation P_{VI}

Theorem 2.10. *Equation P_{VI} (1.1f) has solutions expressible in terms of hypergeometric functions if and only if*

$$h_1 + h_2 + h_3 + h_4 = 2n + 1,$$

with $\varepsilon_j = \pm 1$, $j = 1, 2, 3, 4$, independently where $h_1 = \varepsilon_1\sqrt{2A}$, $h_2 = \varepsilon_2\sqrt{-2B}$, $h_3 = \varepsilon_3\sqrt{2C}$, $h_4 = \varepsilon_4\sqrt{1 - 2D}$.

Proof. See Fokas and Yortsos [22], Lukashovich and Yablonskii [43], Okamoto [56] and also Gromak, Laine and Shimomura [34]. \square

To obtain a special function solution of a P_{VI} (1.1f) we need to substitute (2.17) into P_{VI} (1.1f) which yields a rather large expression. However, by equating powers of w and solving gives

$$p_2(z) = \frac{a}{z(z-1)}, \quad p_1(z) = \frac{(b+1-a)z-c}{z(z-1)}, \quad p_0(z) = \frac{c-b-1}{z-1}, \quad \varepsilon_3^2 = 1,$$

with parameters $A = \frac{1}{2}a^2$, $B = -\frac{1}{2}(b+1-c)^2$, $C = \frac{1}{2}(c-a)^2$ and $D = \frac{1}{2}(1-b^2)$. So for P_{VI} (1.1f) the associated Riccati equation is

$$z(z-1)\frac{dw}{dz} = aw^2 + ((b+1-a)z-c)w - z(b+1-c). \quad (2.38)$$

To solve (2.38) we have to make the following transformation:

$$w(z) = \frac{1}{a} \left\{ c - (b+1)z - z(z-1)\frac{d}{dz} \ln \psi \right\}.$$

This yields

$$z(z-1)\frac{d^2\psi}{dz^2} + (c+1-(a+b+3)z)\frac{d\psi}{dz} - (a+1)(b+1)\psi = 0, \quad (2.39)$$

where ψ is the general hypergeometric function $F(a+1, b+1, c+1; z)$.

Theorem 2.11. Let $\tau_n(\psi_{a,b,c})$ be the determinant given by

$$\tau_n(\psi_{a,b,c}) := |\hat{\tau}_n(\psi_{a,b,c})| = \begin{vmatrix} \psi_{a,b,c} & \delta(\psi_{a,b,c}) & \dots & \delta^{(n-1)}(\psi_{a,b,c}) \\ \delta(\psi_{a,b,c}) & \delta^{(2)}(\psi_{a,b,c}) & \dots & \delta^{(n)}(\psi_{a,b,c}) \\ \vdots & \vdots & \ddots & \vdots \\ \delta^{(n-1)}(\psi_{a,b,c}) & \delta^{(n)}(\psi_{a,b,c}) & \dots & \delta^{(2n-2)}(\psi_{a,b,c}) \end{vmatrix}, \quad (2.40)$$

$\delta = z(z-1)\frac{d}{dz}$. Define

$$\mathcal{W}_n(\psi_{a,b,c}) := z^{n(1-n-2b)/2}(z-1)^{n(1-n)/2}\tau_n(\psi_{a,b,c}). \quad (2.41)$$

Then for $n \geq 0$ the special function solutions for P_{VI} (1.1f) in the form

$$w_n(z; A, B, C, D),$$

are given by the following:

$$w_n(z; A, B, C, D) = \frac{1}{a} \left\{ n + c - (2n + b + 1)z - z(z-1)\frac{d}{dz} \ln \frac{\mathcal{W}_{n+1}(\psi_{a+1,b+1,c+1})}{\mathcal{W}_n(\psi_{a-1,b+1,c})} \right\},$$

for the parameters

$$\{A, B, C, D\} = \left\{ \frac{1}{2}a^2, -\frac{1}{2}(b-c+n+1)^2, \frac{1}{2}(a-c-n)^2, \frac{1}{2}(1-b^2) \right\},$$

with $\psi_{a,b,c}$ the hypergeometric function and a polynomial of degree b

$$\psi_{a,b,c} = {}_2F_1(a, b, c; z)z^b,$$

Proof. See Okamoto [56] and Forrester and Whitte [28]. □

2.3.11 The zeros of the hypergeometric function

Consider the hypergeometric function $F(a, b, c; z)$. If a, b, c are real, $a, b, c-a, c-b \neq 0, -1, -2, \dots$, $b \geq a$ and $c \geq a+b$ then the hypergeometric function has no real

zeros when $a > 0$. The special function solutions to P_{VI} (1.1f) have no bounded solutions due to the fact that the asymptotic behaviour around $\pm\infty$ is linear. However, if an appropriate linear transformation is made we can have bounded solutions. This is precisely when $a > -1$, assuming that all the conditions above have been met.

2.4 Special function solutions to the σ -equations

In the following section we shall be defining all the special function solutions to all the Painlevé equations associated sigma equations.

2.4.1 The second Painlevé σ -equation

Theorem 2.12. *Let $\tau_{n,\nu}(z, \varepsilon)$ be the bi-directional Wronskian determinant given by*

$$\tau_n(z) := \begin{vmatrix} \psi & \delta(\psi) & \dots & \delta^{(n-1)}(\psi) \\ \delta(\psi) & \delta^{(2)}(\psi) & \dots & \delta^{(n)}(\psi) \\ \vdots & \vdots & \ddots & \vdots \\ \delta^{(n-1)}(\psi) & \delta^{(n)}(\psi) & \dots & \delta^{(2n-2)}(\psi) \end{vmatrix}, \quad \delta = \frac{d}{dz}.$$

Then for $n \geq 0$ the special function solutions for S_{II} (1.7b) in the form $\sigma(z; \alpha)$, are given by the following:

$$\sigma(z; \alpha) = \frac{d}{dz} \ln \tau_{n+1}(z), \quad (2.42)$$

for the parameter $\alpha = n + \frac{1}{2}$, where $\psi(z) = C_1 \text{Ai}(\zeta) + C_2 \text{Bi}(\zeta)$, with $\text{Ai}(\zeta)$ and $\text{Bi}(\zeta)$ Airy functions and $\zeta = -2^{-1/3}z$. Also note that $\sigma(z; -n - \frac{1}{2}) = \sigma(z; n + \frac{1}{2})$.

Proof. See Okamoto [58]. □

As with the Painlevé case it is interesting to note that these special functions will have no bounded solutions. The lack of bounded solutions is evident because the solutions can always be written in partial fractions with respect to the logarithmic derivative; this clearly shows an infinite number of singularities at $z = 0$ for all solutions in the hierarchy. Studying the zeros of the Airy type plots in figure 2.2 yields the same conclusion. However, there is one exception to this rule which allows the removal of all possibilities for poles on the real line. This can be done by restricting n to be odd and the constants $C_1 = \cos(\vartheta)$ and $C_2 = \sin(\vartheta)$ where ϑ is 0 or π .

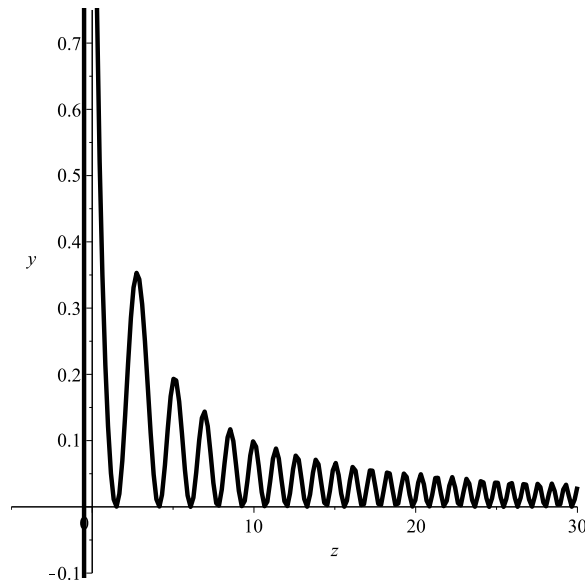


Figure 2.8: Special function solutions to S_{II} (1.1b) with $n = 1$, $C_1 = C_2 = \pi/2$.

2.4.2 The third Painlevé σ -equation

Theorem 2.13. *Let $\mathcal{F}_n(f)$ be the determinant given by*

$$\mathcal{F}_n(\psi) := \begin{vmatrix} \psi & \delta(\psi) & \dots & \delta^{(n-1)}(\psi) \\ \delta(\psi) & \delta^{(2)}(\psi) & \dots & \delta^{(n)}(\psi) \\ \vdots & \vdots & \ddots & \vdots \\ \delta^{(n-1)}(\psi) & \delta^{(n)}(\psi) & \dots & \delta^{(2n-2)}(\psi) \end{vmatrix}, \quad \delta = z \frac{d}{dz}$$

and $\mathcal{K}_n(\psi_\nu)$ be a determinant given by

$$\mathcal{K}_n(\psi_\nu) = \begin{vmatrix} \psi_\nu & \psi_{\nu-1} & \dots & \psi_{\nu-n+1} \\ \psi_{\nu+1} & \psi_\nu & \dots & \psi_{\nu-n+2} \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{\nu+n-1} & \psi_{\nu+n} & \dots & \psi_\nu \end{vmatrix},$$

with $\delta(f) = z \frac{d}{dz} f$.

Then for $n \geq 0$ the special function solutions of S_{III} (1.7c) in the form

$$\sigma_{\nu,n}^{[N]}(z; \vartheta_0^{[N]}, \vartheta_\infty^{[N]}, \varepsilon_1, \varepsilon_2),$$

for $N = 1, \dots, 4$, are given by the following:

$$\begin{aligned} \sigma_{\nu,n}^{[1]}(z; \vartheta_0^{[1]}, \vartheta_\infty^{[1]}, 1, 1) &= F + z \frac{d}{dz} \ln \mathcal{F}_n(\psi_\nu) = F + n(n-1) + z \frac{d}{dz} \ln \mathcal{K}_n(\psi_\nu), \\ \sigma_{\nu,n}^{[2]}(z; \vartheta_0^{[2]}, \vartheta_\infty^{[2]}, -1, 1) &= F + z \frac{d}{dz} \ln \mathcal{F}_n(\psi_\nu) = F + n(n-1) + z \frac{d}{dz} \ln \mathcal{K}_n(\psi_\nu), \\ \sigma_{\nu,n}^{[3]}(z; \vartheta_0^{[3]}, \vartheta_\infty^{[3]}, 1, -1) &= F + z \frac{d}{dz} \ln \mathcal{F}_n(\psi_\nu) = F + n(n-1) + z \frac{d}{dz} \ln \mathcal{K}_n(\psi_\nu), \\ \sigma_{\nu,n}^{[4]}(z; \vartheta_0^{[4]}, \vartheta_\infty^{[4]}, -1, -1) &= F + z \frac{d}{dz} \ln \mathcal{F}_n(\psi_\nu) = F + n(n-1) + z \frac{d}{dz} \ln \mathcal{K}_n(\psi_\nu), \end{aligned}$$

for the parameters

$$\begin{aligned}\{\vartheta_0^{[1]}, \vartheta_\infty^{[1]}\} &= \{\nu^2 + n^2, \nu^2 - n^2\}, \\ \{\vartheta_0^{[2]}, \vartheta_\infty^{[2]}\} &= \{\nu^2 + n^2, n^2 - \nu^2\}, \\ \{\vartheta_0^{[3]}, \vartheta_\infty^{[3]}\} &= \{\nu^2 + n^2, n^2 - \nu^2\}, \\ \{\vartheta_0^{[4]}, \vartheta_\infty^{[4]}\} &= \{\nu^2 + n^2, \nu^2 - n^2\},\end{aligned}$$

where $F = \frac{1}{2}(\varepsilon_1\varepsilon_2z^2/2 + \nu^2 - 2\varepsilon_1n\nu - n(n-1))$

for

$$\psi_\nu(z) = \begin{cases} z^\nu \{C_1J_\nu(z) + C_2Y_\nu(z)\}, & \text{if } \varepsilon_1 = 1, \quad \varepsilon_2 = 1, \\ z^{-\nu} \{C_1J_\nu(z) + C_2Y_\nu(z)\}, & \text{if } \varepsilon_1 = -1, \quad \varepsilon_2 = -1, \\ z^\nu \{C_1I_\nu(z) + C_2K_\nu(z)\}, & \text{if } \varepsilon_1 = 1, \quad \varepsilon_2 = -1, \\ z^{-\nu} \{C_1I_\nu(z) + C_2K_\nu(z)\}, & \text{if } \varepsilon_1 = -1, \quad \varepsilon_2 = 1. \end{cases}$$

Proof. See Okamoto [58]; also Forrester and Witte [26]. □

Then for $n \geq 0$ the special function solutions of $S_{III'}$ in the form

$$\sigma_{\nu,n}^{[N]}(z; \vartheta_0^{[N]}, \vartheta_\infty^{[N]}, \varepsilon_1, \varepsilon_2),$$

for $N = 1, \dots, 4$, are given by the following:

$$\begin{aligned}\sigma_{\nu,n}^{[1]}(z; \vartheta_0^{[1]}, \vartheta_\infty^{[1]}, 1, 1) &= F + t \frac{d}{dz} \ln \mathcal{F}_n(\psi_\nu) = F + \frac{n}{2}(n-1) + t \frac{d}{dz} \ln \mathcal{K}_n(\psi_\nu), \\ \sigma_{\nu,n}^{[2]}(z; \vartheta_0^{[2]}, \vartheta_\infty^{[2]}, -1, 1) &= F + t \frac{d}{dz} \ln \mathcal{F}_n(\psi_\nu) = F + \frac{n}{2}(n-1) + t \frac{d}{dz} \ln \mathcal{K}_n(\psi_\nu), \\ \sigma_{\nu,n}^{[3]}(z; \vartheta_0^{[3]}, \vartheta_\infty^{[3]}, 1, -1) &= F + t \frac{d}{dz} \ln \mathcal{F}_n(\psi_\nu) = F + \frac{n}{2}(n-1) + t \frac{d}{dz} \ln \mathcal{K}_n(\psi_\nu), \\ \sigma_{\nu,n}^{[4]}(z; \vartheta_0^{[4]}, \vartheta_\infty^{[4]}, -1, -1) &= F + t \frac{d}{dz} \ln \mathcal{F}_n(\psi_\nu) = F + \frac{n}{2}(n-1) + t \frac{d}{dz} \ln \mathcal{K}_n(\psi_\nu),\end{aligned}$$

for the parameters

$$\begin{aligned}\{\vartheta_0^{[1]}, \vartheta_\infty^{[1]}\} &= \{\nu + n, \nu - n\}, \\ \{\vartheta_0^{[2]}, \vartheta_\infty^{[2]}\} &= \{\nu + n, n - \nu\}, \\ \{\vartheta_0^{[3]}, \vartheta_\infty^{[3]}\} &= \{\nu + n, n - \nu\}, \\ \{\vartheta_0^{[4]}, \vartheta_\infty^{[4]}\} &= \{\nu + n, \nu - n\},\end{aligned}$$

where $F = \frac{1}{2}(\varepsilon_1\varepsilon_2t + \nu^2/2 + n(1 - \varepsilon_1\nu) - n^2/2)$ for

$$\psi_\nu(t) = \begin{cases} t^{\nu/2} \{C_1 J_\nu(2\sqrt{t}) + C_2 Y_\nu(2\sqrt{t})\}, & \text{if } \varepsilon_1 = 1, \quad \varepsilon_2 = 1, \\ t^{-\nu/2} \{C_1 J_\nu(2\sqrt{t}) + C_2 Y_\nu(2\sqrt{t})\}, & \text{if } \varepsilon_1 = -1, \quad \varepsilon_2 = -1, \\ t^{\nu/2} \{C_1 I_\nu(2\sqrt{t}) + C_2 K_\nu(2\sqrt{t})\}, & \text{if } \varepsilon_1 = 1, \quad \varepsilon_2 = -1, \\ t^{-\nu/2} \{C_1 I_\nu(2\sqrt{t}) + C_2 K_\nu(2\sqrt{t})\}, & \text{if } \varepsilon_1 = -1, \quad \varepsilon_2 = 1. \end{cases}$$

Proof. See Okamoto [58]; also Forrester and Witte [26]. □

The special function solutions to S_{III} (1.1c) and $S_{III'}$ (1.24) have no bounded solutions due to the fact that the asymptotic behaviour around $\pm\infty$ is quadratic. However, if an appropriate quadratic transformation is made we can have bounded solutions. This is precisely when $\varepsilon_1 = -1$, $\nu > 0$ and $C_1 C_2 > 0$, for all n , where n is the number of the solution in the hierarchy. The alternative case is when $\varepsilon_2 = 1$, $\nu < 0$ and $C_1 C_2 > 0$, for all n , where n is the number of the solution in the hierarchy.

2.4.3 The fourth Painlevé σ -equation

Theorem 2.14. *Let $\tau_{n,\nu}(z, \varepsilon)$ be the bi-directional Wronskian determinant given by*

$$\tau_{n,\nu}(z, \varepsilon) := \begin{vmatrix} \psi & \delta(\psi) & \dots & \delta^{(n-1)}(\psi) \\ \delta(\psi) & \delta^{(2)}(\psi) & \dots & \delta^{(n)}(\psi) \\ \vdots & \vdots & \ddots & \vdots \\ \delta^{(n-1)}(\psi) & \delta^{(n)}(\psi) & \dots & \delta^{(2n-2)}(\psi) \end{vmatrix}, \quad \delta = \frac{d}{dz}.$$

Then for $n \geq 0$ the special function solutions of S_{IV} (1.7d) in the form

$$\sigma_{n,\nu}^{[N]}(z; \vartheta_0^{[N]}, \vartheta_\infty^{[N]}),$$

for $N = 1, 2, 3$, are given by the following:

$$\sigma_{n,\nu}^{[1]}(z; \vartheta_0^{[1]}, \vartheta_\infty^{[1]}) = \frac{d}{dz} \ln \tau_{\nu,n}(z; \varepsilon), \quad (2.47a)$$

$$\sigma_{n,\nu}^{[2]}(z; \vartheta_0^{[2]}, \vartheta_\infty^{[2]}) = \frac{d}{dz} \ln \tau_{\nu,n}(z; \varepsilon) - 2\varepsilon n z, \quad (2.47b)$$

$$\sigma_{n,\nu}^{[3]}(z; \vartheta_0^{[3]}, \vartheta_\infty^{[3]}) = \frac{d}{dz} \ln \tau_{\nu,n}(z; \varepsilon) + 2\varepsilon(\nu - n + 1)z. \quad (2.47c)$$

for the parameters

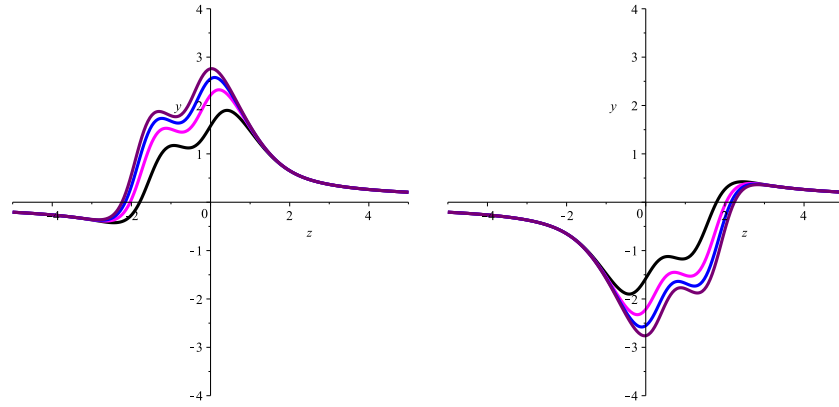
$$\{\vartheta_0^{[1]}, \vartheta_\infty^{[1]}\} = \{\varepsilon(\nu - n + 1), -\varepsilon n\},$$

$$\{\vartheta_0^{[2]}, \vartheta_\infty^{[2]}\} = \{\varepsilon n, \varepsilon(\nu + 1)\},$$

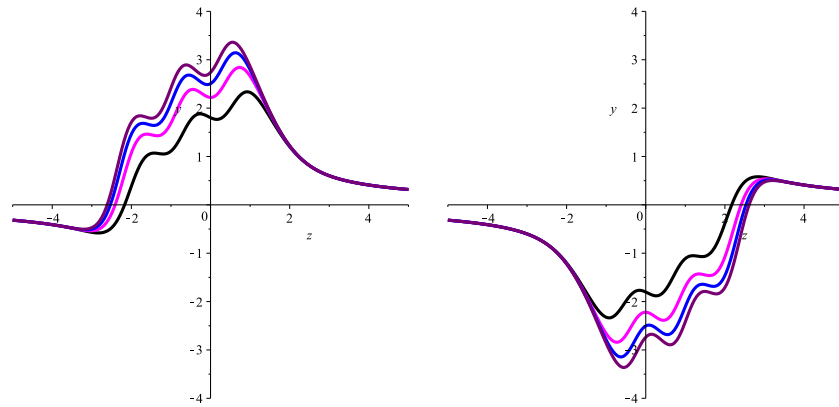
$$\{\vartheta_0^{[3]}, \vartheta_\infty^{[3]}\} = \{-\varepsilon(\nu + 1), -\varepsilon(\nu - n + 1)\},$$

Proof. See Okamoto [58]; also Forrester and Witte [26]. □

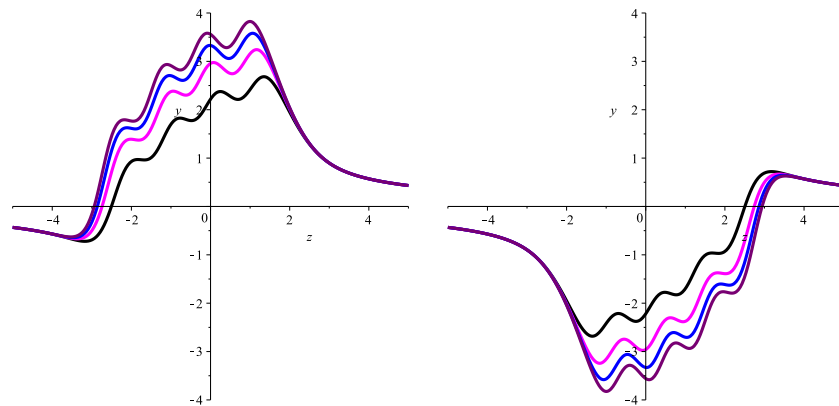
The special function solutions of S_{IV} (1.7d) are only bounded when either: $\nu < 0$ and $C_1 C_2 > 0$ when $\varepsilon = 1$ or $\nu > -1$ and $C_1 C_2 > 0$ when $\varepsilon = -1$ [60, §12.11].



(a) $n=2, C_1=1, C_2=\{10,20,30,40\}$ (b) $n=2, C_2=1, C_1=\{10,20,30,40\}$

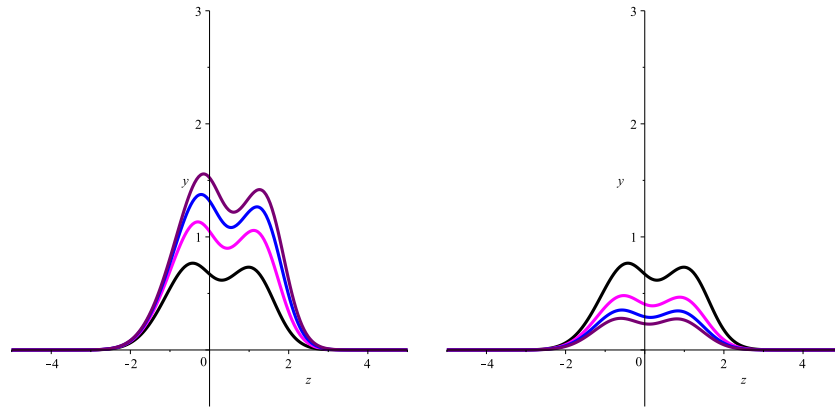


(c) $n=3, C_1=1, C_2=\{10,20,30,40\}$ (d) $n=3, C_2=1, C_1=\{10,20,30,40\}$

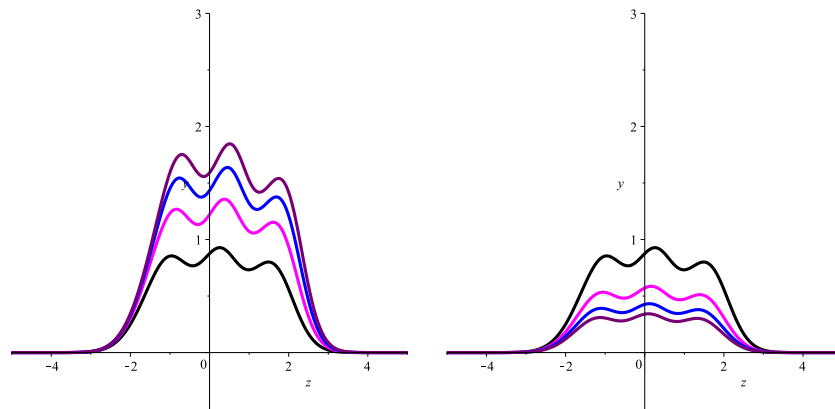


(e) $n=4, C_1=1, C_2=\{10,20,30,40\}$ (f) $n=4, C_2=1, C_1=\{10,20,30,40\}$

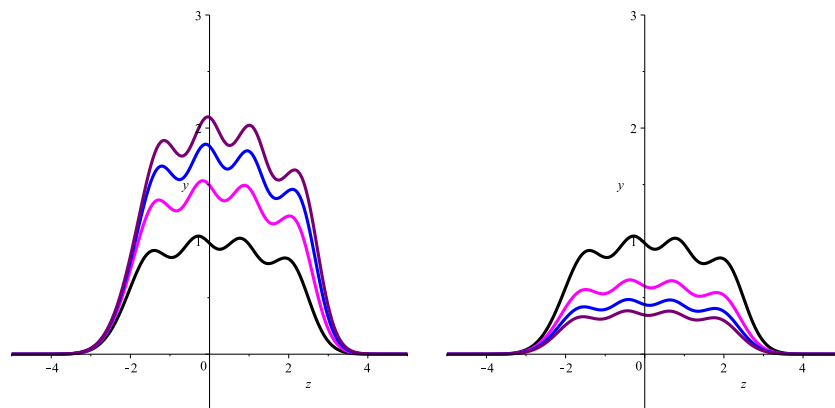
Figure 2.9: Parabolic cylinder function solution plots of $\sigma^{[2]}(z; \nu = -\frac{3}{2}, \varepsilon = 1)$.



(a) $n=2, \nu=1, C_1=1, C_2=\{1,2,3,4\}$ (b) $n=2, \nu=1, C_2=1, C_1=\{1,2,3,4\}$

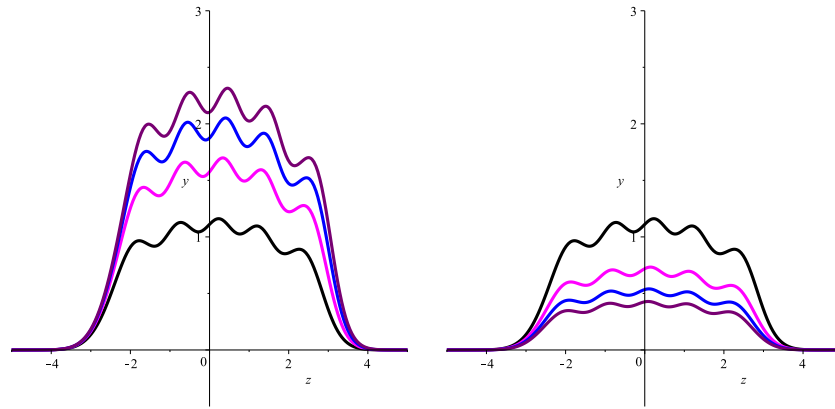


(c) $n=3, \nu=2, C_1=1, C_2=\{1,2,3,4\}$ (d) $n=3, \nu=2, C_2=1, C_1=\{1,2,3,4\}$

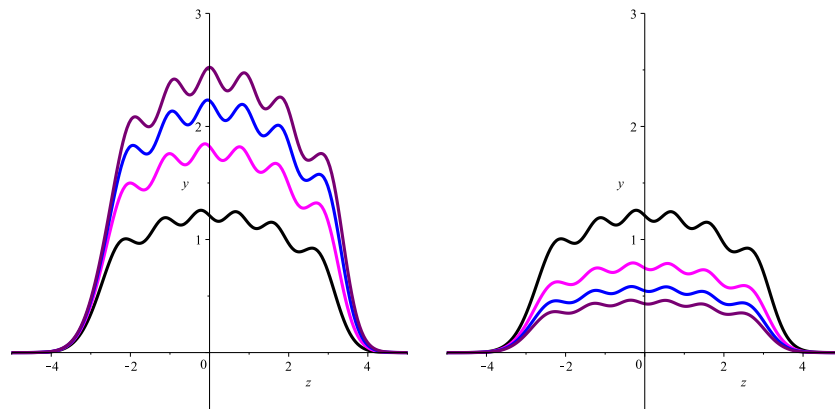


(e) $n=4, \nu=3, C_1=1, C_2=\{1,2,3,4\}$ (f) $n=4, \nu=3, C_2=1, C_1=\{1,2,3,4\}$

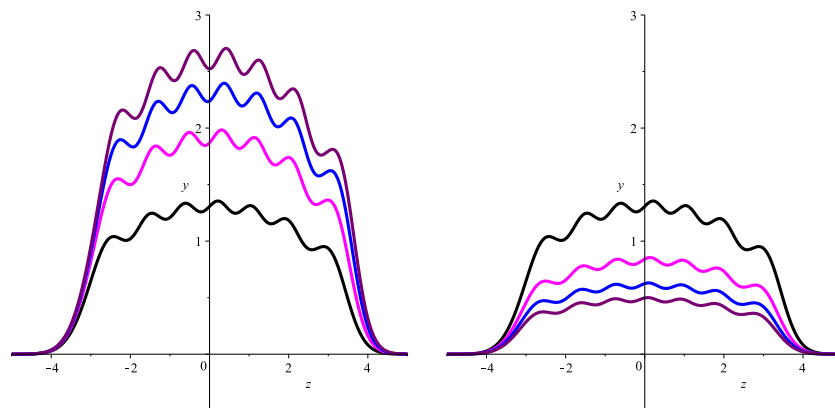
Figure 2.10: Error function solution plots of $\sigma_{n,\nu}^{[1]}(z; n - \nu - 1, n)$.



(a) $n=5, \nu=4, C_1=1, C_2=\{1,2,3,4\}$ (b) $n=5, \nu=4, C_2=1, C_1=\{1,2,3,4\}$



(c) $n=6, \nu=5, C_1=1, C_2=\{1,2,3,4\}$ (d) $n=6, \nu=5, C_2=1, C_1=\{1,2,3,4\}$



(e) $n=7, \nu=6, C_1=1, C_2=\{1,2,3,4\}$ (f) $n=7, \nu=6, C_2=1, C_1=\{1,2,3,4\}$

Figure 2.11: Error function solution plots of $\sigma_{n,\nu}^{[1]}(z; n - \nu - 1, n)$.

2.4.4 The fifth Painlevé σ -equation

Theorem 2.15. *Let $\mathcal{F}_n(\psi)$ be the determinant given by*

$$\mathcal{F}_n(\psi) := \begin{vmatrix} \psi & \delta(\psi) & \dots & \delta^{(n-1)}(\psi) \\ \delta(\psi) & \delta^{(2)}(\psi) & \dots & \delta^{(n)}(\psi) \\ \vdots & \vdots & \ddots & \vdots \\ \delta^{(n-1)}(\psi) & \delta^{(n)}(\psi) & \dots & \delta^{(2n-2)}(\psi) \end{vmatrix}, \quad \delta = z \frac{d}{dz},$$

$\tau_n(\psi)$ be the bi-directional Wronskian determinant given by

$$\tau_n(\psi) := \begin{vmatrix} \psi & \delta(\psi) & \dots & \delta^{(n-1)}(\psi) \\ \delta(\psi) & \delta^{(2)}(\psi) & \dots & \delta^{(n)}(\psi) \\ \vdots & \vdots & \ddots & \vdots \\ \delta^{(n-1)}(\psi) & \delta^{(n)}(\psi) & \dots & \delta^{(2n-2)}(\psi) \end{vmatrix}, \quad \delta = \frac{d}{dz}$$

and

$$\psi_{a,b}(z; a, b) = C_1 M(a, b, z) + C_2 U(a, b, z), \quad (2.49)$$

with C_1 and C_2 arbitrary constants.

Then for $n \geq 0$ the special function solutions of S_V (1.7e) in the form

$$\sigma_n^{[N]}(z; \kappa_0^{[N]}, \kappa_1^{[N]}, \kappa_2^{[N]}, \kappa_3^{[N]}, \varepsilon_3),$$

for $N = 1, 2$ are given by the following:

$$\begin{aligned}
\sigma_n^{[1]}(z; \kappa_0^{[1]}, \kappa_1^{[1]}, \kappa_2^{[1]}, \kappa_3^{[1]}, 1) &= z \frac{d}{dz} \ln \mathcal{F}_{n+1}(e^{-z} \psi_{\alpha, \beta}) + \frac{1}{4} (2\alpha + 1 + \beta + 3z) (n + 1) \\
&\quad - \frac{5}{8} (n + 1)^2 + \frac{1}{8} (2\alpha + 1 - \beta) (-2\alpha - 1 - 2z + \beta) \\
&= z \frac{d}{dz} \ln \tau_{n+1}(\psi_{\alpha-n, \beta-n}) + \frac{1}{4} (2\alpha - 1 + \beta - z) (n + 1) \\
&\quad - \frac{1}{8} (n + 1)^2 + \frac{1}{8} (2\alpha + 1 - \beta) (-2\alpha - 1 - 2z + \beta), \\
\sigma_n^{[2]}(z; \kappa_0^{[2]}, \kappa_1^{[2]}, \kappa_2^{[2]}, \kappa_3^{[2]}, -1) &= z \frac{d}{dz} \ln \mathcal{F}_n(z^\lambda \psi_{\alpha+1, \beta+1}) + \frac{1}{4} (3\beta + 2 - 2\alpha - 3z - 4\lambda) n \\
&\quad - \frac{5}{8} n^2 - \frac{1}{8} (2\alpha - \beta) (2\alpha + 2z - \beta) \quad (2.50a) \\
&= z \frac{d}{dz} \ln \tau_n(\psi_{\alpha+1, \beta-n+2}) + \frac{1}{4} (3\beta - 2\alpha - 3z) n \\
&\quad - \frac{1}{8} n^2 - \frac{1}{8} (2\alpha - \beta) (2\alpha + 2z - \beta), \quad (2.50b)
\end{aligned}$$

for the parameters

$$\begin{aligned}
\{\kappa_0^{[1]}, \kappa_1^{[1]}, \kappa_2^{[1]}, \kappa_3^{[1]}\} &= \frac{1}{4} \{2\alpha - \beta + n + 2, n + 2 - 2\alpha - \beta, 2\alpha - \beta - 3n - 2, 3\beta + n \\
&\quad - 2\alpha - 2\}, \quad (2.51a)
\end{aligned}$$

$$\begin{aligned}
\{\kappa_0^{[2]}, \kappa_1^{[2]}, \kappa_2^{[2]}, \kappa_3^{[2]}\} &= -\frac{1}{4} \{2\alpha + \beta + n, \beta - 3n - 2\alpha, n + 2\alpha - 3\beta, \beta + n - 2\alpha\}. \\
&\quad (2.51b)
\end{aligned}$$

Proof. These results can be inferred from previous work done by Forrester and Witte, [27] and Okamoto [59]. \square

It is interesting to note there is a mapping between the parameters (2.51a) and (2.51b) which gives rise to the following corollary:

Corollary 2.1. *The two determinants $\mathcal{F}_n(\psi_{a,b})$ and $\mathcal{F}_n(e^{-z} \psi_{c,d})$ have the following relation:*

$$\frac{d}{dz} \ln \mathcal{F}_n(\psi_{\nu, \mu}) = nz + \frac{d}{dz} \ln \mathcal{F}_n(e^{-z} \psi_{\nu+n-1, \mu}). \quad (2.52)$$

We can also utilise the fact that (2.50b) is equal to (2.50a) which gives rise to the following corollary:

Corollary 2.2. *The two determinants $\mathcal{F}_n(\psi_{a,b})$ and $\tau_n(\psi_{c,d})$ have the following relation:*

$$\frac{d}{dz} \ln \mathcal{F}_n(\psi_{a,b}) = \frac{d}{dz} \ln \tau_n(\psi_{a,b+1-n}) - \frac{n}{2z} (1-n). \quad (2.53)$$

Corollary 2.3. *The two determinants $\tau_n(\Psi)$ and $\tau_n(e^{-z}\Psi)$ have the following relation:*

$$\tau_n(\Psi e^{-z}) = \tau_n(\Psi) e^{-nz}. \quad (2.54)$$

The special function solutions to S_V (1.1e) have no bounded solutions due to the fact that the asymptotic behaviour around $\pm\infty$ is linear. However, if an appropriate linear transformation is made we can have bounded solutions. This is precisely when $a = 0, -1, -2, -3, \dots$ and n is even.

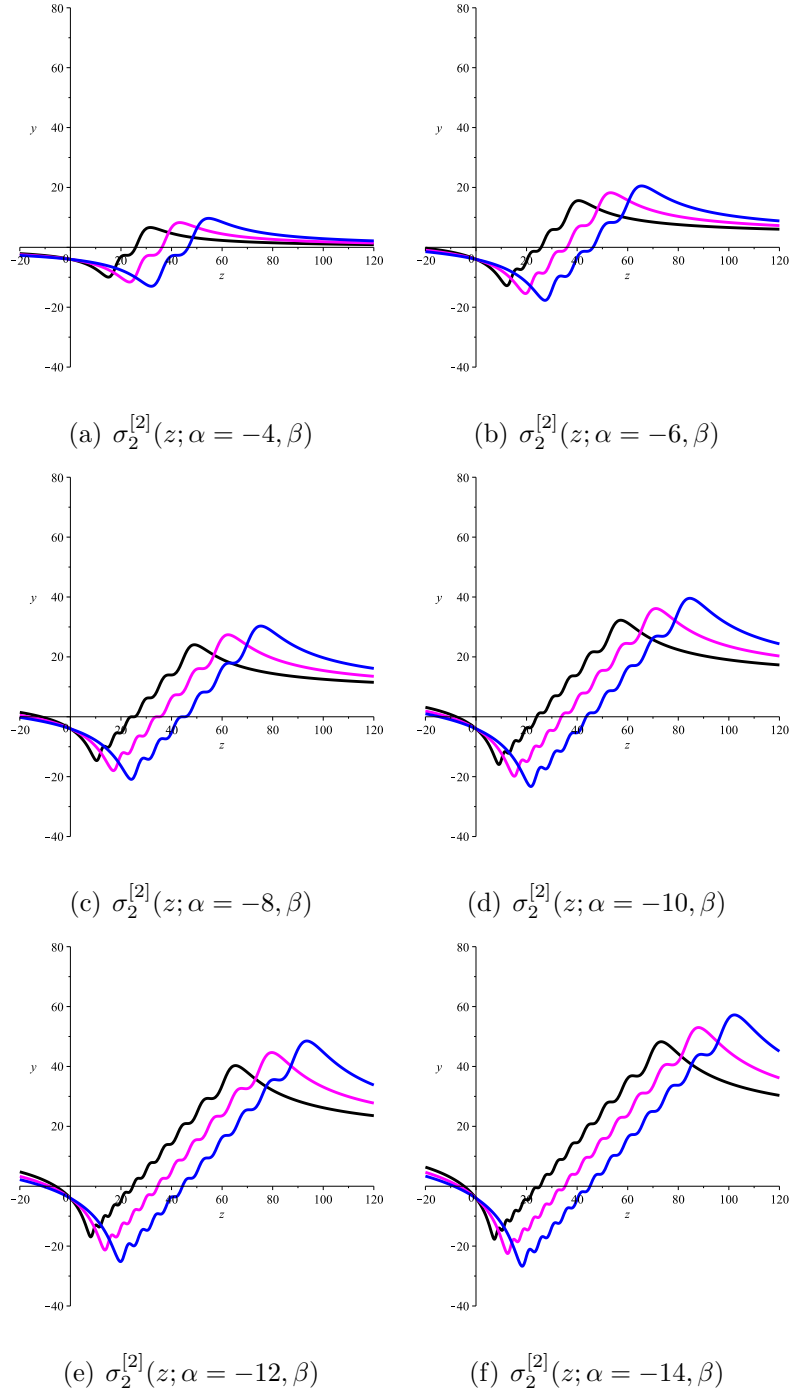


Figure 2.12: Some special function solutions to S_V (1.7e) with $\beta = \{20, 30, 40\}$.

It is interesting to note that the number of “kinks” in the solutions is equal to $-\alpha - n - 1$.

2.4.5 The sixth Painlevé σ -equation

Theorem 2.16. *Let $\tau_n(\psi_{a,b,c})$ be the determinant given by*

$$\tau_n(\psi_{a,b,c}) := |\hat{\tau}_n(\psi_{a,b,c})| = \begin{vmatrix} \psi_{a,b,c} & \delta(\psi_{a,b,c}) & \dots & \delta^{(n-1)}(\psi_{a,b,c}) \\ \delta(\psi_{a,b,c}) & \delta^{(2)}(\psi_{a,b,c}) & \dots & \delta^{(n)}(\psi_{a,b,c}) \\ \vdots & \vdots & \ddots & \vdots \\ \delta^{(n-1)}(\psi_{a,b,c}) & \delta^{(n)}(\psi_{a,b,c}) & \dots & \delta^{(2n-2)}(\psi_{a,b,c}) \end{vmatrix},$$

$\delta = z(z-1)\frac{d}{dz}$. Define

$$\mathcal{W}_n(\psi_{a,b,c}) := z^{n(1-n-2b)/2}(z-1)^{n(1-n)/2}\tau_n(\psi_{a,b,c}).$$

Then for $n \geq 0$ the special function solutions for S_{VI} (1.7f) in the form

$$\sigma_n(z; \kappa_1, \kappa_2, \kappa_3, \kappa_4),$$

are given by the following:

$$\begin{aligned} \sigma_n(z; \kappa_1, \kappa_2, \kappa_3, \kappa_4) &= \frac{1}{4}(n+1)(4az - a + b - 2c + 1) - \frac{1}{4}(a-b+1)^2 z \\ &\quad + \frac{1}{4}(a^2 + a + b^2 - b - ac - bc) + z(z-1)\frac{d}{dz} \ln \mathcal{W}_{n+1}(\psi_{a,b,c}), \end{aligned}$$

for the parameters

$$\{\kappa_1, \kappa_2, \kappa_3, \kappa_4\} = \left\{ -\frac{1}{2}(a-b-2n-1), \frac{1}{2}(a+b-2c+1), -\frac{1}{2}(a-b+1), \frac{1}{2}(a+b-1) \right\}.$$

Proof. See Okamoto [56] and Forrester and Whitte [28]. \square

The special function solutions to P_{VI} (1.1f) have no bounded solutions due to the fact that the asymptotic behaviour around $\pm\infty$ is linear. However, if an appropriate linear transformation is made we can have bounded solutions. This is precisely when $a > 0$, assuming all the conditions we set before in the section “The zeros of the hypergeometric function”.

3 Rational function solutions

In the following section we will define the rational function solutions to the Painlevé equations and their associated sigma equations. However, first we must define some new polynomials that we will need.

3.1 Special polynomials

Now we will define all the special polynomials that we need in order to define the rational solutions themselves.

3.1.1 The Yablonskii-Vorob'ev polynomials

The Yablonskii-Vorob'ev polynomials are defined to be the solutions to the differential-difference equation

$$Q_{n+1}Q_{n-1} = zQ_n^2 - 4 \left[Q_n \frac{d^2 Q_n}{dz^2} - \left(\frac{dQ_n}{dz} \right)^2 \right], \quad (3.1)$$

with $Q_0(z) = 1$ and $Q_1(z) = z$. The Yablonskii-Vorob'ev polynomials are monic polynomials with degree $\frac{1}{2}n(n+1)$. A table of The Yablonskii-Vorob'ev polynomials is shown below.

Table 3.1: Table of the Yablonskii-Vorob'ev polynomials

$$Q_2(z) = z^3 + 4$$

$$Q_3(z) = z^6 + 20z^3 - 80$$

$$Q_4(z) = (z^9 + 60z^6 + 11200)z$$

$$Q_5(z) = z^{15} + 140z^{12} + 2800z^9 + 78400z^6 - 3136000z^3 - 6272000$$

$$Q_6(z) = z^{21} + 280z^{18} + 18480z^{15} + 627200z^{12} - 17248000z^9$$

$$+1448832000z^6 + 19317760000z^3 - 38635520000$$

$$Q_7(z) = (z^{27} + 504z^{24} + 75600z^{21} + 5174400z^{18} + 62092800z^{15} + 13039488000z^{12}$$

$$-828731904000z^9 - 49723914240000z^6 - 3093932441600000)z$$

Some root plots of The Yablonskii-Vorob'ev polynomials are shown below.

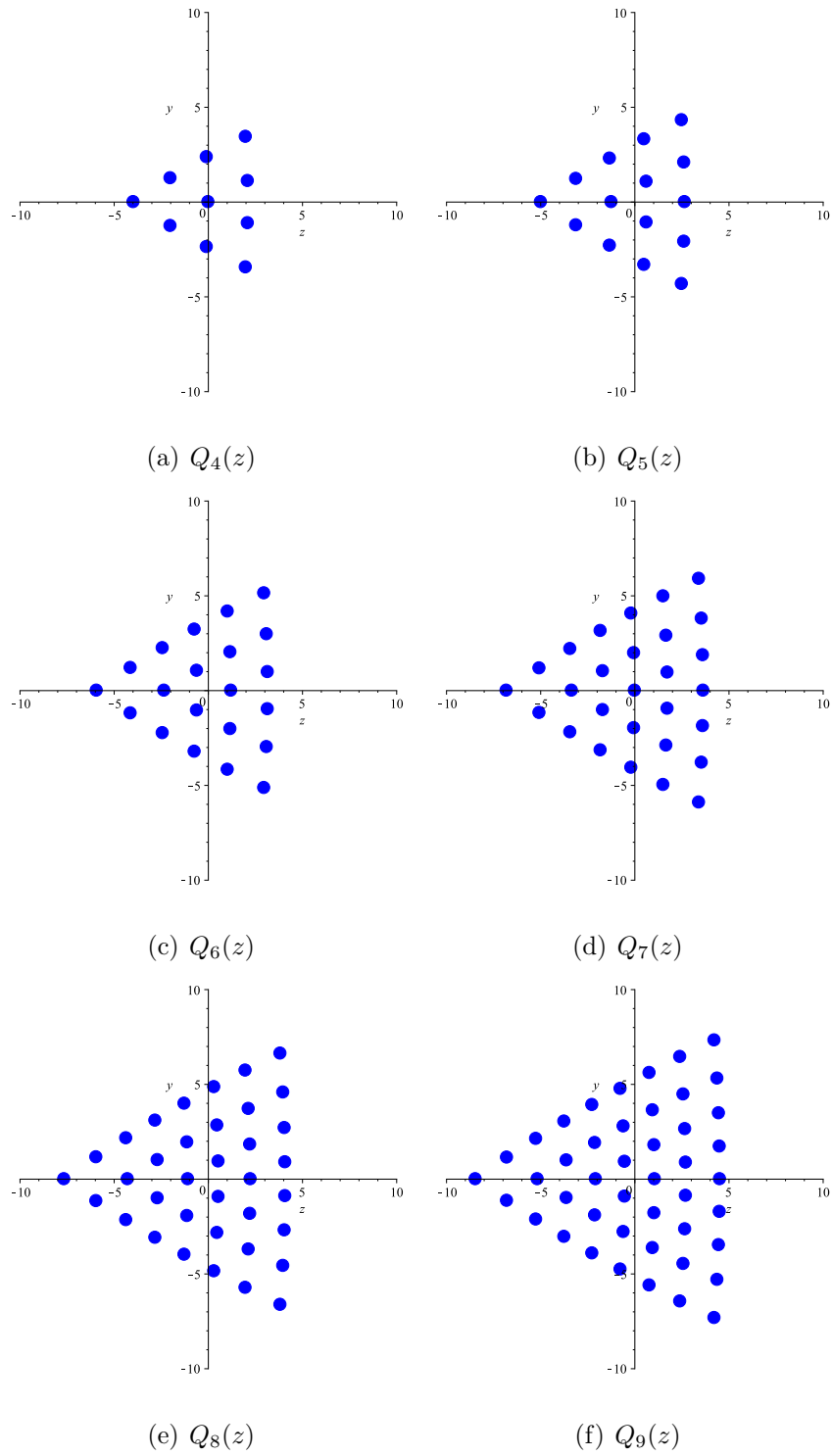


Figure 3.1: Roots of some Yablonskii-Vorob'ev polynomials.

The following remarks are based on observations:

Remark 3.1.

- i) The Yablonskii-Vorob'ev polynomials always have real roots, regardless of n . This will be an important point later in this thesis.*
- ii) The Yablonskii-Vorob'ev polynomials take the form of "triangles", though these are only approximate triangles since the roots lie on arcs rather than straight lines.*
- iii) The Yablonskii-Vorob'ev polynomials have degree equal to $\frac{1}{2}n(n+1)$.*
- iv) The Yablonskii-Vorob'ev polynomials always have $\frac{1}{2}n(n+1)$ roots.*

3.1.2 The generalised associated Laguerre with $\delta = \frac{d^2}{dz^2}$

Suppose that $\tau_{\mu,n}(z)$ satisfies the recursion relation

$$(2n+1)\tau_{\mu,n+1}\tau_{\mu,n-1} = -z \left[\tau_{\mu,n} \frac{d^2\tau_{\mu,n}}{dz^2} - \left(\frac{d\tau_{\mu,n}}{dz} \right)^2 \right] - \tau_{\mu,n} \frac{d\tau_{\mu,n}}{dz} + (z+\mu)\tau_{\mu,n}^2, \quad (3.2)$$

with $\tau_{\mu,-1}(z) = \tau_{\mu,0}(z) = 1$. It is interesting to note that setting $\mu = 1$ recovers the Laguerre polynomials. A table and root plots of $\tau_{\mu,n}(z)$ polynomials is shown below.

Table 3.2: The generalised associated Laguerre with $\delta = \frac{d^2}{dz^2}$

$$\tau_1(z; \mu) = z + \mu$$

$$\tau_2(z; \mu) = -\frac{1}{3} \{ (z + \mu)^3 - \mu \}$$

$$\tau_3(z; \mu) = -\frac{1}{3^2 \cdot 5} \{ (z + \mu)^6 - 5 \mu (z + \mu)^3 + 9 \mu (z + \mu) - 5 \mu^2 \}$$

$$\tau_4(z; \mu) = \frac{1}{3^3 \cdot 5^2 \cdot 7} \{ (z + \mu)^{10} - 15 \mu (z + \mu)^7 + 63 \mu (z + \mu)^5 - 225 \mu (z + \mu)^3 + 315 \mu^2 (z + \mu)^2 - 175 \mu^3 (z + \mu) + 36 \mu^2 \}$$

$$\begin{aligned} \tau_5(z; \mu) = & \frac{1}{3^4 \cdot 5^3 \cdot 7^2 \cdot 9} \{ (z + \mu)^{15} - 35 \mu (z + \mu)^{12} + 252 \mu (z + \mu)^{10} + 175 \mu^2 (z + \mu)^9 \\ & - 2025 \mu (z + \mu)^8 + 945 \mu^2 (z + \mu)^7 - 1225 \mu (\mu - 3) (\mu + 3) (z + \mu)^6 \\ & - 26082 \mu^2 (z + \mu)^5 + 33075 \mu^3 (z + \mu)^4 - 350 \mu^2 (35 \mu^2 + 36) (z + \mu)^3 \\ & + 11340 \mu^3 (z + \mu)^2 - 225 \mu^2 (7 \mu - 6) (7 \mu + 6) (z + \mu) \\ & + 7 \mu^3 (875 \mu^2 - 828) \} \end{aligned}$$

The polynomials $\tau_{\mu,n}(z)$ also have a determinantal representation

$$\tau_{\mu,n}(z) := \begin{vmatrix} \psi_{\mu,n} & \delta^2(\psi_{\mu,n}) & \dots & \delta^{(2n-2)}(\psi_{\mu,n}) \\ \delta(\psi_{\mu,n}) & \delta^{(3)}(\psi_{\mu,n}) & \dots & \delta^{(2n-1)}(\psi_{\mu,n}) \\ \vdots & \vdots & \ddots & \vdots \\ \delta^{(n-1)}(\psi_{\mu,n}) & \delta^{(n+1)}(\psi_{\mu,n}) & \dots & \delta^{(3n-3)}(\psi_{\mu,n}) \end{vmatrix}, \quad \delta = \frac{d}{dz}.$$

where $\psi_{\mu,n} = L_{\mu+1-2n}^{2n-1}(-z)$ and where L_b^a is an associated Laguerre polynomial.

Some roots plots for $\tau_{\mu,n}(z)$ are shown below.

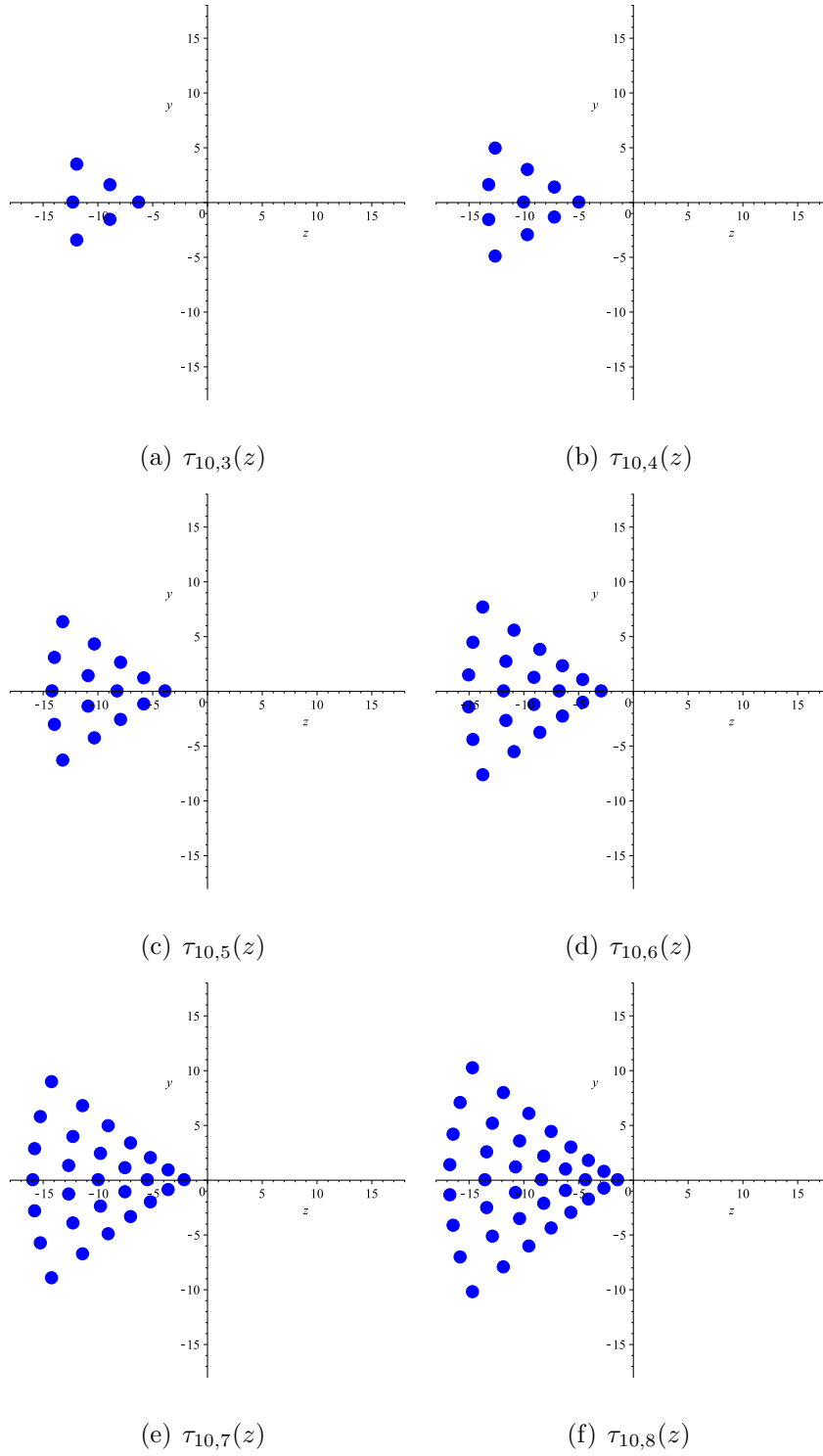


Figure 3.2: Roots of $\tau_{10,n}(z)$ polynomials.

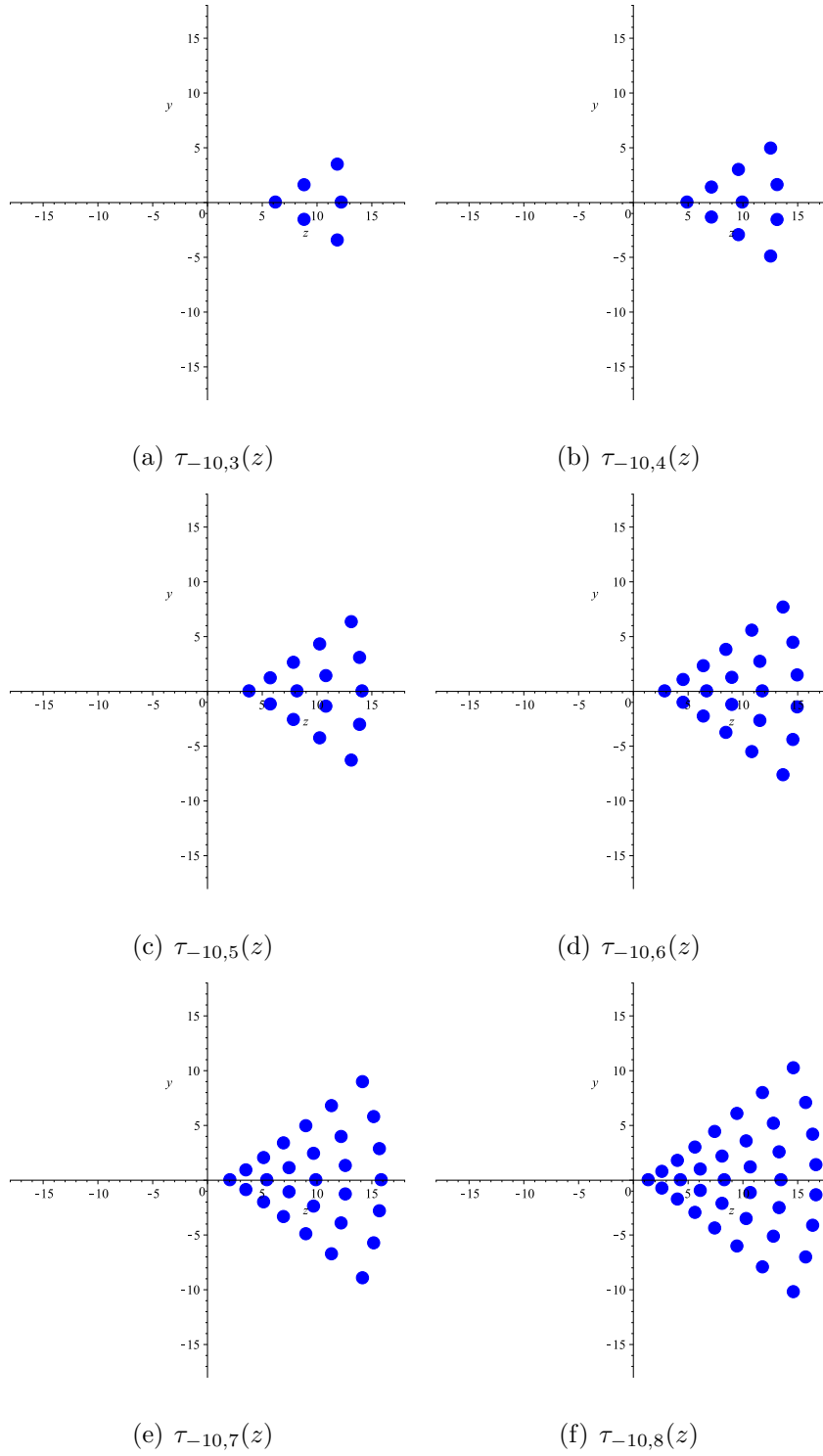


Figure 3.3: Roots of $\tau_{-10,n}(z)$ polynomials.

Note that by varying the parameter μ we can animate these roots. See the “Animations” folder on the supplied USB flash drive.

The following remarks are based on observations:

Remark 3.2.

- i) The $\tau_{\mu,n}(z)$ polynomials always have real roots, regardless of n .*
- ii) The $\tau_{\mu,n}(z)$ polynomials take the form of “triangles”, though these are only approximate triangles since the root “columns” lie on arcs rather than straight lines.*
- iii) The $\tau_{\mu,n}(z)$ polynomials have degree equal to $\frac{1}{2}n(n+1)$.*
- iv) The $\tau_{\mu,n}(z)$ polynomials always have $\frac{1}{2}n(n+1)$ roots.*
- v) The $\tau_{\mu,n}(z)$ polynomials have the property that $\tau_{\mu,n}(z) = \tau_{-\mu,n}(-z)$.*

Note again that the $\tau_{\mu,n}(z)$ polynomials always have real roots therefore removing the chance for bounded solutions. The $\tau_{\mu,n}(z)$ polynomials are very similar to the Umemura polynomials, which Clarkson in [15] essentially redefined so they were no longer polynomials in $1/z$ but in z . We will define these polynomials now.

Suppose that $S_n(z; \mu)$ satisfies the recursion relation

$$S_{n+1}S_{n-1} = -z \left[S_n \frac{d^2 S_n}{dz^2} - \left(\frac{dS_n}{dz} \right) \right] - S_n \frac{dS_n}{dz} + (z + \mu) S_n^2, \quad (3.3)$$

with $S_{-1}(z; \mu) = S_0(z; \mu) = 1$. These polynomials are related to our $\tau_{\mu,n}(z)$ polynomials in the following way:

$$\tau_{\mu,n}(z) = \prod_{j=1}^n \left\{ \prod_{i=1}^j (2i-1) \right\} S_n. \quad (3.4)$$

The similarity is highlighted when comparing the polynomials directly with the following:

Table 3.3: The $S_n(z; \mu)$ polynomials

$$S_1(z; \mu) = z + \mu$$

$$S_2(z; \mu) = (z + \mu)^3 - \mu$$

$$S_3(z; \mu) = (z + \mu)^6 - 5\mu(z + \mu)^3 + 9\mu(z + \mu) - 5\mu^2$$

$$S_4(z; \mu) = (z + \mu)^{10} - 15\mu(z + \mu)^7 + 63\mu(z + \mu)^5 - 225\mu(z + \mu)^3 + 315\mu^2(z + \mu)^2 - 175\mu^3(z + \mu) + 36\mu^2$$

$$S_5(z; \mu) = (z + \mu)^{15} - 35\mu(z + \mu)^{12} + 252\mu(z + \mu)^{10} + 175\mu^2(z + \mu)^9 - 2025\mu(z + \mu)^8 + 945\mu^2(z + \mu)^7 - 1225\mu(\mu - 3)(\mu + 3)(z + \mu)^6 - 26082\mu^2(z + \mu)^5 + 33075\mu^3(z + \mu)^4 - 350\mu^2(35\mu^2 + 36)(z + \mu)^3 + 11340\mu^3(z + \mu)^2 - 225\mu^2(7\mu - 6)(7\mu + 6)(z + \mu) + 7\mu^3(875\mu^2 - 828)$$

3.1.3 The generalised Hermite polynomials

Here we consider the generalised Hermite polynomials $H_{m,n}$ which are defined by the following differential recurrence relations:

$$2mH_{m+1,n}H_{m-1,n} = H_{m,n} \frac{d^2 H_{m,n}}{dz^2} - \left(\frac{dH_{m,n}}{dz} \right)^2 + 2mH_{m,n}^2, \quad (3.5a)$$

$$2nH_{m,n+1}H_{m,n-1} = -H_{m,n} \frac{d^2 H''_{m,n}}{dz^2} - \left(\frac{dH'_{m,n}}{dz} \right)^2 + 2nH_{m,n}^2, \quad (3.5b)$$

with $H_{0,0} = H_{0,1} = H_{1,0} = 1$ and $H_{1,1} = 2z$. The polynomials $H_{m,n}$ defined by (3.5b) are known to be the generalised Hermite polynomials $H_{m,n}$ since $H_{m,1}(z) = H_m(z)$ and $H_{1,m}(z) = i^{-m}H_m(iz)$, where $H_m(z)$ is the well known Hermite polynomial defined by

$$H_m(z) = (-1)^m \exp(z^2) \frac{d^m}{dz^m} \{ \exp(-z^2) \}.$$

A table of generalised Hermite polynomials $H_{m,n}$ and various root plots are shown below.

Table 3.4: Table of the generalised Hermite polynomials $H_{m,n}$.

$$H_{2,1} = 4z^2 - 2$$

$$H_{2,2} = 16z^4 + 12$$

$$H_{3,2} = 64z^6 - 96z^4 + 144z^2 + 72$$

$$H_{3,3} = 512z^9 + 2304z^5 - 4320z$$

$$H_{4,3} = 4096z^{12} - 12288z^{10} + 46080z^8 - 30720z^6 - 57600z^4 - 172800z^2 + 43200$$

$$H_{4,3} = 65536z^{16} + 983040z^{12} - 1843200z^8 + 32256000z^4 + 6048000$$

$$H_{5,5} = 33554432z^{25} + 1258291200z^{21} + 3303014400z^{17} + 115605504000z^{13} - 2059223040000z^9 - 3413975040000z^5 + 2133734400000z$$

Some roots plots for generalised Hermite polynomials $H_{m,n}$ are shown below.

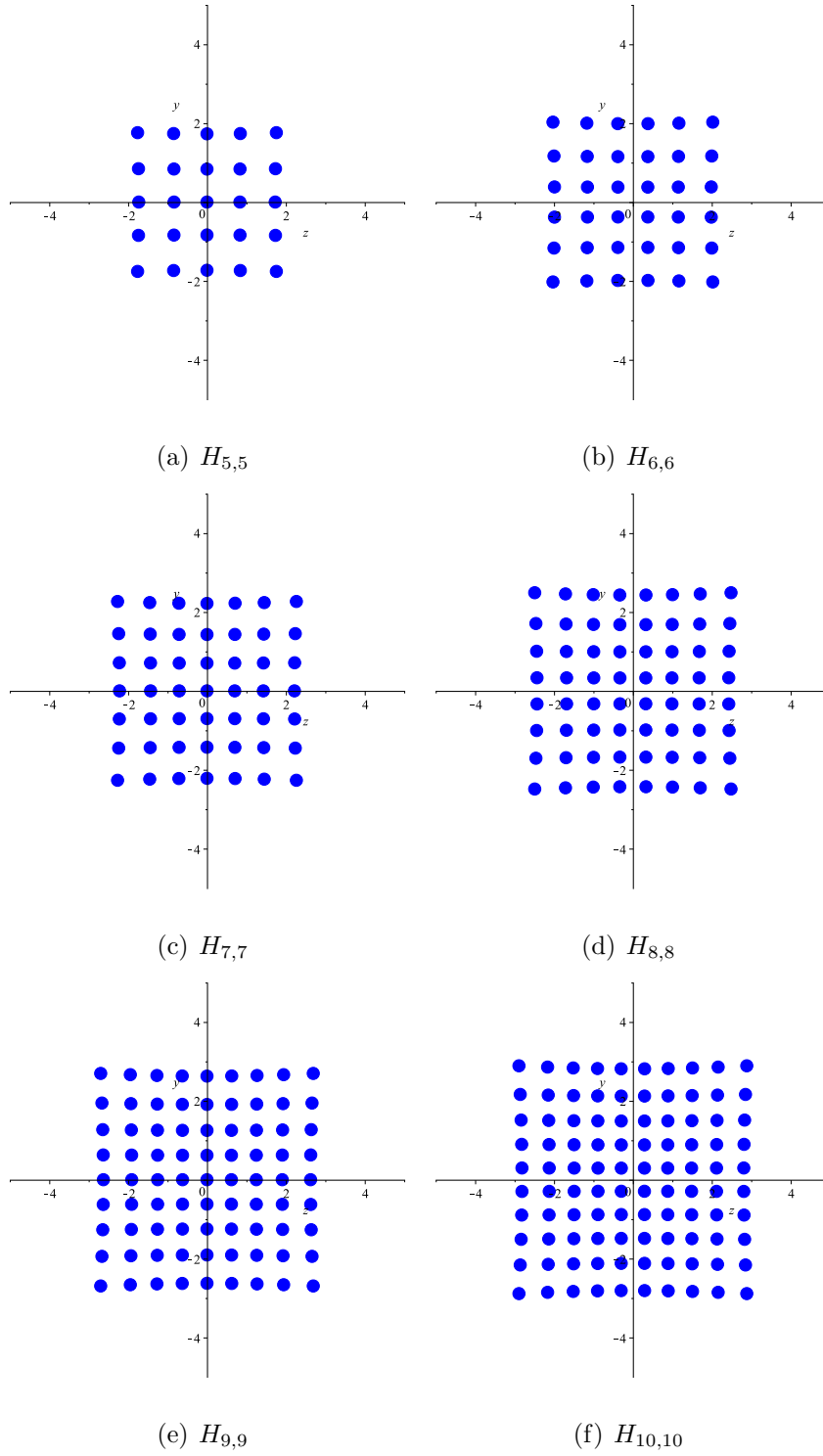


Figure 3.4: Roots of the generalised Hermite polynomials $H_{m,n}$.

The generalised Hermite polynomials $H_{m,n}$ can also be expressed in determinantal form

$$H_{m,n} = c_{m,n} \begin{vmatrix} H_{m+n-1} & H'_{m+n-1} & \cdots & H_{m+n-1}^{(n-1)} \\ H'_{m+n-1} & H''_{m+n-1} & \cdots & H_{m+n-1}^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ H_{m+n-1}^{(n-1)} & H_{m+n-1}^{(n)} & \cdots & H_{m+n-1}^{(2n-2)} \end{vmatrix}, \quad ' = \frac{d}{dz},$$

where $H_n(z)$ is the n th Hermite polynomial and $c_{m,n}$ is a constant.

The following remarks are based on observations:

Remark 3.3.

- i) The generalised Hermite polynomials $H_{m,n}$ take the form of $m \times n$ “rectangles”, though these are only approximate rectangles since the roots lie on arcs rather than straight lines. The parameters m and n govern the number of “columns” and “rows” of the roots, respectively.*
- ii) The generalised Hermite polynomials $H_{m,n}$ will only have real roots when n is an odd integer. This is due to the symmetry in the root formation.*
- iii) The generalised Hermite polynomials $H_{m,n}$ have degree equal to mn .*
- iv) The generalised Hermite polynomials $H_{m,n}$ always have mn roots.*

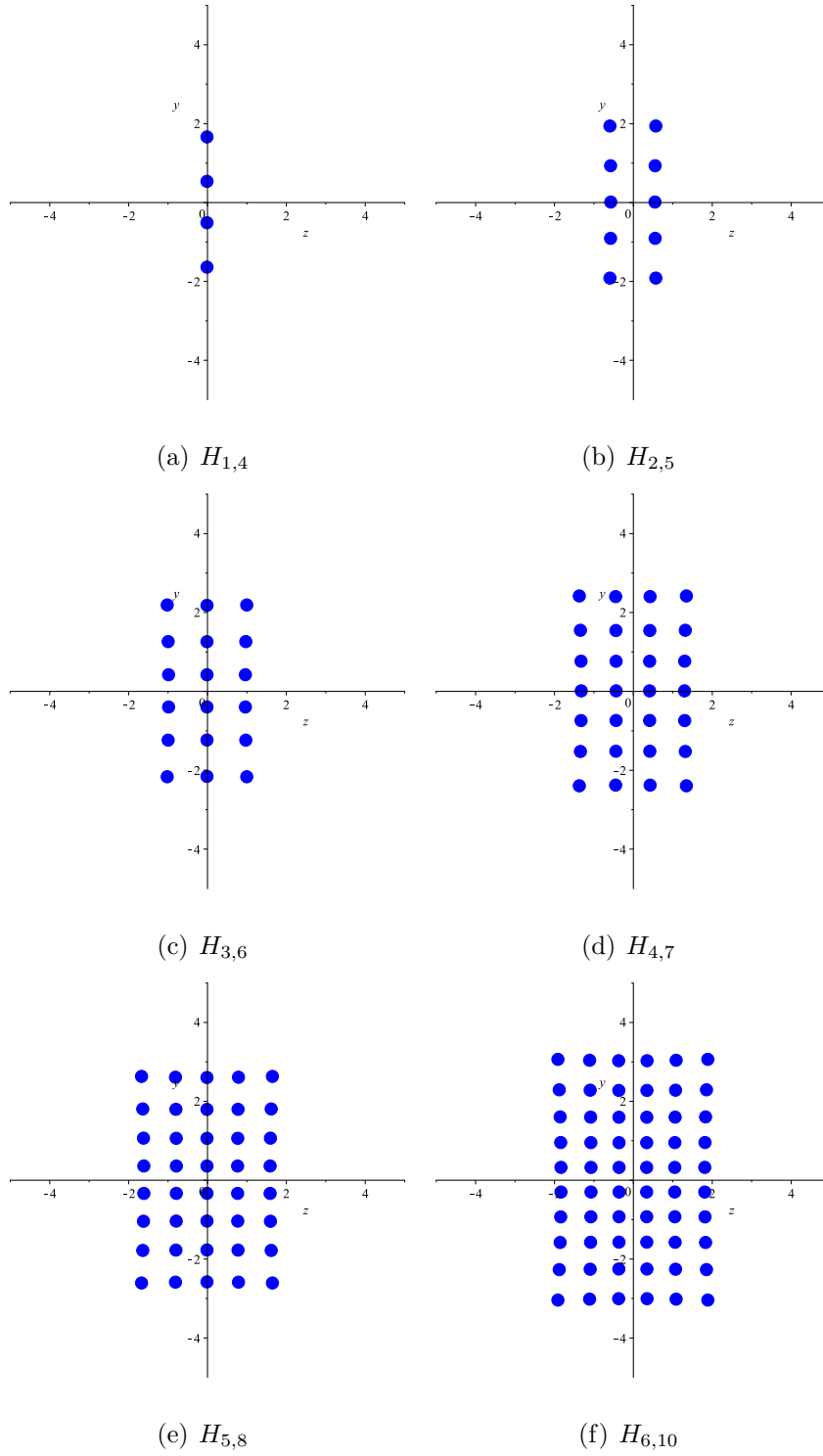


Figure 3.5: Roots of the generalised Hermite polynomials $H_{m,n}$.

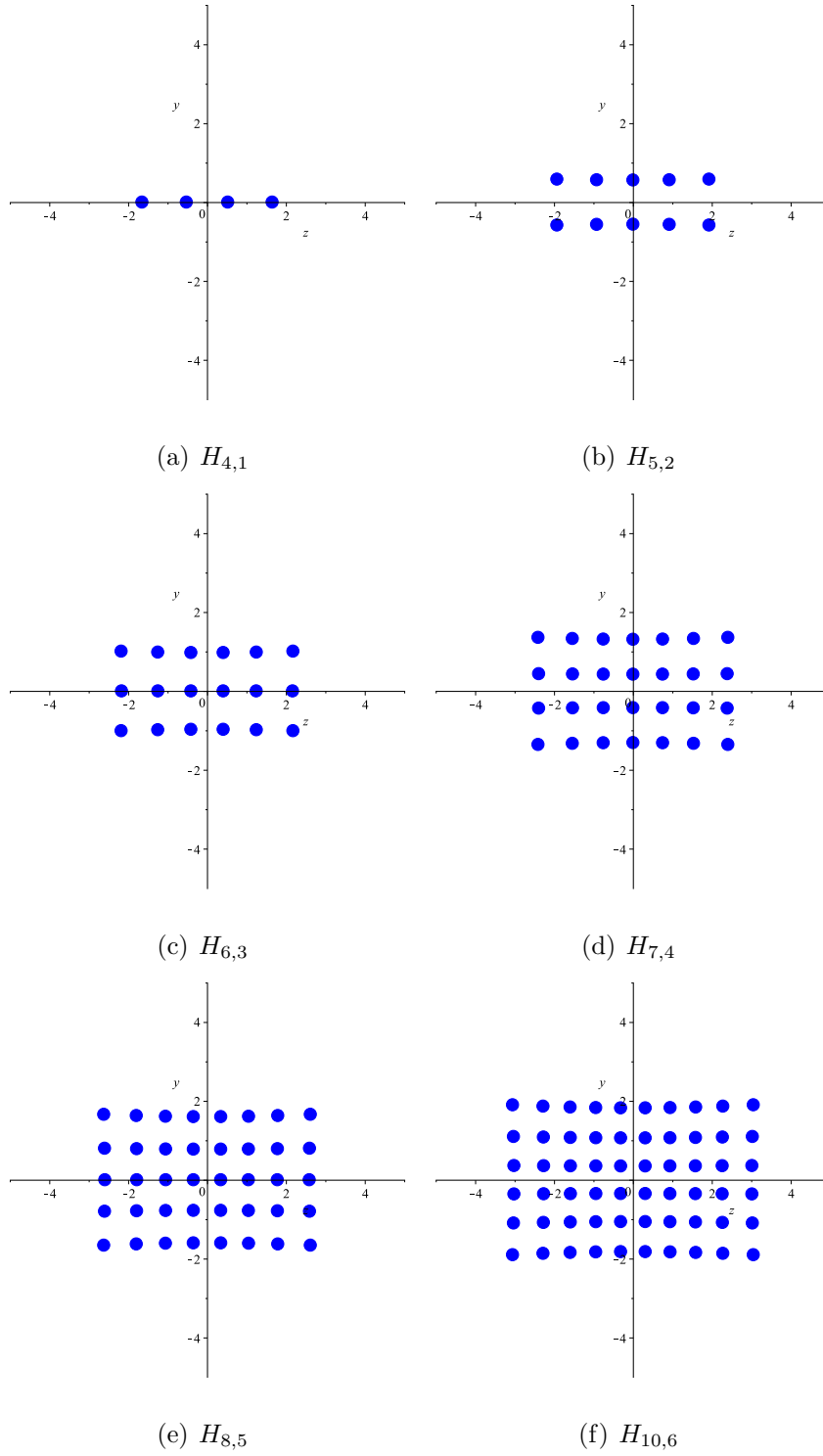


Figure 3.6: Roots of the generalised Hermite polynomials $H_{m,n}$.

It is also interesting to note that due to the symmetric nature of these roots it is easy to see the conditions when we have real roots. This is precisely when the number of rows, n , is odd.

3.1.4 The generalised Okamoto polynomials

Here we consider the generalised Okamoto polynomials $Q_{m,n}$, which are defined by the following differential recurrence relations:

$$Q_{m+1,n}Q_{m-1,n} = \frac{9}{2} \left\{ Q_{m,n} \frac{d^2 Q_{m,n}}{dz^2} - (Q_{m,n})^2 \right\} + \{2z^2 + 3(2m + n - 1)\} Q_{m,n}^2, \quad (3.6a)$$

$$Q_{m,n+1}Q_{m,n-1} = \frac{9}{2} \left\{ Q_{m,n} \frac{d^2 Q_{m,n}}{dz^2} - (Q_{m,n})^2 \right\} + \{2z^2 + 3(1 - m - 2n)\} Q_{m,n}^2. \quad (3.6b)$$

with $Q_{0,0} = Q_{1,0} = 1$ and $Q_{1,1} = \sqrt{2}z$ [52].

Table 3.5: Table of the generalised Okamoto polynomials $Q_{m,n}$

$Q_{2,1} = 4z^4 + 12z^2 - 9$
$Q_{2,2} = \frac{16}{7} (16z^8 - 504z^4 - 567)$
$Q_{3,1} = \frac{4}{35} z (16z^8 + 192z^6 + 504z^4 - 2835)$
$Q_{3,2} = -\frac{128}{1225} (128z^{14} + 1344z^{12} - 6048z^{10} - 75600z^8 - 158760z^6$ $+ 238140z^4 - 1071630z^2 - 535815)$
$Q_{4,2} = \frac{512}{2786875} (2048z^{22} + 64512z^{20} + 483840z^{18} - 3144960z^{16}$ $- 61689600z^{14} - 297198720z^{12} - 445798080z^{10} + 1114495200z^8$ $- 5851099800z^6 - 43883248500z^4 - 13164974550z^2 - 19747461825)$

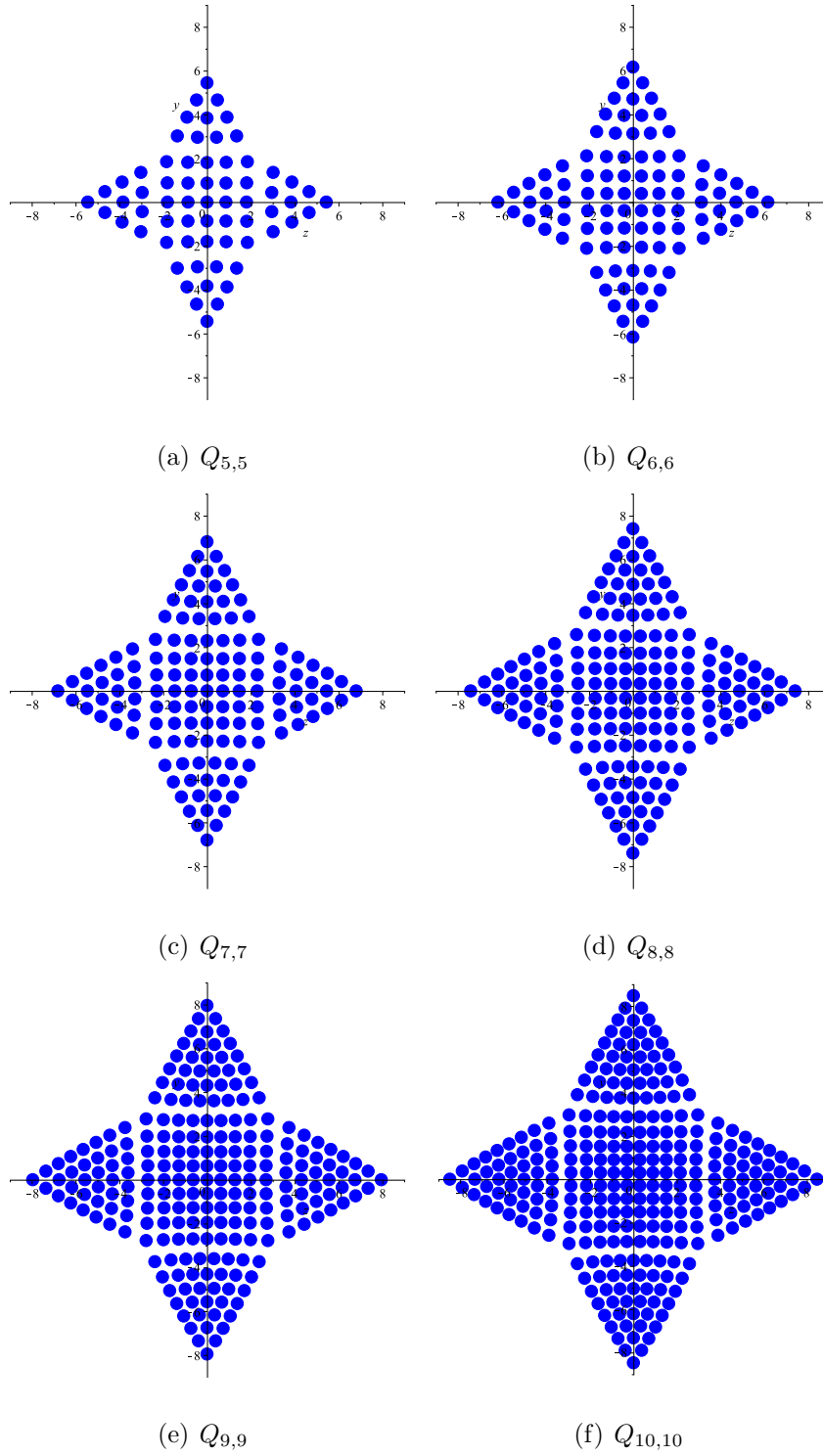


Figure 3.7: Roots of generalised Okamoto polynomials $Q_{m,n}$.

The following remarks are based on observations:

Remark 3.4.

- i) The generalised Okamoto polynomials $Q_{m,n}$ take the form of $m \times n$ “rectangles” with an “equilateral triangle” which either have $m - 1$ or $n - 1$ roots on each of the four sides. These are only approximate rectangles and triangles since the roots lie on arcs rather than straight lines. The parameters m and n govern the number of “columns” and “rows” of the rectangle, respectively.*
- ii) The generalised Okamoto polynomials $Q_{m,n}$ will only have a real root when n is an odd integer. This is due to the symmetry in the root formation.*
- iii) The generalised Okamoto polynomials $Q_{m,n}$ have degree equal to $m^2 + n^2 + mn - m - n$.*
- iv) The generalised Okamoto polynomials $Q_{m,n}$ always have $m^2 + n^2 + mn - m - n$ roots.*

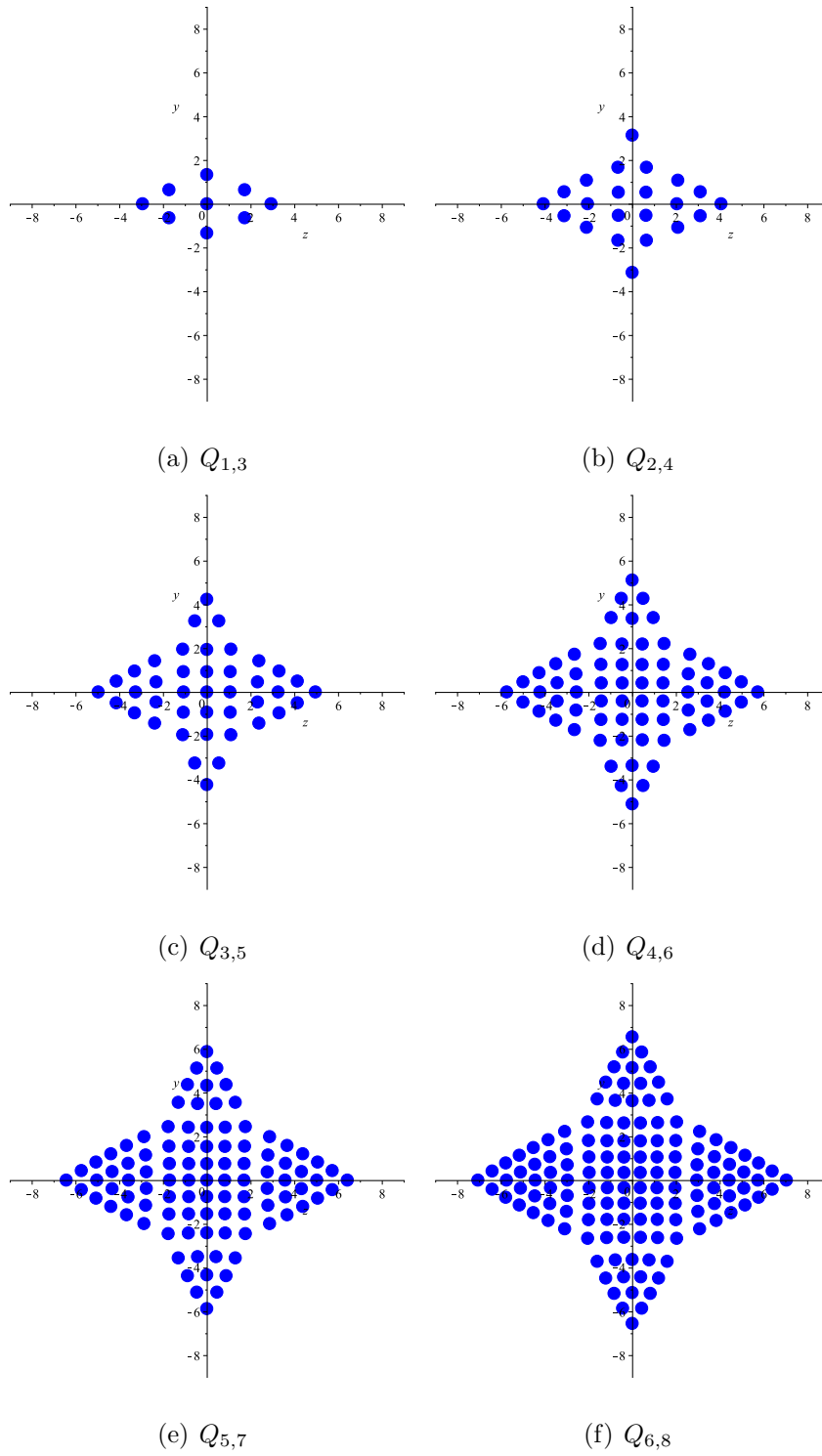


Figure 3.8: Roots of generalised Okamoto polynomials $Q_{m,n}$.

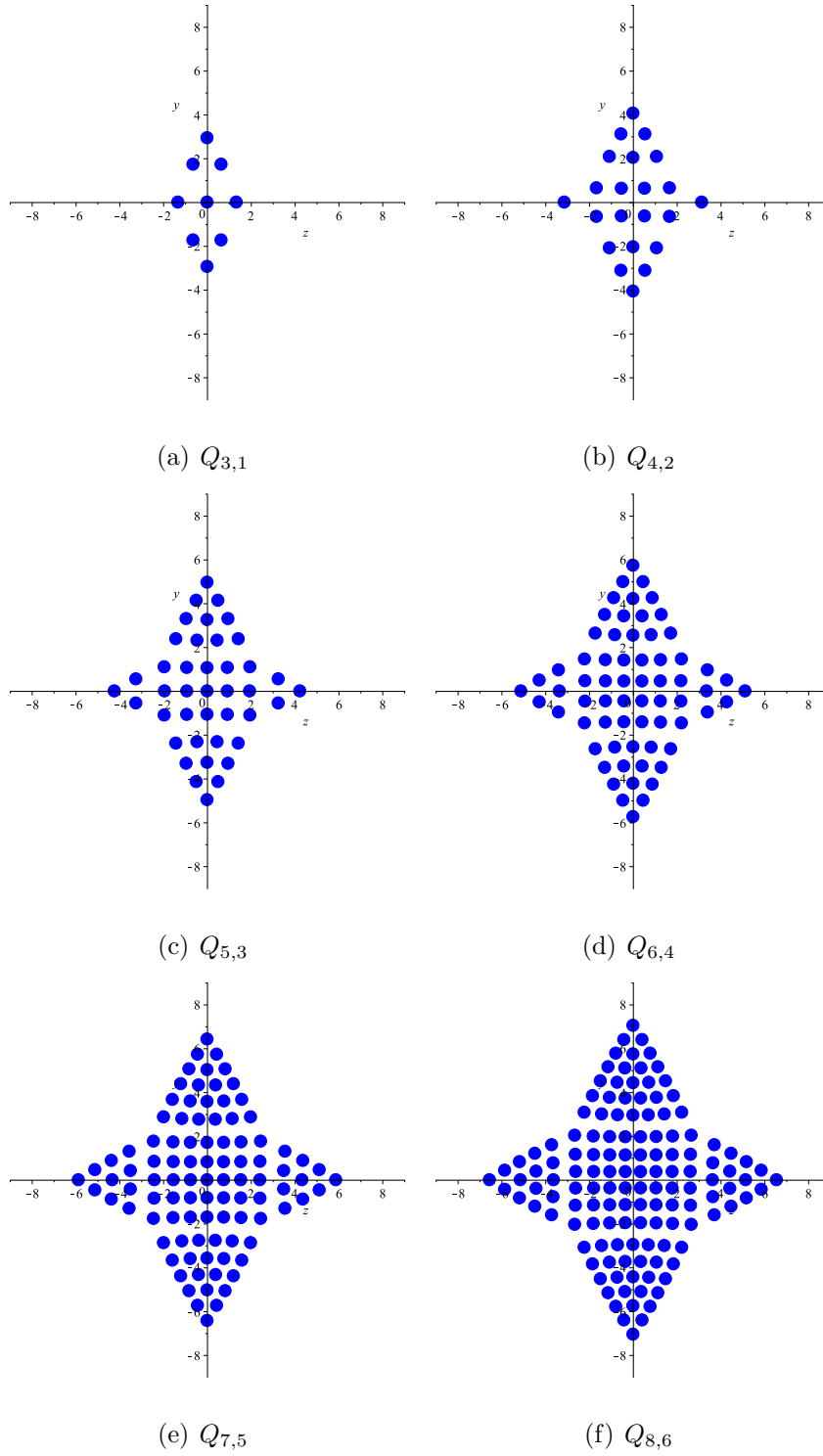


Figure 3.9: Roots of generalised Okamoto polynomials $Q_{m,n}$.

The generalised Okamoto polynomials $Q_{m,n}$ can also be expressed in determinantal form

$$Q_{m,n} = c_{m,n} \begin{vmatrix} \psi_1 & \psi_4 & \cdots & \psi_{3m+3n-5} & \psi_2 & \psi_5 & \cdots & \psi_{3n-4} \\ \psi'_1 & \psi'_4 & \cdots & \psi'_{3m+3n-5} & \psi'_2 & \psi'_5 & \cdots & \psi'_{3n-4} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \psi_1^{(n-1)} & \psi_4^{(n-1)} & \cdots & \psi_{3m+3n-5}^{(n-1)} & \psi_2^{(n-1)} & \psi_5^{(n-1)} & \cdots & \psi_{3n-4}^{(n-1)} \end{vmatrix}$$

and

$$Q_{-m,-n} = \widetilde{c}_{m,n} \begin{vmatrix} \psi_1 & \psi_4 & \cdots & \psi_{3n-2} & \psi_2 & \psi_5 & \cdots & \psi_{3m+3n-1} \\ \psi'_1 & \psi'_4 & \cdots & \psi'_{3n-2} & \psi'_2 & \psi'_5 & \cdots & \psi'_{3m+3n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \psi_1^{(n-1)} & \psi_4^{(n-1)} & \cdots & \psi_{3n-2}^{(n-1)} & \psi_2^{(n-1)} & \psi_5^{(n-1)} & \cdots & \psi_{3m+3n-1}^{(n-1)} \end{vmatrix},$$

where $' = \frac{d}{dz}$, $\psi_n(z) = (-3)^{n/2} H_n(\frac{1}{3}\sqrt{3}iz)$ and $\widetilde{c}_{m,n}$, $c_{m,n}$ are constants [52].

3.1.5 The generalised associated Laguerre with $\delta = z \frac{d}{dz}$

Here we consider the generalised Laguerre polynomials $\widetilde{\mathcal{L}}_{\alpha,\beta}^{(n)}$, which are defined by the following differential recurrence relation:

$$4z^2 \widetilde{\mathcal{L}}_{\alpha,\beta}^{(n)} \frac{d^2 \widetilde{\mathcal{L}}_{\alpha,\beta}^{(n)}}{dz^2} - 4z^3 \left(\frac{d \widetilde{\mathcal{L}}_{\alpha,\beta}^{(n)}}{dz} \right)^2 - 4z(n^2 z - n^2 - nz - z^2 + n) \widetilde{\mathcal{L}}_{\alpha,\beta}^{(n)} \frac{d \widetilde{\mathcal{L}}_{\alpha,\beta}^{(n)}}{dz} - n(z-1)(n-1)(n^2 - n - 2z - 2) (\widetilde{\mathcal{L}}_{\alpha,\beta}^{(n)})^2 - 4z^2 \widetilde{\mathcal{L}}_{\alpha,\beta}^{(n+1)} \widetilde{\mathcal{L}}_{\alpha,\beta}^{(n-1)}, \quad (3.7)$$

where $\widetilde{\mathcal{L}}_{\alpha,\beta}^{(-1)} = 0$, $\widetilde{\mathcal{L}}_{\alpha,\beta}^{(0)} = 1$.

Table 3.6: Table of the generalised associated Laguerre polynomials $\tilde{\mathcal{L}}_{\alpha,\beta}^{(n)}$

$$\tilde{\mathcal{L}}_{-1,\beta}^{(1)} = \beta - z$$

$$\tilde{\mathcal{L}}_{-2,\beta}^{(1)} = \frac{1}{2}(\beta^2 - 2\beta z + z^2 + \beta - 2z)$$

$$\tilde{\mathcal{L}}_{-2,\beta}^{(2)} = -\frac{1}{2}(\beta + 1)(\beta^2 - 2\beta z + z^2 + \beta)$$

$$\begin{aligned} \tilde{\mathcal{L}}_{-3,\beta}^{(2)} = & -\frac{1}{12}(\beta + 2)(\beta^4 - 4\beta^3 z + 6\beta^2 z^2 - 4\beta z^3 + z^4 + 4\beta^3 - 12\beta^2 z \\ & + 12\beta z^2 - 4z^3 + 5\beta^2 - 8\beta z + 6z^2 + 2\beta) \end{aligned}$$

$$\tilde{\mathcal{L}}_{-3,\beta}^{(3)} = -\frac{1}{6}(\beta + 1)(\beta + 2)^2(\beta^3 - 3\beta^2 z + 3\beta z^2 - z^3 + 3\beta^2 - 3\beta z + 2\beta)$$

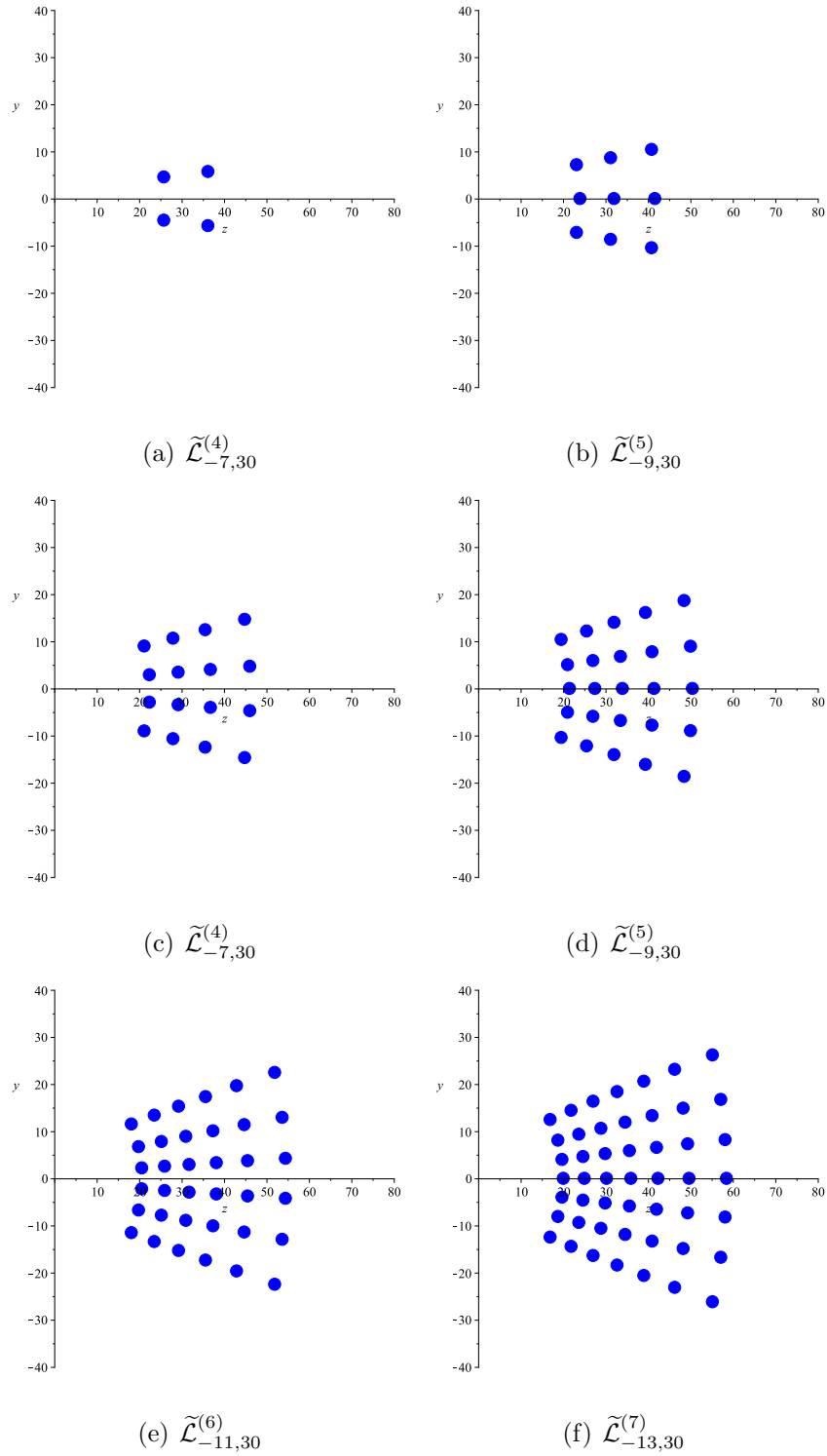


Figure 3.10: Roots of some generalised associated Laguerre polynomials $\tilde{\mathcal{L}}_{\alpha,\beta}^{(n)}$.

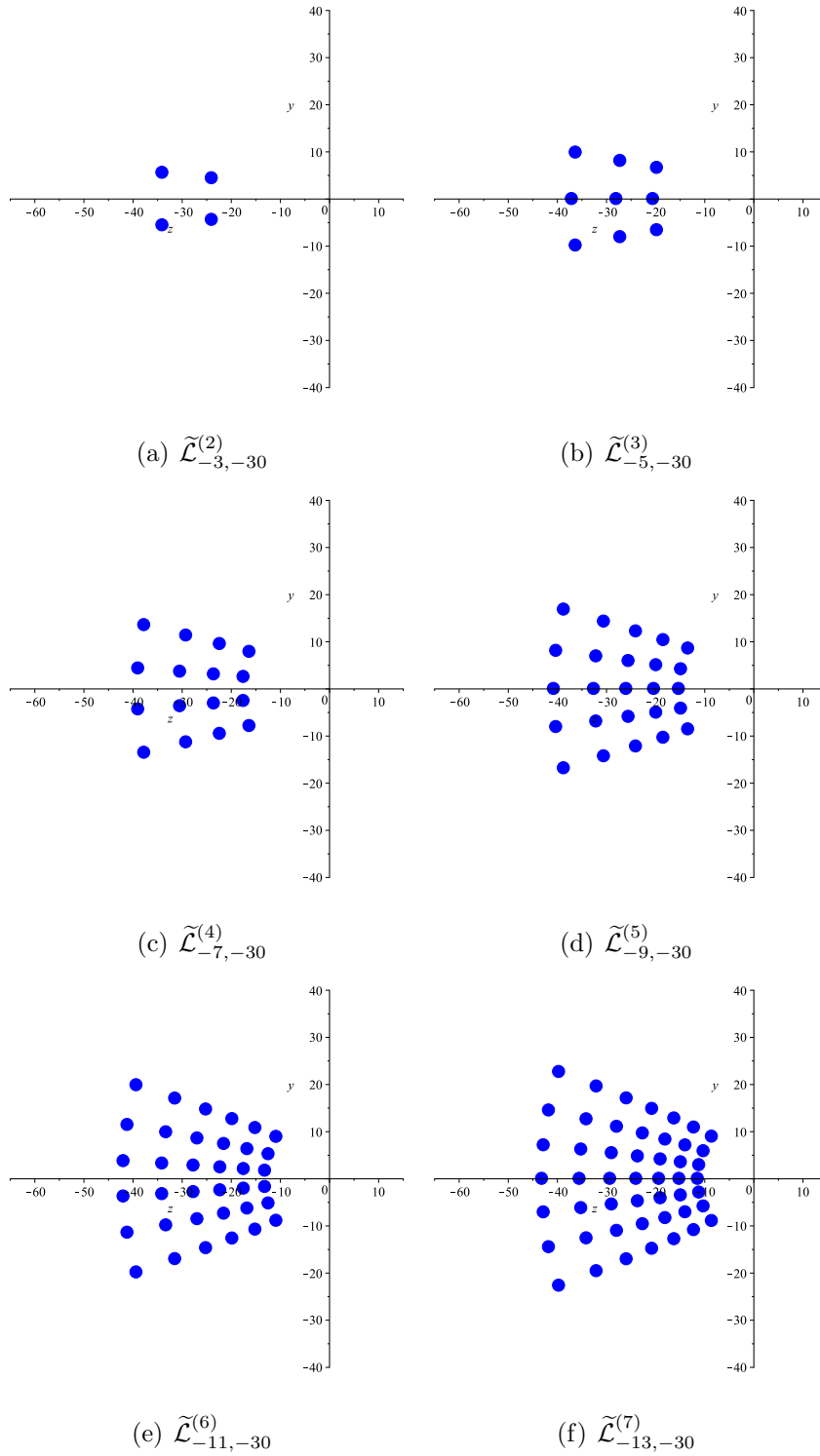


Figure 3.11: Roots of some generalised associated Laguerre polynomials $\tilde{\mathcal{L}}_{\alpha,\beta}^{(n)}$.

Note that by varying the parameter β we can animate these roots. See the “Animations” folder on the supplied USB flash drive.

They also have the following determinantal representation:

$$\tilde{\mathcal{L}}_{\alpha,\beta}^{(n)} := z^{\frac{n}{2}(1-n)} \begin{vmatrix} L_{-\alpha}^{\beta-1} & \delta(L_{-\alpha}^{\beta-1}) & \dots & \delta^{(n-1)}(L_{-\alpha}^{\beta-1}) \\ \delta(L_{-\alpha}^{\beta-1}) & \delta^{(2)}(L_{-\alpha}^{\beta-1}) & \dots & \delta^{(n)}(L_{-\alpha}^{\beta-1}) \\ \vdots & \vdots & \ddots & \vdots \\ \delta^{(n-1)}(L_{-\alpha}^{\beta-1}) & \delta^{(n)}(L_{-\alpha}^{\beta-1}) & \dots & \delta^{(2n-2)}(L_{-\alpha}^{\beta-1}) \end{vmatrix}, \quad \delta = z \frac{d}{dz}, \quad (3.8)$$

where L_b^a is the associated Laguerre polynomial. Using the corollary 2.1 we can write down an entirely new formulation of these polynomials in an alternative form to this determinantal representation which will be useful when we look at the rational function solutions to P_V (1.1e) and S_V (1.7e).

$$\tilde{\mathcal{L}}_{\alpha,\beta,e^{-z}}^{(n)} := z^{\frac{n}{2}(1-n)} \begin{vmatrix} L_{-\alpha}^{\beta-1} e^{-z} & \delta(L_{-\alpha}^{\beta-1} e^{-z}) & \dots & \delta^{(n-1)}(L_{-\alpha}^{\beta-1} e^{-z}) \\ \delta(L_{-\alpha}^{\beta-1} e^{-z}) & \delta^{(2)}(L_{-\alpha}^{\beta-1} e^{-z}) & \dots & \delta^{(n)}(L_{-\alpha}^{\beta-1} e^{-z}) \\ \vdots & \vdots & \ddots & \vdots \\ \delta^{(n-1)}(L_{-\alpha}^{\beta-1} e^{-z}) & \delta^{(n)}(L_{-\alpha}^{\beta-1} e^{-z}) & \dots & \delta^{(2n-2)}(L_{-\alpha}^{\beta-1} e^{-z}) \end{vmatrix}, \quad (3.9)$$

where L_b^a is again the associated Laguerre polynomial and $\delta = z \frac{d}{dz}$.

The following remarks are based on observations:

Remark 3.5.

- i) The generalised associated Laguerre polynomials take the form of $m \times n$ “trapeziums”, though these are only approximate trapeziums since the roots lie on arcs rather than straight lines. The parameters α and n govern the number of “columns”*

and “rows” of the roots in the following ways:

$$\begin{array}{llll} \text{columns} \rightarrow 1 - \alpha - n, & \text{rows} \rightarrow n & \text{for} & \beta > 0, \\ \text{columns} \rightarrow n, & \text{rows} \rightarrow 1 - \alpha - n & \text{for} & \beta < 0. \end{array}$$

ii) The generalised associated Laguerre polynomials will have a real root when the number of rows is an odd integer. This is due to the symmetry in the root formation. This is precisely when

$$\begin{array}{ll} n \text{ is odd for} & 0 < \beta, \\ \text{whenever} & \alpha + 1 \leq \beta \leq 0, \\ 1 - \alpha - n \text{ is odd for} & \beta < \alpha + 1. \end{array}$$

Note that it is possible for all of these cases to be satisfied for all β .

iii) The generalised associated Laguerre polynomials have degree equal to $n(1 - \alpha - n)$.

iv) The generalised associated Laguerre polynomials always have $n(1 - \alpha - n)$ roots.

These polynomials are closely related to another type of polynomial that satisfy

$$\mathcal{L}_{\alpha,\beta}^{(n+1)} \mathcal{L}_{\alpha,\beta}^{(n-1)} + \left(\frac{d\mathcal{L}_{\alpha,\beta}^{(n)}}{dz} \right)^2 = \left(\frac{d^2\mathcal{L}_{\alpha,\beta}^{(n)}}{dz^2} \right),$$

with the following determinantal representation:

$$\mathcal{L}_{\alpha,\beta}^{(n)} := \begin{vmatrix} L_{-\alpha}^{\beta-1} & \delta(L_{-\alpha}^{\beta-1}) & \dots & \delta^{(n-1)}(L_{-\alpha}^{\beta-1}) \\ \delta(L_{-\alpha}^{\beta-1}) & \delta^{(2)}(L_{-\alpha}^{\beta-1}) & \dots & \delta^{(n)}(L_{-\alpha}^{\beta-1}) \\ \vdots & \vdots & \ddots & \vdots \\ \delta^{(n-1)}(L_{-\alpha}^{\beta-1}) & \delta^{(n)}(L_{-\alpha}^{\beta-1}) & \dots & \delta^{(2n-2)}(L_{-\alpha}^{\beta-1}) \end{vmatrix}, \quad \delta = \frac{d}{dz}. \quad (3.10)$$

Theorem 3.1. *The two determinants $\tilde{\mathcal{L}}_{\alpha,\beta}^{(n)}$ and $\mathcal{L}_{\alpha,\beta}^{(n)}$ have the following relation:*

$$\frac{d}{dz} \ln \tilde{\mathcal{L}}_{\alpha,\beta}^{(n)} = \frac{d}{dz} \ln \mathcal{L}_{\alpha,\beta+1-n}^{(n)} - \frac{n}{2z} (1-n). \quad (3.11)$$

Proof. This result can be shown by using both of these determinantal representations in our rational solutions. We can therefore combine them and simplify to acquire this relation exactly. \square

3.1.6 The generalised Jacobi polynomials

Here we consider the generalised Jacobi polynomials $\mathcal{J}_{\alpha,\beta}^{(n)}$ which are defined by the following differential recurrence relation:

$$\begin{aligned} & \left(\hat{\mathcal{J}}_{a,b,c}^{(n)} \right)^2 z^2 (z-1)^2 \frac{d^2}{dz^2} \hat{\mathcal{J}}_{a,b,c}^{(n)} + \left(\hat{\mathcal{J}}_{a,b,c}^{(n)} \right)^2 z (2z-1) (z-1) \frac{d}{dz} \hat{\mathcal{J}}_{a,b,c}^{(n)} \\ & - \hat{\mathcal{J}}_{a,b,c}^{(n+1)} \hat{\mathcal{J}}_{a,b,c}^{(n-1)} = 0, \end{aligned} \quad (3.12)$$

where $\hat{\mathcal{J}}_{a,b,c}^{(n)} = \mathcal{J}_{a,b,c}^{(n)} z^{-1/2 n(1-n-2b)} (z-1)^{-1/2 n(1-n)}$.

Table 3.7: Table of the generalised Jacobi polynomials $\hat{\mathcal{J}}_{a,b,c}^{(n)}$

$$\begin{aligned} \hat{\mathcal{J}}_{a,b,c}^{(n)} &= 4z^4 + 12z^2 - 9 \\ \hat{\mathcal{J}}_{a,b,c}^{(n)} &= -1152z^4 + \frac{256}{7}z^8 - 1296 \\ \hat{\mathcal{J}}_{a,b,c}^{(n)} &= -\frac{36864}{5}z^5 + \frac{98304}{35}z^7 - \frac{8192}{35}z^9 + 41472z \end{aligned}$$

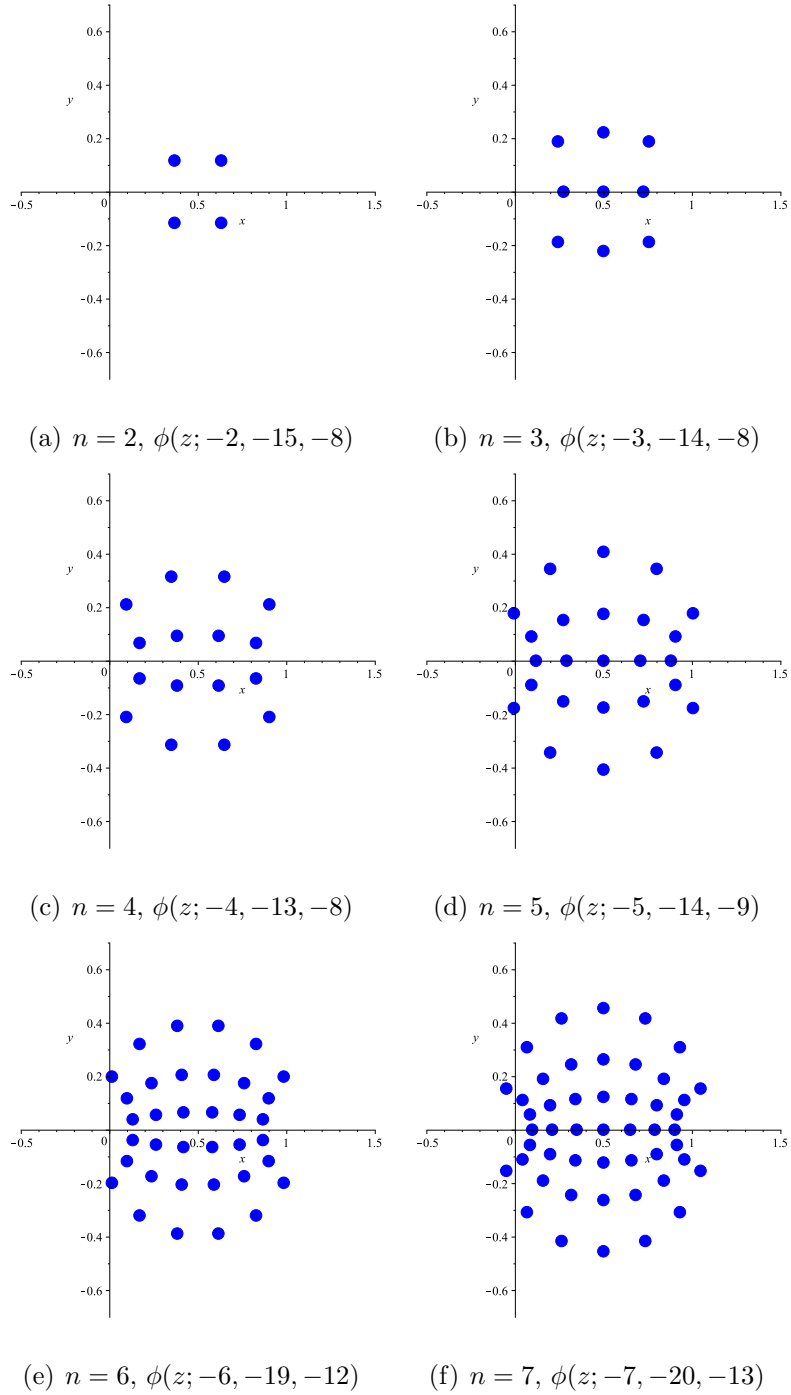


Figure 3.12: Roots of some generalised Jacobi polynomials $\widehat{\mathcal{J}}_{a,b,c}^{(n)}$.

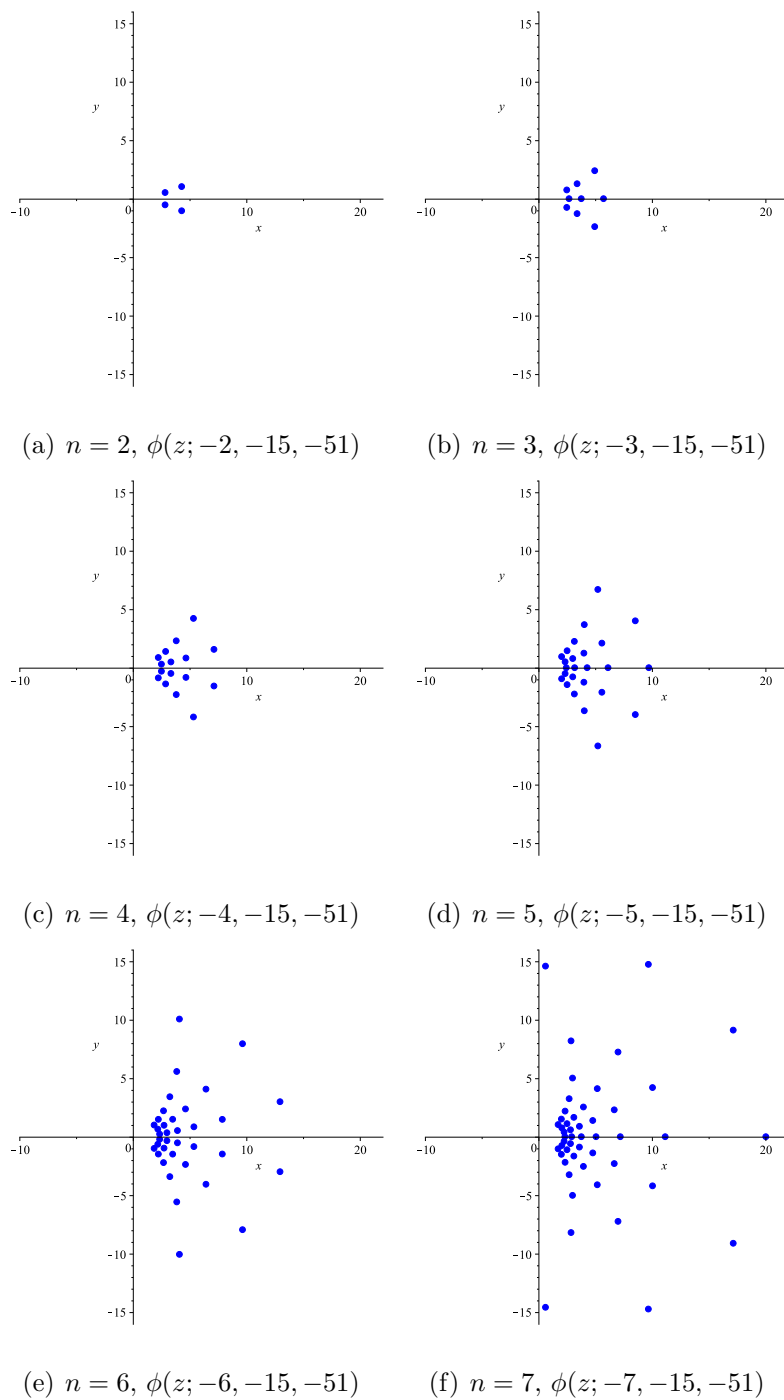


Figure 3.13: Roots of some generalised Jacobi polynomials $\widehat{\mathcal{J}}_{a,b,c}^{(n)}$.

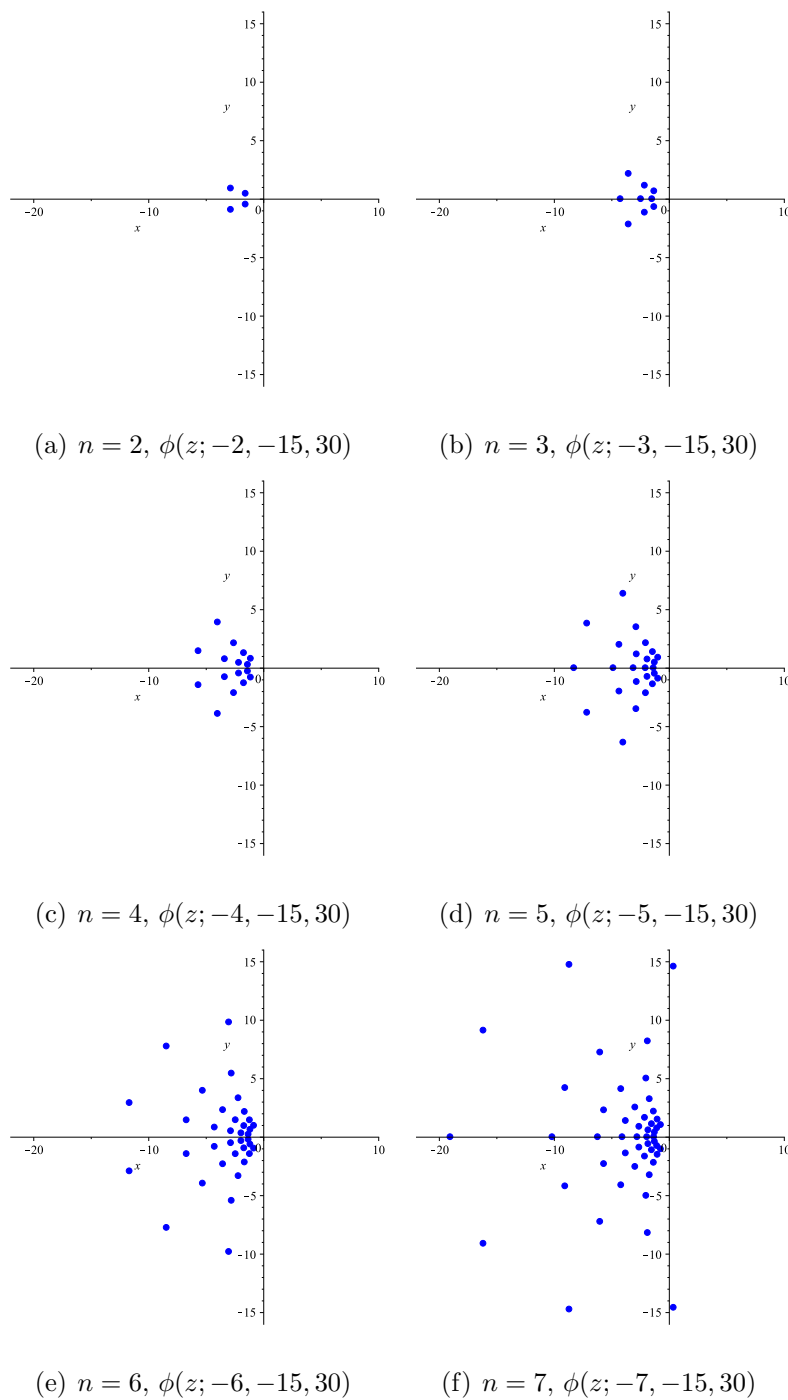


Figure 3.14: Roots of some generalised Jacobi polynomials $\widehat{\mathcal{J}}_{a,b,c}^{(n)}$.

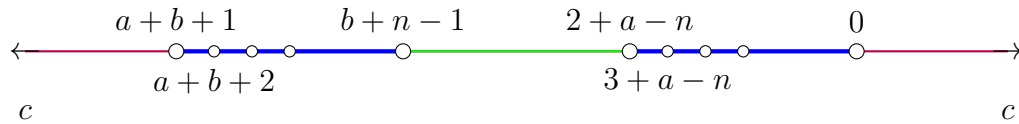
Note that by varying the parameter c we can animate these roots. See the “Animations” folder on the supplied USB flash drive.

They also have the following determinantal representation:

$$\widehat{\mathcal{J}}_{a,b,c}^{(n)} := \begin{vmatrix} P & \delta(P) & \dots & \delta^{(n-1)}(P) \\ \delta(P) & \delta^{(2)}(P) & \dots & \delta^{(n)}(P) \\ \vdots & \vdots & \ddots & \vdots \\ \delta^{(n-1)}(P) & \delta^{(n)}(P) & \dots & \delta^{(2n-2)}(P) \end{vmatrix},$$

where $\delta = z(z - 1)\frac{d}{dz}$ and $P = P_{-a}^{(c-1, a+b-c)}(1 - 2z)$.

The following c line describes the roots of the generalised Jacobi polynomials $\widehat{\mathcal{J}}_{a,b,c}^{(n)}$ as c changes.



The following remarks are based on observations:

Remark 3.6.

- i) When $c < a + b + 1$ or $0 < c$ the generalised Jacobi polynomials $\widehat{\mathcal{J}}_{a,b,c}^{(n)}$ roots take the form of $-a \times n$ “axe”. The parameters $-a$ and n govern the number of “columns” and “rows” of the roots, respectively. If $c < a + b + 1$ then the “axe” will have its longer side directed to the right. However, if $0 < c$ then the longer side of the “axe” will be directed to the left.
- ii) When $b + n - 1 < c < 2 + a - n$ the generalised Jacobi polynomials $\widehat{\mathcal{J}}_{a,b,c}^{(n)}$ roots take the form of $n \times -a$ “double axe”. The parameters n and $-a$ govern the number of “columns” and “rows” of the roots, respectively.

iii) When $c = \{a+b+1, a+b+2, \dots, b+n-1\}$ or $c = \{2+a+n, 3+a+n, \dots, 0\}$ the generalised Jacobi polynomials $\widehat{\mathcal{J}}_{a,b,c}^{(n)}$ will have at least double roots. For example: Take $a = -5$, $b = -14$, $n = 5$. The collision points (at least double roots) are $c = \{-18, -17, \dots, -10\}$ and $c = \{-8, -7, \dots, 0\}$. If c is inside these sets of points we know the roots will form the shapes we saw in figure 3.12.

iv) The generalised Jacobi polynomials $\widehat{\mathcal{J}}_{a,b,c}^{(n)}$ have degree equal to $-an$.

v) The generalised Jacobi polynomials $\widehat{\mathcal{J}}_{a,b,c}^{(n)}$ always have $-an$ roots.

3.2 Rational function solutions to the Painlevé equations

In this section the solutions that are of real interest are the bounded solutions. This is because the applications involving orthogonal always involve the bounded solutions. These bounded solutions are easy to find if we think about the real roots of the polynomials that comprise them. The following theorem explains that a real root, a , in our polynomial solutions, regardless of its order, will result in an explosion at $x = a$. Assuming there is no common root between the comprising polynomials, there is no way in which the logarithmic derivative can cancel with the corresponding root.

Theorem 3.2. *Given a solution of the form*

$$w = A(z) + B(z) \frac{d}{dz} \ln \frac{C(z)}{D(z)},$$

where $A(z)$, $B(z)$, $C(z)$ and $D(z)$ are polynomials in z . If there exists a real root of $C(z)$ or $D(z)$ we will have an unbounded solution along the real axis, assuming the root is not common between them.

Proof. The first thing to consider is that the solution can be rewritten as

$$w = A(z) + B(z) \frac{d}{dz} \left(\ln C(z) - \ln D(z) \right),$$

Then we assume $(z - a)^n$ divides $C(z)$ or $D(z)$. Does this imply $(z - a)$ divides $\frac{d}{dz} \ln C(z)$? This implies that $C(z)$ can be written in the form $C(z) = (z - a)^n g(z)$.

Differentiating yields

$$\frac{dC(z)}{dz} = n(z - a)^{n-1} g(z) + (z - a)^n \frac{dg(z)}{dz}.$$

So,

$$\frac{d}{dz} \ln C(z) = \frac{n(z - a)^{n-1} g(z) + (z - a)^n \frac{dg(z)}{dz}}{(z - a)^n g(z)} \quad (3.13)$$

$$= \frac{n}{z - a} + \frac{d}{dz} \ln g(z). \quad (3.14)$$

This implies that the order n of a root is not important and that if a real root exists the root cannot vanish with the logarithmic derivative. If there is a single uncommon real root the solution will not be bounded. Some of the root plots that follow show some unbounded solutions, however, this is purely an illustration that these cases are easily identifiable using the associated root plots. \square

3.2.1 The second Painlevé equation

Theorem 3.3. *The rational solutions to P_{II} (1.1b) exist if and only if $A = n \in \mathbb{Z}$, which are unique. Suppose $Q_n(z)$ is the Yablonskii-Vorob'ev polynomial defined by (3.1), then the rational solutions of P_{II} (1.1b) in the form $w_n(z; A)$ are given by the following:*

$$w_n(z; A) = \frac{d}{dz} \left\{ \ln \left[\frac{Q_{n-1}(z)}{Q_n(z)} \right] \right\}, \quad (3.15)$$

for the parameter $A = n$.

Proof. See [67, 70, 30, 64].

□

The rational solutions to P_{II} (1.1b) have no bounded solutions due to the polynomials that generate them always having real roots. This can be seen clearly in the following plots.

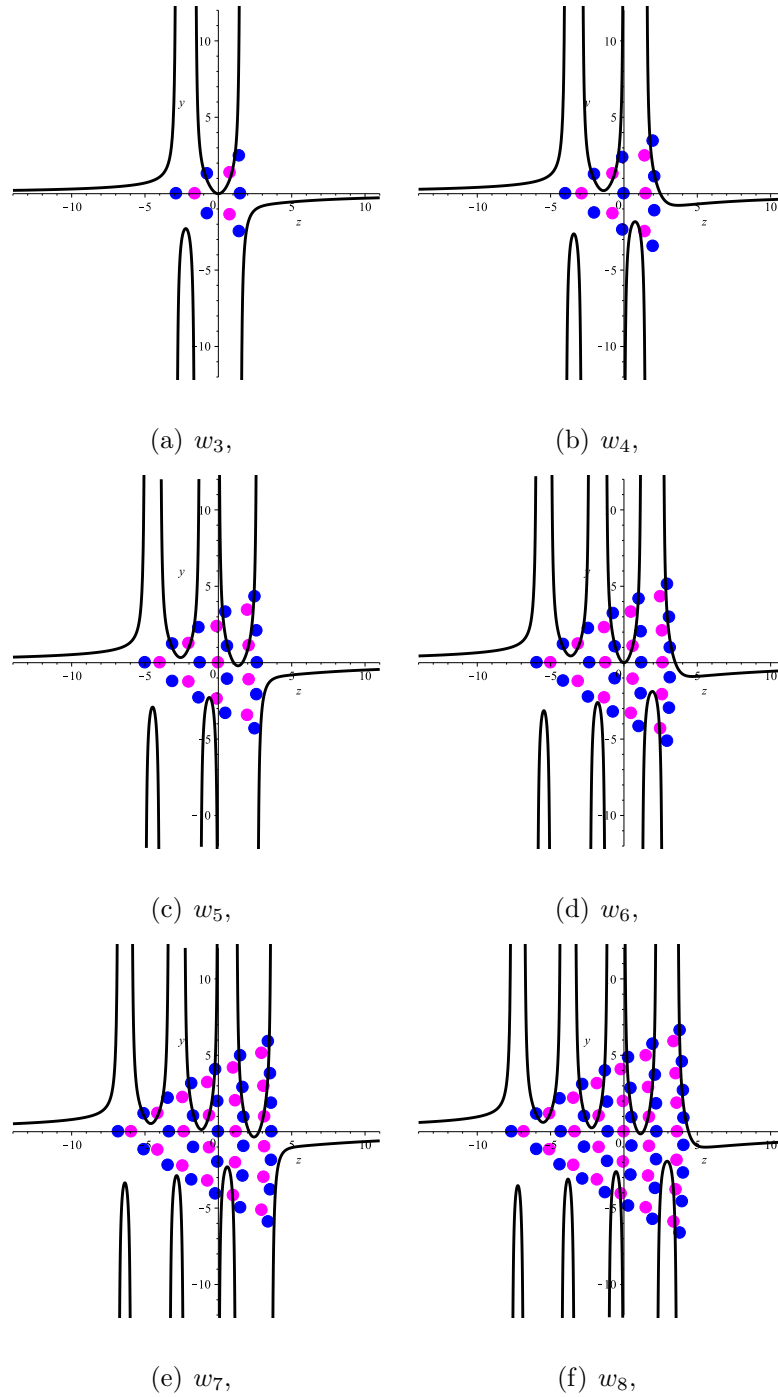


Figure 3.15: Some rational solutions to P_{II} (1.1b) super imposed with the complex roots of the corresponding Yablonskii-Vorob'ev polynomials which comprise the solutions.

3.2.2 The third Painlevé equation

The locations of the rational solutions of P_{III} (1.1c) solutions are stated the following theorem:

Theorem 3.4. P_{III} (1.1c) with $\gamma = -\delta = 1$, has rational solutions if and only if $\alpha + \varepsilon\beta = 4n$, with $n \in \mathbb{Z}$ and $\varepsilon = \pm 1$. These rational solutions have the form

$$w = \frac{P_m(z)}{Q_m(z)},$$

where $P_m(z)$, $Q_m(z)$ are polynomials of degree m with no common roots.

Proof. See Lukashevich [41]; also Milne, Clarkson and Bassom [49] and Murata [51]. □

Suppose $\tau_{a,b}(z)$ is the generalised associated Laguerre polynomial defined by (3.2), then the rational solutions of P_{III} (1.1c) in the form

$$w_n^{[N]}(z; A^{[N]}, B^{[N]}, C^{[N]}, D^{[N]}),$$

for $N = 1, \dots, 4$ are given by the following:

$$w_n^{[1]}(z; A^{[1]}, B^{[1]}, C^{[1]}, D^{[1]}) = 1 - \frac{d}{dz} \ln \frac{\tau_{\mu+1, n+1}(z)}{\tau_{\mu, n}(z)}, \quad (3.16a)$$

$$w_n^{[2]}(z; A^{[2]}, B^{[2]}, C^{[2]}, D^{[2]}) = 1 + \frac{d}{dz} \ln \frac{\tau_{\mu, n+1}(z)}{\tau_{\mu+1, n}(z)}, \quad (3.16b)$$

for the parameters

$$\{A^{[1]}, B^{[1]}, C^{[1]}, D^{[1]}\} = \{2(n + \mu) + 3, 2(n - \mu) + 1, 1, -1\}, \quad (3.17a)$$

$$\{A^{[2]}, B^{[2]}, C^{[2]}, D^{[2]}\} = \{-2(n - \mu) - 1, -2(n + \mu) - 3, 1, -1\}. \quad (3.17b)$$

These rational solutions are identical to the Umemura rational solutions. However, the Umemura polynomials are actually polynomials in $1/z$ rather than polynomials in z . Clarkson in [15] determined special polynomials associated with the rational solutions of P_{III} (1.1c), which were polynomials in z . The rational solutions are shown below in terms of these polynomials.

Theorem 3.5. *Suppose $S_n(z; \mu)$ is the Umemura polynomial defined by*

$$S_{n+1}S_{n-1} = -z \left[S_n \frac{d^2 S_n}{dz^2} - \left(\frac{dS_n}{dz} \right)^2 \right] - S_n \frac{dS_n}{dz} + (z + \mu) S_n^2$$

with $S_{-1}(z; \mu) = S_0(z; \mu) = 1$ [15] (but with the transformation $z \rightarrow 1/z$) then the rational solutions of P_{III} (1.1c) are the following:

$$w_n = w(z; A^{[1]}, B^{[1]}, C^{[1]}, D^{[1]}) = 1 + \frac{d}{dz} \left\{ \ln \left[\frac{S_{n-1}(z; \mu - 1)}{S_n(z; \mu)} \right] \right\} \quad (3.18a)$$

$$= \frac{S_n(z; \mu - 1) S_{n-1}(z; \mu)}{S_n(z; \mu) S_{n-1}(z; \mu - 1)}, \quad (3.18b)$$

$$\hat{w}_n = w(z; A^{[2]}, B^{[2]}, C^{[2]}, D^{[2]}) = 1 + \frac{d}{dz} \left\{ \ln \left[\frac{S_{n-1}(z; \mu)}{S_n(z; \mu - 1)} \right] \right\} \quad (3.18c)$$

$$= \frac{S_n(z; \mu) S_{n-1}(z; \mu - 1)}{S_n(z; \mu - 1) S_{n-1}(z; \mu)}, \quad (3.18d)$$

for the parameters

$$\{A^{[1]}, B^{[1]}, C^{[1]}, D^{[1]}\} = \{2n + 2\mu - 1, 2n - 2\mu + 1, 1, -1\}, \quad (3.19a)$$

$$\{A^{[2]}, B^{[2]}, C^{[2]}, D^{[2]}\} = \{-2n + 2\mu - 1, -2n - 2\mu + 1, 1, -1\}. \quad (3.19b)$$

Proof. See [15], which generalizes the work of Kajiwara and Masuda [37]. \square

The rational solutions of P_{III} have no bounded solutions due to the polynomials that generate them always having real roots. This can be seen clearly in the following plots:

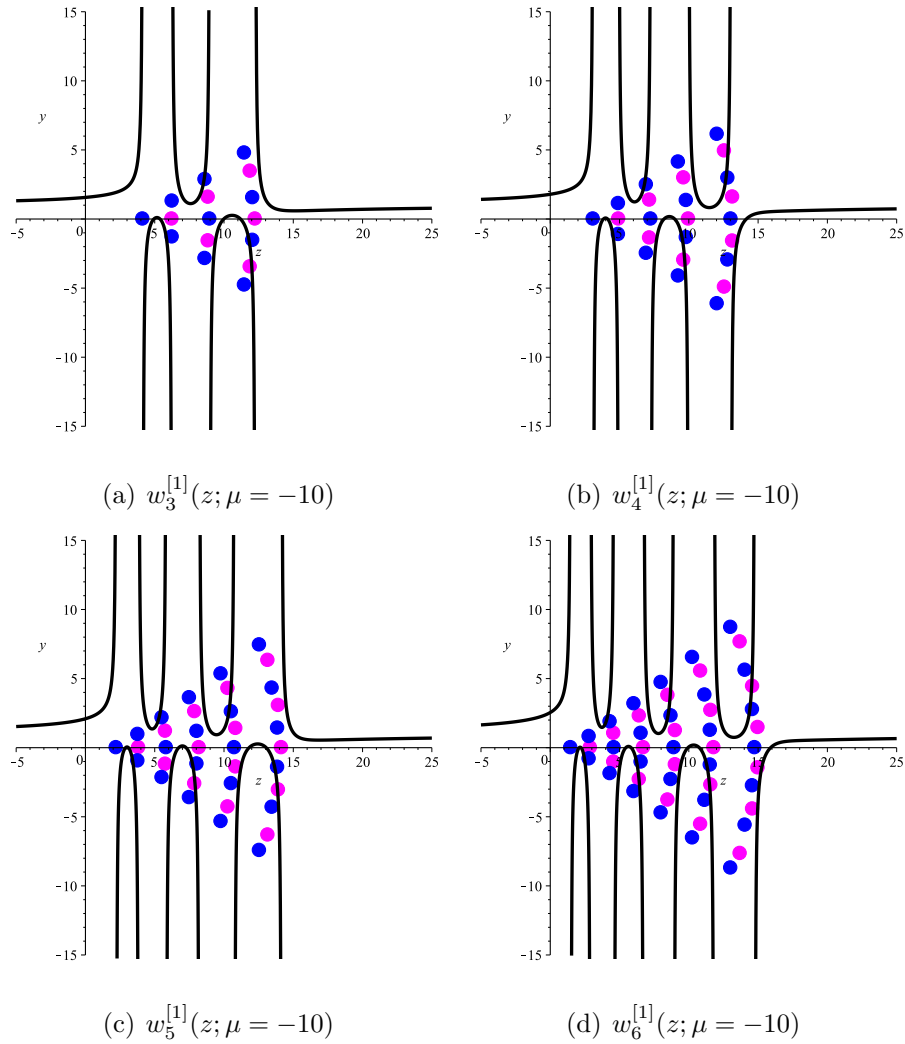


Figure 3.16: Some rational solutions to P_{III} (1.1c) with the complex roots of the corresponding generalised associated Laguerre polynomials which comprise the solutions.

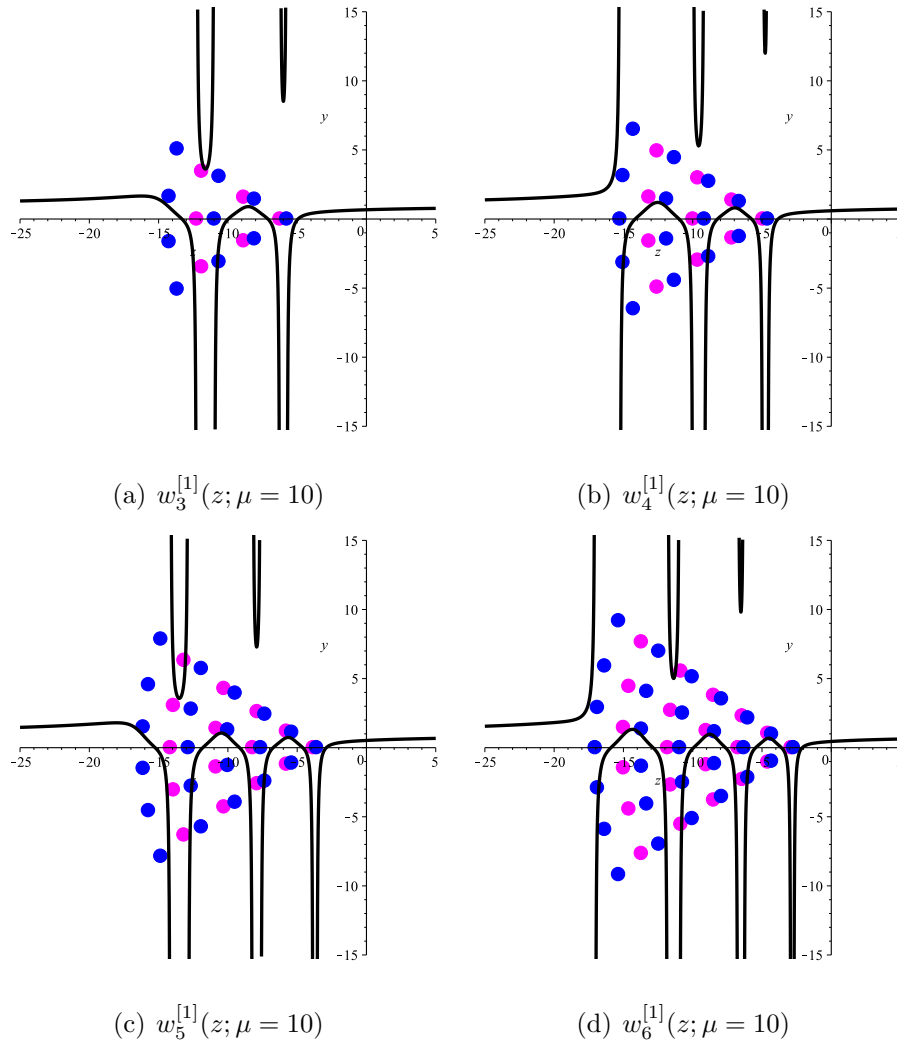


Figure 3.17: Some rational solutions to P_{III} (1.1c) super imposed with the complex roots of the corresponding generalised associated Laguerre polynomials which comprise the solutions.

3.2.3 The fourth Painlevé equation

Theorem 3.6. P_{IV} has rational solutions if and only if the parameters A and B are given by

$$A = m, \quad B = -2(2n - m + 1)^2, \quad (3.20)$$

or

$$A = m, \quad B = -2(2n - m + \frac{1}{3})^2, \quad (3.21)$$

with $m, n \in \mathbb{Z}$.

Proof. See Lukashevich [42], Gromak [32] and Murata [50]; also see Bassom, Clarkson and Hicks [5], Gromak, Laine and Shimomura [34], Umemura and Watanabe [66]. \square

Theorem 3.7. Suppose $H_{m,n}(z)$ is the generalised Hermite polynomial defined by (3.5b), then the rational solutions of P_{IV} (1.1d) in the form

$$w_{m,n}^{[N]} = w(z; A^{[N]}, B^{[N]}),$$

for $N = 1, 2, 3$ are given by the following:

$$\begin{aligned} w_{m,n}^{[1]} &= w(z; A^{[1]}, B^{[1]}) = \frac{d}{dz} \ln \left(\frac{H_{m+1,n}(z)}{H_{m,n}(z)} \right), \\ w_{m,n}^{[2]} &= w(z; A^{[2]}, B^{[2]}) = \frac{d}{dz} \ln \left(\frac{H_{m,n}(z)}{H_{m,n+1}(z)} \right), \\ w_{m,n}^{[3]} &= w(z; A^{[3]}, B^{[3]}) = -2z + \frac{d}{dz} \ln \left(\frac{H_{m,n+1}(z)}{H_{m+1,n}(z)} \right). \end{aligned}$$

for the parameters

$$\begin{aligned}\{A^{[1]}, B^{[1]}\} &= \{2m + n + 1, -2n^2\}, \\ \{A^{[2]}, B^{[2]}\} &= \{-(m + 2n + 1), -2m^2\}, \\ \{A^{[3]}, B^{[3]}\} &= \{n - m, -2(m + n + 1)^2\}.\end{aligned}$$

Proof. See Noumi and Yamada [52]; and also Theorem 3.1 in Clarkson [16]. \square

All the rational solutions of P_{IV} (1.1d) with parameters given by (3.20) can be expressed in terms of determinants whose entries are Hermite polynomials.

The generalised Hermite polynomials rational solutions of P_{IV} (1.1d) have bounded solutions when n is even. However, they are only bounded from the first hierarchy $w^{[1]}$ due to the formation of the root structures. From the other two hierarchies you cannot get real roots simultaneously from both the polynomials in the rational functions.

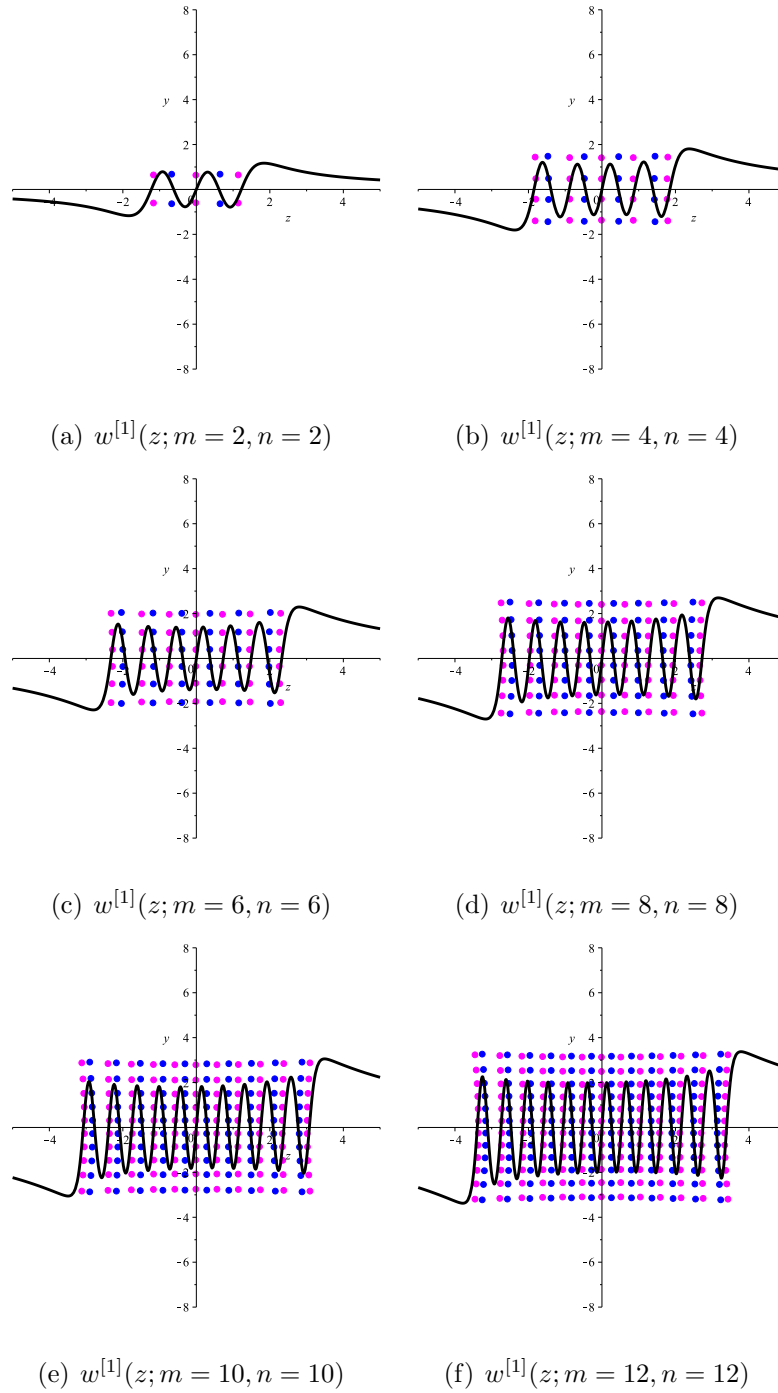


Figure 3.18: Some rational solutions to P_{IV} (1.1d) super imposed with the complex roots of the corresponding generalised Hermite polynomials which comprise the solutions.

Now we consider the generalised Okamoto polynomials.

Theorem 3.8. *Suppose $Q_{m,n}(z)$ is the generalised Okamoto polynomial defined by (3.6b), then the rational solutions of P_{IV} (1.1d) in the form*

$$\hat{w}_{m,n}^{[N]} = w(z; A^{[N]}, B^{[N]}),$$

are the following:

$$\begin{aligned}\hat{w}_{m,n}^{[1]} &= w(z; A^{[1]}, B^{[1]}) = -\frac{2}{3}z + \frac{d}{dz} \left(\frac{Q_{m+1,n}(z)}{Q_{m,n}(z)} \right), \\ \hat{w}_{m,n}^{[2]} &= w(z; A^{[2]}, B^{[2]}) = -\frac{2}{3}z + \frac{d}{dz} \left(\frac{Q_{m,n}(z)}{Q_{m,n+1}(z)} \right), \\ \hat{w}_{m,n}^{[3]} &= w(z; A^{[3]}, B^{[3]}) = -\frac{2}{3}z + \frac{d}{dz} \left(\frac{Q_{m,n+1}(z)}{Q_{m+1,n}(z)} \right),\end{aligned}$$

for the parameters

$$\begin{aligned}\{A^{[1]}, B^{[1]}\} &= \{2m + n, -2(n - \frac{1}{3})^2\}, \\ \{A^{[2]}, B^{[2]}\} &= \{-m - 2n, -2(m - \frac{1}{3})^2\}, \\ \{A^{[3]}, B^{[3]}\} &= \{n - m, -2(m + n + \frac{1}{3})^2\}.\end{aligned}$$

Proof. See Noumi and Yamada [52]; and also Theorem 4.1 in Clarkson [16]. \square

The generalised Okamoto rational solutions of P_{IV} (1.1d) have no bounded solutions due to the polynomials that generate them always having real roots. This can be seen clearly in the following plots:

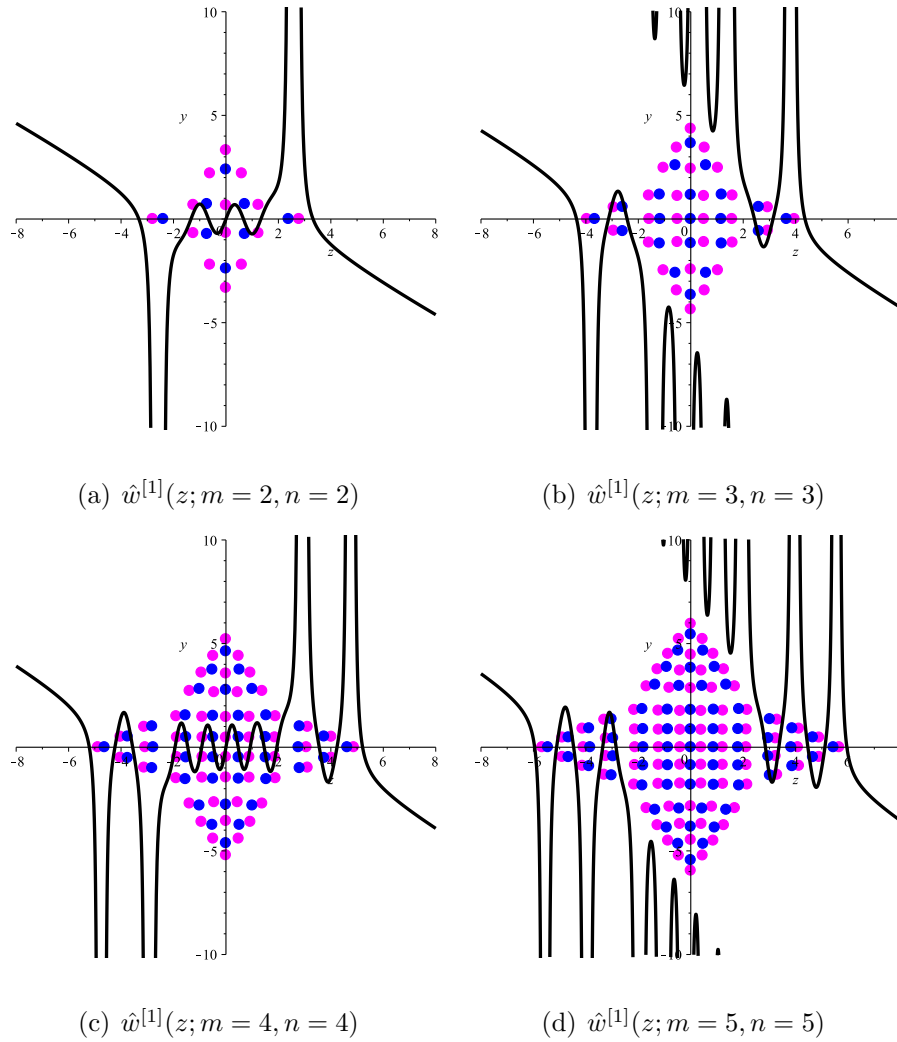


Figure 3.19: Some rational solutions to P_{IV} (1.1d) super imposed with the complex roots of the corresponding generalised Okamoto polynomials which comprise the solutions.

3.2.4 The fifth Painlevé equation

Theorem 3.9. P_V (1.1e) has rational solutions if and only if one of the following holds with $m, n \in \mathbb{Z}$ and $\varepsilon = \pm 1$:

i) $A = \frac{1}{2}(m + \varepsilon)^2$ and $B = -\frac{1}{2}n^2$, where $n > 0$, $m + n$ is odd and $A \neq 0$ where $|m| < n$,

ii) $A = \frac{1}{2}n^2$ and $B = -\frac{1}{2}(m + \varepsilon)^2$, where $n > 0$, $m + n$ is odd and $B \neq 0$ when $|m| < n$,

iii) $A = \frac{1}{2}a$ and $B = -\frac{1}{2}(a + n)^2$ and $C = m$, where $m + n$ is even and a arbitrary,

iv) $A = \frac{1}{2}(b + n)$, $B = -\frac{1}{2}(b)^2$ and $C = m$, where $m + n$ is even and b arbitrary,

v) $A = \frac{1}{8}(2m + 1)^2$ and $B = -\frac{1}{8}(2n + 1)^2$.

Proof. See Kitaev, Law and McLeod [39]; also Gromak and Lukashevich [35]; Gromak, Laine and Shimomura [34]. \square

Theorem 3.10. Suppose $\tilde{\mathcal{L}}_{\alpha,\beta}^{(n)}(z)$, $\tilde{\mathcal{L}}_{\alpha,\beta,e^{-z}}^{(n)}(z)$ and $\mathcal{L}_{\alpha,\beta}^{(n)}(z)$ are the generalised associated Laguerre polynomials defined by (3.8), (3.9) and (3.10) then the rational solutions of P_V (1.1e) in the form $w_n^{[N]}(z; A^{[N]}, B^{[N]}, C^{[N]}, D^{[N]}, \varepsilon_3)$ for $N = 1, 2$

are given by the following:

$$\begin{aligned}
w_n^{[1]}(z; A^{[1]}, B^{[1]}, C^{[1]}, D^{[1]}, 1) &= 1 + \frac{1}{\alpha - \beta - n} \left\{ \beta + z + z \frac{d}{dz} \ln \frac{\tilde{\mathcal{L}}_{\alpha, \beta+1, e^{-z}}^{(n+1)}(z)}{\tilde{\mathcal{L}}_{\alpha, \beta, e^{-z}}^{(n)}(z)} \right\} \\
&= 1 + \frac{1}{\alpha - \beta - n} \left\{ \beta + z \frac{d}{dz} \ln \frac{\tilde{\mathcal{L}}_{\alpha-n, \beta+1}^{(n+1)}(z)}{\tilde{\mathcal{L}}_{\alpha-n+1, \beta}^{(n)}(z)} \right\} \\
&= 1 + \frac{1}{\alpha - \beta - n} \left\{ \beta + n + z \frac{d}{dz} \ln \frac{\mathcal{L}_{\alpha-n+1, \beta-n+2}^{(n+1)}(z)}{\mathcal{L}_{\alpha-n+1, \beta-n+1}^{(n)}(z)} \right\}, \\
w_n^{[2]}(z; A^{[2]}, B^{[2]}, C^{[2]}, D^{[2]}, -1) &= 1 + \frac{1}{\alpha + n} \left\{ z - \beta - z \frac{d}{dz} \ln \frac{\tilde{\mathcal{L}}_{\alpha+1, \beta+1}^{(n+1)}(z)}{\tilde{\mathcal{L}}_{\alpha, \beta}^{(n)}(z)} \right\} \\
&= 1 + \frac{1}{\alpha + n} \left\{ 2z - \beta - z \frac{d}{dz} \ln \frac{\tilde{\mathcal{L}}_{\alpha+n+1, \beta+1, e^{-z}}^{(n+1)}(z)}{\tilde{\mathcal{L}}_{\alpha+n-1, \beta, e^{-z}}^{(n)}(z)} \right\} \\
&= 1 + \frac{1}{\alpha + n} \left\{ z - \beta - n - z \frac{d}{dz} \ln \frac{\mathcal{L}_{\alpha+1, \beta-n+1}^{(n+1)}(z)}{\mathcal{L}_{\alpha, \beta-n+1}^{(n)}(z)} \right\},
\end{aligned}$$

for the parameters

$$\{A^{[1]}, B^{[1]}, C^{[1]}, D^{[1]}\} = \left\{ \frac{1}{2}(-\alpha + \beta + n)^2, -\frac{1}{2}\alpha^2, 1 + n - \beta, -\frac{1}{2} \right\}, \quad (3.27a)$$

$$\{A^{[2]}, B^{[2]}, C^{[2]}, D^{[2]}\} = \left\{ \frac{1}{2}(\alpha + n)^2, -\frac{1}{2}(\beta - \alpha)^2, \beta - n - 1, -\frac{1}{2} \right\}. \quad (3.27b)$$

Proof. Taking α to be a negative integer we can apply the polynomial reduction of $U(a, b; z)$ and $M(a, b; z)$ the Kummer functions (2.12), to the special function solutions of P_V with a suitable transformation of the parameters yields the rational solutions: \square

The rational solutions to P_V (1.1e) have bounded solutions from the second hierarchy precisely when $1 + \alpha - n$ is even for $\beta < 0$ with $C_1 = 0$ or $C_2 = 0$. This is the exact condition which removes the possibility for real roots.

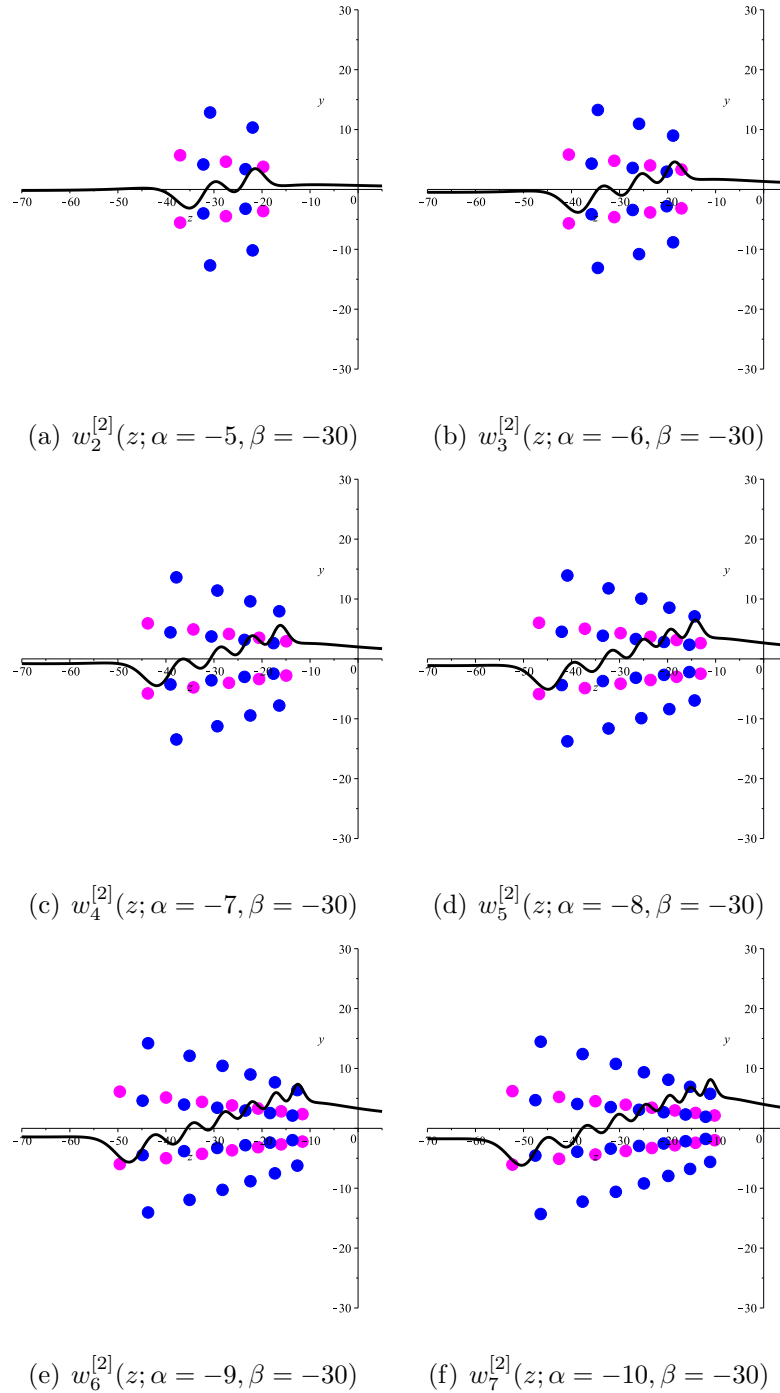


Figure 3.20: Some rational solutions to P_V (1.1e) super imposed with the complex roots of the corresponding generalised Laguerre polynomials which comprise the solutions.

3.2.5 The sixth Painlevé equation

Theorem 3.11. P_{VI} (1.1f) has rational solutions if and only if

$$h_1 + h_2 + h_3 + h_4 = 2n + 1,$$

with $\varepsilon_j = \pm 1$, $j = 1, 2, 3, 4$ independently where $h_1 = \varepsilon_1\sqrt{2A}$, $h_2 = \varepsilon_2\sqrt{-2B}$, $h_3 = \varepsilon_3\sqrt{2C}$, $h_4 = \varepsilon_4\sqrt{1 - 2D}$ and $a \in \mathbb{Z}$.

Proof. See Mazzocco [47]. These are special cases of the special function solutions which we discussed earlier. \square

Suppose $\widehat{\mathcal{J}}_{a,b,c}^{(n)}(z)$ is the generalised Jacobi polynomial defined by (3.12), then the rational solutions of P_{VI} (1.1f) in the form $w_n(z; A, B, C, D)$ are given by the following:

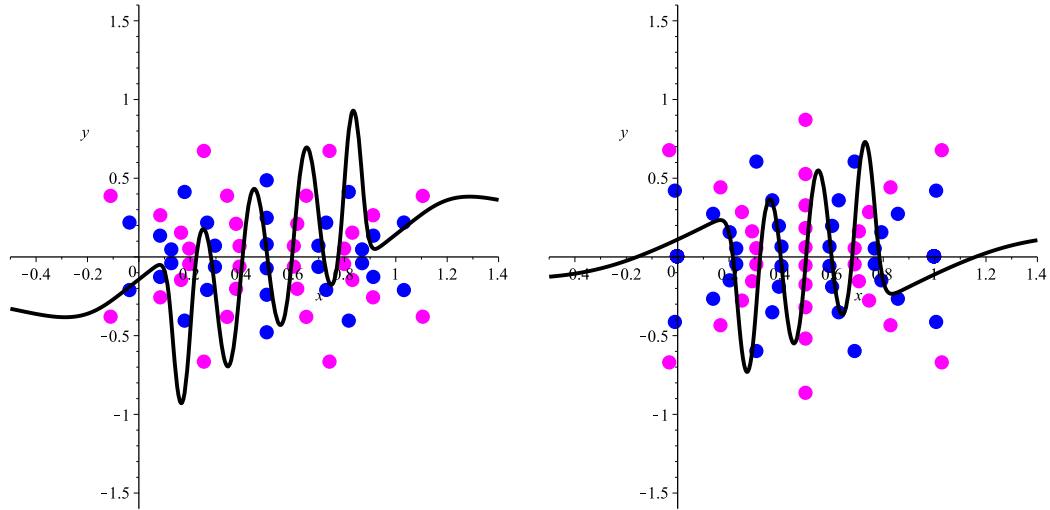
$$w_n(z; A, B, C, D) = \frac{1}{a} \left\{ n + c - (2n + b + 1)z - z(z - 1) \frac{d}{dz} \ln \frac{\mathcal{J}_{a+1,b+1,c+1}^{(n+1)}(z)}{\mathcal{J}_{a-1,b+1,c}^{(n)}(z)} \right\},$$

for the parameters

$$\{A, B, C, D\} = \left\{ \frac{a^2}{2}, -\frac{1}{2}(b - c + n + 1)^2, \frac{1}{2}(a - c - n)^2, \frac{1}{2}(1 - b^2) \right\},$$

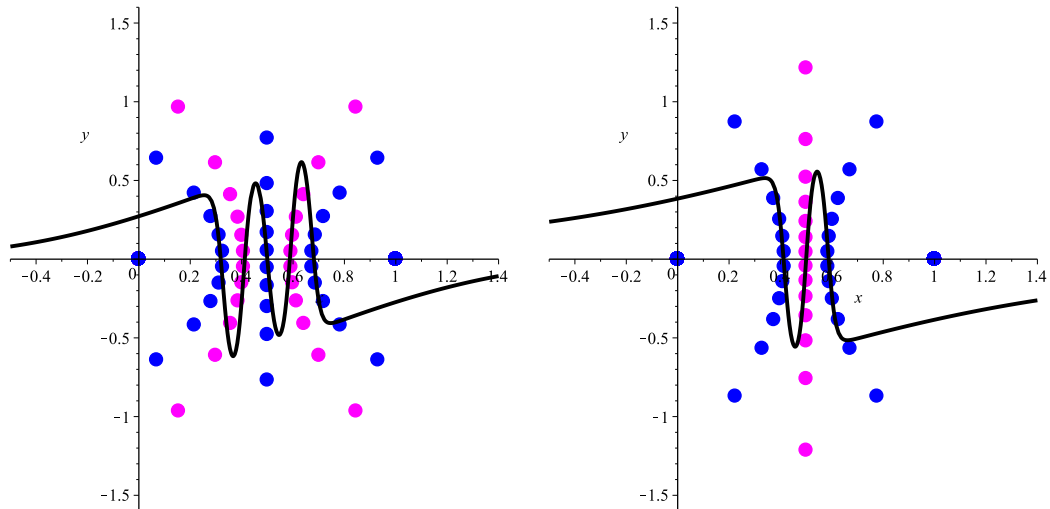
with

$$\mathcal{J}_{a,b,c}^{(n)} = z^{n/2(1-n-2b)}(z - 1)^{n(1-n)/2} \widehat{\mathcal{J}}_{a,b,c}^{(n)}.$$



(a) $w_4(z; a=-7, b=-16, c=-11) - L(z)$

(b) $w_5(z; a=-9, b=-18, c=-13) - L(z)$



(c) $w_6(z; a=-11, b=-20, c=-15) - L(z)$

(d) $w_7(z; a=-13, b=-22, c=-17) - L(z)$

Figure 3.21: Some rational solutions to P_{VI} (1.1f) super imposed with the complex roots of the corresponding generalised Jacobi polynomials which comprise the solutions, where $L(z)$ is the asymptotic behaviour around $\pm\infty$.

The rational solutions to P_{VI} have bounded solutions precisely when $b+n-1 <$

$c < 2 + a - n$ and $-a$ is even. This is exactly the condition that removes the possibility for real roots.

3.3 Rational function solutions to the σ -equations

3.3.1 The second Painlevé σ -equation

Theorem 3.12. *Suppose $Q_n(z)$ is the Yablonskii-Vorob'ev polynomial defined by (3.1), then the rational solutions of S_{II} (1.7b) in the form $\sigma_n(z; \alpha)$ are given by the following:*

$$\sigma_n(z; \alpha) = -\frac{1}{8}z^2 + \frac{d}{dz} \ln Q_n(z), \quad (3.28)$$

for the parameter $\alpha = n$.

Proof. See [34]. □

The rational solutions to S_{II} (1.7b) have no bounded solutions due to the polynomials that generate them always having real roots.

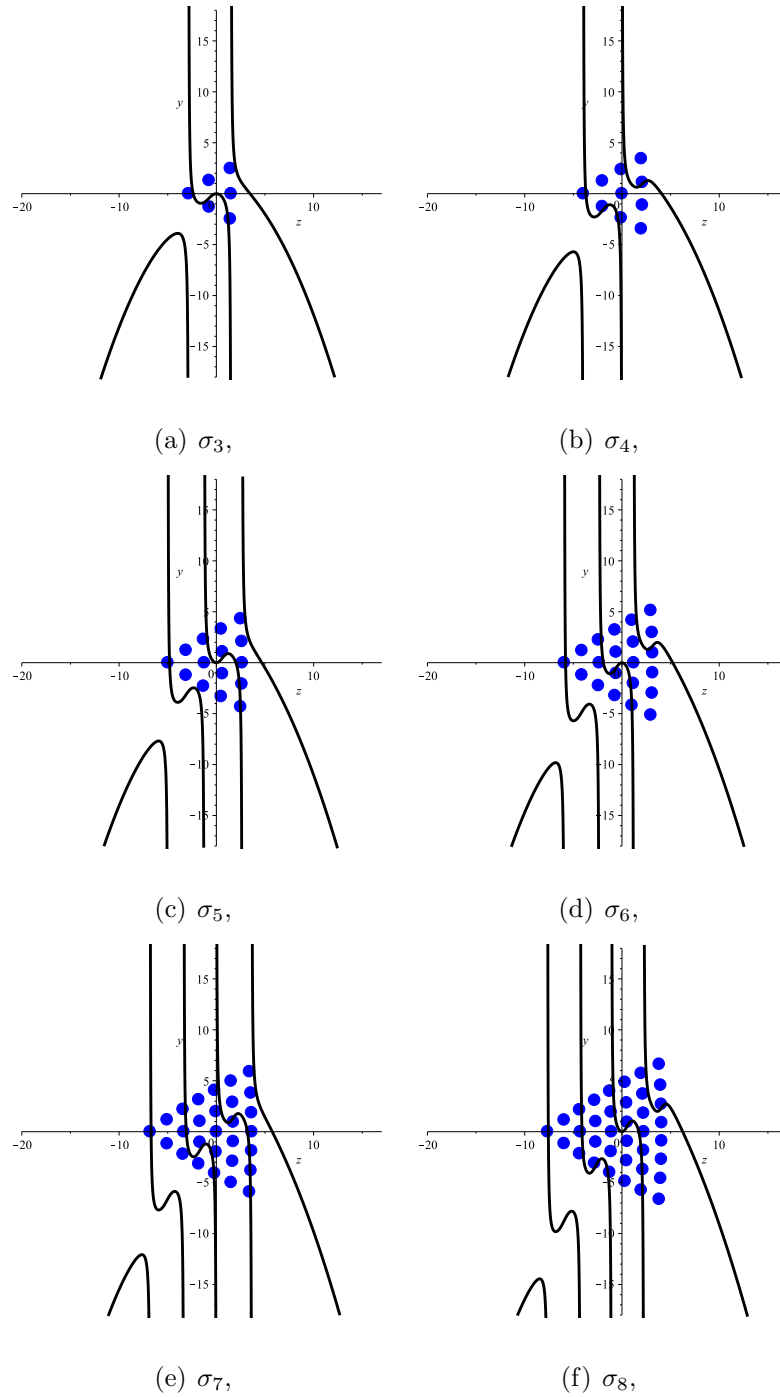


Figure 3.22: Some rational solutions to S_{II} (1.7b) super imposed with the complex roots of the corresponding Yablonskii-Vorob'ev polynomials which comprise the solutions.

3.3.2 The third Painlevé σ -equation

Theorem 3.13. *Suppose $\tau_{a,b}(z)$ is the generalised associated Laguerre polynomial defined by (3.2), then the rational solutions of S_{III} (1.7c) in the form $\sigma_{\mu,n}(z; \vartheta_0, \vartheta_\infty)$ are given by the following:*

$$\sigma_{\mu,n}(z; \vartheta_0, \vartheta_\infty) = -\frac{1}{4}z^2 - \mu z + \frac{1}{8} + z \frac{d}{dz} \ln \tau_{\mu,n}(z), \quad (3.29)$$

for the parameters

$$\{\vartheta_0, \vartheta_\infty\} = \{\mu^2 + (n + \frac{1}{2})^2, \mu^2 - (n + \frac{1}{2})^2\}.$$

Proof. These results can be inferred from the work of Clarkson in [15]. □

The rational solutions to S_{III} (1.7c) have no bounded solutions due to the polynomials that generate them always having real roots and their symmetric triangular structure.

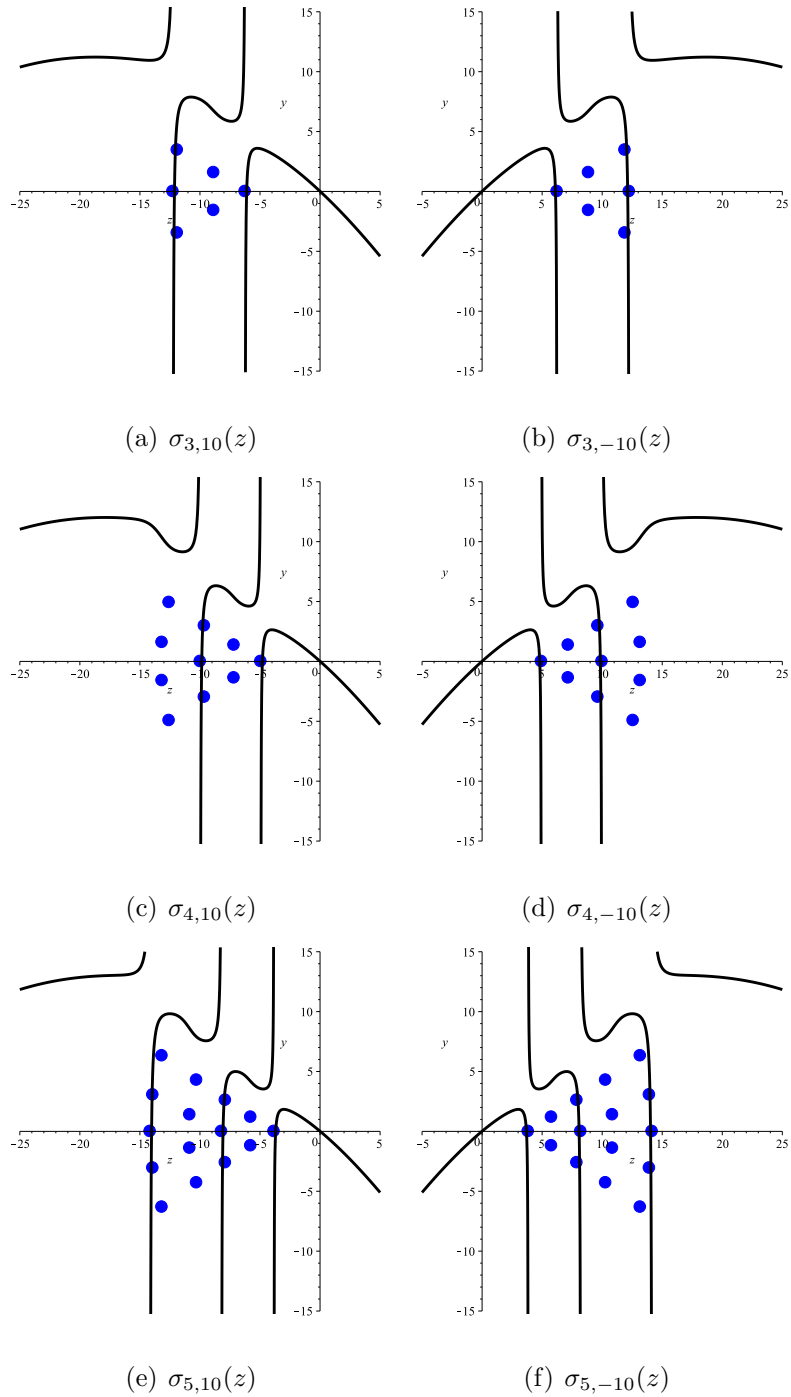


Figure 3.23: Some rational solutions to S_{III} (1.7c) super imposed with the complex roots of the corresponding generalised associated Laguerre which comprise the solutions.

3.3.3 The fourth Painlevé σ -equation

Theorem 3.14. *Suppose $H_{m,n}(z)$ is the generalised Hermite polynomial defined by (3.5b), then the rational solutions of S_{IV} (1.7d) in the form $\sigma_{m,n}^{[N]}(z; \vartheta_0^{[N]}, \vartheta_\infty^{[N]})$ for $N = 1, 2, 3$ are given by the following:*

$$\sigma_{m,n}^{[1]}(z; \vartheta_0^{[1]}, \vartheta_\infty^{[1]}) = \frac{d}{dz} \ln H_{m,n}(z), \quad (3.30a)$$

$$\sigma_{m,n}^{[2]}(z; \vartheta_0^{[2]}, \vartheta_\infty^{[2]}) = \frac{d}{dz} \ln H_{m,n}(z) - 2nz, \quad (3.30b)$$

$$\sigma_{m,n}^{[3]}(z; \vartheta_0^{[3]}, \vartheta_\infty^{[3]}) = \frac{d}{dz} \ln H_{m,n}(z) + 2mz, \quad (3.30c)$$

for the parameters

$$\{\vartheta_0^{[1]}, \vartheta_\infty^{[1]}\} = \{-n, m\}, \quad (3.31a)$$

$$\{\vartheta_0^{[2]}, \vartheta_\infty^{[2]}\} = \{n, m+n\}, \quad (3.31b)$$

$$\{\vartheta_0^{[3]}, \vartheta_\infty^{[3]}\} = \{-m, -m-n\}. \quad (3.31c)$$

Proof. Taking ν to be an integer, m , we can apply the polynomial reduction of $D_\nu(z)$ the parabolic cylinder function (2.7a), (2.7b) and (2.7c) to the special function solutions of S_{IV} with a suitable transformation of the parameters yields the rational solutions: Also see Okamoto [58]; also Forrester and Witte [26]. \square

The rational solutions to S_{IV} have bounded solutions precisely when n is even. However, only (3.31c) omits bounded solutions due to the linear component cancelling out the asymptotic behaviour around $\pm\infty$.

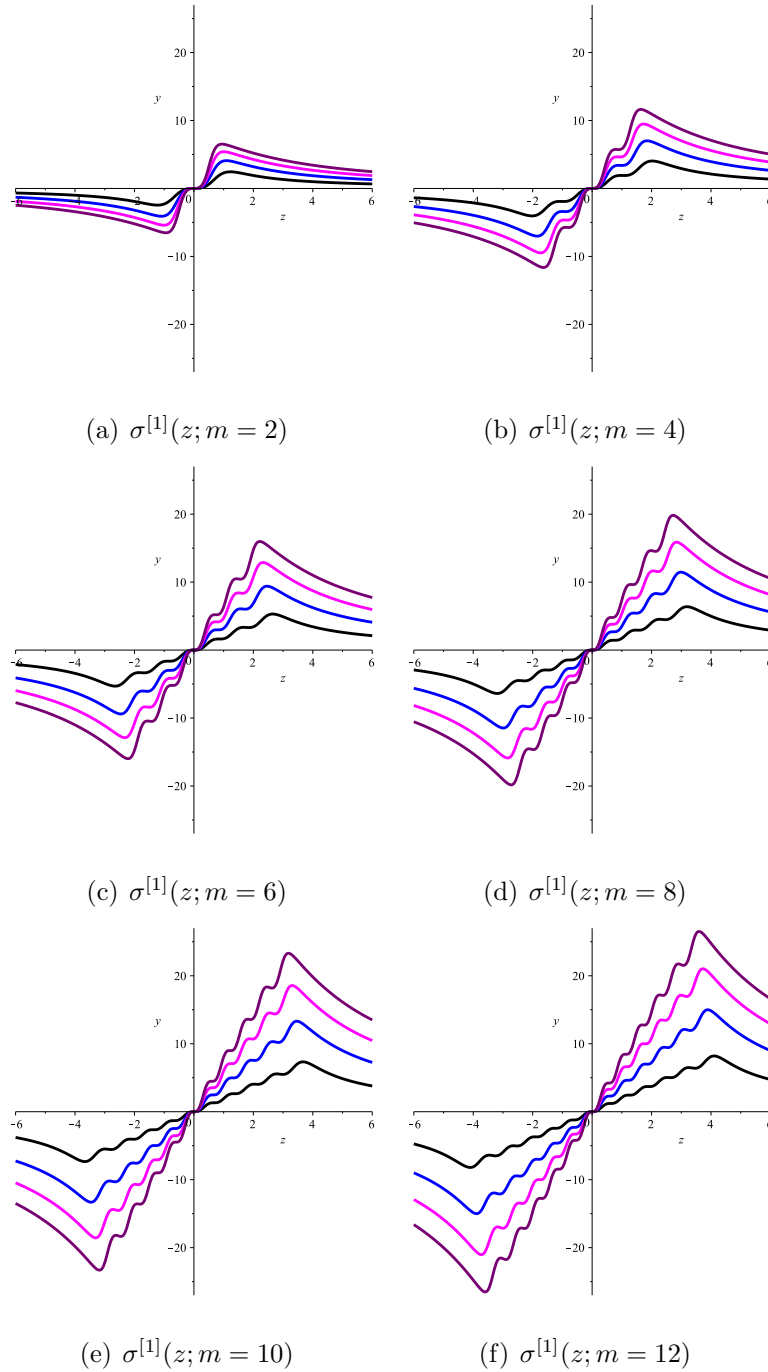
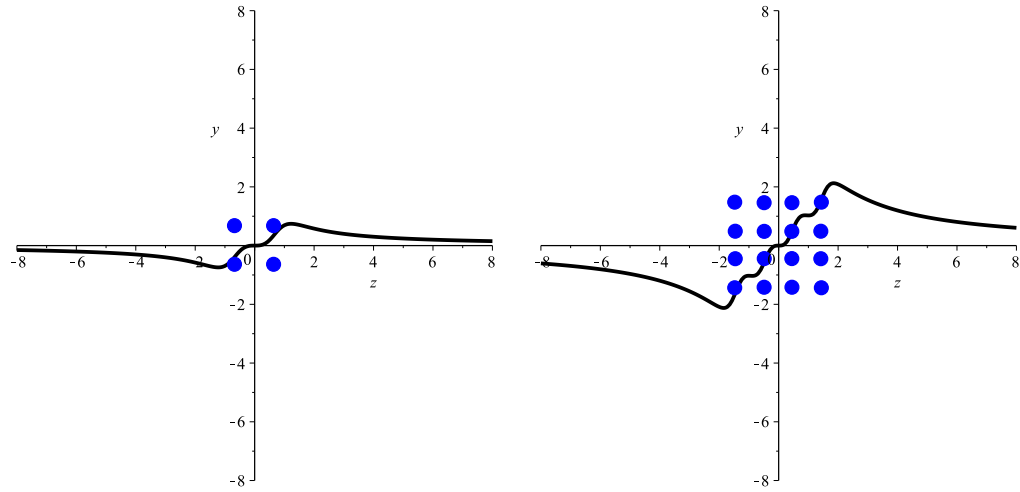


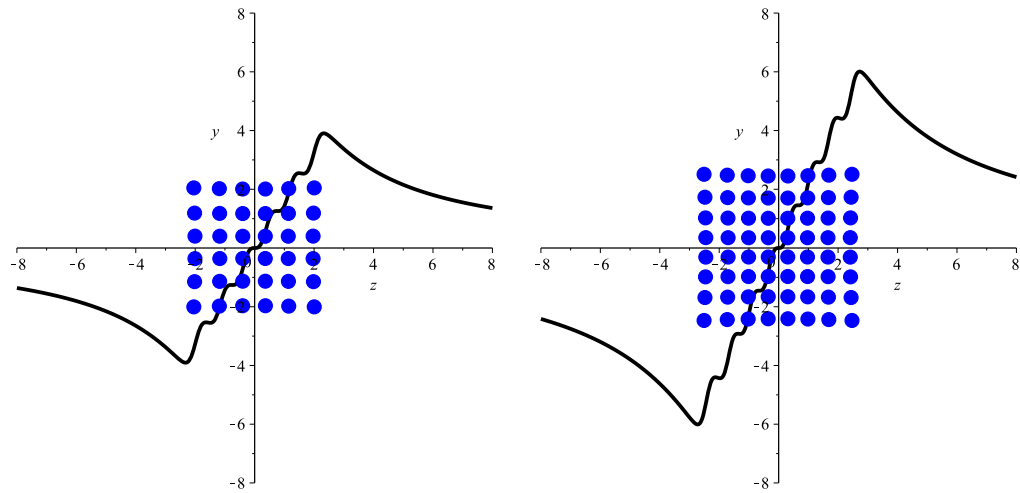
Figure 3.24: Some rational solutions to S_{IV} (1.7d) super imposed with the complex roots of the corresponding generalised Hermite polynomials which comprise the solutions, with $n = \{2, 4, 6, 8\}$.

It is interesting to note that the number of "kinks" in these solutions is equal to $m - 1$.



(a) $\sigma^{[1]}(z; m = 2, n = 2)$

(b) $\sigma^{[1]}(z; m = 4, n = 4)$



(c) $\sigma^{[1]}(z; m = 6, n = 6)$

(d) $\sigma^{[1]}(z; m = 8, n = 8)$

Figure 3.25: Some rational solutions to S_{IV} (1.7d) super imposed with the complex roots of the corresponding generalised Hermite polynomials which comprise the solutions.

Theorem 3.15. *Suppose $Q_{m,n}(z)$ is the generalised Okamoto polynomial defined by (3.6b), then the rational solutions of S_{IV} (1.7d) in the form $\sigma_{m,n}^{[1]}(z; \vartheta_0^{[1]}, \vartheta_\infty^{[1]})$ for $N = 1, 2, 3$ are given by the following:*

$$\sigma_{m,n}^{[1]}(z; \vartheta_0^{[1]}, \vartheta_\infty^{[1]}) = \frac{4}{27}z^3 + \frac{2}{3}(n-m)z + \frac{d}{dz} \ln Q_{m,n}(z), \quad (3.32a)$$

$$\sigma_{m,n}^{[2]}(z; \vartheta_0^{[2]}, \vartheta_\infty^{[2]}) = \frac{4}{27}z^3 - \frac{2}{3}(m+2n-1)z + \frac{d}{dz} \ln Q_{m,n}(z), \quad (3.32b)$$

$$\sigma_{m,n}^{[3]}(z; \vartheta_0^{[3]}, \vartheta_\infty^{[3]}) = \frac{4}{27}z^3 + \frac{2}{3}(2m+n-1)z + \frac{d}{dz} \ln Q_{m,n}(z), \quad (3.32c)$$

for the parameters

$$\{\vartheta_0^{[1]}, \vartheta_\infty^{[1]}\} = \{m - \frac{1}{3}, \frac{1}{3} - n\}, \quad (3.33a)$$

$$\{\vartheta_0^{[2]}, \vartheta_\infty^{[2]}\} = \{n - \frac{1}{3}, m + n - \frac{2}{3}\}, \quad (3.33b)$$

$$\{\vartheta_0^{[3]}, \vartheta_\infty^{[3]}\} = \{\frac{2}{3} - m - n, \frac{1}{3} - m\}. \quad (3.33c)$$

Proof. These solutions can be inferred from the P_{IV} (1.1d) Okamoto rational function solutions using the Hamiltonian structure of P_{IV} (1.1d). \square

3.3.4 The fifth Painlevé σ -equation

Theorem 3.16. *Suppose $\tilde{\mathcal{L}}_{\alpha,\beta}^{(n)}(z)$, $\tilde{\mathcal{L}}_{\alpha,\beta,e^{-z}}^{(n)}(z)$ and $\mathcal{L}_{\alpha,\beta}^{(n)}(z)$ are the generalised associated Laguerre polynomials defined by (3.8), (3.9) and (3.10), then the rational solutions of S_V (1.7e) in the form $\sigma_n(z; \kappa_0^{[N]}, \kappa_1^{[N]}, \kappa_2^{[N]}, \kappa_3^{[N]}, \varepsilon_3)$ for $N = 1, 2$ are*

given by the following:

$$\begin{aligned}
\sigma_n^{[1]}(z; \kappa_0, \kappa_1^{[1]}, \kappa_2^{[1]}, \kappa_3^{[1]}, 1) &= z \frac{d}{dz} \ln \tilde{\mathcal{L}}_{\alpha, \beta, e^{-z}}^{(n+1)}(z) - \frac{5}{8}(n+1)^2 + \frac{1}{4}(2\alpha + 1 + \beta + 3z) \\
&\quad (n+1) - \frac{1}{8}(-2\alpha - 1 + \beta)(-2\alpha - 1 - 2z + \beta) \\
&= z \frac{d}{dz} \ln \mathcal{L}_{\alpha-n, \beta-n}^{(n+1)}(z) - \frac{1}{8}(n+1)^2 + \frac{1}{4}(2\alpha - 1 + \beta - z) \\
&\quad (n+1) - \frac{1}{8}(-2\alpha - 1 + \beta)(-2\alpha - 1 - 2z + \beta), \\
\sigma_n^{[2]}(z; \kappa_0, \kappa_1^{[2]}, \kappa_2^{[2]}, \kappa_3^{[2]}, -1) &= z \frac{d}{dz} \ln \tilde{\mathcal{L}}_{\alpha+1, \beta+1}^{(n)}(z) - \frac{5}{8}n^2 + \frac{1}{4}(3\beta + 2 - 2\alpha - 3z)n \\
&\quad - \frac{1}{8}(2\alpha - \beta)(2\alpha + 2z - \beta) \\
&= z \frac{d}{dz} \ln \mathcal{L}_{\alpha+1, \beta-n+2}^{(n)}(z) - \frac{1}{8}n^2 + \frac{1}{4}(3\beta - 2\alpha - 3z)n \\
&\quad - \frac{1}{8}(2\alpha - \beta)(2\alpha + 2z - \beta),
\end{aligned}$$

for the parameters

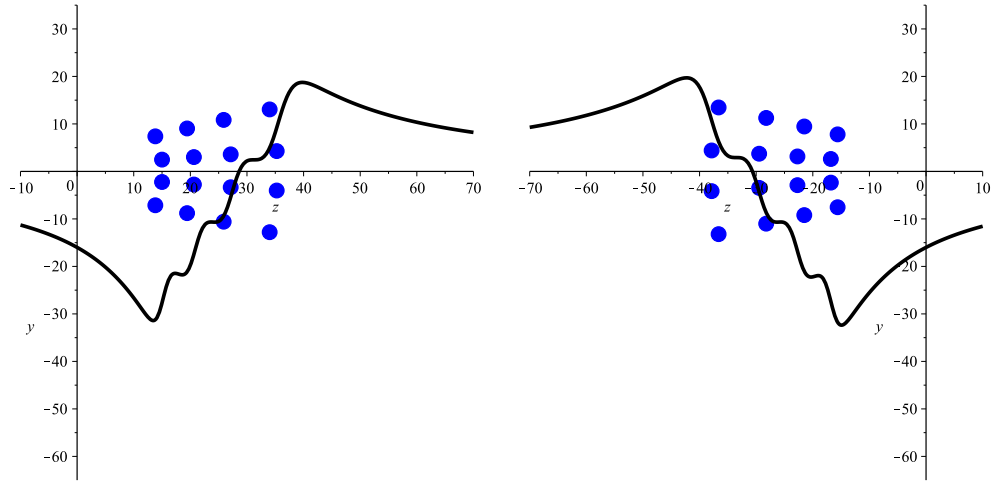
$$\begin{aligned}
\{\kappa_0^{[1]}, \kappa_1^{[1]}, \kappa_2^{[1]}, \kappa_3^{[1]}\} &= \frac{1}{4}\{2\alpha - \beta + n + 2, n + 2 - 2\alpha - \beta, 2\alpha - \beta - 3n - 2, 3\beta + n \\
&\quad - 2\alpha - 2\}, \tag{3.35a}
\end{aligned}$$

$$\begin{aligned}
\{\kappa_0^{[2]}, \kappa_1^{[2]}, \kappa_2^{[2]}, \kappa_3^{[2]}\} &= -\frac{1}{4}\{2\alpha + \beta + n, \beta - 3n - 2\alpha, n + 2\alpha - 3\beta, \beta + n - 2\alpha\}. \tag{3.35b}
\end{aligned}$$

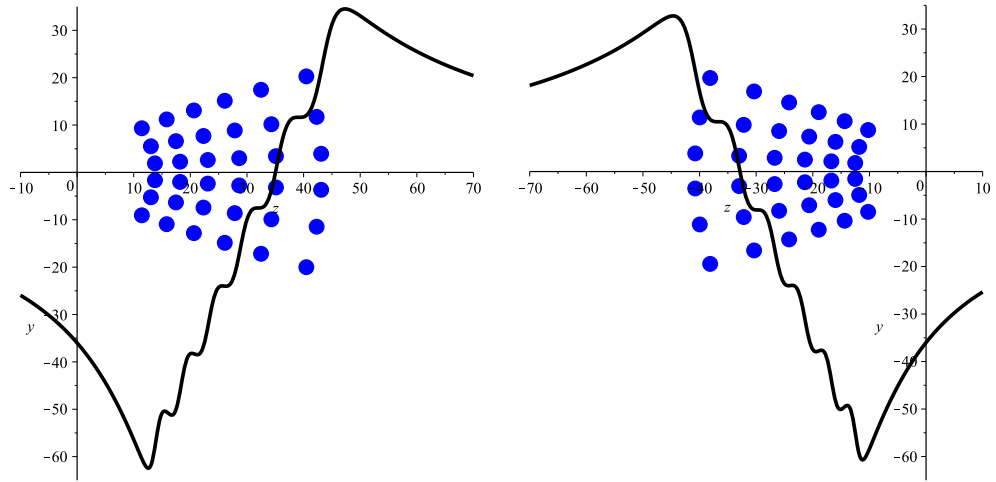
Proof. Taking α to be a negative integer we can apply the polynomial reduction of $U(a, b; z)$ and $M(a, b; z)$ the Kummer functions (2.12), to the special function solutions of S_V with a suitable transformation of the parameters yields the rational solutions: \square

The rational solutions of S_V have no bounded solutions due to the asymptotic behaviour around $\pm\infty$ being linear. However, if an appropriate linear transformation is made we can have bounded solutions and this is precisely when the

polynomials have no real roots. The conditions for this are specified in the generalised associated Laguerre polynomials chapter with $C_1 = 0$ or $C_2 = 0$.



(a) $\sigma_4^{[1]}(z; \alpha = -8, \beta = 20) - A - \frac{n}{2}(3n - 1)$ (b) $\sigma_4^{[1]}(z; \alpha = -8, \beta = -30) - A - \frac{n}{2}(3n - 1)$



(c) $\sigma_6^{[1]}(z; \alpha = -12, \beta = 20) - A - \frac{n}{2}(3n - 1)$ (d) $\sigma_6^{[1]}(z; \alpha = -12, \beta = -30) - A - \frac{n}{2}(3n - 1)$

Figure 3.26: Some rational solutions to S_V (1.7e) super imposed with the complex roots of the corresponding generalised associated Laguerre polynomials which comprise the solutions, where

$$A = -\frac{5}{8}(n+1)^2 + \frac{1}{4}(2\alpha + 1 + \beta + 3z)(n+1) \\ - \frac{1}{8}(-2\alpha - 1 + \beta)(-2\alpha - 1 - 2z + \beta).$$

3.3.5 The sixth Painlevé σ -equation

Theorem 3.17. *Suppose $\mathcal{J}_{a,b,c}^{(n)}(z)$ is the generalised Jacobi polynomial defined by (3.12), then the rational solutions of S_{VI} (1.7f) in the form $\sigma(z; \kappa_1, \kappa_2, \kappa_3, \kappa_4)$ are given by the following:*

$$\sigma(z; \kappa_1, \kappa_2, \kappa_3, \kappa_4) = \frac{1}{4}(n+1)(4az - a + b - 2c + 1) - \frac{1}{4}(a - b + 1)^2 z \\ + \frac{1}{4}(a^2 + a + b^2 - b - ac - bc) + z(z-1)\frac{d}{dz} \ln \mathcal{J}_{a,b,c}^{(n+1)}(z),$$

for the parameters

$$\{\kappa_1, \kappa_2, \kappa_3, \kappa_4\} = \left\{-\frac{1}{2}(a-b-2n-1), \frac{1}{2}(a+b-2c+1), -\frac{1}{2}(a-b+1), \frac{1}{2}(a+b-1)\right\}.$$

Proof. Taking a to be a negative integer we can apply the polynomial reduction of the general hypergeometric function $F(a, b, c; z)$ via an appropriate choice of the parameters. Applying this to the special function solutions of S_{VI} (1.7f) gives the desired result. \square

The rational solutions of S_{VI} have no bounded solutions due to the asymptotic behaviour around $\pm\infty$ being linear. However, if an appropriate linear transformation is made we can have bounded solutions and this is precisely when the polynomials have no real roots. The conditions for this are specified in the generalised Jacobi polynomials chapter.

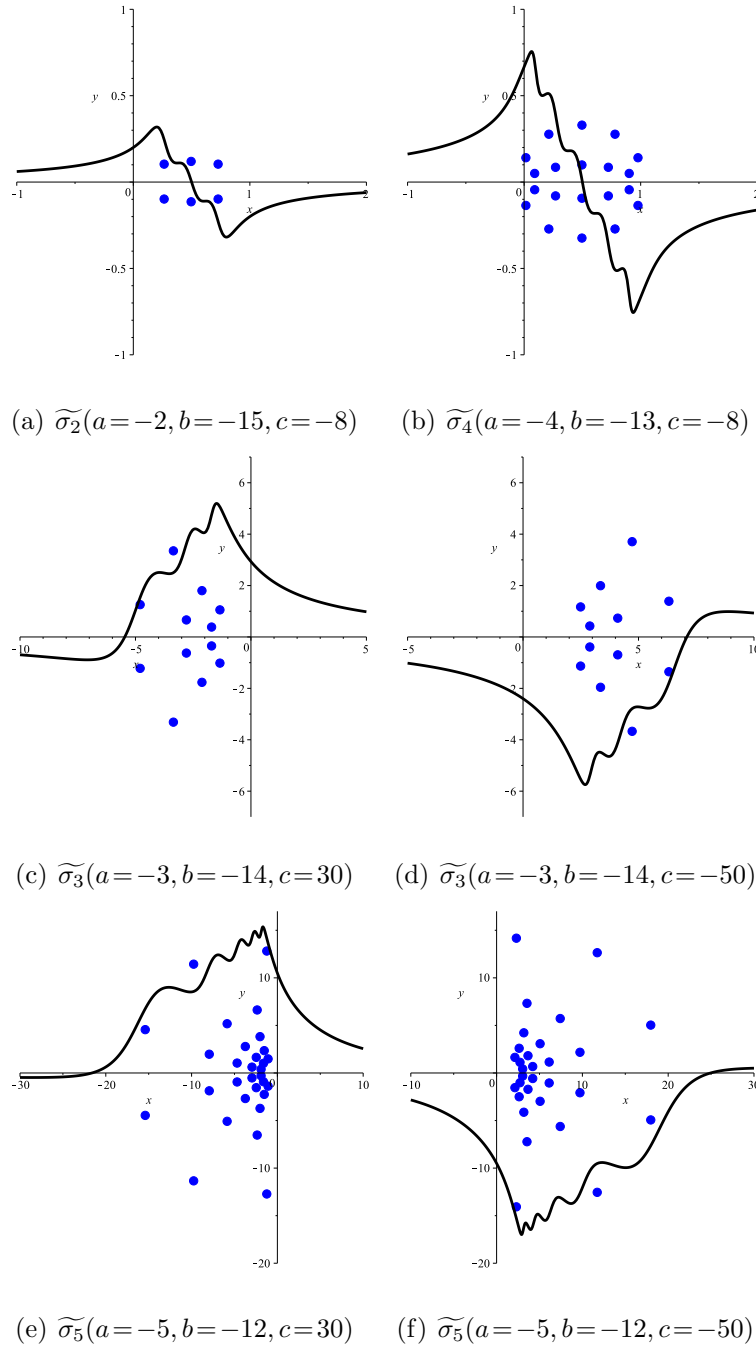


Figure 3.27: Some rational solutions to S_{VI} (1.7f) super imposed with the complex roots of the corresponding generalised Jacobi polynomials which comprise the solutions, with $\widetilde{\sigma}_n(z) = \sigma_n(z) - A(z)$, where $A(z)$ is the asymptotic expansion of the solution around $\pm\infty$.

4 Monic orthogonal polynomials

In the following chapter we will be introducing the concept of monic orthogonal polynomials. We will also be discussing classical and semi-classical orthogonal polynomials and their differences.

4.1 Continuous orthogonal polynomials

Monic orthogonal polynomials are a certain type of polynomial $p_n(x)$, defined over a range $[a, b]$, that satisfy an orthogonality relation

$$\int_a^b w(x)p_m(x)p_n(x) dx = h_n\delta_{mn}, \quad h_n > 0,$$

with $n \in \mathbb{N}$, δ_{mn} the Kroneker delta and $p_n(x)$ an orthogonal polynomial of degree n with respect to a positive weight $w(x)$. An important property of orthogonal polynomials is that they must satisfy a three term recurrence relation of the following form:

$$xp_n(x) = p_{n+1}(x) + \alpha_n p_n(x) + \beta_n p_{n-1}(x),$$

where the coefficients α_n and β_n are given by

$$\alpha_n = \frac{1}{h_n} \int_a^b xp_n^2(x)w(x) dx, \quad \beta_n = \frac{1}{h_{n-1}} \int_a^b xp_{n-1}(x)p_n w(x) dx,$$

with $p_{-1} = 0$ and $p_0 = 1$. One of our aims in this thesis is to find an alternative method for calculating these coefficients using determinants of moments that are produced from our associated orthogonal polynomial weight. The coefficients α_n and β_n can be rewritten in the following way:

$$\alpha_n = \frac{\tilde{\Delta}_{n+1}}{\Delta_{n+1}} - \frac{\tilde{\Delta}_n}{\Delta_n}, \quad \beta_n = \frac{\Delta_{n+1}\Delta_{n-1}}{\Delta_n^2}, \quad (4.1)$$

where Δ_n and $\tilde{\Delta}_n$ are the determinants given by

$$\Delta_n := \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} \\ \mu_1 & \mu_2 & \cdots & \mu_n \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-2} \end{vmatrix}, \quad n \geq 1, \quad (4.2)$$

$$\tilde{\Delta}_n := \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-2} & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n-1} & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-3} & \mu_{2n-1} \end{vmatrix}, \quad n \geq 1, \quad (4.3)$$

and the initial states are $\Delta_0 = 1$ and $\Delta_{-1} = \tilde{\Delta}_0 = 1$. The individual moments can be calculated as follows:

$$\mu_k = \int_a^b x^k w(x) dx. \quad (4.4)$$

We remark that the Hankel determinant $\Delta_n(z)$ (4.2) also has the integral representation

$$\Delta_n(x) = \frac{1}{n!} \int_a^b \cdots \int_a^b \prod_{l=1}^n w(x_l) \prod_{1 \leq j < k \leq n} (x_j - x_k)^2 dx_1, \dots, dx_n, \quad n \geq 1. \quad (4.5)$$

This arises in Random matrix theory as the partition function for the unitary ensemble with eigenvalue distribution. See Mehta [48] for full details.

It is a well known fact that the monic polynomial $p_n(x)$ can be uniquely ex-

pressed as the following:

$$p_n(x) = \frac{1}{\Delta_n} \begin{vmatrix} \mu_0 & \mu_1 & \dots & \mu_{n-1} \\ \mu_1 & \mu_2 & \dots & \mu_n \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \dots & \mu_{2n-1} \\ 1 & x & \dots & x^n \end{vmatrix}, \quad n \geq 1 \quad (4.6)$$

and the normalisation constant is

$$h_n = \frac{\Delta_{n+1}}{\Delta_n}, \quad h_0 = \Delta_1 = \mu_0.$$

Now suppose the weight has the following form:

$$w(x; z) = w_0(x) \exp(xz), \quad (4.7)$$

where z is a parameter with finite moments for all $z \in \mathbb{R}$. If the weight has the form (4.7) then suddenly the polynomials $p_n(x)$, the recurrence coefficients α_n and β_n , the determinants Δ_n , $\tilde{\Delta}_n$ and the moments μ_k are all now functions of z . For certain weights, a consequence of (4.7) is the following:

$$\mu_k = \pm \frac{d}{dz} \mu_{k \pm 1} \quad (4.8)$$

and the recurrence relation has the form

$$xp_n(x; z) = p_{n+1}(x; z) + \alpha_n(z)p_n(x; z) + \beta_n(z)p_{n-1}(x; z). \quad (4.9)$$

The implementation of (4.8) in the determinants Δ_n and $\tilde{\Delta}_n$ given by (4.2) has the following effect:

$$\Delta_n = \begin{vmatrix} \mu_0 & \mu'_0 & \dots & \mu_0^{(n-1)} \\ \mu'_0 & \mu''_0 & \dots & \mu_0^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_0^{(n-1)} & \mu_0^{(n)} & \dots & \mu_0^{(2n-2)} \end{vmatrix}, \quad ' = \frac{d}{dz},$$

$$\tilde{\Delta}_n = \begin{vmatrix} \mu_0 & \mu_0' & \cdots & \mu_0^{(n-2)} & \mu_0^{(n)} \\ \mu_0' & \mu_0'' & \cdots & \mu_0^{(n-1)} & \mu_0^{(n+1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mu_0^{(n-1)} & \mu_0^{(n)} & \cdots & \mu_0^{(2n-3)} & \mu_0^{(2n-1)} \end{vmatrix}, \quad ' = \frac{d}{dz}.$$

Hence we can construct the following theorem:

Theorem 4.1. *As before, we will denote τ_n as the bi-directional Wronskian*

$$\tau_n(f) = \mathcal{W}\left(f, \frac{df}{dz}, \dots, \frac{d^{n-2}f}{dz^{n-2}}, \frac{d^{n-1}f}{dz^{n-1}}\right). \quad (4.10)$$

If the moment $\mu_k(z)$ has the form (4.8) then the determinants Δ_n and $\tilde{\Delta}_n$ can be written in the form

$$\Delta_n(z) = \tau_n(\mu_0), \quad \tilde{\Delta}_n(z) = \frac{d}{dz}\tau_n(\mu_0). \quad (4.11)$$

Proof. See [17, 38, 59, 63]. □

Also, since $\mu_k = \frac{d^k \mu_j}{dz^k}$, the determinants $\Delta_n(z)$ and $\tilde{\Delta}_n(z)$ can be written in

the form

$$\begin{aligned}
\Delta_n &= \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} \\ \mu_1 & \mu_2 & \cdots & \mu_n \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-2} \end{vmatrix} = \mathcal{W}\left(\mu_0, \frac{d}{dz}\mu_0, \dots, \frac{d^{n-1}\mu_0}{dz^{n-1}}\right) \\
&= \tau_n(\mu_0), \\
\tilde{\Delta}_n &= \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-2} & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n-1} & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-3} & \mu_{2n-1} \end{vmatrix} = \mathcal{W}\left(\mu_0, \frac{d}{dz}\mu_0, \dots, \frac{d^{n-2}\mu_0}{dz^{n-1}}, \frac{d^n\mu_0}{dz^n}\right) \\
&= \frac{d}{dz}\mathcal{W}\left(\mu_0, \frac{d}{dz}\mu_0, \dots, \frac{d^{n-2}\mu_0}{dz^{n-1}}, \frac{d^{n-1}\mu_0}{dz^{n-1}}\right) \\
&= \frac{d}{dz}\tau_n(\mu_0).
\end{aligned}$$

Theorem 4.2. *The Hankel determinant $\Delta_n(z)$ given by (4.2) satisfies the Toda equation*

$$\frac{d^2}{dz^2} \ln \Delta_n(z) = \frac{\Delta_{n+1}(z)\Delta_{n-1}(z)}{\Delta_n^2(z)}. \quad (4.12)$$

Proof. See [12, 61, 63]. □

Theorem 4.3. *As long as the condition (4.8) is satisfied the recurrence coefficients $\alpha_n(z)$ and $\beta_n(z)$ in (4.9) can be expressed in the form*

$$\alpha_n(z) = \frac{d}{dz} \ln \left(\frac{\tau_{n+1}(\mu_0)}{\tau_n(\mu_0)} \right), \quad \beta_n(z) = \frac{d^2}{dz^2} \ln \{ \tau_n(\mu_0) \}. \quad (4.13)$$

Proof. The proof is actually straightforward; applying the theorem 4.1 to (4.1) and using that Δ_n satisfies the Toda equation (4.12) gives the desired result. This is shown in detail below.

Recall that $\alpha_n(z)$ and $\beta_n(z)$ are defined by

$$\alpha_n = \frac{\tilde{\Delta}_{n+1}}{\Delta_{n+1}} - \frac{\tilde{\Delta}_n}{\Delta_n}, \quad \beta_n = \frac{\Delta_{n+1}\Delta_{n-1}}{\Delta_n^2},$$

where $\Delta_n(z)$ and $\tilde{\Delta}_n(z)$ are defined by (4.2) and (4.3). Using (4.11) and (4.12) we can deduce the following:

$$\begin{aligned} \alpha_n(z) &= \frac{\tilde{\Delta}_{n+1}(z)}{\Delta_{n+1}(z)} - \frac{\tilde{\Delta}_n(z)}{\Delta_n(z)} = \frac{1}{\Delta_{n+1}(z)} \frac{d\Delta_{n+1}(z)}{dz} - \frac{1}{\Delta_n(z)} \frac{d\Delta_n(z)}{dz} \\ &= \frac{d}{dz} \ln \Delta_{n+1}(z) - \frac{d}{dz} \ln \Delta_n(z) \\ &= \frac{d}{dz} \left\{ \ln \Delta_{n+1}(z) - \ln \Delta_n(z) \right\} \\ &= \frac{d}{dz} \ln \frac{\Delta_{n+1}(z)}{\Delta_n(z)}, \end{aligned}$$

$$\beta_n(z) = \frac{\Delta_{n+1}\Delta_{n-1}}{\Delta_n^2} = \frac{d^2}{dz^2} \ln \Delta_n(z),$$

as required. □

Some motivation for this work is that the recurrence coefficients of semi-classical orthogonal polynomials can often be expressed in terms of solutions of the Painlevé equations. For example; all the recurrence coefficients can be expressed in terms of solutions of P_{II} (1.1b) for semi-classical orthogonal polynomials with respect to an Airy weight

$$w(x; z) = \exp\left(\frac{1}{3}x^3 + zx\right), \quad x^3 < 0, \quad (4.14)$$

with $z \in \mathbb{R}$ a parameter [44]. In terms of solutions of P_{III} (1.1c) for the deformed Laguerre weight

$$w(x; z) = x^\alpha \exp(-x - z/x), \quad x \in \mathbb{R}^+,$$

with $\alpha > 0$ and $z \in \mathbb{R}^+$ parameters [10]. In terms of solutions of P_V (1.1e) for the weights

$$w(x; z) = (1 - x)^\alpha (1 + x)^\beta e^{-zx}, \quad x \in [-1, 1],$$

$$w(x; z) = x^\alpha (1 - x)^\beta e^{-z/x}, \quad x \in [0, 1],$$

$$w(x; z) = x^\alpha (x + z)^\beta e^{-x}, \quad x \in \mathbb{R}^+,$$

with $\alpha, \beta > 0$ and $t \in \mathbb{R}^+$ parameters [2, 20, 11, 13, 29]. In terms of solutions of P_{VI} (1.1f) for the weight

$$w(x; z) = x^\alpha (1 - x)^\beta (z - x)^\gamma, \quad x \in [0, 1],$$

with $\alpha, \beta, \gamma > 0$ and $z \in \mathbb{R}^+$ parameters [3, 13, 19, 44].

4.1.1 Example - Hermite polynomials

Hermite polynomials are orthogonal with respect to the weight

$$w(x) = \exp(-x^2), \quad x \in \mathbb{R}.$$

In this case

$$\mu_{2k} = \int_{-\infty}^{\infty} x^{2k} \exp(-x^2) dx = \frac{\sqrt{\pi(2k)!}}{2^{2k} k!}, \quad \mu_{2k+1} = \int_{-\infty}^{\infty} x^{2k+1} \exp(-x^2) dx = 0, \quad (4.15)$$

so

$$\Delta_n = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} \\ \mu_1 & \mu_2 & \cdots & \mu_n \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-2} \end{vmatrix} = \left(\frac{1}{2}\right)^{n(n-1)/2} \prod_{k=1}^{n-1} (k!), \quad \widetilde{\Delta}_n = 0$$

and therefore the recurrence coefficients are the following:

$$\alpha_n = 0, \quad \beta_n = \frac{\Delta_{n+1}\Delta_{n-1}}{\Delta_n^2} = \frac{1}{2}n,$$

which gives the three-term recurrence relation

$$p_{n+1}(x) = xp_n(x) - \frac{1}{2}np_{n-1}(x),$$

where

$$p_n(x) = 2^{-n}H_n(x),$$

with $H_n(x)$ the Hermite polynomial.

4.1.2 Example - Associated Laguerre polynomials

Associated Laguerre polynomials are orthogonal with respect to the weight

$$w(x) = x^\nu \exp(-x), \quad x \in \mathbb{R}^+, \quad \nu > -1.$$

In this case

$$\mu_k = \int_0^\infty x^{k+\nu} \exp(-x) dx = \Gamma(k + \nu + 1), \quad (4.16)$$

so

$$\Delta_n = \prod_{j=1}^n (j-1)! \Gamma(\nu + j), \quad \widetilde{\Delta}_n = n(n+\nu) \prod_{j=1}^n (j-1)! \Gamma(\nu + j)$$

and therefore the recurrence coefficients are the following:

$$\alpha_n = \frac{\widetilde{\Delta}_{n+1}}{\Delta_{n+1}} - \frac{\widetilde{\Delta}_n}{\Delta_n} = 2n + \nu + 1, \quad \beta_n = \frac{\Delta_{n+1}\Delta_{n-1}}{\Delta_n^2} = n(n + \nu),$$

which gives the three-term recurrence relation

$$p_{n+1}(x) = (x - 2n - 1 - \nu)p_n(x) - n(n + \nu)p_{n-1}(x),$$

where

$$p_n(x) = (-1)^n n! L_n^{(\nu)}(x),$$

with $L_n^{(\nu)}(x)$ the associated Laguerre polynomial.

4.2 Semi-classical orthogonal polynomials

Suppose $p_n(x)$, for $n \in \mathbb{N}$, is a sequence of classical orthogonal polynomials; such as Hermite, Laguerre and Jacobi polynomials; then $p_n(x)$ is a solution of a second-order, ordinary differential equation of the form

$$\sigma(x) \frac{d^2 p_n}{dx^2} + \tau(x) \frac{dp_n}{dx} = \lambda_n p_n, \quad (4.17)$$

where $\tau(x)$ is a polynomial with degree 1, $\sigma(x)$ is a monic polynomial with degree ≤ 2 and λ_n is a real number which is related to the polynomials. A condition on the weights of classical orthogonal polynomials is that they must satisfy the Pearson equation

$$\frac{d}{dx}[\sigma(x)w(x)] = \tau(x)w(x), \quad (4.18)$$

where $\tau(x)$ and $\sigma(x)$ are the same polynomials as above. However, if we look at the semi-classical case the weight function still satisfies the Pearson equation (4.18), with one of the following true: Either the degree of $\sigma(x)$ is > 2 or the degree of $\tau(x)$ is $>$ than 1.

- Classical orthogonal polynomials: $\sigma(x)$ and $\tau(x)$ are polynomials with $\deg(\sigma) \leq 2$ and $\deg(\tau) = 1$.

	$w(x)$	$\sigma(x)$	$\tau(x)$
Hermite	$\exp(-x^2)$	1	$-2x$
Associated Laguerre	$x^\nu \exp(-x)$	x	$1 + \nu - x$
Jacobi	$(1-x)^\alpha(1+x)^\beta$	$1-x^2$	$\beta - \alpha - (2 + \alpha + \beta)x$

- Semi-classical orthogonal polynomials: $\sigma(x)$ and $\tau(x)$ are polynomials with either $\deg(\sigma) > 2$ or $\deg(\tau) > 1$.

	$w(x)$	$\sigma(x)$	$\tau(x)$
semi-classical Laguerre	$x^\nu \exp(-x^2 + zx)$	x	$1 + \nu + tx - 2x^2$
Freud	$\exp(-\frac{1}{4}x^4 - zx^2)$	1	$-2zx - x^3$

4.2.1 Example - Semi-classical Hermite Weight

Consider the semi-classical Hermite weight [18]

$$w(x; z) = |x|^\nu \exp(-x^2 + zx), \quad x \in \mathbb{R}, \quad \nu > -1.$$

The moment $\mu_k(z; \nu)$ is given by

$$\mu_k(z; \nu) = \int_{-\infty}^{\infty} x^k |x|^\nu \exp(-x^2 + zx) dx \quad (4.19)$$

$$= \frac{d^k}{dz^k} \left(\int_{-\infty}^{\infty} |x|^\nu \exp(-x^2 + zx) dx \right) = \frac{d^k \mu_0}{dz^k}. \quad (4.20)$$

The Hankel determinant $\Delta_n(z)$ is given by

$$\Delta_n(z) = \det [\mu_{j+k}(z)]_{j,k=0}^{n-1} = \mathcal{W} \left(\mu_0, \frac{d\mu_0}{dz}, \dots, \frac{d^{n-1}\mu_0}{dz^{n-1}} \right),$$

where

$$\mu_0(z; \nu) = \begin{cases} \frac{\Gamma(\nu+1) \exp(\frac{1}{8}t^2)}{2^{\nu+1/2}} \{D_{-\nu-1}(-\frac{1}{2}\sqrt{2}z) + D_{-\nu-1}(\frac{1}{2}\sqrt{2}z)\}, & \text{if } \nu \notin \mathbb{N}, \\ \sqrt{\pi} (-\frac{1}{2}i)^{2N} H_{2N}(\frac{1}{2}iz) \exp(\frac{1}{4}z^2), & \text{if } \nu = 2N, \\ \sqrt{\pi} \frac{d^{2N+1}}{dz^{2N+1}} \{\operatorname{erf}(\frac{1}{2}z) \exp(\frac{1}{4}z^2)\}, & \text{if } \nu = 2N + 1. \end{cases}$$

5 Painlevé V and continuous orthogonal polynomials

5.1 Time-dependent Jacobi application

The Jacobi polynomials are a class of classical orthogonal polynomials. They are orthogonal with respect to the weight

$$w_0(x) = (1-x)^a(1+x)^b.$$

The Jacobi polynomials can be found in the study of rotation groups; they are also found to be the solutions of equations of motion of the symmetric top. In this case, however, we are not going to explore the original Jacobi polynomials weight but $w_0(x)e^{-tx}$ instead. This deformation will allow the recurrence relations α_n and β_n to become time-dependent and therefore will depend on t . This means that the all important recurrence coefficients are now dependent on t and can be related explicitly to solutions of S_V (1.7e). So, the time-dependent Jacobi polynomials are a class of semi-classical orthogonal polynomials which are orthogonal with respect to the weight

$$w(x;t) = (1-x)^a(1+x)^b e^{-tx}, \quad (5.1)$$

on the interval $[-1, 1]$ where $a, b > -1$. This weight satisfies the Pearson equation (4.18) with the following $\sigma(x)$ and $\tau(x)$:

$$\sigma(x) = -x^3 + x, \quad \tau(x) = tx^3 - (a+b+3)x^2 - (a+t-b)x + 1.$$

Previously, this weight was explored by Basor, Chen and Ehrhardt in [2]. The methods used in this paper are known to be the ladders methods; which are longer

and more convoluted than the direct method that we are going to use here. The key idea of the method that we are about to explore is the recognition of the initial moment as a special function via the appropriate integral representation and also that the following moments are differential variants of the initial one. This in turn makes it possible to write the matrix of moments as a bi-directional Wronskian which we can then compare easily and directly with the special function solutions of S_V (1.7e). Establishing this connection means we can simply read off the recurrence coefficients and therefore calculate new sequences of orthogonal polynomials quickly with little time complexity. For the time-dependent Jacobi polynomial weight (5.1), using (4.4), the general moment μ_k is given by

$$\mu_k(t) = \int_{-1}^1 x^k (1-x)^a (1+x)^b \exp(-tx) dx. \quad (5.2)$$

First we obtain explicit expressions for the moment $\mu_0(t)$.

Theorem 5.1. *For the time-dependent Jacobi polynomial weight (5.1) the initial moment $\mu_0(t)$ is given by*

$$\mu_0(t) = 2^{a+b+1} \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)} \exp(-t) M(a+1, a+b+2, 2t).$$

Proof. Using (2.9a), (2.10a) and the substitution $x = 2u - 1$ we can calculate $\mu_0(t)$ in terms of Kummer functions

$$\begin{aligned} \mu_0 &= \int_{-1}^1 (1-x)^a (1+x)^b \exp(-xt) dx \\ &= 2^{a+b+1} \exp(t) \int_0^1 (1-u)^a u^b \exp(-2ut) du \\ &= 2^{a+b+1} \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)} \exp(-t) M(a+1, a+b+2, 2t). \end{aligned}$$

□

Theorem 5.2. *For the time-dependent Jacobi polynomial weight (5.1) the general moment $\mu_k(t)$ can be given by*

$$\mu_k(t) = 2^{a+b+1} \Gamma(a+1) e^t \sum_{r=0}^k \binom{k}{r} (-1)^{k-r} 2^r \widehat{M}(b). \quad (5.3)$$

where $\widehat{M}(b)$ is defined as the following:

$$\widehat{M}(b) := \frac{M(b+r+1, a+b+r+2, -2t) \Gamma(b+r+1)}{\Gamma(a+b+r+2)}.$$

Proof. This result can be produced by making a suitable transformation in (5.2), then using the binomial expansion formula to then use (2.9a). This gives the desired result. \square

Theorem 5.3. *For the time-dependent Jacobi polynomial weight (5.1) the general moment $\mu_k(t)$ can also be given by*

$$\mu_k(t) = (-1)^k \frac{d^k}{dt^k} \mu_0, \quad k = 0, 1, 2, 3, \dots$$

Proof. This result can be shown by differentiating (5.3), using (2.11a), then showing this is exactly equal to $-\mu_{k+1}$.

$$\begin{aligned} \frac{d\mu_k}{dt} = 2^{a+b+1} \Gamma(a+1) e^t \sum_{r=0}^k \left\{ \binom{k}{r} (-1)^{k-r} 2^r \widehat{M}(b) \right. \\ \left. - \binom{k}{r} (-1)^{k-r} 2^{r+1} \widehat{M}(b+1) \right\}. \end{aligned}$$

Expanding both parts inside this sum and comparing term by term directly with $-\mu_{k+1}$ we can see they are, in fact, equal. \square

This is the point when we branch away from the work done previously by Basor, Chen and Ehrhardt in [2] and some original research is conducted.

We have μ_k in the form (4.8). Using theorem (4.1) we can make the following simplifications inside the Hankel determinant and begin to write Δ_n in the form of a bi-directional Wronskian.

$$\Delta_n = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} \\ \mu_1 & \mu_2 & \cdots & \mu_n \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-2} \end{vmatrix} = \begin{vmatrix} \mu_0 & \mu'_0 & \cdots & \mu_0^{(n-1)} \\ \mu'_0 & \mu''_0 & \cdots & \mu_0^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_0^{(n-1)} & \mu_0^{(n)} & \cdots & \mu_0^{(2n-2)} \end{vmatrix}, \quad ' = \frac{d}{dt}.$$

Therefore we can write $\Delta_n = \tau_n(\mu_0)$, where

$$\mu_0(t) = 2^{a+b+1} \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)} \exp(-t) M(a+1, a+b+2, 2t).$$

We now have Δ_n in the form that is similar to our special function solutions of S_V (1.7e). This means we can write down exact expressions for the recurrence coefficients $\alpha_n(z)$ and $\beta_n(z)$.

Theorem 5.4. *The function*

$$H_n(t; a, b) = t \frac{d}{dt} \ln \tau_n(\mu_0), \quad (5.4)$$

with τ_n given by (4.10) and Δ_n given by (4.2), satisfies the second-order, second-degree equation

$$\begin{aligned} \left(t \frac{d^2 H_n}{dt^2} \right)^2 &= 4 \left\{ \frac{1}{2} (a + 2n + b + 2t) \frac{dH_n}{dt} - n(a + n) - H_n \right\}^2 \\ &\quad - 8 \frac{dH_n}{dt} \left(t \frac{dH_n}{dt} - H_n \right) \left(b + \frac{1}{2} \frac{dH_n}{dt} \right). \end{aligned} \quad (5.5)$$

Proof. Equation (5.5) is equivalent to S_V (1.7e) through the linear transformation

$$H_n(t; a, b) = \sigma(z) - \frac{1}{2}n^2 + \frac{1}{4}(a - b + 2n)t - \frac{1}{2}(a + b) + \frac{1}{8}(a - b)^2, \quad (5.6)$$

where $t \rightarrow z/2$, for the parameters

$$\{\kappa_0, \kappa_1, \kappa_2, \kappa_3\} = \left\{ \frac{1}{4}(a-2n-b), \frac{1}{4}(3b+a+2n), \frac{1}{4}(a+2n-b), -\frac{1}{4}(b+3a+2n) \right\}. \quad (5.7)$$

This is easily verified by comparing (5.6) (with H_n given by (5.4)) with (2.50b). \square

Remark 5.1.

- If we consider the solution to S_V (1.7e) using the corollary (2.3)

$$\begin{aligned} \sigma(z; a, b) = & z \frac{d}{dz} \ln \tau_n(M(a+1, a+b+2, z)) - nz \\ & + \frac{1}{2}n^2 - \frac{1}{4}(a-b+2n)t + \frac{1}{2}(a+b) - \frac{1}{8}(a-b)^2, \end{aligned}$$

for the parameters (5.7). These parameters can be mapped to our original set of parameters (2.51b) by the mapping $a \rightarrow \alpha - 1$ and $b \rightarrow \beta - \alpha - n$. Due to the symmetric form of (1.7e) the choice of $\kappa_1, \kappa_2, \kappa_3$ and κ_4 is not unique.

- In terms of $H_n(t; a, b)$ given by (5.4), the coefficients $\alpha_n(t)$ and $\beta_n(t)$ in the recurrence relation (4.9) have the form

$$\alpha_n(t) = \frac{1}{t} \left\{ H_{n+1} - H_n \right\}, \quad \beta_n(t) = \frac{1}{t^2} \left\{ t \frac{dH_n}{dt} - H_n \right\}.$$

5.2 Pollaczek-Jacobi type polynomials

The Pollaczek-Jacobi type polynomials are similar to the time-dependent Jacobi polynomials in that they are deformations of the classical Jacobi polynomials. Pollaczek-Jacobi type polynomials are a class of semi-classical orthogonal polynomials. They are orthogonal with respect to the weight

$$w(x; z) = \exp(-z/x) x^a (1-x)^b, \quad (5.8)$$

on the interval $[0, 1]$ with $a, b > 0$. This weight satisfies the Pearson equation (4.18) with the following $\sigma(x)$ and $\tau(x)$:

$$\begin{aligned}\sigma(x) &= x^4 + x^3 - 2x^2, \\ \tau(x) &= (a + b + 4)x^3 + (z + a + 2b + 3)x^2 + (z - 2a - 4)x - 2z.\end{aligned}$$

Previously this weight was explored by Chen and Dai in [11] and they conclude that the logarithmic derivative of Δ_n satisfies a second-order, non-linear ODE. As before, the methods used in this paper are known to be the ladders methods which, as we said, are longer and more convoluted than the direct method that we are going to use here. The key idea of the method that we are about to explore is the recognition of the initial moment as a special function via the appropriate integral representation and that the following moments are differential variants of the initial one. Just as we did with the previous weight, this makes it possible to write the matrix of moments as a bi-directional Wronskian which we can then compare easily and directly with the special function solutions of S_V (1.7e). Again, establishing this connection means we can simply read off the recurrence coefficients and therefore calculate new sequences of orthogonal polynomials quickly with little time complexity.

For the Pollaczek-Jacobi type polynomial weight (5.8), using (4.4) the general moment $\mu_k(z)$ is given by

$$\mu_k(z) = \int_0^1 x^k \exp(-z/x) x^a (1-x)^b dx.$$

First we obtain explicit expressions for the moment $\mu_{2n-2}(z)$.

Theorem 5.5. *For the Pollaczek-Jacobi type polynomial weight (5.8), the last moment $\mu_{2n-2}(z)$ is given by*

$$\mu_{2n-2}(z) = \Gamma(b+1)e^{-z}U(b+1, 2-a-2n, z).$$

Proof. Using (2.9b) and the substitution $x = \frac{1}{u+1}$ we can calculate $\mu_{2n-2}(z)$ in terms of Kummer functions.

$$\begin{aligned}\mu_{2n-2} &= \int_0^1 \exp(-z/x) x^{a+2n-2} (1-x)^b dx \\ &= \int_0^\infty e^{-z} e^{-uz} (u+1)^{-a-2n} \left(\frac{u}{u+1}\right)^b du \\ &= e^{-z} \Gamma(b+1) U(b+1, 2-a-2n, z).\end{aligned}$$

□

Theorem 5.6. *For the Pollaczek-Jacobi type polynomial weight (5.8), the general moment $\mu_k(z)$ can be given by*

$$\mu_k(z) = \Gamma(b+1) e^{-z} U(b+1, -a-k, z). \quad (5.9)$$

Proof. This result can be obtained by setting $k = 2n - 2$ in the calculation above.

□

Theorem 5.7. *For the Pollaczek-Jacobi type polynomial weight (5.8), the general moment $\mu_k(z)$ can also be given by*

$$\mu_{2n-2-k}(z) = \frac{d^k}{dz^k} \mu_{2n-2}(z), \quad k = 0, 1, 2, 3, \dots$$

Proof. This result can be shown directly by (2.11b) and (5.9).

□

This is the point when we branch away from the work done previously by Chen and Dai in [11] and some original research is conducted.

We have μ_k in the form (4.8) with only one slight difference in that the differentiation steps down k rather than up. Using theorem (4.1) we can make the

following simplifications inside the Hankel determinant and begin to write Δ_n in the form of a bi-directional Wronskian.

$$\Delta_n = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} \\ \mu_1 & \mu_2 & \cdots & \mu_n \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-2} \end{vmatrix} = \begin{vmatrix} \mu_{2n-2} & \mu'_{2n-2} & \cdots & \mu_{2n-2}^{(n-1)} \\ \mu'_{2n-2} & \mu''_{2n-2} & \cdots & \mu_{2n-2}^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{2n-2}^{(n-1)} & \mu_{2n-2}^{(n)} & \cdots & \mu_{2n-2}^{(2n-2)} \end{vmatrix}, \quad ' = \frac{d}{dz}.$$

Therefore, we can write $\Delta_n = \tau_n(\mu_{2n-2})$ where

$$\mu_{2n-2}(z) = \Gamma(b+1) \exp(-z) U(b+1, -a-2n+2, z).$$

We now have Δ_n in the form that is similar to our special function solutions of S_V (1.7e). This means we can write down exact expressions for the recurrence coefficients $\alpha_n(z)$ and $\beta_n(z)$.

Theorem 5.8. *The function*

$$H_n(z; a, b) = z \frac{d}{dz} \ln \tau_n(\mu_{2n-2}), \quad (5.10)$$

with τ_n given by (4.10), satisfies the second-order, second-degree equation

$$\left(z \frac{d^2 H_n}{dz^2} \right)^2 = \left[n(n+a+b) - H_n + (a+z) \frac{dH_n}{dz} \right]^2 + 4 \frac{dH_n}{dz} \left(z \frac{dH_n}{dz} - H_n \right) \left(b - \frac{dH_n}{dz} \right). \quad (5.11)$$

Proof. Equation (5.11) is equivalent to S_V (1.7e) through the linear transformation

$$H_n(z; a, b) = \sigma + n^2 + (b+a)n + \frac{1}{8}(2b+a)(a+2z+2b), \quad (5.12)$$

for the parameters

$$\{\kappa_0, \kappa_1, \kappa_2, \kappa_3\} = \left\{ -\frac{1}{4}(4n+3a+2b), \frac{1}{4}(4n+a+2b), \frac{1}{4}(a+2b), \frac{1}{4}(a-2b) \right\}. \quad (5.13)$$

This is easily verified by comparing (5.12) (with H_n given by (5.10)) with (2.50b).

□

Remark 5.2.

- If we consider the solution to S_V (1.7e) using the corollary (2.3)

$$\begin{aligned} \sigma_n(z; a, b) = & -n^2 - (b + a + z)n - \frac{1}{8}(2b + a)(a + 2z + 2b) \\ & + z \frac{d}{dz} \tau_n(U(b + 1, -a + 2 - 2n, z)), \end{aligned}$$

for the parameters (5.13). These parameters can be mapped to our original set of parameters (2.51b) by the mapping $a \rightarrow 1 - \beta - n$ and $b \rightarrow \alpha - 1$. Due to the symmetric form of (1.7e) the choice of $\kappa_1, \kappa_2, \kappa_3$ and κ_4 is not unique.

- In terms of $H_n(z; a, b)$ given by (5.10), the coefficients $\alpha_n(z)$ and $\beta_n(z)$ in the recurrence relation (4.9) have the form

$$\alpha_n(z) = \frac{1}{z} \left\{ H_{n+1} - H_n \right\}, \quad \beta_n(z) = \frac{1}{z^2} \left\{ z \frac{dH_n}{dz} - H_n \right\}.$$

5.3 Deformed Laguerre polynomials

The Deformed Laguerre polynomials are a class of semi-classical, orthogonal polynomials which are orthogonal with respect to the weight

$$w(x; z) = x^a (x + z)^b e^{-x}, \tag{5.14}$$

on the interval $[0, \infty)$ for $a > -1$. This weight satisfies the Pearson equation (4.18) with the following $\sigma(x)$ and $\tau(x)$:

$$\begin{aligned}\sigma(x) &= \frac{1}{a+1} \left\{ \frac{(az+z-1)x^3}{z^2} + x^2 + x \right\}, \\ \tau(x) &= 1 + \left\{ \frac{1}{z^2(a+1)} - \frac{1}{z} \right\} x^3 + \left\{ \frac{a+b+3}{z} - \frac{a+b+3}{z^2(a+1)} - 1 \right\} x^2 \\ &\quad + \left\{ \frac{a^2+3a+1}{a+1} + \frac{b}{z(a+1)} \right\} x.\end{aligned}$$

This weight was previously explored by Chen, Basor and McKay in [20] and by Chen and McKay in [13]. It is interesting to note that if $b = 0$ we will get back to the classical Laguerre weight. Again, the methods used are convoluted compared with the direct method that will follow shortly.

For the polynomial weight (5.14), the general moment μ_k is given by

$$\mu_k(z) = \int_0^\infty x^{a+k} (x+z)^b e^{-x} dx. \quad (5.15)$$

First we obtain explicit expressions for the moment $\mu_0(z)$.

Theorem 5.9. *For the deformed Laguerre polynomial weight (5.14) the initial moment $\mu_0(z)$ is given by*

$$\mu_0(z) = z^{a+b+1} \Gamma(a+1) U(a+1, a+b+2, z). \quad (5.16)$$

Proof. Using (2.9a) and the substitution $x = uz$ we can calculate $\mu_0(z)$ in terms of Kummer functions

$$\begin{aligned}\mu_0 &= \int_0^\infty x^a (x+z)^b e^{-x} dx \\ &= z^{a+b+1} \int_0^\infty u^a (u+1)^b e^{-zu} du \\ &= z^{a+b+1} \Gamma(a+1) U(a+1, a+b+2, z).\end{aligned}$$

□

Theorem 5.10. *For the deformed Laguerre polynomial weight (5.14), the general moment is given by*

$$\mu_k(z) = z^{a+b+k+1} \Gamma(a+k+1) U(a+k+1, a+b+k+2, z). \quad (5.17)$$

Proof. The result can be inferred by repeating the above calculation with $b = b+k$. □

Theorem 5.11. *For the deformed Laguerre polynomial weight (5.14), the general moment $\mu_k(z)$ can also be given by*

$$\mu_k(z) = \frac{d^k}{dz^k} \left\{ \mu_0 / z^{a+b+1} \right\} z^{a+b+k+1}, \quad k = 0, 1, 2, 3, \dots$$

Proof. This result can be shown directly from (2.11d) and (5.17). □

This is the point when we branch away from the work done previously by Chen, Basor and McKay in [20] and by Chen and McKay in [13] and some original research is conducted.

Define

$$\Psi := \frac{\mu_0}{z^{(a+b+1)}} = \Gamma(a+1) U(a+1, a+b+2, z).$$

As the moment here is not in the form (4.8) we must write Δ_n in the form of the following Wronskian by factoring out appropriate powers of z . Our goal here is

to write Δ_n in the form of a bi-directional Wronskian.

$$\begin{aligned}
\Delta_n &= \begin{vmatrix} \mu_0 & \mu_1 & \dots & \mu_{n-1} \\ \mu_1 & \mu_2 & \dots & \mu_n \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \dots & \mu_{2n-2} \end{vmatrix} \\
&= z^{n(a+b+n)} \begin{vmatrix} \mu_0/z^{(a+b+1)} & \mu_1/z^{(a+b+2)} & \dots & \mu_{n-1}/z^{(a+b+n)} \\ \mu_1/z^{(a+b+2)} & \mu_2/z^{(a+b+3)} & \dots & \mu_n/z^{(a+b+n+1)} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1}/z^{(a+b+n)} & \mu_n/z^{(a+b+n+1)} & \dots & \mu_{2n-2}/z^{(a+b+2n-1)} \end{vmatrix} \\
&= z^{n(a+b+n)} \begin{vmatrix} \Psi & \Psi' & \dots & \Psi^{(n-1)} \\ \Psi' & \Psi'' & \dots & \Psi^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ \Psi^{(n-1)} & \Psi^{(n)} & \dots & \Psi^{(2n-2)} \end{vmatrix}, \quad ' = \frac{d}{dz}.
\end{aligned}$$

Therefore, we can write

$$\Delta_n = \tau_n(\Psi) z^{n(a+b+n)}. \quad (5.18)$$

We now have Δ_n in the form that is similar to our special function solutions of S_V (1.7e). This means we can write down exact expressions for the recurrence coefficients $\alpha_n(z)$ and $\beta_n(z)$.

Theorem 5.12. *The function*

$$H_n(z) = z \frac{d}{dz} \ln \tau_n(\Psi) z^{n(a+b+n)}, \quad (5.19)$$

with τ_n given by (4.10), satisfies the second-order, second-degree equation

$$\left(z \frac{d^2 H_n}{dz^2}\right)^2 = \left\{ H_n + n(a+n) - (a+b+2n+z) \frac{dH_n}{dz} \right\}^2 - 4 \frac{dH_n}{dz} \left(z \frac{dH_n}{dz} - H_n \right) \left(b + \frac{dH_n}{dz} \right). \quad (5.20)$$

Proof. Equation (6.7) is equivalent to S_V (1.7e) through the linear transformation

$$H_n(z; a, b) = \sigma - \frac{1}{2} n^2 + \frac{1}{2} (z - a - b) n + \frac{1}{8} (a - b) (a - b + 2z), \quad (5.21)$$

for the parameters

$$\{\kappa_0, \kappa_1, \kappa_2, \kappa_3\} = \left\{ \frac{1}{4}(a - b - 2n), \frac{1}{4}(3b + a + 2n), \frac{1}{4}(a - b + 2n), -\frac{1}{4}(3a + b + 2n) \right\}. \quad (5.22)$$

This is easily verified by comparing (6.8) (with H_n given by (5.19)) with (2.50b). □

Remark 5.3.

- If we consider the solution to S_V (1.7e)

$$\begin{aligned} \sigma(z; a, b) = & z \frac{d}{dz} \ln \tau_n(U(a+1, a+b+2, z)) + \frac{1}{2} n^2 - \frac{1}{2} (z - a - b) n \\ & - \frac{1}{8} (a - b) (a - b + 2z), \end{aligned}$$

for the parameters (5.22). These parameters can be mapped to one of our original set of parameters (2.51b) by the mapping $a \rightarrow \alpha - 1$, and $b \rightarrow \beta - \alpha - n$. Due to the symmetric form of (1.7e) the choice of $\kappa_1, \kappa_2, \kappa_3$ and κ_4 is not unique.

- As $\tilde{\Delta}_n \neq \frac{d}{dz} \Delta_n$ we need to calculate the recurrence coefficients directly using (4.1). This can be done by substituting (5.18) into (4.1), as follows:

Note that:

$$\widetilde{\Delta}_n = -\tau_n(\Psi)z^{n(a+b+n)+1}. \quad (5.23)$$

So, to find an expression $\widetilde{\Delta}_n$ in terms of Δ_n and its derivatives it just remains to differentiate (5.18)

$$\frac{d}{dz}\Delta_n = \frac{d}{dz}\tau_n(\Psi)z^{n(a+b+n)} + n(a+b+n)z^{n(a+b+n)-1}\tau_n(\Psi) \quad (5.24)$$

$$= \widetilde{\Delta}_n + n(a+b+n)\Delta_n. \quad (5.25)$$

Rearranging this gives

$$\widetilde{\Delta}_n = -z\frac{d}{dz}\Delta_n + n(a+b+n)\Delta_n,$$

substituting this and (5.19) into (4.1) yields $\alpha_n(z)$ and $\beta_n(z)$ in terms of $H_n(z; a, b)$:

$$\alpha_n(z) = H_n - H_{n+1} + a + b + 2n + 1, \quad \beta_n(z) = n(a+b+n) + z\frac{dH_n}{dz} - H_n. \quad (5.26)$$

- If we consider the original weight (5.14) again but with $b \in \mathbb{Z}$ there is an interesting simplification to note

$$\Delta_n = [(-1)^b b! \Gamma(a+1) z^{a+b+n}]^n \tau_n \left(z^{-(a+b+1)} L_b^{(-a-b-1)} \right), \quad (5.27)$$

where $L_a^{(b)}$ is an associated Laguerre polynomial. This result can be applying (2.12) to (5.18). This simplifies the coefficients $\alpha_n(z)$ and $\beta_n(z)$ to Laguerre polynomials

$$\alpha_n(z) = z \frac{d}{dz} \ln \left[\frac{\tau_n \left(z^{-(a+b+1)} L_b^{(-a-b-1)} \right)}{\tau_{n+1} \left(z^{-(a+b+1)} L_b^{(-a-b-1)} \right)} \right],$$

$$\beta_n(z) = z^2 \frac{d^2}{dz^2} \ln \left[\tau_n \left(z^{-(a+b+1)} L_b^{(-a-b-1)} \right) \right].$$

Proof. This result can be shown by applying (5.27) to (5.19) to obtain

$$H_n = n(a + b + n) + z \frac{d}{dz} \ln \tau_n(\Psi), \quad (5.28)$$

and then applying this to (5.26). \square

With this result we can now generate entirely new sequences of orthogonal polynomials using $\alpha_n(z)$ and $\beta_n(z)$. Notice that they are polynomials in x with rational coefficients in z .

Table 5.1: Table of new orthogonal polynomials $p_n(\alpha, \beta; x)$

$$\begin{aligned}
 p_2(1, 0; x) &= x^2 - 4x - 2z^{-2} \\
 p_2(1, 1; x) &= x^2 - \frac{4(z^3 + 9z^2 + 21z + 15)x}{(z^2 + 6z + 6)(z + 2)} - 2 \frac{z^2 + 6z + 6}{(z + 2)z^3} \\
 p_3(1, 0; x) &= x^3 - 10x^2 + 18x + \frac{-2x + 12}{z^2} \\
 p_3(3, 0; x) &= x^3 - 2 \frac{(5z^4 + 75z^3 + 336z^2 + 552z + 288)x^2}{(z^3 + 12z^2 + 36z + 24)(z + 2)} \\
 &\quad + \frac{2(9z^7 + 144z^6 + 683z^5 + 1134z^4 + 534z^3 - 312z^2 - 360z - 144)x}{(z^3 + 12z^2 + 36z + 24)(z + 2)z^3} \\
 &\quad + 12 \frac{z^5 + 19z^4 + 124z^3 + 348z^2 + 408z + 168}{(z^3 + 12z^2 + 36z + 24)(z + 2)z^3}
 \end{aligned}$$

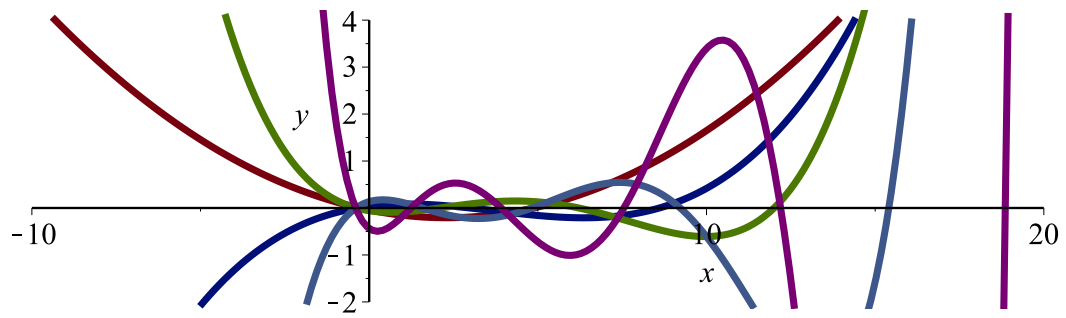
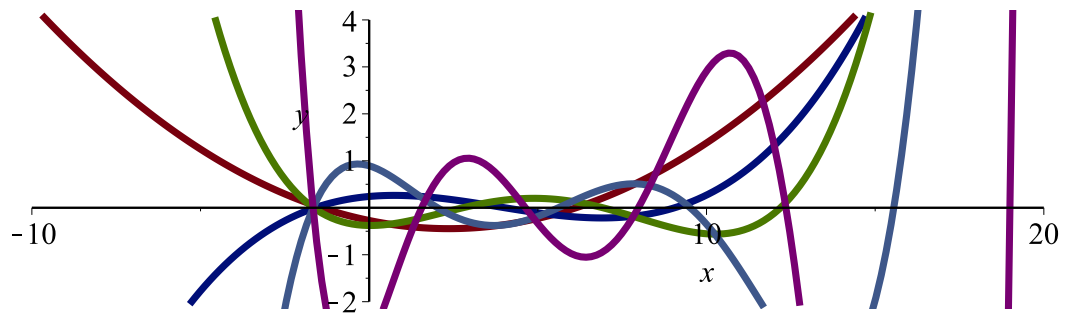
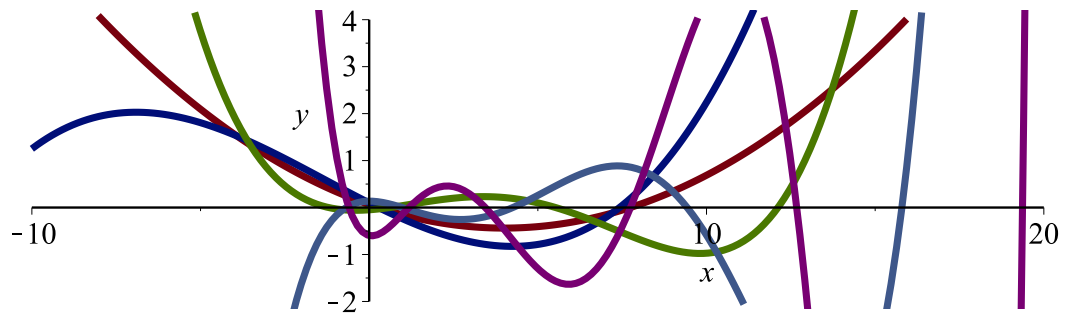
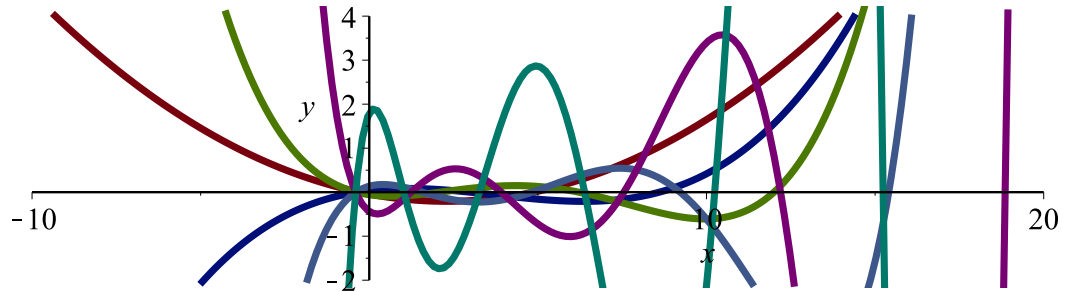
(a) $z = 2$ (b) $z = 1$ (c) $z = -1$

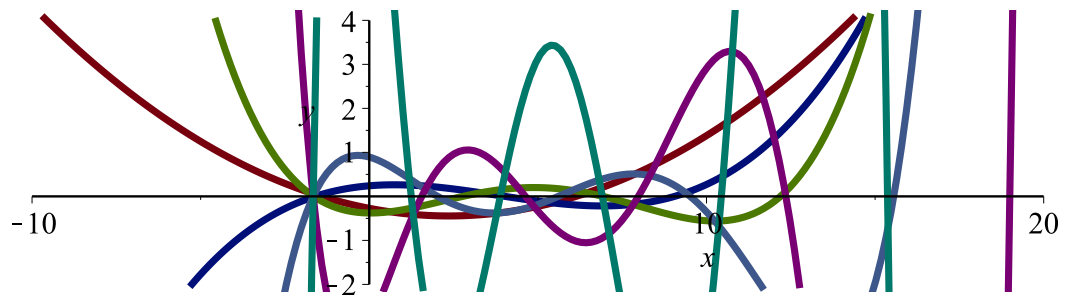
Figure 5.1: Plots of new orthogonal polynomials $p_2(1, 1; z)$, $p_3(1, 1; z)$, $p_4(1, 1; z)$, $p_5(1, 1; z)$, $p_6(1, 1; z)$.

It is interesting to note that the polynomials interlace as we would expect from

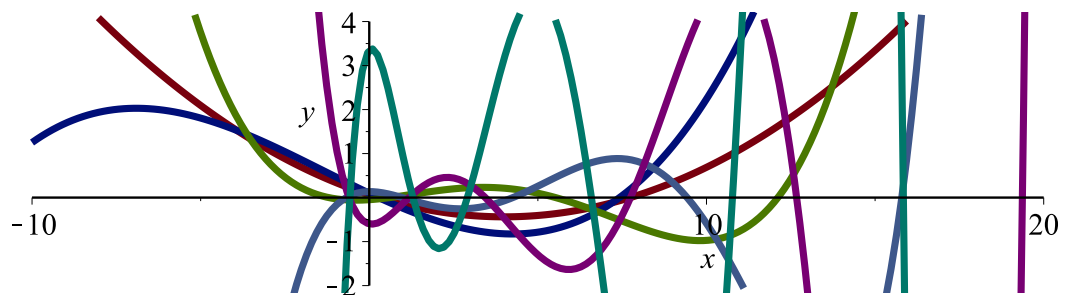
orthogonal polynomials.



(a) $z = 2$



(b) $z = 1$



(c) $z = -1$

Figure 5.2: Plots of new orthogonal polynomials $p_2(1, 1; z)$, $p_3(1, 1; z)$, $p_4(1, 1; z)$, $p_5(1, 1; z)$, $p_6(1, 1; z)$, $p_7(1, 1; z)$.

6 Painlevé V and discontinuous orthogonal polynomials

In this section will be exploring the connection between discontinuous orthogonal polynomial weight and S_V (1.7e). This particular weight has been separated into a different chapter because of the Heaviside function which is 1 for $y > 0$.

6.1 Deformed Laguerre polynomials

The Laguerre polynomials are a class of classical orthogonal polynomials which are orthogonal with respect to the weight

$$w_0(x) = e^{-x}x^b.$$

However, here we will be looking at the deformed Laguerre polynomials. This weight was previously explored by Forrester and Ormerod in [23]. Again, the methods used are convoluted compared with the direct method that will follow shortly. These are a class of semi-classical orthogonal polynomials which are orthogonal with respect to the weight

$$w(x; z) = [1 - \zeta\vartheta(x - z)](x - z)^a x^b e^{-x}, \quad (6.1)$$

on the interval $[0, \infty)$, with $a, b > 0$ and where $\vartheta(y)$ is the Heaviside function $\vartheta(y) = 1$ for $y > 0$, otherwise $\vartheta(y) = 0$. In fact, if we set $\zeta = 0$ and $a = 0$ we get straight back to the Laguerre polynomial weight. This weight satisfies the

Pearson equation (4.18) with the following $\sigma(x)$ and $\tau(x)$:

$$\begin{aligned}\sigma(x) &= \frac{(x-z)x\{(b+x+3)z+(x+1)(a+b+3)\}}{z^2+(2a+b+3)z+(a+b+3)(a+b+2)}, \\ \tau(x) &= \frac{1}{z^2+(2a+b+3)z+(a+b+3)(a+b+2)} \left\{ -(a+b+z+3)x^3 \right. \\ &\quad + (z^2+(2a+b+3)z+(a+b+3)(a+b+2))x^2 \\ &\quad + (z^2+(2a+b+3)z+(a+b+3)(a+b+2))x \\ &\quad \left. - (b+3)(b+1)z^2 - (b+1)(a+b+3)z \right\}.\end{aligned}$$

For the deformed Laguerre polynomial weight (6.1), using

$$\mu_k(z) = \int_a^b x^k w(x; z) dx.$$

the general moment μ_k is given by

$$\mu_k(z) = \int_0^\infty x^k [1 - \zeta \vartheta(x-z)] (x-z)^a x^b e^{-x} dx. \quad (6.2)$$

First we obtain explicit expressions for the moment $\mu_0(z)$.

Theorem 6.1. *For the deformed Laguerre polynomial weight (6.1), the initial moment, μ_0 , is given by*

$$\begin{aligned}\mu_0 &= \Gamma(a+1)z^{a+b+1}e^{-z} \left\{ \frac{\Gamma(b+1)}{\Gamma(a+b+2)} M(a+1, a+b+2, z) \right. \\ &\quad \left. + (1-\zeta)U(a+1, a+b+2, z) \right\}.\end{aligned} \quad (6.3)$$

Proof. Consider

$$\mu_0(z) = \int_0^\infty [1 - \zeta \vartheta(x-z)] (x-z)^a x^b e^{-x} dx.$$

This can be separated out into two integrals that will be much easier to deal with

$$\begin{aligned}\mu_0 &= \int_0^\infty [1 - \zeta \vartheta(x-z)] (x-z)^a x^b e^{-x} dx \\ &= \int_0^z (x-z)^a x^b e^{-x} dx + (1-\zeta) \int_z^\infty (x-z)^a x^b e^{-x} dx.\end{aligned}$$

Now we make the substitutions $x = z(1 - u)$ and $x = z(u + 1)$.

$$\begin{aligned}\mu_0 &= \int_0^z (x - z)^a x^b e^{-x} dx + (1 - \zeta) \int_z^\infty (x - z)^a x^b e^{-x} dx \\ &= e^{-z} z^{a+b+1} \int_0^1 (1 - u)^a u^b e^{-uz} du + (1 - \zeta) e^{-z} z^{a+b+1} \int_0^\infty u^a (1 + u)^b e^{-uz} du \\ &= \Gamma(a + 1) z^{a+b+1} e^{-z} \left\{ \frac{\Gamma(b+1)}{\Gamma(a+b+2)} M(a + 1, a + b + 2, z) \right. \\ &\quad \left. + (1 - \zeta) U(a + 1, a + b + 2, z) \right\}.\end{aligned}$$

□

Theorem 6.2. *For the deformed Laguerre polynomial weight (6.1), the general moment is given by*

$$\begin{aligned}\mu_k(z) &= \Gamma(a + 1) z^{a+b+k+1} e^{-z} \left\{ \frac{\Gamma(b+k+1)}{\Gamma(a+b+k+2)} M(a + 1, a + b + k + 2, z) \right. \\ &\quad \left. + (1 - \zeta) U(a + 1, a + b + k + 2, z) \right\}.\end{aligned}\tag{6.4}$$

Proof. The result can be inferred by repeating the above calculation with $b = b + k$. □

Theorem 6.3. *For the deformed Laguerre polynomial weight (6.1), the general moment $\mu_k(z)$ can also be given by*

$$\mu_k(z) = \frac{d^k}{dz^k} \left\{ \mu_0 / z^{a+b+1} \right\} z^{a+b+k+1}, \quad k = 0, 1, 2, 3, \dots$$

Proof. This result can be shown directly from from (2.11c), (2.11d) and (6.4). □

This is the point when we branch away from the work done previously by Forrester and Ormerod in [23] and some original research is conducted.

Define

$$\Psi := \frac{\mu_0}{z^{(a+b+1)}} = \Gamma(a+1)e^{-z} \left\{ \frac{\Gamma(b+1)}{\Gamma(a+b+2)} M(a+1, a+b+2, z) + (1-\zeta)U(a+1, a+b+2, z) \right\}.$$

As the moment here is not in the form $\mu_k = \frac{d^k}{dz^k} \mu_0$ we have to write Δ_n in the form of the following Wronskian by factoring out appropriate powers of z . Our goal here is to write Δ_n in the form of a bi-directional Wronskian.

$$\begin{aligned} \Delta_n &= \begin{vmatrix} \mu_0 & \mu_1 & \dots & \mu_{n-1} \\ \mu_1 & \mu_2 & \dots & \mu_n \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \dots & \mu_{2n-2} \end{vmatrix} \\ &= z^{n(a+b+n)} \begin{vmatrix} \mu_0/z^{(a+b+1)} & \mu_1/z^{(a+b+2)} & \dots & \mu_{n-1}/z^{(a+b+n)} \\ \mu_1/z^{(a+b+2)} & \mu_2/z^{(a+b+3)} & \dots & \mu_n/z^{(a+b+n+1)} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1}/z^{(a+b+n)} & \mu_n/z^{(a+b+n+1)} & \dots & \mu_{2n-2}/z^{(a+b+2n-1)} \end{vmatrix} \\ &= z^{n(a+b+n)} \begin{vmatrix} \Psi & \Psi' & \dots & \Psi^{(n-1)} \\ \Psi' & \Psi'' & \dots & \Psi^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ \Psi^{(n-1)} & \Psi^{(n)} & \dots & \Psi^{(2n-2)} \end{vmatrix}, \quad ' = \frac{d}{dz}. \end{aligned}$$

Therefore, we can write

$$\Delta_n = \tau_n(\Psi) z^{n(a+b+n)}. \quad (6.5)$$

We now have Δ_n in the form that is similar to our special function solutions of

S_V (1.7e). This means we can write down exact expressions for the recurrence coefficients $\alpha_n(z)$ and $\beta_n(z)$.

Theorem 6.4. *The function*

$$H_n(z; a, b) = z \frac{d}{dz} \ln \tau_n(\Psi) z^{n(a+b+n)}, \quad (6.6)$$

satisfies the second-order, second-degree equation

$$\begin{aligned} \left(z \frac{d^2 H_n}{dz^2} \right)^2 &= \left[(z - a - b - 2n) \frac{dH_n}{dz} - n(n+b) - H_n \right]^2 \\ &+ 4 \frac{dH_n}{dz} \left(z \frac{dH_n}{dz} - H_n \right) \left(a - \frac{dH_n}{dz} \right). \end{aligned} \quad (6.7)$$

Proof. Equation (6.7) is equivalent to S_V (1.7e) through the linear transformation

$$H_n(z; a, b) = \sigma - \frac{1}{2} n^2 - \frac{1}{2} (a + b + z)n + \frac{1}{8} (a - b)(a - b + 2z), \quad (6.8)$$

for the parameters

$$\{\kappa_0, \kappa_1, \kappa_2, \kappa_3\} = \left\{ \frac{1}{4}(a-b-2n), \frac{1}{4}(3b+a+2n), \frac{1}{4}(a-b+2n), -\frac{1}{4}(3a+b+2n) \right\}. \quad (6.9)$$

This is easily verified by comparing (6.8) (with H_n given by (6.6)) with (2.50b). \square

Remark 6.1.

- *If we consider the solution to S_V (1.7e) using the corollary (2.3)*

$$\sigma(z; a, b) = \frac{1}{2} n^2 + \frac{1}{2} (a + b - z)n - \frac{1}{8} (a - b)(a - b + 2z) + z \frac{d}{dt} \ln \tau_n(e^z \Psi),$$

for the parameters (6.9). These parameters can be mapped to one of our original set of parameters (2.51b) by the mapping $a \rightarrow \alpha - 1$, and $b \rightarrow \beta - \alpha - n$. Due to the symmetric form of (1.7e) the choice of $\kappa_1, \kappa_2, \kappa_3$ and κ_4 is not unique.

- As $\widetilde{\Delta}_n \neq \frac{d}{dz}\Delta_n$ we need to calculate what the recurrence coefficients are directly using (4.1). This is done by substituting (6.5) into (4.1) as follows.

Note that:

$$\widetilde{\Delta}_n = -\tau_n(\Psi)z^{n(a+b+n)+1}. \quad (6.10)$$

Then, to find an expression $\widetilde{\Delta}_n$ in terms of Δ_n and its derivatives it just remains to differentiate (6.5)

$$\frac{d}{dz}\Delta_n = \frac{d}{dz}\tau_n(\Psi)z^{n(a+b+n)} + n(a+b+n)z^{n(a+b+n)-1}\tau_n(\Psi) \quad (6.11)$$

$$= \widetilde{\Delta}_n + n(a+b+n)\Delta_n. \quad (6.12)$$

Rearranging this gives

$$\widetilde{\Delta}_n = -z\frac{d}{dz}\Delta_n + n(a+b+n)\Delta_n,$$

substituting this and (6.6) into (4.1) yields $\alpha_n(z)$ and $\beta_n(z)$ in terms of $H_n(z; a, b)$:

$$\alpha_n(z) = H_n - H_{n+1} - z + a + b + 2n + 1, \quad \beta_n(z) = n(a+b+n) + z\frac{dH_n}{dz} - H_n.$$

7 Orthogonal polynomials on the unit circle

Now we are going to look at a slight variation in orthogonal polynomials. Rather than working with polynomials over the real line we are going to be working with polynomials over the unit circle. That is, we are now integrating around the unit circle rather than over the real line [60, §18.33].

A sequence of polynomials $\phi_n(z)$, $n = 0, 1, \dots$, where $\phi_n(z)$ is of degree n , is orthonormal on the unit circle with respect to the weight function $w(z)$ (> 0) if [60, §18.33]

$$\frac{1}{2\pi i x} \int_{|z|=1} \phi_n(z) \overline{\phi_m(z)} w(z) z^{-1} dz = \delta_{m,n}.$$

For a simplified evaluation of certain weights, we can make an appropriate transformation back to the real line, thus making the implementation of the integral representation far easier.

Consider the following weight

$$w(x; z) = (1 + x)^b (1 + 1/x)^a e^{zx}. \tag{7.1}$$

We will now consider orthogonal polynomials with respect to a complex weight function. This polynomial weight defines a class of semi-classical orthogonal polynomials with general moment μ_k , given by

$$\mu_k = \int_T \frac{1}{2\pi i x} x^k w(x) dx, \tag{7.2}$$

where T denotes the unit circle $|x| = 1$, appropriately deformed in order to not cross the cut and $x = e^{2i\vartheta}$, $\vartheta \in (-\pi/2, \pi/2]$. This weight satisfies the Pearson

equation (4.18) with the following $\sigma(x)$ and $\tau(x)$

$$\begin{aligned}\sigma(x) &= \frac{x}{a-1} \left\{ ax^2 + (a-1)x - 1 \right\}, \\ \tau(x) &= \frac{1}{a-1} \left\{ azx^3 + (ba + za + 3a - z)x^2 - (a^2 - 2a + b + z + 2)x + a - 1 \right\}.\end{aligned}$$

For the polynomial weight (7.1), using (7.2), the general moment μ_k is given by

$$\mu_k = \int_T \frac{1}{2\pi i x} x^k (1+x)^b (1+1/x)^a e^{zx} dx.$$

This weight was previously explored by Forrester and Witte in [25, 24]. In this paper Forrester and Witte explain that this weight can be evaluated as an ${}_1F_1$ function which is equivalent to a Kummer function. They then conclude that this satisfies a second-order ODE using a logarithmic derivative. Again, the methods used are convoluted compared with the direct method. First we obtain explicit expressions for the moment $\mu_0(z)$.

Theorem 7.1. *For the polynomial weight (7.1) the initial moment $\mu_0(z)$ is given by*

$$\mu_0(z) = \frac{\Gamma(a+b+1)}{\Gamma(a+1)\Gamma(b+1)} M(-a, b+1, -z).$$

Proof. By expanding the exponential term in the polynomial weight, noting the identity

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)},$$

using (2.8) and

$$\int_0^\pi (\sin t)^{a-1} e^{ibt} dt = \frac{\pi}{2^{a-1}} \frac{e^{i\pi b/2}}{aB(\frac{1}{2}(a+b+1), \frac{1}{2}(a-b+1))},$$

where $B(a, b)$ is the beta function, we can calculate $\mu_0(z)$ in terms of Kummer

functions

$$\begin{aligned}
\mu_0 &= \int_T \frac{1}{2\pi i x} (1+x)^b (1+1/x)^a e^{zx} dx \\
&= \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1+e^{2i\vartheta})^b (1+e^{-2i\vartheta})^a \exp(ze^{2i\vartheta}) d\vartheta \\
&= \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2\cos(\vartheta))^{a+b} e^{i\vartheta(b-a+2n)} d\vartheta \\
&= \frac{2^{a+b}}{\pi} \sum_{n=0}^{\infty} \frac{z^n e^{-\pi i(b-a+2n)/2}}{n!} \int_0^{\pi} \sin^{a+b}(\tilde{\vartheta}) e^{i\tilde{\vartheta}(b-a+2n)} d\tilde{\vartheta}, \quad \vartheta = \tilde{\vartheta} + \frac{\pi}{2} \\
&= \sum_{n=0}^{\infty} \frac{z^n \Gamma(a+b+2)}{n!(a+b+1)\Gamma(a-n+1)\Gamma(b+n+1)} \\
&= \sum_{n=0}^{\infty} \frac{z^n \Gamma(a+b+1)}{n!\Gamma(a-n+1)\Gamma(b+n+1)} \\
&= \sum_{n=0}^{\infty} \frac{z^n \Gamma(a+b+1)\Gamma(n-a)}{n!\Gamma(a-n+1)\Gamma(b+n+1)\Gamma(n-a)} \\
&= \sum_{n=0}^{\infty} \frac{z^n \Gamma(a+b+1)\Gamma(n-a) \sin(\pi(n-a))}{n!\Gamma(b+n+1)\pi} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n z^n \Gamma(a+b+1)\Gamma(n-a)}{n!\Gamma(b+n+1)\Gamma(-a)\Gamma(a+1)} \\
&= \frac{\Gamma(a+b+1)}{\Gamma(a+1)\Gamma(b+1)} \sum_{n=0}^{\infty} \frac{(-1)^n z^n \Gamma(n-a)\Gamma(b+1)}{n!\Gamma(b+n+1)\Gamma(-a)} \\
&= \frac{\Gamma(a+b+1)}{\Gamma(a+1)\Gamma(b+1)} \sum_{n=0}^{\infty} \frac{(-z)^n (-a)_n}{n!(b+1)_n} \\
&= \frac{\Gamma(a+b+1)}{\Gamma(a+1)\Gamma(b+1)} M(-a, b+1, -z).
\end{aligned}$$

□

Theorem 7.2. *For the polynomial weight (7.1) the general moment $\mu_k(z)$ can be given by*

$$\mu_k(z) = \frac{\Gamma(a+b+1)}{\Gamma(a-k+1)\Gamma(b+k+1)} M(-a+k, b+k+1, -z). \quad (7.3)$$

Proof. The result can be inferred by repeating the above calculation with $a = a - k$ and $b = b + k$. □

Theorem 7.3. *For the polynomial weight (7.1), the general moment $\mu_k(z)$ can also be given by*

$$\mu_k(z) = \frac{d^k}{dz^k} \mu_0, \quad k = 0, 1, 2, 3, \dots$$

Proof. This result can be shown directly from from (2.10a), (2.11c) and (8.3). □

This is the point when we branch away from the work done previously by Forrester and Witte in [25, 24] and some original research is conducted.

We have μ_k in the form (4.8). Using theorem (4.1) we can make the following simplifications inside the Hankel determinant. Our goal here is to write Δ_n in the form of a bi-directional Wronskian.

$$\Delta_n = \begin{vmatrix} \mu_0 & \mu_1 & \dots & \mu_{n-1} \\ \mu_1 & \mu_2 & \dots & \mu_n \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \dots & \mu_{2n-2} \end{vmatrix} = \begin{vmatrix} \mu_0 & \mu'_0 & \dots & \mu_0^{(n-1)} \\ \mu'_0 & \mu''_0 & \dots & \mu_0^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_0^{(n-1)} & \mu_0^{(n)} & \dots & \mu_0^{(2n-2)} \end{vmatrix}, \quad ' = \frac{d}{dz}.$$

Therefore, we can write

$$\Delta_n = \tau_n(\mu_0), \tag{7.4}$$

where

$$\mu_0(z) = \frac{\Gamma(a + b + 1)}{\Gamma(a + 1)\Gamma(b + 1)} M(-a, b + 1, -z).$$

Theorem 7.4. *We now have Δ_n in the form that is similar to our special function solutions of P_V (1.1e). This means we can write down exact expressions for the recurrence coefficients $\alpha_n(z)$ and $\beta_n(z)$. The function*

$$H_n(z; a, b) = z \frac{d}{dz} \ln \tau_n(\mu_0), \tag{7.5}$$

with τ_n given by (4.10), satisfies the second-order, second-degree equation

$$\begin{aligned} \left(z \frac{d^2 H_n}{dz^2} \right)^2 &= \left\{ H_n - n(a+1-n) + (2n-z+b-1) \frac{dH_n}{dz} \right\}^2 \\ &+ 4 \frac{dH_n}{dz} \left(z \frac{dH_n}{dz} - H_n \right) \left(b+a - \frac{dH_n}{dz} \right). \end{aligned} \quad (7.6)$$

Proof. Equation (7.6) is equivalent to S_V (1.7e) through the linear transformation

$$H_n(z; a, b) = \sigma - \frac{1}{2}n^2 + \frac{1}{2}(1-b-z)n + \frac{1}{4}(a+b+1) + \frac{1}{8}(2a+b+1)^2, \quad (7.7)$$

for the parameters

$$\{\kappa_0, \kappa_1, \kappa_2, \kappa_3\} = \frac{1}{4}\{2a-2n+b+1, 1-3b-2a-2n, 2a+1+b+2n, b+2n-2a-3\}. \quad (7.8)$$

This is easily verified by comparing (7.7) (with H_n given by (7.5)) with (2.50b). \square

Remark 7.1.

- If we consider the solution to S_V (1.7e) using transformation (2.10a) and the corollary (2.3)

$$\begin{aligned} \sigma(z; a, b) &= z \frac{d}{dz} \ln \tau_n(M(a+b+1, b+1, z)) + \frac{1}{2}n^2 - \frac{1}{2}(1-b+z)n \\ &\quad - \frac{1}{4}(a+b+1) - \frac{1}{8}(2a+b+1)^2, \end{aligned}$$

for the parameters (7.8). These parameters can be mapped to our original set of parameters (2.51b) by the mapping $a \rightarrow \alpha + n - \beta - 1$ and $b \rightarrow \beta - n$. Due to the symmetric form of (1.7e) the choice of $\kappa_1, \kappa_2, \kappa_3$ and κ_4 is not unique.

- In terms of $H_n(z; a, b)$ given by (7.5), the coefficients $\alpha_n(z)$ and $\beta_n(z)$ in the recurrence relation have the form

$$\alpha_n(z) = \frac{1}{z} \left\{ H_{n+1} - H_n \right\}, \quad \beta_n(z) = \frac{1}{z^2} \left\{ z \frac{dH_n}{dz} - H_n \right\}.$$

- If we consider the original weight (7.1) again but with $a \in \mathbb{Z}$ there is an interesting simplification to note:

$$\Delta_n(z) = \left[\frac{a!}{\Gamma(a+1)} \right]^n \tau_n(L_a^{(b)}(-z)), \quad (7.9)$$

where $L_a^{(b)}$ is an associated Laguerre polynomial. This result can be shown if (2.12) is applied to (7.4). This simplifies the coefficients $\alpha_n(z)$ and $\beta_n(z)$ to Laguerre polynomials

$$\alpha_n(z) = \frac{d}{dz} \ln \frac{\tau_{n+1}(L_a^{(b)}(-z))}{\tau_n(L_a^{(b)}(-z))}, \quad \beta_n(z) = \frac{d^2}{dz^2} \ln \tau_n(L_a^{(b)}(-z)).$$

Proof. This result can be shown by applying (7.9) to (4.13). □

With this result we can now generate entirely new sequences of orthogonal polynomials using $\alpha_n(z)$ and $\beta_n(z)$. Notice that they are polynomials in x with rational coefficients in z .

Table 7.1: Table of new orthogonal polynomials $p_n(\alpha, \beta; x)$

$$\begin{aligned}
 p_2(1, 1; x) &= x^2 + \frac{x}{2+z} + \frac{1}{2+z} \\
 p_2(2, 1; x) &= x^2 + \frac{12(3+z)x}{(z^2+6z+12)(z^2+6z+6)} + \frac{z^2+6z+12}{z^2+6z+6} \\
 p_3(2, 2; x) &= x^3 + \frac{2(4+z)x^2}{(z+6)(z+2)} + \frac{(z^2+8z+22)x}{(z+6)(z+2)} + \frac{2(4+z)}{(z+6)(z+2)} \\
 p_3(3, 2; x) &= x^3 + \frac{60(z^3+15z^2+75z+120)x^2}{(z^3+15z^2+90z+210)(z^3+15z^2+60z+60)} \\
 &\quad + \frac{(z^7+35z^6+540z^5+4746z^4+25620z^3+84600z^2+158040z+131400)x}{2(z^3+15z^2+90z+210)(z^3+15z^2+60z+60)} \\
 &\quad + \frac{z^6+30z^5+420z^4+3480z^3+17100z^2+45000z+46800}{2(z^3+15z^2+90z+210)(z^3+15z^2+60z+60)}
 \end{aligned}$$

8 Painlevé VI and continuous orthogonal polynomials

8.1 Deformed Jacobi polynomials

The Jacobi polynomials are a class of classical orthogonal polynomials which are orthogonal with respect to the weight

$$w_0(x) = (1-x)^\alpha(1+x)^\beta.$$

However, here we will be studying the semi-classical deformed Jacobi polynomials with respect to the weight

$$w(x; z) = (x-z)^\gamma x^{\alpha+k}(1-x)^\beta, \tag{8.1}$$

on the interval $(0,1)$, with $\alpha, \beta > 0$, $z < 0$ and $\gamma \in \mathbb{R}$. In 2010 Dai and Zhang showed in [19] that the Δ_n generated by the matrix of moments satisfies the sixth Painlevé equation in the following way:

$$\begin{aligned} H_n(z) := & z(z-1) \frac{d}{dz} \ln \Delta_n - nz \left((n + \alpha + \beta + \gamma) - \frac{1}{4}(\alpha + \beta)^2 \right) \\ & + \frac{1}{4} \{ 2n(n + \alpha + \beta + \gamma) + \beta(\alpha + \beta) - \gamma(\alpha - \beta) \}. \end{aligned}$$

Then $H_n(z)$ satisfies

$$\begin{aligned} & \frac{dH_n}{dz} \left(z(z-1) \frac{d^2 H_n}{dz^2} \right)^2 + \left(\frac{dH_n}{dz} \left\{ 2H_n - (2z-1) \frac{dH_n}{dz} \right\} + \nu_1 \nu_2 \nu_3 \nu_4 \right)^2 \\ & = \prod_{j=1}^4 \left(\frac{dH_n}{dz} + \nu_j^2 \right), \end{aligned}$$

with parameters

$$\{\nu_1, \nu_2, \nu_3, \nu_4\} = \left\{ \frac{1}{2}(\alpha + \beta), \frac{1}{2}(\beta - \alpha), \frac{1}{2}(2n + \alpha + \beta), \frac{1}{2}(2n + \alpha + \beta + 2\gamma) \right\}.$$

As before, the methods used in this paper are known to be the ladders methods which, as we said, are longer and more convoluted than the direct method that we are going to use here. The key idea of the method that we are about to explore is the recognition of the initial moment as a special function via the appropriate integral representation and that the following moments are differential variants of the initial one. Just as we did with the previous weight, this makes it possible to write the matrix of moments as a bi-directional Wronskian which we can then compare easily and directly with the special function solutions of S_V (1.7e). Again, establishing this connection means we can simply read off the recurrence coefficients and therefore calculate new sequences of orthogonal polynomials quickly and with little time complexity. Lets compare Δ_n with our special function solutions.

For the deformed Jacobi polynomial weight (8.1), using

$$\mu_k(z) = \int_a^b x^k w(x) dx,$$

the general moment μ_k is given by

$$\mu_k(z) = \int_0^1 (x-z)^\gamma x^{\alpha+k} (1-x)^\beta dx. \quad (8.2)$$

Rather than obtaining explicit expressions for μ_0 , this time we will calculate $\mu_k(z)$ first. This can be done easily using the integral representation.

Theorem 8.1. *For the polynomial weight (8.1) the general moment $\mu_k(z)$ can be given by*

$$\mu_k = (-1)^\gamma B(\alpha+k+1, \beta+1) z^\gamma F(-\gamma, \alpha+k+1, \alpha+\beta+k+2; 1/z). \quad (8.3)$$

Proof. The result can be inferred by applying (2.14) to (8.2). □

Theorem 8.2. *The general moment $\mu_k(z)$ given by (8.3) satisfies the following second-order ODE.*

$$z(z-1)\frac{d^2\mu_k}{dz^2} + (\alpha + \gamma + k - z(\alpha + \beta + k + 2\gamma))\frac{d\mu_k}{dz} + \gamma(\alpha + \beta + k + \gamma + 1)\mu_k = 0. \quad (8.4)$$

Proof. The result can be inferred by applying (8.3) to (2.13). □

This is the point when we branch away from the work done previously by Dai and Zhang showed in [19] and some original research is conducted.

8.1.1 Hypergeometric relations

In this section we will prove some of the essential hypergeometric relations that we will need for some of the proofs later in this thesis.

Theorem 8.3. *Given the hypergeometric function $F(a, b, c; z)$, the following recurrence relation holds:*

$$(b - c + 1)F(a, b, c; z) + (c - 1)F(a, b, c - 1; z) - bF(a, b + 1, c; z) = 0.$$

Proof. Consider [60, §15.5.12] and [60, §15.5.15].

$$(b - a)F(a, b, c; z) + aF(a + 1, b, c; z) - bF(a, b + 1, c; z) = 0, \quad (8.5a)$$

$$(c - a - 1)F(a, b, c; z) + aF(a + 1, b, c; z) - (c - 1)F(a, b, c - 1; z) = 0. \quad (8.5b)$$

Computing (8.5a)-(8.5b) gives

$$(b - c + 1)F(a, b, c; z) + (c - 1)F(a, b, c - 1; z) - bF(a, b + 1, c; z) = 0.$$

□

Theorem 8.4. *Given the hypergeometric function $F(a, b, c; z)$ the following differential relation holds:*

$$\frac{d}{dz}(F(a, b, c; z)z^b) = bF(a, b + 1, c; z)z^{b-1}. \quad (8.6)$$

Proof. Consider [60, §15.5.4]

$$\frac{d}{dz}(z^{c-1}F(a, b, c; z) = (c - 1)z^{c-2}F(a, b, c - 1; z)). \quad (8.7)$$

Then compute the following using (8.7) and theorem 8.3:

$$\begin{aligned} \frac{d}{dz}(F(a, b, c; z)z^b) &= \frac{d}{dz}(z^{c-1}F(a, b, c; z)z^{b-c+1}) \\ &= z^{c-2}(c - 1)F(a, b, c - 1; z)z^{b-c+1} \\ &\quad + z^{c-1}F(a, b, c; z)(b - c + 1)z^{b-c} \\ &= z^{b-1}((c - 1)F(a, b, c - 1; z) + F(a, b, c; z)(b - c + 1)) \\ &= bF(a, b + 1, c; z)z^{b-1}. \end{aligned}$$

□

In order to compare Δ_n with something similar to our P_{VI} special function solutions we need control of α and β within μ_k . To do this we will make the following transformation of parameter inside μ_k :

$$\{\alpha, \beta, \gamma\} = \{a + 1 - c - n, c - b - n, -a\},$$

where the inverse transformation, simply for completeness, is

$$\{a, b, c\} = -\{\gamma, \alpha + \beta + \gamma + 2n + 1, \alpha + \gamma + n - 1\}.$$

We now have direct control of n and k . Alternatively, we can think of $\mu_{n,k}$ as the k th moment in a matrix of size n . This means $\mu_{n,k}$ now takes the following form:

$$\mu_{n,k} = (-1)^{-a} B(2 - c - n + a + k, c + 1 - b - n) z^{-a} \quad (8.8)$$

$$\times F(a, 2 - c - n + a + k, a + k + 3 - b - 2n; 1/z). \quad (8.9)$$

Theorem 8.5. *The following differential recurrence equation holds for $\mu_{n+1,n}$ and $\mu_{n,n-1}$:*

$$(b + n - c) \frac{d\mu_{n+1,n}}{dz} = \frac{d^2\mu_{n,n-1}}{dz^2} z + (b + n)\mu_{n,n-1}. \quad (8.10)$$

Proof. Substituting what we have for $\mu_{n,k}$ into theorem 8.4 we get the following:

$$\begin{aligned} (b + n - c)\mu_{n+1,n} &= \frac{d}{dz}(\mu_{n,n-1} z^{b+n-1}) z^{2-b-n} \\ &= \left(\frac{d\mu_{n,n-1}}{dz} z^{b+n-1} + (b + n - 1)\mu_{n,n-1} z^{b+n-2} \right) z^{2-b-n} \\ &= \frac{d\mu_{n,n-1}}{dz} z + (b + n - 1)\mu_{n,n-1}. \end{aligned}$$

Finally, differentiating the last line yields the desired result. \square

Theorem 8.6. *Consider the recurrence relation for $\mu_{n,k}$*

$$\mu_{n,k} - \mu_{n,k+1} - \mu_{n-1,k-1} = 0. \quad (8.11)$$

Proof. Substituting what we have for $\mu_{n,k}$ into theorem 8.3 gives the desired result. \square

In the next section we want to show that $\Delta_n(z)$ is directly equal to $\tau_n(\psi_{a,b,c})$ multiplied by some other matrices. We are going to show this in a very ‘‘brute force’’ way by showing the equivalence of each individual matrix entry.

8.1.2 Proof of main theorem

Theorem 8.7. *Suppose the following is true:*

$$\Delta_n = |\mathcal{P}_n \hat{\tau}_n(\psi_{a,b,c}) \mathcal{E}_n \mathcal{Q}_n|, \quad (8.12)$$

where $\hat{\tau}_n(\psi_{a,b,c})$ is given by (2.40),

$$\mathcal{P}_n := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{b}{z} & z^{-1} & 0 & 0 & 0 \\ \frac{b(b+1)}{z^2} & \frac{2b+1}{z^2} & z^{-2} & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{(b)_{n-1}}{\Gamma(0)x^{n-1}} & \frac{d}{db}(b)_{n-1} & \cdots & \frac{d^{n-1}}{db^{n-1}}(b)_{n-1} & z^{1-n} \end{bmatrix},$$

$$\mathcal{E}_n := \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & S(1,1) & S(2,1) & \cdots & S(n-2,1) \\ 0 & 0 & S(2,2) & \cdots & S(n-2,2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & S(n-2,n-2) \end{bmatrix},$$

where $S(m,n)$ are the Stirling numbers of the first kind and

$$\mathcal{Q}_n := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{b(n-2,n-1)}{(c-b-1)(z-1)} & \frac{1}{(z-1)} & 0 & 0 & 0 \\ \frac{b(n-3,n-1)}{(c-b-1)_2(z-1)^2} & \frac{b(n-3,n-2)}{(c-b-2)(z-1)^2} & \frac{1}{(z-1)^2} & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{b(1,n-1)}{(c-b-1)_{n-1}(z-1)^{n-1}} & \frac{b(1,n-2)}{(c-b-1)_{n-2}(z-1)^{n-1}} & \cdots & \frac{b(1,1)}{(c-b-1)(z-1)^{n-1}} & \frac{1}{(z-1)^{n-1}} \end{bmatrix},$$

where $(a)_n$ is the Pochhammer symbol.

In order to prove that these matrices are indeed equal we must first look at the very top right hand corner of (8.12).

8.1.3 Step 1

Proof. Multiplying out the general form of the right hand side of (8.12) yields

$$C_n(\Delta_n)_{1,n}z^b = C_n\mu_{n,n-1}z^b = (z-1)^{n-1} \sum_{k=1}^{n-1} S(n-1, k)\delta^{(k)}\phi, \quad (8.13)$$

where $C_n = \frac{(-1)^n}{(c)_a}((b)_{1-a}(a-b-n+2)_{n-2})$. It will be useful to note that

$$\frac{C_{n+1}}{C_n} = b + n - c.$$

Applying induction to (8.13), where the base case is trivial, we can show that

this result is true

$$\begin{aligned}
C_{n+1}\mu_{n+1,n}z^b &= (z-1)^{-n} \sum_{k=1}^n S(n,k)\delta^{(k)}\phi \\
&= (z-1)^{-n} \left\{ \sum_{k=1}^{n-1} S(n,k)\delta^{(k)}\phi + S(n,n)\delta^{(n)}\phi \right\}, \quad (S(n,n) = 1) \\
&= (z-1)^{-n} \left\{ \sum_{k=1}^{n-1} \{S(n-1,k-1) - (n-1)S(n-1,k)\}\delta^{(k)}\phi + \delta^{(n)}\phi \right\}, \\
&\quad \text{as } S(n,k) = S(n-1,k-1) - (n-1)S(n-1,k) \\
&= -C_n \frac{(n-1)}{(z-1)} \mu_{n,n-1} z^b + (z-1)^{-n} \left\{ \sum_{k=1}^{n-1} S(n-1,k-1)\delta^{(k)}\phi + \delta^{(n)}\phi \right\} \\
&= -C_n \frac{(n-1)}{(z-1)} \mu_{n,n-1} z^b + (z-1)^{-n} \left\{ \delta \sum_{k=1}^{n-1} S(n-1,k)\delta^{(k)}\phi \right. \\
&\quad \left. + \delta S(n-1,0) - S(n-1,n-1)\delta^{(n)}\phi + \delta^{(n)}\phi \right\} \\
&= -C_n \frac{(n-1)}{(z-1)} \mu_{n,n-1} z^b + (z-1)^n \left\{ \delta \sum_{k=1}^{n-1} S(n-1,k)\delta^{(k)}\phi \right\} \\
&= -C_n \frac{(n-1)}{(z-1)} \mu_{n,n-1} z^b + (z-1)^{-n} \left\{ C_n \delta \mu_{n,n-1} z^b (z-1)^{n-1} \right\} \\
&= -C_n \frac{(n-1)}{(z-1)} \mu_{n,n-1} z^b + C_n z (z-1)^{1-n} \left\{ \frac{d\mu_{n,n-1}}{dz} z^b (z-1)^{n-1} \right. \\
&\quad \left. + b\mu_{n,n-1} z^{b-1} (z-1)^{n-1} + \mu_{n,n-1} z^b (n-1)(z-1)^{n-2} \right\},
\end{aligned}$$

then dividing both sides by z^b

$$\begin{aligned}
C_{n+1}\mu_{n+1,n} &= C_n z \left\{ \frac{d\mu_{n,n-1}}{dz} + b\mu_{n,n-1}z^{-1} + \mu_{n,n-1}(n-1)(z-1)^{-1} \right\} \\
&\quad - C_n \frac{(n-1)}{(z-1)}\mu_{n,n-1} \\
&= C_n \left\{ z \frac{d\mu_{n,n-1}}{dz} + b\mu_{n,n-1} + (n-1)(z-1)^{-1}\mu_{n,n-1}(z-1) \right\} \\
&= C_n \left\{ z \frac{d\mu_{n,n-1}}{dz} + \mu_{n,n-1}(b+n-1) \right\}.
\end{aligned}$$

Differentiation of the last line shows

$$C_{n+1} \frac{d\mu_{n+1,n}}{dz} = C_n \left\{ z \frac{d^2\mu_{n,n-1}}{dz^2} + \frac{d\mu_{n,n-1}}{dz}(b+n) \right\}$$

and finally the division by C_n

$$(b+n-c) \frac{d\mu_{n+1,n}}{dz} = z \frac{d^2\mu_{n,n-1}}{dz^2} + \frac{d\mu_{n,n-1}}{dz}(b+n).$$

Implementing (8.10) gives the desired result.

This proves that the top right hand corner of the matrix $\Delta_n(z)$ is always equal to the right hand corner of (8.12) for all n . Mathematically speaking this can be written down in the following way:

$$C_n(\Delta_n)_{1,n}z^b = C_n\mu_{n,n-1}z^b = (z-1)^{n-1} \sum_{k=1}^{n-1} S(n-1, k)\delta^{(k)}\phi. \quad (8.14)$$

Now we need to extend this to show the matrices are equal everywhere.

8.1.4 Step 2

The following formula gives the $(n-1-j)$ th entry in the first row. Again, we are simply trying to show the equivalence of the individual matrix entries once all the

multiplication of the right hand side of (8.12) has been computed:

$$\begin{aligned} D_{n,j}(\Delta_n)_{1,j}z^b &= D_{n,j}\mu(n, n+j-2)z^b \\ &= \sum_{i=1}^j \frac{b(j-1, i-1)(z-1)^{j+1-i-n} \sum_{k=1}^{n-1} \delta^{(k)} S(n+i-j-1, k)}{(c-b-n-i+j+1)_{i-1}}, \end{aligned}$$

with $D_{n,j} = \frac{C(n)}{(1-b-n+c)_{j-1}}$. It is worth noting that

$$\frac{D_{n,j+1}}{D_{n-1,j}} = -1$$

and

$$\frac{D_{n,j}}{D_{n-1,j}} = b+n-c-j. \quad (8.15)$$

Setting $n := \tilde{n} + i - j$ in (8.14) gives

$$\begin{aligned} C_{\tilde{n}+i-j}\mu_{\tilde{n}+i-j, \tilde{n}+i-j-1}z^b &= (z-1)^{\tilde{n}+i-j-1} \sum_{k=1}^{\tilde{n}+i-j-1} S(\tilde{n}+i-j-1, k)\delta^{(k)}\phi \\ &= (z-1)^{\tilde{n}+i-j-1} \sum_{k=1}^{\tilde{n}-1} S(\tilde{n}+i-j-1, k)\delta^{(k)}\phi, \end{aligned}$$

where $S(n+i-j-1, k) = 0$ when $k > n+i-j-1$ ($i < j$). This transforms (8.14) into

$$D_{n,j}\mu_{n, n+j-2}z^b = \sum_{i=1}^j \frac{b(j-1, i-1)C_{n+i-j}\mu_{n+i-j, n+i-j-1}z^b}{(c-b-n-i+j+1)_{i-1}}. \quad (8.16)$$

Applying induction to (8.16), where the base case is trivial, we can show that this result is true

$$\begin{aligned}
D_{n,j+1}\mu_{n,n+j-1}z^b &= \sum_{i=1}^{j+1} \frac{b(j,i-1)C_{n+i-j-1}\mu_{n+i-j-1,n+i-j-2}z^b}{(c-b-n-i+j+2)_{i-1}} \\
&= C_{n-j}\mu_{n-j,n-j-1}z^b + \sum_{i=1}^j \frac{b(j,i)C_{n+i-j}\mu_{n+i-j,n+i-j-1}z^b}{(c-b-n-i+j+1)_i} \\
&= C_{n-j}\mu_{n-j,n-j-1}z^b \\
&\quad + \sum_{i=1}^j \frac{\{b(j-1,i-1) + b(j-1,i)\}C_{n+i-j}\mu_{n+i-j,n+i-j-1}z^b}{(c-b-n-i+j+1)_i} \\
&= C_{n-j}\mu_{n-j,n-j-1}z^b + \frac{D_{n,j}\mu_{n,n+j-2}z^b}{(c-b-n+j)} \\
&\quad + \sum_{i=1}^j \frac{b(j-1,i)C_{n+i-j}\mu_{n+i-j,n+i-j-1}z^b}{(c-b-n-i+j+1)_i} \\
&= C_{n-j}\mu_{n-j,n-j-1}z^b + \frac{D_{n,j}\mu_{n,n+j-2}z^b}{(c-b-n+j)} - C_{n-j}\mu_{n-j,n-j-1}^{-1}z^b \\
&\quad + \sum_{i=1}^j \frac{b(j-1,i-1)C_{n+i-j-1}\mu_{n+i-j-1,\tilde{n}+i-j-2}z^b}{(c-b-n-i+j+2)_{i-1}} \\
&= C_{n-j}\mu_{n-j,n-j-1}z^b + \frac{D_{n,j}\mu_{n,n+j-2}z^b}{(c-b-n+j)} + D_{n-1,j}\mu_{n-1,n+j-3}z^b \\
&\quad - C_{n-j}\mu_{n-j,n-j-1}z^b.
\end{aligned}$$

It therefore suffices to show

$$D_{n,j+1}\mu_{n,n+j-1} = (c-b-n+j)^{-1}D_{n,j}\mu_{n,n+j-2} + D_{n-1,j}\mu_{n-1,n+j-3},$$

which is easily verified using $\frac{D_{n,j+1}}{D_{n-1,j}} = -1$, $\frac{D_{n,j}}{D_{n-1,j}} = b+n-c-j$ and (8.11).

8.1.5 Step 3

We now have to show that the matrix moments of Δ_n and the right hand column of (8.12) are equal when we move in the downwards direction from the top right

hand corner.

$$E_{n,j}(\Delta_n)_{n,n-j}z^b = E_{n,j}\mu_{n,n-j}z^b = \sum_{k=1}^j \frac{\left(\frac{1}{\Gamma(k)} \frac{d^{k-1}}{db^{k-1}}(b)_{j-1}\right) \left(\sum_{i=1}^{n-1} \delta^{(k+i-1)} S(n-1, i)\right)}{(z-1)^{n-1} z^{j-1}},$$

where $E_{n,j} = C(n)(c-a)_{j-1}$. It also is useful to note that

$$\frac{E_{n,j+1}}{E_{n,j}} = c - a + j - 1.$$

We can also remove this double sum here by making use of the identity we have already proved (8.13)

$$\begin{aligned} E_{n,j}z^b\mu_{n,n-j} &= \sum_{k=1}^j \frac{\left(\frac{1}{\Gamma(k)} \frac{d^{k-1}}{db^{k-1}}(b)_{j-1}\right) \left(\sum_{i=1}^{n-1} \delta^{(k+i-1)} S(n-1, i)\right)}{(z-1)^{n-1} z^{j-1}} \\ &= \sum_{k=1}^j \frac{1}{z^{j-1}(z-1)^{n-1}} \left(\frac{1}{\Gamma(k)} \frac{d^{k-1}}{db^{k-1}}(b)_{j-1}\right) \delta^{(k-1)} (C_n \mu_{n,n-1} z^b (z-1)^{n-1}), \end{aligned} \tag{8.17}$$

Applying induction to (8.17), where the base case is trivial, we can show that this result is true

$$\begin{aligned}
E_{n,j+1}z^b\mu_{n,n-j-1} &= \sum_{k=1}^{j+1} \frac{1}{z^j(z-1)^{n-1}} \left(\frac{1}{\Gamma(k)} \frac{d^{k-1}}{db^{k-1}}(b)_j \right) \delta^{(k-1)}(C_n\mu_{n,n-1}z^b(z-1)^{n-1}) \\
&= \sum_{k=1}^j \frac{1}{z^j(z-1)^{n-1}} \left(\frac{1}{\Gamma(k)} \frac{d^{k-1}}{db^{k-1}}(b)_j \right) (\delta^{(k-1)}(C_n\mu_{n,n-1}z^b(z-1)^{n-1}) \\
&\quad + \frac{1}{z^j(z-1)^{n-1}\Gamma(j+1)} \frac{d^j}{db^j}(b)_{j+1} \delta^{(j)}\mu_{n,n-1}) \\
&= \sum_{k=1}^j \frac{1}{z^j(z-1)^{n-1}} \left(\frac{1}{\Gamma(k)} \left\{ (b+j-1) \frac{d^{k-1}}{db^{k-1}}(b)_{j-1} \right. \right. \\
&\quad \left. \left. + (k-1) \frac{d^{k-2}}{db^{k-2}}(b)_{j-1} \right\} \right) \delta^{(k-1)}(C_n\mu_{n,n-1}z^b(z-1)^{n-1}) \\
&\quad + \frac{C_n\delta^{(j)}\mu_{n,n-1}}{z^j(z-1)^{n-1}} \\
&= (b+j-1)E_{n,j}z^{b-1}\mu_{n,n-j} + \sum_{k=1}^j \frac{1}{z^j(z-1)^{n-1}} \left(\frac{(k-1)}{\Gamma(k)} \frac{d^{k-2}}{db^{k-2}}(b)_{j-1} \right) \\
&\quad \delta^{(k-1)}(C_n\mu_{n,n-1}z^b(z-1)^{n-1}) + \frac{C_n\delta^{(j)}\mu_{n,n-1}}{z^j(z-1)^{n-1}} \\
&= (b+j-1)E_{n,j}z^{b-1}\mu_{n,n-j} + \frac{1}{z^j(z-1)^{n-1}} \delta \sum_{k=1}^j \left(\frac{1}{\Gamma(k)} \frac{d^{k-1}}{db^{k-1}}(b)_{j-1} \right) \\
&\quad (\delta^{(k-1)}C_n\mu_{n,n-1}z^b(z-1)^{n-1}) \\
&= (b+j-1)E_{n,j}z^{b-1}\mu_{n,n-j} \\
&\quad + \frac{1}{z^j(z-1)^{n-1}} \delta (E_{n,j}z^{b+j-1}(z-1)^{n-1}\mu_{n,n-j}) \\
&= (b+j-1)E_{n,j}z^{b-1}\mu_{n,n-j} \\
&\quad + \frac{1}{z^{j-1}(z-1)^{n-2}} \frac{d}{dz} (E_{n,j}z^{b+j-1}(z-1)^{n-1}\mu_{n,n-j}).
\end{aligned}$$

So it just remains to show

$$(c-a+j-1)\mu_{n,n-j-1} = (z-1) \frac{d\mu_{n,n-j}}{dz} + (b+j+n-2)\mu_{n,n-j}.$$

Using [60, §15.5.1] and substituting $\mu_{n,k}$ into [60, §15.5.13] this result is verified. \square

We have now shown that each element of Δ_n and the right hand side of (8.12) are in fact equal and therefore (8.1.3) has been proved. Now we can continue with simplifying Δ_n in the following way:

$$\begin{aligned}\Delta_n &= |\mathcal{P}_n \hat{\tau}_n(\psi_{a,b,c}) \mathcal{E}_n \mathcal{Q}_n| \\ &= |\mathcal{P}_n| |\hat{\tau}_n(\psi_{a,b,c})| |\mathcal{E}_n| |\mathcal{Q}_n| \\ &= C_n z^{n(1-n-2b)/2} \tau_n(\psi_{a,b,c}) (z-1)^{n(1-n)/2} \\ &= C_n \mathcal{W}_n(\psi_{a,b,c}),\end{aligned}$$

where C_n is irrelevant due to the logarithmic derivative and $\mathcal{W}_n(\psi_{a,b,c})$ is given by (2.41).

Theorem 8.8. *We now have Δ_n in the form that is similar to our special function solutions of S_{VI} (1.7f). This means we can write down exact expressions for the recurrence coefficients $\alpha_n(z)$ and $\beta_n(z)$. The function*

$$H_n(z) := z(z-1) \frac{d}{dz} \ln \mathcal{W}_n(\psi_{a,b,c}), \quad (8.18)$$

satisfies the second-order, second degree equation

$$\begin{aligned}z^2(z-1)^2 \left(\frac{d^2}{dz^2} H_n \right)^2 \left(\frac{d}{dz} H_n + A \right) - \prod_{j=1}^4 \left(\frac{dH_n}{dz} + A + \nu_j^2 \right) \\ + \left\{ \left(\frac{d}{dz} H_n + A \right) \left[2H_n + 2Az + 2B - (2z-1) \left(\frac{d}{dz} H_n + A \right) \right] + \nu_1 \nu_2 \nu_3 \nu_4 \right\}^2,\end{aligned} \quad (8.19)$$

where $A = an - \frac{1}{4}(a-b+1)^2$ and $B = \frac{1}{4}(n(1+b-a-2c) + a^2 - ac + b^2 - bc + a - b + c)$.

Proof. Equation (8.19) is equivalent to S_{VI} (1.7f) through the linear transformation

$$H_n(z; a, b, c) = \sigma - Az - B, \quad (8.20)$$

for the parameters

$$\{\nu_1, \nu_2, \nu_3, \nu_4\} = \left\{ \frac{1}{2}(1 - b - 2n + a), -\frac{1}{2}(1 - 2c + b + a), \frac{1}{2}(1 + a - b), \frac{1}{2}(1 - a - b) \right\}. \quad (8.21)$$

□

Now we must calculate $\tilde{\Delta}$ in terms of Δ_n and its derivatives. In order to do this we must use the following theorem:

Theorem 8.9. *The function μ_k has the following differential relation:*

$$\mu_k z^{C-k} (z-1)^{B+k} = \delta^{(k)}(\mu_0 z^C (z-1)^B),$$

where $\delta = \frac{(z-1)^2}{k-b-2n+2} \frac{d}{dz}$, $C = n + c - 2$, and $B = a - n - c + 2$.

Proof. This can be shown easily using [60, §15.5(ii)].

□

Theorem 8.10. *The function $\tilde{\Delta}_n(z)$ is related to $\Delta_n(z)$ in the following way:*

$$\tilde{\Delta}_n = \frac{1}{b-1} \left\{ \Delta_n n(1 + C - n - z(B + C)) - \frac{d\Delta_n}{dz} z(z-1) \right\}.$$

Proof. Using theorem 8.9 we can see that

$$\Delta_n = z^{n(n-C-1)} (z-1)^{n(1-B-n)} \tilde{\mathcal{H}}_n(\mu_0 z^C (z-1)^B), \quad (8.22)$$

where

$$\tilde{\mathcal{H}}_n(\phi) := \begin{vmatrix} \phi & \delta(\phi) & \dots & \delta^{(n-1)}(\phi) \\ \delta(\phi) & \delta^{(3)}(\phi) & \dots & \delta^{(n)}(\phi) \\ \vdots & \vdots & \ddots & \vdots \\ \delta^{(n-1)}(\phi) & \delta^{(n)}(\phi) & \dots & \delta^{(2n-2)}(\phi) \end{vmatrix}$$

and from this we can write down an expression for $\tilde{\Delta}_n$

$$\tilde{\Delta}_n = \frac{1}{1-b} z^{n(n-C-1)+1} (z-1)^{n(1-B-n)+1} \frac{d}{dz} \tilde{\mathcal{H}}_n(\mu_0 z^C (z-1)^B). \quad (8.23)$$

Rearranging (8.22) for $\tilde{\mathcal{H}}_n(\mu_0 z^C (z-1)^B)$, substituting into (8.23) and simplifying gives the desired result. \square

8.1.6 The recurrence coefficients

Theorem 8.11. *In terms of H_n given by (8.18) the recurrence coefficients $\alpha_n(z)$ and $\beta_n(z)$ have the following form:*

$$\begin{aligned} \alpha_n(z) &= C - 2n - z(B + C) + H_{n+1} - H_n, \\ \beta_n(z) &= \frac{1}{(2-b-2n)(3-b-2n)} \left\{ z^2(z-1)^2 \left[\frac{dH_n}{dz} + 2H_n \right] \right. \\ &\quad \left. + n(Bz^2 + Cz^2 - C + n - 1) \right\}. \end{aligned}$$

Proof. Substituting (8.22) and (8.23) into α_n and β_n gives

$$\begin{aligned}
\alpha_n &= \frac{\tilde{\Delta}_{n+1}}{\Delta_{n+1}} - \frac{\tilde{\Delta}_n}{\Delta_n} \\
&= (n+1)(C-n-z(B+C)) - z(z-1)\frac{\Delta'_{n+1}}{\Delta_{n+1}} - n(1+C-n-z(B+C)) \\
&\quad + z(z-1)\frac{\Delta'_n}{\Delta_n} \\
&= C-2n-z(B+C) + z(z-1)\frac{d}{dz}\ln\frac{\Delta_{n+1}}{\Delta_n} \\
&= C-2n-z(B+C) + H_{n+1} - H_n, \\
\beta_n &= \frac{\Delta_{n+1}\Delta_{n-1}}{\Delta_n^2} \\
&= \frac{z^2\tilde{H}_{n+1}\tilde{H}_{n-1}}{(z-1)^2(\tilde{H}_n)^2} \\
&= \frac{z^2}{(2-b-2n)(3-b-2n)}\frac{d}{dz}\left\{(z-1)^2\frac{d}{dz}\ln(\Delta_n z^{-n(n-C-1)}(z-1)^{-n(1-B-n)})\right\} \\
&= \frac{z^2}{(2-b-2n)(3-b-2n)}\frac{d}{dz}\left\{(z-1)^2\left(\frac{d}{dz}\ln\Delta_n\right.\right. \\
&\quad \left.\left.-\frac{d}{dz}\ln(z^{n(n-C-1)}(z-1)^{n(1-B-n)})\right)\right\} \\
&= \frac{z^2}{(2-b-2n)(3-b-2n)}\frac{d}{dz}\left\{(z-1)^2\left(H_n - \frac{d}{dz}\ln(z^{n(n-C-1)}(z-1)^{n(1-B-n)})\right)\right\} \\
&= \frac{1}{(2-b-2n)(3-b-2n)}\left\{z^2(z-1)^2\left[\frac{dH_n}{dz} + 2H_n\right]\right. \\
&\quad \left.+ n(Bz^2 + Cz^2 - C + n - 1)\right\}.
\end{aligned}$$

□

9 Conclusion

In this thesis we have new formulations for the special function solutions in order for them to be viewed and used in a more manageable form. This new formulation meant we could improve upon some previous work; specifically we have improved upon the “ladder methods”, as they are known in the literature. Along side this new formulation we also included the rational function solutions. As we have seen, some of the rational function solutions form a subset of the special function solutions. This reduction of some of the special functions to polynomials gives the applications we looked at much more usability and diversity with regards to plotting and analysis.

This thesis not only re-formulates the special function solutions, but also utilises these new solution forms in order to simplify the overall comparison between the Painlevé equations and orthogonal polynomials. We have seen this explicitly for various cases involving P_V (1.1e) and one non-trivial example connecting P_{VI} (1.1f) with orthogonal polynomials. In all of these cases we have been able to generate new orthogonal polynomials with coefficients that are special functions.

In certain cases we have been able to improve upon this even further with the reduction of the special functions to polynomials, in the case of P_V (1.1e) this was to Laguerre polynomials and in the case of P_{VI} (1.1f) this is Jacobi polynomials. This provided a computationally beneficial simplification of the special function solutions and meant we could analyse them quickly and efficiently by computing plots and comparing different aspects of the polynomials, such as the interlacing root properties etc. This thesis has also given us an efficient way of generating

new orthogonal polynomials.

This thesis has answered a lot of questions that I originally wanted to answer concerning the already well known connection between Painlevé equations and orthogonal polynomials. However, during the study of this connection I have also uncovered numerous branches of mathematics that I still wish to investigate further. Some of these areas include some unknown root structure that the rational function solutions possess; for example: In the limiting case, the corners of Yablonskii-Vorob'ev polynomials tends towards a finite angle. Recently there was some research carried out into this by Buckingham, Miller, Bertola and Bothner in 2014 [6, 8, 9]; this result could easily be applied to the remaining polynomials that comprise the rational function solutions of the Painlevé equations. This is just one area of the root structure that we could investigate; there is also the unanswered question of why these polynomial roots actually form these patterns with such structure. This is a question I have given a lot of thought but, so far, I have been unable to answer.

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