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# *Continued fractions for some transcendental numbers*

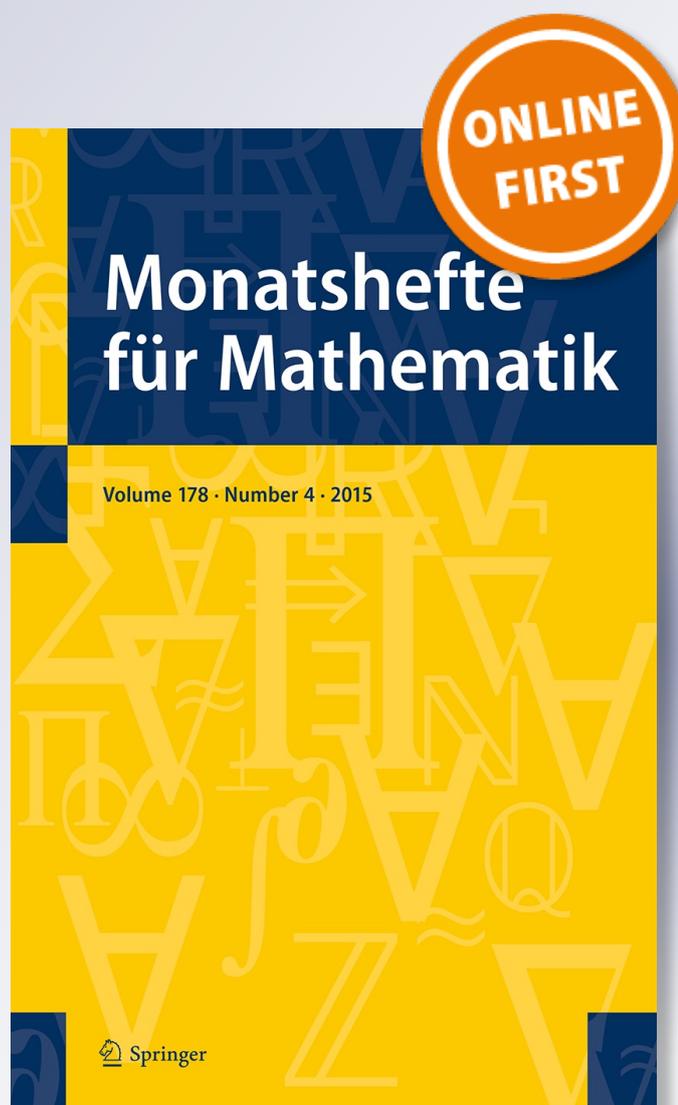
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# Continued fractions for some transcendental numbers

Andrew N. W. Hone<sup>1</sup> 

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**Abstract** We consider series of the form

$$\frac{p}{q} + \sum_{j=2}^{\infty} \frac{1}{x_j},$$

where  $x_1 = q$  and the integer sequence  $(x_n)$  satisfies a certain non-autonomous recurrence of second order, which entails that  $x_n | x_{n+1}$  for  $n \geq 1$ . It is shown that the terms of the sequence, and multiples of the ratios of successive terms, appear interlaced in the continued fraction expansion of the sum of the series, which is a transcendental number.

**Keywords** Continued fraction · Non-autonomous recurrence · Transcendental number

**Mathematics Subject Classification** Primary 11J70; Secondary 11B37

## 1 Introduction

In recent work [5], we considered the integer sequence

$$1, 1, 2, 12, 936, 68408496, 342022190843338960032, \dots \quad (1.1)$$

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(sequence A112373 in Sloane's Online Encyclopedia of Integer Sequences), which is generated from the initial values  $x_0 = x_1 = 1$  by the nonlinear recurrence relation

$$x_{n+2} x_n = x_{n+1}^2 (x_{n+1} + 1), \quad (1.2)$$

and proved some observations of Hanna, namely that the sum

$$\sum_{j=1}^{\infty} \frac{1}{x_j} \quad (1.3)$$

has the continued fraction expansion

$$[x_0; y_0, x_1, y_1, x_2, \dots, y_{j-1}, x_j, \dots], \quad (1.4)$$

where  $y_j = x_{j+1}/x_j \in \mathbb{N}$  and we use the notation

$$[a_0; a_1, a_2, a_3, \dots, a_n, \dots] = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots \frac{1}{a_n + \dots}}}}$$

for continued fractions. Furthermore, we generalized this result by obtaining the explicit continued fraction expansion for the sum of reciprocals (1.3) in the case of a sequence  $(x_n)$  generated by a nonlinear recurrence of the form

$$x_{n+1} x_{n-1} = x_n^2 F(x_n), \quad (1.5)$$

with  $F(x) \in \mathbb{Z}_{\geq 0}[x]$  and  $F(0) = 1$ ; so (1.2) corresponds to the particular case  $F(x) = x + 1$ .

All of the recurrences (1.5) exhibit the Laurent phenomenon [4], and starting from  $x_0 = x_1 = 1$  they generate a sequence of positive integers satisfying  $x_n | x_{n+1}$ . The latter fact means that the sum (1.3) is an Engel series (see Theorem 2.3 in Duverney's book [3], for instance).

The purpose of this note is to present a further generalization of the results in [5], by considering a sum

$$S = \frac{p}{q} + \sum_{j=2}^{\infty} \frac{1}{x_j}, \quad (1.6)$$

with the terms  $x_n$  satisfying the recurrence

$$x_{n+1} x_{n-1} = x_n^2 (z_n x_n + 1), \quad (1.7)$$

for  $n \geq 2$ , where  $(z_n)$  is a sequence of positive integers,  $x_1 = q$ , and  $x_2$  is specified suitably. Observe that, in contrast to (1.5), the recurrence (1.7) can be viewed as a

non-autonomous dynamical system for  $x_n$ , because the coefficient  $z_n$  can vary independently (unless it is taken to be  $G(x_n)$ , for some function  $G$ ). The same argument as used in [5], based on Roth’s theorem, shows the transcendence of any number  $S$  defined by a sum of the form (1.6) with such a sequence  $(x_n)$ .

## 2 The main result

We start with a rational number written in lowest terms as  $p/q$ , and suppose that the continued fraction of this number is given as

$$\frac{p}{q} = [a_0; a_1, a_2, a_3, \dots, a_{2k}] \tag{2.1}$$

for some  $k \geq 0$ . Note that, in accordance with a comment on p. 230 of [7], there is no loss of generality in assuming that the index of the final coefficient is even. For the convergents we denote numerators and denominators by  $p_n$  and  $q_n$ , respectively, and use the correspondence between matrix products and continued fractions, which says that

$$\mathbf{M}_n := \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.2}$$

yielding the determinantal identity

$$\det \mathbf{M}_n = p_n q_{n-1} - p_{n-1} q_n = (-1)^{n+1}. \tag{2.3}$$

Now for a given sequence  $(z_n)$  of positive integers, we define a new sequence  $(x_n)$  by

$$x_1 = q, \quad x_{n+1} = x_n y_{n-1} (x_n z_n + 1) \quad \text{for } n \geq 1, \tag{2.4}$$

where

$$y_0 = q_{2k-1} + 1, \quad y_n = \frac{x_{n+1}}{x_n} \quad \text{for } n \geq 1. \tag{2.5}$$

It is clear from (2.4) and (2.5) that  $(x_n)$  is an increasing sequence of positive integers such that  $x_n | x_{n+1}$  for all  $n \geq 1$ ;  $(y_n)$  also consists of positive integers, and is an increasing sequence as well. The recurrence (1.7) for  $n \geq 2$  follows immediately from (2.4) and (2.5).

**Theorem 2.1** *The partial sums of (1.6) are given by*

$$S_n := \frac{p}{q} + \sum_{j=2}^n \frac{1}{x_j} = [a_0; a_1, \dots, a_{2(k+n-1)}]$$

for all  $n \geq 1$ , where the coefficients appearing after  $a_{2k}$  are

$$a_{2k+2j-1} = y_{j-1} z_j, \quad a_{2k+2j} = x_j \quad \text{for } j \geq 1.$$

*Proof* For  $n = 1$ ,  $S_1$  is just (2.1), and we note that  $q_{2k-1} = y_0 - 1$  and  $q_{2k} = q = x_1$ . Proceeding by induction, we suppose that  $q_{2k+2n-3} = y_{n-1} - 1$  and  $q_{2k+2n-2} = x_n$ , and calculate the product

$$\begin{aligned} \mathbf{M}_{2k+2n} &= \mathbf{M}_{2k+2n-2} \begin{pmatrix} a_{2k+2n-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{2k+2n} & 1 \\ 1 & 0 \end{pmatrix} \\ &= \mathbf{M}_{2k+2n-2} \begin{pmatrix} y_{n-1}z_n & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_n & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} p_{2k+2n-2} & p_{2k+2n-3} \\ q_{2k+2n-2} & q_{2k+2n-3} \end{pmatrix} \begin{pmatrix} x_n y_{n-1} z_n & y_{n-1} z_n \\ x_n & 1 \end{pmatrix}. \end{aligned}$$

By making use of (2.4) and (2.5), this gives  $p_{2k+2n} = (x_n y_{n-1} z_n + 1)p_{2k+2n-2} + x_n p_{2k+2n-3}$ ,

$$\begin{aligned} q_{2k+2n-1} &= y_{n-1} z_n q_{2k+2n-2} + q_{2k+2n-3} = x_n y_{n-1} z_n + y_{n-1} - 1 \\ &= \frac{x_{n+1}}{x_n} - 1 = y_n - 1, \end{aligned}$$

and

$$\begin{aligned} q_{2k+2n} &= (x_n y_{n-1} z_n + 1)q_{2k+2n-2} + x_n q_{2k+2n-3} \\ &= (x_n y_{n-1} z_n + 1)x_n + x_n(y_{n-1} - 1) = x_{n+1}, \end{aligned}$$

which are the required denominators for the  $(2k + 2n - 1)$ th and  $(2k + 2n)$ th convergents. Thus we have

$$S_{n+1} = S_n + \frac{1}{x_{n+1}} = \frac{p_{2k+2n-2}}{q_{2k+2n-2}} + \frac{1}{q_{2k+2n}} = \frac{1}{q_{2k+2n}} \left( \frac{x_{n+1}}{x_n} p_{2k+2n-2} + 1 \right).$$

From (2.3) and (2.4), the bracketed expression above can be rewritten as

$$\begin{aligned} &\left( y_{n-1}(x_n z_n + 1) - q_{2n+2k-3} \right) p_{2k+2n-2} + q_{2n+2k-2} p_{2k+2n-3} \\ &= \left( y_{n-1}(x_n z_n + 1) - y_{n-1} + 1 \right) p_{2k+2n-2} + x_n p_{2k+2n-3}, \end{aligned}$$

giving

$$S_{n+1} = \frac{1}{q_{2k+2n}} \left( (x_n y_{n-1} z_n + 1) p_{2k+2n-2} + x_n p_{2k+2n-3} \right) = \frac{p_{2k+2n}}{q_{2k+2n}},$$

which is the required result.  $\square$

Upon taking the limit  $n \rightarrow \infty$  we obtain the infinite continued fraction expansion for the sum  $S$ , which is clearly irrational. To show that  $S$  is transcendental, we need the following growth estimate for  $x_n$ :

**Lemma 2.2** *The terms of a sequence defined by (2.4) satisfy*

$$x_{n+1} > x_n^{5/2}$$

for all  $n \geq 3$ .

*Proof* Since  $(x_n)$  is an increasing sequence, the recurrence relation (1.7) gives

$$x_{n+1} > \frac{x_n^3}{x_{n-1}} > x_n^2$$

for  $n \geq 2$ . Hence  $x_{n-1} < x_n^{1/2}$  for  $n \geq 3$ , and putting this back into the first inequality above yields  $x_{n+1} > x_n^3/x_n^{1/2} = x_n^{5/2}$ , as required.  $\square$

The preceding growth estimate for  $x_n$  means that  $S$  can be well approximated by rational numbers.

**Theorem 2.3** *The sum*

$$S = \frac{p}{q} + \sum_{j=2}^{\infty} \frac{1}{x_j} = [a_0; a_1, \dots, a_{2k}, y_0z_1, x_1, y_1z_2, \dots, y_{j-1}z_j, x_j, \dots]$$

is a transcendental number.

*Proof* This is the same as the proof of Theorem 4 in [5], which we briefly outline here. Let  $P_n = p_{2k+2n-2}$  and  $Q_n = q_{2k+2n-2}$ . Approximating the irrational number  $S$  by the partial sum  $S_n = P_n/Q_n$ , then using Lemma 2.2 and a comparison with a geometric sum, gives the upper bound

$$\left| S - \frac{P_n}{Q_n} \right| = \sum_{j=n+1}^{\infty} \frac{1}{x_j} < \frac{1}{x_n^{5/2-\epsilon}} = \frac{1}{Q_n^{5/2-\epsilon}}$$

for any  $\epsilon > 0$ , whenever  $n$  is sufficiently large. Roth’s theorem [6] (see also chapter VI in [1]) says that, for an arbitrary fixed  $\kappa > 2$ , an irrational algebraic number  $\alpha$  has only finitely many rational approximations  $P/Q$  for which  $\left| \alpha - \frac{P}{Q} \right| < \frac{1}{Q^\kappa}$ ; so  $S$  is transcendental.  $\square$

For other examples of transcendental numbers whose continued fraction expansion is explicitly known, see [2] and references therein.

### 3 Examples

The autonomous recurrences (1.5) considered in [5], where the polynomial  $F$  has positive integer coefficients and  $F(0) = 1$ , give an infinite family of examples. In that case, one has  $p = 1$  and  $x_1 = q = 1$ , so that  $k = 0$ ,  $y_0 = 1$  and  $z_n =$



$(F(x_n) - 1)/x_n$ . More generally, one could take  $z_n = G(x_n)$  for any non-vanishing arithmetical function  $G$ .

In general, it is sufficient to take the initial term in (1.6) lying in the range  $0 < p/q \leq 1$ , since going outside this range only alters the value of  $a_0$ . As a particular example, we take

$$\frac{p}{q} = \frac{2}{7} = [0; 3, 2], \quad z_n = n \quad \text{for } n \geq 1,$$

so that  $k = 1$ , and  $q_1 = 3$  which gives  $y_0 = 2$ . Hence  $x_1 = 7$ ,  $x_2 = 112$ , and the sequence  $(x_n)$  continues with

403200, 1755760043520000, 53695136666462381094317154204367872000000, . . .

The sum  $S$  is the transcendental number

$$\frac{2}{7} + \frac{1}{112} + \frac{1}{403200} + \frac{1}{1755760043520000} + \dots \approx 0.2946453373015879,$$

with continued fraction expansion

$[0; 3, 2, 2, 7, 32, 112, 10800, 403200, 17418254400, 1755760043520000, \dots]$ .

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