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Abstract

Many systems contain bottlenecks, critical linkages and key facilities. Such components when lost due to a man-made or natural disaster may imperil a system in performing its intended function. Loss of critical components of a system can lead to increased risk to the health and safety of populations, decreased sense of well-being, eliminated or degraded levels of services, costs to the economy, and can require significant time to recover to normal operations. This paper focuses on the impact on response and supply services due to the loss of one or more facilities, based upon a man-made disaster, where such facilities can be fortified in order to prevent such events. It is assumed that fortification resources are limited and must be used in the most efficient manner. This paper presents a model for allocating fortification resources among a set of facilities so that the impact of man-made/terrorist strikes is minimized. In a recent paper, Church et al. (2004) introduced the *r*-interdiction median problem that can be used to identify worst-case losses of facilities. This model was developed to identify critical infrastructure. In this paper we extend that model to address the option of fortifying such sites against possible interdiction. We present a new integer linear programming model that optimally allocates fortification resources in order to minimize the impact of interdiction. Computational results are presented in using this model for several hypothetical problems. We also prove a general property associated with interdiction and fortification.

1. INTRODUCTION

Since September 11, 2001 there has been a heightened concern for terrorism and the losses that a terrorist group may cause. The notion of large intentional, man-made disasters has a long history, starting with King Sennacherib (689 B.C.) who built a dam on the Euphrates so that he could let loose a resulting flood on the city of Babylon (Biswas, 1970). As a part of homeland security planning, there has been an interest in identifying critical infrastructure. Critical infrastructure may be defined as those elements that when lost result in significant disruption of the system in its ability to perform its function. These elements can include transportation linkages (e.g. bridges, tunnels, rail, etc.), facilities (e.g. port terminals, production facilities, warehouses, operations centers, emergency response facilities, hospitals, etc.), critical stock piles (e.g. smallpox vaccine, drugs, food, etc.), key personnel (e.g. water system operators) and landmarks that may contribute to loss of well-being. In a recent publication, Grubesic, O'Kelly, and Murray (2003) studied internet survivability by calculating network connectivity given specific node or link failure. This type of analysis can be used to identify critical system components. It is important to note that many systems have built in redundancy so that a system continues to operate (e.g. backup pumps in a sewage collection system and backup batteries in telephone switching centers, etc.) in the event of a component failure. There is a mature literature on system design under component failure. Unfortunately, such design does not normally take into account the possibility of intentional strikes, like the strike against the world trade center.

The military has had a long term interest in identifying critical elements of supply lines, so that they can identify targets that when hit (or interdicted), will result in decreased supplies or delays in getting supplies to an area of conflict (see for example, Ghare, et al. 1971; Whiteman, 1999; Wood, 1993). Models have been developed to allocate strike resources along supply routes in order to inflict the greatest harm on an enemy. The literature on this type of modeling is reviewed in Church, Scaparra, and Middleton (2004). For all practical purposes the focus in such models is directed to the interdiction of transportation elements. Our focus in this paper is on the possible loss of one or more facilities in a system. Facilities, such as hospitals, emergency response facilities, and power plants can serve critical functions in protecting people and providing important services. For example, one major railway in the U.S. has emergency response equipment placed along their rail system so that all areas of track are within 500 miles of such a facility and equipment. If one intentionally destroyed one of these facilities, then both the equipment would be lost as well as a significantly decreased capability to respond in the region served by any lost facility. We can measure the impact of this service loss in several ways (e.g. degraded response capability, recovery time, increased operation costs, increased damages, etc.). Increases in response time to an accident, like a hazardous materials spill, could lead to longer system downtime as well as increases in damages due to delayed spill containment. Thus, we can calculate the added costs of such events with an anticipated facility loss. If one assumes that such accidents are not too common, then the equipment at the closest remaining facility would be dispatched to a hazardous materials spill. By assigning each track segment to its closest facility, it is possible to calculate the average distance of the system of facilities to the system of tracks. Then one can compare the average response distance for the entire system to the average response distance when one or more facilities are lost due to an intentional strike. In a recent paper, Church et al. (2004) presented a model called the *r*-interdiction median model. This model identifies a subset *r* out of a total *p* facilities, which if lost, results in the greatest impact on average response distance or total weighted distance. Thus, the *r*-interdiction median model can be used to identify the facilities that are considered to be the most critical in terms of providing efficient service. The objective of this paper is to extend the *r*-interdiction median problem where there exists the possibility of fortifying one or more facilities, so that they can be protected from an attack. We assume that fortification resources are limited and only a subset of facilities can be protected. The question we pose is what facilities should be protected so that the impact of interdiction on the remaining facilities is minimized.

In the next section we begin with describing the *r*-interdiction median model. This model is based upon the assumption that all facilities are vulnerable to a strike. Following that we discuss the notion of fortification and present a new model that minimizes the greatest possible system disruption by allocating a limited amount of fortification resources. We prove that at least one of the facilities identified by the *r*-interdiction median problem must be fortified. However, the optimal fortification set does not necessarily include all interdicted facilities of the *r*-interdiction median model as those sites are identified without the option of fortification. That is, the option of fortification changes what remains critical. We also present some computational experience associated with the fortification problem on several hypothetical datasets.

2. BACKGROUND

One of the most widely used and studied location models is the *p*-median problem. The *p*-median problem involves locating *p* facilities on a network in such a manner that the total weighted distance of supplying each demand from their closest facility is minimized (Hakimi 1964 1965). Total weighted distance is calculated as the sum of all demand-facility interactions where each demand is assigned to its closest facility. For a given demand, the distance to that demand's closest facility is multiplied (weighted) by some measure of demand (e.g. population, the number of truck trips needed to supply that demand from a facility, the total tonnage needing transport, or the cost per unit distance in supplying a given demand). If total weighted distance is minimized, then the configuration is as accessible as possible to the points of demand. The p-median model is based upon the assumption that each demand can be served by its closest facility. Thus, it is assumed that the capacity of any facility will exceed the demands placed upon it. This location problem has been applied in a wide variety of settings, including post offices, school districting, salt storage locations, public clinics, transit garages, etc. There has also been a concerted effort at developing efficient solution techniques for the p-median problem starting with the classic works of Teitz and Bart (1968) and ReVelle and Swain (1970).

The p-median problem deals with finding the locations that decrease weighted distance the most. It is obvious that a planner would employ such a model in making siting decisions. In contrast to locating facilities, the focus could be turned in an opposite direction by asking which facilities of an existing configuration would disrupt accessibility the most if lost, intentionally or by accident. That is, we seek to identify those facilities, which, if lost, impact the resulting system efficiency the most. Such facilities could be defined as the most critical to efficient operation. We can define this in a more formal manner as:

> Of the p different locations of supply or emergency response, find the subset of r facilities, which when removed, yields the highest level of weighted distance.

This problem was originally defined by Church et al. (2004) and called the r-interdiction median problem. This problem can be cast as a mixed integer-linear programming problem. To present this model formulation, consider the following notation:

i index representing places of demand

j index representing existing facility locations

$$s_j = \begin{cases} 1, \text{ if a facility located at } j \text{ is eliminated by interdiction} \\ 0, \text{ otherwise} \end{cases}$$

F = the set of existing facilities j

$$x_{ij} = \begin{cases} 1, \text{ if demand } i \text{ assigns to a facility at } j \\ 0, \text{ otherwise} \end{cases}$$

a_i = a measure of demand (e.g. number of supply trips) needed at demand i

d_{ij} = the shortest distance between the supply/service facility at j and demand i

r = the number of facilities to be interdicted or eliminated

T_{ij} $= \{k \in F \mid k \neq j \text{ and } d_{ik} > d_{ij}\}$, the set of existing sites (not including *j*) that are farther than *j* is from demand *i*.

We can now formulate the r-interdiction median (RIM) problem as (Church et al. 2004):

$$Max \quad Z = \sum_i \sum_{j \in F} a_i d_{ij} x_{ij} \tag{1}$$

Subject to:

$$\sum_{j \in F} x_{ij} = 1 \quad \text{for each demand } i \tag{2}$$

$$\sum_{j \in F} s_j = r \tag{3}$$

$$\sum_{k \in T_{ij}} x_{ik} \leq s_j \quad \text{for each } i \text{ and each } j \in F \tag{4}$$

$$\begin{aligned} x_{ij} &= 0,1 \quad \text{for each } i \text{ and each } j \in F \\ s_j &= 0,1 \quad \text{for each } j \in F \end{aligned} \tag{5}$$

The RIM model objective seeks to maximize the weighted distance associated with the facilities that remain after interdiction of *r*-facilities. This model identifies the worst configuration of *p*-*r* facilities among the set of *p* existing facility locations. Constraint (2) requires that each demand assigns to a facility that is still available after interdiction. Constraint (3) establishes that exactly *r* facilities will be interdicted. Constraints (4) ensure that demand *i* assigns to the closest remaining facility to *i*. Since the objective involves maximizing the weighted distance of demand assignment, it is necessary to ensure that each demand assigns to its closest remaining facility. Constraint (4) prevents demand *i* from assigning to a facility farther than what *j* is from *i*, unless the facility at *j* has been interdicted. Essentially, demand *i* is forced to assign to its closest remaining facility. Gerrard and Church (1996) have shown that some versions of closest assignment constraints that have been defined in the literature have logical problems when there exist two or more facilities that are at the exact same distance from a demand. The manner in which constraints (4) are structured requires that for assignment to take place beyond such equidistant facilities, all would have to be interdicted as well as any closer facilities to that demand. Thus, distance ties do not pose a conflict as identified in Gerrard and Church (1996). The last set of constraints (5) maintains the integer restrictions on the decision variables. It is important to recognize that if all of the site interdiction variables s_j are zero-one in value, then at optimality the demand assignment variables, x_{ij}, will be zero-one as well. Thus, the above model can be solved using only one integer variable for each existing facility. The RIM model is the antithesis of the original p-median model formulation of ReVelle and Swain (1970), in that the p-median model locates facilities in order to minimize weighted distance and the RIM model eliminates facilities in order to maximize weighted distance (Church et al. 2004). Church et al. (2004) used this model formulation on a hypothetical problem to identify "critical facilities" and demonstrated that significant loss of system efficiency could result from optimal interdiction.

3. PROTECTING FACILITIES FROM INTERDICTION

Simply accounting for maximum possible losses to a system due to a limited number of interdicted elements may be a bit shortsighted. The solution to the RIM model helps illuminate what is critical, assuming that nothing is done to prevent such losses. In general there are a number of possible methods in which to disrupt worst-case losses of facilities. They include approaches such as:

1) site fortification,
2) enhanced security and detection,
3) general area defense capabilities like a missile battery,
4) general intelligence gathering and possible event preemption,
5) stated policy of retaliatory response, and
6) stock piles of critical components.

The first two alternatives are local facility-based actions, whereas the last four alternatives are wider reaching types of actions. For example, one might place vehicle barriers to prevent an unauthorized vehicle from moving within an area that needs to be safeguarded. Another example of fortification involves, making a building element capable of withstanding an attack, like the containment buildings at nuclear power plants. It may also be possible to protect a site by attempting to detect an event at a significant distance from the planned target with enough time to disrupt the event. For example, some facilities have a perimeter fence, surveillance cameras, and security patrols. Such measures are designed to detect and interrupt a disruptive event before major harm has occurred.

Somewhat wider reaching defensive systems can also be used to protect assets, like an air base or a defense battery. Such facilities will probably not deter terrorists who are likely to use methods that are not thwarted by ground-to-air missiles or air-to-ground missiles. Another approach might be to stock pile critical components so that if some are lost in a strike (or the capability to produce them is lost in a strike), they can be readily replaced without significant lags in time or be supplied for the over time that it takes to restore the capability to produce such critical components. Critical stockpiles could include drugs and vaccines that might not be manufactured when a facility is lost.

The threat of retaliation may also be a viable plan for increasing safety of facilities and infrastructure, in general. As long as the perpetrator can be readily identified and can be targeted, and if retaliation would inflict substantial harm, then such a policy could bring a type of blanket defense across a wide-ranging infrastructure. At issue is the fact that it is possible to develop two types of defense capabilities for important infrastructure: 1) wide ranging defense approaches, like intelligence gathering, retaliatory strikes and stockpiles, and 2) local types of defense strategies like fortification, surveillance and limited/controlled access. For the remainder of this paper we will concentrate on the possibility of protecting one or more facilities from interdiction in terms of local-based defense. This concept of hardening or fortifying infrastructure against terrorism was recently discussed by Salmeron, Wood, and Baldick (2004) within the context of the electrical power grid. In the next section, we model the option of fortifying or hardening facilities against interdiction. Specifically, we will assume here that we have the option of "fortifying" or hardening a facility so that it cannot be easily interdicted or that the probability of being successfully interdicted is decreased to the point that an intentional strike would be either thwarted entirely or aimed at an easier target. Thus, we assume that fortification can protect a site from interdiction. We also assume that facility "fortification" costs money and there is a limit to the amount of infrastructure that can be fortified. In fact, we assume that we have resources to fortify or protect q facilities from interdiction. If we have enough resources to fortify q facilities from interdiction, then the logical question is which facilities should be protected? We can formalize this as:

> Of the p different locations of supply or emergency response, find the subset of q facilities, which when fortified, provides the best protection to a subsequent optimal r-interdiction strike.

The basic premise is that we want to identify which subset of facilities to fortify or protect, such that when the remaining unprotected facilities are subjected to an optimal r-interdiction, the resulting total weighted distance is minimized. In essence, we seek the best way to thwart interdiction. We will call this problem the interdiction median problem with fortification (IMF).

Let us assume that we have a system of p facilities serving n demand nodes. The RIM model can be applied to this system to identify the optimal interdiction set of r facilities, assuming that no facilities have been fortified. Call this optimal interdiction set U. Now consider the following:

Theorem: Optimal fortification of q of the p facilities must include at least one site of the interdiction set U.

The proof of this theorem is quite simple. If we do not fortify any site that is identified in the unrestricted interdiction set U, then it is still possible to interdict every site in the set U and the worst possible case of interdiction for the p sites has not been thwarted. Thus, in order to prevent the worst case of interdiction from being possible at least one site in the interdiction set U must be fortified. *Q.E.D.*

In some instances, the best fortification plan may involve fortifying the entire interdiction set U. However, there are circumstances where the optimal strategy involves the interdiction of only one site of the set U. We will give an example of this in a subsequent section. In the next section we will present a model that optimally distributes fortification resources, in order to thwart the worst-case interdiction pattern on a configuration of facilities.

4. FORMULATING A MODEL THAT OPTIMIZES THE USE OF FORTIFICATION RESOURCES

In this section we develop a formulation for the *r*-interdiction median problem with fortification (IMF). This problem involves two competing elements, interdiction and fortification. The basic idea is to determine which subset of sites to fortify, so that the resulting partially fortified pattern is the least vulnerable to interdiction.

It is important to note from the outset that depending upon the size of a specific IMF problem, enumeration might be a viable solution technique. The number of cases possible for a specific set of values *p, q* and *r* is:

$$\binom{p}{q} \times \binom{p-q}{r} = \frac{p!}{q!(p-q)!} \times \frac{(p-q)!}{r!(p-q-r)!} \qquad (6)$$

or the number of combinations of fortifying q facilities out of p times the number of combinations of interdicting r of the remaining p-q unprotected facilities. Even what might seem to be a small problem could have a significant number of cases to enumerate before identifying an optimal solution. For example, consider a problem involving 20 facilities where 7 facilities are fortified and 4 facilities are interdicted. This problem has over 55 million cases. Enumerating this number of possible fortification patterns and cases of interdiction can take considerable computation time. Consequently, it makes sense to pursue the development of a formal optimization model. It is important to note that the reason why the number of cases can be large is that it represents the product of two different values, each representing a number of combinations.

Let us assume that the number of interdicted facilities will always be relatively small in number, especially in relation to the total number of facilities. If this is the case we can then assume that the number of interdicted patterns will be relatively small, say in the thousands. For example, if we have a 20 facility pattern and the level of interdiction is $r = 4$, then there are:

$$\binom{20}{4} = 4{,}845 \text{ interdiction patterns} \qquad (7)$$

This means that we have to be concerned with 4,845 possible responses to a possible fortification plan. We can test a given fortification plan in terms of its efficacy, based upon which interdiction patterns have been thwarted. As long as the number of such patterns is not too large, we can generate these possible patterns in advance and use them to calculate the value of fortification.

For example consider the 4-node network given in Figure 1. For this problem, we will start off with all four nodes housing a facility. To make this simple, assume that each demand nodes has a demand weight of 1. This means that each demand node has a facility and that the weighted distance associated with the 4-facility pattern is zero.

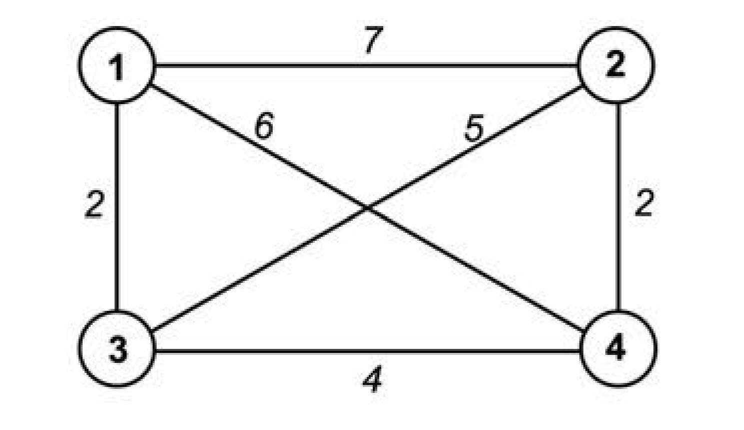

Figure 1: A simple four-node network, starting with each node housing a facility.

For this example, we will assume that we have the resources to fortify 2 facilities and that the interdiction will remove the 2 facilities not fortified. That is, either a facility is fortified or it will be interdicted. There are 6 possible ways in which the above network can be interdicted, when there is no fortification. We can calculate the impact of each pattern of interdiction. The worst will be the optimal RIM solution for $r = 2$. The set of possibilities are given in Table 1.

Table 1: Enumerating all possible interdiction patterns

Interdiction pattern	Remaining facilities	Resulting weighted distance
1, 2	3, 4	4
1, 3	2, 4	10
1, 4	2, 3	4
2, 3	1, 4	4
2, 4	1, 3	9
3, 4	1, 2	4

The worst-case interdiction pattern involves eliminating facilities at nodes 1 and 3, leaving facilities at nodes 2 and 4 and a total weighted distance of 10. The theorem given in section 4 implies that at least one of these two interdicted facilities must be part of the optimal fortification plan. But what is the optimal fortification plan? To identify the optimal fortification plan, we can enumerate the possibilities for this simple problem as well. Results of enumerating fortification patterns of 2 facilities followed by interdiction of the remaining 2 facilities are given in Table 2.

Table 2: Enumerating patterns of fortification

Fortification Pattern	Interdicted Facilities	Remaining facilities	Resulting weighted distance
1, 2	3, 4	1, 2	4
1, 3	2, 4	1, 3	9
1, 4	2, 3	1, 4	4
2, 3	1, 4	2, 3	4
2, 4	1, 3	2, 4	10
3, 4	1, 2	3, 4	4

From Table 2, we observe that there are several patterns of fortification that yield the lowest weighted distance after interdiction. For example, if we fortify sites 1 and 2 leaving facilities at nodes 3 and 4 to be interdicted, then the resulting weighted distance is 4. Note that one of these sites is in the optimal RIM solution of Table 1. If our pattern of fortification is sites 1 and 3, leaving facilities at 2 and 4 to be interdicted, then the resulting weighted distance is 9. Thus, fortifying all sites in the optimal interdiction set U consisting of sites 1 and 3 (see Table 1) yields a solution that is worse than fortifying only one site in the optimal interdiction set (e.g. fortify sites 1 & 2). Note for this simple problem, that there are 4 fortification patterns that yield the same lowest possible weighted distance (i.e. 4) after interdiction of the remaining facilities. Each one of these optimal fortification solutions uses at least one site from the optimal interdiction set without fortification. This property was proven in the theorem given in section 4.

Suppose that we identify in advance all possible interdiction patterns, like what we did in the previous example. We can do this given our earlier assumption that the number of interdictions is relatively small in number. Consider the following notation:

h index used to represent specific interdiction patterns

H set of all possible interdiction patterns

d_i^h shortest distance between demand i and its closest non-interdicted facility after the interdiction of pattern h

$B_i^h = \{ j \in F \mid d_{ij} < d_i^h \}$, the set of closest sites to i that have been interdicted in pattern h.

q number of facilities to be fortified

$z_j = \begin{cases} 1, & \text{if a facility located at } j \text{ is fortified} \\ 0, & \text{otherwise} \end{cases}$

For each interdiction pattern, we can calculate the weighted distance that would result without any fortification being made. Without fortification, the weighted distance for a given interdiction pattern can be calculated by assigning each demand to its closest non-interdicted facility. This is:

$$WD_h = \sum_{i=1}^{n} a_i d_i^h \qquad (8)$$

For the moment, let us assume that we must protect against interdiction pattern h. Without fortification of any facilities of pattern h, the weighted distance would remain unchanged. Thus, to affect the value of WD_h, we must fortify at least one or more sites in the interdiction pattern h. Each demand assigns to its closest non-interdicted site at a distance of d_i^h. In fact all facility locations $j \in F$ where $d_{ij} < d_i^h$ must be interdicted, or the value of d_i^h would be smaller. If we fortify any one of the sites $j \in B_i^h$, then weighted distance can be improved by:

$$a_i(d_i^h - d_{ij}) \qquad (9)$$

depending upon which site $j \in B_i^h$, we fortify. If two sites in B_i^h are fortified, then the improvement in weighted distance should be calculated based upon the closest fortified facility in the set B_i^h. In order to track such possibilities, consider the following decision variable:

$$x_{ij}^h = \begin{cases} 1, \text{ if demand } i \text{ assigns to fortified facility } j \text{ in interdiction pattern } h \\ \\ 0, \text{ otherwise} \end{cases}$$

This type of decision variable will be used for demand i and interdiction pattern h only for those $j \in B_i^h$. Improvement in weighted distance under interdiction pattern h can be made only for those demands i where the closest facility(ies) is(are) interdicted (i.e. $j \in B_i^h$). A specific interdiction pattern can be partially or completely thwarted depending upon which facilities are fortified. When a facility j is fortified, then the fortification of j will disrupt its interdiction and allow demand to assign to it, if it results in an improvement in weighted distance (for any interdiction pattern h including site j). With this basis, we can now formulate a novel form of the IMF model as follows:

$$\min\ W \tag{10}$$

Subject to:

$$x_{ij}^h \le z_j \qquad \text{for all } i \in N, \text{ for all } h \in H \text{ and for all } j \in B_i^h \tag{11}$$

$$\sum_{j \in B_i^h} x_{ij}^h \le 1 \qquad \text{for all } i \in N, \text{ and for all } h \in H \text{ and where } \left|B_i^h\right| \ge 2 \tag{12}$$

$$W \ge \sum_{i \in N} a_i d_i^h - \sum_{i \in N} \sum_{j \in B_i^h} a_i (d_i^h - d_{ij}) x_{ij}^h \qquad \text{for all } h \in H \tag{13}$$

$$\sum_{j \in F} z_j = q \tag{14}$$

$$z_j = 0,1 \qquad \text{for all } j \in F \tag{15}$$

$$x_{ij}^h = 0,1 \qquad \text{for all } i \in N, \text{ for all } h \in H \text{ and for all } j \in B_i^h \tag{16}$$

where:

$W =$ the weighted distance resulting from the fortification of q facilities (assuming that the most disruptive interdiction of r non-fortified sites will occur).

This model optimizes the fortification of q facilities, given that any interdiction pattern can happen involving r facilities. The objective is to allocate fortification resources in such a manner that the impact of interdiction is reduced the most. Basically, weighted distance is minimized by fortifying sites, assuming that the worst case of interdiction of r non-fortified sites will take place. Constraints of type (11) ensure that demand i cannot assign to a facility $j \in B_i^h$ under interdiction pattern h, unless that site j has been fortified. Constraints of type (12) ensure that a demand assigns to at most one facility assuming that interdiction pattern h occurs. This assignment is made either to the closest non-interdicted site or to the closest fortified site. Constraints of type (13) calculate the weighted distance of interdiction pattern h, given fortification. If for a given pattern h, some sites in that pattern have been fortified, then that interdiction pattern has been partially thwarted, and the weighted distance is computed associated with this fortification. If none of the sites in the interdiction pattern h have been fortified, then the weighted distance remains unchanged and is calculated as the following sum:

$$\sum_{i=1}^{n} a_i d_i^h \qquad (17)$$

in the constraint associated with interdiction pattern h. The final structural constraint ensures that exactly q sites will be fortified. Basically the above model keeps track of all possible interdiction patterns, and fortifies those sites so that the lowest weighted distance is achieved after any possible interdiction.

The above model can be reduced for several cases involving the size of the membership of set B_i^h given an interdiction pattern h and a demand i. If the closest facility to demand i is not interdicted in pattern h, then the set B_i^h will be empty and there will be no need for any variables x_{ij}^h and no associated constraints (11) and (12). If for a given i and h, the set B_i^h contains only one member, then a different type of reduction is possible. For this case let $\overline{j}$ be the lone member of B_i^h. There is no need for the variable $x_{i\overline{j}}^h$, as the only option for improving weighted distance for this i (given interdiction pattern h) is to fortify site $\overline{j}$. If site $\overline{j}$ is fortified, then the weighted distance for demand i can be improved. This can be represented by the variable demand $z_{\overline{j}}$. Thus, for the second case where B_i^h has only one member $\overline{j}$, we can eliminate both constraints (11) and (12) and use the variable $z_{\overline{j}}$ in place of $x_{i\overline{j}}^h$ in constraint

(13). This second case results in the elimination of one constraint and one variable. Finally, consider the case where the membership of the set B_i^h is greater than or equal to 2. For this last case, it is necessary to maintain constraints (11) and (12), but it might be possible to eliminate specific x_{ij}^h variables. This last concept is based upon the properties of the COBRA formulation of the p-median problem presented by Church (2003). In the p-median problem, Church proved that certain variables can be combined under specific circumstances. For example, if site 10 is the 6th closest facility for demands 2 and 5, and the set of closer sites is the same for both demands, then it can be proved that if 2 assigns to site 10 as its closest facility then demand 5 will assign to site 10 as its closest facility. For such cases in the p-median problem, only one assignment variable is needed rather than two individual assignment variables. For possible interdiction patterns that include site 10, improvement in weighted distance by fortifying site 10 will accrue to both demands 2 and 5. In fact, given the same closeness for facility 10 in regard to demands 2 and 5, the theorem given in Church (2003) applies and this means that $x_{2\,10}^h = x_{5\,10}^h$ for those patterns h that involve interdiction of facility at 10. Consequently, variables $x_{2\,10}^h$ and $x_{5\,10}^h$ can be combined for those specific cases of h that involve the interdiction of site 10. Combining such variables will result in a reduction of the number of needed variables as well the reduction of constraints (11) (and possibly some of (12) as well). The model applications used in the next section invoke all such reductions when they apply.

The IMF model formulation is a classic integer-linear programming model and can be solved by the use of general-purpose integer-linear programming optimization software. In the next section, we present computational results using this model applied to several geographical data sets.

5. SOLVING THE INTERDICTION MEDIAN WITH FORTIFICATION MODEL

We utilized two different geographical data sets to test the IMF: the 55 node problem of Swain (1971) and the 150 node London, Ontario data set (Goodchild and Noronha, 1983). Both data sets have been used extensively in the literature to test location model constructs. The IMF model was set up using Optimization Programming Language (OPL) within OPL studio and solved by Cplex. The results presented here are based upon setting a priority on feasibility and a priority branching strategy involving the set of variables representing the best interdiction pattern. Results were generated using OPL studio 3.5 and Cplex 7.0 on a Intel Pentium 4 (1.8Ghtz with 512 MB of ram).

Table 3 presents results of the IMF model applied to the two different data sets for a variety of parameter values (i.e. number of existing facilities, p; number of facilities to be fortified, q; and the number of unfortified facilities to be interdicted, r). For a given problem of p-facilities, we assumed that the original configuration was the optimal p-facility configuration. For this pattern of p facilities, we solved the IMF model in order to determine the set of q facilities to fortify so that the impact of a worst-case interdiction of r facilities was minimized. For each problem solved, Table 3 gives the original weighted distance of the optimal p-median configuration, as well as the weighted distance after fortification and worst-case interdiction (i.e. F/I). The table also gives for each problem the computational time needed to solve the problem using the IMF model, the number of variables, and the number of constraints. For example, one of the problems solved for the London, Ontario data set involved 10 facilities, 2 fortifications and 2 interdictions. For this problem, the weighted distance of the starting p-median configuration was 189,440.37. Optimal fortification based upon worst-case interdiction results in a weighted distance of 246,353.28. The model was solved in 0.13 seconds and involved 73 variables and 140 constraints. The result means that weighted distance after worst-case interdiction can be no lower than 246,353.28 after fortification. In fact, without optimal fortification weighted distance after interdiction is likely to be considerably larger than 246,353.28. Reviewing the problem data and model results in Table 3, reveals that most problems were solved in less than a few seconds, although a few problems took more than 10 minutes of solution time. Overall, the larger the value of r (the amount of interdiction), the larger the composite model and the longer it takes to solve using integer linear programming and branch and bound.

Table 4 presents results that are focused on a configuration of 15 facilities on the London, Ontario data set. Given the optimal 15 facility median solution, we solved the IMF model for a variety of values of r and q. We used values of r ranging from 1 to 4 and q ranging from 1 to 15-r. For each value of interdiction we also solved the original RIM model. The RIM solution is listed with a level of 0 fortifications. The optimal objective values for IMF (and RIM) and the solution times are given in Table 4. The results of table 4 are presented graphically in Figure 2. Figure 2 depicts the tradeoff in weighted distance based upon the amount of fortification, given a certain level of interdiction. For example, if the level of interdiction is r=4, the tradeoff curve is plotted as a set of connected circles. The curve starts at a high of 254,601.02 when no fortification is used. Changing the level of fortification from 0 to 1 facility reduces the weighted distance after interdiction to 246,584.54. The reduction in weighted distance is attributable to the fortification of 1 site. Increasing fortification to 2 facilities from 1 reduces weighted distance after interdiction to 223,037.02. The marginal gain in improvement in weighted distance for the second fortification is greater than the marginal gain in weighted distance improvement due to the fortification of only one site. This means that for this case the combination of 2 fortified sites provides considerably more protection than single fortification of either site alone. That is, the combined value of fortification can be greater than the sum of the individual fortification contributions. As fortification increases, weighted distance after interdiction continues to drop over the entire range of fortification values. One should note that in general, as the level of fortification increases, the marginal reductions in weighted distance tend to decrease. From the 4 curves, one can also observe that the greatest return to an investment in fortification occurs for the first few units of fortification.

Figures 3, 4, and 5 depict a configuration of 5 facilities on the Swain data set. Figure 3 depicts the optimal 5 median solution, which has an associated weighted distance of 2,950.41. Optimal interdiction of 2 facilities of this pattern is depicted in Figure 4. This pattern was generated by the RIM model and is depicted in Church et al. (2004). The weighted distance after the loss of the most critical 2 facilities (given no fortification), results in a weighted distance of 6,124.53. Figure 5 presents an optimal solution to the IMF model where 2 facilities are fortified. This solution after interdiction is 4,072.74. Fortification included one of the original interdicted sites of figure 4. These three figures depict the spatial complexities of weakness to interdiction. First, the weighted distance of an unprotected system increased by 100% after interdiction. Fortifying only 2 of the 5 sites provided a substantial improvement in system performance after interdiction. For this case, the weighted distance increases by only 35% due to interdiction, when optimally fortifying 2 facilities.

6. CONCLUSIONS

Facilities and other types of infrastructure are at risk to intentional disruption caused by possible military or terrorist activities. An intentional strike to take out a facility is called interdiction. A number of models have been developed for military purposes that involve interdiction of transport supply (e.g. McMuster and Mustin, 1970; Ghare et al. 1971; Wood 1993). Interdiction modeling has also been extended to supply and response facilities (Church, et al. 2004). Such models can be used to identify the weakest parts of a service or supply system. The weakest part of a system can be designated as “critical.” In this paper, we address the possibility of fortifying a limited set of facilities against interdiction. We assume that “fortified” facilities are protected against intentional strikes or encourage interdiction to focus on something without such protection. When there are resources to fortify a subset of existing facilities, then the obvious question is: which facilities should be fortified? This is the main question addressed in this paper.

In earlier work, Church et al. (2004) presented a model called the r-interdiction median problem. The r-interdiction median problem involves identifying a subset of existing facilities which, when removed, results in the greatest increase in weighted distance. This model is the antitheses of the p-median model. Whereas the p-median problem involves finding the p locations for facilities that create the most efficient system, the r-interdiction median involves finding the r out of p existing facilities that when removed, results in the least efficient system. This paper introduces the option of fortification, where a subset of q facilities can be protected from interdiction. The fortification problem involves identifying which set of q facilities should be fortified so that optimal interdiction of this system is thwarted the most. We have presented an integer-linear programming model that can be used to identify an optimal fortification strategy. We have tested this model on two different geographical data sets, using general-purpose software. Optimal solutions were generated for all of the problem instances solved.

We have also presented a theorem and proof concerning one of the properties of the r-interdiction median problem with fortification (IMF). Specifically, we have shown that a solution to the fortification problem includes at least one site that is a member of the optimal solution to the r-interdiction median model. We conjecture that this property can be generalized to other fortification problems as well. Although optimal solutions to the IMF model utilize at least one site found in an optimal r-interdiction median solution, results tend to show that fortification resources are also dispersed beyond the sites that are part of the optimal r-interdiction median solution.

The concepts of interdiction and fortification are new concepts in location science and have been motivated by recent world events. Protecting valuable resources should be an important aspect in public safety. Through the development of models such as RIM and IMF, we can begin to focus on possible impacts of intentional strikes as well as how best to protect existing infrastructure from such possibilities. Even though traditional designs can include failure analysis and may involve the use of redundant elements (e.g. backup pumps in a sewage lift station), such methods are not based upon the complete and catastrophic loss of a facility, but on the failure rates of components within a facility. Repair of components often takes hours or days, but the loss of a facility may not be replaceable until months or even years. The RIM and IMF models focus on such catastrophic events.

Research is needed to expand these two models in two principal areas. The first need involves the development of efficient algorithms and heuristics. Although we have developed a new model and demonstrated the use of general-purpose integer programming software in solving this new model applied to moderately sized problems, the capability to handle large problems needs to be developed. The second major need is associated with extending model elements, such as allowing for partial protection/fortification where there is a probability that a facility will survive an intentional strike. We hope that this paper encourages the development of this new area of location science and leads to the protection of important services and lifelines.

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8. ACKNOWLEDGEMENTS

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Table 3: Results from the application of the IMF model to two different data sets

Pr. Set	*p*	*q*	*r*	Obj Values PMP	F/I	Time (sec.)	Vars	Constrs
Swain	5	2	2	2,950.41	4,072.74	0.03	26	42
Swain	7	2	2	2,427.41	3,462.92	0.03	42	74
Swain	7	2	3	2,427.41	3,820.16	0.11	184	312
Swain	7	3	2	2,427.41	3,252.89	0.05	42	74
Swain	7	3	3	2,427.41	3,639.41	0.13	184	312
Swain	9	2	2	2,067.97	2,877.58	0.05	48	95
Swain	9	2	3	2,067.97	3,207.41	0.22	291	520
Swain	9	3	2	2,067.97	2,811.04	0.08	48	95
Swain	9	3	3	2,067.97	3,096.71	0.23	291	520
Swain	9	3	4	2,067.97	3,448.54	1.23	923	1,568
Swain	9	4	3	2,067.97	2,894.39	0.23	291	520
Swain	9	4	4	2,067.97	3,148.65	1.22	923	1,568
London	10	2	2	189,440.37	246,353.28	0.13	73	140
London	10	2	3	189,440.37	278,117.28	0.50	507	918
London	10	2	4	189,440.37	315,642.64	2.49	1816	3275
London	10	3	2	189,440.37	238,765.24	0.13	73	140
London	10	3	3	189,440.37	271,858.78	0.70	507	918
London	10	3	4	189,440.37	302,606.11	3.69	1,816	3,275
London	10	4	2	189,440.37	237,389.35	0.13	73	140
London	10	4	3	189,440.37	266,155.05	0.73	507	918
London	10	4	4	189,440.37	286,432.60	5.09	1,816	3,275
London	10	5	2	189,440.37	235,407.72	0.16	73	140
London	10	5	3	189,440.37	255,685.27	0.75	507	918
London	10	5	4	189,440.37	275,642.12	5.14	1,816	3,275
London	15	2	2	146,601.37	182,539.41	0.23	108	245
London	15	2	3	146,601.37	202,165.68	1.75	1,216	2,302
London	15	2	4	146,601.37	223,037.02	12.33	7,336	12,858
London	15	3	2	146,601.37	178,885.48	0.23	108	245
London	15	3	3	146,601.37	194,776.53	2.22	1,216	2,302
London	15	3	4	146,601.37	218,329.62	23.70	7,336	12,858
London	15	4	2	146,601.37	177,390.35	0.23	108	245
London	15	4	3	146,601.37	192,454.70	2.19	1,216	2,302
London	15	4	4	146,601.37	212,163.47	32.55	7,336	12,858
London	15	5	2	146,601.37	175,068.51	0.27	108	245
London	15	5	3	146,601.37	191,346.09	2.73	1,216	2,302
London	15	5	4	146,601.37	207,597.07	59.28	7,336	12,858
London	15	6	2	146,601.37	172,258.51	0.23	108	245

Table 3 Continued:

Pr. Set	***p***	***q***	***r***	**Obj Values PMP**	**F/I**	**Time (sec.)**	**Vars**	**Constrs**
London	15	6	3	146,601.37	190,210.89	3.61	1,216	2,302
London	15	6	4	146,601.37	204,985.42	87.97	7,336	12,858
London	15	7	2	146,601.37	171,676.68	0.28	108	245
London	15	7	3	146,601.37	187,599.23	5.30	1,216	2,302
London	15	7	4	146,601.37	198,808.38	105.75	7,336	12,858
London	20	6	2	122,360.47	143,120.20	0.42	129	354
London	20	6	3	122,360.47	153,263.89	8.08	1,984	4,121
London	20	6	4	122,360.47	164,855.27	246.06	16,973	30,840
London	20	7	2	122,360.47	141,722.36	0.41	129	354
London	20	7	3	122,360.47	151,219.26	9.94	1,984	4,121
London	20	7	4	122,360.47	163,083.32	380.94	16,973	30,840
London	20	8	2	122,360.47	140,661.26	0.42	129	354
London	20	8	3	122,360.47	150,890.24	10.09	1,984	4,121
London	20	8	4	122,360.47	161,448.25	611.08	16,973	30,840
London	20	9	2	122,360.47	139,808.49	0.44	129	354
London	20	9	3	122,360.47	150,396.96	14.95	1,984	4,121
London	20	9	4	122,360.47	160,219.78	898.52	16,973	30,840
London	20	10	2	122,360.47	139,075.35	0.47	129	354
London	20	10	3	122,360.47	149,219.04	15.86	1,984	4,121
London	20	10	4	122,360.47	156,646.17	714.61	16,973	30,840

Table 4: Results of fortifying an optimal 15 facility median configuration on the London, Ontario data set (results generated by RIM and IMF models).

	Obj Values				**Time (sec.)**			
q	**r = 1**	**r = 2**	**r = 3**	**r = 4**	**r = 1**	**r = 2**	**r = 3**	**r = 4**
0	165,624.30	204,319.87	226,305.37	254,601.02	0.03	0.02	0.05	0.16
1	164,781.84	183,010.48	205,003.46	246,584.54	0.09	0.22	1.44	6.08
2	163,987.55	182,539.41	202,165.68	223,037.02	0.11	0.23	2.03	13.84
3	161,499.30	178,885.48	194,776.53	218,329.62	0.11	0.23	1.97	25.28
4	161,007.95	177,390.35	192,454.70	212,163.47	0.11	0.25	2.30	35.47
5	160,709.16	175,068.51	191,346.09	207,597.07	0.13	0.25	2.86	62.88
6	159,145.96	172,258.51	190,210.89	204,985.42	0.14	0.25	3.69	93.52
7	159,132.09	171,676.68	187,599.23	198,808.38	0.13	0.33	4.72	98.80
8	156,756.26	171,512.69	183,910.45	196,521.46	0.14	0.28	4.42	114.75
9	156,507.61	169,663.94	181,831.58	192,429.85	0.14	0.31	4.61	96.05
10	156,163.10	169,286.98	179,273.52	188,198.80	0.14	0.36	4.23	81.47
11	155,526.64	166,662.51	175,587.79	183,523.02	0.13	0.28	3.25	79.55
12	155,005.01	164,566.74	172,111.42	--	0.14	0.27	2.31	–
13	154,536.59	163,403.46	--	--	0.16	0.23	--	–
14	154,044.22	--	--	--	0.16	--	--	–

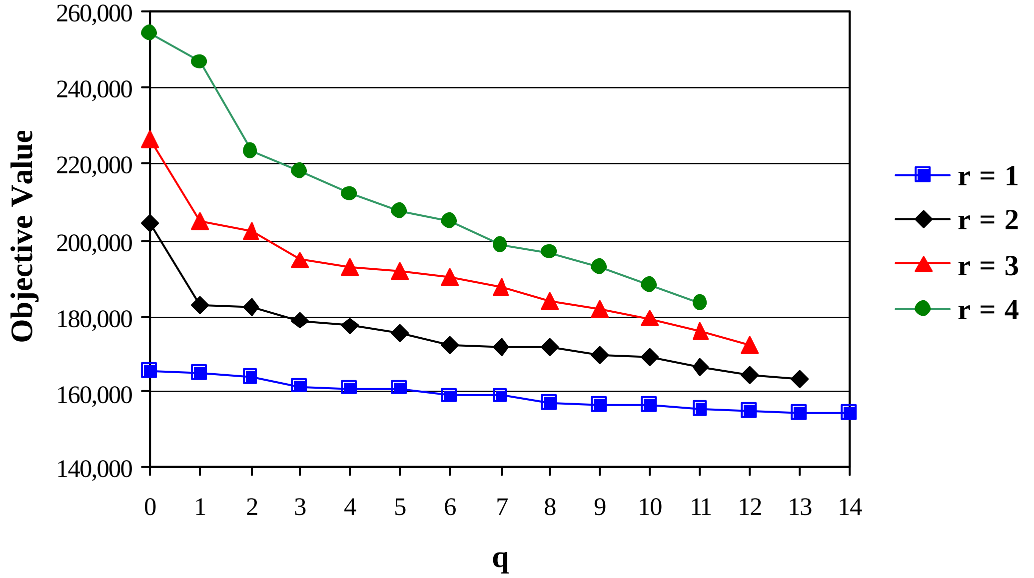

Figure 2: Tradeoff curves that depict the marginal values to adding fortification resources given a level of interdiction

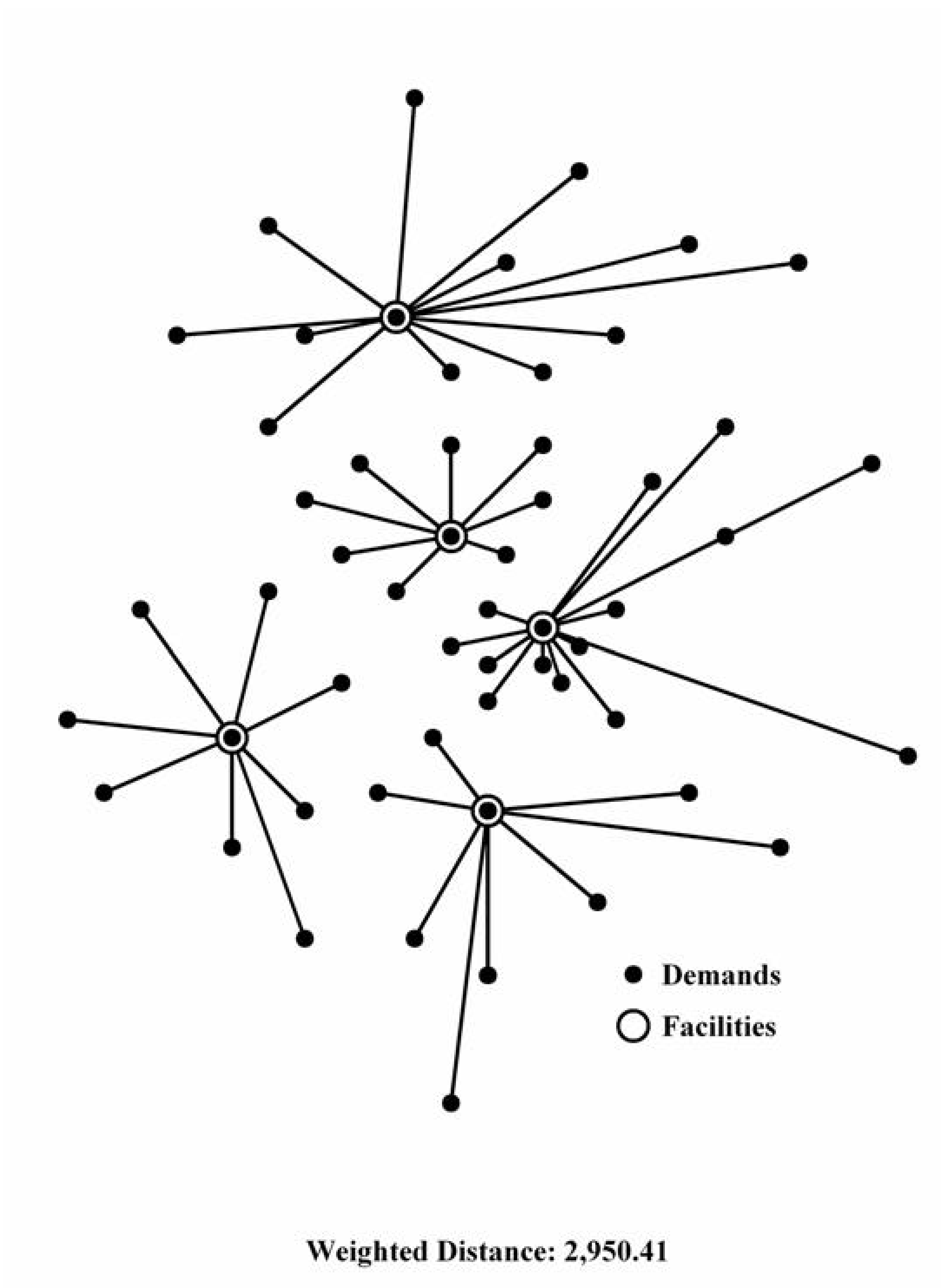

Figure 3: Optimal five facility median configuration serving a set of 55 demand points.

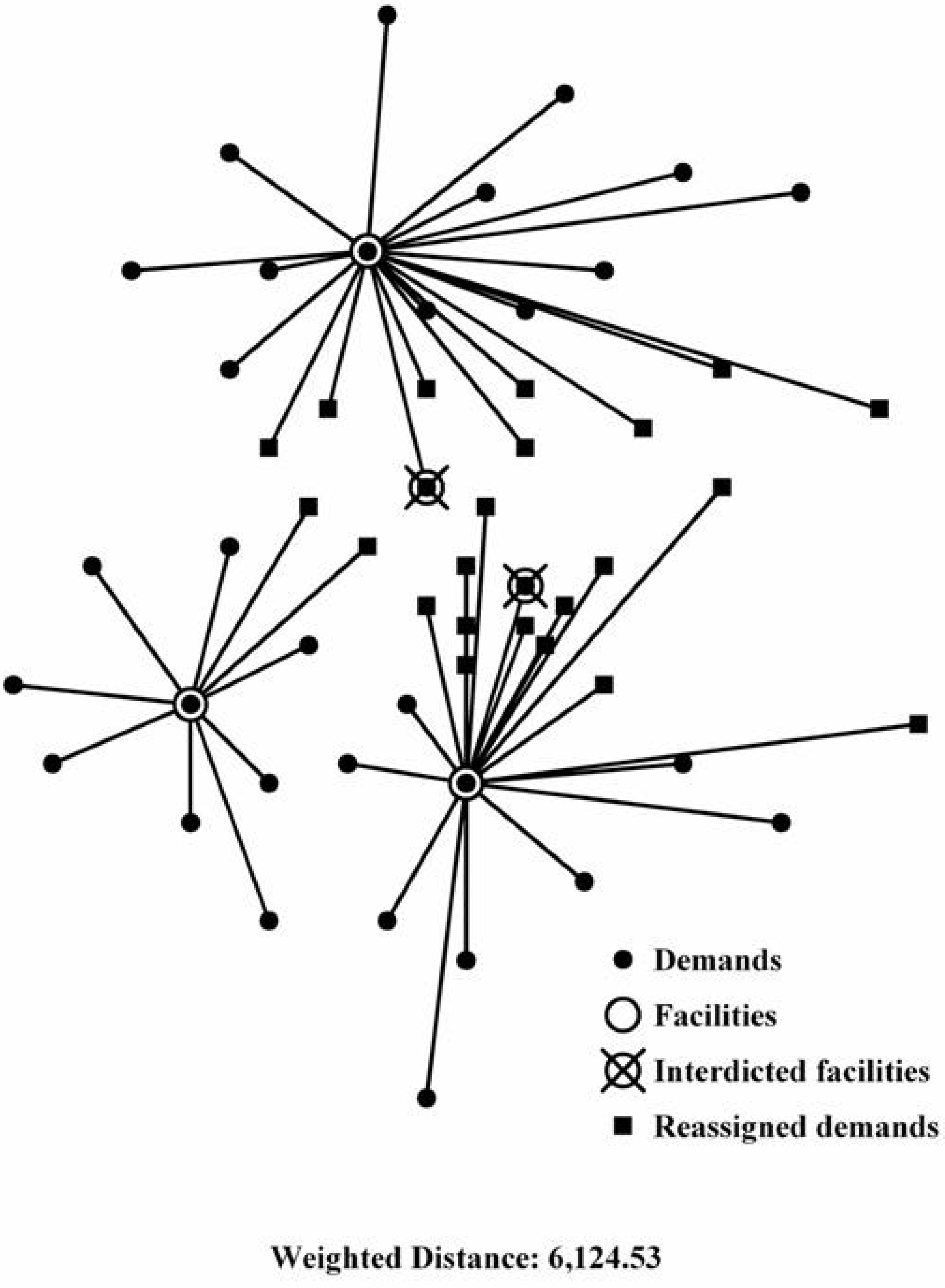

Figure 4: Five facility configuration with worst case interdiction of 2 sites (results of the RIM model).

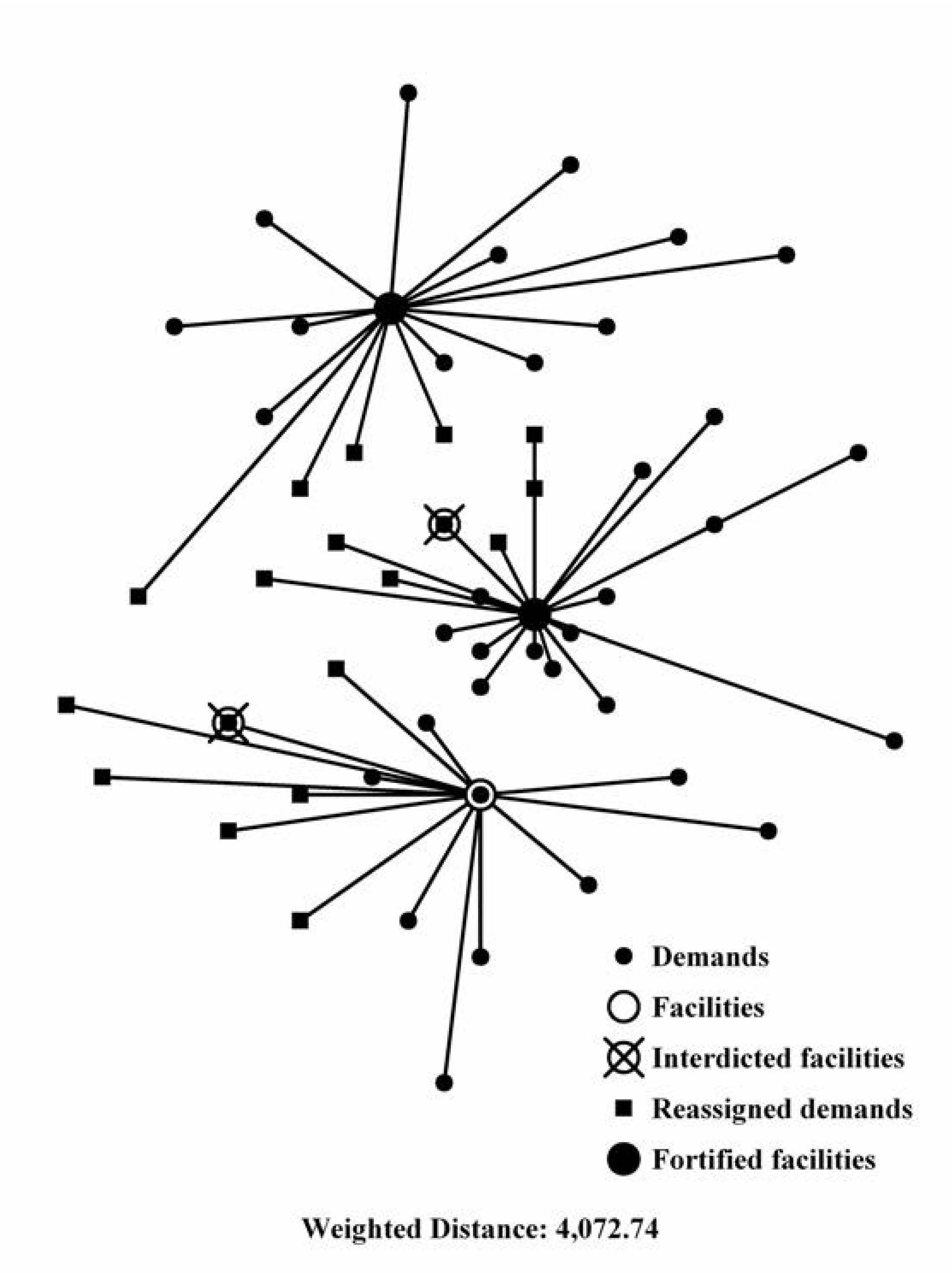

Figure 5: Five facility configuration with optimal fortification of 2 sites subject to worst case interdiction of 2 sites (results of the IMF model).