Large degree asymptotics of orthogonal polynomials with respect to an oscillatory weight on a bounded interval

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Abstract

In this paper we consider polynomials $p_n^\omega(x)$ orthogonal with respect to the oscillatory weight $w(x) = \mathrm{e}^{\mathrm{i}\omega x}$ on $[-1,1]$, where $\omega > 0$ is a real parameter. A first analysis of $p_n^\omega(x)$ for large values of ω was carried out in [5], in connection with complex Gaussian quadrature rules with uniform good properties in ω. In this contribution we study the existence, asymptotic behavior and asymptotic distribution of the roots of $p_n^\omega(x)$ in the complex plane as $n \to \infty$. The parameter ω can be fixed or grow with n at most linearly. The tools used are logarithmic potential theory and the S-property, together with the Riemann–Hilbert formulation and the Deift–Zhou steepest descent method.

1 Introduction

We are concerned with a family of polynomials orthogonal on the interval $[-1,1]$ with respect to an oscillatory weight function. More precisely, we consider the weight function

$$w(x) = \mathrm{e}^{\mathrm{i}\omega x}, \tag{1.1}$$

on $[-1,1]$, where $\omega > 0$ is a real parameter, and we define formal orthogonal polynomials, depending on two parameters ω and n, in the following sense:

$$\int_{-1}^{1} p_n^\omega(x) x^k w(x) \mathrm{d}x = 0, \qquad k = 0, 1, \ldots, n-1. \tag{1.2}$$

The purpose of this paper is to investigate the large n asymptotic behavior of $p_n^\omega(x)$, covering two different regimes: the case where ω is fixed and the case where $\omega = \omega_n$ is allowed to grow coupled with n and at a linear rate at most.

Polynomials orthogonal with respect to complex weight functions have been considered in the literature in connection with rational approximation of analytic functions, see for example the work of Aptekarev [3]. In that reference, the author considers more general complex weight functions, holomorphic in a neighborhood of the curve where the orthogonality is defined. The weight (1.1) is a particular case of that analysis, although its simplicity allows for more explicit results. We should mention that polynomials with respect to a complex exponential weight have also been considered recently in the work of Suetin [28]. In this reference the author studies (using different techniques) the case $\omega = 1$ in our notation, modified with a Chebyshev factor $(1 - x^2)^{-1/2}$.

The analysis of this family of orthogonal polynomials was motivated in [5] by the problem of constructing complex quadrature rules of Gaussian type for oscillatory integrals of the type

$$I[f] = \int_{-1}^{1} f(x) \mathrm{e}^{\mathrm{i}\omega x} \mathrm{d}x, \tag{1.3}$$

which is a numerical challenge when the frequency ω is large. For this and more general Fourier–type integrals, several possibilities have been contemplated in the literature: one option is an application of a Filon–type rule, as exposed for example in [17], which is based on interpolation of $f(x)$ and its derivatives (or approximations) at the endpoints. Another possibility is the application of the classical method of steepest descent, see for instance [4, 16], which leads to complex quadrature rules which are optimal for large ω, and whose convergence properties can be improved used interpolation at carefully selected points, see [14]. In this case, this approach would lead to deformation of the path of integration into the upper complex plane and application of Gauss–Laguerre quadrature to the resulting contour integrals after a suitable parametrization. Another possibility is the so–called exponentially fitted rules, see [21], which are restricted to be real and not directly connected to orthogonal polynomials.

The main purpose of analyzing the family of orthogonal polynomials $p_n^\omega(x)$ in (1.2) is that a complex quadrature rule based on them would have good asymptotic properties both for large ω, resembling the performance of the steepest descent method in that respect, and also for small ω, meaning that it reduces to the classical Gauss–Legendre rule when $\omega \to 0$.

In [5] the authors consider several properties of $p_n^\omega(x)$, in particular the large ω asymptotic behavior. The fact that the weight function $w(x)$ is not positive poses a problem of existence of $p_n^\omega(x)$ from the outset, since the standard Gram–Schmidt procedure to generate the family of orthogonal

polynomials may fail for some values of n and/or ω. In fact, one of the conclusions in [5] is that given $n \geq 1$, there exists a countable set of values of ω for which the polynomial of odd degree $p_{2n+1}^{\omega}(x)$ is not defined. This corresponds to zeros of the bilinear form

$$(f, g) = \int_{-1}^{1} f(x)g(x)w(x)\mathrm{d}x \tag{1.4}$$

applied to the polynomial $p_{2n}^{\omega}(x)$. For instance, in the case of $p_1^{\omega}(x)$, that sequence is given by $\omega = k\pi$, with $k \in \mathbb{Z}$. In contrast, the polynomials of even degree $p_{2n}^{\omega}(x)$ are conjectured to exist for all n, and under this assumption, their roots tend to the roots of the product of two rotated and scaled Laguerre polynomials as $\omega \to \infty$.

In order to state asymptotic results for large n, we write

$$\lambda_n = \frac{\omega_n}{n}, \tag{1.5}$$

and we rewrite the weight as varying with n:

$$w_n(x) = \mathrm{e}^{-nV_n(x)}, \qquad V_n(x) = -\mathrm{i}\lambda_n x. \tag{1.6}$$

We assume that λ_n has the following asymptotic behavior as $n \to \infty$:

$$\lambda_n = \lambda + \varepsilon_n, \tag{1.7}$$

where $\lambda \geq 0$, and $\varepsilon_n = \mathcal{O}(n^{-\alpha})$, with $\alpha > 0$, as $n \to \infty$. We also define

$$V(x) := \lim_{n\to\infty} V_n(x) = -\mathrm{i}\lambda x. \tag{1.8}$$

We highlight two important situations:

- When ω is fixed, then we have $\lambda = 0$ and $\varepsilon_n = \omega$ is independent of n.
- When $\omega = \lambda n$, with $\lambda > 0$, then $\lambda_n = \lambda$ and $\varepsilon_n \equiv 0$.

The asymptotic behavior of the roots of $p_n^{\omega}(x)$ as $n \to \infty$ will be obtained using logarithmic potential theory and the notion of S-curve in a polynomial external field, that goes back to the works of H. Stahl [26], see also the recent work of Rakhmanov [24]. These ideas have been used recently in connection with non–Hermitian orthogonality in the complex plane, see for instance [19] for a general formulation of the theory with more general exponential weight functions, see [2, 9, 15] for an analysis in the case when $V(x)$ is a cubic polynomial, or the recent contribution [6] in connection with Laguerre polynomials with arbitrary complex parameters.

The results on the large n asymptotic behavior of $p_n^{\omega}(x)$ will be obtained using the Riemann–Hilbert formulation and the nonlinear steepest descent

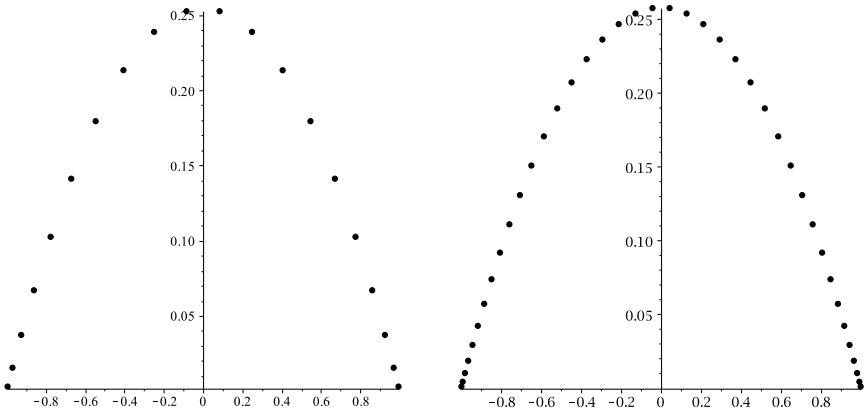

Figure 1: Zeros of $p_{20}^{10}(z)$ (left) and $p_{40}^{20}(z)$ (right). Here $\lambda_n = \lambda = 1/2$.

method. This approach also gives asymptotic information about the coefficients $a_n^2 = a_{n,\omega}^2$ and $b_n = b_{n,\omega}$ of the three term recurrence relation

$$xp_n^\omega(x) = p_{n+1}^\omega(x) + b_n p_n^\omega(x) + a_n^2 p_{n-1}^\omega(x), \tag{1.9}$$

which holds provided that $p_n^\omega(x)$ and $p_{n\pm1}^\omega(x)$ are well defined. Here we have omitted the possible dependence of ω on n, of course in this situation the parameter ω_n would be shifted as well. We also remark that in [5, Theorem 3.3], the deformation equations for these coefficients in terms of ω are given.

Figures 1, 2 and 3 show the zeros of the polynomials $p_n^\omega(z)$ in the complex plane, computed with MAPLE using extended precision of 50 digits, and taking $n = 20$, $n = 40$ and then $\lambda_n = \lambda = 1/2$ in Figure 1, $\lambda_n = \lambda = 1$ in Figure 2 and $\lambda_n = \lambda = 3/2$ in Figure 3. These numerical experiments indicate that one should expect a transition in the behavior of the zeros of $p_n^\omega(x)$, from being supported on a single curve joining $z = -1$ and $z = 1$ to distributing along two disjoint arcs in the complex plane, as the coupling parameter λ goes through a critical value, say λ_0, that seems to be located between 1 and 3/2. In the next section we make this statement precise.

2 Statement of main results

Consider λ as defined in (1.7), and the function

$$h(\lambda) = 2\log\left(\frac{2+\sqrt{\lambda^2+4}}{\lambda}\right) - \sqrt{\lambda^2+4}, \tag{2.1}$$

and let λ_0 be the (unique) positive solution to the equation $h(\lambda) = 0$, i.e.

$$2\log\left(\frac{2+\sqrt{\lambda_0^2+4}}{\lambda_0}\right) - \sqrt{\lambda_0^2+4} = 0. \tag{2.2}$$

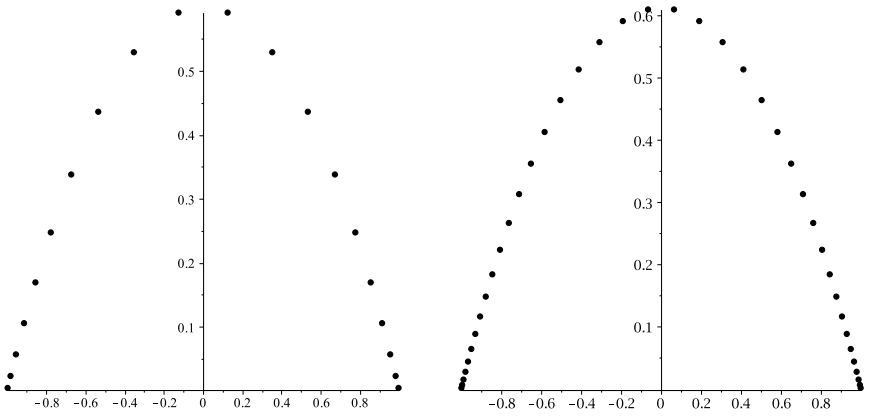

Figure 2: Zeros of $p_{20}^{20}(z)$ (left) and $p_{40}^{40}(z)$ (right). Here $\lambda_n = \lambda = 1$.

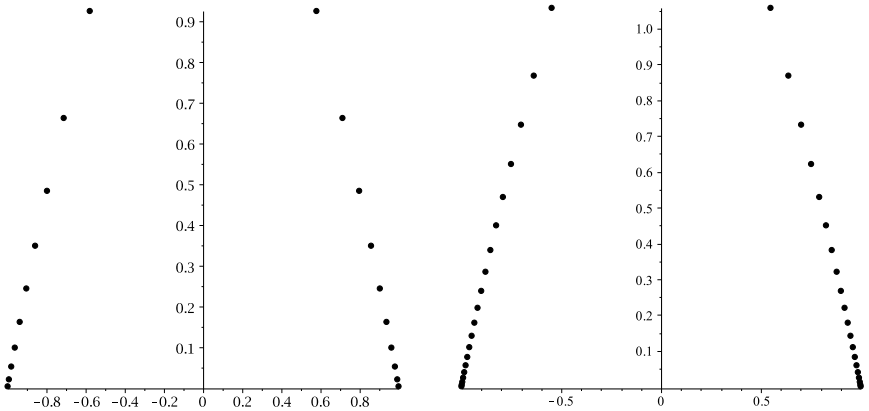

Figure 3: Zeros of $p_{20}^{30}(z)$ (left) and $p_{40}^{60}(z)$ (right). Here $\lambda_n = \lambda = 3/2$.

The fact that λ_0 is unique is a consequence of $h(\lambda)$ being a map from $[0,\infty)$ onto $\mathbb{R}$ and a decreasing function of λ. The value of λ_0 can be computed numerically, and we have $\lambda_0 = 1.325486839\ldots$

In our results, we will work with the following function:

$$\varphi(z) = z + (z^2-1)^2, \tag{2.3}$$

which is analytic in $\mathbb{C}\setminus[-1,1]$ and maps $\mathbb{C}\setminus[-1,1]$ onto the exterior of the unit circle.

We can prove the following result:

Theorem 2.1. *Let $V(z) = -i\lambda z$ with $0 \le \lambda < \lambda_0$, then*

1. *there exists a smooth curve γ_λ joining $z=1$ and $z=-1$ that is a part of the level set given by*
$$\operatorname{Re}\phi(z) = 0, \tag{2.4}$$
where
$$\phi(z) = 2\log\varphi(z) + i\lambda(z^2-1)^{1/2} \tag{2.5}$$
and the cut of the square root is taken on γ_λ,

2. *the measure*
$$d\mu_\lambda(z) = \psi_\lambda(z)\,dz = -\frac{1}{2\pi i}\frac{2+i\lambda z}{(z^2-1)^{1/2}}\,dz, \tag{2.6}$$
with a branch cut taken on γ_λ, is the equilibrium measure on γ_λ in the external field Re $V(z)$,

3. *the curve γ_λ joining $z=-1$ and $z=1$ has the S-property in the external field* Re $V(z)$,

4. *if we consider the normalized zero counting measure of $p_n^\omega(z)$, then*
$$\mu_n = \frac{1}{n}\sum_{p_n^\omega(\zeta)=0}\delta(\zeta) \xrightarrow{*} \mu_\lambda, \tag{2.7}$$
as $n\to\infty$, in the sense of weak convergence of measures.

In particular, if $\lambda = 0$, we get

$$\operatorname{Re}\left[\log(z+(z^2-1)^{1/2})\right] = 0, \tag{2.8}$$

and the curve γ_0 coincides with the interval $[-1,1]$. Note that the theorem is consistent with what can be observed in the numerical experiments before.

Regarding the asymptotic behavior of the orthogonal polynomials $p_n^\omega(z)$, we have the following result:

Theorem 2.2. *Let $0 \leq \lambda < \lambda_0$ with $0 \leq \lambda < \lambda_0$, then the following holds true:*

1. *For large enough n the orthogonal polynomial $p_n^\omega(z)$ defined by (1.2) exists uniquely, and its zeros accumulate on γ_λ, as $n \to \infty$.*

2. *For $z \in \mathbb{C}\backslash\gamma_\lambda$, the monic orthogonal polynomial $p_n^\omega(z)$ has the following asymptotic behavior:*

$$p_n^\omega(z) = \frac{\varphi(z)^{n+1/2}}{2^{n+1/2}(z^2-1)^{1/4}} \exp\left(-\frac{in\lambda_n}{2\varphi(z)}\right)\left(1+\mathcal{O}\left(\frac{1}{n}\right)\right), \quad n \to \infty. \tag{2.9}$$

3. *Fix a neighborhood U of γ_λ in the complex plane, and two discs*

$$D(\pm 1, \delta) = \{z \in \mathbb{C} : |z \mp 1| < \delta\}, \tag{2.10}$$

with $\delta > 0$. For $z \in U \setminus (D(1,\delta) \cup D(-1,\delta))$, we have as $n \to \infty$,

$$p_n^\omega(z) = \frac{2^{1/2-n} e^{\frac{n}{2}V_n(z)}}{(1-z^2)^{1/4}} \times \left[\cos\left(\left(n+\frac{1}{2}\right)\arccos z + \frac{n\lambda_n}{2}(z^2-1)^{1/2} - \frac{\pi}{4}\right) + \mathcal{O}(1/n)\right]. \tag{2.11}$$

4. *For $z \in \gamma_\lambda \cap D(1,\delta)$ we have*

$$p_n^\omega(z) = 2^{-n}(2n\pi)^{1/2} f(z)^{1/4}\, \mathrm{e}^{\frac{nV_n(z)}{2}} \times \left[\cos\xi(z) J_0\left(-\frac{in\phi(z)}{2}\right) + \sin\xi(z) J_0'\left(-\frac{in\phi(z)}{2}\right) + \mathcal{O}(1/n)\right], \tag{2.12}$$

as $n \to \infty$, in terms of standard Bessel functions, with $f(z) = \phi(z)^2/16$ and $\phi(z)$ given by (2.5). Here

$$\xi(z) = -\frac{n}{2}(\lambda - \lambda_n)(z^2-1)^{1/2} + \frac{1}{2}\arccos z.$$

Remark 2.3. *The asymptotic results are consistent with Theorem 1.4 in [18], taking $\alpha = \beta = 0$ and $h(z) = w_n(z)$, and can be seen as a complex generalization of the Jacobi–type weight function considered in that reference. Note that the definition of the phase function $\psi(z)$ in [18, formula (1.34)], see also [20, formula (3.9)] can be adapted to this case. Namely, we have*

$$\psi(z) = \frac{(1-z^2)^{1/2}}{4\pi i} \oint_\gamma \frac{\log w_n(t)}{(t^2-1)^{1/2}} \frac{dt}{t-z} = \frac{n\lambda_n}{2}(z^2-1)^{1/2}, \tag{2.13}$$

using residue calculation, which fits well with the estimates in Section 3 of the previous theorem. Here γ is a smooth curve that encircles $[-1,1]$ once in the positive direction.

Remark 2.4. *The previous result is also consistent with the strong asymptotics given in Theorem 2 in [3], with notation* $Q(z) = -\frac{i\lambda_n z}{2}$ *and* $\tilde{h}_n(z) \equiv 1$.

Regarding the recurrence coefficients in (1.9) and the norm of the orthogonal polynomials, we can deduce the following result from the Riemann–Hilbert formulation and the steepest descent analysis:

Theorem 2.5. *Let* $0 \leq \lambda < \lambda_0$ *with* $0 \leq \lambda < \lambda_0$, *then the coefficients* a_n^2 *and* b_n *in the three term recurrence relation*

$$xp_n^\omega(x) = p_{n+1}^\omega(x) + b_n p_n^\omega(x) + a_n^2 p_{n-1}^\omega(x), \tag{2.14}$$

exist for large enough n, *and they satisfy*

$$\begin{aligned} a_n^2 &= \frac{1}{4} + \frac{4-\lambda^2}{4(4+\lambda^2)^2}\frac{1}{n^2} + \mathcal{O}\left(\frac{1}{n^3}\right), \\ b_n &= -\frac{2i\lambda}{(4+\lambda^2)^2}\frac{1}{n^2} + \mathcal{O}\left(\frac{1}{n^3}\right). \end{aligned} \tag{2.15}$$

Remark 2.6. *Higher order coefficients in the previous asymptotic expansion can in principle be computed by iterating the last step in the steepest descent analysis, see [18, Section 8] and also Section 5 below.*

Following this approach, it is also possible to obtain asymptotic estimates for related quantities, such as leading coefficients of the orthogonal polynomials or Hankel determinants, but we omit these results for brevity.

3 Proof of Theorem 2.1

The proof of Theorem 2.1 will make use of the following tools: first we recall some standard ideas from logarithmic potential theory in the complex plane and the S-property, which is a key tool in the analysis of non–Hermitian orthogonality in the complex plane. In order to construct such a curve with the S-property, which will attract the zeros of $p_n^\omega(z)$ as $n \to \infty$, we need to study the local and global trajectories of a certain quadratic differential $-Q_\lambda(z)\mathrm{d}z^2$, that turns out to be explicit in this case.

3.1 Potential theory and the S-property

From the work of Gonchar and Rakhmanov, [13, 24], it is known that a key element in the analysis of the large n behavior of zeros of orthogonal polynomials is the equilibrium measure in an external field, which in this case is given by $\operatorname{Re} V(z)$, with $V(z) = -\mathrm{i}\lambda z$.

We take the set $\mathcal{T}$ of smooth curves joining $z = -1$ with $z = 1$, with orientation from -1 to 1. For any $\gamma \in \mathcal{T}$ we consider the space of probability

measures on γ, denoted here $\mathcal{M}_1(\gamma)$, and we pose the following equilibrium problem:

$$\inf\{I(\mu) : \mu \in \mathcal{M}_1(\gamma)\}, \tag{3.1}$$

where

$$I(\mu) = \int_\gamma U^\mu(z)\mathrm{d}\mu(z) + \mathrm{Re}\int_\gamma V(z)\mathrm{d}z \tag{3.2}$$

is the energy functional and

$$U^\mu(z) = \int_\gamma \log\frac{1}{|z-y|}\mathrm{d}\mu(y) \tag{3.3}$$

is the logarithmic potential of the measure μ. This problem has a unique solution for each $\gamma \in \mathcal{T}$, following the standard theory, see for instance [25]. Furthermore, the equilibrium measure μ is characterized by the so–called variational conditions. Define

$$g(z) = \int_\gamma \log(z-s)\mathrm{d}\mu(s), \tag{3.4}$$

which is analytic in $\mathbb{C}\setminus\gamma$. Note that $\mathrm{Re}\, g(z) = -U^\mu(z)$. Then there exists a constant ℓ such that

$$\begin{aligned} &\mathrm{Re}\,(-g_+(z) - g_-(z) + V(z)) = \ell, && z \in \mathrm{supp}\,\mu, \\ &\mathrm{Re}\,(-g_+(z) - g_-(z) + V(z)) \geq \ell, && z \in \gamma. \end{aligned} \tag{3.5}$$

When working with non–Hermitian orthogonality in the complex plane, we have an additional freedom to choose γ without changing the orthogonality condition (1.2). In order to find the precise curve that describes the asymptotic behavior of the zeros of $p_n^\omega(x)$ as $n \to \infty$, we need an extra condition, which is called the S-property in the literature. We refer the reader to [26, 24, 19] for more details.

Definition 3.1. *We say that a contour γ has the S-property in the external field* $\mathrm{Re}\, V(x)$ *if the following two conditions are satisfied:*

1. *there exists a set E of zero capacity such that for any $\zeta \in \gamma\setminus E$, there exists a neighborhood $D = D(\zeta)$ such that the set $D \cap \mathrm{supp}\,\mu$ is an analytic arc,*

2. *for any point on this analytic arc, the logarithmic potential of μ satisfies*

$$\frac{\partial}{\partial n_+}\left(U^\mu + \frac{1}{2}\,\mathrm{Re}\; V\right) = \frac{\partial}{\partial n_-}\left(U^\mu + \frac{1}{2}\,\mathrm{Re}\; V\right), \tag{3.6}$$

where $\frac{\partial}{\partial n_\pm}$ indicates normal derivatives in opposite directions, and μ is the equilibrium measure for γ in the external field $\mathrm{Re}\, V$.

If one is able to find such a contour with the S-property in the external field $\operatorname{Re} V(z)$, then it follows from the work of Gonchar and Rakhmanov, cf. [24, Theorem 1.3], that the zero counting measure on the zeros of $p_n^\omega(x)$ satisfies

$$\mu_n = \frac{1}{n} \sum_{p_n^\omega(\zeta)=0} \delta(\zeta) \xrightarrow{*} \mu, \tag{3.7}$$

where μ is the equilibrium measure of γ in the external field $\operatorname{Re} V$. Here $\xrightarrow{*}$ indicates convergence in the weak topology, in the sense that

$$\int_\gamma f(x)\mathrm{d}\mu_n(x) \to \int_\gamma f(x)\mathrm{d}\mu(x), \qquad n \to \infty, \tag{3.8}$$

for any continuous function f.

In the sequel we will denote the curve with the S-property as γ_λ, depending on the parameter λ, and μ_λ will be the equilibrium measure on γ_λ.

An alternative characterization of the curve γ_λ with the S-property is a consequence of the Cauchy–Riemann equations, and has a form similar to the variational equations (3.5), but involving the imaginary part of the external field:

$$\operatorname{Im}\left(-g_+(z) - g_-(z) + \frac{1}{2}V(z)\right) = \tilde{\ell}, \qquad z \in \operatorname{supp}\mu, \tag{3.9}$$

where the constant $\tilde{\ell}$ might be different in each component of $\operatorname{supp}\mu$, if this consists of several pieces. Hence, combining (3.9) and (3.5), if γ_λ has the S-property in the external field $\operatorname{Re} V(z)$, we have

$$-g_+(z) - g_-(z) + V(z) = \ell + \mathrm{i}\tilde{\ell}, \qquad z \in \operatorname{supp}\mu_\lambda. \tag{3.10}$$

Furthermore, it is known, see for example [24], that the support of the equilibrium measure μ_λ is a union of analytic arcs that are trajectories of the quadratic differential $-Q_\lambda(z)\mathrm{d}z^2$, which are given by the condition

$$-Q_\lambda(z)\mathrm{d}z^2 > 0. \tag{3.11}$$

This function $Q_\lambda(z)$ is defined as

$$Q_\lambda(z) = \left(\int \frac{\mathrm{d}\mu_\lambda(x)}{x-z} + \frac{V'(z)}{2}\right)^2. \tag{3.12}$$

This equation is a key element in order to obtain properties of the equilibrium measure and its Cauchy transform, see for example [22, 19], but in order to obtain some explicit formula for the quadratic differential, one normally has to make some assumptions on μ_λ, to be proved later. Assume for

the moment that μ_λ is supported on a single arc γ_λ that connects $z=-1$ and $z=1$ in the complex plane, then the function

$$w_\lambda(z) = \int_{\gamma_\lambda} \frac{\mathrm{d}\mu_\lambda(x)}{z-x} \tag{3.13}$$

is analytic in $\mathbb{C} \setminus \gamma_\lambda$, and it satisfies

$$\begin{aligned} w_\lambda(z) &= \frac{1}{z} + \mathcal{O}\left(\frac{1}{z^2}\right), \qquad z \to \infty, \\ w_{\lambda+}(z) + w_{\lambda-}(z) &= V'(z) = -\mathrm{i}\lambda, \qquad z \in \gamma_\lambda, \end{aligned} \tag{3.14}$$

where $w_\pm(z)$ denote the boundary values from the left (right) of the curve γ_λ. The second equation is a direct consequence of the formula (3.10), since $w_\lambda(z) = g'(z)$.

Consequently, we look for $w_\lambda(z)$ in the form

$$w_\lambda(z) = -\frac{\mathrm{i}\lambda}{2} + \frac{p_1(z)}{(z^2-1)^{1/2}}, \tag{3.15}$$

where $p_1(z) = a_0 + a_1 z$ is a polynomial of degree 1, because of the first condition in (3.14), and the square root is taken with a cut on γ_λ. From the first equation in (3.14), we obtain $a_0 = 1$ and $a_1 = \mathrm{i}\lambda/2$, so

$$w_\lambda(z) = -\frac{\mathrm{i}\lambda}{2} + \frac{2+\mathrm{i}\lambda z}{2(z^2-1)^{1/2}}. \tag{3.16}$$

Then, for $z \in \gamma_\lambda$, we have

$$w_{\lambda\pm}(z) = -\frac{\mathrm{i}\lambda}{2} \mp \mathrm{i}\,\frac{2+\mathrm{i}\lambda z}{2\sqrt{1-z^2}}, \tag{3.17}$$

where the branch of the square root that is real and positive on γ_λ. The density of the equilibrium measure can be recovered as

$$\mathrm{d}\mu_\lambda(z) = \psi_\lambda(z)\mathrm{d}z = -\frac{1}{2\pi\mathrm{i}}\left(w_{\lambda+}(z) - w_{\lambda-}(z)\right)\mathrm{d}z = \frac{1}{2\pi}\frac{2+\mathrm{i}\lambda z}{\sqrt{1-z^2}}\mathrm{d}z, \tag{3.18}$$

using the Sokhotski–Plemelj formula, see for instance [12, §1.4.2]. The density can be extended to the complex plane with a cut on γ_λ:

$$\mathrm{d}\mu_\lambda(z) = \psi_\lambda(z)\mathrm{d}z = -\frac{1}{2\pi\mathrm{i}}\frac{2+\mathrm{i}\lambda z}{(z^2-1)^{1/2}}\mathrm{d}z. \tag{3.19}$$

Finally, the quadratic differential $Q_\lambda(z)\mathrm{d}z^2$ has the form

$$Q_\lambda(z)\mathrm{d}z^2 = \frac{(2+\mathrm{i}\lambda z)^2}{4(z^2-1)}\mathrm{d}z^2. \tag{3.20}$$

In general, this argument will break down if the support of μ_λ has several components, in particular (3.15) will cease to be valid. For this reason, in the next section we will analyze this quadratic differential in more detail, with the goal of showing that there is indeed one trajectory of $-Q_\lambda(z)\mathrm{d}z^2$ joining $z=-1$ and $z=1$, for small values of λ. This will prove the previous heuristic argument.

3.2 Trajectories of the quadratic differential $-Q_\lambda(z)\mathrm{d}z^2$

3.2.1 Local trajectories near critical points

From (3.20), it is clear that the finite critical points of the quadratic differential $-Q_\lambda(z)\mathrm{d}z^2$ are two poles located at $z = \pm 1$, a double zero at $z = z^* = 2\mathrm{i}/\lambda$. At infinity, the quadratic differential has a pole of order 4, since using the local parameter $\xi = 1/w$, we get

$$-\frac{1}{w^4}Q_\lambda\left(\frac{1}{w}\right)\mathrm{d}w^2 = \left(\frac{\lambda^2}{4w^4} - \frac{\mathrm{i}\lambda}{w^3} + \frac{\lambda^2-4}{4w^2} - \frac{\mathrm{i}\lambda}{w} + \mathcal{O}(1)\right)\mathrm{d}w^2, \qquad w \to 0. \tag{3.21}$$

Naturally, if $\lambda = 0$, the double zero disappears and the pole at infinity becomes of order 2, with negative residual.

In a neighborhood of a critical point the condition $-Q_\lambda(z)\mathrm{d}z^2 > 0$ is known to be equivalent, see [22, Section 4], to $\operatorname{Im}\xi_\lambda(z) = \text{const}$, in terms of the local parameter ξ_λ, given by

$$\xi_\lambda = \xi_\lambda(z) = \int^z \sqrt{-Q_\lambda(s)}\mathrm{d}s. \tag{3.22}$$

In our case, this local parameter can be computed explicitly:

$$\xi_\lambda(z) = \mathrm{i}\log(z + (z^2-1)^{1/2}) - \frac{\lambda(z^2-1)^{1/2}}{2}, \tag{3.23}$$

with a branch cut on $(-\infty, -1] \cup \gamma_\lambda$ and the principal value of the root and the logarithm.

From (3.23), we have the following local behavior near $z = 1$:

$$\xi_\lambda(z) - \xi_\lambda(1) = \frac{\sqrt{2}}{2\pi}(-\lambda + 2\mathrm{i})(z-1)^{1/2} + \mathcal{O}((z-1)^{3/2}). \tag{3.24}$$

If we write $z = 1 + r\mathrm{e}^{\mathrm{i}\theta}$, the argument of the leading term is

$$\arg(-\lambda + 2\mathrm{i})(z-1)^{1/2} = \frac{\theta}{2} - \arctan\frac{2}{\lambda}. \tag{3.25}$$

If this is to be real and positive, we get

$$\theta \approx 2\arctan\frac{2}{\lambda}, \tag{3.26}$$

taking principal values of the argument. If $\lambda \to 0^+$ we get $\theta \to \pi^-$, so the trajectory follows the real axis from $z = 1$. The angle decreases as we increase the value of λ, so the trajectory enters the upper half plane. By symmetry, we have a similar behavior near $z = -1$.

If $\lambda > 0$, the quadratic differential has a double zero, located at

$$z^* = \frac{2\mathrm{i}}{\lambda}. \tag{3.27}$$

Locally, four trajectories emanate from $z = z^*$. Note that

$$\xi_\lambda(z) - \xi_\lambda(z^*) = \frac{\mathrm{i}\lambda^2}{4\sqrt{\lambda^2+4}}(z-z^*)^2 + \mathcal{O}((z-z^*)^3). \tag{3.28}$$

So, if we write $z = z^* + r\mathrm{e}^{\mathrm{i}\theta}$ and we take arguments in the previous equality, we get

$$\frac{\pi}{2} + 2\theta \equiv 0 \pmod{\pi}, \tag{3.29}$$

which gives the possible angles

$$\theta = \pm\frac{\pi}{4}, \qquad \theta = \pm\frac{3\pi}{4}. \tag{3.30}$$

Regarding infinity, the pole of order 4 attracts the trajectories in two directions with angle π.

3.2.2 Global trajectories

The main tool that we will use to analyze the global behavior of trajectories of $-Q_\lambda(z)\mathrm{d}z^2$ is the Teichmüller lemma, [27, Theorem 14.1]. First we define the following:

Definition 3.2. *A Q_λ-polygon is a curve Γ composed of open straight arcs, along which $\arg Q_\lambda(z)dz^2 = \theta = const$, with $0 \le \theta \le 2\pi$ and their endpoints, which can be critical points.*

In particular, we can use horizontal ($\theta = 0$) and vertical ($\theta = \pi$) trajectories of the quadratic differential as boundaries of a Q_λ-polygon.

Let us denote the zeros and poles of $Q_\lambda(z)$ as z_i, and assign a number n_i to each of them, so that n_i is positive and equal to the order of the zero if z_i is a zero, and n_i is negative and equal to the order of the pole if z_i is a pole. Then, we have the following result, see [27, Theorem 14.1] for the proof:

Theorem 3.3. *Let D be the interior of a simple closed Q_λ-polygon Γ, with sides Γ_j and interior angles θ_j at its vertices, $0 \le \theta \le 2\pi$, and suppose that $Q_\lambda(z)$ is meromorphic in $\overline{D}$ (the only points which can be critical on the boundary of the polygon are the vertices). Then*

$$\sum_j \left(1 - \theta_j \frac{n_j+2}{2\pi}\right) = 2 + \sum_i n_i, \tag{3.31}$$

where the index i runs over all critical points inside the sector.

In the present situation, consider first the four trajectories emanating from the double zero $z^* = 2\mathrm{i}/\lambda$, with starting angles equal to $\pi/2$, see Figure 4.

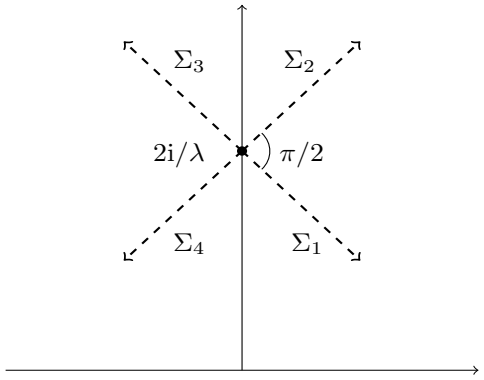

Figure 4: Local trajectories emanating from $z = 2\mathrm{i}/\lambda$.

Assume first that the four trajectories from $z = z^*$ diverge to infinity, and consider one of the sectors delimited by two adjacent trajectories, say Ω. The internal angle at the double zero $z = z^*$ is $\pi/2$, so we have

$$1 + \frac{\theta}{\pi} = 2 + \sum_i n_i, \tag{3.32}$$

where θ is the angle at infinity. If both poles $z = \pm 1$ are in Ω, then the right hand side of the equation is 0, which does not give a valid value for θ. If there are no poles in Ω then the right hand side is 2, which is possible, but clearly cannot happen in the four sectors, since the poles must be in one of them.

Therefore, we can have two situations:

1. The trajectories define four infinite sectors, two of which are free of poles, one contains $z = -1$ and another one contains $z = 1$, symmetric with respect to the imaginary axis.

2. The trajectories define two infinite sectors and a finite one. If we have a closed trajectory from $z = z^*$, then we get

$$0 = 2 + \sum_i n_i, \tag{3.33}$$

so both poles $z = \pm 1$ must be inside the loop. So in this case, the finite trajectory encircles the two poles and the other two regions are infinite.

An intermediate situation can take place: two trajectories go to infinity and the other two connect $z = z^*$ with $z = 1$ and with $z = -1$. This should

take place when $\operatorname{Im}\xi(1) = \operatorname{Im}\xi(z^*)$, and since $\operatorname{Im}\xi(1) = 0$, we get a value of λ, say λ_0, that must satisfy the equation

$$2\log\left(\frac{2+\sqrt{\lambda_0^2+4}}{\lambda_0}\right) - \sqrt{\lambda_0^2+4} = 0. \tag{3.34}$$

Thus, this formula defines the value λ_0 that separates the two different cases in the behavior of the zeros of $p_n^\omega(x)$. Namely, as a consequence of the previous analysis, we have the following result:

Lemma 3.4. *Let λ_0 be defined as the unique positive solution of* (3.34)*. Then*

- *If $0 \le \lambda < \lambda_0$, then there exists a trajectory of the quadratic differential $-Q_\lambda(z)dz^2$ that joins the points $z = 1$ and $z = -1$.*
- *If $\lambda > \lambda_0$, then the trajectories of the quadratic differential $-Q_\lambda(z)dz^2$ emanating from $z = 1$ and $z = -1$ diverge to infinity in opposite directions.*

Proof. In the case $\lambda = 0$ it is clear that $\operatorname{Im}\xi_0(z) = \operatorname{Im}\xi_0(1) = 0$ is equivalent to $\operatorname{Re}(\log(z+(z^2-1)^{1/2})) = 0$. Since the function $\varphi(z) = z+(z^2-1)^{1/2}$ maps the interval $[-1,1]$ onto the unit circle, so $\log\varphi(z)$ is purely imaginary for $z \in [-1,1]$. This comes as no surprise, because the case $\lambda = 0$ corresponds to the classical Legendre polynomials.

If $\lambda > 0$, let us consider the function $\operatorname{Im}(\xi(z))$ when $z = x \ge 0$ is on the positive real axis. From (3.23), we get that if $x \ge 1$, then

$$\operatorname{Im}(\xi(x)) = \log(x + \sqrt{x^2-1}), \tag{3.35}$$

which takes the value 0 at $x = 1$ and is increasing with x. If $0 \le x \le 1$, then $\varphi(x) = x + \sqrt{x^2-1}$ maps the interval $[0,1]$ onto the arc of the unit circle $\mathrm{e}^{\mathrm{i}\,\theta}$, with $0 \le \theta \le \pi/2$, so $\mathrm{i}\log(x+\sqrt{x^2-1}) \in \mathbb{R}$, and therefore

$$\operatorname{Im}(\xi(x)) = -\frac{\lambda\sqrt{1-x^2}}{2}, \qquad 0 \le x \le 1, \tag{3.36}$$

which takes the value $-\lambda/2 < 0$ at $x = 0$ and increases to 0 when $x = 1$. As a consequence, this imaginary part is a continuous and increasing function of x on the positive real axis.

Now consider $\operatorname{Im}(\xi(z^*))$, which is equal to

$$\operatorname{Im}\,\xi\left(\frac{2\,\mathrm{i}}{\lambda}\right) = \log\left(\frac{\sqrt{\lambda^2+4}+2}{\lambda}\right) - \frac{\sqrt{\lambda^2+4}}{2}. \tag{3.37}$$

This function maps $[0,\infty)$ onto $\mathbb{R}$, with value 0 at λ_0, and it is a decreasing function of λ.

Therefore, because of continuity and monotonicity, if we fix a value of λ, there exists a (unique) real and positive value x such that $\mathrm{Im}(\xi(x)) = \mathrm{Im}(\xi(z^*))$. If $\lambda > \lambda_0$, then $0 < x < 1$, and if $0 < \lambda < \lambda_0$, then $x > 1$.

If $0 < \lambda < \lambda_0$ then the trajectory Σ_1 crosses the real axis at a point $x > 1$, and we have a closed trajectory starting and ending at z^* and encircling the two poles $z = \pm 1$. In this case, there is a trajectory emanating from $z = 1$, and since trajectories cannot intersect, it must end at the other pole $z = -1$. This proves that if $0 < \lambda < \lambda_0$, there is a curve joining the two critical points $z = \pm 1$.

If $\lambda > \lambda_0$, the trajectory γ_1 from z^* crosses the real axis at a point $x < 1$, and we must be in the first case before. We apply the Teichmüller lemma to the sectors bounded by Σ_2 and Σ_3 and by Σ_4 and Σ_1. In both cases, we get that the angle at infinity is equal to π, so the two trajectories diverge to infinity in opposite directions. In the two remaining sectors, the angle at infinity is 0, so the trajectory emanating from $z = \pm 1$ must diverge to infinity within these sectors.

□

This proves the first part of Theorem 2.1, since a trajectory of the quadratic differential satisfies $-Q_\lambda(z)\mathrm{d}z^2 > 0$, which is equivalent to $\mathrm{Im}\,\xi(z) = \text{const}$, the constant being equal to $\mathrm{Im}\,\xi(1) = 0$ in this case. Bearing in mind (3.23), we get the desired result.

Next, because of the explicit construction of $\mathrm{d}\mu_\lambda$ before, see (3.19) and the discussion leading to this formula, it follows that (3.10) are satisfied, so γ_λ has the S-property in the external field $\mathrm{Re}\,V(z)$. Finally, the weak convergence of the zero counting measure for $p_n^\omega(z)$ is a direct consequence of the work of Gonchar and Rakhmanov, and this completes the proof of Theorem 2.1.

Figure 5 illustrates the trajectories of the quadratic differential $-Q_\lambda(z)\mathrm{d}z^2$ for different values of $\lambda < \lambda_0$.

4 Proof of Theorem 2.2

In order to give the asymptotic behavior of the orthogonal polynomials $p_n^\omega(z)$, we will use the Riemann–Hilbert approach, together with the Deift–Zhou nonlinear steepest descent method. We take the weight function $w_n(x) = \mathrm{e}^{-nV_n(x)}$, with $V_n(x) = -\mathrm{i}\lambda_n x$ and $0 \le \lambda < \lambda_0$, where λ is defined in (1.7) and λ_0 in (2.2).

4.1 Riemann–Hilbert (RH) problem

We seek $Y(z) = Y_{n,\omega}(z) \in \mathbb{C}^{2\times 2}$ such that

1. $Y(z)$ is analytic (entrywise) in $\mathbb{C} \setminus \gamma_\lambda$.

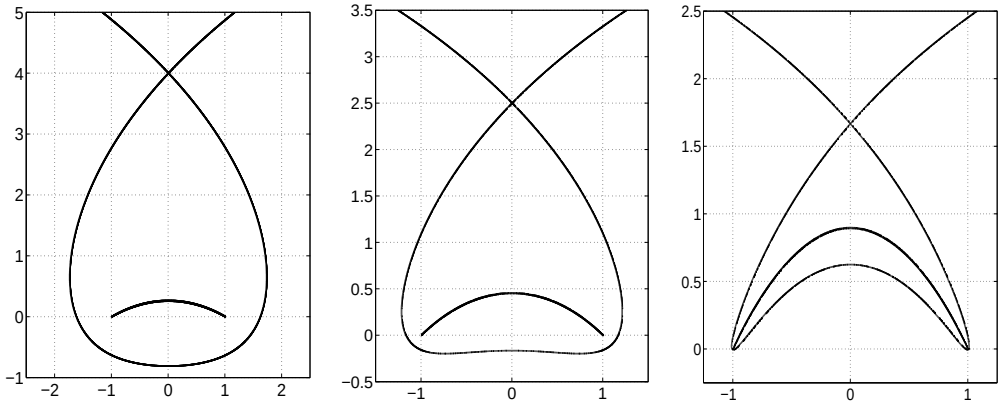

Figure 5: Trajectories corresponding to the critical points $z = \pm 1$ (in black) and to the point $z = z^*$ (in red) for $\lambda = 0.5$ (left), $\lambda = 0.8$ (center) and $\lambda = 1.2$ (right). The black trajectory on the top of the figure (right) is omitted in the other two plots for clarity, as it does not play any significant role in the analysis.

2. On γ_λ, we have the jump

$$Y_+(x) = Y_-(x) \begin{pmatrix} 1 & w_n(x) \\ 0 & 1 \end{pmatrix}. \tag{4.1}$$

3. As $z \to \infty$:

$$Y(z) = \left(I + \mathcal{O}\left(\frac{1}{z}\right) \right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}. \tag{4.2}$$

4. As $z \to \pm 1$ we have

$$Y(z) = \mathcal{O} \begin{pmatrix} 1 & \log|z \mp 1| \\ 1 & \log|z \mp 1| \end{pmatrix}. \tag{4.3}$$

It is known that if this RH problem has a solution, it is unique, see for instance [10, §3.2 and §7.1]. Moreover, in this case the solution is given by

$$Y(z) = \begin{pmatrix} p_n^\omega(z) & (\mathcal{C}p_n^\omega w_n)(z) \\ -2\pi \mathrm{i} \kappa_{n-1}^2 p_{n-1}^\omega(z) & -2\pi \mathrm{i} \kappa_{n-1}^2 (\mathcal{C}p_{n-1}^\omega w_n)(z) \end{pmatrix} \tag{4.4}$$

where

$$(\mathcal{C}f)(z) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma_\lambda} \frac{f(s)}{s - z} \mathrm{d}s \tag{4.5}$$

is the Cauchy transform of the function $f(z)$, and κ_{n-1} is the leading coefficient of the orthonormal polynomial, i.e. $\pi_{n-1}^\omega(x) = \kappa_{n-1} p_{n-1}^\omega(x)$.

The RH formulation of the orthogonal polynomials is due to Fokas, Its and Kitaev, [11], see also the monograph of Deift, [10, §3.2]. The conditions at $z = \pm 1$ follow from a general result on the logarithmic behavior of the Cauchy transform, see [12, §1.8.1].

The existence of $p_n^\omega(z)$ and $p_{n-1}^\omega(z)$ would be guaranteed for all n if the weight function was positive, via the standard Gram–Schmidt orthogonalization procedure applied to the basis of monomials. In this case the weight function is not positive, so the existence of $p_n^\omega(z)$ is not clear a priori, however the Deift–Zhou steepest descent analysis will provide a proof of existence for large enough n. This analysis consists of the following sequence of (explicit and invertible) transformations:

$$Y(z) \mapsto T(z) \mapsto S(z) \mapsto R(z). \tag{4.6}$$

A global estimate of $R(z)$ for large n will give the asymptotic behavior of $Y(z)$, and in particular of $p_n^\omega(z)$ and $p_{n-1}^\omega(z)$, and this will imply the existence of these polynomials, at least for large n.

4.1.1 First transformation

In order to normalize the RH problem at infinity, we need the g-function corresponding to the potential $V(z)$. This function is analytic in $\mathbb{C}\backslash((-\infty, -1]\cup \gamma_\lambda)$, and on this contour it has the following jumps:

$$g_+(x) - g_-(x) = \begin{cases} 2\pi\mathrm{i} \displaystyle\int_x^1 \psi_\lambda(s)\mathrm{d}s, & x \in \gamma_\lambda \\ 2\pi\mathrm{i}, \qquad x \in (-\infty, -1], \end{cases} \tag{4.7}$$

where integration is taken along γ_λ, and $\psi_\lambda(s)$ is given by (3.18). We consider the function

$$\phi(z) = 2g(z) - V(z) - l, \tag{4.8}$$

which is analytic in $\mathbb{C} \setminus ((-\infty, -1] \cup \gamma_\lambda)$ and satisfies

$$\phi_+(x) = -\phi_-(x) = g_+(x) - g_-(x), \qquad x \in (-\infty, -1] \cup \gamma_\lambda. \tag{4.9}$$

Using the analytic extension of $\psi(z)$, it is possible to write

$$\phi(z) = 2\pi\mathrm{i} \int_z^1 \psi(s)\mathrm{d}s, \tag{4.10}$$

for $z \in \mathbb{C} \setminus ((-\infty, -1] \cup \gamma)$. Integration is taken along the a smooth curve joining the points z and 1 in the complex plane, without crossing the cut γ_λ.

Direct evaluation of the integral gives $\phi(z)$ as in (2.5):

$$\phi(z) = 2\log\varphi(z) + \mathrm{i}\lambda(z^2 - 1)^{1/2}, \tag{4.11}$$

again with the cut of the square root taken on γ_λ, and $\varphi(z)$ given by (2.3).

Furthermore, we know from the integral representation that $g(z) = \log z + \dots$ as $z \to \infty$, and

$$g(z) = \frac{1}{2}\left(\phi(z) + V(z) + l\right) = \log 2 + \log z + \frac{l}{2} + \mathcal{O}(1/z), \tag{4.12}$$

using the explicit formulas for $\phi(z)$ and $V(z)$, so $l = -2\log 2$. The function $g(z)$ can be given explicitly as well:

$$g(z) = \log \varphi(z) + \frac{\mathrm{i}\lambda}{2}(z^2-1)^{1/2} - \frac{\mathrm{i}\lambda z}{2} - \log 2, \tag{4.13}$$

and it is analytic in $\mathbb{C} \setminus (-\infty, -1] \cup \gamma_\lambda)$.

The first transformation reads

$$T(z) = \mathrm{e}^{-\frac{nl}{2}\sigma_3} Y(z) \mathrm{e}^{-n\left[g(z)-\frac{l}{2}\right]\sigma_3} = 2^{n\sigma_3} Y(z) \mathrm{e}^{-n\left[g(z)-\frac{l}{2}\right]\sigma_3}, \tag{4.14}$$

where we have use the standard notation for the Pauli matrix $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Note that when $\lambda = 0$, we get

$$g(z) - \frac{l}{2} = g(z) + \log 2 = \log \varphi(z), \tag{4.15}$$

so

$$\mathrm{e}^{-n\left[g(z)-\frac{l}{2}\right]} = \varphi(z)^{-n}, \tag{4.16}$$

which coincides with the definition in [18].

Then the matrix $T(z)$ satisfies the following RH problem:

1. $T(z)$ is analytic (entrywise) in $\mathbb{C} \setminus \gamma_\lambda$
2. On γ_λ, we have the jump

$$T_+(z) = T_-(z) \begin{pmatrix} \mathrm{e}^{-n\phi_+(z)} & W_n(z) \\ 0 & \mathrm{e}^{n\phi_+(z)} \end{pmatrix}, \qquad z \in \gamma_\lambda. \tag{4.17}$$

3. As $z \to \infty$:

$$T(z) = I + \mathcal{O}\left(\frac{1}{z}\right) \tag{4.18}$$

4. As $z \to \pm 1$, $T(z)$ has the same behavior as $Y(z)$.

Here we have used the variational equations (3.10), and we define

$$W_n(z) = \mathrm{e}^{n(V(z)-V_n(z))} = \mathrm{e}^{-n\mathrm{i}z(\lambda-\lambda_n)}. \tag{4.19}$$

4.1.2 Second transformation

We note the following factorization of the jump matrix on γ_λ:

$$\begin{pmatrix} \mathrm{e}^{-n\phi_+(z)} & W_n(z) \\ 0 & \mathrm{e}^{n\phi_+(z)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{\mathrm{e}^{n\phi_+(z)}}{W_n(z)} & 1 \end{pmatrix} \begin{pmatrix} 0 & W_n(z) \\ -\frac{1}{W_n(z)} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{\mathrm{e}^{-n\phi_+(z)}}{W_n(z)} & 1 \end{pmatrix} \tag{4.20}$$

In the next transformation we open a lens around γ_λ, see Figure 6, and we consider the following matrix function:

$$S(z) = \begin{cases} T(z), & z \text{ outside the lens} \\ T(z) \begin{pmatrix} 1 & 0 \\ -\frac{\mathrm{e}^{-n\phi(z)}}{W_n(z)} & 1 \end{pmatrix}, & z \text{ in the upper part of the lens} \\ T(z) \begin{pmatrix} 1 & 0 \\ \frac{\mathrm{e}^{-n\phi(z)}}{W_n(z)} & 1 \end{pmatrix}, & z \text{ in the lower part of the lens.} \end{cases} \tag{4.21}$$

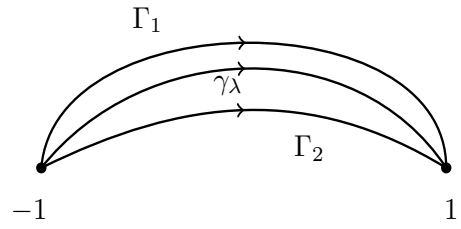

Figure 6: Lens–shaped contour Σ_S around the curve γ_λ.

The matrix $S(z)$ satisfies the following RH problem:

1. $S(z)$ is analytic (entrywise) in $\mathbb{C} \setminus \Sigma_S$, where $\Sigma_S = \Gamma_1 \cup \gamma_\lambda \cup \Gamma_2$, see Figure 6

2. On $\Sigma_S \setminus \{-1, 1\}$ we have the following jumps:

$$S_+(z) = S_-(z) \begin{cases} \begin{pmatrix} 1 & 0 \\ \frac{\mathrm{e}^{-n\phi(z)}}{W_n(z)} & 1 \end{pmatrix}, & z \in \Gamma_1 \cup \Gamma_2, \\ \begin{pmatrix} 0 & \mathcal{W}_n(z) \\ -\frac{1}{W_n(z)} & 0 \end{pmatrix}, & z \in \gamma_\lambda. \end{cases} \tag{4.22}$$

3. As $z \to \infty$:

$$S(z) = I + \mathcal{O}\left(\frac{1}{z}\right). \tag{4.23}$$

4. As $z \to 1$ we have

$$S(z) = \begin{cases} \mathcal{O}\begin{pmatrix} 1 & \log|z-1| \\ 1 & \log|z-1| \end{pmatrix}, & z \text{ outside the lens} \\ \mathcal{O}\begin{pmatrix} \log|z-1| & \log|z-1| \\ \log|z-1| & \log|z-1| \end{pmatrix}, & z \text{ inside the lens} \end{cases} \tag{4.24}$$

5. As $z \to -1$ we have the same behavior as before, replacing $|z-1|$ with $|z+1|$.

Since the function $\phi(z)$ is purely imaginary on γ_λ and $\operatorname{Im}\phi(z)$ is decreasing along γ_λ, the Cauchy–Riemann equations imply that $\operatorname{Re}\phi(z)$ increases locally in the complex plane as we move up from the curve γ_λ. Observe that

$$\frac{\mathrm{e}^{-n\phi(z)}}{W_n(z)} = \mathrm{e}^{-n(\phi(z)-\mathrm{i}z(\lambda-\lambda_n))},$$

and if we write $z = x + \mathrm{i}y$, we have

$$\operatorname{Re}(\phi(z) - \mathrm{i}z(\lambda-\lambda_n)) = \operatorname{Re}\phi(z) + y(\lambda-\lambda_n).$$

Since y is independent of n and $|\lambda - \lambda_n|$ is arbitrary small for large n because of (1.7), it follows that the lens can be opened in such a way that this real part is positive, and therefore the off–diagonal element in the jump matrix for $S(z)$ on Γ_1 decays to 0 exponentially fast as $n \to \infty$. A similar reasoning applies on Γ_2.

4.1.3 Model RH problem

Ignoring all jumps that are exponentially close to the identity as $n \to \infty$, we seek a matrix $N(z)$ that satisfies the following RH problem:

1. $N(z)$ is analytic in $\mathbb{C} \setminus \gamma_\lambda$.

2. On γ_λ, we have the following jump:

$$N_+(x) = N_-(x)\begin{pmatrix} 0 & \mathcal{W}_n(z) \\ -\frac{1}{\mathcal{W}_n(z)} & 0 \end{pmatrix}, \tag{4.25}$$

3. As $z \to \infty$:

$$N(z) = I + \mathcal{O}\left(\frac{1}{z}\right). \tag{4.26}$$

It is known that the solution of this RH problem is constructed with the Szegő function coresponding to the weight $W_n(z)$:

$$D(z) = \exp\left(\frac{(z^2-1)^{1/2}}{2\pi}\int_{\gamma_\lambda} \frac{\log W_n(s)}{\sqrt{1-s^2}}\frac{\mathrm{d}s}{z-s}\right), \tag{4.27}$$

which is analytic in $\mathbb{C} \setminus \gamma_\lambda$ and verifies $D_+(z)D_-(z) = W_n(z)$ on γ_λ. Using residue calculation, in this case we can write explicitly

$$D(z) = \exp\left(-\frac{in(\lambda - \lambda_n)}{2}\left(z - (z^2-1)^{1/2}\right)\right), \tag{4.28}$$

taking the cut of the square root on γ_λ. If we further define

$$D_\infty := \lim_{z\to\infty} D(z) = 1, \tag{4.29}$$

then the matrix

$$N(z) = D_\infty^{\sigma_3} M(z) D(z)^{-\sigma_3} = M(z)D(z)^{-\sigma_3} \tag{4.30}$$

solves the RH problem stated before. The matrix $M(z)$ solves the model RH problem given by

1. $M(z)$ is analytic in $\mathbb{C} \setminus \gamma_\lambda$.

2. On γ_λ, we have the following jump:

$$M_+(z) = M_-(z)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{4.31}$$

3. As $z \to \infty$:

$$M(z) = I + \mathcal{O}\left(\frac{1}{z}\right). \tag{4.32}$$

It can be given explicitly as follows:

$$M(z) = \frac{1}{\sqrt{2}(z^2-1)^{1/4}}\begin{pmatrix} \varphi(z)^{1/2} & i\varphi(z)^{-1/2} \\ -i\varphi(z)^{-1/2} & \varphi(z)^{-1/2} \end{pmatrix}, \tag{4.33}$$

in terms of the function $\varphi(z) = z + (z^2-1)^{1/2}$, taken analytic in $\mathbb{C} \setminus \gamma_\lambda$.

4.1.4 Local parametrices

We consider a disc $D(1,\delta)$ around $z = 1$, with radius $\delta > 0$ fixed. We seek a matrix P that satisfies the following RH problem:

1. $P(z)$ is analytic in $D(1,\delta) \setminus \Sigma_S$.

2. On $D(1,\delta) \cap \Sigma_S$, we have the following jumps:

$$P_+(x) = P_-(x)\begin{cases} \begin{pmatrix} 1 & 0 \\ \frac{e^{-n\phi(z)}}{W_n(z)} & 1 \end{pmatrix}, & z \in D(1,\delta) \cap (\Gamma_1 \cup \Gamma_2), \\ \begin{pmatrix} 0 & W_n(z) \\ -\frac{1}{W_n(z)} & 0 \end{pmatrix}, & z \in D(1,\delta) \cap \gamma_\lambda. \end{cases} \tag{4.34}$$

3. Uniformly for $z \in \partial D(1,\delta)$, we have the matching

$$P(z) = N(z)\left(I + \mathcal{O}\left(\frac{1}{n}\right)\right), \qquad n \to \infty. \tag{4.35}$$

4. As $z \to 1$ we have

$$P(z) = \begin{cases} \mathcal{O}\begin{pmatrix} 1 & \log|z-1| \\ 1 & \log|z-1| \end{pmatrix}, & z \text{ outside the lens} \\ \mathcal{O}\begin{pmatrix} \log|z-1| & \log|z-1| \\ \log|z-1| & \log|z-1| \end{pmatrix}, & z \text{ inside the lens} \end{cases} \tag{4.36}$$

We construct the local parametrix in the following way: define

$$P(z) = E_n(z)Q(z)W_n(z)^{-\sigma_3/2}\mathrm{e}^{-\frac{n}{2}\phi(z)\sigma_3}, \tag{4.37}$$

where $E_n(z)$ is an analytic factor, to be determined later to get the matching condition with $N(z)$ on the boundary of the disk. Then $Q(z)$ satisfies the following RH problem:

1. $Q(z)$ is analytic in $D(1,\delta) \setminus \Sigma_S$.

2. On $D(1,\delta) \cap \Sigma_S$, we have the following jumps:

$$Q_+(z) = Q_-(z) \begin{cases} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & z \in D(1,\delta) \cap (\Gamma_1 \cup \Gamma_2), \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & z \in D(1,\delta) \cap \gamma_\lambda. \end{cases} \tag{4.38}$$

3. As $z \to 1$ we have

$$Q(z) = \begin{cases} \mathcal{O}\begin{pmatrix} 1 & \log|z-1| \\ 1 & \log|z-1| \end{pmatrix}, & z \text{ outside the lens} \\ \mathcal{O}\begin{pmatrix} \log|z-1| & \log|z-1| \\ \log|z-1| & \log|z-1| \end{pmatrix}, & z \text{ inside the lens} \end{cases} \tag{4.39}$$

Now the matrix $Q(z)$ is constructed using Bessel functions, following the ideas in [18, Section 6].

$$Q(z) = \Psi(n^2 f(z)), \tag{4.40}$$

where $\zeta = f(z)$ is a conformal mapping from a neighborhood of $z = 1$ onto a neighborhood of the origin, and $\Psi(\zeta)$ solves the following auxiliary RH problem: in the ζ plane, consider the contour Σ_Ψ as illustrated in Figure 7, then

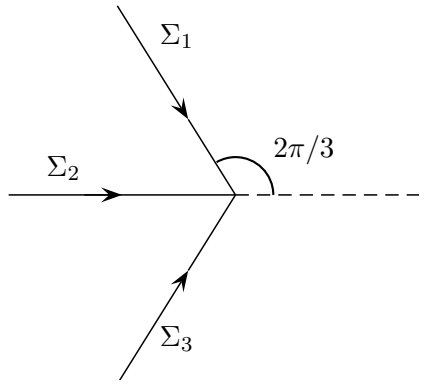

Figure 7: Contour Σ_Ψ in the ζ plane

1. $\Psi(\zeta)$ is analytic in $\mathbb{C} \setminus \Sigma_\Psi$.

2. On Σ_Ψ, we have the following jumps:

$$\Psi_+(\zeta) = \Psi_-(\zeta) \begin{cases} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & \zeta \in \gamma_1 \cup \gamma_3, \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \zeta \in \gamma_2. \end{cases} \tag{4.41}$$

3. As $\zeta \to 0$ we have the following behavior:

$$\Psi(\zeta) = \begin{cases} \mathcal{O}\begin{pmatrix} 1 & \log|\zeta| \\ 1 & \log|\zeta| \end{pmatrix}, & |\arg\zeta| < \frac{2\pi}{3} \\ \mathcal{O}\begin{pmatrix} \log|\zeta| & \log|\zeta| \\ \log|\zeta| & \log|\zeta| \end{pmatrix}, & \text{elsewhere} \end{cases} \tag{4.42}$$

The solution of this RH problem is given in terms of modified and ordinary Bessel functions, see [18, Section 6]:

$$\Psi(\zeta) = \begin{pmatrix} \frac{1}{2}H_0^{(2)}(2(-\zeta)^{1/2}) & -\frac{1}{2}H_0^{(1)}(2(-\zeta)^{1/2}) \\ -\pi\zeta^{1/2}(H_0^{(2)})'(2(-\zeta)^{1/2}) & \pi\zeta^{1/2}(H_0^{(1)})'(2(-\zeta)^{1/2}) \end{pmatrix} \tag{4.43}$$

if $-\pi < \arg\zeta < -\frac{2\pi}{3}$,

$$\Psi(\zeta) = \begin{pmatrix} I_0(2\zeta^{1/2}) & \frac{i}{\pi}K_0(2\zeta^{1/2}) \\ 2\pi i\zeta^{1/2}I_0'(2\zeta^{1/2}) & -2\zeta^{1/2}K_0'(2\zeta^{1/2}) \end{pmatrix} \tag{4.44}$$

if $|\arg\zeta| < \frac{2\pi}{3}$ and

$$\Psi(\zeta) = \begin{pmatrix} \frac{1}{2}H_0^{(1)}(2(-\zeta)^{1/2}) & \frac{1}{2}H_0^{(2)}(2(-\zeta)^{1/2}) \\ \pi\zeta^{1/2}(H_0^{(1)})'(2(-\zeta)^{1/2}) & \pi\zeta^{1/2}(H_0^{(2)})'(2(-\zeta)^{1/2}) \end{pmatrix} \tag{4.45}$$

if $\frac{2\pi}{3} < \arg\zeta < \pi$. Consider the asymptotic behavior of this $\Psi(\zeta)$ function. If $|\arg\zeta| < 2\pi/3$, we have

$$\Psi(\zeta) = \frac{1}{\sqrt{2}}(2\pi\zeta^{1/2})^{-\sigma_3/2}\begin{pmatrix}1 & \mathrm{i}\\ \mathrm{i} & 1\end{pmatrix}\left(I + \mathcal{O}(\zeta^{-1/2})\right)\mathrm{e}^{2\zeta^{1/2}\sigma_3}, \qquad \zeta\to\infty, \tag{4.46}$$

and the same result can be checked in the other two sectors, using known connection formulas for Bessel functions. The behavior at the origin also follows from standard expansions of Hankel and Bessel functions.

The square root takes the principal value, with a cut on the negative real axis.

First, we want to match the last factors in $Q(z)$ and $\Psi(z)$, so we set

$$\zeta = n^2 f(z) = \frac{n^2}{16}\phi(z)^2. \tag{4.47}$$

Hence, $f(z) = \phi(z)^2/16$, and since

$$\phi(z) = \sqrt{2}\left(1 + \frac{\mathrm{i}\lambda}{2}\right)(z-1)^{1/2} + \mathcal{O}((z-1)^{3/2}), \qquad z\to 1, \tag{4.48}$$

we obtain

$$f(z) = \frac{(2+\mathrm{i}\lambda)^2}{8}(z-1) + \mathcal{O}((z-1)^2), \qquad z\to 1, \tag{4.49}$$

so $f(z)$ is indeed a conformal mapping locally. Moreover,

$$\Psi(n^2 f(z)) = \frac{1}{\sqrt{2}}(2\pi n)^{-\sigma_3/2} f(z)^{-\sigma_3/4}\begin{pmatrix}1 & \mathrm{i}\\ \mathrm{i} & 1\end{pmatrix}\left(I + \mathcal{O}(n^{-1})\right)\mathrm{e}^{2nf(z)^{1/2}\sigma_3}, \tag{4.50}$$

so we define

$$E_n(z) = N(z)W_n(z)^{\sigma_3/2}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -\mathrm{i}\\ -\mathrm{i} & 1\end{pmatrix} f(z)^{\sigma_3/4}(2\pi n)^{\sigma_3/2}, \tag{4.51}$$

and thus the matching of $P(z)$ and $N(z)$ required in (4.35) is achieved on the boundary of $D(1,\delta)$.

In a neighborhood of $z=-1$, say $D(-1,\delta)$, we have a similar construction: the local parametrix $\tilde{P}(z)$ is given by

$$\tilde{P}(z) = \tilde{E}_n(z)\tilde{Q}(z)W_n(z)^{-\sigma_3/2}\mathrm{e}^{-\frac{n}{2}\tilde{\phi}(z)\sigma_3}, \tag{4.52}$$

where

$$\tilde{\phi}(z) = 2\pi\mathrm{i}\int_z^{-1}\psi(s)\mathrm{d}s = \phi(z) - 2\pi\mathrm{i} \tag{4.53}$$

and

$$\tilde{Q}(z) = \tilde{\Psi}(n^2\tilde{f}(z)) = \sigma_3\Psi(n^2\tilde{f}(z))\sigma_3, \tag{4.54}$$

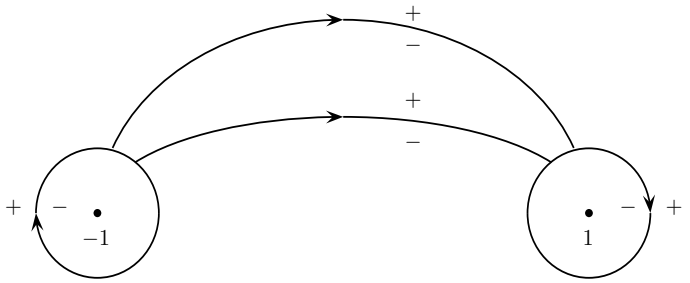

Figure 8: Final contour Σ_R in the steepest descent analysis

with $\tilde{f}(z) = \tilde{\phi}(z)^2/16$ a conformal mapping from a neighborhood of $z = -1$ onto a neighborhood of $\zeta = 0$. The analytic factor in this case is

$$\tilde{E}_n(z) = N(z)W_n(z)^{\sigma_3/2}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & \mathrm{i}\\ \mathrm{i} & 1\end{pmatrix}\tilde{f}(z)^{\sigma_3/4}(2\pi n)^{\sigma_3/2}. \tag{4.55}$$

4.1.5 Final transformation

Using the global and local parametrices, we define

$$R(z) = S(z)\begin{cases} N^{-1}(z), & z \in \mathbb{C} \setminus (D(1,\delta) \cup D(-1,\delta) \cup \Sigma_S),\\ P^{-1}(z), & z \in D(1,\delta) \setminus \Sigma_S,\\ \tilde{P}^{-1}(z), & z \in D(-1,\delta) \setminus \Sigma_S, \end{cases} \tag{4.56}$$

Following an argument similar to the one in [18, §7], it can be shown that this matrix $R(z)$ is analytic off the contour Σ_R shown in Figure 8. Furthermore, as $n \to \infty$ the jumps are

$$R_+(z) = R_-(z)\begin{cases} I + \mathcal{O}(\mathrm{e}^{-2cn}), & z \in \Sigma_R \setminus (\partial D(1,\delta) \cup \partial D(-1,\delta)),\\ I + \mathcal{O}(n^{-1}), & z \in \partial D(1,\delta) \cup \partial D(-1,\delta). \end{cases} \tag{4.57}$$

From this result and the fact that $R(z) \to I$ as $z \to \infty$, it can be proved that

$$R(z) = I + \mathcal{O}(n^{-1}), \qquad n \to \infty, \tag{4.58}$$

uniformly for $z \in \mathbb{C} \setminus \Sigma_R$, see for example [7, Section 11]. Reversing the transformations, we can obtain the asymptotic behavior of $Y(z)$ as $n \to \infty$, and in particular of the $(1,1)$ entry, that contains the polynomial $p_n^{\omega}(z)$. Actually, the matrix $R(z)$ admits a full asymptotic expansion in inverse powers of n:

$$R(z) \sim I + \sum_{k=1}^{\infty} \frac{R^{(k)}(z)}{n^k}, \qquad n \to \infty, \tag{4.59}$$

where the functions $R^{(k)}(z)$ are analytic in $z \in \mathbb{C} \setminus (\partial D(1,\delta) \cup \partial D(-1,\delta))$. Furthermore, it is true that

$$R^{(k)}(z) = \mathcal{O}\left(\frac{1}{z}\right), \qquad z \to \infty, \tag{4.60}$$

and, as in [18, Lemma 8.3], the expansion (4.59) is uniformly valid for large values of z as well. This double asymptotic property, for large n and large z, will be used later to obtain the asymptotic expansion of the recurrence coefficients.

The outer asymptotics for $p_n^\omega(z)$, for z in compact subsets of $\mathbb{C} \setminus \gamma_\lambda$ (outside of the lens and away from the endpoints), can be worked out from

$$p_n^\omega(z) = \begin{pmatrix} 1 & 0 \end{pmatrix} Y(z) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathrm{e}^{ng(z)} \left(R_{11}(z) N_{11}(z) + R_{12}(z) N_{21}(z) \right). \tag{4.61}$$

We apply the first transformation (4.14), together with the fact that in the outer region $T(z) = S(z) = R(z)N(z) = R(z)M(z)D(z)^{-\sigma_3}$, so

$$\begin{aligned} p_n^\omega(z) &= \frac{\mathrm{e}^{ng(z)}}{D(z)} \begin{pmatrix} 1 & 0 \end{pmatrix} R(z) M(z) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \frac{\mathrm{e}^{ng(z)}}{D(z)} \left(R_{11}(z) M_{11}(z) + R_{12}(z) M_{21}(z) \right) \end{aligned} \tag{4.62}$$

Now we use the fact that $R_{11}(z) = 1 + \mathcal{O}(1/n)$ and $R_{12}(z) = \mathcal{O}(1/n)$, and the explicit form of $M_{11}(z)$, see (4.33). Finally, note that

$$\frac{\mathrm{e}^{ng(z)}}{D(z)} = \left(\frac{\varphi(z)}{2}\right)^n \exp\left(-\frac{\mathrm{i}n\lambda_n}{2} (z - (z^2-1)^{1/2}) \right) = \left(\frac{\varphi(z)}{2}\right)^n \exp\left(-\frac{\mathrm{i}n\lambda_n}{2\varphi(z)} \right),$$

in terms of the function $\varphi(z) = z + (z^2-1)^{1/2}$, and using the explicit formula for $g(z)$, see (4.13) and $D(z)$, see (4.28). Thus we get the asymptotic expansion (2.9).

If z is in the upper (lower) part of the lens, we get

$$\begin{aligned} p_n^\omega(z) &= \begin{pmatrix} 1 & 0 \end{pmatrix} Y(z) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{e}^{\frac{nl}{2}} & 0 \end{pmatrix} S(z) \begin{pmatrix} 1 & 0 \\ \pm \frac{\mathrm{e}^{-n\phi(z)}}{W_n(z)} & 1 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{n\left(g(z) - \frac{l}{2}\right)} \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{e}^{\frac{n\ell}{2}} & 0 \end{pmatrix} R(z) N(z) \begin{pmatrix} \mathrm{e}^{n\left(g(z) - \frac{l}{2}\right)} \\ \pm \frac{\mathrm{e}^{n\left(g(z) - \phi(z) - \frac{\ell}{2}\right)}}{W_n(z)} \end{pmatrix} \end{aligned} \tag{4.63}$$

Since

$$g(z) - \frac{l}{2} = \frac{V(z)}{2} - \frac{\phi(z)}{2},$$

we obtain

$$p_n^{\omega}(z) = \mathrm{e}^{\frac{n}{2}(V(z)+l)} \begin{pmatrix} 1 & 0 \end{pmatrix} R(z) M(z) D(z)^{-\sigma_3} \begin{pmatrix} \mathrm{e}^{\frac{n}{2}\phi(z)} \\ \pm \frac{\mathrm{e}^{-\frac{n}{2}\phi(z)}}{W_n(z)} \end{pmatrix}$$
$$= \frac{\mathrm{e}^{\frac{n}{2}(V(z)+\ell)}}{W_n(z)^{1/2}} \left(M_{11}(z) \frac{W_n(z)^{1/2}}{D(z)} \mathrm{e}^{\frac{n\phi(z)}{2}} \pm M_{12}(z) \frac{D(z)}{W_n(z)^{1/2}} \mathrm{e}^{-\frac{n\phi(z)}{2}} + \mathcal{O}(1/n) \right),$$

bearing in mind again that $R_{11}(z) = 1 + \mathcal{O}(1/n)$ and $R_{12}(z) = \mathcal{O}(1/n)$. Observe that if we take boundary values on γ_λ, we can use the fact that $\phi_+(z) = -\phi_-(z)$, and also $N_{11+}(z) = -N_{12-}(z)$ and $N_{12+}(z) = N_{11-}(z)$, which come from the model Riemann–Hilbert problem. As a result, both boundary values coincide on the curve γ_λ.

The reason to write the formula in that symmetric form is because

$$\frac{W_n(z)^{1/2}}{D(z)} = \mathrm{e}^{-\frac{\mathrm{i}n}{2}(\lambda-\lambda_n)(z^2-1)^{1/2}}, \tag{4.64}$$

using (4.19) and (4.28). Now we combine this with the explicit form of $\phi(z)$, see (4.11), and with the formulas for $M_{11}(z)$ and $M_{12}(z)$. Note also that on γ_λ we can write $\varphi_+(z) = \mathrm{e}^{\mathrm{i}\arccos z}$, using the standard definition of the arccosine function, with a cut on $(-\infty, -1] \cup [1, \infty)$, see [1, §4.23.22]. Thus, substituing the value of l and simplifying, we obtain the asymptotic result (2.11) on a neighborhood of the curve γ_λ.

For $z \in D(1, \delta)$ and in the upper part of the lens, we use the relation between $T(z)$ and $S(z)$ in (4.21), the connection between $S(z)$ and $R(z)$ in (4.56), and the the expression for $P(z)$ in (4.37) to write

$$\begin{aligned} p_n^{\omega}(z) &= \mathrm{e}^{\frac{nl}{2}} \begin{pmatrix} 1 & 0 \end{pmatrix} R(z) P(z) \begin{pmatrix} \mathrm{e}^{n\left(g(z)-\frac{l}{2}\right)} \\ \frac{\mathrm{e}^{n\left(g(z)-\frac{l}{2}-\phi(z)\right)}}{W_n(z)} \end{pmatrix} \\ &= \frac{\mathrm{e}^{\frac{n}{2}(V(z)+l)}}{W_n(z)^{1/2}} \begin{pmatrix} 1 & 0 \end{pmatrix} \left(I + \mathcal{O}\left(\frac{1}{n}\right) \right) E_n(z) \Psi(n^2 f(z)) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= \mathrm{e}^{\frac{n}{2}(V_n(z)+l)} \begin{pmatrix} 1 & 0 \end{pmatrix} \left(I + \mathcal{O}\left(\frac{1}{n}\right) \right) E_n(z) \Psi(n^2 f(z)) \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \end{aligned} \tag{4.65}$$

Now we observe that

$$\begin{aligned} E_n(z) &= \frac{1}{\sqrt{2}} N(z) W_n(z)^{\sigma_3/2} \begin{pmatrix} 1 & -\mathrm{i} \\ -\mathrm{i} & 1 \end{pmatrix} f(z)^{\sigma_3/4} (2\pi n)^{\sigma_3/2} \\ &= \frac{1}{\sqrt{2}} M(z) D(z)^{-\sigma_3} W_n(z)^{\sigma_3/2} \begin{pmatrix} 1 & -\mathrm{i} \\ -\mathrm{i} & 1 \end{pmatrix} f(z)^{\sigma_3/4} (2\pi n)^{\sigma_3/2} \\ &= \frac{1}{(z^2-1)^{1/4}} \begin{pmatrix} \cos \xi_+(z) & \sin \xi_+(z) \\ -\mathrm{i} \cos \xi_-(z) & -\mathrm{i} \sin \xi_-(z) \end{pmatrix} f(z)^{\sigma_3/4} (2\pi n)^{\sigma_3/2}, \end{aligned} \tag{4.66}$$

using the expression for $M(z)$ in terms pf $\varphi(z)$, see (4.33), and also (4.64). Here

$$\xi_{\pm}(z) = -\frac{n}{2}(\lambda - \lambda_n)(z^2-1)^{1/2} \pm \frac{1}{2}\arccos z,$$

using (4.64) and the fact that $\varphi(z) = \mathrm{e}^{\mathrm{i}\arccos z}$.

Also, bearing in mind that $f(z) = \phi^2(z)/16$, we have

$$\Psi(n^2 f(z)) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} J_0\left(2n(-f(z))^{1/2}\right) \\ \pi n f(z)^{1/2}(J_0)'\left(2n(-f(z))^{1/2}\right) \end{pmatrix},$$

using the standard connection between Hankel and Bessel functions, see for instance [1, 10.4.3]. As a consequence,

$$f(z)^{\sigma_3/4}(2\pi n)^{\sigma_3/2}\Psi(n^2 f(z)) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = f(z)^{1/4}(2\pi n)^{1/2} \begin{pmatrix} J_0\left(2n(-f(z))^{1/2}\right) \\ (J_0)'\left(2n(-f(z))^{1/2}\right) \end{pmatrix},$$

and putting all together we get

$$\begin{aligned} p_n^{\omega}(z) = \mathrm{e}^{\frac{n}{2}(V_n(z)+l)} & \frac{f(z)^{1/4}(2\pi n)^{1/2}}{(z^2-1)^{1/4}} \begin{pmatrix} 1 & 0 \end{pmatrix} \left(I + \mathcal{O}\left(\frac{1}{n}\right)\right) \\ & \times \begin{pmatrix} \cos\xi_+(z) & \sin\xi_+(z) \\ -\mathrm{i}\cos\xi_-(z) & -\mathrm{i}\sin\xi_-(z) \end{pmatrix} \begin{pmatrix} J_0\left(2n(-f(z))^{1/2}\right) \\ (J_0)'\left(2n(-f(z))^{1/2}\right) \end{pmatrix} \end{aligned}$$

Finally, the argument of the Bessel function is equal to $-\frac{\mathrm{i}n}{2}\phi(z)$, using the relation between $f(z)$ and $\phi(z)$. The sign of the square root is determined by the fact that in that sector of the lens we have $\frac{2\pi}{3} < \arg\phi(z) < \pi$. Substituting $l = -2\log 2$, we obtain the asymptotic expansion (2.12). A similar computation can be carried out in the disc around $z = -1$, but we omit it for brevity.

It is clear that using the properties of $R(z)$, in particular its asymptotic expansion for large n, it is possible to compute higher order terms in the asymptotic expansion for the orthogonal polynomials if one can obtain the terms $R^{(k)}(z)$ in (4.59). Thus, substitution of a more refined asymptotic estimate for $R(z)$ in the previous formulas will lead to more accurate estimates. We omit this in the present paper for brevity, but we will present a possible approach in the next section, taken from [18] and that is exploited in [8].

5 Proof of Theorem 2.5

The recurrence coefficients of the three term recursion can be written in terms of the matrices in the Riemann–Hilbert analysis:

$$\begin{aligned} a_n^2 &= [Y_1]_{12}[Y_1]_{21}, \\ b_n &= \frac{[Y_2]_{12}}{[Y_1]_{12}} - [Y_1]_{22}, \end{aligned} \tag{5.1}$$

where

$$Y(z)z^{-n\sigma_3} = I + \frac{Y_1}{z} + \frac{Y_2}{z^2} + \dots. \tag{5.2}$$

Hence, the solvability of the Riemann–Hilbert problem for large n proves the existence of the recurrence coefficients for large n. In order to obtain the asymptotic estimates, we want to write the coefficients in terms of $R(z)$, so we denote

$$T(z) = I + \frac{T_1}{z} + \frac{T_2}{z^2} + \dots, \qquad z \to \infty. \tag{5.3}$$

We use the fact that

$$g(z) = \log z - \frac{c_1}{z} - \frac{c_2}{z^2} + \mathcal{O}(z^{-3}), \qquad z \to \infty, \tag{5.4}$$

where the constants c_1 and c_2 are given by

$$c_1 = \int_{\gamma_\lambda} s \, \mathrm{d}\mu_\lambda(s), \qquad c_2 = \int_{\gamma_\lambda} s^2 \, \mathrm{d}\mu_\lambda(s). \tag{5.5}$$

These constants can be computed explicitly in this case, using residues, but this is not really needed. Consequently,

$$\mathrm{e}^{-ng(z)} = z^{-n}\left(1 + \frac{nc_1}{z} + \frac{2n^2c_1^2 + nc_2}{2z^2} + \mathcal{O}(z^{-3})\right), \qquad z \to \infty, \tag{5.6}$$

Next, because of the relation between $Y(z)$ and $T(z)$, see (4.14), and expanding for large z, we have

$$\begin{aligned} T(z) = \mathrm{e}^{\frac{-nl}{2}\sigma_3}&\left(I + \frac{Y_1}{z} + \frac{Y_2}{z^2} + \dots\right) \times \\ &\left(I + \frac{nc_1\sigma_3}{z} + \frac{(2nc_1^2 + c_2)n\sigma_3}{2z^2} + \dots\right)\mathrm{e}^{\frac{nl}{2}\sigma_3}, \end{aligned} \tag{5.7}$$

so

$$\begin{aligned} T_1 &= \mathrm{e}^{\frac{-nl}{2}\sigma_3} Y_1 \mathrm{e}^{\frac{nl}{2}\sigma_3} + c_1 n\sigma_3, \\ T_2 &= \mathrm{e}^{\frac{-nl}{2}\sigma_3} Y_2 \mathrm{e}^{\frac{nl}{2}\sigma_3} + c_1 n \mathrm{e}^{-\frac{nl}{2}\sigma_3} Y_1 \mathrm{e}^{\frac{nl}{2}\sigma_3} + \frac{(2nc_1^2 + c_2)n\sigma_3}{2} \end{aligned} \tag{5.8}$$

It follows that

$$[T_1]_{12} = \mathrm{e}^{-nl}[Y_1]_{12}, \qquad [T_1]_{21} = \mathrm{e}^{nl}[Y_1]_{21}, \qquad [Y_1]_{22} = [T_1]_{22} - c_1 n, \tag{5.9}$$

and also

$$[T_2]_{12} = \mathrm{e}^{-nl}\left(-c_1 n[Y_1]_{12} + [Y_2]_{12}\right). \tag{5.10}$$

As a consequence, we have in terms of $T(z)$:

$$\begin{aligned} a_n^2 &= [T_1]_{12}[T_1]_{21}, \\ b_n &= \frac{[T_2]_{12}}{[T_1]_{12}} - [T_1]_{22}, \end{aligned} \tag{5.11}$$

Next, away from the curve γ_λ, we have $T(z) = S(z) = R(z)N(z)$, and we define

$$N(z) = I + \frac{N_1}{z} + \frac{N_2}{z^2} + \dots, \qquad R(z) = I + \frac{R_1}{z} + \frac{R_2}{z^2} + \dots, \tag{5.12}$$

as $z \to \infty$. Then,

$$\begin{aligned} T_1 &= R_1 + N_1, \\ T_2 &= N_2 + R_1 N_1 + R_2. \end{aligned} \tag{5.13}$$

From (4.30), we have

$$N(z) = I - \frac{1}{2z}\sigma_2 + \frac{1}{8z^2} I + \dots, \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \tag{5.14}$$

so

$$\begin{aligned} T_1 &= R_1 - \frac{1}{2}\sigma_2, \\ T_2 &= \frac{1}{8} I - \frac{1}{2} R_1 \sigma_2 + R_2, \end{aligned} \tag{5.15}$$

and then

$$\begin{aligned} a_n^2 &= \left([R_1]_{12} + \frac{\mathrm{i}}{2} \right) \left([R_1]_{21} - \frac{\mathrm{i}}{2} \right), \\ b_n &= \frac{\mathrm{i}[R_1]_{11} + 2[R_2]_{12}}{\mathrm{i} + 2[R_1]_{12}} - [R_1]_{22}. \end{aligned} \tag{5.16}$$

In order to compute the terms R_1 and R_2, we need to obtain additional terms in the large n asymptotics of $R(z)$ first, that is, the coefficients $R^{(k)}(z)$ in (4.59). Following [18, §8], we write the jump matrix for $R(z)$ on Σ_R as a perturbation of the identity, so

$$R_+(z) = R_-(z) \left(I + \Delta(z) \right), \qquad z \in \Sigma_R, \tag{5.17}$$

where $\Delta(z)$ admits an expansion in inverse powers of n:

$$\Delta(z) \sim \sum_{k=1} \frac{\Delta_k(z)}{n^k}, \qquad n \to \infty. \tag{5.18}$$

Since the jump is exponentially close to the identity on $\Sigma_R \setminus (\partial D(1,\delta) \cup \partial D(-1,\delta))$, the coefficients $\Delta_k(z)$ are identically 0 there. For $z \in \partial D(1,\delta) \cup \partial D(-1,\delta)$, they can be computed from the asymptotic expansion of the Bessel functions in the local parametrices. The outcome is the following:

$$\begin{aligned} \Delta_k(z) = \frac{(-1)^{k-1}}{4^{k-1}(k-1)!\phi(z)^k} \prod_{j=1}^{k-1} (2j-1)^2 \times \\ N(z) \begin{pmatrix} \frac{(-1)^k}{k}\left(\frac{k}{2} - \frac{1}{4}\right) & -\left(k - \frac{1}{2}\right)\mathrm{i} \\ (-1)^k \left(k - \frac{1}{2}\right)\mathrm{i} & \frac{1}{k}\left(\frac{k}{2} - \frac{1}{4}\right) \end{pmatrix} N^{-1}(z), \qquad z \in \partial D(1,\delta) \end{aligned} \tag{5.19}$$

and

$$\Delta_k(z) = \frac{(-1)^{k-1}}{4^{k-1}(k-1)!\tilde{\phi}(z)^k} \prod_{j=1}^{k-1}(2j-1)^2 \times \\ N(z) \begin{pmatrix} \frac{(-1)^k}{k}\left(\frac{k}{2}-\frac{1}{4}\right) & \left(k-\frac{1}{2}\right)\mathrm{i} \\ (-1)^{k+1}\left(k-\frac{1}{2}\right)\mathrm{i} & \frac{1}{k}\left(\frac{k}{2}-\frac{1}{4}\right) \end{pmatrix} N^{-1}(z), \qquad z \in \partial D(-1,\delta), \tag{5.20}$$

for $k \geq 1$, cf. [18, formulas (8.5) and (8.6)].

Additionally, as proved in [18, Lemma 8.2], these functions Δ_k have analytic continuations to bigger discs around $z = \pm 1$ as meromorphic functions with poles at $z = \pm 1$ of order at most $[(k+1)/2]$.

The importance of the functions $\Delta_k(z)$ is that they appear in an additive Riemann–Hilbert problem for $R_k(z)$ on the boundary of the discs. Namely, we have

$$R_+^{(k)}(z) = R_-^{(k)}(z) + \sum_{j=1}^{k} R_-^{(k-j)}(z)\Delta_j(z), \qquad z \in \partial D(1,\delta) \cup \partial D(-1,\delta), \tag{5.21}$$

where $+$ indicates the boundary value from the exterior and $-$ from the interior, recall Figure 8.

For $k = 1$, (5.21) gives

$$R_+^{(1)}(z) = R_-^{(1)}(z) + \Delta_1(z), \qquad z \in \partial D(1,\delta) \cup \partial D(-1,\delta). \tag{5.22}$$

Furthermore, we can write

$$\begin{aligned} \Delta_1(z) &= \frac{A^{(1)}}{z-1} + \mathcal{O}(1), \qquad z \to 1, \\ \Delta_1(z) &= \frac{B^{(1)}}{z+1} + \mathcal{O}(1), \qquad z \to -1, \end{aligned} \tag{5.23}$$

where $A^{(1)}$ are $B^{(1)}$ are constant matrices. From (5.19) and (5.20), and the behavior of $N(z)$, $\phi(z)$ and $\tilde{\phi}(z)$, we get

$$A^{(1)} = -\frac{1}{8(2+\mathrm{i}\lambda)}\begin{pmatrix} -1 & \mathrm{i} \\ \mathrm{i} & 1 \end{pmatrix}, \qquad B^{(1)} = -\frac{1}{8(2-\mathrm{i}\lambda)}\begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & -1 \end{pmatrix}. \tag{5.24}$$

The Riemann–Hilbert problem for $R^{(1)}(z)$ can be solved as follows:

$$R^{(1)}(z) = \begin{cases} \dfrac{A^{(1)}}{z-1} + \dfrac{B^{(1)}}{z+1}, & z \in \mathbb{C} \setminus (\overline{D(1,\delta)} \cup \overline{D(-1,\delta)}), \\ \dfrac{A^{(1)}}{z-1} + \dfrac{B^{(1)}}{z+1} - \Delta_1(z), & z \in \overline{D(1,\delta)} \cup \overline{D(-1,\delta)}. \end{cases} \tag{5.25}$$

Hence, we have

$$R^{(1)}(z) = -\frac{1}{8(2+\mathrm{i}\lambda)}\begin{pmatrix}-1 & \mathrm{i}\\ \mathrm{i} & 1\end{pmatrix}\frac{1}{z-1} - \frac{1}{8(2-\mathrm{i}\lambda)}\begin{pmatrix}1 & \mathrm{i}\\ \mathrm{i} & -1\end{pmatrix}\frac{1}{z+1}, \quad (5.26)$$

for $z \in \mathbb{C} \setminus (\partial D(1,\delta) \cup \partial D(-1,\delta))$. Expanding for large z, we obtain

$$R^{(1)}(z) = -\frac{\mathrm{i}}{4(4+\lambda^2)}\begin{pmatrix}\lambda & 2\\ 2 & -\lambda\end{pmatrix}\frac{1}{z} + \frac{1}{4(4+\lambda^2)}\begin{pmatrix}2 & -\lambda\\ -\lambda & -2\end{pmatrix}\frac{1}{z^2} + \mathcal{O}\left(\frac{1}{z^3}\right). \quad (5.27)$$

For $k = 2$, equation (5.21) reads

$$R_+^{(2)}(z) = R_-^{(2)}(z) + R_-^{(1)}(z)\Delta_1(z) + \Delta_2(z), \qquad z \in \partial D(1,\delta) \cup \partial D(-1,\delta). \quad (5.28)$$

Now we have

$$\begin{aligned} R_-^{(1)}(z)\Delta_1(z) + \Delta_2(z) &= \frac{A^{(2)}}{z-1} + \mathcal{O}(1), \qquad z \to 1,\\ R_-^{(1)}(z)\Delta_1(z) + \Delta_2(z) &= \frac{B^{(2)}}{z+1} + \mathcal{O}(1), \qquad z \to -1, \end{aligned} \quad (5.29)$$

where $A^{(2)}$ are $B^{(2)}$ are constant matrices. From (5.19) and (5.20), and the behavior of $N(z)$, $\phi(z)$ and $\tilde{\phi}(z)$, we get after collecting all relevant terms,

$$\begin{aligned} A^{(2)} &= \frac{1}{64(\lambda-2\mathrm{i})^2(\lambda+2\mathrm{i})}\begin{pmatrix}\lambda-2\mathrm{i} & 4\mathrm{i}(2\lambda+5)\\ -4\mathrm{i}(2\lambda+5) & \lambda-2\mathrm{i}\end{pmatrix},\\ B^{(2)} &= \frac{1}{64(\lambda+2\mathrm{i})^2(\lambda-2\mathrm{i})}\begin{pmatrix}\lambda+2\mathrm{i} & -4\mathrm{i}(2\lambda-5)\\ 4\mathrm{i}(2\lambda-5) & \lambda+2\mathrm{i}\end{pmatrix}. \end{aligned} \quad (5.30)$$

The Riemann–Hilbert problem for $R^{(2)}(z)$ can be solved as follows:

$$R^{(2)}(z) = \begin{cases} \dfrac{A^{(2)}}{z-1} + \dfrac{B^{(2)}}{z+1}, & z \in \mathbb{C} \setminus (\overline{D(1,\delta)} \cup \overline{D(-1,\delta)}),\\ \dfrac{A^{(2)}}{z-1} + \dfrac{B^{(2)}}{z+1} - R^{(1)}(z)\Delta_1(z) - \Delta_2(z), & z \in \overline{D(1,\delta)} \cup \overline{D(-1,\delta)}. \end{cases} \quad (5.31)$$

Reexpanding at infinity, we get

$$\begin{aligned} R^{(2)}(z) = &\frac{1}{4(4+\lambda^2)^2}\begin{pmatrix}0 & -\mathrm{i}(\lambda^2-5)\\ \mathrm{i}(\lambda^2-5) & 0\end{pmatrix}\frac{1}{z}\\ &- \frac{1}{32(4+\lambda^2)^2}\begin{pmatrix}\lambda^2+4 & 36\lambda\\ -36\lambda & \lambda^2+4\end{pmatrix}\frac{1}{z^2} + \mathcal{O}\left(\frac{1}{z^3}\right). \end{aligned} \quad (5.32)$$

If we now pick the contributions from $R^{(1)}(z)$ and $R^{(2)}(z)$ in terms of n, we obtain

$$\begin{aligned} a_n^2 &= \frac{1}{4} + \frac{4-\lambda^2}{4(4+\lambda^2)^2n^2} + \mathcal{O}\left(\frac{1}{n^3}\right),\\ b_n &= -\frac{2\mathrm{i}\lambda}{(4+\lambda^2)^2n^2} + \mathcal{O}\left(\frac{1}{n^3}\right). \end{aligned} \quad (5.33)$$

This proves the leading terms in Theorem 2.5, and higher order coefficients can be computed if the functions $R^{(k)}(z)$ are available for $k \geq 3$. This can be accomplished in general following the idea just presented and using (5.21), see also [18], although the computation becomes increasingly complicated. A general algorithm to carry out this kind of computations for Jacobi–type weights will be available in [8].

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