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The cellular structure of wreath product algebras

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Abstract

We review the definitions and basic theory of cellular algebras as developed in the papers of Graham and Lehrer and of König and Xi. We then introduce a reformulation of the concept of an iterated inflation of cellular algebras (a concept due originally to König and Xi), which we use to show that the Brauer algebra is cellular (following the work of König and Xi). We then review the notion of the wreath product of an algebra with a symmetric group, and apply our work on iterated inflations to prove that the wreath product of a cellular algebra with a symmetric group is in all cases cellular, and we obtain a description of the cell modules of such a wreath product.

Acknowledgements

Firstly and foremostly, I am very grateful for the generous and patient advice and support of my supervisor, Dr. Rowena Paget. I am also, of course, very grateful to my parents, Ian and Louise Green, without whom neither I nor this thesis would exist.

Contents

Introduction

The main topics of this thesis are *cellular algebras* and *wreath products* of algebras with symmetric groups. Cellular algebras are a class of algebras first introduced by Graham and Lehrer in [5], and subsequently studied by many other authors; in the original definition, a cellular algebra is defined to be an associative, unital algebra over a commutative ring R equipped with an R-linear bijection on the algebra with certain properties, called an *anti-involution*, and an R-basis, called a *cellular basis*, which interacts in an especially simple and convenient way with the multiplication of the algebra and the anti-involution. The wreath product of an algebra with a symmetric group S_n is a well-known construction which arises, for example, in the study of the representation theory of symmetric groups, and which may be regarded in some sense as a generalisation of the more familiar wreath product of groups. Informally, if n is a positive integer, and A an algebra over a field $\Bbbk$, then the wreath product of A with S_n is the vector space tensor product of one copy of the group algebra of S_n and n copies of A, with multiplication based on the natural "placewise" definition of multiplication in a tensor product, but "twisted" by the natural action of S_n on the n copies of A by place permutations. The main result presented in this thesis is a proof that if A is a cellular algebra over a field $\Bbbk$, then the wreath product of A with S_n is also a cellular algebra; this proof uses the concept of an *iterated inflation of cellular algebras*, which was introduced by König and Xi in [9].

Graham and Lehrer showed in [5] that a number of interesting algebras admit such a cellular structure, for example the Brauer algebra and the Temperley-Lieb algebra, and also certain kinds of Hecke algebra, including as a special case the group ring RS_n of the symmetric group S_n. Graham and Lehrer also showed how the existence of a cellular structure on an algebra allows us to study its representation theory, especially in the case where the underlying ring R is in fact a field. For any cellular algebra, we may define a family of modules called the *cell modules*, and if R is a field then the simple modules of the algebra may be obtained as quotients of the cell modules in a standard way which is to some extent independent of the field R.

König and Xi then gave an alternative, but equivalent, definition of cellular algebras in [8]; this definition avoided the use of a basis, and instead characterised cellular algebras in terms of a chain of ideals whose quotients possess certain properties involving the anti-involution, so that we may simply speak of an algebra being cellular with respect to some anti-involution, rather than with respect to an anti-involution *and* a particular basis. Subsequently, in the papers [9], [10], and [15], König and Xi developed the notion of an *iterated inflation* of cellular algebras, a construction by which known cellular algebras may be combined to produce new cellular algebras. By showing that a given algebra may be constructed as such an iterated inflation of cellular algebras, one can conclude that this algebra is itself cellular, and moreover one may obtain information about its cellular structure and its cell modules. König and Xi used this method in [10] to give another proof that the Brauer algebra is cellular, and Xi applied the method to the partition algebra in [15]. The concept of an iterated inflation of cellular algebras has since been used by a number of authors.

The wreath product of a cellular algebra with a symmetric group has been studied by Geetha and Goodman in [4], where in particular they drew on the work of Dipper, James, and Mathas in [3] and of Murphy in [14] to show that if we make the extra assumption that every cell module of the cellular algebra A is cyclic (they define the term *cyclic cellular* to describe such an algebra), then the wreath product of A and S_n is again cellular (and moreover is again cyclic cellular). The proof of this result is quite combinatorial in nature, and does not make use of iterated inflations of cellular algebras.

In this thesis, we shall only consider cellular algebras where the underlying ring is in fact a field. In Chapter 1 we review some of the basic theory of cellular algebras over a field, drawing mainly on [5] and [8], while in Chapter 2 we shall study iterated inflations of cellular algebras, and in particular their application to proving that the Brauer algebra is cellular. In Chapter 3, we review the definition of the wreath product of an algebra with a symmetric group, and then prove that the wreath product of a cellular algebra with a symmetric group may be exhibited as an iterated inflation of cellular algebras.

Chapter 1 begins by considering the original definition of a cellular algebra from [5] (our Definition 1.2) in Section 1.1, and giving some well-known examples to illustrate it, including the algebra $M_n(\Bbbk)$ of $n \times n$ matrices over a field $\Bbbk$ and the Temperley-Lieb algebra $\mathrm{TL}_{\Bbbk}(r, \delta)$, for which we give a detailed definition. In Section 1.2 we continue to review the ideas in [5] by constructing the cell modules of a cellular algebra, which are fundamental to the representation theory of cellular algebras. Next, Section 1.3 gives the basis-free definition of a cellular algebra introduced by König and Xi in [8] (our Definition 1.10), and a proof that the two definitions of a cellular algebra are equivalent. Section 1.4 briefly outlines (without proofs) how a complete set of non-isomorphic simple modules of a cellular algebra may be obtained as certain quotients of the cell modules, as explained in [5]. In Section 1.5, we show that the tensor product of two (or more) cellular algebras is again cellular, and describe its cell modules (these results will be used in Chapter 3); all of this was given by Geetha and Goodman in [4].

Section 2.1 begins by briefly introducing the symmetric group S_n and some associated combinatorics, and then explains how the group algebra $\Bbbk S_n$ (where $\Bbbk$ is any field) may be exhibited as a cellular algebra (see Theorem 2.1); we do not give any proofs, but rather rely on the work of Mathas in [12]. Section 2.2 gives a well-known construction of the Brauer algebra as a diagram algebra, and then gives a decomposition of the Brauer algebra as a direct sum of vector spaces, used by König and Xi in [9], where each subspace has a natural decomposition as a tensor product of two copies of a certain vector space and one copy of a group algebra of a symmetric group. In Section 2.3, we study iterated inflations of cellular algebras; the results given in this section are a reformulation of the work of König and Xi. In particular, Theorem 2.2 provides the definition of an iterated inflation of cellular algebras which we shall use in this thesis; essentially, Theorem 2.2 allows us to show that an algebra with a subspace decomposition like the one given for the Brauer algebra in Section 2.2 is cellular with respect to some anti-involution, provided that certain conditions governing the interaction between the decomposition, the multiplication, and the anti-involution are satisfied. In Section 2.4, we apply Theorem 2.2 to complete our proof that the Brauer algebra is cellular, while in Section 2.5 we show how the cell modules of an iterated inflation may be obtained from the subspace decomposition which exhibits it as an iterated inflation (Corollary 2.7), and apply this to the Brauer algebra; this result is implied in the work of König and Xi.

Section 3.1 recalls the notion of the opposite algebra of an associative unital algebra over a field, and proves that the opposite algebra of a cellular algebra is again cellular; we need to make use of opposite algebras in order to overcome some technical differences between our definition of the wreath product in Section 3.2 and the definition used in much of the literature. Section 3.2 defines the wreath product of an algebra with a symmetric group. As noted above, this definition is different from the definition used in much of the literature, for example in [1] and [11]; this difference occurs because we have adopted different conventions on the symmetric group S_n in order that our work on the Brauer algebra in Chapter 2 agrees with the work of König and Xi. However, there is a straightforward connection between the two versions of the wreath product, as Section 3.2 explains. Section 3.3 reviews some standard combinatorics related to the symmetric group, in particular Young subgroups. Section 3.4 then describes a well-known method of obtaining modules for the wreath product from modules of the algebra and certain symmetric groups; the description is based closely on Section 3 of [1]. Section 3.5 shows how the wreath product of a cellular algebra and a symmetric group may be exhibited as an iterated inflation of cellular algebras, and hence proved to be cellular, and Section 3.6 explains how the cell modules of such a wreath product may be constructed from the cell modules of the original cellular algebra and the Specht modules of certain symmetric groups using the method of Section 3.4.

In [15], Xi gave a lemma characterising iterated inflations of cellular algebras, which has since been used by a number of authors. However, this lemma is in fact incorrect, and in Appendix A we give a counterexample to demonstrate this.

At the end of each chapter, I have included a brief paragraph indicating which results from that chapter are "new". When I claim that a result is new, I mean that I have obtained it myself and that, as far as I know, it has not previously been published.

Throughout this thesis, except where otherwise indicated, $\Bbbk$ will denote an arbitrary field. By an *algebra* over $\Bbbk$, we shall always mean an associative unital $\Bbbk$-algebra unless stated otherwise, and in fact all of the algebras we shall consider will be finite-dimensional over $\Bbbk$. We shall usually write the multiplicative identity element of a $\Bbbk$-algebra as 1, and further we shall demand that $1 \neq 0$ in our algebras; thus our $\Bbbk$-algebras must have $\Bbbk$-dimension at least one.

A remark about tensor products

In this thesis, we shall often need to consider the tensor products of various algebraic structures, for example $\Bbbk$-algebras or modules for $\Bbbk$-algebras. In all cases the objects whose tensor product is being taken have the underlying structure of a vector space over the field $\Bbbk$, and in almost all cases the desired tensor product is constructed by taking the tensor product of the two objects *as vector spaces over* $\mathbb{k}$, and then equipping the resulting $\mathbb{k}$-vector space with whatever additional operations are required. As an example, let A and B be finite-dimensional $\mathbb{k}$-algebras. Then the *tensor product algebra* of A and B is defined to be the $\mathbb{k}$-vector space tensor product $A \otimes_{\mathbb{k}} B$ of A and B, which is made into an algebra over $\mathbb{k}$ by equipping it with the multiplication defined by the formula

$$(a_1 \otimes_k b_1)(a_2 \otimes_k b_2) = (a_1 a_2) \otimes_k (b_1 b_2) \tag{0.1}$$

where $a_1, a_2 \in A$ and $b_1, b_2 \in B$. For convenience, we shall in this thesis adopt the convention that the plain symbol $\otimes$ always denotes a tensor product $\otimes_{\mathbb{k}}$ taken over *the field* $\mathbb{k}$; tensor products over any other ring R will be indicated as usual by a subscript, as $\otimes_R$. However, there is an important issue to be considered in Formula (0.1): well-definedness. Indeed, the elements $a \otimes b$ of $A \otimes B$ where $a \in A$ and $b \in B$, called *pure tensors*, do *not* (except in trivial cases) form a basis of $A \otimes B$, and not all elements of $A \otimes B$ are pure tensors. Thus it is not immediate that Formula (0.1) does yield a well-defined operation on $A \otimes B$, nor that this operation is $\mathbb{k}$-bilinear. In the case of a tensor product over an arbitrary commutative ring R, one would have to justify the definition given in Formula (0.1) by appealing to the *universal property* of the tensor product; however, since we are working with tensor products over a field, there is a simpler justification. Let $\alpha_1, \dots, \alpha_n$ be a $\mathbb{k}$-basis of A and $\beta_1, \dots, \beta_m$ a $\mathbb{k}$-basis of B. Then the pure tensors $\alpha_i \otimes \beta_j$ for all $i = 1, \dots, n$ and all $j = 1, \dots, m$ form a $\mathbb{k}$-basis for $A \otimes B$. We then replace the pure tensors in Formula (0.1) with elements of this basis, to obtain the formula

$$(\alpha_i \otimes \beta_j)(\alpha_p \otimes \beta_q) = (\alpha_i \alpha_p) \otimes_k (\beta_j \beta_q) \tag{0.2}$$

where $i, p \in \{1, \dots, n\}$ and $j, q \in \{1, \dots, m\}$. Now since the pure tensors $\alpha_i \otimes \beta_j$ are a $\mathbb{k}$-basis of $A \otimes B$, it is immediate that Formula (0.2) *does* yield a well-defined $\mathbb{k}$-bilinear operation on $A \otimes B$ when it is extended $\mathbb{k}$-bilinearly to the whole of $A \otimes B$; further, it is trivial to show that this multiplication does indeed satisfy Formula (0.1) for all $a_1, a_2 \in A$ and $b_1, b_2 \in B$, and so we conclude that Formula (0.1) *does* yield a well-defined $\Bbbk$-bilinear operation on $A \otimes B$ after all.

In the course of this thesis, we shall on numerous occasions define operations, actions, functions, etc. on tensor products over $\Bbbk$ via formulae like Formula (0.1) which give a definition "on pure tensors" (for example, Section 3.4 contains several such definitions). Formally, such definitions must be justified via an argument similar to the one given above; however, all of these arguments are indeed *very* similar to the one given above, and their inclusion would be both tedious and unnecessary. Thus we shall just give the formulae involving pure tensors, and state that the operation (or whatever it is) which is being defined is indeed "well-defined"; in all such cases it is possible to prove that this is so by taking bases over $\Bbbk$ and applying an argument like the one above.

The only place where we shall work with tensor products over a ring which is not a field is in the proof of Proposition 3.7, and separate arguments will be provided there.

1 Cellular algebras

Let $\Bbbk$ be any field. The following standard definition is fundamental to all of the material in this thesis:

Definition 1.1. Let A be an associative unital $\Bbbk$-algebra. An **anti-involution** on A is a $\Bbbk$-linear map $\iota : A \to A$, such that $\iota(ab) = \iota(b)\iota(a)$ for all $a, b \in A$ and $\iota^2 = \mathrm{id}_A$.

Notice that the requirement that $\iota^2 = \mathrm{id}_A$ implies that ι is a bijection. Further, if ι is an anti-involution on the associative unital $\Bbbk$-algebra A, we may easily show that

$$a\iota(1) = \iota(1)a = a$$

for any $a \in A$, and hence, by uniqueness of the multiplicative identity element of A, we have $\iota(1) = 1$.

Note that some authors, in particular König and Xi, often use the term *involution* instead of anti-involution.

1.1 First definition and examples

We begin with the original definition of a cellular algebra.

Definition 1.2. (Graham and Lehrer — Definition 1.1 in [5]) A **cellular algebra** over the field $\Bbbk$ is an associative unital $\Bbbk$-algebra A equipped with a tuple (Λ, M, C, ι) of **cellular data**, such that

(C1) The set Λ is finite and non-empty, with a partial order $\leq$, and for each $\lambda \in \Lambda$, $M(\lambda)$ is a finite set. Further, C is a family of mappings C^λ, indexed by the elements λ of Λ, such that

$$C^\lambda : M(\lambda) \times M(\lambda) \longrightarrow A$$

and such that the collection of all elements $C^\lambda(S,T)$ for all $\lambda \in \Lambda$ and all $S, T \in M(\lambda)$ is a basis of A. We shall henceforth write the image of $(S,T) \in M(\lambda) \times M(\lambda)$ under C^λ as $C^\lambda_{S,T}$.

(C2) The map ι is an anti-involution on A, and we have $\iota(C^\lambda_{S,T}) = C^\lambda_{T,S}$ for all $\lambda \in \Lambda$ and all $S, T \in M(\lambda)$.

(C3) For any $a \in A$ and any basis element $C^\lambda_{S,T}$ we have

$$aC^\lambda_{S,T} = \sum_{U \in M(\lambda)} r_a(U,S) C^\lambda_{U,T} \; + \; L$$

where the coefficients $r_a(U,S) \in \Bbbk$ are independent of T, and L is a $\Bbbk$-linear combination of basis elements $C^\mu_{V,W}$ where $\mu < \lambda$ and $V, W \in M(\mu)$.

Note that Graham and Lehrer did not explicitly require that the set Λ be finite or non-empty. However, our convention that a $\Bbbk$-algebra must have $\Bbbk$-dimension at least one implies that Λ is non-empty. Further, since we shall only be interested in finite-dimensional $\Bbbk$-algebras, we lose no generality by assuming Λ to be finite (indeed, König and Xi introduced this requirement when they recalled this definition in [8]). Also, Graham and Lehrer write their anti-involution as $* : a \mapsto a^*$.

If A is as in Definition 1.2, then the basis of A consisting of all elements $C^\lambda_{S,T}$ is called a **cellular basis** of A. We shall presently illustrate this somewhat technical-looking definition with examples, but first we shall follow Graham and Lehrer in [5] and give an equation describing products of the form $C^\lambda_{S,T}a$ which parallels the equation in (C3) for products $aC^\lambda_{S,T}$. Let A be a cellular algebra as in Definition 1.2. Let $a \in A$, $\lambda \in \Lambda$ and $S, T \in M(\lambda)$. By the properties of ι given in (C2), we have

$$\begin{aligned} C^\lambda_{S,T}a &= \iota^2 \left(C^\lambda_{S,T}a \right) \\ &= \iota \left(\iota\left(a\right) C^\lambda_{T,S} \right) \\ &= \iota \left(\sum_{U \in M(\lambda)} r_{\iota(a)}(U,T) C^\lambda_{U,S} \; + \; L' \right) \end{aligned}$$

by (C3), where L' is a $\Bbbk$-linear combination of basis elements $C^\mu_{V,W}$ for $\mu < \lambda$

and $V, W \in M(\mu)$. Therefore,

$$C^\lambda_{S,T} a = \sum_{U \in M(\lambda)} r_{\iota(a)}(U, T) C^\lambda_{S,U} \; + \; \iota\left(L'\right).$$

Now by the linearity of ι and the fact that, for any $\mu \in \Lambda$ and $V, W \in M(\mu)$, we have $\iota\left(C^\mu_{V,W}\right) = C^\mu_{W,V}$, we see that $\iota(L')$ is again a $\Bbbk$-linear combination of basis elements $C^\mu_{V,W}$ for $\mu < \lambda$ and $V, W \in M(\mu)$. We have thus proved that, for any $a \in A$, $\lambda \in \Lambda$ and $S, T \in M(\lambda)$, we have

$$(\mathrm{C3})' \qquad C^\lambda_{S,T} a = \sum_{U \in M(\lambda)} r_{\iota(a)}(U, T) C^\lambda_{S,U} \; + \; L$$

where L is a $\Bbbk$-linear combination of basis elements $C^\mu_{V,W}$ with $\mu < \lambda$ and $V, W \in M(\mu)$ (which will in general be different from the element L in the expansion of the product $aC^\lambda_{S,T}$ given by (C3)). Notice that the coefficients $r_{\iota(a)}(U, T)$ are independent of S.

We shall now give some well-known examples of cellular algebras. We begin with a trivial example: the field $\Bbbk$ is a cellular algebra over itself with respect to the tuple of data (Λ, M, C, ι), where $\Lambda = \{1\}$, $M(1) = \{1\}$, $C^1_{1,1} = 1$ and ι is the identity map on $\Bbbk$. We shall call this the **trivial cellular algebra** over $\Bbbk$.

Our second example of a cellular algebra is the algebra $M_n(\Bbbk)$ of $n \times n$ matrices over $\Bbbk$. We define:

- the set Λ to be $\{n\}$ with the trivial ordering;
- the set $M(n)$ to be $\{1, \ldots, n\}$;
- the matrix $C^n_{i,j}$ (where $i, j \in \{1, \ldots, n\}$) to be the elementary $n \times n$ matrix $E_{i,j}$ which has (i, j)-th entry 1 and all other entries 0;
- the map ι to take a matrix to its transpose.

Proposition 1.3. *$M_n(\Bbbk)$ is a cellular algebra with respect to the data (Λ, M, C, ι).*

Proof. Condition (C1) is immediate, and (C2) follows by well-known properties of the transpose matrix. To prove (C3), we first recall that

$$E_{l,m}E_{i,j} = \delta_{mi}E_{l,j}$$

for any $l, m, i, j \in \{1, \dots, n\}$ (where δ_{mi} is the Kronecker delta). Let X be a matrix in $M_n(\Bbbk)$ and $i, j \in \{1, \dots, n\} = M(n)$. Then we have

$$X = \sum_{l=1}^{n}\sum_{m=1}^{n} x_{l,m}E_{l,m}$$

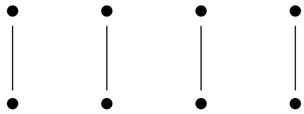
where the coefficient $x_{l,m} \in \Bbbk$ is the (l, m)-th entry of X. Thus,

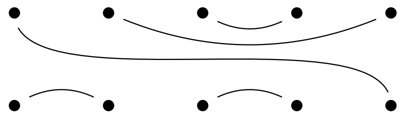
$$\begin{aligned}
XC^n_{i,j} &= XE_{i,j} \\
&= \left(\sum_{l=1}^{n}\sum_{m=1}^{n} x_{l,m}E_{l,m}\right) E_{i,j} \\
&= \sum_{l=1}^{n}\sum_{m=1}^{n} x_{l,m}E_{l,m}E_{i,j} \\
&= \sum_{l=1}^{n}\sum_{m=1}^{n} x_{l,m}\delta_{mi}E_{l,j} \\
&= \sum_{l=1}^{n} x_{l,i}E_{l,j} \\
&= \sum_{l\in M(n)} x_{l,i}C^n_{l,j}
\end{aligned}$$

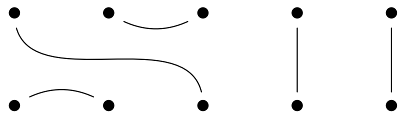
which is of the form required by (C3), because the coefficients $x_{l,i}$ are independent of j. □

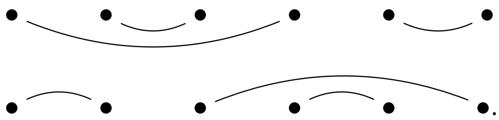
Our third example of a cellular algebra, the *Temperley-Lieb algebra*, has a rather more interesting cellular structure. Before we can define this well-known algebra, we must define the notion of a *planar diagram*.

Let r be a positive integer. A **planar diagram** with $2r$ nodes consists of two rows of r **nodes**, one above the other, and exactly r **edges** between the nodes, such that each node is connected via an edge to exactly one other node (which may be either on the same row or the other row), with the additional restrictions that no two edges are allowed to cross, and the edges must lie entirely within the rectangular area between the rows. For example, the following are all planar diagrams, with $r = 4, 5, 5$, and 6, respectively:

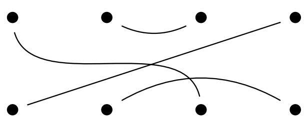


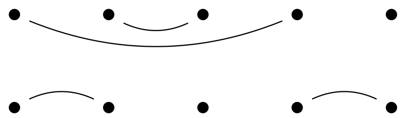


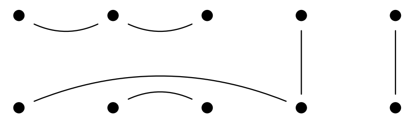


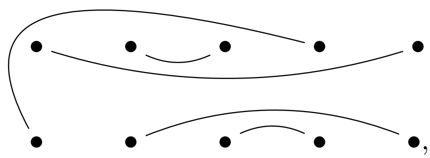


On the other hand, the following are *not* planar diagrams:

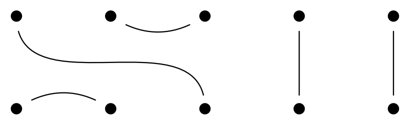




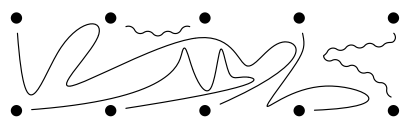




since the first has edges which cross, the second contains nodes which are not connected to any other nodes, the third contains nodes which are connected to more than one other node, and the fourth has an edge which passes outside the area between the rows. Because each node of a planar diagram is connected to exactly one other node, any planar diagram defines a partition of its nodes into pairs. We consider any two planar diagrams with the same number of nodes to be equal if they define the same partition of their nodes, regardless of the exact shape of their edges. Intuitively, we only care about which nodes of a planar diagram are connected, not about the path taken by the edges between them. Thus, for example, the planar diagrams



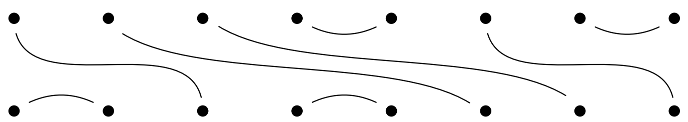
and



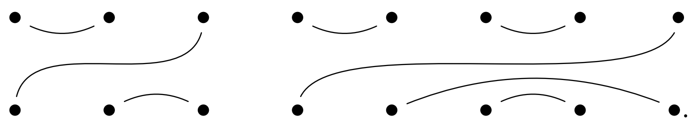
are considered to be equal.

Now fix a positive integer r and some $\delta \in \Bbbk$, and let $\mathrm{TL}_{\Bbbk}(r, \delta)$ denote the $\Bbbk$-vector space with basis the set of all planar diagrams with $2r$ nodes. To define a multiplication on $\mathrm{TL}_{\Bbbk}(r, \delta)$, it is enough to define the product of any pair of planar diagrams in $\mathrm{TL}_{\Bbbk}(r, \delta)$.

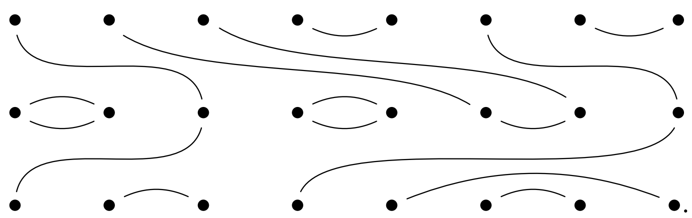
We shall first give an example of how such a product is computed, which will hopefully clarify the subsequent formal definition. So take $r = 8$ and let d_1 be the diagram



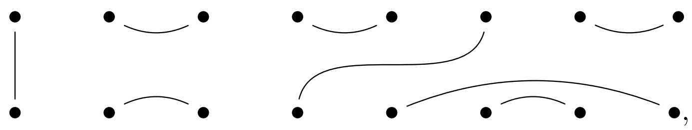
and d_2 be the diagram



To compute the product $d_1 d_2$, we first *concatenate* the two planar diagrams into a single diagram with three rows of 8 nodes, by first drawing d_1 and then drawing d_2 immediately below it, using the nodes on the bottom row of d_1 as the nodes of the top row of d_2. We thus obtain the *concatenated diagram*


(1.1)

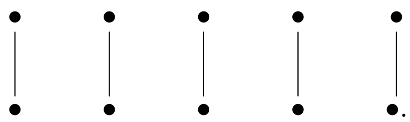
We now modify this concatenated diagram by firstly deleting each of the nodes on the middle row and joining together the two ends of the edges which meet at that node, and secondly removing the two "closed loops" which are thus formed. The diagram we obtain is equivalent to the planar diagram



which we shall call p. Thus we have taken our two planar diagrams d_1 and d_2, and combined them to produce a third planar diagram p. However, we do not wish to completely ignore the two closed loops in the concatenated diagram, and so we define the product $d_1 d_2$ to be the scalar multiple $\delta^2 p$ of p in $\mathrm{TL}_{\Bbbk}(r, \delta)$, with the factor δ^2 occurring because we have *two* closed loops.

Returning to the general case, let d_1 and d_2 be planar diagrams with $2r$ nodes. To compute the product $d_1 d_2$, we follow the procedure described in the above example. We **concatenate** the two planar diagrams into a single **concatenated diagram** with three rows of r nodes, by first drawing d_1 and then drawing d_2 immediately below it, using the nodes on the bottom row of d_1 as the nodes of the top row of d_2. We may consider the edges and nodes in this concatenated diagram to be grouped together to form **paths**, each of which consists of one or more edges linked end-to-end with nodes between them; these paths may be **open paths** with two "ends", each consisting of a node connected to only one edge, or else they may be **closed loops** where each node is connected to exactly two edges (for example, the concatenated diagram (1.1) has two closed loops and eight open paths). Next we modify the concatenated diagram: firstly we remove the closed loops, and then for each open path we replace all the edges and nodes of the path, except the two end nodes, with a single edge. We thus obtain a diagram with two rows of r nodes and r edges connecting them, which we may see must in fact be a planar diagram. As above, we shall call this planar diagram p. We define the product $d_1 d_2$ to be $\delta^n p$, where $n \geq 0$ is the number of closed loops in the concatenated diagram (and where δ^0 is to be interpreted as 1 for any value of δ in $\Bbbk$).

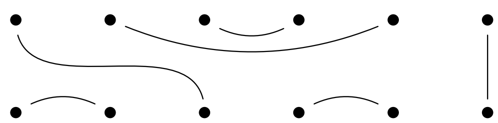
The product which we have now defined on $\mathrm{TL}_{\Bbbk}(r,\delta)$ may easily be seen to be associative on planar diagrams, and hence on the whole of $\mathrm{TL}_{\Bbbk}(r,\delta)$. Further, let e be the planar diagram where each node in the top row is connected to the node directly below it on the bottom row, so that for example if $r = 5$ then e is



It is easy to see that e is a two-sided identity for the multiplication on $\mathrm{TL}_{\Bbbk}(r,\delta)$. We have now established that $\mathrm{TL}_{\Bbbk}(r,\delta)$ is an associative, unital $\Bbbk$-algebra when equipped with this multiplication, which we shall call the **Temperley-Lieb** algebra with parameters r and δ. Note that $\mathrm{TL}_{\Bbbk}(r,\delta)$ is not commutative for $r \geq 3$.

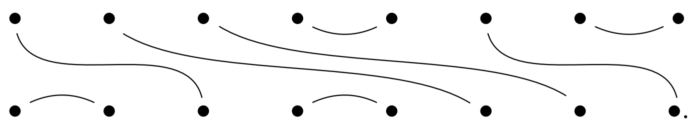
We shall next equip $\mathrm{TL}_{\Bbbk}(r,\delta)$ with a cellular structure. In order to do this, we shall first introduce some further definitions and ideas about the structure of planar diagrams.

Firstly, a **through string** of a planar diagram is an edge which connects a node in the top row to a node in the bottom row. For example, the planar diagram



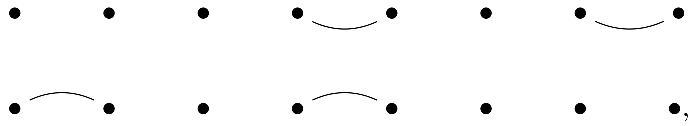
has two through strings. If s is a through string of some planar diagram d, connecting the node x on the top row of d to the node y on the bottom row, then we shall call x and y the **northern node** and the **southern node** of s, respectively. Any edge in a planar diagram which is not a through string will be called a **horizontal edge**.

Next, consider the planar diagram



(1.2)

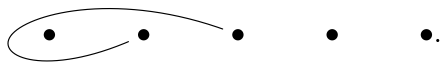
If we erase the through strings of this planar diagram, we obtain a new diagram



(1.3)

which consists of two rows of nodes, where some of the nodes in each row are linked together in pairs by horizontal edges. However, we may recover the original planar diagram (1.2) from the "erased" diagram (1.3) by simply drawing the through strings back in again; the restriction that no two edges may cross in a planar diagram means that there is only one way of drawing four through strings into the diagram (1.3) to yield a valid planar diagram. Thus we may regard the operation of erasing the through strings of a planar diagram as a way of "splitting" it into two "half planar diagrams" (each consisting of a single row of nodes where some of the nodes are connected in pairs by horizontal edges); these "half planar diagrams" may then be "reconnected" by drawing in through strings in a unique way to yield the original planar diagram. We shall now state these ideas more precisely.

Formally, if r is a positive integer and l is an integer with $0 \leq l \leq r$ such that $r - l$ is even, then a **half planar diagram** with r nodes and l **free nodes** is a row of r nodes and exactly $\frac{r-l}{2}$ edges between the nodes, such that each node is the end point of at most one edge; it follows that exactly l of the nodes are not an end point of any edge — we shall call these nodes the **free nodes** of the half planar diagram. Further, we require that no two edges may cross, that two nodes may not be connected if there is a free node between them, and that no edge may cross the (infinitely extended) line defined by the row of nodes. So for example, the following are *not* valid half planar diagrams:

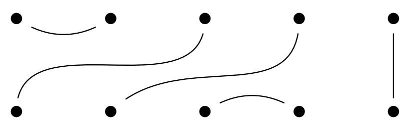


Indeed, the first diagram has edges which cross; in the second diagram, the first and fifth node are connected by an edge, but the second node, which is free, lies between them; the third diagram contains an edge which crosses the line defined by the row of nodes. As with planar diagrams, we consider two half planar diagrams with the same number of nodes and the same number of free nodes to be equal if their free nodes are in the same positions and the edges of both half planar diagrams give rise to the same partition into pairs of their non-free nodes: the exact shape of the edges is not important. In particular, we are free to draw the edges of a half planar diagram either above or below the row of nodes, so that for example

and

are considered to be the same half planar diagram; we shall make frequent use of this freedom below.

Returning to our above idea of splitting a planar diagram into two half planar diagrams, we see that indeed if we take a planar diagram d and delete its through strings, then we shall obtain two half planar diagrams, one from the top row of d (and its associated horizontal edges), and one from the bottom. We define the half diagrams so formed to be the **top** and **bottom** of d, respectively. For example, the planar diagram



has top

and bottom

Note that the number of free nodes in both the top and bottom of a planar diagram will always be equal to the number of through strings of the planar diagram.

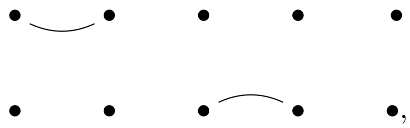
Conversely, if we are given a pair of half planar diagrams with the same number of nodes and the same number of free nodes, we may place one of them above the other and then connect them by drawing in through strings to form a single planar diagram. For example, consider the half planar diagrams obtained in the previous example, and for convenience let us call them S and T, say

$S =$ 

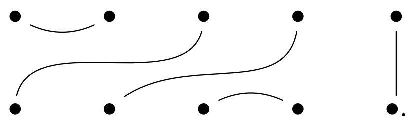
and

$T =$ 

We can combine them together by first drawing S above T



and then (going along the two rows of nodes from left to right) adding edges to join the first free node of S to the first free node of T, the second free node of S to the second free node of T, and so on, until each free node of S is joined via an edge to the corresponding free node of T, thus forming a planar diagram:



Note that, due to the restriction that edges may not cross in a planar diagram, this is the *unique* planar diagram with top S and bottom T; there is no other way to draw in three through strings linking the free nodes of S to the free nodes of T without violating this restriction. Of course, the planar diagram we have obtained is the same as the original planar diagram which we split above to obtain S and T; notice too that if we had started by drawing T above S, we would have obtained a different planar diagram. It is easy to see how the above procedure may be applied to any pair of half planar diagrams to yield a unique planar diagram, provided that they have the same number of nodes and the same number of free nodes; the number of through strings in the resulting planar diagram is clearly the same as the number of free nodes in each of the initial half planar diagrams. Further, it is clear that the two operations of splitting a planar diagram into its top and bottom on the one hand, and connecting two half planar diagrams to yield a planar diagram on the other, are mutually inverse. We summarise this in a lemma:

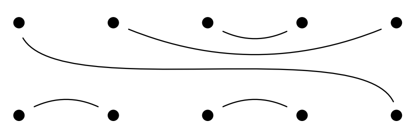
Lemma 1.4. *Let r be a positive integer, and l an integer $0 \leq l \leq r$ such that $r - l$ is even. Then there is a bijective correspondence between planar diagrams with $2r$ nodes and l through strings on the one hand, and pairs of half planar diagrams with r nodes and l free nodes on the other. This correspondence is witnessed by the operation of splitting a planar diagram with $2r$ nodes and l through strings into its top and bottom, and by its inverse operation of connecting two half planar diagrams with r nodes and l free nodes (in a unique way) to yield a planar diagram.*

The next result explains how through strings of planar diagrams interact with the operation of multiplication on $\mathrm{TL}_{\Bbbk}(r, \delta)$.

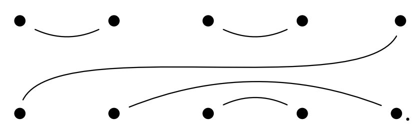
Lemma 1.5. *Let d_1 and d_2 be planar diagrams in $\mathrm{TL}_{\Bbbk}(r, \delta)$ with l_1 and l_2 through strings respectively, and let p be the planar diagram formed when computing the product $d_1 d_2$, as explained above, so that $d_1 d_2 = \delta^n p$ for some non-negative integer n. Then the number of through strings in p is at most $\min(l_1, l_2)$, and furthermore this number depends only on the bottom of d_1 and the top of d_2: if d_1' and d_2' are planar diagrams such that the bottom of d_1' equals the bottom of d_1 and the top of d_2' equals the top of d_2, then the number of through strings in the planar diagram p' formed when computing the product $d_1' d_2'$ is equal to the number of through strings in p.*

Proof. Recall from the above definition of the product $d_1 d_2$ that each edge in the planar diagram p arises from an open path in the concatenated diagram formed by placing d_1 above d_2. It is easy to see that each through string of p must arise from an open path with a through string of d_1 at one end and a through string of d_2 at the other, and that any through string of either d_1 or d_2 can be part of at most one such open path. Thus p has at most $\min(l_1, l_2)$ through strings. Further, any given pair of through strings s_1 in d_1 and s_2 in d_2 form the ends of such an open path if and only if, in the concatenated diagram, the southern node of s_1 is linked via a "chain" of horizontal edges to the northern node of s_2 (note that this "chain" will be empty if the southern node of s_1 equals the northern node of s_2). Thus the number of through strings of p depends only on the arrangement of the southern nodes of the through strings of d_1, the arrangement of the northern nodes of the through strings of d_2, and the arrangement of the horizontal edges on the bottom row of d_1 and the top row of d_2; it is clear that all of these are completely determined by the bottom of d_1 and the top of d_2. $\square$

Next, let us note that if d is any planar diagram then the diagram formed by reflecting d in the line parallel to and halfway between its two rows of nodes is again planar; let us henceforth refer to this operation as "flipping d upside-down". For example, if d is



then flipping d upside down yields the planar diagram



Finally, for any positive integer r let us define I_r to be the set

$$\{r, r-2, r-4, \ldots, 1 \text{ or } 0\}$$

with the natural order. In other words, I_r is the set of all l with $0 \leq l \leq r$ such that $r - l$ is even.

We can now define a cellular structure on the Temperley-Lieb algebra $\mathrm{TL}_{\Bbbk}(r, \delta)$ as follows:

- let Λ be I_r;

- for each $l \in \Lambda$, let $M(l)$ be the set of all half planar diagram with r nodes and l free nodes;

- for each $l \in \Lambda$ and each pair of half diagrams $S, T \in M(l)$, let $C^l_{S,T}$ to be the unique planar diagram formed by putting S above T and connecting them as in Lemma 1.4 (notice that $C^l_{S,T}$ thus has exactly l through strings);

- let ι be the $\Bbbk$-linear map defined on $\mathrm{TL}_{\Bbbk}(r, \delta)$ by mapping each planar diagram d to the planar diagram obtained by flipping d upside down, as explained above.

Proposition 1.6. *(Graham and Lehrer — Example 1.4 in [5]) The data* (Λ, M, C, ι) *exhibits* $\mathrm{TL}_{\Bbbk}(r, \delta)$ *as a cellular algebra.*

Proof. We must check conditions (C1), (C2) and (C3).

(C1): We need only show that the collection of all elements $C^l_{S,T}$ for $l \in \Lambda$ and $S, T \in M(l)$ is a basis of $\mathrm{TL}_{\Bbbk}(r, \delta)$. Indeed, it is the set of all planar diagrams in $\mathrm{TL}_{\Bbbk}(r, \delta)$, by Lemma 1.4.

(C2): It is immediate from the definition of ι that $\iota^2(d) = d$ for any planar diagram d, and hence that ι^2 is the identity map on $\mathrm{TL}_{\Bbbk}(r, \delta)$. This in turn implies that ι must be a bijection. The fact that $\iota(d_1 d_2) = \iota(d_2)\iota(d_1)$ for any planar diagrams d_1 and d_2 follows from the definition of the product of planar diagrams: flipping the 3-row concatenated diagram formed by placing d_1 above d_2 upside down yields the same result as flipping both d_1 and d_2 upside down and then concatenating them in reverse order. Finally, we see from the definition of $C^l_{S,T}$ that $\iota\left(C^l_{S,T}\right) = C^l_{T,S}$.

(C3): The fact that planar diagrams form a basis of $\mathrm{TL}_{\Bbbk}(r, \delta)$ means that it is enough to prove that the product

$$dC^l_{S,T}$$

has the required form for any $C^l_{S,T}$ and any planar diagram d. Indeed, let d have m through strings. Since $C^l_{S,T}$ is a planar diagram, we know from the definition of multiplication in $\mathrm{TL}_{\Bbbk}(r,\delta)$ that

$$dC^l_{S,T} = \delta^n p$$

for some $n \geq 0$ and some planar diagram p. From the proof of (C1) above, it follows that $p = C^{l'}_{S',T'}$ for some $l' \in \Lambda$ and some $S', T' \in M(l')$, so that

$$dC^l_{S,T} = \delta^n C^{l'}_{S',T'}.$$

Since l' is the number of through strings of $C^{l'}_{S',T'}$, we have by Lemma 1.5 that $l' \leq l$, and moreover that l' depends only on the bottom of d and the top of $C^l_{S,T}$, which is of course S. So l' is independent of T. We consider the cases $l' < l$ and $l' = l$ separately. If $l' < l$ then we set $r_d(U,S) = 0$ for all $U \in M(l)$ and we define

$$L = dC^l_{S,T} = \delta^n C^{l'}_{S',T'}.$$

If $l' = l$, then T' is a half planar diagram with l free nodes, and further by considering the edges between nodes on the bottom row of the concatenated diagram formed by joining d above $C^l_{S,T}$, we see that every edge in T must be "inherited" by T', and hence we must have $T = T'$, since T also has l free nodes. Further, the index n is the number of closed loops in the concatenated diagram formed from d and $C^l_{S,T}$, and the edges contained in such closed loops come solely from the bottom of d and the top of $C^l_{S,T}$, so n is independent of T. Thus for the case $l' = l$, we define $L = 0$ and for each $U \in M(l)$ we let

$$r_d(U,S) = \begin{cases} \delta^n & \text{if } U = S' \\ 0 & \text{otherwise.} \end{cases}$$

With the above definitions, we see that for any value of l', we have

$$dC^l_{S,T} = \sum_{U \in M(l)} r_d(U,S) C^l_{U,T} \,+\, L$$

and moreover that the coefficients $r_d(U,S)$ are indeed independent of T. $\square$

1.2 Cell modules and the structure of a cellular algebra

We shall now continue to review the work of Graham and Lehrer in [5] by defining the *cell modules* of a cellular algebra.

Let A be as in Definition 1.2. For each $\lambda \in \Lambda$, we define

$$A(\lambda) = \mathrm{span}_{\Bbbk}\{C^{\lambda}_{S,T} : S, T \in M(\lambda)\},$$

a vector subspace of A. From (C1), we have

$$A = \bigoplus_{\lambda \in \Lambda} A(\lambda),$$

([5], Lemma 2.2, (ii)) and so we see that the cellular structure affords a decomposition of the cellular algebra as a $\Bbbk$-vector space. For example, in the case of the Temperley-Lieb algebra $\mathrm{TL}_{\Bbbk}(r, \delta)$, recall that Λ is the set I_r; for each $l \in \Lambda$, $A(l)$ is then the $\Bbbk$-span of all planar diagrams with exactly l through strings.

However, as A is not just a vector space, but rather an algebra, we are more interested in ideals of A than subspaces. In general, $A(\lambda)$ is not an ideal of A for $\lambda \in \Lambda$. However, we can associate two two-sided ideals of A to each $\lambda \in \Lambda$. Indeed, for each $\lambda \in \Lambda$, let

$$A(< \lambda) = \bigoplus_{\mu < \lambda} A(\mu)$$

and

$$A(\leq \lambda) = A(< \lambda) \oplus A(\lambda).$$

Then we have

$$A(< \lambda) = \mathrm{span}_{\Bbbk}\{C^{\mu}_{P,Q} : \mu < \lambda,\ P, Q \in M(\mu)\}$$

and

$$A(\leq \lambda) = \mathrm{span}_{\Bbbk}\{C^{\mu}_{P,Q} : \mu \leq \lambda,\ P, Q \in M(\mu)\}.$$

Further, from (C3) and (C3)$'$ above, we see that $A(< \lambda)$ and $A(\leq \lambda)$ are two-sided ideals of A ([5], Lemma 1.5). For example, in $\mathrm{TL}_{\Bbbk}(r, \delta)$, $A(\leq l)$

is the $\Bbbk$-span of all planar diagrams with *at most* l through strings, while $A(< l)$ is the $\Bbbk$-span of all planar diagrams with *strictly fewer than* l through strings. Now, for any subspace V of the algebra A and $a, b \in A$, let us introduce the notation

$$a \equiv b \pmod{V}$$

to mean that $a - b \in V$ (in other words, the cosets $a + V$ and $b + V$ are equal); it is easy to show that this is an equivalence relation on A. We may thus restate (C3) and (C3)$'$ as follows:

(C3) For any $a \in A$ and any basis element $C^\lambda_{S,T}$ we have

$$aC^\lambda_{S,T} \equiv \sum_{U \in M(\lambda)} r_a(U,S) C^\lambda_{U,T} \qquad \big(\text{mod } A(<\lambda)\big)$$

where the coefficients $r_a(U,S) \in \Bbbk$ are independent of T.

(C3)$'$ For any $a \in A$ and any basis element $C^\lambda_{S,T}$ we have

$$C^\lambda_{S,T} a \equiv \sum_{U \in M(\lambda)} r_{\iota(a)}(U,T) C^\lambda_{S,U} \qquad \big(\text{mod } A(<\lambda)\big).$$

We shall make frequent use of these restatements below. Further, from the fact that $\iota\left(C^\lambda_{S,T}\right) = C^\lambda_{T,S}$ for all $S, T \in M(\lambda)$, we note that

$$\iota\big(A(\lambda)\big) = A(\lambda)$$

for all $\lambda \in \Lambda$, and hence that

$$\iota\big(A(\leq \lambda)\big) = A(\leq \lambda)$$

and

$$\iota\big(A(< \lambda)\big) = A(< \lambda),$$

that is, the subspace $A(\lambda)$ and the ideals $A(\leq \lambda)$ and $A(< \lambda)$ are *ι-invariant* for all $\lambda \in \Lambda$.

Now for any $\lambda \in \Lambda$, we have

$$A(< \lambda) \subseteq A(\leq \lambda),$$

and so the quotient $A(\leq \lambda)/A(< \lambda)$ is a two-sided ideal of the quotient algebra $A/A(< \lambda)$, and hence an A-bimodule, which we shall denote by $Q(\lambda)$ ([5], Definition 1.6). Notice that $Q(\lambda)$ is thus isomorphic as a $\Bbbk$-vector space to $A(\lambda)$, via the linear map given by

$$\begin{aligned} A(\lambda) &\longrightarrow Q(\lambda) \\ C^{\lambda}_{S,T} &\longmapsto C^{\lambda}_{S,T} + A(< \lambda). \end{aligned}$$

Let us now fix some $\lambda \in \Lambda$, and examine the structure of $Q(\lambda)$ in more detail. We shall first show how $Q(\lambda)$ may be decomposed as a left A-module into a direct sum of isomorphic copies of a particular left A-module. For $a \in A$, let us write $\overline{a}$ for the element $a + A(< \lambda)$ of the quotient algebra $A/A(< \lambda)$. By the definition of $A(\lambda)$, we know that

$$A(\lambda) = \bigoplus_{T \in M(\lambda)} \mathrm{span}_{\Bbbk}\{C^{\lambda}_{S,T} : S \in M(\lambda)\}$$

and so

$$Q(\lambda) = \bigoplus_{T \in M(\lambda)} \Delta^{\lambda}(T) \tag{1.4}$$

where

$$\Delta^{\lambda}(T) = \mathrm{span}_{\Bbbk}\{\overline{C}^{\lambda}_{S,T} : S \in M(\lambda)\}.$$

Recall from (C3) above that for any $a \in A$ and any $S, T \in M(\lambda)$ we have

$$aC^{\lambda}_{S,T} \equiv \sum_{U \in M(\lambda)} r_a(U,S) C^{\lambda}_{U,T} \qquad \big(\mathrm{mod}\ A(< \lambda)\big)$$

where the coefficients $r_a(U,S) \in \Bbbk$ are independent of T. So for the left action of A on $Q(\lambda)$, we have

$$a\overline{C}^{\lambda}_{S,T} = \sum_{U \in M(\lambda)} r_a(U,S) \overline{C}^{\lambda}_{U,T} \tag{1.5}$$

and so each $\Delta^{\lambda}(T)$ is a left A-submodule of $Q(\lambda)$. Further, the fact that the coefficients $r_a(U,S)$ are independent of T implies that for any $T, T' \in M(\lambda)$, the $\Bbbk$-linear bijection from $\Delta^{\lambda}(T)$ to $\Delta^{\lambda}(T')$ induced by mapping

$$\overline{C}^{\lambda}_{S,T} \longmapsto \overline{C}^{\lambda}_{S,T'}$$

for each $S \in M(\lambda)$ is in fact an isomorphism of left A-modules, and so the modules $\Delta^\lambda(T)$ for $T \in M(\lambda)$ are all mutually isomorphic as left A-modules. To emphasise the fact that the isomorphism type of $\Delta^\lambda(T)$ is independent of T, let us define Δ^λ to be the $\Bbbk$-vector space with a basis consisting of symbols C_S for all $S \in M(\lambda)$, with a left action of A on Δ^λ given by

$$aC_S = \sum_{U \in M(\lambda)} r_a(U, S) C_U \tag{1.6}$$

for $a \in A$. Then Δ^λ is a left A-module, and we have $\Delta^\lambda \cong \Delta^\lambda(T)$ for all $T \in M(\lambda)$. This module Δ^λ is called the **cell module labelled by** λ. Note that in [5], Graham and Lehrer denote this module by $W(\lambda)$ ([5], Definition 2.1); our notation Δ^λ is based on that of König and Xi in [8].

These cell modules Δ^λ for $\lambda \in \Lambda$ play a fundamental role in the representation theory of the cellular algebra A, and we shall make extensive use of them below. For the moment, notice that we have shown that, as a left A-module, $Q(\lambda)$ is a direct sum of $|M(\lambda)|$ isomorphic copies of Δ^λ ([8], Proposition 3.3). We shall next show that there is a corresponding decomposition of $Q(\lambda)$ as a right A-module. From the fact that the ideal $A(< \lambda)$ is invariant under ι, we see that ι induces a well-defined anti-involution on the quotient algebra $A/A(< \lambda)$, which we shall also call ι; from the fact that $\iota\left(C^\lambda_{S,T}\right) = C^\lambda_{T,S}$, this new map ι has the property that

$$\iota\left(\overline{C}^\lambda_{S,T}\right) = \overline{C}^\lambda_{T,S} \tag{1.7}$$

for all $S, T \in M(\lambda)$. We apply this map ι to (1.4) to find that

$$\iota\big(Q(\lambda)\big) = \bigoplus_{S \in M(\lambda)} \iota\left(\Delta^\lambda(S)\right),$$

where the sum of the subspaces $\iota\left(\Delta^\lambda(S)\right)$ must indeed be a *direct* sum because ι is a bijection. Now from the definition of $Q(\lambda)$, we may see that

$$\iota\big(Q(\lambda)\big) = Q(\lambda)$$

and so

$$Q(\lambda) = \bigoplus_{S \in M(\lambda)} \iota\left(\Delta^{\lambda}(S)\right).$$

Now by (1.7),

$$\iota\left(\Delta^{\lambda}(S)\right) = \operatorname{span}_{\Bbbk}\{\overline{C}^{\lambda}_{S,T} : T \in M(\lambda)\},$$

and from (C3)$'$, we know that any $a \in A$, $\lambda \in \Lambda$ and $S, T \in M(\lambda)$, we have

$$C^{\lambda}_{S,T} a \equiv \sum_{U \in M(\lambda)} r_{\iota(a)}(U,T) C^{\lambda}_{S,U} \qquad \big(\operatorname{mod} A(< \lambda)\big)$$

where the coefficients $r_{\iota(a)}(U,T)$ are independent of S. So for the right action of A on $Q(\lambda)$, we have

$$\overline{C}^{\lambda}_{S,T} a = \sum_{U \in M(\lambda)} r_{\iota(a)}(U,T) \overline{C}^{\lambda}_{S,U}. \tag{1.8}$$

Hence, each $\iota\left(\Delta^{\lambda}(S)\right)$ is a right A-submodule of $Q(\lambda)$. Further, each $\iota\left(\Delta^{\lambda}(S)\right)$ is isomorphic to the right A-module which can be defined as the $\Bbbk$-vector space with a basis consisting of symbols C_T for all $T \in M(\lambda)$ and a right action of A given by

$$C_T a = \sum_{U \in M(\lambda)} r_{\iota(a)}(U,T) C_U \tag{1.9}$$

for $a \in A$. By an abuse of notation, we shall denote this right A-module by $\iota\left(\Delta^{\lambda}\right)$ (again, this is based on the notation of König and Xi in [8]; in [5], Graham and Lehrer denote this module by $W(\lambda)^*$). We have shown that, as a right A-module, $Q(\lambda)$ is a direct sum of $|M(\lambda)|$ isomorphic copies of $\iota\left(\Delta^{\lambda}\right)$. Note that while Δ^{λ} and $\iota\left(\Delta^{\lambda}\right)$ are formally equal as $\Bbbk$-vector spaces (since we used the same basis to define both of them), their respective actions of A do *not* in general commute and so we have not defined an A-bimodule.

Now if we have a left A-module X, then for any $a \in A$ and any $x \in X$, let us define xa to be the element $\iota(a)x$ of X; it is easy to use the fact that ι is an anti-involution on A to show that this defines a right A-module structure on X, and we shall denote this right A-module by $\iota(X)$. It is now easy to show that this definition agrees with our above definition of $\iota\left(\Delta^\lambda\right)$, in the sense that $\iota\left(\Delta^\lambda\right)$ is exactly the right A-module obtained from Δ^λ by defining

$$xa = \iota(a)x \tag{1.10}$$

for any $a \in A$ and any $x \in \Delta^\lambda$. Conversely, it is also easy to see that Δ^λ may be considered to be the left A-module obtained from the right A-module $\iota\left(\Delta^\lambda\right)$ by defining

$$ax = x\iota(a) \tag{1.11}$$

for any $a \in A$ and any $x \in \iota\left(\Delta^\lambda\right)$.

We have now produced decompositions of $Q(\lambda)$ as both a left and a right A-module. But $Q(\lambda)$ is an A-bimodule, and we would like a decomposition of $Q(\lambda)$ which respects this. Such a decomposition is provided by the next proposition, which will also help to motivate the second definition of a cellular algebra (due to König and Xi) which will be given in the next section. Firstly, note that the $\mathbb{k}$-vector space tensor product $\Delta^\lambda \otimes \iota\left(\Delta^\lambda\right)$ is an A-bimodule, with left and right actions of A well-defined on pure tensors by $a(x \otimes y) = (ax) \otimes y$ and $(x \otimes y)a = x \otimes (ya)$ for any $a \in A$, any $x \in \Delta^\lambda$ and any $y \in \iota\left(\Delta^\lambda\right)$ (and recall that we are writing $\otimes$ for $\otimes_{\mathbb{k}}$).

Proposition 1.7. *Let A be a cellular algebra as in Definition 1.2. For each $\lambda \in \Lambda$, the $\mathbb{k}$-linear map*

$$\alpha : Q(\lambda) \longrightarrow \Delta^\lambda \otimes \iota\left(\Delta^\lambda\right)$$

given by

$$\alpha : \overline{C}^\lambda_{S,T} \longmapsto C_S \otimes C_T$$

is an isomorphism of A-bimodules (Graham and Lehrer, [5], Lemma 2.2, (i)). Further, recall that Δ^λ and $\iota\left(\Delta^\lambda\right)$ are equal as vector spaces, so that for any *pure tensor $x \otimes y$ in $\Delta^\lambda \otimes \iota\left(\Delta^\lambda\right)$, the pure tensor $y \otimes x$ is also an element of $\Delta^\lambda \otimes \iota\left(\Delta^\lambda\right)$. We may show that the map*

$$\phi : \Delta^\lambda \otimes \iota\left(\Delta^\lambda\right) \longrightarrow \Delta^\lambda \otimes \iota\left(\Delta^\lambda\right)$$

defined on pure tensors by

$$\phi : x \otimes y \mapsto y \otimes x$$

is indeed a well-defined $\Bbbk$-linear map. Then the diagram

$$\begin{tikzcd}
Q(\lambda) \arrow[r, "\alpha"] \arrow[d, "\iota"'] & \Delta^\lambda \otimes \iota\left(\Delta^\lambda\right) \arrow[d, "\phi"] \\
Q(\lambda) \arrow[r, "\alpha"] & \Delta^\lambda \otimes \iota\left(\Delta^\lambda\right)
\end{tikzcd}$$

commutes (König and Xi, in Section 3 of [8]).

Proof. Since the collection of all elements $\overline{C}^\lambda_{S,T}$ for all $S, T \in M(\lambda)$ is a basis of $Q(\lambda)$, and the set of all symbols C_S for $S \in M(\lambda)$ is a basis of both Δ^λ and $\iota\left(\Delta^\lambda\right)$, we see at once that α is a $\Bbbk$-linear bijection. The fact that α preserves both the left and right actions of A on $Q(\lambda)$ follows from Equations (1.5), (1.6), (1.8) and (1.9).

To prove that the diagram

$$\begin{tikzcd}
Q(\lambda) \arrow[r, "\alpha"] \arrow[d, "\iota"'] & \Delta^\lambda \otimes \iota\left(\Delta^\lambda\right) \arrow[d, "\phi"] \\
Q(\lambda) \arrow[r, "\alpha"] & \Delta^\lambda \otimes \iota\left(\Delta^\lambda\right)
\end{tikzcd}$$

commutes, it is enough to prove that

$$\phi\left(\alpha\left(\overline{C}^\lambda_{S,T}\right)\right) = \alpha\left(\iota\left(\overline{C}^\lambda_{S,T}\right)\right)$$

for all $S, T \in M(\lambda)$. Indeed,

$$\begin{aligned}
\phi\left(\alpha\left(\overline{C}^{\lambda}_{S,T}\right)\right) &= \phi\left(C_S \otimes C_T\right) \\
&= C_T \otimes C_S \\
&= \alpha\left(\overline{C}^{\lambda}_{T,S}\right) \\
&= \alpha\left(\iota\left(\overline{C}^{\lambda}_{S,T}\right)\right).
\end{aligned}$$

□

1.3 An alternative definition of cellular algebras

We now turn to a second definition of a cellular algebra, which was introduced by König and Xi in [8], and proved to be equivalent to the definition of Graham and Lehrer. We shall give a slightly more detailed proof of this equivalence than that given in [8].

First, let A be a cellular algebra as in Definition 1.2. Now any partial order on a finite set may be extended (in general non-uniquely) to a total order. Thus we may assume without loss of generality that the order on Λ is in fact a total order. So we may list the elements of Λ in order as

$$\lambda_1 < \lambda_2 < \ldots < \lambda_n$$

where $n = |\Lambda|$, and the two-sided ideals $A(\leq \lambda_i)$ for $i = 1, \ldots, n$ now form a chain

$$0 \subseteq A(\leq \lambda_1) \subseteq A(\leq \lambda_2) \subseteq \ldots \subseteq A(\leq \lambda_n) = A. \tag{1.12}$$

For example, in the case of $\mathrm{TL}_{\Bbbk}(r, \delta)$, the ideals $A(\leq l)$ for $l \in I_r$ form a chain

$$0 \subseteq A(\leq 0) \text{ or } A(\leq 1) \subseteq \ldots \subseteq A(\leq r-2) \subseteq A(\leq r) = A$$

where, as noted above, $A(\leq l)$ is the $\Bbbk$-span of all planar diagrams with at most l through strings.

We now give a definition which captures the structural properties of the ideal $Q(\lambda)$ expressed in Proposition 1.7.

Definition 1.8. (König and Xi — Definition 3.2 in [8]) Let A be a finite-dimensional associative unital algebra over $\Bbbk$ and let ι be an anti-involution on A. A **cell ideal** of A with respect to ι is a two-sided ideal J of A such that $\iota(J) = J$, equipped with a left ideal Δ of A contained in J (which implies that $\iota(\Delta) \subseteq J$, where $\iota(\Delta)$ must be a right ideal of A by the properties of ι) and an isomorphism

$$\alpha : J \longrightarrow \Delta \otimes \iota(\Delta)$$

of A-bimodules which makes the diagram

$$\begin{tikzcd} J \arrow[r, "\alpha"] \arrow[d, "\iota"'] & \Delta\otimes\iota(\Delta) \arrow[d, "x\otimes y \mapsto \iota(y)\otimes\iota(x)"] \\ J \arrow[r, "\alpha"] & \Delta\otimes\iota(\Delta) \end{tikzcd}$$

commute; note that the formula

$$x\otimes y \mapsto \iota(y)\otimes\iota(x)$$

for $x \in \Delta$ and $y \in \iota(\Delta)$ does indeed yield a well-defined $\Bbbk$-linear map from $\Delta\otimes\iota(\Delta)$ to itself.

Proposition 1.9. *Let A be a cellular algebra with cellular data (Λ, M, C, ι), as in Definition 1.2. Let $\lambda \in \Lambda$ and fix an element $X \in M(\lambda)$. Then, with respect to the anti-involution ι induced on $A/A(< \lambda)$ by the anti-involution ι on A, $Q(\lambda)$ is a cell ideal of the quotient algebra $A/A(< \lambda)$ when equipped with the left ideal $\Delta^\lambda(X)$ and the $\Bbbk$-linear map α from $Q(\lambda)$ to*

$$\Delta^\lambda(X) \otimes \iota\left(\Delta^\lambda(X)\right)$$

given by

$$\alpha : \overline{C}^\lambda_{S,T} \longmapsto \overline{C}^\lambda_{S,X} \otimes \overline{C}^\lambda_{X,T}$$

where, as above, we have written $\overline{a}$ for the coset $a + A(< \lambda)$ of $a \in A$ in $A/A(< \lambda)$.

Proof. This is immediate from Proposition 1.7 and the fact that Δ^λ and $\iota\left(\Delta^\lambda\right)$ are isomorphic to $\Delta^\lambda(X)$ and $\iota\left(\Delta^\lambda(X)\right)$ respectively, via the maps given by

$$C_S \longmapsto \overline{C}^\lambda_{S,X}$$

and

$$C_T \longmapsto \overline{C}^\lambda_{X,T}.$$

□

We now introduce the second definition of a cellular algebra.

Definition 1.10. (König and Xi — Definition 3.2 in [8]) A finite-dimensional unital associative $\Bbbk$-algebra A with an anti-involution ι is **cellular** with respect to ι if it can be equipped with a decomposition

$$A = \bigoplus_{j=1}^{n} J'_j$$

of A as a direct sum of vector subspaces J'_j such that $\iota(J'_j) = J'_j$ for each j, and such that if we let $J_j = \bigoplus_{l=1}^{j} J'_l$ then

$$0 = J_0 \subseteq J_1 \subseteq \ldots \subseteq J_n = A$$

is a chain of two-sided ideals of A, and for each $j = 1, \ldots, n$, the quotient J_j/J_{j-1} (which is isomorphic as a vector space to J'_j) is a cell ideal of A/J_{j-1} with respect to the anti-involution induced by ι. Such a chain of ideals is called a **cell chain** for A.

We shall prove that Definition 1.10 is equivalent to the definition of Graham and Lehrer. One direction is now almost immediate.

Theorem 1.11. *(König and Xi, in Section 3 of [8]) Let A be a cellular algebra with cellular data (Λ, M, C, ι), as in Definition 1.2. Extend the partial* *order on Λ to a total order as above, so that we may list the elements of Λ in order as*

$$\lambda_1 < \lambda_2 < \ldots < \lambda_n$$

where $n = |\Lambda|$. Then A is a cellular algebra with respect to ι in the sense of Definition 1.10 when equipped with the subspace decomposition

$$A = \bigoplus_{j=1}^{n} A(\lambda_j).$$

Proof. We have noted above that $\iota\left(A(\lambda_j)\right) = A(\lambda_j)$ for all $\lambda_j \in \Lambda$. Further, we have by the definition of the two-sided ideal $A(\leq \lambda_j)$ that

$$A(\leq \lambda_j) = \bigoplus_{l=1}^{j} A(\lambda_l),$$

and so by (1.12) we have the required chain of two-sided ideals, where each quotient is indeed a cell ideal by Proposition 1.9. □

It remains only to prove the other direction of the equivalence.

Theorem 1.12. *(König and Xi, in Section 3 of [8]) Suppose that A is a cellular algebra in the sense of Definition 1.10 with respect to an anti-involution ι, so that in particular A is equipped with a direct sum decomposition into vector subspaces*

$$A = \bigoplus_{j=1}^{n} J'_j$$

such that $\iota(J'_j) = J'_j$. Then there exists a tuple of data (Λ, M, C, ι) which exhibits A as a cellular algebra in the sense of Definition 1.2, where in particular ι is the original anti-involution and Λ is the set $\{1, \ldots, n\}$ with the natural order.

Proof. For each $j \in \{1, \ldots, n\}$, let $J_j = \bigoplus_{l=1}^{j} J'_l$. Then by Definition 1.10, we have a chain of two-sided ideals of A,

$$0 = J_0 \subseteq J_1 \subseteq \ldots \subseteq J_n = A,$$

and for each $j = 1, \ldots, n$, we have a finite-dimensional left ideal Δ_j of A/J_{j-1} such that $\Delta_j \subseteq J_j/J_{j-1}$ and also an isomorphism

$$\alpha_j : J_j/J_{j-1} \longrightarrow \Delta_j \otimes \iota(\Delta_j)$$

of (A/J_{j-1})-bimodules which gives a commutative diagram as in Definition 1.8.

Let Λ be $\{1, \ldots, n\}$ with the natural ordering. We shall now define the elements M and C of the tuple (Λ, M, C, ι). Indeed, for each $j \in \Lambda$, let us write the coset $a + J_{j-1}$ of $a \in A$ in A/J_{j-1} as $\overline{a}$, so that in particular the map $a \mapsto \overline{a}$ is a $\Bbbk$-linear bijection from J'_j to J_j/J_{j-1}. Now choose elements

$$x_1^j, x_2^j, \ldots, x_{m_j}^j$$

of A such that

$$\overline{x}_1^j, \overline{x}_2^j, \ldots, \overline{x}_{m_j}^j$$

is a basis of Δ_j (where m_j is the $\Bbbk$-dimension of Δ_j). Then let $M(j)$ be $\{1, 2, \ldots, m_j\}$, and for all pairs $s, t \in M(j)$ let $C_{s,t}^j$ be an element of J'_j such that

$$\overline{C}_{s,t}^j = \alpha_j^{-1}(\overline{x}_s^j \otimes \iota(\overline{x}_t^j)).$$

We now check that conditions (C1), (C2), and (C3) of Definition 1.2 hold.

(C1): We need only check that the collection of all elements $C_{s,t}^j$ for all $j \in \Lambda$ and all $s, t \in M(j)$ is a basis of A. Indeed, for any $j \in \Lambda$, it is clear that the collection of all elements $C_{s,t}^j$ for $s, t \in M(j)$ is a basis of J'_j, and since

$$A = \bigoplus_{j=1}^{n} J'_j$$

as a vector space, we see that the elements $C_{s,t}^j$ do indeed form a basis of A.

(C2): It suffices to check that $\iota(C_{s,t}^j) = C_{t,s}^j$ for all $j \in \Lambda$ and all

$s, t \in M(j)$. Indeed, we know that the diagram

$$
\begin{tikzcd}
J_j/J_{j-1} \arrow[r, "\alpha_j"] \arrow[d, "\iota"'] & \Delta_j \otimes \iota(\Delta_j) \arrow[d, "\phi : x\otimes y \mapsto \iota(y)\otimes\iota(x)"] \\
J_j/J_{j-1} \arrow[r, "\alpha_j"] & \Delta_j \otimes \iota(\Delta_j)
\end{tikzcd}
$$

commutes. So

$$
\begin{aligned}
\alpha_j\left(\iota\left(\overline{C}^j_{s,t}\right)\right) &= \phi\left(\alpha_j\left(\overline{C}^j_{s,t}\right)\right) \\
&= \phi\left(\overline{x}^j_s \otimes \iota(\overline{x}^j_t)\right) \\
&= \overline{x}^j_t \otimes \iota(\overline{x}^j_s) \\
&= \alpha_j\left(\overline{C}^j_{t,s}\right),
\end{aligned}
$$

and since α_j is a bijection and $\iota(\overline{a}) = \overline{\iota(a)}$ for any $a \in A$, it follows that

$$\iota\left(C^j_{s,t}\right) = C^j_{t,s}.$$

(C3): Let $j \in \Lambda$, $s, t \in M(j)$ and $a \in A$. Now Δ_j is a left ideal of A/J_{j-1}. So we have

$$\overline{a}\,\overline{x}^j_s = \sum_{u \in M(j)} r_a(u,s)\overline{x}^j_u$$

for some elements $r_a(u,s) \in \Bbbk$, which are of course independent of t. Then we have

$$
\begin{aligned}
\overline{a\,C^j_{s,t}} &= \overline{a}\alpha_j^{-1}\left(\overline{x}^j_s \otimes \iota(\overline{x}^j_t)\right) \\
&= \alpha_j^{-1}\left((\overline{a}\,\overline{x}^j_s) \otimes \iota(\overline{x}^j_t)\right) \\
&= \sum_{u \in M(j)} r_a(u,s)\alpha_j^{-1}\left(\overline{x}^j_u \otimes \iota(\overline{x}^j_t)\right) \\
&= \sum_{u \in M(j)} r_a(u,s)\overline{C}^j_{u,t},
\end{aligned}
$$

from which it follows that

$$aC^j_{s,t} \equiv \sum_{u \in M(j)} r_a(u,s)C^j_{u,t} \pmod{J_{j-1}},$$

as required. $\square$

1.4 Bilinear forms and the simple modules of a cellular algebra

In [5], Graham and Lehrer define a symmetric bilinear form on each cell module of a cellular algebra, and then use this to obtain a complete classification of the simple modules of the cellular algebra. We shall now review the definition of these symmetric bilinear forms (for use in the next chapter), and then briefly describe how they may be used to find the simple modules.

Recall that the cell module Δ^λ associated to λ can be considered to be the $\Bbbk$-vector space with a basis consisting of symbols C_S for all $S \in M(\lambda)$. Thus we may define a $\Bbbk$-valued bilinear form $\langle \cdot\,, \cdot \rangle$ on Δ^λ by simply giving the value of $\langle C_S, C_T \rangle$ for each pair $S, T \in M(\lambda)$. Indeed, choose $S, T \in M(\lambda)$ and let $c = C^\lambda_{T,S}$. Then on the one hand, we have by (C3) that

$$C^\lambda_{T,S} C^\lambda_{T,S} \equiv \sum_{U \in M(\lambda)} r_c(U, T) C^\lambda_{U,S} \pmod{A(< \lambda)},$$

while on the other hand, we have by (C3)$'$ that

$$C^\lambda_{T,S} C^\lambda_{T,S} \equiv \sum_{U \in M(\lambda)} r_{\iota(c)}(U, S) C^\lambda_{T,U} \pmod{A(< \lambda)}.$$

Comparing these two equivalences, we see that we must have

$$C^\lambda_{T,S} C^\lambda_{T,S} \equiv \langle C_S, C_T \rangle C^\lambda_{T,S} \pmod{A(< \lambda)},$$

where $\langle C_S, C_T \rangle \in \Bbbk$ is defined to be the common value of $r_c(T, T)$ and $r_{\iota(c)}(S, S)$; from these values $\langle C_S, C_T \rangle$ we may now define our bilinear form on Δ^λ. Further, if we choose any pair of elements X, Y of $M(\lambda)$, then by applying the above argument using (C3) and (C3)$'$ to the product $C^\lambda_{X,S} C^\lambda_{T,Y}$, we may see that

$$C^\lambda_{X,S} C^\lambda_{T,Y} \equiv \langle C_S, C_T \rangle C^\lambda_{X,Y} \pmod{A(< \lambda)}. \tag{1.13}$$

Note that in [5], this bilinear form is called ϕ_λ ([5], Definition 2.3). To show that this bilinear form is symmetric, we apply ι to both sides of the equivalence (1.13) to obtain

$$\iota\left(C^{\lambda}_{X,S}C^{\lambda}_{T,Y}\right) \equiv \iota\left(\langle C_S, C_T\rangle C^{\lambda}_{X,Y}\right) \pmod{A(<\lambda)}$$

(where we have used the fact that $A(<\lambda)$ is invariant under ι), from which we have by the properties of ι that

$$C^{\lambda}_{Y,T}C^{\lambda}_{S,X} \equiv \langle C_S, C_T\rangle C^{\lambda}_{Y,X} \pmod{A(<\lambda)}.$$

But by using (1.13) again, we have

$$C^{\lambda}_{Y,T}C^{\lambda}_{S,X} \equiv \langle C_T, C_S\rangle C^{\lambda}_{Y,X} \pmod{A(<\lambda)},$$

and so we conclude that $\langle C_S, C_T\rangle = \langle C_T, C_S\rangle$.

Now in [5], Graham and Lehrer define

$$\mathrm{rad}(\lambda) = \{x \in \Delta^{\lambda} : \langle x, y\rangle = 0 \text{ for all } y \in \Delta^{\lambda}\},$$

([5], Definition 3.1), which they show to be a submodule of Δ^{λ} ([5], Proposition 3.2, (i)). They then define L^{λ} to be the quotient module $\Delta^{\lambda}/\mathrm{rad}(\lambda)$, and show that for each $\lambda \in \Lambda$, L^{λ} is either a simple module, or it is zero. They then prove that these modules L^{λ} provide a complete list of the isomorphism classes of simple left A-modules.

Theorem 1.13. *(Graham and Lehrer — Theorem 3.4 in [5]) Let A be a cellular algebra with data (Λ, M, C, ι) as in Definition 1.2. Let*

$$\Lambda_0 = \{\lambda \in \Lambda : L^{\lambda} \neq 0\}.$$

Then the collection of all modules L^{λ} for $\lambda \in \Lambda_0$ is a complete list of the simple left A-modules, and if λ and μ are distinct elements of Λ_0, then the modules L^{λ} and L^{μ} are not isomorphic.

1.5 Tensor products of cellular algebras

In this section, we shall prove that the tensor product algebra of two (or more) cellular algebras is again cellular, and describe its cell modules; we shall use these results in Chapter 3.

Recall that if A and B are $\Bbbk$-algebras, then the **tensor product algebra** $A \otimes B$ is defined to be their tensor product as $\Bbbk$-vector spaces, with multiplication (well-) defined on pure tensors by the formula

$$(a \otimes b)(c \otimes d) = (ac) \otimes (bd)$$

(See "A remark about tensor products" on page 8 for more details). One may easily verify that this multiplication makes $A \otimes B$ an associative unital $\Bbbk$-algebra, which is finite-dimensional if both A and B are. We shall show that if both A and B are cellular, then $A \otimes B$ is again cellular, and that the cell modules of $A \otimes B$ may be easily obtained as tensor products of the cell modules of A and B; these results were stated by Geetha and Goodman in Section 3.2 of [4].

Firstly, recall that if $\phi : A \to A$ and $\varphi : B \to B$ are $\Bbbk$-linear maps, then one may easily show that there is a well-defined $\Bbbk$-linear map

$$\phi \otimes \varphi : A \otimes B \longrightarrow A \otimes B$$

given by linearly extending

$$\phi \otimes \varphi : a \otimes b \longmapsto \phi(a) \otimes \varphi(b),$$

where $a \in A$ and $b \in B$. This map is called the **tensor product** of ϕ and φ.

Proposition 1.14. *(Geetha and Goodman, in Section 3.2 of [4]) Let A and B be cellular algebras over the field $\Bbbk$ as in Definition 1.2, with cellular data $(\Lambda_A, M, C, \iota_A)$ and $(\Lambda_B, M, C, \iota_B)$ respectively (we need to distinguish the sets Λ_A and Λ_B and the maps ι_A and ι_B by notation, but there will be no confusion if we use the same notation for the other items of cellular data). Then*

- *let Λ be $\Lambda_A \times \Lambda_B$, with the partial order defined by setting*

$$(\lambda_1, \mu_1) \leq (\lambda_2, \mu_2)$$

if and only if

$$\lambda_1 \leq \lambda_2 \text{ and } \mu_1 \leq \mu_2$$

(it is easy to check that this does indeed define a partial order on Λ*);*

- *for each* $(\lambda,\mu) \in \Lambda$*, let* $M\big((\lambda,\mu)\big)$ *be the set* $M(\lambda) \times M(\mu)$*;*
- *for each pair* $(S,U),(T,V) \in M\big((\lambda,\mu)\big)$*, let*

$$C^{(\lambda,\mu)}_{(S,U),(T,V)} = C^\lambda_{S,T} \otimes C^\mu_{U,V};$$

- *let* $\iota : A \otimes B \longrightarrow A \otimes B$ *be the tensor product* $\iota_A \otimes \iota_B$*.*

Then the data (Λ, M, C, ι) *exhibits* $A \otimes B$ *as a cellular algebra.*

Proof. We verify the conditions (C1), (C2), and (C3) of Definition 1.2.

For (C1), it is enough to prove that the elements $C^{(\lambda,\mu)}_{(S,U),(T,V)}$ form a basis of $A \otimes B$; this follows from the fact that the cellular bases of A and B are indeed bases.

For (C2), the fact that $\iota^2 = \mathrm{id}_{A\otimes B}$ follows immediately from the fact that $\iota_A^2 = \mathrm{id}_A$ and $\iota_B^2 = \mathrm{id}_B$. For any $a, c \in A$ and $b, d \in B$, we have

$$\begin{aligned}
\iota\big((a \otimes b)(c \otimes d)\big) &= \iota\big((ac) \otimes (bd)\big) \\
&= \big(\iota_A(ac)\big) \otimes \big(\iota_B(bd)\big) \\
&= \big(\iota_A(c)\iota_A(a)\big) \otimes \big(\iota_B(d)\iota_B(b)\big) \\
&= \big(\iota_A(c) \otimes \iota_B(d)\big)\big(\iota_A(a) \otimes \iota_B(b)\big) \\
&= \iota(c \otimes d)\iota(a \otimes b),
\end{aligned}$$

and so ι is indeed an anti-involution on $A \otimes B$. Now let $\lambda \in \Lambda_A$ and $S, T \in M(\lambda)$, and $\mu \in \Lambda_B$ and $U, V \in M(\mu)$. Then

$$\begin{aligned}
\iota\left(C^{(\lambda,\mu)}_{(S,U),(T,V)}\right) &= \iota\left(C^\lambda_{S,T} \otimes C^\mu_{U,V}\right) \\
&= \iota_A\left(C^\lambda_{S,T}\right) \otimes \iota_B\left(C^\mu_{U,V}\right) \\
&= C^\lambda_{T,S} \otimes C^\mu_{V,U}
\end{aligned}$$

$$= C^{(\lambda,\mu)}_{(T,V),(S,U)}$$

as required.

Now let us verify that (C3) holds; that is, that for any $x \in A \otimes B$ and any element $C^{(\lambda,\mu)}_{(S,U),(T,V)}$ as above, we have

$$xC^{(\lambda,\mu)}_{(S,U),(T,V)} \equiv \sum_{(X,Y)\in M((\lambda,\mu))} r_x\big((X,Y),(S,U)\big)C^{(\lambda,\mu)}_{(X,Y),(T,V)} \quad \Big(\text{mod } I\big((\lambda,\mu)\big)\Big)$$

where the coefficients $r_x\big((X,Y),(S,U)\big)$ do not depend on (T,V) and we define $I\big((\lambda,\mu)\big)$ to be the subspace of $A \otimes B$ which is spanned over $\Bbbk$ by all elements $C^{(\lambda',\mu')}_{(S',U'),(T',V')}$, where $(\lambda',\mu') < (\lambda,\mu)$ and

$$S', T' \in M(\lambda'),\ U', V' \in M(\mu').$$

Since the pure tensors $a \otimes b$ span $A \otimes B$, it is sufficient to show that this holds for $x = a \otimes b$. So let $a \in A$ and $b \in B$. Then we have

$$aC^{\lambda}_{S,T} = \sum_{X\in M(\lambda)} r_a(X,S)C^{\lambda}_{X,T} \ + \ L_1$$

and

$$bC^{\mu}_{U,V} = \sum_{Y\in M(\mu)} r_b(Y,U)C^{\mu}_{Y,V} \ + \ L_2$$

where the coefficients $r_a(X,S)$ and $r_b(Y,U)$ are all independent of both T and V, and $L_1 \in A(< \lambda)$ and $L_2 \in B(< \mu)$. Now

$$\begin{aligned}
(a \otimes b)C^{(\lambda,\mu)}_{(S,U),(T,V)} &= (a \otimes b)\left(C^{\lambda}_{S,T} \otimes C^{\mu}_{U,V}\right) \\
&= \left(aC^{\lambda}_{S,T}\right) \otimes \left(bC^{\mu}_{U,V}\right) \\
&= \sum_{X\in M(\lambda)} \sum_{Y\in M(\mu)} r_a(X,S)r_b(Y,U)\left(C^{\lambda}_{X,T} \otimes C^{\mu}_{Y,V}\right) \ + \ L
\end{aligned}$$

where we define L to be

$$\left(\sum_{X\in M(\lambda)} r_a(X,S)C^{\lambda}_{X,T}\right) \otimes L_2$$

$$+ L_1 \otimes \left(\sum_{Y \in M(\mu)} r_b(Y,U) C^{\mu}_{Y,V} \right) + L_1 \otimes L_2.$$

The fact that $L_1 \in A(< \lambda)$ and $L_2 \in B(< \mu)$ implies that L is a $\Bbbk$-linear combination of elements

$$C^{\lambda'}_{S',T'} \otimes C^{\mu'}_{U',V'}$$

where $\lambda' \leq \lambda$ and $\mu' \leq \mu$, with at least one of these inequalities being strict. So indeed $L \in I\big((\lambda,\mu)\big)$. Finally, we have that

$$\sum_{X \in M(\lambda)} \sum_{Y \in M(\mu)} r_a(X,S) r_b(Y,U) \left(C^{\lambda}_{X,T} \otimes C^{\mu}_{Y,V} \right)$$
$$= \sum_{(X,Y) \in M((\lambda,\mu))} r_{a\otimes b}\big((X,Y),(S,U)\big) C^{(\lambda,\mu)}_{(X,Y),(T,V)}$$

where we have defined

$$r_{a\otimes b}\big((X,Y),(S,U)\big) = r_a(X,S) r_b(Y,U) \tag{1.14}$$

for each $(X,Y) \in M\big((\lambda,\mu)\big)$, which we note is independent of (T,V). $\square$

Next, we shall describe the cell modules of a tensor product of cellular algebras. Recall that if A and B are $\Bbbk$-algebras, and V, W are left modules for A and B respectively, then the tensor product $V \otimes W$ of V and W as $\Bbbk$-vector spaces becomes a left $A \otimes B$ module when equipped with the action which is well-defined by the formula

$$(a \otimes b)(v \otimes w) = (av) \otimes (bw)$$

for $a \in A$, $b \in B$, $v \in V$, and $w \in W$.

Proposition 1.15. *(Geetha and Goodman, in Section 3.2 of [4]) Let A and B be cellular algebras with cellular data as in Proposition 1.14, so that the algebra $A \otimes B$ is cellular as described in that proposition. Then the cell module $\Delta^{(\lambda,\mu)}$ of $A \otimes B$ is isomorphic to the $A \otimes B$ module $\Delta^{\lambda} \otimes \Delta^{\mu}$, via the map given by $\Bbbk$-linearly extending*

$$\Phi : \Delta^{(\lambda,\mu)} \longrightarrow \Delta^{\lambda} \otimes \Delta^{\mu}$$

$$C_{(S,U)} \longmapsto C_S \otimes C_U.$$

Proof. To prove that Φ is an isomorphism of $A \otimes B$ modules, it is enough to prove that

$$\Phi\big((a \otimes b)C_{(S,U)}\big) = (a \otimes b)\Phi\big(C_{(S,U)}\big)$$

for all $a \in A$, $b \in B$, $S \in M(\lambda)$, and $U \in M(\mu)$. Indeed, let us recall from Equation (1.14) in the proof of Proposition 1.14 that for a pure tensor $a \otimes b$ in $A \otimes B$, we have

$$r_{a\otimes b}\big((X,Y),(S,U)\big) = r_a(X,S)r_b(Y,U)$$

for any $X, S \in M(\lambda)$ and any $Y, T \in M(\mu)$, for any $\lambda \in \Lambda_A$ and any $\mu \in \Lambda_B$. Then we have

$$\begin{aligned}
\Phi\big((a \otimes b)C_{(S,U)}\big) &= \Phi\left(\sum_{(X,Y)\in M((\lambda,\mu))} r_{a\otimes b}\big((X,Y),(S,U)\big)C_{(X,Y)}\right) \\
&\qquad \text{(by the definition of the action on the cell module } \Delta^{(\lambda,\mu)}\text{)} \\
&= \sum_{X\in M(\lambda)} \sum_{Y\in M(\mu)} r_a(X,S)r_b(Y,U)\,(C_X \otimes C_Y) \\
&= \left(\sum_{X\in M(\lambda)} r_a(X,S)C_X\right) \otimes \left(\sum_{Y\in M(\mu)} r_b(Y,U)C_Y\right) \\
&= (aC_S) \otimes (bC_U) \\
&= (a \otimes b)\,(C_S \otimes C_U) \\
&= (a \otimes b)\Phi\,(C_{S,U})\,.
\end{aligned}$$

□

We may generalise the above definition of the tensor product of two algebras in the obvious way to define the tensor product of n algebras $A_1, A_2, \ldots, A_n$ for any $n \geq 1$, which we may denote by $\bigotimes_{j=1}^{n} A_j$. In particular, for any algebra A, we may define the tensor product $A^{\otimes n}$ of n copies of A. Further, we shall adopt the convention that $A^{\otimes 0}$ is just the field $\Bbbk$ for any

$\Bbbk$-algebra A. By using Propositions 1.14 and 1.15 together with induction, we have the following theorem:

Theorem 1.16. *(Geetha and Goodman, in Section 3.2 of [4]) Let $\Bbbk$ be any field, n a positive integer, and $A_1, A_2, \ldots, A_n$ be cellular algebras over $\Bbbk$, where each A_j has cellular data $(\Lambda_j, M, C, \iota_j)$ as in Definition 1.2 (we need to distinguish the different partially ordered sets Λ_j and the different maps ι_j by notation, but there will be no confusion if we use the same notation for the other items of cellular data).*

Then the tensor product algebra $\bigotimes_{j=1}^{n} A_j$ is cellular with respect to the data (Λ, M, C, ι), where:

- *we denote by Λ the set $\Lambda_1 \times \Lambda_2 \times \ldots \times \Lambda_n$ with the partial order defined by setting*
$$(\lambda_1, \lambda_2, \ldots, \lambda_n) \leq (\mu_1, \mu_2, \ldots, \mu_n)$$
to mean that
$$\lambda_j \leq \mu_j \text{ for all } j = 1, \ldots, n;$$
- *for each element $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \Lambda$, we define $M\big((\lambda_1, \lambda_2, \ldots, \lambda_n)\big)$ to be the set*
$$M(\lambda_1) \times M(\lambda_2) \times \ldots \times M(\lambda_n);$$
- *for each element $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \Lambda$ and each pair*
$$\boldsymbol{S} = (S_1, S_2, \ldots, S_n),\ \boldsymbol{T} = (T_1, T_2, \ldots, T_n) \in M\big(\boldsymbol{\lambda}\big),$$
we define
$$C^{\boldsymbol{\lambda}}_{\boldsymbol{S},\boldsymbol{T}} = C^{\lambda_1}_{S_1,T_1} \otimes C^{\lambda_2}_{S_2,T_2} \otimes \cdots \otimes C^{\lambda_n}_{S_n,T_n};$$
- *we define ι to be the tensor product map $\iota_1 \otimes \iota_2 \otimes \cdots \otimes \iota_n$ (it is clear how we may extend the definition of the tensor product of maps to the case of an n-fold tensor product).*

Further, for any $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \Lambda$, we have an isomorphism

$$\Delta^{\boldsymbol{\lambda}} \cong \Delta^{\lambda_1} \otimes \Delta^{\lambda_2} \otimes \cdots \otimes \Delta^{\lambda_n}$$

of left modules over $\bigotimes_{j=1}^{n} A_j$.

New material in Chapter 1: There is no new material in Chapter 1. As far as I am aware, however, proofs of the results in Section 1.5 have not previously been published.

2 The symmetric group, the Brauer algebra, and iterated inflations

In this chapter we shall consider another type of diagram algebra which is constructed in a very similar way to the Temperley-Lieb algebra, called the *Brauer algebra*. This algebra has an important place in group representation theory, and has been extensively studied. In particular, Graham and Lehrer have shown in [5] that the Brauer algebra is cellular; we shall present a proof of this fact based on the proof given by König and Xi in [10], by exhibiting it as an *iterated inflation* of known cellular algebras.

2.1 The symmetric group and its group algebra

We shall begin this chapter by considering the symmetric group S_n on n letters. In particular, we shall see that the group algebra $\Bbbk S_n$ is cellular over any field $\Bbbk$; not only is this fact interesting in its own right, but it is a vital ingredient in our proof that the Brauer algebra is cellular.

For any positive integer n, let us write S_n for the symmetric group of all permutations on the set $\{1, \ldots, n\}$. We shall adopt the convention that S_n acts on the *right*, so that for $\pi, \sigma \in S_n$, the product $\pi\sigma$ is the permutation obtained by first applying π and then applying σ. Consequently, for $i \in \{1, \ldots, n\}$ we shall write $(i)\pi$ for the image of i under the permutation π, so that we have the formula $(i)(\pi\sigma) = ((i)\pi)\sigma$, as expected. We shall also find it convenient below to define S_0 to be the trivial group, so that $S_0 \cong S_1$.

The group S_n and its representation theory are of great importance across several branches of mathematics, and in many related areas. In order to discuss the representation theory of S_n, we must first develop some standard combinatorics.

Let n be a positive integer. A **partition** of n is a finite tuple

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$$

where $m \geq 1$ and each λ_i is a positive integer such that $\lambda_i \geq \lambda_{i+1}$ for $i = 1, \ldots, m-1$, and such that

$$\lambda_1 + \lambda_2 + \ldots + \lambda_m = n.$$

The number λ_i is called the i-th **part** of λ, and we may refer to λ as a **partition of n with m parts**. We shall find it convenient to define the **empty partition** $\varnothing$ to be the unique partition of zero, and to adopt the convention that $\varnothing$ has 0 parts. For any non-negative integer n, we shall adopt the notation $\lambda \vdash n$ to mean that λ is a partition of n. Further, if we again take

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$$

to be a partition of the positive integer n, and we let $m' = \lambda_1$, and for each $i = 1, \ldots, m'$, we define

$$\lambda'_i = |\{\lambda_j : 1 \leq j \leq m \text{ and } \lambda_j \geq i\}|,$$

then it is easy to show that

$$\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_{m'})$$

is also a partition of n, the **conjugate partition** λ' of λ. We also define the conjugate of the empty partition $\varnothing$ to be $\varnothing$. It is clear that this operation of conjugation $\lambda \mapsto \lambda'$ is a self-inverse bijection on the set of all partitions of each $n \geq 0$.

For a non-negative integer n, let us define Λ_n to be the set of all partitions of n; we wish to equip Λ_n with an ordering. Assume $n \geq 1$. Let

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m) \text{ and } \mu = (\mu_1, \mu_2, \ldots, \mu_r)$$

be partitions in Λ_n. We define $\lambda \trianglelefteq \mu$ to mean that

$$\sum_{j=1}^{i} \lambda_j \;\leq\; \sum_{j=1}^{i} \mu_j$$

for every $i = 1, \ldots, \min(m, r)$. It is easy to check that this defines a partial order on Λ_n, the well-known **dominance order**. The order which we shall require on Λ_n is in fact the **reverse dominance order**, which we shall denote by $\trianglelefteq^{\mathrm{op}}$ and which is defined by setting $\mu \trianglelefteq^{\mathrm{op}} \lambda$ to mean $\lambda \trianglelefteq \mu$; again, this is a partial order on Λ_n. For the case $n = 0$ we have $\Lambda_0 = \{\varnothing\}$ and so we define both $\trianglelefteq$ and $\trianglelefteq^{\mathrm{op}}$ to be the trivial order on Λ_0.

We now consider the group algebra $\Bbbk S_n$ of S_n over the field $\Bbbk$. It turns out that $\Bbbk S_n$ is in fact a cellular algebra for any field $\Bbbk$ and any non-negative integer n; this fact is a special case of a result proved by Graham and Lehrer (Example 1.2 in [5]). We shall not give a proof of the cellularity of $\Bbbk S_n$ here, but rather refer to the work of Mathas in [12]. Now the definition of a cellular algebra used by Mathas ([12], page 16, 2.1) is slightly different from, but equivalent to, our Definition 1.2. Indeed, let A be a $\Bbbk$-algebra which is cellular in the sense of Mathas's definition with respect to a tuple of data (Λ, M, C, ι) (note that Mathas writes $\mathcal{T}$ where we write M, and further he writes the anti-involution as $* : a \mapsto a^*$). Then A is cellular in the sense of our Definition 1.2 with respect to the tuple (Λ', M, C, ι), where Λ' is the partially ordered set obtained from Λ by reversing the ordering (that is, we replace each relation $\lambda < \mu$ on Λ with the relation $\mu <^{\mathrm{op}} \lambda$). Further, the cell module C^λ associated by Mathas to $\lambda \in \Lambda$ ([12], page 17), which is a *right* module, is isomorphic to the right module $\iota(\Delta^\lambda)$ obtained as in Section 1.2 from the cellular data (Λ', M, C, ι). Thus from Equation (1.11), we have for $\lambda \in \Lambda$, that the (left) cell module Δ^λ obtained from the cellular data (Λ', M, C, ι) as in Section 1.2 is isomorphic to the left A-module obtained from Mathas's (right) cell module C^λ by equipping C^λ with the left action defined by

$$ax = x\iota(a)$$

for all $x \in C^\lambda$ and all $a \in A$.

Now, in Chapter 3 of [12], Mathas (following the work of Murphy in [13] and [14]) gives a detailed proof that for any field $\Bbbk$ and any positive integer n, the *Iwahori-Hecke algebra* $\mathcal{H}_{\Bbbk,q}(S_n)$ of S_n over $\Bbbk$ with parameter $q \in \Bbbk \setminus \{0\}$ is cellular ([12], page 37, Theorem 3.20). Mathas also explains how the (right) cell modules he obtains for $\mathcal{H}_{\Bbbk,q}(S_n)$ relate to the *Specht modules* of $\mathcal{H}_{\Bbbk,q}(S_n)$ given by Dipper and James in [2] ([12], page 54, Note 2, as corrected by the author's errata). We are interested in $\Bbbk S_n$ rather than $\mathcal{H}_{\Bbbk,q}(S_n)$, and so we shall consider only the case where $q = 1$, because $\mathcal{H}_{\Bbbk,1}(S_n)$ is isomorphic to $\Bbbk S_n$ ([12], page 5; note that Mathas also adopts the convention that S_n acts on the right ([12], page 1), so that our notion of $\Bbbk S_n$ agrees with his). Further, since the definition of the Specht modules of $\mathcal{H}_{\Bbbk,q}(S_n)$ given by Dipper and James in [2] is a generalisation of the definition of the Specht modules of $\Bbbk S_n$ given by James in [6], we shall refer to [6] rather than [2]. Indeed, recall that in [6] (page 13, 4.3), James defines for each $\lambda \in \Lambda_n$ a right $\Bbbk S_n$-module, the *Specht module* of λ, for which he writes S^λ; however, we shall follow the notation of Mathas in [12] and denote this module by $\mathbb{S}^\lambda$ (Mathas uses S^λ to denote the (right) cell modules which he obtains for $\mathcal{H}_{\Bbbk,q}(S_n)$, but we shall have a different use for this notation, see Theorem 2.1, below). Finally, recall that for any right $\Bbbk S_n$-module E, the *dual module* of E is the right $\Bbbk S_n$-module formed by equipping $\mathrm{Hom}_\Bbbk(E, \Bbbk)$ with the action defined by letting

$$(\varphi\pi)(x) = \varphi(x\pi^{-1})$$

for any $\varphi \in \mathrm{Hom}_\Bbbk(E, \Bbbk)$, $\pi \in S_n$, and $x \in E$ (see [6], 1.4, pages 2 and 3; note that we are writing linear functionals on the *left* of their arguments). Again, we shall follow the notation of Mathas and denote this dual module by $E^\diamond$ ([12], page 24, Exercise 7). We can now use the results in Chapter 3 of [12] which we have discussed above to give the following theorem on the cellularity of $\Bbbk S_n$:

Theorem 2.1. *For any field $\Bbbk$ and integer $n \geq 0$, the group algebra $\Bbbk S_n$ is cellular in the sense of Definition 1.2 with respect to a tuple (Λ_n, M, C, ι) of* *cellular data, where Λ_n is as above the set of all partitions of n equipped with the reverse dominance order, and ι is the anti-involution on $\Bbbk S_n$ induced by mapping each $\pi \in S_n$ to π^{-1}. We denote by S^λ the (left) cell module associated to $\lambda \in \Lambda_n$ by the cellular data (Λ_n, M, C, ι) as described in Section 1.2. If we denote by $\mathbb{S}^\lambda$ the (right) Specht module associated to $\lambda \in \Lambda_n$ as defined by James in [6], then S^λ is isomorphic to the left $\Bbbk S_n$-module obtained by equipping the right $\Bbbk S_n$-module $(\mathbb{S}^\lambda)^\diamond$ (see above) with the action*

$$ax = x\iota(a)$$

where $a \in \Bbbk S_n$ and $x \in (\mathbb{S}^\lambda)^\diamond$.

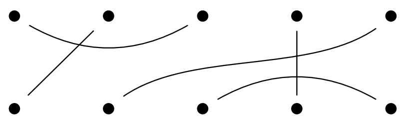
Note that we have included the case $n = 0$ in Theorem 2.1. Indeed, recall that we have adopted the convention that S_0 denotes the trivial group, so that $\Bbbk S_0$ may be identified with the field $\Bbbk$; it is now easy to see that all of the claims made in Theorem 2.1 are trivially true for $n = 0$. Further, note that we shall not require any details of the cellular basis of $\Bbbk S_n$ which is given in Mathas's result (the *Murphy basis* or *standard basis* as it is called in [12]).

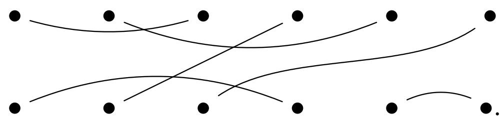
2.2 The Brauer algebra

In Chapter 1 we showed how the Temperley-Lieb algebra $\mathrm{TL}_\Bbbk(r, \delta)$ may be constructed as the $\Bbbk$-vector space with a basis consisting of all planar diagrams on $2r$ nodes, and with a multiplication based on the notion of "concatenating" pairs of planar diagrams. The Brauer algebra with parameters r (a positive integer) and $\delta \in \Bbbk$ is constructed in exactly the same way, except that in defining the set of diagrams which will form the basis of the algebra, we remove the restriction that no two edges of a diagram may cross. We thus obtain the set of *Brauer diagrams* on $2r$ nodes, which contains the set of all planar diagrams on $2r$ nodes. The multiplication is again based on the concatenation of diagrams as for the Temperley-Lieb algebra, and so in fact

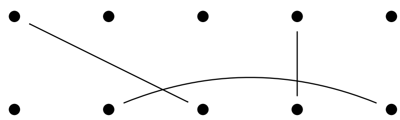
$\mathrm{TL}_{\Bbbk}(r, \delta)$ is a subalgebra of the Brauer algebra.

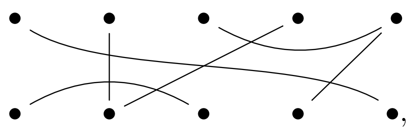
Formally, then, let r a be positive integer. A **Brauer diagram** with $2r$ nodes consists of two rows of r nodes, one above the other, and exactly r **edges** between the nodes, such that each node is connected via an edge to exactly one other node (which may be either on the same row or the other row), where the edges must lie entirely within the rectangular area between the rows. For example, the following two diagrams are Brauer diagrams, with $r = 5$ and $r = 6$ respectively:





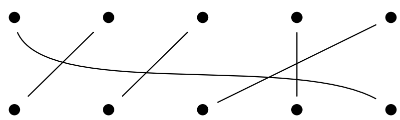
As with planar diagrams we insist that each node in a Brauer diagram is the endpoint of *exactly one* edge, and so the following are *not* Brauer diagrams



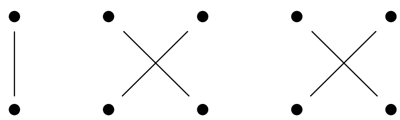


as the first contains nodes which are not connected to any other node, and the second contains nodes connected to more than one other node. Because of this requirement, any Brauer diagram defines a partition of its nodes into pairs. As with planar diagrams, we consider any two Brauer diagrams with the same number of nodes to be equal if they define the same partition of their nodes, regardless of the exact shape of their edges.

It is immediate that any planar diagram is also a Brauer diagram. Another special kind of Brauer diagram is a **permutation diagram**, which is a Brauer diagram which satisfies the additional restriction that there are no "horizontal" edges between nodes on the same row, so that each node on the top row is connected to exactly one node on the bottom row. For example, the diagrams



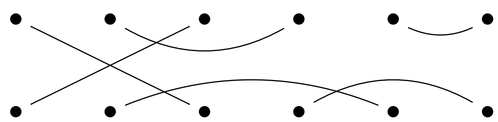
and



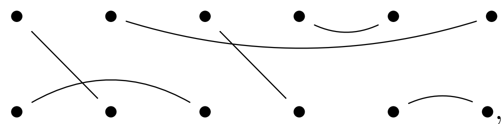
are permutation diagrams, while the two previous examples of Brauer diagrams given above are not. Now if $\pi \in S_r$, then we may construct a permutation diagram on $2r$ nodes by connecting the i-th node on the top row to the $(i)\pi$-th node on the bottom row for each $i = 1, \ldots, r$. It is clear that this construction sets up a bijective correspondence between S_r and the set of permutation diagrams on $2r$ nodes; for example, the two permutation diagrams above correspond to the elements $(1\,5\,3\,2)$ and $(2\,3)(4\,5)$ of S_5, respectively.

Now let $\Bbbk$ be any field, and fix $\delta \in \Bbbk$. We define $\mathrm{B}_{\Bbbk}(r, \delta)$ to be the $\Bbbk$-vector space with a basis consisting of all Brauer diagrams on $2r$ nodes. To define a multiplication on $\mathrm{B}_{\Bbbk}(r, \delta)$, it is enough to define the product of two Brauer diagrams, and we define such a product in *exactly* the same way as for the Temperley-Lieb algebra $\mathrm{TL}_{\Bbbk}(r, \delta)$: given two Brauer diagrams d_1

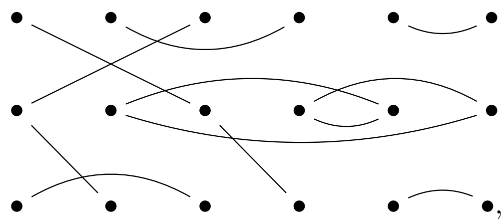
and d_2, the product d_1d_2 is computed by forming a *concatenated diagram* by first drawing d_1 and then drawing d_2 immediately below it, using the nodes on the bottom row of d_1 as the nodes of the top row of d_2; we then let p be the diagram with two rows of r nodes which are connected by edges in the same way that the nodes on the top and bottom rows of the concatenated diagram are connected by *open paths* consisting of one or more edges linked end-to-end with nodes between them. It is easy to see that p must again be a Brauer diagram. Finally, we multiply p by δ^n where n is the number of *closed loops* in the concatenated diagram (if there are no such closed loops then the product is just p). For example, let $r = 6$ and let d_1, d_2 be the Brauer diagrams



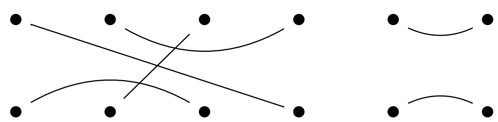
and

,

respectively. To calculate d_1d_2, we form the concatenated diagram

,

from which we see that $d_1d_2 = \delta p$, where p is the diagram



(note that there is only *one* closed loop in the concatenated diagram, even though that loop crosses itself and thus produces two "lobes", one inside the other; we are not concerned with such self-crossing).

As with $\mathrm{TL}_{\Bbbk}(r,\delta)$, this product can easily be seen to be associative on Brauer diagrams and hence is associative on the whole of $\mathrm{B}_{\Bbbk}(r,\delta)$. Further, we define e to be the Brauer diagram where each node on the top row is connected to the node directly below it, and as for $\mathrm{TL}_{\Bbbk}(r,\delta)$, e is then a two-sided identity. Thus we have now established that $\mathrm{B}_{\Bbbk}(r,\delta)$ equipped with this multiplication is an associative unital $\Bbbk$-algebra, the **Brauer algebra** with parameters r and δ.

It is immediate that the Temperley-Lieb algebra $\mathrm{TL}_{\Bbbk}(r,\delta)$ occurs as the subalgebra of $\mathrm{B}_{\Bbbk}(r,\delta)$ spanned by all planar diagrams. Further, if we identify each permutation $\pi \in S_r$ with the associated permutation diagram as described above, we may see by considering the way the multiplication rule applies to permutation diagrams that in fact the $\Bbbk$-span of all permutation diagrams in $\mathrm{B}_{\Bbbk}(r,\delta)$ is a subalgebra of $\mathrm{B}_{\Bbbk}(r,\delta)$ isomorphic to the group algebra $\Bbbk S_r$.

Having defined $\mathrm{B}_{\Bbbk}(r,\delta)$, we now wish to prove that it is a cellular algebra. However, the proof of this is not as straightforward as the proof that $\mathrm{TL}_{\Bbbk}(r,\delta)$ is cellular: in particular the basis of $\mathrm{B}_{\Bbbk}(r,\delta)$ consisting of all Brauer diagrams is *not* a cellular basis in general. Graham and Lehrer gave a rather computational proof that $\mathrm{B}_{\Bbbk}(r,\delta)$ is cellular in Section 4 of [5], but we shall follow the approach introduced by König and Xi in Section 5 of [9] and expanded upon in Section 5 of [10], by exhibiting $\mathrm{B}_{\Bbbk}(r,\delta)$ as an *iterated inflation* of cellular algebras. In the remainder of this section, we shall lay the foundations of this proof by constructing a well-known decomposition of $\mathrm{B}_{\Bbbk}(r,\delta)$ as a direct sum of subspaces which themselves admit a further natural decomposition as tensor products; in the next section we shall define iterated inflations of cellular algebras and apply this definition to our decomposition of $\mathrm{B}_{\Bbbk}(r,\delta)$ to conclude that $\mathrm{B}_{\Bbbk}(r,\delta)$ is cellular.

Let us start by defining the anti-involution on $\mathrm{B}_{\Bbbk}(r,\delta)$ as in Theorem 4.10 of [5]. We do this in exactly the same way as for $\mathrm{TL}_{\Bbbk}(r,\delta)$: given any Brauer diagram d, we define $\iota(d)$ to be the diagram formed by "flipping d upside down" (formally, by reflecting d in the line parallel to and halfway between its two rows of nodes), which we may easily see must also be a Brauer diagram. As for planar diagrams, it is clear that for any Brauer diagrams d, d' in $\mathrm{B}_{\Bbbk}(r,\delta)$, we have $\iota^2(d) = d$ and $\iota(dd') = \iota(d')\iota(d)$. It follows that ι is an anti-involution on $\mathrm{B}_{\Bbbk}(r,\delta)$.

Recall that a *through string* of a planar diagram is simply an edge which connects a node on the top row of the diagram to a node on the bottom row. We may define a through string of a Brauer diagram in exactly the same way, and we may carry over the definitions of the *northern node* and *southern node* of a through string. We shall call any edge of a Brauer diagram which is not a through string a *horizontal edge*, as for planar diagrams. Then for any Brauer diagram d in $\mathrm{B}_{\Bbbk}(r,\delta)$, there exists a unique l in the set

$$I_r = \{r, r-2, r-4, \ldots, 1 \text{ or } 0\}$$

such that d has l through strings. Thus we may partition the basis of $\mathrm{B}_{\Bbbk}(r,\delta)$ consisting of all Brauer diagrams in $\mathrm{B}_{\Bbbk}(r,\delta)$ according to the number of through strings each diagram has, and hence if we define

$$D_l = \operatorname{span}_{\Bbbk}\{d : \ d \text{ is a Brauer diagram with exactly } l \text{ through strings}\}$$

for each $l \in I_r$, then we have a direct sum decomposition

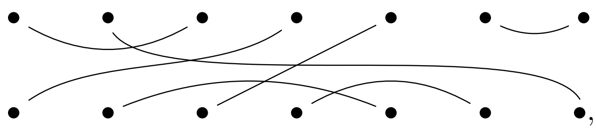
$$\mathrm{B}_{\Bbbk}(r,\delta) = \bigoplus_{l \in I_r} D_l$$

of $\mathrm{B}_{\Bbbk}(r,\delta)$ as a $\Bbbk$-vector space.

Next, recall that we have defined the notion of a *half planar diagram* on r nodes; we shall now make a corresponding definition for Brauer diagrams.

Indeed, if r is a positive integer and $l \in I_r$, then a **half Brauer diagram** with r nodes and l **free nodes** is a row of r nodes and exactly $\frac{r-l}{2}$ edges between the nodes, such that each node is the end point of at most one edge; it follows that exactly l of the nodes are not an end point of any edge — we shall call these nodes the **free nodes** of the half Brauer diagram. As for planar diagrams, we require that no edge may cross the (infinitely extended) line defined by the row of nodes, but (since edges are allowed to cross each other in a Brauer diagram) we do not require that no two edges may cross or that two nodes may not be connected if there is a free node between them. As with planar diagrams, the idea behind this definition is that half Brauer diagrams are exactly the diagrams which may be obtained by taking a Brauer diagram and erasing its through strings, and then taking one of the resulting two single-row diagrams.

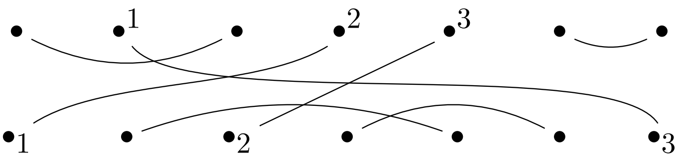
Now let $l \in I_r$ and S, T be two half *planar* diagrams with r nodes and l free nodes. Recall how the restriction that edges may not cross each other in a planar diagram means that there is a *unique* planar diagram with top S and bottom T (see Lemma 1.4). The situation is, however, more complicated for $\mathrm{B}_{\mathbb{k}}(r, \delta)$. If we erase the through strings of a Brauer diagram, we get two half Brauer diagrams; as for planar diagrams, we shall call these its *top* and *bottom*. However, a Brauer diagram with more than one through string is *not* uniquely determined by its top and its bottom: given two half Brauer diagrams S and T with l free nodes, there are in fact exactly $l!$ distinct Brauer diagrams with top S and bottom T, since there are $l!$ ways of connecting the l free nodes of S to the l free nodes of T by drawing in l through strings. So rather than just decomposing a Brauer diagram into its top and bottom, we must also record the way in which the through strings are arranged, as we shall now explain. Let d be a Brauer diagram with $2r$ nodes which has l through strings, with top S and bottom T. Number the northern nodes of the through strings of d with the numbers 1 to l, going from left to right, and do likewise for the southern nodes of the through strings of d. We may define an element π of S_l by letting $(i)\pi$ be the number labelling the southern node of the through string whose northern node has label i. For example, if we take d to be



then $l = 3$ and S and T are

and

respectively. Numbering the northern and southern nodes of the through strings of d as described above gives us



from which we see that π is $(1\,3\,2) \in S_3$.

It is easy to see that the triple (S, π, T) uniquely determines the Brauer diagram d, and hence we have established a bijective correspondence between the set of Brauer diagrams with $2r$ nodes and precisely l through strings on the one hand, and on the other hand the set

$$\Omega_l \times S_l \times \Omega_l$$

where Ω_l is the set of all half Brauer diagrams with r nodes and l free nodes. If we define V_l to be the $\Bbbk$-vector space with basis Ω_l, then this correspondence induces a $\Bbbk$-linear bijection

$$D_l \longleftrightarrow V_l \otimes \Bbbk S_l \otimes V_l \tag{2.1}$$

and hence, up to isomorphism of $\Bbbk$-vector spaces, we have obtained the well-known decomposition

$$\mathrm{B}_{\Bbbk}(r,\delta) \cong \bigoplus_{l \in I_r} V_l \otimes \Bbbk S_l \otimes V_l, \tag{2.2}$$

where (recall) I_r is the set of all integers l such that $0 \leq l \leq r$ and $r-l$ is even. This is our desired decomposition of $\mathrm{B}_{\Bbbk}(r,\delta)$, and in the next section we shall explain how this allows us to prove the cellularity of $\mathrm{B}_{\Bbbk}(r,\delta)$ as given by König and Xi in [9]. Informally, we shall show that the multiplication of $\mathrm{B}_{\Bbbk}(r,\delta)$ and the anti-involution ι interact in a "nice" way with this decomposition, and in particular with the multiplication of the symmetric group algebras $\Bbbk S_l$ which appear in the decomposition. This will allow us to exploit the known cellularity of the algebras $\Bbbk S_l$ to produce a cellular structure on $\mathrm{B}_{\Bbbk}(r,\delta)$.

2.3 Iterated inflations of cellular algebras

In [9], König and Xi introduced the concept of an *iterated inflation* of cellular algebras, and we shall now briefly review this work. Firstly, König and Xi defined two methods of *inflating* an algebra (Sections 3.1 and 3.2 in [9]). In the first method, we take a $\Bbbk$-algebra A, a $\Bbbk$-vector space V, and a bilinear form $\varphi : V \times V \to A$, and we define an associative multiplication on the tensor product $V \otimes A \otimes V$ via the formula

$$(x \otimes a \otimes w)(y \otimes b \otimes z) = x \otimes a\varphi(w,y)b \otimes z, \tag{2.3}$$

thus making $V \otimes A \otimes V$ an "algebra", potentially without a unit, called an *inflation of A along V*. In the second method, we take two $\Bbbk$ algebras A and B, where B is not assumed to have a unit element, and define some associative multiplication on the (external) direct sum $A \oplus B$ such that the multiplication on B is preserved, B is a two-sided ideal of $A \oplus B$, and the quotient algebra $(A \oplus B)/B$ is isomorphic to A; this construction is called an *inflation of A along B*. An *iterated inflation* is then defined by using these two constructions repeatedly (Section 3.3 in [9]): one starts with some algebra A_1, then one takes an algebra B_1, a vector space V_1 and a bilinear form φ_1 as above, and one forms the inflation $V_1 \otimes B_1 \otimes V_1$; one then forms an inflation A_2 of A_1 along $V_1 \otimes B_1 \otimes V_1$. One can then take another algebra B_2, a vector space V_2 and a bilinear form φ_2, and form the inflation A_3 of A_2 along the inflation $V_2 \otimes B_2 \otimes V_2$, and so on. Further, König and Xi showed that if the algebras A_1 and $B_1, B_2, \ldots$ are cellular, then provided that certain technical conditions are satisfied, all of the algebras $A_2, A_3, \ldots$ will also be cellular.

In Section 4 of [9], König and Xi showed that the class of cellular algebras over a field $\Bbbk$ is exactly the class of algebras obtained via the method of iterated inflations as above, by taking A_1 to be a matrix algebra over $\Bbbk$, and each B_i to be the field $\Bbbk$ ([9], Theorem 4.1). In Section 5 of the same paper, they outlined how the Brauer algebra $\mathrm{B}_{\Bbbk}(r, \delta)$ may be exhibited as an iterated inflation constructed from cellular algebras, thus giving a new proof that it is indeed cellular. Other authors have since applied the idea of an iterated inflation of cellular algebras in various contexts.

We have now outlined the basic concept of an iterated inflation of cellular algebras as given by König and Xi. However, as mentioned above, there are various technical details and extra conditions which must be satisfied in order for the construction to work. In [9] and [10], these conditions are not presented in a single definition or result, but rather are developed as needed in the course of the text (see, for example, Lemmas 5.2–5.5 in [10]). In this section, we shall present a reformulation of the concept of an iterated inflation of cellular algebras, derived from the work of König and Xi, but with a somewhat simpler construction. Further, we shall present our version via a single theorem which explicitly gives all of the necessary conditions and which may thus be more easily applied in practice. We shall show below that the decomposition (2.2) of $\mathrm{B}_{\Bbbk}(r,\delta)$ satisfies the hypotheses of this theorem and hence that $\mathrm{B}_{\Bbbk}(r,\delta)$ is cellular (all of this work was essentially given by König and Xi in [10]), and we shall make further use of the theorem in the next chapter, when we consider wreath products of cellular algebras with symmetric groups.

Indeed, let us start with our theorem on iterated inflations of cellular algebras.

Theorem 2.2. *Let A be an associative, unital, finite-dimensional $\Bbbk$-algebra, with an anti-involution ι. Suppose that we have, up to isomorphism of $\Bbbk$-vector spaces, a $\Bbbk$-vector space decomposition*

$$A \cong \bigoplus_{i \in I} V_i \otimes B_i \otimes V_i \tag{2.4}$$

of A, where I is some finite partially ordered set and where each V_i is some $\Bbbk$-vector space and each B_i is a cellular algebra over $\Bbbk$. We shall henceforth consider A to be identified with this direct sum of tensor products.

Suppose that for each $i \in I$, we have a basis Ω_i for V_i and a basis $\mathcal{B}_i$ for B_i, such that the following conditions hold:

1. *For each $i \in I$, we have for any $u, v \in \Omega_i$ and any $b \in \mathcal{B}_i$ that*

$$\iota(u \otimes b \otimes v) = v \otimes \sigma_i(b) \otimes u \tag{2.5}$$

where σ_i is the anti-involution on B_i.

2. *Let $\mathcal{A}$ be the basis of A consisting of all elements $u \otimes b \otimes v$ for all $u, v \in \Omega_i$ and all $b \in \mathcal{B}_i$, as i ranges over I. Then for any $i \in I$ and any $u, v \in \Omega_i$ and any $b \in \mathcal{B}_i$, we have for any $a \in \mathcal{A}$ that*

$$a \cdot (u \otimes b \otimes v) \equiv x(a,u) \otimes c(a,u) b \otimes v \qquad \big(\mathrm{mod}\ J(< i)\big) \tag{2.6}$$

where

$$J(< i) = \bigoplus_{l<i} V_l \otimes B_l \otimes V_l$$

and $x(a,u) \in V_i$ *and* $c(a,u) \in B_i$ *depend only on* a *and* u*, as indicated by the notation.*

Then A *may be exhibited as a cellular algebra, via a tuple of data* (Λ, M, C, ι) *which we shall now define. Indeed, for* $i \in I$*, let* $(\Lambda_i, M, C, \sigma_i)$ *be cellular data for* B_i *(we need to distinguish the different partially ordered sets* Λ_i *and the different maps* σ_i *for all* $i \in I$ *by notation, but there will be no confusion if we use the same notation for the other items of cellular data). Then:*

- *let* Λ *be the set* $\{(i,\lambda) : i \in I \text{ and } \lambda \in \Lambda_i\}$*, with the partial order defined by setting*

  $$(i,\lambda) < (j,\mu) \text{ if } i < j$$

  and

  $$(i,\lambda) < (i,\mu) \text{ if } \lambda < \mu$$

  (that is, lexicographic order);

- *for* $(i,\lambda) \in \Lambda$*, let* $M(i,\lambda)$ *be the set* $\Omega_i \times M(\lambda)$*;*

- *for* $(i,\lambda) \in \Lambda$ *and* $(x,X),(y,Y) \in M(i,\lambda)$*, let*

  $$C^{(i,\lambda)}_{(x,X),(y,Y)} = x \otimes C^{\lambda}_{X,Y} \otimes y.$$

We shall call an algebra A satisfying the conditions of Theorem 2.2 an **iterated inflation**; more specifically, we might call it an iterated inflation **of** the algebras B_i for $i \in I$, or an iterated inflation of the algebras B_i for $i \in I$ **along** the vector spaces V_i for $i \in I$.

Proof of Theorem 2.2. We show that the claimed cellular data satisfies properties (C1), (C2), and (C3) of Definition 1.2.

For (C1), it suffices to note that our order on Λ is indeed a partial order, and that the elements $C^{(i,\lambda)}_{(x,X),(y,Y)}$ do indeed form a basis of A.

For (C2), first note that by linearity of ι and of each map σ_i, we may easily show that in fact Equation (2.5) holds for any $u, v \in V_i$ and any $b \in B_i$. Then to prove (C2), it is enough to note that for any $(i,\lambda) \in \Lambda$ and $(x,X),(y,Y) \in M(i,\lambda)$, we have

$$\begin{aligned}
\iota\left(C^{(i,\lambda)}_{(x,X),(y,Y)}\right) &= \iota\left(x \otimes C^{\lambda}_{X,Y} \otimes y\right) \\
&= y \otimes \sigma_i\left(C^{\lambda}_{X,Y}\right) \otimes x \\
&= y \otimes C^{\lambda}_{Y,X} \otimes x \\
&= C^{(i,\lambda)}_{(y,Y),(x,X)}.
\end{aligned}$$

It remains only to prove (C3). Firstly, note that by linearity, we may easily show that in fact Equation (2.6) holds for any $a \in \mathcal{A}$, any $u, v \in \Omega_i$ and any $b \in B_i$ (that is, not just for $b \in \mathcal{B}_i$), and so, in particular, it holds when b is taken to be any element $C^{\lambda}_{X,Y}$ of the cellular basis of B_i. Next, for any $(i,\lambda) \in \Lambda$, let $A\big(< (i,\lambda)\big)$ be the subspace of A spanned by all the elements

$$C^{(j,\mu)}_{(w,W),(z,Z)}$$

for all $(j,\mu) \in \Lambda$ with $(j,\mu) < (i,\lambda)$, and all pairs $(w,W),(z,Z)$ in $M(j,\mu)$ (since we have not yet proved that A is cellular with respect to the given data, we do not yet know that $A\big(< (i,\lambda)\big)$ is an ideal of A — we only know that it is a subspace). Notice in particular that $J(< i) \subseteq A\big(< (i,\lambda)\big)$.

Now let $(i,\lambda) \in \Lambda$ and $(u,X),(v,Y) \in M(i,\lambda)$, and $a \in \mathcal{A}$. We have

$$\begin{aligned}
aC^{(i,\lambda)}_{(u,X),(v,Y)} &= a \cdot \left(u \otimes C^{\lambda}_{X,Y} \otimes v\right) \\
&\equiv x \otimes c\, C^{\lambda}_{X,Y} \otimes v \qquad \big(\mathrm{mod}\ J(< i)\big)
\end{aligned}$$

for some $x = x(a,u)$ and some $c = c(a,u)$ as in Equation (2.6). Thus we have

$$aC^{(i,\lambda)}_{(u,X),(v,Y)} = \left(x \otimes c\, C^{\lambda}_{X,Y} \otimes v\right) + L_1 \tag{2.7}$$

for some $L_1 \in J(< i)$; notice that we have $L_1 \in A\big(< (i, \lambda)\big)$. Then we have by the cellularity of B_i that

$$x \otimes c\, C^{\lambda}_{X,Y} \otimes v = x \otimes \left(\sum_{U \in M(\lambda)} r_c(U, X) C^{\lambda}_{U,Y} \;+\; L_2 \right) \otimes v$$

for some $L_2 \in B_i(< \lambda)$, and hence

$$x \otimes c\, C^{\lambda}_{X,Y} \otimes v = x \otimes \left(\sum_{U \in M(\lambda)} r_c(U, X) C^{\lambda}_{U,Y} \right) \otimes v \;+\; x \otimes L_2 \otimes v. \quad (2.8)$$

It is easy to see that $x \otimes L_2 \otimes v$ lies in $A\big(< (i, \lambda)\big)$. Now write x as a $\Bbbk$-linear combination

$$x = \sum_{w \in \Omega_i} \gamma_w w$$

and note that these coefficients $\gamma_w \in \Bbbk$ depend only on a, u and w. We have

$$\begin{aligned} x \otimes \left(\sum_{U \in M(\lambda)} r_c(U, X) C^{\lambda}_{U,Y} \right) \otimes v &= \sum_{w \in \Omega_i} \sum_{U \in M(\lambda)} \gamma_w r_c(U, X) \left(w \otimes C^{\lambda}_{U,Y} \otimes v \right) \\ &= \sum_{(w,U) \in M(i,\lambda)} r_a\big((w, U), (u, X)\big) \left(w \otimes C^{\lambda}_{U,Y} \otimes v \right) \end{aligned}$$

where we have defined $r_a\left((w, U), (u, X)\right) = \gamma_w r_c(U, X)$, which is independent of Y and v (recall that c depends only on a and u). Thus by (2.7) and (2.8), we have

$$a C^{(i,\lambda)}_{(u,X),(v,Y)} = \sum_{(w,U) \in M(i,\lambda)} r_a((w, U), (u, X)) C^{(i,\lambda)}_{(w,U),(v,Y)} \;+\; L$$

where $L = L_1 + x \otimes L_2 \otimes v$, which is an element of $A\big(< (i, \lambda)\big)$. The right hand side of this equation is indeed of the form required by (C3), and (C3) now follows by the fact that $\mathcal{A}$ is a basis of A. $\square$

As mentioned above, [9] and [10] do not present iterated inflations via a single result like Theorem 2.2. However, in [15], Xi offered the following lemma to characterise iterated inflations of cellular algebras, which has been cited by several subsequent authors:

Lemma. *(Xi — Lemma 3.3 in [15]) Let A be an algebra with an anti-involution ι. Suppose there is a decomposition*

$$A = \bigoplus_{j=1}^{m} V_j \otimes B_j \otimes V_j \qquad \text{(direct sum of vector spaces),}$$

where V_j is a vector space and B_j is a cellular algebra with respect to an anti-involution σ_j and a cell chain

$$J_1^{(j)} \subseteq \cdots \subseteq J_{s_j}^{(j)} = B_j$$

for each j. Define

$$J_t = \bigoplus_{j=1}^{t} V_j \otimes B_j \otimes V_j.$$

Assume that

(i) the restriction of ι on $V_j \otimes B_j \otimes V_j$ is given by

$$w \otimes b \otimes v \longmapsto v \otimes \sigma_j(b) \otimes w$$

(ii) for each j, there is a bilinear form $\phi_j : V_j \times V_j \to B_j$ such that $\sigma_j(\phi_j(w, v)) = \phi_j(v, w)$ for all $v, w \in V_j$

(iii) the multiplication of two elements in $V_j \otimes B_j \otimes V_j$ is governed by ϕ_j modulo J_{j-1}, that is, for $x, y, u, v \in V_j$ and $b, c \in B_j$, we have

$$(x \otimes b \otimes y)(u \otimes c \otimes v) = x \otimes b\phi_j(y, u)c \otimes v$$

modulo the ideal J_{j-1}

(iv) $\left(V_j \otimes J_l^{(j)} \otimes V_j\right) + J_{j-1}$ is an ideal in A for all l and j.

Then A is a cellular algebra.

Note that in the context in which this lemma is given in [15], the claim "Then A is a cellular algebra" in the last line of the lemma means (in terms of our definitions from Chapter 1, in particular Definition 1.10) "Then A is cellular *with respect to the anti-involution* ι *on* A" (see [15]; in particular the wording of Definition 3.2 and the proof of the above lemma). However, this lemma is incorrect; see Appendix A for a counterexample. Essentially, the lemma imposes conditions on the multiplication within each "layer" $V_j \otimes B_j \otimes V_j$ of the algebra A, by demanding that it is "governed" by a bilinear form as in point *(iii)*; this condition is, of course, derived from the formula (2.3) which defines the multiplication in an inflation of an algebra along a vector space. However, in order to ensure that we can construct cellular data for the algebra A from the cellular data of the algebras B_i, it is also necessary to control how the multiplication behaves "between" layers, and this is why we require the condition 2 in Theorem 2.2; this condition was given (for the Brauer algebra) by König and Xi in Lemma 5.5 in [10].

Although the condition given in point *(iii)* of Xi's proposed lemma is not strong enough to ensure cellularity by itself, it is nonetheless the case that if A is an algebra which satisfies the hypotheses of Theorem 2.2, then the multiplication within each "layer" of A is indeed governed by a bilinear form in exactly this way. This fact demonstrates the link between the original version of iterated inflations as given by König and Xi, and our reformulation in Theorem 2.2. Before we prove it, recall from the proof of Theorem 2.2 that we can easily use the linearity of ι and Equation (2.5) to show that

$$\iota(u \otimes b \otimes v) = v \otimes \sigma_i(b) \otimes u \tag{2.9}$$

for any $u, v \in V_i$ and any $b \in B_i$ (where σ_i is the anti-involution on B_i). Similarly, recall that we may use Equation (2.6) to show that

$$a \cdot (u \otimes b \otimes v) \equiv x(a,u) \otimes c(a,u)b \otimes v \qquad \big(\text{mod } J(< i)\big) \tag{2.10}$$

for any $a \in \mathcal{A}$, any $u, v \in \Omega_i$ and any $b \in B_i$. Now let us state the result we wish to prove concerning the multiplication in an iterated inflation.

Proposition 2.3. *Let A be an algebra satisfying all the hypotheses of Theorem 2.2, with notation as in that theorem. Then for each $i \in I$ there exists a* *unique* $\Bbbk$*-bilinear form*

$$\phi_i : V_i \times V_i \longrightarrow B_i$$

such that for any $u, v, w, z \in V_i$ *and* $b, d \in B_i$, *we have*

$$(w \otimes d \otimes z)(u \otimes b \otimes v) \equiv w \otimes d\phi_i(z, u)b \otimes v \qquad \big(\text{mod } J(< i)\big). \tag{2.11}$$

Further, we have

$$\phi_i(z, u) = \sigma_i\left(\phi_i(u, z)\right) \tag{2.12}$$

for all $z, u \in V_i$.

In order to prove this result, we shall require the following technical lemma.

Lemma 2.4. *Let* A *be an algebra satisfying all the hypotheses of Theorem 2.2, with notation as in that theorem. Fix some index* $i \in I$, *and enumerate the elements of the basis* $\mathcal{B}_i$ *of* B_i *as* $b_1, \ldots, b_N$. *For any* $w, z, u \in \Omega_i$ *and any* $k \in \{1, \ldots, N\}$, *define* $\alpha(w, z, u, k)$ *to be the coefficient of the basis element* w *in the expansion of the element* $x(w \otimes b_k \otimes z, u)$ *of the vector space* V_i *over the basis* Ω_i *(this* $x(w \otimes b_k \otimes z, u)$ *is of course obtained by taking* $a = w \otimes b_k \otimes z$ *in Equation 2.10). Then we have for any* $l \in \{1, \ldots, N\}$ *that*

$$\alpha(w, z, u, k)c(w \otimes b_k \otimes z, u)\sigma_i(b_l) = \alpha(u, u, z, l)b_k\sigma_i\big(c(u \otimes b_l \otimes u, z)\big).$$

Further, if

$$c(w \otimes b_k \otimes z, u) \neq 0$$

then we must have

$$x(w \otimes b_k \otimes z, u) = \alpha(w, z, u, k)w.$$

Proof. For any $l \in \{1, \ldots, N\}$ we have by Equation (2.10) that

$$\begin{aligned}&(w \otimes b_k \otimes z)(u \otimes \sigma_i(b_l) \otimes u)\\ &\qquad \equiv x(w \otimes b_k \otimes z, u) \otimes c(w \otimes b_k \otimes z, u)\sigma_i(b_l) \otimes u \qquad \big(\text{mod } J(< i)\big).\end{aligned}$$

But we can also apply $\iota^2 = \mathrm{id}$ to the product $(w \otimes b_k \otimes z)(u \otimes \sigma_i(b_l) \otimes u)$, and use Equation (2.9) and the properties of the anti-involution ι to find that $(w \otimes b_k \otimes z)(u \otimes \sigma_i(b_l) \otimes u)$ is equal to $\iota\big((u \otimes b_l \otimes u)(z \otimes \sigma_i(b_k) \otimes w)\big)$. Then we have

$$
\begin{aligned}
&\iota\big((u \otimes b_l \otimes u)(z \otimes \sigma_i(b_k) \otimes w)\big) \\
&\qquad \equiv \iota\big(x(u \otimes b_l \otimes u, z) \otimes c(u \otimes b_l \otimes u, z)\sigma_i(b_k) \otimes w\big) \quad \big(\mathrm{mod}\ J(< i)\big) \\
&\qquad\qquad \text{(by Equation (2.10) and the fact that } \iota \text{ preserves } J(< i), \\
&\qquad\qquad \text{which follows from Equation (2.9))} \\
&\qquad = w \otimes b_k\sigma_i(c(u \otimes b_l \otimes u, z)) \otimes x(u \otimes b_l \otimes u, z) \\
&\qquad\qquad \text{(by Equation (2.9)).}
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
&x(w \otimes b_k \otimes z, u) \otimes c(w \otimes b_k \otimes z, u)\sigma_i(b_l) \otimes u \\
&\qquad \equiv w \otimes b_k\sigma_i(c(u \otimes b_l \otimes u, z)) \otimes x(u \otimes b_l \otimes u, z) \qquad \big(\mathrm{mod}\ J(< i)\big)
\end{aligned}
$$

and in fact since both sides lie in $V_i \otimes B_i \otimes V_i$ (which has trivial intersection with $J(< i)$), we have

$$
\begin{aligned}
&x(w \otimes b_k \otimes z, u) \otimes c(w \otimes b_k \otimes z, u)\sigma_i(b_l) \otimes u \\
&\qquad\qquad = w \otimes b_k\sigma_i(c(u \otimes b_l \otimes u, z)) \otimes x(u \otimes b_l \otimes u, z). \qquad (2.13)
\end{aligned}
$$

When we expand the right-hand side of this equation as a $\Bbbk$-linear combination over the basis $\mathcal{A}$ of A, we see that it is in fact a $\Bbbk$-linear combination of elements of $\mathcal{A}$ of the form $w \otimes * \otimes *$. Similarly, when we expand the left-hand side as a $\Bbbk$-linear combination over the basis $\mathcal{A}$ of A, we see that it is in fact a $\Bbbk$-linear combination of elements of $\mathcal{A}$ of the form $* \otimes * \otimes u$. It follows that left-hand side of the equation must be equal to

$$\alpha(w, z, u, k)w \otimes c(w \otimes b_k \otimes z, u)\sigma_i(b_l) \otimes u \qquad (2.14)$$

and that the right-hand side must be equal to

$$w \otimes b_k \sigma_i(c(u \otimes b_l \otimes u, z)) \otimes \alpha(u, u, z, l)u.$$

Thus we have

$$\begin{aligned}\alpha(w, z, u, k)w \otimes c(w \otimes b_k \otimes z, u)\sigma_i(b_l) \otimes u \\ = w \otimes b_k \sigma_i(c(u \otimes b_l \otimes u, z)) \otimes \alpha(u, u, z, l)u.\end{aligned}$$

and so

$$\begin{aligned}w \otimes \alpha(w, z, u, k)c(w \otimes b_k \otimes z, u)\sigma_i(b_l) \otimes u \\ = w \otimes \alpha(u, u, z, l)b_k \sigma_i(c(u \otimes b_l \otimes u, z)) \otimes u. \quad (2.15)\end{aligned}$$

Since both w and u are non-zero, it follows that

$$\alpha(w, z, u, k)c(w \otimes b_k \otimes z, u)\sigma_i(b_l) = \alpha(u, u, z, l)b_k \sigma_i(c(u \otimes b_l \otimes u, z))$$

as claimed.

Recall from (2.14) our simplified form of the left-hand side of Equation (2.13): we have

$$\begin{aligned}x(w \otimes b_k \otimes z, u) \otimes c(w \otimes b_k \otimes z, u)\sigma_i(b_l) \otimes u \\ = \alpha(w, z, u, k)w \otimes c(w \otimes b_k \otimes z, u)\sigma_i(b_l) \otimes u \quad (2.16)\end{aligned}$$

for any $l \in \{1, \ldots, N\}$. Now suppose that $c(w \otimes b_k \otimes z, u) \neq 0$. Let us take the expansion of $1 \in B_i$ over the basis $\mathcal{B}_i$, to obtain

$$1 = \sum_{l=1}^{N} \beta_l b_l$$

for some $\beta_1, \ldots, \beta_N \in \Bbbk$. By applying σ_i to both sides of this equation and using the fact that $\sigma_i(1) = 1$, we have

$$1 = \sum_{l=1}^{N} \beta_l \sigma_i(b_l).$$

We now have

$$
\begin{aligned}
0 &\neq c(w \otimes b_k \otimes z, u) \\
&= c(w \otimes b_k \otimes z, u)1 \\
&= c(w \otimes b_k \otimes z, u) \sum_{l=1}^{N} \beta_l \sigma_i(b_l) \\
&= \sum_{l=1}^{N} \beta_l \, c(w \otimes b_k \otimes z, u)\sigma_i(b_l).
\end{aligned}
$$

It follows that

$$\beta_l \, c(w \otimes b_k \otimes z, u)\sigma_i(b_l) \neq 0$$

for some $l \in \{1, \dots, N\}$, and hence that for this l we have

$$c(w \otimes b_k \otimes z, u)\sigma_i(b_l) \neq 0.$$

Then by using Equation (2.16) with this value of l, and the fact that $u \neq 0$, we have

$$x(w \otimes b_k \otimes z, u) = \alpha(w, z, u, k)w.$$

$\square$

Proof of Proposition 2.3. Choose some $i \in I$. We shall first prove that such a bilinear form ϕ_i exists, then that is is unique, and finally that it interacts with σ_i as claimed.

Existence: Let us enumerate the elements of $\mathcal{B}_i$ as $b_1, \dots, b_N$ as in Lemma 2.4. To prove that there exists a $\Bbbk$-bilinear form ϕ_i such that Equation (2.11) holds for all $u, v, w, z \in V_i$ and $b, d \in B_i$, it is enough (by linearity) to find some function $\phi_i : \Omega_i \times \Omega_i \to B_i$ such that

$$(w \otimes b_k \otimes z)(u \otimes b_r \otimes v) \equiv w \otimes b_k \phi_i(z, u) b_r \otimes v \qquad \big(\mathrm{mod}\ J(< i)\big)$$

for any $u, v, w, z \in \Omega_i$ and $k, r \in \{1, \dots, N\}$, and then extend this ϕ_i bilinearly. So let $u, v, w, z \in \Omega_i$ and $k, r \in \{1, \dots, N\}$. We know by Equation (2.10) that

$$(w \otimes b_k \otimes z)(u \otimes b_r \otimes v) \equiv x(w \otimes b_k \otimes z, u) \otimes c(w \otimes b_k \otimes z, u) b_r \otimes v \quad \big(\mathrm{mod}\ J(< i)\big)$$

and so we shall seek to prove that the right-hand side of this equivalence has the desired form.

As in Lemma 2.4, let us define $\alpha(w, z, u, k)$ to be the coefficient of the basis element w in the expansion of the element $x(w \otimes b_k \otimes z, u)$ of the vector space V_i over the basis Ω_i. Further, let us express $1 \in B_i$ as a $\Bbbk$-linear combination over $\mathcal{B}_i$, say

$$1 = \sum_{l=1}^{N} \beta_l b_l,$$

and recall that $\sigma_i(1) = 1$. We thus have

$$\begin{aligned}
&\alpha(w, z, u, k)c(w \otimes b_k \otimes z, u) \\
&\qquad = \alpha(w, z, u, k)c(w \otimes b_k \otimes z, u)\sigma_i\left(\sum_{l=1}^{N} \beta_l b_l\right) \\
&\qquad = \sum_{l=1}^{N} \beta_l \alpha(w, z, u, k)c(w \otimes b_k \otimes z, u)\sigma_i(b_l) \\
&\qquad = \sum_{l=1}^{N} \beta_l \alpha(u, u, z, l) b_k \sigma_i(c(u \otimes b_l \otimes u, z)) \\
&\qquad\qquad \text{(by Lemma 2.4)} \\
&\qquad = b_k \sum_{l=1}^{N} \beta_l \alpha(u, u, z, l)\sigma_i(c(u \otimes b_l \otimes u, z)).
\end{aligned}$$

Now let us define

$$\phi_i(z, u) = \sum_{l=1}^{N} \beta_l \alpha(u, u, z, l)\sigma_i(c(u \otimes b_l \otimes u, z))$$

which depends (as required) only on z and u. Then we have

$$\alpha(w, z, u, k)c(w \otimes b_k \otimes z, u) = b_k \phi_i(z, u). \tag{2.17}$$

Now suppose that $c(w \otimes b_k \otimes z, u) \neq 0$. Then we have

$$(w \otimes b_k \otimes z)(u \otimes b_r \otimes v) \equiv x(w \otimes b_k \otimes z, u) \otimes c(w \otimes b_k \otimes z, u)b_r \otimes v \pmod{J(< i)}$$

$$= \alpha(w, z, u, k) w \otimes c(w \otimes b_k \otimes z, u) b_r \otimes v$$

(by Lemma 2.4)

$$= w \otimes \alpha(w, z, u, k) c(w \otimes b_k \otimes z, u) b_r \otimes v$$

$$= w \otimes b_k \phi_i(z, u) b_r \otimes v$$

(by Equation (2.17)).

If we have $c(w \otimes b_k \otimes z, u) = 0$, then we have on the one hand that

$$(w \otimes b_k \otimes z)(u \otimes b_r \otimes v) \equiv x(w \otimes b_k \otimes z, u) \otimes c(w \otimes b_k \otimes z, u) b_r \otimes v \quad \big(\mathrm{mod}\ J(< i)\big)$$

$$= 0,$$

while on the other hand we have

$$w \otimes b_k \phi_i(z, u) b_r \otimes v = w \otimes \alpha(w, z, u, k) c(w \otimes b_k \otimes z, u) b_r \otimes v$$

(by Equation (2.17))

$$= 0,$$

so that indeed

$$(w \otimes b_k \otimes z)(u \otimes b_r \otimes v) \equiv w \otimes b_k \phi_i(z, u) b_r \otimes v \quad \big(\mathrm{mod}\ J(< i)\big).$$

Uniqueness: Suppose we have two $\Bbbk$-bilinear forms ϕ_i and ϕ_i' mapping $V_i \times V_i$ to B_i, such that

$$(w \otimes d \otimes z)(u \otimes b \otimes v) \equiv w \otimes d\phi_i(z, u) b \otimes v \qquad \big(\mathrm{mod}\ J(< i)\big)$$

and

$$(w \otimes d \otimes z)(u \otimes b \otimes v) \equiv w \otimes d\phi_i'(z, u) b \otimes v \qquad \big(\mathrm{mod}\ J(< i)\big)$$

for any $u, v, w, z \in V_i$ and any $b, d \in B_i$. So for any $z, u \in V_i$, we may take b and d to be 1 and w and v to be u in these equivalences, and then since the right-hand side of both equivalences lies in $V_i \otimes B_i \otimes V_i$, we have

$$u \otimes \phi_i(z, u) \otimes u = u \otimes \phi_i'(z, u) \otimes u$$

and thus (since $u \neq 0$)

$$\phi_i(z,u) = \phi_i'(z,u)$$

so that indeed

$$\phi_i = \phi_i'.$$

Interaction with σ_i: It is enough to prove that

$$\phi_i(z,u) = \sigma_i\big(\phi_i(u,z)\big)$$

for any $z, u \in \Omega_i$. Indeed, let $z, u \in \Omega_i$. On the one hand, we have

$$(z \otimes 1 \otimes z)(u \otimes 1 \otimes u) \equiv z \otimes \phi_i(z,u) \otimes u \pmod{J(< i)}$$

but on the other hand we can apply $\iota^2 = \mathrm{id}$ to the product $(z \otimes 1 \otimes z)(u \otimes 1 \otimes u)$, and use Equation (2.9) and the properties of the anti-involution ι to find that $(z \otimes 1 \otimes z)(u \otimes 1 \otimes u)$ is equal to $\iota\big((u \otimes 1 \otimes u)(z \otimes 1 \otimes z)\big)$. Then we have

$$\begin{aligned}
\iota\big((u \otimes 1 \otimes u)(z \otimes 1 \otimes z)\big) &\equiv \iota(u \otimes \phi_i(u,z) \otimes z) \pmod{J(< i)} \\
&\qquad \text{(using the fact that } \iota \text{ preserves } J(< i)\text{)} \\
&= z \otimes \sigma_i\big(\phi_i(u,z)\big) \otimes u \\
&\qquad \text{(by Equation (2.9)).}
\end{aligned}$$

Thus by the fact that $z \otimes \phi_i(z,u) \otimes u$ and $z \otimes \sigma_i\big(\phi_i(u,z)\big) \otimes u$ both lie in $V_i \otimes B_i \otimes V_i$, we have in fact shown that

$$z \otimes \phi_i(z,u) \otimes u = z \otimes \sigma_i\big(\phi_i(u,z)\big) \otimes u$$

and hence (because z and u are both non-zero) we have

$$\phi_i(z,u) = \sigma_i\big(\phi_i(u,z)\big)$$

as required. $\square$

2.4 The Brauer algebra is an iterated inflation

We shall now complete our proof that the Brauer algebra $\mathrm{B}_{\Bbbk}(r, \delta)$ may be exhibited as an iterated inflation of cellular algebras. Recall from Equation (2.2) that we have (up to isomorphism of $\Bbbk$-vector spaces) a direct sum decomposition

$$\mathrm{B}_{\Bbbk}(r, \delta) \cong \bigoplus_{l \in I_r} V_l \otimes \Bbbk S_l \otimes V_l$$

where for each $l \in I_r$, V_l is the $\Bbbk$-vector space with basis the set Ω_l of all half Brauer diagrams with r nodes and l free nodes. Recall further that this isomorphism is witnessed by the correspondences given for each $l \in I_r$ by the isomorphism (2.1), where a Brauer diagram d with exactly l through strings corresponds to the element $S \otimes \pi \otimes T$ of $V_l \otimes \Bbbk S_l \otimes V_l$, where S and T are the top and bottom of d respectively, and $\pi \in S_l$ is the permutation describing the arrangement of the through strings of d. In the notation of Theorem 2.2, we let V_l and Ω_l be as above, we define the algebra B_l to be $\Bbbk S_l$ (which is cellular by Theorem 2.1), and we define $\mathcal{B}_l$ to be the basis S_l of $\Bbbk S_l$. Then the basis $\mathcal{A}$ of $\mathrm{B}_{\Bbbk}(r, \delta)$ as in Theorem 2.2 is the basis of $\mathrm{B}_{\Bbbk}(r, \delta)$ consisting of all Brauer diagrams with $2r$ nodes. We shall now show that Equations (2.5) and (2.6) are satisfied.

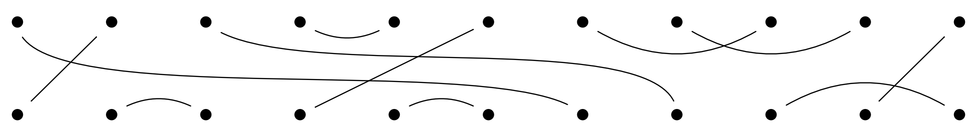
Firstly, we show that ι interacts with the decomposition as required by Equation (2.5) (as shown by König and Xi in Lemma 5.4 of [10]). For this, it is enough to prove that if d is a Brauer diagram corresponding to $S \otimes \pi \otimes T$ as above, then $\iota(d)$ corresponds to $T \otimes \pi^{-1} \otimes S$ (since the anti-involution on $\Bbbk S_l$ maps π to π^{-1}). Indeed, $\iota(d)$ is the Brauer diagram obtained by flipping d upside down, so it certainly has top T and bottom S. Further, if we number the northern nodes of the through strings of d with the numbers 1 to l from left to right, and likewise for the southern nodes, then by the definition of π, the node labelled i on the top row of d is connected to the node labelled $(i)\pi$ on the bottom row. Hence, the node labelled $(i)\pi$ on the *top* row of $\iota(d)$ is connected to the node labelled i on the *bottom* row of $\iota(d)$. This is equivalent to saying that the node labelled i on the top row of $\iota(d)$ is connected to the node labelled $(i)\pi^{-1}$ on the bottom row of $\iota(d)$. Thus indeed $\iota(d)$ corresponds to $T \otimes \pi^{-1} \otimes S$.

Now let us show that the multiplication in $\mathrm{B}_{\Bbbk}(r, \delta)$ behaves as required by Equation (2.6). In fact we can prove a much more precise result:

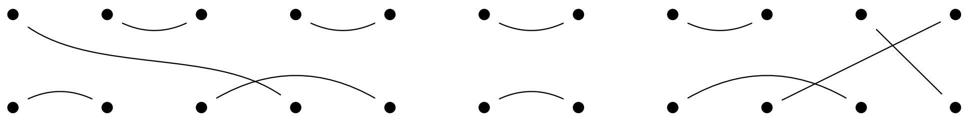
Proposition 2.5. *(König and Xi — Lemma 5.5 in [10]; see also Proposition 4.7 in [5]) Let r be a positive integer and $m, l \in I_r$. Let d_1 and d_2 be Brauer diagrams in $B_{\Bbbk}(r, \delta)$ with m and l through strings, respectively. Then we have half Brauer diagrams $P, Q \in \Omega_m$ and $U, V \in \Omega_l$, and permutations $\pi \in S_m$ and $\sigma \in S_l$ and such that under the isomorphism* (2.1)*, d_1 corresponds to $P \otimes \pi \otimes Q$ and d_2 corresponds to $U \otimes \sigma \otimes V$. Let p be the Brauer diagram formed when computing the product $d_1 d_2$ as explained above, so that $d_1 d_2 = \delta^n p$ for some integer $n \geq 0$ (with δ^0 taken to be 1). Then p has at most l through strings, and whether p has exactly l through strings or not depends only on Q and U.*

If p has exactly l through strings, then p corresponds under the isomorphism (2.1) *to $Z \otimes \theta\sigma \otimes V$, where Z is a half Brauer diagram with l free nodes depending only on P, Q, π and U, and $\theta \in S_l$ depends only on Q, π and U, while the index n of δ (as above) depends only on Q and U.*

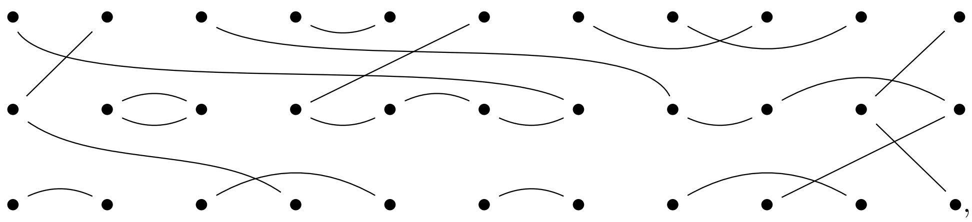
We shall give a formal proof of this proposition presently; first let us consider an example of such a calculation, which will hopefully clarify the proof. Take $r = 11$ and let d_1 be the diagram



with 5 through strings, and d_2 be the diagram

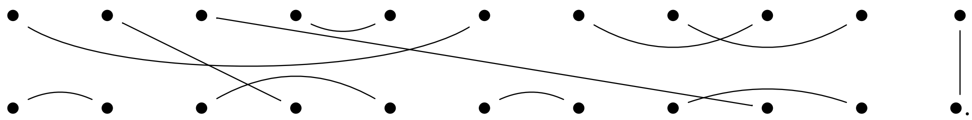


with 3 through strings. To calculate the product $d_1 d_2$, we form the concatenated diagram



(2.18)

from which we see that the product $d_1 d_2$ is δp, where p is



Let us now consider this product in terms of the claims made in Proposition 2.5.

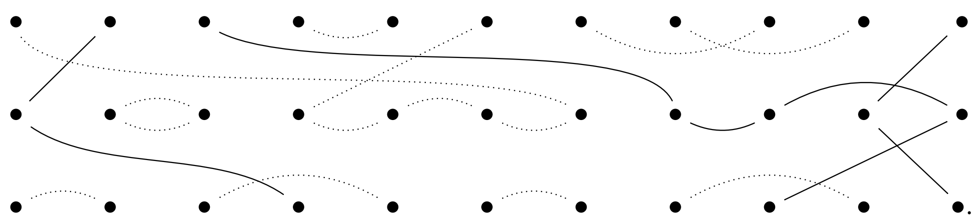
Firstly, notice that p has 3 through strings, which is the same number as d_2. This is so because, in the concatenated diagram (2.18), there are three open paths which have a through string of d_1 "at one end" and a through string of d_2 "at the other end", and in turn this is so because (in the concatenated diagram) the northern node of each through string of d_2 is "connected" to the southern node of a through string of d_1, either because the two nodes coincide in the concatenated diagram, or via a "chain" of horizontal edges; this fact is in turn a consequence purely of the arrangement of free nodes and horizontal edges in the bottom of d_1 and the top of d_2

(these are the half diagrams Q and U of Proposition 2.5, respectively), as claimed in the proposition.

Now the bottom of p is the same as the bottom of d_2 (this half diagram corresponds to the half diagram V of Proposition 2.5); whatever diagram we had chosen for d_1, the horizontal edges present in the bottom of d_2 would have been "inherited" by the bottom row of the concatenated diagram (2.18) and hence by the bottom of p; thus if (with a different choice of d_1) the bottom of p had *not* been equal to the bottom of d_2, then the bottom of p would necessarily have had *more* horizontal edges than the bottom of d_2, and hence p would necessarily have had *fewer* than 3 through strings.

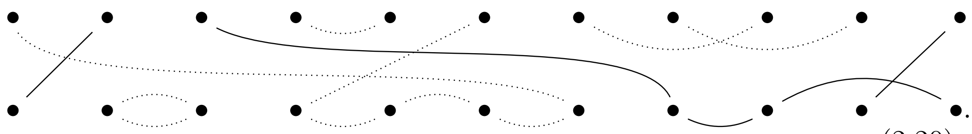
Next, we see that the top of p (corresponding to the half diagram Z in the proposition) has four edges: three are "inherited" from the top of d_1, and the fourth arises from an open path in the concatenated diagram (2.18) which consists of a pair of through strings of d_1 whose southern nodes are connected by a "chain" of horizontal edges from the bottom row of d_1 and the top row of d_2; note in particular that neither the through strings of d_2 nor the horizontal edges in the bottom of d_2 are involved (as claimed in Proposition 2.5).

Now let us return to the three open paths in the concatenated diagram (2.18) which give rise to the through strings of p. Indeed, let us pick them out in the concatenated diagram:



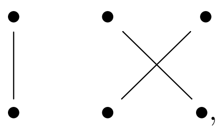
(2.19)

Now recall that we may describe the arrangement of the through strings of a Brauer diagram via a permutation: for d_2 this permutation is $(2\,3) \in S_3$ (this is the permutation σ of Proposition 2.5); for p it is id $\in S_3$. If we extract the "upper part" of the diagram (2.19) by removing the edges corresponding to through strings of d_2 and also the lowest row of nodes together with its horizontal edges, we get



(2.20)

Let us now remove from this diagram all of the dotted edges and the nodes to which they are connected; further, for each of the three remaining open paths, let us replace all of the nodes and edges in the path, except the two "end nodes", with a single edge. We are left with the permutation diagram



(2.21)

which corresponds to the permutation $(2\,3) \in S_3$ in the manner explained above when we defined permutation diagrams; further notice that the diagram (2.20), and hence this permutation $(2\,3)$ obtained from it, depends only on the diagram d_1 and the top of the diagram d_2 (this permutation $(2\,3)$ is the θ of Proposition 2.5). Now if we multiply this permutation $(2\,3)$ obtained from the diagram (2.20) with the permutation $(2\,3)$ describing the layout of the through strings of d_2, we get the permutation id which describes the layout of the through strings of p, and the diagram (2.20) shows why this is so (all of this is in agreement with the claim made in Proposition 2.5 that the permutation describing the arrangement of the through strings of p is equal to $\theta\sigma$).

Finally, the single factor of δ appearing in the product $d_1 d_2$ arises from the single closed loop in the concatenated diagram (2.18). This closed loop is formed entirely from horizontal edges in the bottom of d_1 and the top of d_2 (in the notation of Proposition 2.5, we have $n = 1$, and indeed this value is determined solely by the half diagrams Q and U).

Proof of Proposition 2.5. Recall Lemma 1.5, where we showed that if f_1 and f_2 are planar diagrams, then the number of through strings in the planar diagram formed when calculating the product $f_1 f_2$ cannot be more than the number of through strings in either f_1 or f_2, and further that this number depends only on the bottom of f_1 and the top of f_2. Exactly the same argument used in the proof of that lemma can be applied to the two Brauer diagrams d_1 and d_2, and thus indeed p has at most l through strings, and whether p has exactly l through strings or not depends only on Q and U.

From now on, we shall assume that p has exactly l through strings. Thus p corresponds under the isomorphism (2.1) to $Z \otimes \tau \otimes V'$ for some $Z, V' \in \Omega_l$ and some $\tau \in S_l$. Further, we shall denote by $\mathcal{D}$ the concatenated diagram with three rows of nodes formed when computing the product $d_1 d_2$ (so in our example, $\mathcal{D}$ is the diagram 2.18).

Now all of the horizontal edges on the bottom row of d_2 are "inherited" by the bottom row of $\mathcal{D}$, and hence are also "inherited" by the bottom row of p. Thus all of the edges of the bottom V of d_2 are present in the bottom V' of p. Since both V and V' have l free nodes, it follows that $V' = V$.

By considering $\mathcal{D}$, we see that the edges of Z are exactly the edges of P together with any edges which arise from an open path in $\mathcal{D}$ which has a through string of d_1 at each end and one or more horizontal edges from the middle row of $\mathcal{D}$ between them. Since the horizontal edges on the middle row of $\mathcal{D}$ are all "inherited" from either Q or U, we see that Z does not depend on σ or V, so that as claimed Z depends only on P, Q, π and U.

To see that $\tau = \theta\sigma$ for a suitable θ as claimed, we do exactly as in the above example. Firstly we note that, as in the proof of Lemma 1.5, any through string of p must arise from an open path in $\mathcal{D}$ with a through string of d_1 at one end and a through string of d_2 at the other; since p has exactly l through strings, there are exactly l such open paths, which we shall call the **good** paths of $\mathcal{D}$ (in the above example, the paths picked out in the diagram (2.19) are exactly the good paths). We notice that the good paths of $\mathcal{D}$ are exactly those paths which contain a node corresponding to a northern node of a through string of d_2, and that these nodes are exactly the nodes corresponding to a free node of U (when the nodes of U are taken to correspond to the nodes of the middle row of $\mathcal{D}$ in the natural way). We next extract the "upper part" of $\mathcal{D}$ by removing the edges corresponding to through strings of d_2 and also the lowest row of nodes together with its horizontal edges; let us call this diagram $\mathcal{D}'$ (this $\mathcal{D}'$ corresponds to the diagram (2.20) in the example above). Further, we define the **good** paths of $\mathcal{D}'$ to be exactly those paths in $\mathcal{D}'$ which are formed from part of a good path of $\mathcal{D}$ (in the above example, the paths picked out in the diagram (2.20) are exactly the good paths). Now clearly the diagram $\mathcal{D}'$ does not depend on either σ or V, and moreover the good paths of $\mathcal{D}'$ are exactly those paths which contain a node corresponding to a free node of U. Thus if we do as in the above example and use the good paths of $\mathcal{D}'$ to form a permutation diagram on $2l$ nodes, then this permutation diagram does not depend on either σ or V. Define θ to be the permutation in S_l corresponding to this permutation diagram (in the manner explained above when we defined permutation diagrams), which thus depends only on d_1 and U. It is now clear by considering the diagram $\mathcal{D}$ that $\tau = \theta\sigma$ as required.

Finally, the index n is the number of closed loops in the concatenated diagram $\mathcal{D}$, which we may see depends only on the arrangement of horizontal edges in the bottom of d_1 and the top of d_2, so indeed n depends only on Q and U. $\square$

We may now use Proposition 2.5 to prove that Equation (2.6) holds for our decomposition of the Brauer algebra $\mathrm{B}_{\Bbbk}(r, \delta)$. Recall that for each $l \in I_r$, we have defined Ω_l to be the set of all half Brauer diagrams with r nodes and l free nodes, the algebra B_l to be $\Bbbk S_l$, and $\mathcal{B}_l$ to be the basis S_l of $\Bbbk S_l$. Then the basis $\mathcal{A}$ of $\mathrm{B}_{\Bbbk}(r, \delta)$ as in Theorem 2.2 is the basis consisting of all Brauer diagrams with $2r$ nodes. Let $a \in \mathcal{A}$, let $l \in I_r$, let $U, V \in \Omega_l$ and let $\sigma \in S_l$. Let d be the Brauer diagram corresponding to $U \otimes \sigma \otimes V$ under the isomorphism (2.1). Then we have $ad = \delta^n p$ for some Brauer diagram p and some non-negative integer n.

By Proposition 2.5, p has at most l through strings, and whether or not it has exactly l through strings does not depend on σ or V, but only on a and U. If p has fewer than l through strings, put $x(a, U) = 0$ and $c(a, U) = 0$; it is clear that Equation (2.6) now holds. If p has exactly l through strings, then as in Proposition 2.5, p corresponds to $Z \otimes \theta\sigma \otimes V$ under the isomorphism (2.1). Thus if we define $x(a, U) = Z$ and $c(a, u) = \delta^n \theta$, then Equation (2.6) follows by Proposition 2.5. Finally, notice that by Proposition 2.5, our choice of $x(a, U)$ and $c(a, U)$ has depended *only* on a and U. Thus we have now proved the following:

Theorem 2.6. *(König and Xi — Theorem 5.6 in [10]) The decomposition*

$$B_{\Bbbk}(r, \delta) \cong \bigoplus_{l \in I_r} V_l \otimes \Bbbk S_l \otimes V_l$$

of the Brauer algebra $B_{\Bbbk}(r, \delta)$ given in Equation (2.2) *exhibits $B_{\Bbbk}(r, \delta)$ as an iterated inflation of the cellular algebras $\Bbbk S_l$ for $l \in I_r$, in the sense of Theorem 2.2, with Ω_l being the set of all half Brauer diagrams with r nodes and l free nodes, and $\mathcal{B}_l$ being the basis S_l of $\Bbbk S_l$.*

2.5 The cell modules of an iterated inflation

We shall now show how the cell modules of an iterated inflation of cellular algebras may be obtained from the cell modules of the cellular algebras B_i which appear in the decomposition (2.4). While this method of obtaining the cell modules of an iterated inflation is not explicitly given in [9] or [10], it is implicit in the work done there; Xi gave a more explicit formulation of the idea in [15], in particular in the proof given for Lemma 3.3, and in Corollary 4.10. The result we present here is slightly different from the corresponding ideas in the work of König and Xi, due to the different way we have formulated the concept of an iterated inflation of cellular algebras.

Corollary 2.7. *Let A be an iterated inflation of cellular algebras as in Theorem 2.2, with notation as in that theorem. Then the cell module of A corresponding to $(i, \lambda) \in \Lambda$ is (up to isomorphism) $V_i \otimes \Delta^\lambda$, where Δ^λ is the cell module of B_i corresponding to λ, with the action given by*

$$a(u \otimes \xi) = x(a, u) \otimes c(a, u)\xi$$

for $a \in \mathcal{A}$, $u \in \Omega_i$ and $\xi \in \Delta^\lambda$, where $x(a, u)$ and $c(a, u)$ are as in Theorem 2.2.

Proof. Let $(u, X), (v, Y) \in M(i, \lambda)$, and $a \in \mathcal{A}$. Then as in the proof of Theorem 2.2, we have

$$\begin{aligned} aC^{(i,\lambda)}_{(u,X),(v,Y)} &= a \cdot \left(u \otimes C^\lambda_{X,Y} \otimes v\right) \\ &\equiv x \otimes c\, C^\lambda_{X,Y} \otimes v \qquad \left(\bmod J(< i)\right) \end{aligned}$$

for some $x = x(a, u)$ and some $c = c(a, u)$ as in Equation (2.6). Recall also that if we write x as a $\Bbbk$-linear combination

$$x = \sum_{w \in \Omega_i} \gamma_w w$$

over Ω_i, and for any $(w, U) \in M(i, \lambda)$ we define

$$r_a((w, U), (u, X)) = \gamma_w r_c(U, X),$$

then we have

$$aC^{(i,\lambda)}_{(u,X),(v,Y)} = \sum_{(w,U) \in M(i,\lambda)} r_a((w, U), (u, X)) C^{(i,\lambda)}_{(w,U),(v,Y)} + L$$

where L is a linear combination of elements $C^{(i',\lambda')}_{(u',X'),(v',Y')}$ for $(i',\lambda') < (i,\lambda)$. Thus, the cell module of A corresponding to $(i,\lambda) \in \Lambda$ may be constructed as the $\Bbbk$-vector space with basis the set of all symbols $C_{(u,X)}$ for all $(u,X) \in M(i,\lambda)$ and action given by

$$aC_{(u,X)} = \sum_{(w,U)\in M(i,\lambda)} r_a((w,U),(u,X))C_{(w,U)}$$

(by Section 1.2, in particular Equation (1.6)). We can set up a linear bijection from this module to the $\Bbbk$-vector space $V_i \otimes \Delta^\lambda$ by sending each basis element $C_{(u,X)}$ to $u \otimes C_X$ (where C_X is a basis element of Δ^λ as usual). The formula for the action of A induced on $V_i \otimes \Delta^\lambda$ by this isomorphism is

$$\begin{aligned}
a(u \otimes C_X) &= \sum_{(w,U)\in M(i,\lambda)} r_a((w,U),(u,X))(w \otimes C_U) \\
&= \sum_{w\in\Omega_i} \sum_{U\in M(\lambda)} \gamma_w r_c(U,X)(w \otimes C_U) \\
&= \left(\sum_{w\in\Omega_i} \gamma_w w\right) \otimes \left(\sum_{U\in M(\lambda)} r_c(U,X)C_U\right) \\
&= x(a,u) \otimes (c(a,u)\, C_X)\,.
\end{aligned}$$

Since the elements C_X for $X \in M(\lambda)$ form a basis of Δ^λ, we may easily show that this action of A on $V_i \otimes \Delta^\lambda$ agrees with the formula given above. $\square$

As a consequence of Corollary 2.7, we may obtain the cell modules of $\mathrm{B}_\Bbbk(r,\delta)$ with respect to the cellular structure given in Theorem 2.6. This characterisation of the cell modules of $\mathrm{B}_\Bbbk(r,\delta)$ is implicit in Section 4 of [5], and has since been used by a number of authors.

Proposition 2.8. *Let $l \in I_r$ and $\lambda \in \Lambda_l$ (recall that Λ_l is the set of all partitions of l with the reverse dominance order). Then for the cell module $\Delta^{(l,\lambda)}$ of $B_\Bbbk(r,\delta)$, we have an isomorphism*

$$\Delta^{(l,\lambda)} \cong V_l \otimes S^\lambda$$

of $\Bbbk$*-vector spaces, where (recall)* V_l *is the* $\Bbbk$*-vector space with basis the set of all half Brauer diagrams with* r *nodes and* l *free nodes and* S^λ *is a cell module of* $\Bbbk S_n$*, as explained in Theorem 2.1. The action is as given in Corollary 2.7.*

Proof. This is immediate from Theorems 2.1, 2.2, 2.6, and Corollary 2.7. □

New material in Chapter 2: The main new idea in this chapter is our reformulation of the concept of an iterated inflation of cellular algebras in Theorem 2.2. Of course, this reformulation is closely based on the work of König and Xi, and so it is not a completely new result. Similarly, Proposition 2.3 (which shows that multiplication "within a layer" of an iterated inflation is "governed" by a bilinear form) is also "new" in the sense that it is based on our reformulation in Theorem 2.2, but it is really just a demonstration of the link between Theorem 2.2 and the original work of König and Xi; the same applies to Corollary 2.7 (on the cell modules of an iterated inflation). However, I do not believe that a counterexample to Xi's Lemma about iterated inflations (Lemma 3.3 in [15]) has previously been published, so the content of Appendix A is new.

3 Wreath products of cellular algebras with symmetric groups

The wreath product of an algebra with a symmetric group is a well-known construction which arises naturally in certain areas of representation theory; such wreath products and their representation theory have been studied, for example, by Chuang and Tan in [1].

3.1 Opposite algebras

We begin by briefly reviewing the notion of *opposite algebras*, which will be used below to overcome certain minor technical problems we shall encounter in reconciling our work on wreath products with some of the literature on the subject. Recall that for any unital associative $\Bbbk$-algebra A, we define the **opposite algebra** A^{op} to be the unital associative $\Bbbk$-algebra whose underlying vector space is A, with multiplication $*$ defined by

$$a * b = ba,$$

where the product on the right-hand side is the product in the original algebra A; the unit element of A^{op} is the same as the unit element of A. It is immediate that $\left(A^{\mathrm{op}}\right)^{\mathrm{op}} = A$.

Proposition 3.1. *Let A be a cellular algebra over $\Bbbk$ as in Definition 1.2 with cellular data (Λ, M, C, ι). For each $\lambda \in \Lambda$ and each $S, T \in M(\lambda)$, define $\widehat{C}^{\lambda}_{S,T} = C^{\lambda}_{T,S}$. Then A^{op} is cellular with respect to $(\Lambda, M, \widehat{C}, \iota)$.*

Proof. The properties (C1) and (C2) of Definition 1.2 are immediate by the cellularity of A. For (C3), let $a \in A^{\mathrm{op}}$, $\lambda \in \Lambda$ and $S, T \in M(\lambda)$. Then

$$\begin{aligned} a * \widehat{C}^{\lambda}_{S,T} &= C^{\lambda}_{T,S} a \\ &\equiv \sum_{U \in M(\lambda)} r_{\iota(a)}(U, S) C^{\lambda}_{T,U} \qquad (\mathrm{mod}\ A(< \lambda)) \qquad (\text{by (C3)}') \end{aligned}$$

$$= \sum_{U \in M(\lambda)} \hat{r}_a(U,S) \widehat{C}^{\lambda}_{U,T}$$

(where $\hat{r}_a(U,S)$ is defined to be $r_{\iota(a)}(U,S)$).

Thus $a * \widehat{C}^{\lambda}_{S,T}$ has the required form, since the coefficients $\hat{r}_a(U,S)$ are independent of T and $A^{\mathrm{op}}(< \lambda) = A(< \lambda)$. Thus A^{op} is cellular as claimed.

□

3.2 The wreath product

Let $\mathbb{k}$ be any field. We shall begin by defining our main object of study, the *wreath product* of a $\mathbb{k}$-algebra with a symmetric group S_n. The definition which we shall use is slightly different from the one usually used in the literature (for example, in Section 3 of [1] or Section 6 of [11]); this is because we have adopted the convention that the symmetric group S_n acts on the *right* (see Section 2.1), whereas works such as [1] and [11] adopt the convention that the symmetric group acts on the *left*. Thus, with S_n acting on the right as per our convention, the symmetric group on n letters as used in [1] and [11] is the *opposite group* S_n^{op} of S_n, whose group operation we shall write as $*$, so that $\sigma * \pi = \pi\sigma$. There is, however, a simple relationship between the two different definitions of the wreath product, as explained below.

Definition 3.2. Let A be a finite-dimensional unital associative $\mathbb{k}$-algebra, and n a positive integer. The **wreath product** $S_n \wr A$ of A with the symmetric group S_n is defined to be the $\mathbb{k}$-vector space

$$\mathbb{k}S_n \otimes \left(A^{\otimes n}\right)$$

(where $A^{\otimes n}$ denotes the tensor product of n copies of the vector space A), with multiplication well-defined on pure tensors by the formula

$$(\sigma \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_n)(\pi \otimes b_1 \otimes b_2 \otimes \cdots \otimes b_n)$$
$$= \sigma\pi \otimes (a_{(1)\pi^{-1}} b_1) \otimes (a_{(2)\pi^{-1}} b_2) \otimes \cdots \otimes (a_{(n)\pi^{-1}} b_n) \quad (3.1)$$

where $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n \in A$ and $\sigma, \pi \in S_n$. We may easily verify that this formula does indeed yield a well-defined $\Bbbk$-bilinear multiplication on $S_n \wr A$, and moreover that $S_n \wr A$ equipped with this multiplication is a finite-dimensional unital associative $\Bbbk$-algebra; indeed it has dimension $(\dim(A))^n n!$ and unit element

$$e \otimes 1 \otimes 1 \otimes \cdots \otimes 1$$

where 1 is the unit of A and e is the identity permutation in S_n. Further, we shall adopt the convention that the wreath product $S_0 \wr A$ of A with S_0 is just $\Bbbk S_0$, which (recall) is taken to be the field $\Bbbk$; so note carefully that although we regard both S_1 and S_0 to be the trivial group, $S_1 \wr A$ and $S_0 \wr A$ are *not* isomorphic (unless $A = \Bbbk$), since the first is isomorphic to A, while the second is $\Bbbk$.

We shall adopt the following notation for pure tensors in the wreath product $S_n \wr A$: for $a_1, a_2, \ldots, a_n \in A$ and $x \in \Bbbk S_n$, we shall write

$$(x; a_1, a_2, \ldots, a_n)$$

for the pure tensor

$$x \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_n \ \in \ S_n \wr A.$$

So notice that this notation $(\,\cdot\,;\,\cdot\,,\ldots,\,\cdot\,)$ is $\Bbbk$-linear in each place, and that with this notation the formula (3.1) for multiplication in $S_n \wr A$ becomes

$$\begin{aligned}(\sigma; a_1, a_2, \ldots, a_n)&(\pi; b_1, b_2, \ldots, b_n) \\ &= (\sigma\pi;\ a_{(1)\pi^{-1}} b_1, a_{(2)\pi^{-1}} b_2, \ldots, a_{(n)\pi^{-1}} b_n) \qquad (3.2)\end{aligned}$$

where $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n \in A$ and $\sigma, \pi \in S_n$.

As mentioned above, a different definition of the wreath product is found in much of the literature: in both [1] and [11], the wreath product of A and S_n^{op} is defined to be the vector space $(A^{\otimes n}) \otimes \Bbbk S_n^{\mathrm{op}}$ with a product defined on pure tensors by the formula

$$(a_1, a_2, \ldots, a_n; \sigma)(b_1, b_2, \ldots, b_n; \pi) \\ = (a_1 b_{\sigma^{-1}(1)}, a_2 b_{\sigma^{-1}(2)}, \ldots, a_n b_{\sigma^{-1}(n)}; \sigma * \pi) \quad (3.3)$$

for $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n \in A$ and $\sigma, \pi \in S_n^{\mathrm{op}}$ (where we have adopted a notation for pure tensors which is analogous to the one given above). Again, we may check that Equation (3.3) does indeed give a well-defined associative unital multiplication on $(A^{\otimes n}) \otimes \Bbbk S_n^{\mathrm{op}}$; let us denote the algebra so defined by $A \wr S_n^{\mathrm{op}}$, and further let us emphasise that this $A \wr S_n^{\mathrm{op}}$ is *precisely* the wreath product as defined in [1] and [11]. By direct calculation of products, we may easily show that we have an isomorphism

$$S_n \wr A \cong \big((A^{\mathrm{op}}) \wr S_n^{\mathrm{op}}\big)^{\mathrm{op}}$$

of $\Bbbk$-algebras, via the map (well-)defined on pure tensors by

$$(\sigma; a_1, a_2, \ldots, a_n) \longmapsto (a_1, a_2, \ldots, a_n; \sigma).$$

By applying this isomorphism with A^{op} in place of A and then taking the opposite algebra of both sides, we have also

$$A \wr S_n^{\mathrm{op}} \cong \big(S_n \wr (A^{\mathrm{op}})\big)^{\mathrm{op}}, \quad (3.4)$$

with the isomorphism being

$$(a_1, a_2, \ldots, a_n; \sigma) \longmapsto (\sigma; a_1, a_2, \ldots, a_n). \quad (3.5)$$

We shall use this relationship, together with Proposition 3.1, to apply some of the results we shall obtain below for the algebra $S_n \wr A$ to the algebra $A \wr S_n^{\mathrm{op}}$; since $A \wr S_n^{\mathrm{op}}$ is the version of the wreath product most commonly found in the literature, this should make our results more readily usable.

Now if H is a subgroup of S_n, then the subspace of $S_n \wr A$ spanned by all pure tensors of the form

$$(\sigma; a_1, a_2, \ldots, a_n)$$

for $a_1, a_2, \ldots, a_n \in A$ and $\sigma \in H$ may easily be shown to be a subalgebra of $S_n \wr A$, which we shall denote by $H \wr A$. It is easy to see that this subalgebra of $S_n \wr A$ is identical to the algebra which may be obtained by replacing S_n with its subgroup H in the definition of $S_n \wr A$, and thus we shall identify $H \wr A$ with this algebra, and call $H \wr A$ the **wreath product** of A with H. It is easy to see that $H \wr A$ has $\Bbbk$-dimension $(\dim(A))^n |H|$. In particular, consider the subalgebra $\{e\} \wr A$ of $S_n \wr A$, where e is the identity permutation in S_n: it is easy to see that this subalgebra is isomorphic to the n-fold tensor product algebra $A^{\otimes n}$.

3.3 Compositions and Young subgroups of S_n

We shall now review some well-known ideas and facts about the structure of the symmetric group S_n. In particular, we shall recall the familiar method of associating to each composition μ of n a subgroup S_μ of S_n, called a *Young subgroup*.

Let n be a positive integer. A **composition** of n is a finite tuple

$$\mu = (\mu_1, \mu_2, \ldots, \mu_m)$$

where $m \geq 1$ and each μ_i is a non-negative integer, such that

$$\mu_1 + \mu_2 + \ldots + \mu_m = n.$$

The number μ_i is called the i-th **part** of μ, and we may refer to μ as a **composition of n with m parts**. We shall adopt the notation $\mu \vDash n$ to mean that μ is a composition of n. Conversely, if m is a positive integer and

$$\mu = (\mu_1, \mu_2, \ldots, \mu_m)$$

is any finite tuple of non-negative integers which are not all zero, then we may define the **size** of μ to be

$$|\mu| = \mu_1 + \mu_2 + \ldots + \mu_m$$

and it is then clear that $\mu \vDash |\mu|$.

Now let $\mu = (\mu_1, \mu_2, \ldots, \mu_m) \vDash n$ (so we have $m, n \geq 1$). Define $\hat{\mu}_0 = 0$ and for $i = 1, \ldots, m$, define

$$\hat{\mu}_i = \sum_{j=1}^{i} \mu_j.$$

Then for $i = 1, \ldots, m$, define a set Θ_i via

$$\Theta_i = \begin{cases} \{\hat{\mu}_{i-1} + 1, \ldots, \hat{\mu}_i\} & \text{if } \mu_i > 0 \\ \varnothing & \text{if } \mu_i = 0, \end{cases}$$

so that, for example, if $n = 10$ and $\mu = (3, 0, 1, 2, 0, 4)$, then we have

$$\Theta_1 = \{1, 2, 3\},\ \Theta_2 = \varnothing,\ \Theta_3 = \{4\},\ \Theta_4 = \{5, 6\},\ \Theta_5 = \varnothing,\ \Theta_6 = \{7, 8, 9, 10\}.$$

Then for each i we have $\Theta_i \subseteq \{1, \ldots, n\}$ and hence S_{Θ_i} is a subgroup of S_n (recall that we have adopted the convention that $S_\varnothing$ is the trivial group). Further, since the sets Θ_i for $i = 1, \ldots, m$ are pairwise disjoint, it follows that the product

$$S_{\Theta_1} \times S_{\Theta_2} \times \ldots \times S_{\Theta_m} \tag{3.6}$$

of subgroups of S_n is in fact a direct product, which is called the **Young subgroup** of μ in S_n, and which we denote by S_μ. Now for each $i = 1, \ldots, m$, we have

$$|\Theta_i| = \mu_i$$

and hence the group S_{Θ_i} is isomorphic to S_{μ_i}; moreover, if $\mu_i > 0$ then there is a *canonical* identification of Θ_i with the set $\{1, \ldots, \mu_i\}$ by mapping $\hat{\mu}_{i-1} + j$ to j, so that we obtain a canonical isomorphism from S_{Θ_i} to S_{μ_i}; of course if $\mu_i = 0$ then we have a unique (trivial) isomorphism from S_{Θ_i} to S_{μ_i}. For $\sigma \in S_{\mu_i}$, we shall usually write $\hat{\sigma}$ for the corresponding element of S_{Θ_i} under this isomorphism. It now follows that S_μ is canonically isomorphic to the direct product

$$S_{\mu_1} \times S_{\mu_2} \times \ldots \times S_{\mu_m}. \tag{3.7}$$

We thus obtain a decomposition of the group algebra $\Bbbk S_\mu$ as a tensor product of symmetric group algebras

$$\Bbbk S_\mu \cong \Bbbk S_{\mu_1} \otimes \Bbbk S_{\mu_2} \otimes \cdots \otimes \Bbbk S_{\mu_m}. \tag{3.8}$$

Further, since S_μ is a subgroup of S_n, we have as above the subalgebra $S_\mu \wr A$ of $S_n \wr A$; it is easy to show that we have an isomorphism of $\Bbbk$-algebras

$$(S_{\mu_1} \wr A) \otimes (S_{\mu_2} \wr A) \otimes \cdots \otimes (S_{\mu_m} \wr A) \longrightarrow S_\mu \wr A$$

well-defined on pure tensors by mapping

$$(\sigma_1; a^1_1, a^1_2, \ldots, a^1_{\mu_1}) \otimes (\sigma_2; a^2_1, a^2_2, \ldots, a^2_{\mu_2}) \otimes \cdots \otimes (\sigma_m; a^m_1, a^m_2, \ldots, a^m_{\mu_m})$$

to

$$(\hat{\sigma}_1 \cdots \hat{\sigma}_m\, ;\, a^1_1, a^1_2, \ldots, a^1_{\mu_1}, a^2_1, a^2_2, \ldots, a^2_{\mu_2}, \ldots, a^m_1, a^m_2, \ldots, a^m_{\mu_m}) \tag{3.9}$$

where each a^i_j lies in A, and $\sigma_i \in S_{\mu_i}$, and $\hat{\sigma}_i$ represents the image in S_{Θ_i} of σ_i under the canonical isomorphism; note also that if $\mu_i = 0$ then $(\sigma_i; a^i_1, a^i_2, \ldots, a^i_{\mu_i})$ is understood to be just $1 \in S_0 \wr A \cong \Bbbk$, and $\sigma_i = e$.

Let us keep $\mu = (\mu_1, \mu_2, \ldots, \mu_m) \vDash n$ as above. Let $\mathcal{L}^\mu$ be a complete family of left coset representatives of S_μ in S_n, without redundancy (note that by a *left* coset we mean a coset of the form xS_μ for $x \in S_n$). Further, let $\mathcal{T}^\mu$ be the set of all tuples

$$(p_1, p_2, \ldots, p_n)$$

where each p_j lies in $\{1, \ldots, m\}$ and for each $i = 1, \ldots, m$ we have

$$|\{j : p_j = i\}| = \mu_i$$

(that is, each i appears exactly μ_i times in the tuple). There is a natural bijection between $\mathcal{L}^\mu$ and $\mathcal{T}^\mu$, as we shall now show. Indeed, let

$$\omega^\mu = (\underbrace{1, \ldots, 1}_{\mu_1 \text{ places}}, \underbrace{2, \ldots, 2}_{\mu_2 \text{ places}}, \ldots, \underbrace{m, \ldots, m}_{\mu_m \text{ places}}) \in \mathcal{T}^\mu.$$

Now S_n has a natural left action on $\mathcal{T}^\mu$ given by

$$\sigma \cdot (p_1, p_2, \ldots, p_n) = (p_{(1)\sigma}, p_{(2)\sigma}, \ldots, p_{(n)\sigma}).$$

It is clear that this action is transitive, so that

$$\mathcal{T}^\mu = \mathrm{Orbit}(\omega^\mu),$$

and further it is also clear that S_μ is the stabiliser of ω^μ. Hence, by the well-known Orbit-Stabiliser theorem, we have a bijection from $\mathcal{L}^\mu$ to $\mathcal{T}^\mu$, given by

$$x \longmapsto x \cdot \omega^\mu. \tag{3.10}$$

The reverse direction of this bijection is a bijective mapping from $\mathcal{T}^\mu$ to $\mathcal{L}^\mu$ which takes an element $(p_1, p_2, \ldots, p_n) \in \mathcal{T}^\mu$ to an element $x \in \mathcal{L}^\mu$ such that

$$(p_1, p_2, \ldots, p_n) = x \cdot \omega^\mu$$

which implies

$$x^{-1} \cdot (p_1, p_2, \ldots, p_n) = \omega^\mu,$$

and further the fact that the map (3.10) is a bijection implies that this x is the unique element of $\mathcal{L}^\mu$ whose inverse acts on $(p_1, p_2, \ldots, p_n)$ to give ω^μ.

We now summarise the above discussion in a proposition.

Proposition 3.3. *Let n be a positive integer, and $\mu = (\mu_1, \mu_2, \ldots, \mu_m)$ be a composition of n with m parts. Then for any $(p_1, p_2, \ldots, p_n) \in \mathcal{T}^\mu$, there exists a unique $x \in \mathcal{L}^\mu$ such that*

$$(p_{(1)x^{-1}}, p_{(2)x^{-1}}, \ldots, p_{(n)x^{-1}}) = (\underbrace{1, \ldots, 1}_{\mu_1 \text{ places}}, \underbrace{2, \ldots, 2}_{\mu_2 \text{ places}}, \ldots, \underbrace{m, \ldots, m}_{\mu_m \text{ places}})$$

and the mapping so induced from $\mathcal{T}^\mu$ to $\mathcal{L}^\mu$ is a bijection.

Now for any $n \geq 1$, and any $m \geq 1$, let us define $\Lambda_n^{\vDash}(m)$ to be the set of all compositions of n with exactly m parts. A simple order on $\Lambda_n^{\vDash}(m)$ is the **backwards lexicographic order**, which we shall denote simply by $<$, where for compositions

$$\mu = (\mu_1, \mu_2, \dots, \mu_m) \text{ and } \hat{\mu} = (\hat{\mu}_1, \hat{\mu}_2, \dots, \hat{\mu}_m)$$

in $\Lambda_n^{\models}(m)$, we define $\mu < \hat{\mu}$ to mean that there exists some $i \in \{1, \dots, m\}$ such that $\mu_j = \hat{\mu}_j$ for all j such that $m \geq j > i$, and $\mu_i < \hat{\mu}_i$. It is easy to see that this defines a strict total order on $\Lambda_n^{\models}(m)$, and we shall denote the corresponding non-strict order by $\leq$, as usual. Note that the name *backwards lexicographic order* for this order is not standard; this order is sometimes called the *reverse lexicographic order*, but we shall not use that name to avoid confusion with the order which may be obtained by simply reversing the standard lexicographic order (in the same way that we obtained the reverse dominance order from the standard dominance order in Section 2.1), which is *not* the same order as this backwards lexicographic order.

3.4 Construction of modules for $S_n \wr A$

In this section, we shall describe a well-known method of combining modules of a $\Bbbk$-algebra A with modules of certain symmetric group algebras to produce modules for the subalgebra $S_\mu \wr A$ of the wreath product $S_n \wr A$, where μ is some composition of n; we shall also describe how $S_\mu \wr A$-modules may be used to produce $S_n \wr A$-modules. These ideas have been described, for example, in Section 3 of [1] and (for the case of the wreath product of *groups*) in Chapter 4 of [7], although the definitions given in this section are slightly different from the corresponding definitions in those accounts due to the fact that we have a different definition of the wreath product. Note that in this section, we shall make several definitions involving tensor products which are given by a formula "on pure tensors"; all of these definitions may be shown to be well-defined via arguments like those given in "A remark about tensor products" on page 8. Further, we shall not give proofs that the constructions described below do indeed yield modules for the given algebras; all of these proofs consist purely of routine but sometimes lengthy verifications.

For the rest of this section, let us fix some positive integers n and m, and some composition $\mu = (\mu_1, \mu_2, \ldots, \mu_m) \in \Lambda_n^{\vDash}(m)$.

Let $V_1, \ldots, V_m$ be left A-modules. Then we define $(V_1, \ldots, V_m)^{\wr\mu}$ to be the left $S_\mu \wr A$-module with underlying vector space

$$\left(V_1^{\otimes \mu_1}\right) \otimes \cdots \otimes \left(V_m^{\otimes \mu_m}\right)$$

with the action given by

$$(\sigma; a_1, a_2, \ldots, a_n)(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = \\ (a_{(1)\sigma} v_{(1)\sigma}) \otimes (a_{(2)\sigma} v_{(2)\sigma}) \otimes \cdots \otimes (a_{(n)\sigma} v_{(n)\sigma})$$

for $a_1, a_2, \ldots, a_n \in A$, $\sigma \in S_\mu$, and each v_j in the appropriate V_i. Notice that, since $\sigma \in S_\mu$, we have that the right-hand side of the above equation does indeed lie in the underlying vector space of $(V_1, \ldots, V_m)^{\wr\mu}$. Further, if $\mu_i = 0$ for some i, then the factor $V_i^{\otimes \mu_i}$ is taken to be $\Bbbk$ and thus may be ignored when taking the tensor product over $\Bbbk$. Note also that if $\mu = (n)$ (so that $m = 1$ and we may write V_1 as just V), we have that $(V)^{\wr\mu}$ is equal as a vector space to $V^{\otimes n}$; we thus introduce the shorthand notation $V^{\wr n}$ for this module, which is of course a module for $S_n \wr A$ since $S_\mu = S_n$. For convenience, let us define $V^{\wr 0}$ to be the trivial $S_0 \wr A$-module $\Bbbk$, recalling that $S_0 \wr A \cong \Bbbk$.

There is another way of viewing the module $(V_1, \ldots, V_m)^{\wr\mu}$. Recall the isomorphism (3.9) which allows us to identify $S_\mu \wr A$ with the tensor product algebra

$$(S_{\mu_1} \wr A) \otimes (S_{\mu_2} \wr A) \otimes \cdots \otimes (S_{\mu_m} \wr A),$$

for which we may form as usual the tensor product module

$$V_1^{\wr \mu_1} \otimes V_2^{\wr \mu_2} \otimes \cdots \otimes V_m^{\wr \mu_m} \tag{3.11}$$

where from above each $V_i^{\wr\mu_i}$ is a $S_{\mu_i}\wr A$-module. We may thus view the module (3.11) as a $S_\mu\wr A$-module, which we may identify with $(V_1,\ldots,V_m)^{\wr\mu}$ via the mapping given by

$$\begin{aligned}(v_1^1\otimes v_2^1\otimes\cdots\otimes v_{\mu_1}^1)\otimes(v_1^2\otimes v_2^2\otimes\cdots\otimes v_{\mu_2}^2)\otimes\cdots\otimes(v_1^m\otimes v_2^m\otimes\cdots\otimes v_{\mu_m}^m)\\ \longmapsto v_1^1\otimes v_2^1\otimes\cdots\otimes v_{\mu_1}^1\otimes v_1^2\otimes v_2^2\otimes\cdots\otimes v_{\mu_2}^2\otimes\cdots\otimes v_1^m\otimes v_2^m\otimes\cdots\otimes v_{\mu_m}^m\end{aligned}$$

from the module (3.11) to $(V_1,\ldots,V_m)^{\wr\mu}$, which is indeed an isomorphism of $S_\mu\wr A$-modules.

Now let X be a left $S_\mu\wr A$-module and U a left $\Bbbk S_\mu$-module. Then the vector space tensor product $X\otimes U$ is an $S_\mu\wr A$-module when equipped with the action

$$(\sigma;a_1,a_2,\ldots,a_n)(x\otimes u)=((\sigma;a_1,a_2,\ldots,a_n)x)\otimes(\sigma u)$$

for $a_i\in A$, $\sigma\in S_\mu$, $x\in X$ and $u\in U$ (note that in the second factor of this tensor product, we are essentially "inflating" the $\Bbbk S_\mu$-module U to a $S_\mu\wr A$-module by exploiting the fact that $\Bbbk S_\mu$ occurs as the quotient of $S_\mu\wr A$ by the subalgebra $\{e\}\wr A$). We shall denote this module by $X\oslash U$.

Now recall from Equation (3.8) above that we have a decomposition

$$\Bbbk S_\mu\cong\Bbbk S_{\mu_1}\otimes\Bbbk S_{\mu_2}\otimes\cdots\otimes\Bbbk S_{\mu_m}.$$

of $\Bbbk S_\mu$ as a tensor product of algebras. Now if for each $i=1,\ldots,m$ we have a left module U_i for $\Bbbk S_{\mu_i}$, then we may form the (outer) tensor product module $U_1\otimes\cdots\otimes U_m$ for $\Bbbk S_{\mu_1}\otimes\cdots\otimes\Bbbk S_{\mu_m}$ as usual. By the above isomorphism, we may then regard $U_1\otimes\cdots\otimes U_m$ as a $\Bbbk S_\mu$-module. To understand the action of $\Bbbk S_\mu$ on this module, let $\pi\in S_\mu$. By the decomposition (3.6) of S_μ, we have a unique factorisation $\pi=\hat\pi_1\hat\pi_2\cdots\hat\pi_m$ of π, where each $\hat\pi_i\in S_{\Theta_i}$ is the image under the canonical isomorphism of $\pi_i\in S_{\mu_i}$ (see above). Then if $u_i\in U_i$ for $i=1,\ldots,m$, the action of π on the pure tensor $u_1\otimes\cdots\otimes u_m$ in $U_1\otimes\cdots\otimes U_m$ is given by

$$\pi(u_1\otimes\cdots\otimes u_m)=(\pi_1u_1)\otimes\cdots\otimes(\pi_mu_m).\tag{3.12}$$

We shall now show how the above constructions may be used to produce modules for $S_\mu \wr A$ from modules of A and modules of the symmetric group algebras $\Bbbk S_{\mu_i}$, in a way which will prove useful below. Indeed, for each $i = 1, \ldots, m$, let V_i be a left A-module and U_i a left $\Bbbk S_{\mu_i}$-module. By applying the above constructions, we obtain an $S_\mu \wr A$-module

$$(V_1, \ldots, V_m)^{\wr \mu} \oslash (U_1 \otimes \cdots \otimes U_m).$$

We shall now give an alternative method for constructing this module. For each $i = 1, \ldots, m$, we form the $S_{\mu_i} \wr A$-module $V_i^{\wr \mu_i}$, and then the $S_{\mu_i} \wr A$-module $V_i^{\wr \mu_i} \oslash U_i$. We then take the (outer) tensor product of these modules $V_i^{\wr \mu_i} \oslash U_i$ to form a module

$$\left(V_1^{\wr \mu_1} \oslash U_1\right) \otimes \cdots \otimes \left(V_m^{\wr \mu_m} \oslash U_m\right)$$

for the tensor product algebra

$$(S_{\mu_1} \wr A) \otimes \cdots \otimes (S_{\mu_m} \wr A),$$

which by the isomorphism (3.9) we may identify with $S_\mu \wr A$, and hence regard

$$\left(V_1^{\wr \mu_1} \oslash U_1\right) \otimes \cdots \otimes \left(V_m^{\wr \mu_m} \oslash U_m\right)$$

as an $S_\mu \wr A$-module. We may now easily show that there is a well-defined $\Bbbk$-linear mapping from

$$\left(V_1^{\wr \mu_1} \oslash U_1\right) \otimes \cdots \otimes \left(V_m^{\wr \mu_m} \oslash U_m\right)$$

to

$$(V_1, \ldots, V_m)^{\wr \mu} \oslash (U_1 \otimes U_2 \otimes \cdots \otimes U_m)$$

which is given by taking the pure tensor

$$(v_1^1 \otimes v_2^1 \otimes \cdots \otimes v_{\mu_1}^1 \otimes u_1) \otimes (v_1^2 \otimes v_2^2 \otimes \cdots \otimes v_{\mu_2}^2 \otimes u_2) \otimes \cdots$$
$$\cdots \otimes (v_1^m \otimes v_2^m \otimes \cdots \otimes v_{\mu_m}^m \otimes u_m)$$

to

$$(v^1_1 \otimes v^1_2 \otimes \cdots \otimes v^1_{\mu_1} \otimes v^2_1 \otimes v^2_2 \otimes \cdots \otimes v^2_{\mu_2} \otimes \cdots \\ \cdots \otimes v^m_1 \otimes v^m_2 \otimes \cdots \otimes v^m_{\mu_m}) \otimes (u_1 \otimes u_2 \otimes \cdots \otimes u_m).$$

This mapping may then easily be shown to be an isomorphism of $S_\mu \wr A$-modules.

We have so far been concerned with modules for the subalgebra $S_\mu \wr A$ of $S_n \wr A$. For any left $S_\mu \wr A$-module W, we define as usual the **induced module** formed from W to be the left $S_n \wr A$-module

$$(S_n \wr A) \otimes_{S_\mu \wr A} W,$$

which we shall denote by $\mathrm{Ind}^n_\mu W$.

The constructions described in this section arise naturally when working with modules over wreath products. For example, in Chapter 4 of [7], James and Kerber use exactly these constructions to obtain the simple modules of the wreath product group of a finite group with a symmetric group.

3.5 The wreath product of a cellular algebra and a symmetric group

We have now finished our preliminary discussion of the wreath product $S_n \wr A$ for an arbitrary $\Bbbk$-algebra A, and so we turn to the real subject of our work in this chapter: the wreath product $S_n \wr A$ where A is a *cellular* $\Bbbk$-algebra. In this section, we shall show that $S_n \wr A$ is then a cellular algebra; this has been proved by Geetha and Goodman in the case that A is not only cellular but *cyclic cellular*, meaning that all of the cell modules of A are cyclic (Theorem 4.1 in [4]). Their proof is quite combinatorial in nature, and draws on the work of Dipper, James, and Mathas in [3] and of Murphy in [14]. However, we shall prove the cellularity of $S_n \wr A$, where A is any cellular algebra, by exhibiting it as an iterated inflation in the sense of Theorem 2.2. Note also that the version of the wreath product used in [4] is the construction which we have called $A \wr S_n^{\text{op}}$ (see Section 3.2), but as indicated above we may easily transfer results between the two different wreath products $S_n \wr A$ and $A \wr S_n^{\text{op}}$.

Firstly, let us fix some cellular algebra A with cellular data (Λ, M, C, ι) as in Definition 1.2. Further, as noted at the beginning of Section 1.3, we may assume without loss of generality that the partially ordered set Λ is in fact totally ordered. Thus, if we let $r = |\Lambda|$, then we may list the elements of Λ in order, say

$$\lambda_1 < \lambda_2 < \ldots < \lambda_r.$$

Next, recall that if $n = 0$ then $S_n \wr A$ is just $\Bbbk$, which is trivially cellular. So from now on we shall assume that $n \geq 1$.

Now recall that to apply Theorem 2.2 to $S_n \wr A$, we need a subspace decomposition of $S_n \wr A$ as in Equation (2.4), where each subspace has (up to isomorphism of $\Bbbk$-vector spaces) a further decomposition as a tensor product. In the statement of Theorem 2.2, we begin with this decomposition, and then by taking bases for all of the vector spaces V_i and cellular algebras B_i involved in the decomposition, we produce a basis $\mathcal{A}$ for the original algebra. However, we shall obtain our desired decomposition of $S_n \wr A$ by first defining this basis $\mathcal{A}$ of $S_n \wr A$, and then showing how we may use this basis to define the required decomposition. Thus, we define $\mathcal{A}$ to be the basis of $S_n \wr A$ consisting of all elements of the form

$$\Big(\sigma\,;\, C^{\epsilon_1}_{P_1,Q_1}, C^{\epsilon_2}_{P_2,Q_2}, \ldots, C^{\epsilon_n}_{P_n,Q_n}\Big) \tag{3.13}$$

where $\sigma \in S_n$ and for each $i = 1, \ldots, n$, $\epsilon_i \in \Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_r\}$ and $P_i, Q_i \in M(\epsilon_i)$; it is immediate that $\mathcal{A}$ is indeed a basis of $S_n \wr A$ because S_n is a basis of $\Bbbk S_n$ and the cellular basis of A is of course a basis of A. Now take an element of the form (3.13), and for each $i = 1, \ldots, n$ define p_i to be the unique element of $\{1, \ldots, r\}$ such that $\epsilon_i = \lambda_{p_i}$. Now define $\mu_j = |\{i : p_i = j\}|$ for each $j = 1, \ldots, r$, and define $\mu = (\mu_1, \ldots, \mu_r)$. Then clearly μ is a composition of n with r parts, which we shall call the **layer index** of the element (3.13) of $\mathcal{A}$.

For each $\mu \in \Lambda_n^{\vDash}(r)$, let us define $\mathcal{A}^\mu$ to be the set of all elements of $\mathcal{A}$ with layer index μ, and further let us define X^μ to be the $\Bbbk$-span of $\mathcal{A}^\mu$ in $S_n \wr A$. Since the collection of sets $\mathcal{A}^\mu$ as μ varies over $\Lambda_n^{\vDash}(r)$ is clearly a partition of $\mathcal{A}$ by disjoint non-empty sets, it follows that we have a vector space direct sum decomposition

$$S_n \wr A = \bigoplus_{\mu \in \Lambda_n^{\vDash}(r)} X^\mu.$$

This is the decomposition of $S_n \wr A$ which we shall use to exhibit $S_n \wr A$ as an iterated inflation; our next step is to understand how to decompose each subspace X^μ as a tensor product $V_\mu \otimes B_\mu \otimes V_\mu$ as in Theorem 2.2.

Let $\mu \in \Lambda_n^{\vDash}(r)$ and E be an element of $\mathcal{A}^\mu$, so that we have

$$E = \left(\sigma\,;\, C^{\epsilon_1}_{P_1,Q_1}, C^{\epsilon_2}_{P_2,Q_2}, \ldots, C^{\epsilon_n}_{P_n,Q_n}\right)$$

as in Equation (3.13) above, and define the tuple $(p_1, \ldots, p_n)$ over $\{1, \ldots, r\}$ as above. Then clearly we have

$$(p_1, \ldots, p_n) \in \mathcal{T}^\mu,$$

and so by Proposition 3.3, we have a unique $y \in \mathcal{L}^\mu$ such that

$$(p_{(1)y^{-1}}, p_{(2)y^{-1}}, \ldots, p_{(n)y^{-1}}) = (\underbrace{1, \ldots, 1}_{\mu_1 \text{ places}}, \underbrace{2, \ldots, 2}_{\mu_2 \text{ places}}, \ldots, \underbrace{r, \ldots, r}_{\mu_r \text{ places}}),$$

from which it follows that

$$(\epsilon_{(1)y^{-1}}, \epsilon_{(2)y^{-1}}, \ldots, \epsilon_{(n)y^{-1}}) = (\underbrace{\lambda_1, \ldots, \lambda_1}_{\mu_1 \text{ places}}, \underbrace{\lambda_2, \ldots, \lambda_2}_{\mu_2 \text{ places}}, \ldots, \underbrace{\lambda_r, \ldots, \lambda_r}_{\mu_r \text{ places}}).$$

Then we have

$$\begin{aligned} E &= \left((\sigma y)y^{-1}\,;\, C^{\epsilon_1}_{P_1,Q_1}, C^{\epsilon_2}_{P_2,Q_2}, \ldots, C^{\epsilon_n}_{P_n,Q_n}\right) \\ &= \left(\sigma y\,;\, C^{\epsilon_{(1)y^{-1}}}_{P_{(1)y^{-1}},Q_{(1)y^{-1}}}, C^{\epsilon_{(2)y^{-1}}}_{P_{(2)y^{-1}},Q_{(2)y^{-1}}}, \ldots, C^{\epsilon_{(n)y^{-1}}}_{P_{(n)y^{-1}},Q_{(n)y^{-1}}}\right)(y^{-1}; 1, \ldots, 1) \end{aligned}$$

$$= \left(\sigma y\,;\, C^{\eta_1}_{X_1,Y_1}, C^{\eta_2}_{X_2,Y_2}, \ldots, C^{\eta_n}_{X_n,Y_n}\right)(y^{-1}; 1, \ldots, 1)$$

(where we have defined $\eta_i = \epsilon_{(i)y^{-1}}$, $X_i = P_{(i)y^{-1}}$, and $Y_i = Q_{(i)y^{-1}}$).

Further, we have $\sigma y = x\theta$ for unique $x \in \mathcal{L}^\mu$ and unique $\theta \in S_\mu$, so

$$\begin{aligned} E &= \left(x\theta\,;\, C^{\eta_1}_{X_1,Y_1}, C^{\eta_2}_{X_2,Y_2}, \ldots, C^{\eta_n}_{X_n,Y_n}\right)(y^{-1}; 1, \ldots, 1) \\ &= (x; 1, \ldots, 1)\left(\theta\,;\, C^{\eta_1}_{X_1,Y_1}, C^{\eta_2}_{X_2,Y_2}, \ldots, C^{\eta_n}_{X_n,Y_n}\right)(y^{-1}; 1, \ldots, 1), \end{aligned} \tag{3.14}$$

where from above we have

$$(\eta_1, \eta_2, \ldots, \eta_n) = (\underbrace{\lambda_1, \ldots, \lambda_1}_{\mu_1 \text{ places}}, \underbrace{\lambda_2, \ldots, \lambda_2}_{\mu_2 \text{ places}}, \ldots, \underbrace{\lambda_r, \ldots, \lambda_r}_{\mu_r \text{ places}}). \tag{3.15}$$

Further, the fact that x, y and θ were uniquely determined rather than arbitrarily chosen in the above argument implies that the above expression (3.14) is in fact the *unique* expression of E in that form, in the sense that if we have $w, z \in \mathcal{L}^\mu$ and $\pi \in S_\mu$ such that

$$E = (w; 1, \ldots, 1)\left(\pi\,;\, C^{\delta_1}_{S_1,T_1}, C^{\delta_2}_{S_2,T_2}, \ldots, C^{\delta_n}_{S_n,T_n}\right)(z^{-1}; 1, \ldots, 1),$$

(which implies that $\delta_i = \epsilon_{(i)z^{-1}}$, $S_i = P_{(i)z^{-1}}$, and $T_i = Q_{(i)z^{-1}}$) with

$$(\delta_1, \delta_2, \ldots, \delta_n) = (\underbrace{\lambda_1, \ldots, \lambda_1}_{\mu_1 \text{ places}}, \underbrace{\lambda_2, \ldots, \lambda_2}_{\mu_2 \text{ places}}, \ldots, \underbrace{\lambda_r, \ldots, \lambda_r}_{\mu_r \text{ places}}),$$

then we must have $w = x$, $z = y$, and $\pi = \theta$, and hence $\delta_i = \eta_i$, $S_i = X_i$ and $T_i = Y_i$ for each i. Finally, let us note that any element of $S_n \wr A$ of the form (3.14) for some $x, y \in \mathcal{L}^\mu$, some $\theta \in S_\mu$, some $\eta_1, \ldots, \eta_n$ such that (3.15) is satisfied, and some $X_i, Y_i \in M(\eta_i)$ is clearly an element of the basis $\mathcal{A}^\mu$ of X^μ. Summarising, we have shown that taking the collection of all elements of the form (3.14) satisfying (3.15), as x, y range over $\mathcal{L}^\mu$, θ ranges over S_μ and X_i, Y_i range over $M(\eta_i)$, yields exactly the set $\mathcal{A}^\mu$, with no repetitions.

We shall now use Equation (3.14) to decompose the subspace X^μ as a tensor product $V_\mu \otimes B_\mu \otimes V_\mu$ as in Theorem 2.2, where (recall) we require

V_μ to be a $\Bbbk$-vector space and B_μ to be a cellular algebra. Recall that for $i = 1, \ldots, r$, we denote the cell module of A associated to $\lambda_i \in \Lambda$ by Δ^{λ_i}. We define V_μ to be the $\Bbbk$-vector space tensor product

$$(\Bbbk\mathcal{L}^\mu) \otimes \left(\Delta^{\lambda_1}\right)^{\otimes \mu_1} \otimes \left(\Delta^{\lambda_2}\right)^{\otimes \mu_2} \otimes \cdots \otimes \left(\Delta^{\lambda_r}\right)^{\otimes \mu_r}$$

where $\Bbbk\mathcal{L}^\mu$ denotes the $\Bbbk$-vector space with basis $\mathcal{L}^\mu$, and each cell module Δ^{λ_i} is regarded purely as a $\Bbbk$-vector space. Further, we take B_μ to be the group algebra $\Bbbk S_\mu$. We have by Equation (3.8) that $\Bbbk S_\mu$ is isomorphic as a $\Bbbk$-algebra to the tensor product algebra $\Bbbk S_{\mu_1} \otimes \Bbbk S_{\mu_2} \otimes \cdots \otimes \Bbbk S_{\mu_m}$. By Theorem 2.1, each algebra $\Bbbk S_{\mu_i}$ is cellular, and hence $\Bbbk S_\mu$ is indeed cellular by Theorem 1.16. Now recall that each cell module Δ^{λ_i} has a basis consisting of all symbols C_S for $S \in M(\lambda_i)$. Thus V_μ has a basis consisting of all pure tensors of the form

$$\boldsymbol{S} = x \otimes C_{S_1} \otimes C_{S_2} \otimes \cdots \otimes C_{S_n}$$

where we have

$$(\underbrace{S_1, S_2, \ldots, S_{\mu_1}}_{\text{elements of } M(\lambda_1)}, \underbrace{S_{\mu_1+1}, \ldots, S_{\mu_1+\mu_2}}_{\text{elements of } M(\lambda_2)}, \ldots, S_n)$$

and $x \in \mathcal{L}^\mu$. In keeping with the notation of Theorem 2.1, we shall call this basis Ω_μ. Thus the tensor product $V_\mu \otimes \Bbbk S_\mu \otimes V_\mu$ has a $\Bbbk$-basis consisting of all elements $\boldsymbol{S} \otimes \theta \otimes \boldsymbol{T}$ where

$$\boldsymbol{S} = x \otimes C_{S_1} \otimes C_{S_2} \otimes \cdots \otimes C_{S_n}$$

and

$$\boldsymbol{T} = y \otimes C_{T_1} \otimes C_{T_2} \otimes \cdots \otimes C_{T_n}$$

are elements of Ω_μ and $\theta \in S_\mu$. Thus, we may define a $\Bbbk$-linear map

$$\Psi_\mu : V_\mu \otimes \Bbbk S_\mu \otimes V_\mu \longrightarrow X^\mu$$

by defining

$$\Psi_\mu\left(\boldsymbol{S}\otimes\theta\otimes\boldsymbol{T}\right) =$$
$$(x;1,\ldots,1)\left(\theta\,;\,C^{\eta_1}_{S_{(1)\theta^{-1}},T_1},C^{\eta_2}_{S_{(2)\theta^{-1}},T_2},\ldots,C^{\eta_n}_{S_{(n)\theta^{-1}},T_n}\right)(y^{-1};1,\ldots,1)$$

where for each $i=1,\ldots,n$, we have defined $\eta_i\in\Lambda$ such that $T_i\in M(\eta_i)$; the fact that $\theta\in S_\mu$ then implies that $S_{(i)\theta^{-1}}\in M(\eta_i)$ also, so that $C^{\eta_i}_{S_{(i)\theta^{-1}},T_i}$ is indeed defined. Now since the value of $\Psi_\mu\left(\boldsymbol{S}\otimes\theta\otimes\boldsymbol{T}\right)$ is of the form (3.14), and we certainly have

$$(\eta_1,\eta_2,\ldots,\eta_n) = (\underbrace{\lambda_1,\ldots,\lambda_1}_{\mu_1\text{ places}},\underbrace{\lambda_2,\ldots,\lambda_2}_{\mu_2\text{ places}},\ldots,\underbrace{\lambda_r,\ldots,\lambda_r}_{\mu_r\text{ places}}),$$

it follows that $\Psi_\mu\left(\boldsymbol{S}\otimes\theta\otimes\boldsymbol{T}\right)$ is in fact an element of the basis $\mathcal{A}^\mu$ of X^μ. Further, we may easily use the fact that each element of $\mathcal{A}^\mu$ has a *unique* expression of the form (3.14) (subject to (3.15)) to show that in fact any element of $\mathcal{A}^\mu$ may be obtained as an image $\Psi_\mu\left(\boldsymbol{S}\otimes\theta\otimes\boldsymbol{T}\right)$ for a unique choice of $\boldsymbol{S},\boldsymbol{T}\in\Omega_\mu$ and $\theta\in S_\mu$, and from this it follows that the map Ψ_μ is in fact a $\Bbbk$-linear bijection from $V_\mu\otimes\Bbbk S_\mu\otimes V_\mu$ to X^μ. Thus we have our desired decomposition

$$\begin{aligned} S_n\wr A &= \bigoplus_{\mu\in\Lambda_n^{\models}(r)} \Phi_\mu\left(V_\mu\otimes\Bbbk S_\mu\otimes V_\mu\right) \\ &\cong \bigoplus_{\mu\in\Lambda_n^{\models}(r)} V_\mu\otimes\Bbbk S_\mu\otimes V_\mu. \end{aligned}\tag{3.16}$$

Returning to Theorem 2.1, and (in the notation of that theorem) taking the basis $\mathcal{B}_\mu$ of the algebra $B_\mu=\Bbbk S_\mu$ to be simply S_μ, we see that under this decomposition (3.16), our basis $\mathcal{A}$ of $S_n\wr A$ is indeed the basis arising from the bases Ω_μ and $\mathcal{B}_\mu$ just as described in Theorem 2.1.

Our next task will be to prove that the decomposition (3.16) has the properties required by Theorem 2.1, but before we can do that we must define our anti-involution on $S_n\wr A$.

Proposition 3.4. *The formula*

$$\iota\big((\sigma;a_1,a_2,\ldots,a_n)\big) = \big(\sigma^{-1}\,;\,\iota(a_{(1)\sigma}),\iota(a_{(2)\sigma}),\ldots,\iota(a_{(n)\sigma})\big)$$

for $a_1, a_2, \ldots, a_n \in A$ and $\sigma \in S_n$ (where the map ι on the right hand side is the anti-involution on A) yields a well-defined anti-involution ι on $S_n \wr A$; note that we are thus using ι to denote the anti-involutions on both A and $S_n \wr A$, but this should not cause confusion.

Proof. We may easily show that defining ι by the given formula on pure tensors does indeed yield a well-defined $\Bbbk$-linear map. Then for any elements $a_1, a_2, \ldots, a_n \in A$ and any $\sigma \in S_n$, we have

$$\begin{aligned}\iota^2\big((\sigma; a_1, a_2, \ldots, a_n)\big) &= \iota\big(\big(\sigma^{-1}\,;\, \iota(a_{(1)\sigma}), \iota(a_{(2)\sigma}), \ldots, \iota(a_{(n)\sigma})\big)\big)\\ &= \big((\sigma^{-1})^{-1}\,;\, \iota^2(a_{(1)\sigma^{-1}\sigma}), \iota^2(a_{(2)\sigma^{-1}\sigma}), \ldots, \iota^2(a_{(n)\sigma^{-1}\sigma})\big)\\ &= (\sigma; a_1, a_2, \ldots, a_n).\end{aligned}$$

Since such elements $(\sigma; a_1, a_2, \ldots, a_n)$ span $S_n \wr A$ over $\Bbbk$, we have $\iota^2 = \mathrm{id}$. To check that $\iota(\boldsymbol{ab}) = \iota(\boldsymbol{b})\iota(\boldsymbol{a})$ for all $\boldsymbol{a}, \boldsymbol{b} \in S_n \wr A$, it suffices by linearity to prove that for any $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$ and any $\sigma, \pi \in S_n$, we have

$$\iota\big((\sigma; a_1, \ldots, a_n)(\pi; b_1, \ldots, b_n)\big) = \iota\big((\pi; b_1, \ldots, b_n)\big)\iota\big((\sigma; a_1, \ldots, a_n)\big),$$

which may easily be verified by direct calculation. $\square$

Note that this anti-involution ι on $S_n \wr A$ corresponds exactly to the anti-involution used on the wreath product $A \wr S_n^{\mathrm{op}}$ by Geetha and Goodman (see the start of Section 4 in [4]), under the isomorphism (3.5).

We shall now complete our proof that $S_n \wr A$ is an iterated inflation of cellular algebras, by showing that our decomposition (3.16) satisfies Equations (2.5) and (2.6).

Equation (2.5) may be verified by a straightforward calculation. Indeed, let $\mu = (\mu_1, \ldots, \mu_r) \in \Lambda_n^{\models}(r)$ and as above take

$$\boldsymbol{S} = x \otimes C_{S_1} \otimes C_{S_2} \otimes \cdots \otimes C_{S_n}$$

and

$$\boldsymbol{T} = y \otimes C_{T_1} \otimes C_{T_2} \otimes \cdots \otimes C_{T_n}$$

to be elements of Ω_μ and let $\theta \in S_\mu$. Then we have

$$
\begin{aligned}
&\iota\Big(\Psi_\mu\big(\boldsymbol{S} \otimes \theta \otimes \boldsymbol{T}\big)\Big) \\
&\quad = \iota\left((x;1,\ldots,1)\left(\theta\,;\,C^{\eta_1}_{S_{(1)\theta^{-1}},T_1}, C^{\eta_2}_{S_{(2)\theta^{-1}},T_2},\ldots,C^{\eta_n}_{S_{(n)\theta^{-1}},T_n}\right)(y^{-1};1,\ldots,1)\right) \\
&\quad = \iota\big((y^{-1};1,\ldots,1)\big)\,\iota\left(\left(\theta\,;\,C^{\eta_1}_{S_{(1)\theta^{-1}},T_1}, C^{\eta_2}_{S_{(2)\theta^{-1}},T_2},\ldots,C^{\eta_n}_{S_{(n)\theta^{-1}},T_n}\right)\right)\iota\big((x;1,\ldots,1)\big) \\
&\qquad\quad \text{(because } \iota \text{ is an anti-involution)} \\
&\quad = (y;1,\ldots,1)\Big(\theta^{-1}\,;\,\iota\left(C^{\eta_1}_{S_1,T_{(1)\theta}}\right), \iota\left(C^{\eta_2}_{S_2,T_{(2)\theta}}\right),\ldots \\
&\qquad\qquad\qquad\qquad\qquad\qquad \ldots,\iota\left(C^{\eta_n}_{S_n,T_{(n)\theta}}\right)\Big)(x^{-1};1,\ldots,1) \\
&\qquad\quad \text{(where we have used the fact that } \eta_{(i)\theta} = \eta_i, \text{ since } \theta \in S_\mu) \\
&\quad = (y;1,\ldots,1)\Big(\theta^{-1}\,;\,C^{\eta_1}_{T_{(1)\theta},S_1}, C^{\eta_2}_{T_{(2)\theta},S_2},\ldots,C^{\eta_n}_{T_{(n)\theta},S_n}\Big)(x^{-1};1,\ldots,1) \\
&\quad = \Psi_\mu\big(\boldsymbol{T} \otimes \theta^{-1} \otimes \boldsymbol{S}\big) \\
&\quad = \Psi_\mu\big(\boldsymbol{T} \otimes \iota(\theta) \otimes \boldsymbol{S}\big)
\end{aligned}
$$

as required.

Finally, we prove that Equation (2.6) is satisfied. We do this by proving the following slightly more general result:

Proposition 3.5. *Let $\mu = (\mu_1,\ldots,\mu_r) \in \Lambda^{\vDash}_n(r)$ and let*

$$
\begin{aligned}
\boldsymbol{S} &= x \otimes C_{S_1} \otimes C_{S_2} \otimes \cdots \otimes C_{S_n}, \\
\boldsymbol{T} &= y \otimes C_{T_1} \otimes C_{T_2} \otimes \cdots \otimes C_{T_n}
\end{aligned}
$$

be elements of Ω_μ, and $\theta \in S_\mu$. Let $a_1,\ldots,a_n \in A$ and $\sigma \in S_n$. Then we have

$$
\begin{aligned}
&(\sigma; a_1,\ldots,a_n)\Psi_\mu\left(\boldsymbol{S} \otimes \theta \otimes \boldsymbol{T}\right) \equiv \\
&\Psi_\mu\Big(\big(z \otimes a_{(1)\pi x^{-1}} C_{S_{(1)\pi}} \otimes \cdots \otimes a_{(n)\pi x^{-1}} C_{S_{(n)\pi}}\big) \otimes \pi\theta \otimes \boldsymbol{T}\Big) \quad (\mathit{mod}\ J_{<\mu})
\end{aligned}
$$

where $z \in \mathcal{L}^\mu$ and $\pi \in S_\mu$ are the unique elements such that $\sigma x = z\pi$, and we define

$$J_{<\mu} = \bigoplus_{\mu' < \mu} X^{\mu'},$$

so that $J_{<\mu}$ is in fact the $\Bbbk$-span of all elements of the basis $\mathcal{A}$ of $S_n \wr A$ with layer index strictly less than μ.

Proof. The proof is by direct calculation. Indeed, we have

$$(\sigma; a_1, \dots, a_n)\Psi_\mu\left(\boldsymbol{S} \otimes \theta \otimes \boldsymbol{T}\right) =$$
$$(\sigma; a_1, \dots, a_n)(x; 1, \dots, 1)\Big(\theta\,;\, C^{\eta_1}_{S_{(1)\theta^{-1}},T_1}, \dots, C^{\eta_n}_{S_{(n)\theta^{-1}},T_n}\Big)(y^{-1}; 1, \dots, 1). \tag{3.17}$$

Now

$$\begin{aligned}(\sigma; a_1, \dots, a_n)(x; 1, \dots, 1) &= (\sigma x;\, a_{(1)x^{-1}}, \dots, a_{(n)x^{-1}})\\ &= (z\pi;\, a_{(1)x^{-1}}, \dots, a_{(n)x^{-1}})\\ &= (z; 1, \dots, 1)(\pi; a_{(1)x^{-1}}, \dots, a_{(n)x^{-1}}).\end{aligned}$$

Further, we have

$$\begin{aligned}&(\pi; a_{(1)x^{-1}}, \dots, a_{(n)x^{-1}})\Big(\theta\,;\, C^{\eta_1}_{S_{(1)\theta^{-1}},T_1}, \dots, C^{\eta_n}_{S_{(n)\theta^{-1}},T_n}\Big)\\
&\quad= \Big(\pi\theta\,;\, \hat{a}_1 C^{\eta_1}_{S_{(1)\theta^{-1}},T_1}, \dots, \hat{a}_n C^{\eta_n}_{S_{(n)\theta^{-1}},T_n}\Big)\\
&\qquad\qquad\text{(where we have defined } \hat{a}_i = a_{(i)\theta^{-1}x^{-1}}\text{)}\\
&\quad= \Big(\pi\theta\,;\, \sum_{U_1 \in M(\eta_1)} r_{\hat{a}_1}(U_1, S_{(1)\theta^{-1}})C^{\eta_1}_{U_1,T_1} \;+\; L_1, \dots\\
&\qquad\qquad\qquad \dots, \sum_{U_n \in M(\eta_n)} r_{\hat{a}_n}(U_n, S_{(n)\theta^{-1}})C^{\eta_n}_{U_n,T_n} \;+\; L_n\Big)\\
&\qquad\qquad\text{(where each } L_i \text{ lies in } A(<\eta_i)\text{)}\\
&\quad= \Big(\pi\theta\,;\, \sum_{U_1 \in M(\eta_1)} r_{\hat{a}_1}(U_1, S_{(1)\theta^{-1}})C^{\eta_1}_{U_1,T_1}, \dots\\
&\qquad\qquad\qquad \dots, \sum_{U_n \in M(\eta_n)} r_{\hat{a}_n}(U_n, S_{(n)\theta^{-1}})C^{\eta_n}_{U_n,T_n}\Big) \;+\; \mathbb{L}\end{aligned}$$

where $\mathbb{L}$ is a $\Bbbk$-linear combination of elements of the basis $\mathcal{A}$ of $S_n \wr A$ of the form

$$\big(\pi\theta; C^{\delta_1}_{X_1,Y_1}, \dots, C^{\delta_n}_{X_n,Y_n}\big) \tag{3.18}$$

where for each $i = 1, \ldots, n$, we have $\delta_i \le \eta_i$ in Λ, and moreover this inequality must be *strict* for at least one i. It follows that the layer index of each of these elements is strictly less that μ in the backwards lexicographic order on $\Lambda_n^{\vDash}(r)$. It is also clear that if $\boldsymbol{c}$ is an element of the form (3.18), then

$$(z; 1, \ldots, 1)\boldsymbol{c}(y^{-1}; 1, \ldots, 1)$$

has the same layer index as $\boldsymbol{c}$. It now follows from (3.17) that

$$\begin{aligned}
&(\sigma; a_1, \ldots, a_n)\Psi_\mu\left(\boldsymbol{S} \otimes \theta \otimes \boldsymbol{T}\right) \\
&\quad \equiv (z; 1, \ldots, 1)\Big(\pi\theta\,; \sum_{U_1 \in M(\eta_1)} r_{\hat{a}_1}\big(U_1, S_{(1)\theta^{-1}}\big) C^{\eta_1}_{U_1,T_1}, \ldots \\
&\qquad\qquad \ldots, \sum_{U_n \in M(\eta_n)} r_{\hat{a}_n}\big(U_n, S_{(n)\theta^{-1}}\big) C^{\eta_n}_{U_n,T_n}\Big)(y^{-1}; 1, \ldots, 1) \\
&\qquad\qquad\qquad\qquad (\mathrm{mod}\ J_{<\mu}),
\end{aligned}$$

the right hand side of which is equal to

$$\begin{aligned}
&\sum_{U_1 \in M(\eta_1)} \cdots \sum_{U_n \in M(\eta_n)} r_{\hat{a}_1}\big(U_1, S_{(1)\theta^{-1}}\big) \cdots r_{\hat{a}_n}\big(U_n, S_{(n)\theta^{-1}}\big) \cdot \\
&\qquad (z; 1, \ldots, 1)\Big(\pi\theta\,; C^{\eta_1}_{U_1,T_1}, \ldots, C^{\eta_n}_{U_n,T_n}\Big)(y^{-1}; 1, \ldots, 1) \\
&= \sum_{U_1 \in M(\eta_1)} \cdots \sum_{U_n \in M(\eta_n)} r_{\hat{a}_1}\big(U_1, S_{(1)\theta^{-1}}\big) \cdots r_{\hat{a}_n}\big(U_n, S_{(n)\theta^{-1}}\big) \cdot \\
&\qquad \Psi_\mu\left(\left(z \otimes C_{U_{(1)\pi\theta}} \otimes \cdots \otimes C_{U_{(n)\pi\theta}}\right) \otimes \pi\theta \otimes \boldsymbol{T}\right)
\end{aligned}$$

(by the definition of Ψ_μ; notice that we do indeed have $U_{(i)\pi\theta} \in M(\eta_i)$ for each i, since $\pi\theta \in S_\mu$)

$$\begin{aligned}
&= \Psi_\mu\Bigg(\Big(z \otimes \sum_{U'_1 \in M(\eta_1)} r_{a'_1}\big(U'_1, S_{((1)\pi\theta)\theta^{-1}}\big) C_{U'_1} \otimes \cdots \\
&\qquad \cdots \otimes \sum_{U'_n \in M(\eta_n)} r_{a'_n}\big(U'_n, S_{((n)\pi\theta)\theta^{-1}}\big) C_{U'_n}\Big) \otimes \pi\theta \otimes \boldsymbol{T}\Bigg)
\end{aligned}$$

(where we have defined $U'_i = U_{(i)\pi\theta}$ and $a'_i = \hat{a}_{(i)\pi\theta}$)

$$= \Psi_\mu\Big(\big(z \otimes a'_1 C_{S_{((1)\pi\theta)\theta^{-1}}} \otimes \cdots \otimes a'_n C_{S_{((n)\pi\theta)\theta^{-1}}}\big) \otimes \pi\theta \otimes \boldsymbol{T}\Big)$$

$$= \Psi_\mu\Big(\big(z \otimes a_{(1)\pi x^{-1}} C_{S_{(1)\pi}} \otimes \cdots \otimes a_{(n)\pi x^{-1}} C_{S_{(n)\pi}}\big) \otimes \pi\theta \otimes \boldsymbol{T}\Big)$$

(because $a'_i = \hat{a}_{(i)\pi\theta} = a_{(i)\pi\theta\theta^{-1}x^{-1}} = a_{(i)\pi x^{-1}}$).

□

Equation (2.6) now follows from Proposition 3.5, since every element of the basis $\mathcal{A}$ of $S_n \wr A$ is of the form $(\sigma; a_1, \ldots, a_n)$, and it is clear that the elements

$$\big(z \otimes a_{(1)\pi x^{-1}} C_{S_{(1)\pi}} \otimes \cdots \otimes a_{(n)\pi x^{-1}} C_{S_{(n)\pi}}\big)$$

and π depend only on $(\sigma; a_1, \ldots, a_n)$ and $\boldsymbol{S}$ (since z and π are defined by $\sigma x = z\pi$). Thus, we may apply Theorem 2.2 to our decomposition (3.16) and hence prove that $S_n \wr A$ is cellular. Before we give a formal statement of this result, let us introduce one more notational convention. By tracing back through our arguments, we may see that the indexing set of the cell modules of $S_n \wr A$ is the set of all tuples $(\mu, (\zeta_1, \ldots, \zeta_r))$ such that $\mu \in \Lambda_n^{\vDash}(r)$ and $\zeta_i \in \Lambda_{\mu_i}$ (this comes from our use of the theorems 1.16, 2.1 and 2.2). For such a tuple, it is clear that $(|\zeta_1|, \ldots, |\zeta_r|) = \mu$, and so we lose no information if we omit μ from the notation and thus identify the set which indexes the cell modules of $S_n \wr A$ with the set of all r-tuples of partitions $(\zeta_1, \ldots, \zeta_r)$ such that $(|\zeta_1|, \ldots, |\zeta_r|) = \mu$ for some $\mu \in \Lambda_n^{\vDash}(r)$; let us denote this set by $\Lambda_n^{\vdash}(r)$.

Theorem 3.6. *Let $\Bbbk$ be any field and A a cellular $\Bbbk$-algebra with cellular data (Λ, M, C, ι), where $|\Lambda| = r$. Then for any positive integer n, the wreath product algebra $S_n \wr A$ is a cellular algebra with respect to a tuple $(\Lambda_n^{\vdash}(r), \mathcal{M}, \mathcal{C}, \iota)$, where $\Lambda_n^{\vdash}(r)$ is the set of all r-tuples of partitions $(\zeta_1, \ldots, \zeta_r)$ such that $(|\zeta_1|, \ldots, |\zeta_r|) = \mu$ for some composition μ of n with exactly r parts, and further the map ι in the tuple is the anti-involution (well-) defined on $S_n \wr A$ by*

$$\iota\big((\sigma; a_1, a_2, \ldots, a_n)\big) = \big(\sigma^{-1}\,;\, \iota(a_{(1)\sigma}), \iota(a_{(2)\sigma}), \ldots, \iota(a_{(n)\sigma})\big)$$

for $a_1, a_2, \ldots, a_n \in A$ and $\sigma \in S_n$.

Now let us briefly show how this result may be applied to the wreath product $A \wr S_n^{\mathrm{op}}$ as defined, for example, in [1] (see Section 3.2, above). Recall from Equation (3.4) that

$$A \wr S_n^{\mathrm{op}} \cong \big(S_n \wr (A^{\mathrm{op}})\big)^{\mathrm{op}}.$$

By using Proposition 3.1 and Theorem 3.6, we may show that $\big(S_n \wr (A^{\mathrm{op}})\big)^{\mathrm{op}}$ is cellular with respect to a tuple of data including the partially ordered set $\Lambda_n^{\vdash}(r)$ and the map ι, exactly as in Theorem 3.6 (note that ι is indeed a map on $\big(S_n \wr (A^{\mathrm{op}})\big)^{\mathrm{op}}$, since it is clear that $\big(S_n \wr (A^{\mathrm{op}})\big)^{\mathrm{op}}$ and $S_n \wr A$ are equal as vector spaces). Hence by the isomorphism (3.5), $A \wr S_n^{\mathrm{op}}$ is cellular with respect to a tuple of data including the partially ordered set $\Lambda_n^{\vdash}(r)$ and the anti-involution ι which is (well-) defined on $A \wr S_n^{\mathrm{op}}$ by

$$\iota\big((a_1, a_2, \ldots, a_n; \sigma)\big) = \big(\iota(a_{\sigma(1)}), \iota(a_{\sigma(2)}), \ldots, \iota(a_{\sigma(n)})\,;\,\sigma^{-1}\big)$$

for $a_1, a_2, \ldots, a_n \in A$ and $\sigma \in S_n^{\mathrm{op}}$.

3.6 The cell modules of a wreath product

In this section, we shall show how the cell modules of the wreath product $S_n \wr A$ (where n is a positive integer) which arise from the cellular structure obtained in Theorem 3.6 may be obtained from the cell modules of A and S_n via the constructions described in Section 3.4.

Firstly, let us obtain one description of the cell modules by applying Corollary 2.7. Indeed, let $(\zeta_1, \ldots, \zeta_r) \in \Lambda_n^{\vdash}(r)$, and let $\mu = (|\zeta_1|, \ldots, |\zeta_r|) \in \Lambda_n^{\vDash}(r)$. Recall from above that

$$V_\mu = (\Bbbk\mathcal{L}^\mu) \otimes \left(\Delta^{\lambda_1}\right)^{\otimes \mu_1} \otimes \left(\Delta^{\lambda_2}\right)^{\otimes \mu_2} \otimes \cdots \otimes \left(\Delta^{\lambda_r}\right)^{\otimes \mu_r}$$

and that we define Ω_μ to be the basis of V_μ consisting of all pure tensors of the form

$$\boldsymbol{S} = x \otimes C_{S_1} \otimes C_{S_2} \otimes \cdots \otimes C_{S_n}$$

where $x \in \mathcal{L}^\mu$ and we have

$$(\underbrace{S_1, S_2, \ldots, S_{\mu_1}}_{\text{elements of } M(\lambda_1)}, \underbrace{S_{\mu_1+1}, \ldots, S_{\mu_1+\mu_2}}_{\text{elements of } M(\lambda_2)}, \ldots, S_n).$$

Further, recall that by Equation (3.8) and Theorems 1.16 and 2.1, the cell module of $\Bbbk S_\mu$ indexed by $(\zeta_1, \ldots, \zeta_r)$ is the tensor product module

$$S^{\zeta_1} \otimes S^{\zeta_2} \otimes \cdots \otimes S^{\zeta_r}$$

where the action is as given in Equation (3.12). Hence, by applying Corollary 2.7, we have for the cell module $\Delta^{(\zeta_1,\ldots,\zeta_r)}$ of $S_n \wr A$ an isomorphism of $\Bbbk$-vector spaces

$$\Delta^{(\zeta_1,\ldots,\zeta_r)} \cong V_\mu \otimes \left(S^{\zeta_1} \otimes S^{\zeta_2} \otimes \cdots \otimes S^{\zeta_r}\right)$$

so that in fact $\Delta^{(\zeta_1,\ldots,\zeta_r)}$ is isomorphic as a $\Bbbk$-vector space to the tensor product

$$\Bbbk\mathcal{L}^\mu \otimes \left(\Delta^{\lambda_1}\right)^{\otimes\mu_1} \otimes \left(\Delta^{\lambda_2}\right)^{\otimes\mu_2} \otimes \cdots \otimes \left(\Delta^{\lambda_r}\right)^{\otimes\mu_r} \otimes S^{\zeta_1} \otimes S^{\zeta_2} \otimes \cdots \otimes S^{\zeta_r}.$$

Further, by Corollary 2.7, the action of $S_n \wr A$ is as follows: take some $(\sigma; a_1, \ldots, a_n) \in \mathcal{A}$, so that each a_i is in fact some element $C^{\eta_i}_{X_i,Y_i}$ of the cellular basis of A, let $\boldsymbol{S} \in \Omega_\mu$ be as above and let $w_i \in S^{\zeta_i}$ for $i = 1, \ldots, r$; then we have by Proposition 3.5 that

$$\begin{aligned}(\sigma; a_1, \ldots, a_n)&\big(\boldsymbol{S} \otimes (w_1 \otimes \cdots \otimes w_r)\big) = \\ &\big(z \otimes a_{(1)\pi x^{-1}} C_{S_{(1)\pi}} \otimes \cdots \otimes a_{(n)\pi x^{-1}} C_{S_{(n)\pi}}\big) \otimes (\pi_1 w_1 \otimes \cdots \otimes \pi_r w_r)\end{aligned} \tag{3.19}$$

where $z \in \mathcal{L}^\mu$ and $\pi \in S_\mu$ are the unique elements such that $\sigma x = z\pi$, and the elements $\pi_i \in S_{\mu_i}$ arise from the factorisation of π as for Equation (3.12). We may now easily show by linearity that Equation (3.19) holds for any $a_1, \ldots, a_n \in A$, not just elements of the cellular basis. Further, we may similarly show that Equation (3.19) continues to hold when the basis elements

C_{S_i} of the cell modules of A are replaced by general elements of those cell modules, so that in fact we have

$$(\sigma; a_1, \ldots, a_n)\big((x \otimes v_1 \otimes \cdots \otimes v_n) \otimes (w_1 \otimes \cdots \otimes w_r)\big) = \\ \big(z \otimes a_{(1)\pi x^{-1}} v_{(1)\pi} \otimes \cdots \otimes a_{(n)\pi x^{-1}} v_{(n)\pi}\big) \otimes (\pi_1 w_1 \otimes \cdots \otimes \pi_r w_r) \quad (3.20)$$

where $a_1, \ldots, a_n \in A$ and each v_i is an element of the appropriate cell module of A (and the other quantities are all as in Equation (3.19)).

We shall now give an alternative description of the cell modules $\Delta^{(\zeta_1,\ldots,\zeta_r)}$ of $S_n \wr A$ as inflations of $S_\mu \wr A$-modules obtained via the constructions described in Section 3.4; this construction of the cell modules of a wreath product was given (for $A \wr S_n^{\mathrm{op}}$) by Geetha and Goodman for the case where all the cell modules of A are cyclic (Theorem 4.26 in [4]).

Proposition 3.7. *The cell module $\Delta^{(\zeta_1,\ldots,\zeta_r)}$ of $S_n \wr A$ is isomorphic to the $S_n \wr A$-module*

$$\mathrm{Ind}_\mu^n\Big(\big(\Delta^{\lambda_1}, \ldots, \Delta^{\lambda_r}\big)^{\wr\mu} \oslash \big(S^{\zeta_1} \otimes \cdots \otimes S^{\zeta_r}\big)\Big),$$

where $\mu = (|\zeta_1|, \ldots, |\zeta_r|)$.

Proof. We shall explain how the isomorphism is constructed, but we shall omit the rather lengthy but routine calculations needed to verify the various stages of the argument. Recall that

$$\mathrm{Ind}_\mu^n\Big(\big(\Delta^{\lambda_1}, \ldots, \Delta^{\lambda_r}\big)^{\wr\mu} \oslash \big(S^{\zeta_1} \otimes \cdots \otimes S^{\zeta_r}\big)\Big)$$

is defined to be the module

$$\big(S_n \wr A\big) \otimes_\mu \Big(\big(\Delta^{\lambda_1}, \ldots, \Delta^{\lambda_r}\big)^{\wr\mu} \oslash \big(S^{\zeta_1} \otimes \cdots \otimes S^{\zeta_r}\big)\Big)$$

where we have adopted the shorthand notation $\otimes_\mu$ to mean $\otimes_{S_\mu \wr A}$. Further, let us identify the cell module $\Delta^{(\zeta_1,\ldots,\zeta_r)}$ with the vector space tensor product

$$\Bbbk\mathcal{L}^\mu \otimes \big(\Delta^{\lambda_1}\big)^{\otimes\mu_1} \otimes \big(\Delta^{\lambda_2}\big)^{\otimes\mu_2} \otimes \cdots \otimes \big(\Delta^{\lambda_r}\big)^{\otimes\mu_r} \otimes S^{\zeta_1} \otimes S^{\zeta_2} \otimes \cdots \otimes S^{\zeta_r}$$

as explained above. It is then easy to prove that we may define a $\Bbbk$-linear map

$$\psi : \Delta^{(\zeta_1,\dots,\zeta_r)} \longrightarrow \mathrm{Ind}^n_\mu\Big(\big(\Delta^{\lambda_1},\dots,\Delta^{\lambda_r}\big)^{\wr\mu} \oslash \big(S^{\zeta_1}\otimes\cdots\otimes S^{\zeta_r}\big)\Big)$$

via the formula

$$\begin{aligned}\psi : x\otimes v_1\otimes\cdots\otimes v_n\otimes w_1\otimes\cdots\otimes w_r &\longmapsto \\ &(x;1,\dots,1)\otimes_\mu \big((v_1\otimes\cdots\otimes v_n)\otimes(w_1\otimes\cdots\otimes w_r)\big) \quad (3.21)\end{aligned}$$

where $x \in \mathcal{L}^\mu$, each v_i lies in the appropriate cell module of A, and $w_i \in S^{\zeta_i}$ for $i = 1,\dots,r$. Further, we may check by direct calculation using Equation (3.20) that

$$\begin{aligned}(\sigma;a_1,\dots,a_n)\psi(x\otimes v_1\otimes\cdots\otimes v_n\otimes w_1\otimes\cdots\otimes w_r) &= \\ &\psi\big((\sigma;a_1,\dots,a_n)(x\otimes v_1\otimes\cdots\otimes v_n\otimes w_1\otimes\cdots\otimes w_r)\big)\end{aligned}$$

for any $\sigma \in S_n$ and $a_1,\dots,a_n \in A$, which implies that ψ is a homomorphism of $S_n \wr A$-modules. It remains only to prove that ψ is invertible, and to do this we shall construct an inverse map for ψ. Indeed, it is straightforward to check that we may define a map

$$\phi : \big(S_n \wr A\big) \times \Big(\big(\Delta^{\lambda_1},\dots,\Delta^{\lambda_r}\big)^{\wr\mu} \oslash \big(S^{\zeta_1}\otimes\cdots\otimes S^{\zeta_r}\big)\Big) \longrightarrow \Delta^{(\zeta_1,\dots,\zeta_r)}$$

which is $\Bbbk$-linear in both places by the formula

$$\begin{aligned}\phi : \big((\sigma;a_1,\dots,a_n),(v_1\otimes\cdots\otimes v_n)\otimes(w_1\otimes\cdots\otimes w_r)\big) &\longmapsto \\ &x\otimes a_{(1)\theta}v_{(1)\theta}\otimes\cdots\otimes a_{(n)\theta}v_{(n)\theta}\otimes\theta_1 w_1\otimes\cdots\otimes\theta_r w_r \quad (3.22)\end{aligned}$$

where v_i and w_i are as above, $\sigma \in S_n$, $a_i \in A$, $x \in \mathcal{L}^\mu$ and $\theta \in S_\mu$ are the unique elements such that $\sigma = x\theta$, and the elements $\theta_i \in S_{\mu_i}$ are the unique elements such that $\theta = \hat{\theta}_1\cdots\hat{\theta}_r$ where $\hat{\theta}_i$ is as usual the image of θ_i under the canonical isomorphism from S_{μ_i} to S_{Θ_i}. Further, we may check that

$$\phi\big((\sigma;a_1,\dots,a_n)(\pi;b_1,\dots,b_n),(v_1\otimes\cdots\otimes v_n)\otimes(w_1\otimes\cdots\otimes w_r)\big)=$$
$$\phi\Big((\sigma;a_1,\dots,a_n),(\pi;b_1,\dots,b_n)\big((v_1\otimes\cdots\otimes v_n)\otimes(w_1\otimes\cdots\otimes w_r)\big)\Big)$$

for any $\pi\in S_\mu$ and $b_i\in A$ (with the other quantities as above), from which we may conclude that the map ϕ is $S_\mu\wr A$-*balanced*; that is, that $\phi(\boldsymbol{a}\boldsymbol{\alpha},\boldsymbol{d})=\phi(\boldsymbol{a},\boldsymbol{\alpha}\boldsymbol{d})$ for any $\boldsymbol{a}\in S_n\wr A$, and $\boldsymbol{\alpha}\in S_\mu\wr A$ and any

$$\boldsymbol{d}\in\big(\Delta^{\lambda_1},\dots,\Delta^{\lambda_r}\big)^{\wr\mu}\oslash\big(S^{\zeta_1}\otimes\cdots\otimes S^{\zeta_r}\big).$$

Thus by the universal property of the tensor product, we have a well-defined map

$$\hat\phi:\big(S_n\wr A\big)\otimes_\mu\Big(\big(\Delta^{\lambda_1},\dots,\Delta^{\lambda_r}\big)^{\wr\mu}\oslash\big(S^{\zeta_1}\otimes\cdots\otimes S^{\zeta_r}\big)\Big)\longrightarrow\Delta^{(\zeta_1,\dots,\zeta_r)}$$

given by

$$\hat\phi:(\sigma;a_1,\dots,a_n)\otimes_\mu\big((v_1\otimes\cdots\otimes v_n)\otimes(w_1\otimes\cdots\otimes w_r)\big)\longmapsto$$
$$x\otimes a_{(1)\theta}v_{(1)\theta}\otimes\cdots\otimes a_{(n)\theta}v_{(n)\theta}\otimes\theta_1w_1\otimes\cdots\otimes\theta_rw_r$$

where the quantities involved are as in Equation (3.22). By direct calculation, we may now verify that

$$\hat\phi\circ\psi\,(x\otimes v_1\otimes\cdots\otimes v_n\otimes w_1\otimes\cdots\otimes w_r)=$$
$$x\otimes v_1\otimes\cdots\otimes v_n\otimes w_1\otimes\cdots\otimes w_r$$

where the quantities are as in Equation (3.21), and also that

$$\psi\circ\hat\phi\Big((\sigma;a_1,\dots,a_n)\otimes_\mu\big((v_1\otimes\cdots\otimes v_n)\otimes(w_1\otimes\cdots\otimes w_r)\big)\Big)=$$
$$(\sigma;a_1,\dots,a_n)\otimes_\mu\big((v_1\otimes\cdots\otimes v_n)\otimes(w_1\otimes\cdots\otimes w_r)\big)$$

where the quantities involved are as in Equation (3.22). It now follows that $\hat\phi$ is indeed an inverse map to ψ. □

New material in Chapter 3: As far as I am aware, the concept of an iterated inflation of cellular algebras has not previously been applied to the wreath product of a cellular algebra with a symmetric group, and thus our proof of the cellularity of $A \wr S_n$ in Section 3.5 is a new result. Of course, Geetha and Goodman have already shown that the wreath product $A \wr S_n^{\mathrm{op}}$ is cellular if all of the cell modules of A are cyclic, but their proof is quite different from ours. The construction of the cell modules of $A \wr S_n$ given in Proposition 3.7 was given by Geetha and Goodman (for $A \wr S_n^{\mathrm{op}}$), but our proof of it is again based on the decomposition of $A \wr S_n$ as an iterated inflation and thus is new.

A A counterexample to a proposed lemma of Xi

This appendix contains a counterexample to the proposed lemma of Xi on page 68, that is, an algebra A with an anti-involution ι which satisfies all of the hypotheses of the lemma, but is not cellular with respect to ι (in the sense of our Definition 1.10). As far as I know, this counterexample is a new result.

Indeed, let $\Bbbk$ be any field, and define A to be the $\Bbbk$-vector space

$$M_2(\Bbbk) \oplus \Bbbk \oplus \Bbbk.$$

We define a multiplication on A by setting

$$\left(\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \alpha, \beta\right)\left(\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \gamma, \delta\right) = \left(\begin{pmatrix} \gamma a_{11} & \delta a_{12} \\ \delta a_{21} & \gamma a_{22} \end{pmatrix} + \begin{pmatrix} \alpha b_{11} & \beta b_{12} \\ \beta b_{21} & \alpha b_{22} \end{pmatrix}, \alpha\gamma, \beta\delta\right).$$

It is routine to verify that this formula does indeed define a $\Bbbk$-bilinear associative multiplication on A, and moreover that this multiplication is *commutative* (so that the concepts of left ideals, right ideals, and two-sided ideals coincide, and thus we may unambiguously refer to *ideals* of A) and has identity element

$$\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 1, 1\right).$$

Further, let us define a map ι on A by setting

$$\iota\,(M, \alpha, \beta) = \left(M^T, \alpha, \beta\right)$$

for any $\alpha, \beta \in \Bbbk$ and any $M \in M_2(\Bbbk)$; it is routine to verify that this ι is in fact an anti-involution on A. Now we may identify the $\Bbbk$-vector spaces $\Bbbk$ and $M_2(\Bbbk)$ with the tensor products $\Bbbk \otimes \Bbbk \otimes \Bbbk$ and $\Bbbk \otimes M_2(\Bbbk) \otimes \Bbbk$ respectively, by identifying the element $1 \otimes x \otimes 1$ of $\Bbbk \otimes \Bbbk \otimes \Bbbk$ (resp. $\Bbbk \otimes M_2(\Bbbk) \otimes \Bbbk$) with the element x of $\Bbbk$ (resp. $M_2(\Bbbk)$). Let $V_1 = V_2 = V_3 = \Bbbk$ and let $B_1 = M_2(\Bbbk)$ and $B_2 = B_3 = \Bbbk$, where $M_2(\Bbbk)$ is a cellular algebra as in Proposition 1.3 and $\Bbbk$ is taken to be the trivial cellular algebra. Thus we obtain an isomorphism of vector spaces

$$\begin{aligned} A &\cong \big(V_1 \otimes B_1 \otimes V_1\big) \oplus \big(V_2 \otimes B_2 \otimes V_2\big) \oplus \big(V_3 \otimes B_3 \otimes V_3\big) \\ &= \big(\Bbbk \otimes M_2(\Bbbk) \otimes \Bbbk\big) \oplus \big(\Bbbk \otimes \Bbbk \otimes \Bbbk\big) \oplus \big(\Bbbk \otimes \Bbbk \otimes \Bbbk\big) \end{aligned} \tag{A.1}$$

from the mapping

$$(M, \alpha, \beta) \longmapsto (1 \otimes M \otimes 1,\, 1 \otimes \alpha \otimes 1,\, 1 \otimes \beta \otimes 1). \tag{A.2}$$

To show that A satisfies the hypotheses of Xi's lemma when equipped with the anti-involution ι and the decomposition (A.1), we define $\Bbbk$-bilinear forms

$$\phi_1 : V_1 \times V_1 \to B_1,\ \phi_2 : V_2 \times V_2 \to B_2,\ \phi_3 : V_3 \times V_3 \to B_3$$

by setting

$$\phi_1(1,1) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and $\phi_2(1,1) = \phi_3(1,1) = 1$. It is now routine to check that these definitions satisfy all the hypotheses of Xi's lemma; in particular note that for $j = 1, 2, 3$, the cellular algebra B_j has cell chain $\{0\} \subseteq B_j$.

Proposition A.1. *The algebra A is not cellular with respect to the anti-involution ι.*

Proof. Suppose for a contradiction that A is cellular in the sense of Definition 1.10 with respect to ι. Then by Theorem 1.12, A is cellular in the sense of Definition 1.2 with respect to a tuple (Λ, M, C, ι) of cellular data.

We have

$$\dim(A) = \sum_{\lambda \in \Lambda} |M(\lambda)|^2.$$

But $\dim(A) = 6$, and the only ways of writing 6 as a sum of square integers are $6 = 1+1+1+1+1+1$ and $6 = 4+1+1$. It follows that either $|\Lambda| = 6$

or $|\Lambda| = 3$. But if $|\Lambda| = 6$ then we must have $|M(\lambda)| = 1$ for all $\lambda \in \Lambda$, and thus the cellular basis of A would be of the form

$$C^{\lambda_1}_{S_1,S_1},\ C^{\lambda_2}_{S_2,S_2},\ C^{\lambda_3}_{S_3,S_3},\ C^{\lambda_4}_{S_4,S_4},\ C^{\lambda_5}_{S_5,S_5},\ C^{\lambda_6}_{S_6,S_6}$$

where $\Lambda = \{\lambda_1, \ldots, \lambda_6\}$ and $M(\lambda_i) = \{S_i\}$ for each $i = 1, \ldots, 6$. But then by axiom (C2) of Definition 1.2, we must have $\iota\big(C^{\lambda_i}_{S_i,S_i}\big) = C^{\lambda_i}_{S_i,S_i}$ for each i, and it follows that ι must be the identity map on A. Since ι is *not* the identity map, we have a contradiction.

Thus, we must have $\Lambda = \{\lambda_1, \lambda_2, \lambda_3\}$, where we may assume that

$$|M(\lambda_1)| = |M(\lambda_2)| = 1$$

and

$$M(\lambda_3) = \{S, T\}.$$

Note, however, that our indexing of the elements of Λ as $\lambda_1, \lambda_2, \lambda_3$ need not have any relation to the ordering on Λ. Recall that for $\lambda \in \Lambda$, $A(\lambda)$ is defined to be the subspace

$$\mathrm{span}_{\Bbbk}\{C^{\lambda}_{X,Y} : X, Y \in M(\lambda)\}$$

of A, and $A(< \lambda)$ is defined to be the ideal

$$\mathrm{span}_{\Bbbk}\{C^{\mu}_{X,Y} : \mu < \lambda \text{ and } X, Y \in M(\mu)\}$$

of A; we shall now show that

$$A(\lambda_3) = \{(M, 0, 0) : M \in M_2(\Bbbk)\}$$

and that $A(< \lambda_3) = \{0\}$. Indeed, it is clear that $A(< \lambda_3)$ must be equal to exactly one of

$$\{0\},\ A(\lambda_1),\ A(\lambda_2),\ \text{or } A(\lambda_1) \oplus A(\lambda_2)$$

(depending upon the ordering on Λ) and hence $A(< \lambda_3)$ is an ideal of A with dimension *at most 2*.

Now let $(M, \alpha, \beta) \in A$, so that $M \in M_2(\Bbbk)$ and $\alpha, \beta \in \Bbbk$. Then

$$\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, 0, 0\right)(M, \alpha, \beta) = \left(\begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix}, 0, 0\right)$$

and

$$\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, 0, 0\right)(M, \alpha, \beta) = \left(\begin{pmatrix} 0 & 0 \\ 0 & \alpha \end{pmatrix}, 0, 0\right)$$

and if $\alpha \neq 0$, then the elements

$$(M, \alpha, \beta), \left(\begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix}, 0, 0\right), \left(\begin{pmatrix} 0 & 0 \\ 0 & \alpha \end{pmatrix}, 0, 0\right)$$

of A are linearly independent, so that if $\alpha \neq 0$ then the ideal generated by (M, α, β) in A must have dimension at least 3. Similarly, we may use the facts that

$$\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, 0, 0\right)(M, \alpha, \beta) = \left(\begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}, 0, 0\right)$$

and

$$\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, 0, 0\right)(M, \alpha, \beta) = \left(\begin{pmatrix} 0 & 0 \\ \beta & 0 \end{pmatrix}, 0, 0\right)$$

to show that if $\beta \neq 0$ then the ideal generated by (M, α, β) in A must have dimension at least 3. Thus, we have shown that any ideal of A which does not lie in $\{(M, 0, 0) : M \in M_2(\Bbbk)\}$ must have dimension at least 3, and thus $A(< \lambda_3)$ must lie in $\{(M, 0, 0) : M \in M_2(\Bbbk)\}$.

Next, by the fact that A is commutative, we have

$$C^{\lambda_3}_{S,T} C^{\lambda_3}_{T,S} = C^{\lambda_3}_{T,S} C^{\lambda_3}_{S,T}.$$

But by Equation (1.13), we have

$$C^{\lambda_3}_{S,T} C^{\lambda_3}_{T,S} = \langle C_T, C_T \rangle C^{\lambda_3}_{S,S} + L_1$$

and

$$C^{\lambda_3}_{T,S} C^{\lambda_3}_{S,T} = \langle C_S, C_S \rangle C^{\lambda_3}_{T,T} + L_2$$

for some $L_1, L_2 \in A(< \lambda_3)$. It follows (because $A(< \lambda_3) \cap A(\lambda_3) = \{0\}$) that

$$\langle C_S, C_S \rangle C^{\lambda_3}_{T,T} = \langle C_T, C_T \rangle C^{\lambda_3}_{S,S}$$

and since $C^{\lambda_3}_{T,T}$ and $C^{\lambda_3}_{S,S}$ are linearly independent, it follows that

$$\langle C_S, C_S \rangle = \langle C_T, C_T \rangle = 0. \tag{A.3}$$

By applying the same argument to the product $C^{\lambda_3}_{T,T} C^{\lambda_3}_{S,S}$, we find that

$$\langle C_S, C_T \rangle = \langle C_T, C_S \rangle = 0. \tag{A.4}$$

Now consider the element $C^{\lambda_3}_{S,S}$. We have

$$C^{\lambda_3}_{S,S} = (B, \alpha, \beta)$$

for some $B \in M_2(\Bbbk)$ and $\alpha, \beta \in \Bbbk$, and thus

$$\begin{aligned} C^{\lambda_3}_{S,S} C^{\lambda_3}_{S,S} &= (B, \alpha, \beta)(B, \alpha, \beta) \\ &= (X, \alpha^2, \beta^2) \end{aligned}$$

(where X is some 2×2 matrix which we shall not need to consider further). But by Equation (1.13), we have some $L \in A(< \lambda_3)$ such that

$$\begin{aligned} C^{\lambda_3}_{S,S} C^{\lambda_3}_{S,S} &= \langle C_S, C_S \rangle C^{\lambda_3}_{S,S} + L \\ &= L \qquad \text{(since } \langle C_S, C_S \rangle = 0 \text{ by Equation (A.3))} \end{aligned}$$

and since $A(< \lambda_3) \subseteq \{(M, 0, 0) : M \in M_2(\Bbbk)\}$, L is of the form $(Y, 0, 0)$ (for some 2×2 matrix Y), and thus we have $\alpha = \beta = 0$. Similar arguments show that the elements $C^{\lambda_3}_{T,S}$, $C^{\lambda_3}_{S,T}$, $C^{\lambda_3}_{T,T}$ must each be of the form $(M, 0, 0)$ (for some 2×2 matrix M), and hence $A(\lambda_3)$ must be a 4-dimensional subspace of $\{(M, 0, 0) : M \in M_2(\Bbbk)\}$, so in fact

$$A(\lambda_3) = \{(M, 0, 0) : M \in M_2(\Bbbk)\}.$$

Further, we must now have $A(< \lambda_3) \subseteq A(\lambda_3)$ and $A(< \lambda_3) \cap A(\lambda_3) = \{0\}$, so that $A(< \lambda_3) = \{0\}$.

Returning to the elements $C^{\lambda_3}_{S,S}$, $C^{\lambda_3}_{T,S}$, $C^{\lambda_3}_{S,T}$, $C^{\lambda_3}_{T,T}$, we have

$$C^{\lambda_3}_{S,S} = (B, 0, 0)$$
$$C^{\lambda_3}_{S,T} = (C, 0, 0)$$
$$C^{\lambda_3}_{T,S} = (C', 0, 0)$$
$$C^{\lambda_3}_{T,T} = (D, 0, 0)$$

for some $B, C, C', D \in M_2(\Bbbk)$. Further, by axiom (C2) of Definition 1.2 we have

$$\iota\big(C^{\lambda_3}_{S,S}\big) = C^{\lambda_3}_{S,S}$$
$$\iota\big(C^{\lambda_3}_{S,T}\big) = C^{\lambda_3}_{T,S}$$
$$\iota\big(C^{\lambda_3}_{T,T}\big) = C^{\lambda_3}_{T,T}$$

from which we deduce that B and D are symmetric and $C' = C^T$; further since $C^{\lambda_3}_{S,T} \neq C^{\lambda_3}_{T,S}$, C *cannot* be symmetric.

In order to derive a contradiction, let us fix a to be the element

$$\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 1, 0\right)$$

of A. Then by axiom (C3) of Definition 1.2 and the fact that $A(< \lambda_3) = \{0\}$, we have

$$\begin{aligned} a \cdot \big(C^{\lambda_3}_{S,T} - C^{\lambda_3}_{T,S}\big) &= aC^{\lambda_3}_{S,T} \;-\; aC^{\lambda_3}_{T,S} \\ &= r_a(S,S)C^{\lambda_3}_{S,T} + r_a(T,S)C^{\lambda_3}_{T,T} \\ &\qquad - r_a(S,T)C^{\lambda_3}_{S,S} - r_a(T,T)C^{\lambda_3}_{T,S}. \end{aligned}$$

But if we let

$$C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$$

then by direct calculation, we have

$$a \cdot \big(C^{\lambda_3}_{S,T} - C^{\lambda_3}_{T,S}\big) = a \cdot (C - C^T, 0, 0)$$

$$= a \cdot \left(\begin{pmatrix} 0 & c_{12} - c_{21} \\ c_{21} - c_{12} & 0 \end{pmatrix}, 0, 0 \right)$$
$$= \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 0, 0 \right),$$

and so by linear independence of $C^{\lambda_3}_{S,S}$, $C^{\lambda_3}_{T,S}$, $C^{\lambda_3}_{S,T}$, $C^{\lambda_3}_{T,T}$, we have

$$r_a(S,S) = r_a(S,T) = r_a(T,S) = r_a(T,T) = 0. \tag{A.5}$$

Now let

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{pmatrix}$$

and

$$D = \begin{pmatrix} d_{11} & d_{12} \\ d_{12} & d_{22} \end{pmatrix},$$

where we have used the fact the B and D are symmetric. Then by direct calculation we have

$$a \cdot C^{\lambda_3}_{S,S} = \left(\begin{pmatrix} b_{11} & 0 \\ 0 & b_{22} \end{pmatrix}, 0, 0 \right),$$

but by using Equation (A.5), we have

$$a \cdot C^{\lambda_3}_{S,S} = r_a(S,S) C^{\lambda_3}_{S,S} + r_a(T,S) C^{\lambda_3}_{T,S}$$
$$= \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, 0, 0 \right)$$

and so $b_{11} = b_{22} = 0$. By applying the same argument to the product $a \cdot C^{\lambda_3}_{T,T}$, we also have $d_{11} = d_{22} = 0$. But now we have

$$C^{\lambda_3}_{S,S} = (B, 0, 0) = b_{12} \cdot \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, 0, 0 \right)$$
$$C^{\lambda_3}_{T,T} = (D, 0, 0) = d_{12} \cdot \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, 0, 0 \right)$$

which implies that $C^{\lambda_3}_{S,S}$ and $C^{\lambda_3}_{T,T}$ are linearly dependent, and this is a contradiction. Thus indeed A is not cellular with respect to ι. $\square$

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