A comparison of Landau-Ginzburg models for odd dimensional Quadrics

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Abstract

In [Rie08], the second author defined a Landau-Ginzburg model for homogeneous spaces G/P, as a regular function on an affine subvariety of the Langlands dual group. In this paper, we reformulate this LG model $(\check{X}, W_t)$ in the case of the odd-dimensional quadric Q_{2m-1} as a rational function on a Langlands dual projective space, in the spirit of work by R. Marsh and the second author [MR12] for type A Grassmannians and by both authors [PR13] for Lagrangian Grassmannians.

We also compare this LG model with the one obtained independently by Gorbounov and Smirnov in [GS13], and we use this comparison to deduce part of [Rie08, Conj. 8.1] for odd-dimensional quadrics.

1 Introduction

In 2000 Hori and Vafa wrote down a conjectured LG model for any hypersurface in a (weighted) complex projective space [HV00], [Prz07, Rmk. 19]. This is a Laurent polynomial associated to the hypersurface which plays the part of the B-model to the hypersurface in mirror symmetry, meaning its singularities are meant to encode various structures to do with Gromov-Witten theory of the hypersurface. In the case of the smooth quadric Q_3 in $\mathbb{P}^4$ the LG model is

$$Y_1 + Y_2 + \frac{(Y_3 + q)^2}{Y_1 Y_2 Y_3},$$

and in this special case it was written down earlier by Eguchi, Hori, and Xiong [EHX97]. For a quadric Q_{2m-1} the formula of Hori and Vafa reads

$$Y_1 + Y_2 + \ldots + Y_{m-1} + \frac{(Y_m + q)^2}{Y_1 Y_2 \cdots Y_m}.$$

One issue with these Laurent polynomial formulas is that they do not always have the expected number of critical points (at fixed generic value of q) which should be equal to $\dim(H^*(Q_{2m-1}))$. This was already observed in [EHX97], where it was suggested to solve this problem using a partial compactification, and this was carried out for the first time albeit in an ad hoc fashion.

The quadratic hypersurfaces Q_{2m-1} have a large symmetry group. Indeed Q_{2m-1} is a cominuscule homogeneous space for the group $\mathrm{Spin}_{2m+1}(\mathbb{C})$. Therefore there is already

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another LG model on an affine variety generally larger than a torus, which was defined by the second author using a Lie theoretic construction [Rie08]. Namely for any projective homogeneous space G/P of a simple complex algebraic group, [Rie08] constructs a conjectural LG model, which is a regular function on an affine subvariety of the Langlands dual group. It is shown in [Rie08] that this LG model recovers the Peterson variety presentation [Pet97] of the quantum cohomology of G/P. It therefore defines an LG model whose Jacobi ring has the correct dimension.

For odd-dimensional quadrics Q_{2m-1} a recent paper [GS13] of Gorbounov and Smirnov constructed directly a partial compactification of the Hori-Vafa mirrors, without making use of [Rie08]. Moreover they proved a version of mirror symmetry, which identifies the initial data of the Frobenius manifold associated to the LG model with that constructed out of the quantum cohomology of Q_{2m-1}.

The goal of this note is twofold. We first express the LG model from [Rie08] in the case of Q_{2m-1} in terms of natural coordinates on an affine subvariety of a 'mirror homogeneous space' $X^\vee = IG_1(2m) \cong \mathbb{P}^{2m-1}$. For example in the case of Q_3 we obtain

$$W_q = p_1 + \frac{p_2^2}{p_1p_2 - p_3} + q\frac{p_1}{p_3}.$$

The first main result generalises this formula. Define

$$\check{X}^\circ := \check{X} \setminus D, \tag{1}$$

where $D := D_0 + D_1 + \ldots + D_{m-1} + D_m$, the D_i's being given by

$$\begin{aligned} D_0 &:= \{p_0 = 0\}, \\ D_l &:= \left\{p_l p_{2m-1-l} - p_{l-1}p_{2m-l} + \cdots + (-1)^l p_0 p_{2m-1} = 0\right\} \text{ for } 1 \le l \le m-1, \\ D_m &:= \{p_{2m-1} = 0\}. \end{aligned}$$

The divisor D is an anti-canonical divisor. Indeed, the index of $\check{X} = \mathbb{P}(V^*)$ is $2m$.

Theorem 1. *The LG model $\mathcal{F}_q : \mathcal{R} \to \mathbb{C}$ from [Rie08] is isomorphic to $W_q : \check{X}^\circ \to \mathbb{C}$ defined by*

$$W_q = p_1 + \sum_{l=1}^{m-1} \frac{p_{l+1}p_{2m-1-l}}{p_l p_{2m-1-l} - p_{l-1}p_{2m-l} + \cdots + (-1)^l p_{2m-1}} + q\frac{p_1}{p_{2m-1}}. \tag{2}$$

Corollary 2. *There is an isomorphism*

$$\mathbb{C}[\check{X}^\circ \times \mathbb{C}^*]/(\partial W_q) \to QH^*(X)[q^{-1}] \tag{3}$$

defined by sending p_i to the Schubert class $\sigma_i \in H^{2i}(X)$.

This follows from Thm. 1 together with [Rie08]. Indeed the isomorphism in Cor. 2 fits in well with the geometric Satake correspondence (see [Lus83], [Gin95], [MV07]), by which

$$H^*(Q_{2m-1}) = V_{\omega_1}^{\mathrm{PSp}_{2m}}.$$

With this in mind it is natural to identify $\check{X}$ with $\mathbb{P}(H^*(Q_{2m-1})^*)$ and the coordinates $\{p_i\}$ with the Schubert basis $\{\sigma_i\}$ of $H^*(Q_{2m-1})$.

In is interesting to note that under the isomorphism from Cor. 2, the denominators of W_q actually map to something extremely simple inside the quantum cohomology of the quadric :

Corollary 3. *For $1 \leq l \leq m-1$, the denominator $p_l p_{2m-1-l} - p_{l-1}p_{2m-l} + \cdots + (-1)^l p_{2m-1}$ represents an element in the Jacobi ring of W_q which maps to*

$$\sigma_l \sigma_{2m-1-l} - \sigma_{l-1}\sigma_{2m-l} + \cdots + (-1)^l \sigma_{2m-1} = q$$

inside $QH^*(X)$ under the isomorphism (3).

This is an easy consequence of quantum Schubert calculus on the quadric (which can be deduced from the quantum Chevalley formula of [FW04]).

Finally, in Sec. 6 we recall a partial compactification of the Hori-Vafa mirror defined by Gorbounov and Smirnov. We then show the following corollary.

Corollary 4. *The partially compactified LG model defined in Gorbounov and Smirnov is related to the formula* (2) *by a change of coordinates. In particular the Gorbounov and Smirnov LG model is isomorphic to the LG model defined in [Rie08].*

Together with Cor. 4, the work of Gorbounov and Smirnov implies a part of the mirror conjecture stated in [Rie08, Conjecture 8.1] for the groups $\mathrm{Spin}_{2m+1}(\mathbb{C})$ with maximal parabolic $P = P_{\omega_1}$, see Sec. 7.

2 Notations and Definitions

The LG model for $Q_{2m-1} = \mathrm{Spin}_{2m+1}/P_{\omega_1}$ defined in [Rie08] takes place on an open Richardson variety inside the Langlands dual flag variety PSp_{2m}/B_-. We let $G = \mathrm{PSp}_{2m}(\mathbb{C})$, since this is the group we will primarily be working with. Then $G^\vee = \mathrm{Spin}_{2m+1}(\mathbb{C})$ and $Q_{2m-1} = G^\vee/P^\vee$ for the parabolic subgroup $P^\vee$ associated to the first node of the Dynkin diagram of type B_m :

$$\underset{1}{\circ} - \underset{2}{\circ} - \cdots - \circ \Rightarrow \underset{m}{\circ} \; .$$

Let $V = \mathbb{C}^{2m}$ with fixed symplectic form

$$J = \begin{pmatrix} & & & & -1 \\ & & & 1 & \\ & & \iddots & & \\ & -1 & & & \\ 1 & & & & \end{pmatrix}.$$

For $G = \mathrm{PSp}(V, J)$ we fix Chevalley generators $(e_i)_{1\leq i \leq m}$ and $(f_i)_{1 \leq i \leq m}$. To be explicit we embed $\mathfrak{sp}(V, J)$ into $\mathfrak{gl}(V)$ and set

$$e_i = E_{i,i+1} + E_{2m-i,2m-i+1}, \quad \text{for } i = 1, \dots, m-1, \text{ and } \; e_m = E_{m,m+1}.$$

and $f_i := e_i^T$, the transpose matrix, for every $i = 1, \dots, m$. Here $E_{i,j} = (\delta_{i,k}\delta_{l,j})_{k,l}$ is the standard basis of $\mathfrak{gl}(V)$. For elements of the group $\mathrm{PSp}(V)$, we will take matrices to represent their equivalence classes. We have Borel subgroups $B_+ = TU_+$ and $B_- = TU_-$ consisting of upper-triangular and lower-triangular matrices in $\mathrm{PSp}(V)$, respectively. T is the maximal torus of $\mathrm{PSp}(V)$, consisting of diagonal matrices (d_{ij}) with non-zero entries $d_{i,i} = d^{-1}_{2m-i+1,2m-i+1}$.

The parabolic subgroup P we are interested in is the one whose Lie algebra $\mathfrak{p}$ is generated by all of the e_i together with $f_2, \dots, f_m$, leaving out f_1. Let $x_i(a) := \exp(ae_i)$ and $y_i(a) =$

$\exp(af_i)$. The Weyl group W of PSp_{2m} is generated by simple reflections s_i for which we choose representatives

$$\dot{s}_i = y_i(-1)x_i(1)y_i(-1).$$

We let W_P denote the parabolic subgroup of the Weyl group W, namely $W_P = \langle s_2, \dots, s_m \rangle$. The length of a Weyl group element w is denoted by $\ell(w)$. The longest element in W_P is denoted by w_P. We also let w_0 be the longest element in W. Next W^P is defined to be the set of minimal length coset representatives for W/W_P. The minimal length coset representative for w_0 is denoted by w^P. Let $\dot{w}$ denote the representative of $w \in W$ in G obtained by setting $\dot{w} = \dot{s}_{i_1} \cdots \dot{s}_{i_m}$, where $w = s_{i_1} \cdots s_{i_m}$ is a reduced expression.

We consider the open Richardson variety $\mathcal{R} := R_{w_P, w_0} \subset G/B_-$, namely

$$\mathcal{R} := R_{w_P, w_0} = (B_+ \dot{w}_P B_- \cap B_- \dot{w}_0 B_-)/B_-.$$

Let T^{W_P} be the W_P-fixed part of the maximal torus T, and fix $d \in T^{W_P}$. Then we also define

$$Z_d := B_- \dot{w}_0 \cap U_+ d \dot{w}_P U_-.$$

The map

$$\pi_R : Z_d \to \mathcal{R} : z \mapsto zB_-,$$

is an isomorphism from Z_d to the open Richardson variety.

Let q be the coordinate α_1 on the 1-dimensional torus T^{W_P}. The mirror LG model is a regular function on $\mathcal{R}$ depending also on q, so a regular function on $\mathcal{R} \times T^{W_P}$. It is defined as follows [Rie08],

$$\mathcal{F} : (u_1 \dot{w}_P B_-, d) \;\mapsto\; z = u_1 \dot{w}_P d \bar{u}_2 \in Z_d \;\mapsto\; \sum e_i^*(u_1) + \sum f_i^*(\bar{u}_2). \tag{4}$$

The corresponding map from $\mathcal{R}$, when d is fixed, is denoted

$$\mathcal{F}_d : \mathcal{R} \to \mathbb{C} : u_1 \dot{w}_P B_- \mapsto \mathcal{F}(u_1 \dot{w}_P B_-, d).$$

We also define another embedding

$$\pi_L : Z_d \to P\backslash \mathrm{PSp}(V) : \; z \mapsto Pz,$$

which maps Z_d isomorphically to an open subvariety of a big cell in $P\backslash \mathrm{PSp}(V)$. Note that $P\backslash \mathrm{PSp}(V)$ is canonically the isotropic Grassmannian of lines in V^*, when this Grassmannian is viewed as a homogeneous space via the action of $\mathrm{PSp}(V)$ from the right. Moreover the isotropic Grassmannian of lines is just $\mathbb{P}(V^*)$, since any line is automatically isotropic. Therefore the second embedding π_L has an advantage, that it is just an embedding into a projective space.

Definition 2.1 (Plücker coordinates)**.** First we introduce notation for the elements of W^P :

$$w_k = \begin{cases} s_k s_{k-1} \dots s_1 & \text{if } k \le m, \\ s_{2m-k} \dots s_{m-1} s_m s_{m-1} \dots s_1 & \text{if } m+1 \le k \le 2m-1. \end{cases}$$

The associated Plücker coordinates p_k are defined by

$$p_k(g) = \langle v_{\omega_1}^- g, w_k \cdot v_{\omega_1}^- \rangle.$$

Note that the Plücker coordinates are just the homogeneous coordinates on the projective space $\mathbb{P}(V^*)$. For a coset Pg they are given by the bottom row entries of g read from right to left. If $g = u_1 \dot{w}_P d \bar{u}_2$ then

$$(p_0(g) : \dots : p_{2m-1}(g)) = (p_0(\bar{u}_2) : \dots : p_{2m-1}(\bar{u}_2)).$$

Our goal is to express $\mathcal{F}$ as a rational function in the Plücker coordinates and $q = \alpha_1(d)$. We first illustrate our result in the smallest interesting example : that of the three-dimensional quadric Q_3.

3 The mirror to Q_3

A generic element of $Z_d := B_-\dot{w}_0 \cap U_+ d\dot{w}_P U_-$ can be written as $u_1 d\dot{w}_P \bar{u}_2$, where

$$\bar{u}_2 = y_1(a_1)y_2(c)y_1(b_1)$$

and a_1, c, b_1 are non-zero. Hence

$$\bar{u}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a_1+b_1 & 1 & 0 & 0 \\ cb_1 & c & 1 & 0 \\ a_1cb_1 & a_1c & a_1+b_1 & 1 \end{pmatrix}$$

The map $\pi_L : Z_d \to P\backslash \mathrm{PSp}(V) \cong \mathbb{P}(V^*)$ takes $z = u_1 d\dot{w}_P \bar{u}_2$ to $Pz = P\bar{u}_2$. This may be interpreted as taking z to the span of the reverse row vector corresponding to the last row of $\bar{u}_2$ after the identification $P\backslash \mathrm{PSp}(V) \cong \mathbb{P}(V^*)$. The Plücker coordinates of $\bar{u}_2$ are given by $p_0 = 1, p_1 = a_1 + b_1, p_2 = a_1c, p_3 = a_1cb_1$.

If we are interested in the image of Z_d in $\mathbb{P}(V^*)$ then first of all we can observe that it is independent of d. So we may choose for d the identity element, and restrict our attention to $B_-\dot{w}_0 \cap U_+\dot{w}_P U_-$. It turns out that the image of Z_d in $\mathbb{P}(V^*)$ is obtained from $\mathbb{P}(V^*)$ in coordinates

$$(p_0 : p_1 : p_2 : p_3) \in \mathbb{P}(V^*)$$

by removing $\{p_0 = 0\} \cup \{p_3p_0 - p_2p_1 = 0\} \cup \{p_3 = 0\}$. We call this variety $\check{X}^\circ$, and the isomorphism with Z_d in Prop. 9 shows that $\check{X}^\circ$ is also isomorphic to the open Richardson variety $\mathcal{R}$.

Let us denote by $W : \check{X}^\circ \times \mathbb{C}^* \to \mathbb{C}$ the map obtained from $\mathcal{F}$, see (4), after the identifications $\mathcal{R} \cong \check{X}^\circ$ and $(T)^{W_P} \cong \mathbb{C}^*$ via $d \mapsto \alpha_1(d) = q$. In this way we can compute the superpotential $\mathcal{F}$ from [Rie08] in the coordinates on $\mathbb{P}(V^*)$:

$$W = \frac{p_1}{p_0} + \frac{p_2^2}{p_1p_2 - p_0p_3} + q\frac{p_1}{p_3}.$$

This is equivalent to the following Landau-Ginzburg model of [GS13] :

$$g = y + yz + q\frac{x^2}{(xy-1)z}$$

via the change of coordinates :

$$x = \frac{p_0p_2}{p_1p_2 - p_0p_3}; y = \frac{p_1}{p_0}; z = \frac{qp_0}{p_3}.$$

Note that in [GS13] the superpotential denoted $\tilde{f}$ is g where z is replaced by $z+1$.

4 The mirror to Q_{2m-1}

We now write down $W_q = (\pi_L)_* \pi_R^* \mathcal{F}_d$ as a rational function on $\check{X}$, where $d \in (T)^{W_P}$ is such that $\alpha_1(d) = q$. We will then prove in the next section that the locus $\check{X}^\circ$ where it is defined is isomorphic to the open Richardson variety $\mathcal{R}$.

Proposition 5. *As a rational function on* $\check{X}$

$$W_q = \frac{p_1}{p_0} + \sum_{l=1}^{m-1} \frac{p_{l+1}p_{2m-1-l}}{p_lp_{2m-1-l} - p_{l-1}p_{2m-l} + \cdots + (-1)^l p_0p_{2m-1}} + q\frac{p_1}{p_{2m-1}}.$$

To prove the result, we first recall that

$$\pi_R^*\mathcal{F}_d : z = u_1\dot{w}_P d\bar{u}_2 \in Z_d \;\mapsto\; \sum e_i^*(u_1) + \sum f_i^*(\bar{u}_2).$$

Now $\bar{u}_2$ appearing in $u_1 d\dot{w}_P\bar{u}_2 \in Z_d\dot{w}_0$ can be assumed to lie in $U_- \cap B_+(\dot{w}^P)^{-1}B_+$. This is because we have two birational maps

$$\begin{aligned} \Psi_1 : U_- \cap B_+(\dot{w}^P)^{-1}B_+ \to P\backslash G : &\qquad \bar{u}_2 \mapsto P\bar{u}_2, \\ \Psi_2 : B_- \cap U^+ d\dot{w}^P U_- \to P\backslash G : &\quad b_- = u_1 d\dot{w}_P\bar{u}_2 \mapsto Pb_-, \end{aligned}$$

which compose to give $\Psi_1^{-1}\circ\Psi_2 : b_- \mapsto \bar{u}_2$. This gives a birational map

$$\Psi_1^{-1}\circ\Psi_2 : Z_d\dot{w}_0 \to U_- \cap B_+(\dot{w}^P)^{-1}B_+.$$

Now a generic element $\bar{u}_2$ in $U_- \cap B_+(\dot{w}^P)^{-1}B_+$ can be assumed to have a particular factorisation. The smallest representative w^P in W of $[w_0] \in W/W_P$ has the following reduced expression :

$$w^P = s_1 \dots s_{m-1}s_m s_{m-1}\dots s_m.$$

It follows that as a generic element of $U_- \cap B_+(\dot{w}^P)^{-1}B_+$, the element $\bar{u}_2$ can be assumed to be written as

$$\bar{u}_2 = y_1(a_1)\dots y_{m-1}(a_{m-1})y_m(c)y_{m-1}(b_{m-1})\dots y_1(b_1), \tag{5}$$

where $a_i, c, b_j \neq 0$. We have the following standard expression for the p_k on factorized elements, which is a simple consequence of their definition.

Lemma 6. *Fix $0 \le k \le 2m-1$ an integer. Then if $\bar{u}_2$ is of the form (5) we have*

$$p_k(\bar{u}_2) = \begin{cases} 1 & \text{if } k = 0, \\ a_1\dots a_{k-1}(a_k + b_k) & \text{if } 1 \le k \le m-1, \\ a_1\dots a_{m-1}cb_{m-1}\dots b_{2m-k} & \text{otherwise.} \end{cases}$$

□

We will also need the following :

Lemma 7. *If u_1 and $\bar{u}_2$ are as above then we have the following identities*

$$f_i^*(\bar{u}_2) = \begin{cases} a_i + b_i & \text{if } 1 \le i \le m, \\ c & \text{otherwise.} \end{cases} \tag{6}$$

$$e_i^*(u_1) = \begin{cases} 0 & \text{if } 2 \le i \le m, \\ e^t \frac{a_1+b_1}{a_1\dots a_{m-1}cb_{m-1}\dots b_1} & \text{if } i = 1. \end{cases} \tag{7}$$

Proof. Equation (6) is obtained immediately from the definition of $\bar{u}_2$. For Equation (7), notice that

$$\begin{aligned} e_i^*(u_1) &= \frac{\langle u_1^{-1}\cdot v_{\omega_i}^-, e_i \cdot v_{\omega_i}^-\rangle}{\langle u_1^{-1}\cdot v_{\omega_i}^-, v_{\omega_i}^-\rangle} \\ &= \frac{\langle e^h\dot{w}_P\bar{u}_2\cdot v_{\omega_i}^+, e_i\cdot v_{\omega_i}^-\rangle}{\langle e^h\dot{w}_P\bar{u}_2\cdot v_{\omega_i}^+, v_{\omega_i}^-\rangle}. \end{aligned}$$

Assume $2 \leq i \leq m$. Then $e_i^*(u_1) = 0$ if and only if $\langle \bar{u}_2 \cdot v^+_{\omega_i}, \dot{w}_P^{-1} e_i \cdot v^-_{\omega_i} \rangle = 0$. Now the vector $w_P^{-1} e_i \cdot v^-_{\omega_i}$ is in the μ-weight space of the i-th fundamental representation, where $\mu = w_P^{-1} s_i(-\omega_i)$. Moreover, $\bar{u}_2 \in B_+(\dot{w}^P)^{-1}B_+$, hence $\bar{u}_2 \cdot v^+_{\omega_i}$ can have non-zero components only down to the weight space of weight $(w^P)^{-1}(\omega_i) = w_P^{-1}(-\omega_i)$. Since $l(w_P^{-1}s_i) > l(w_P^{-1})$ for $2 \leq i \leq m$, this is higher than μ, which proves that $e_i^*(u_1) = 0$.

Now assume $i = 1$. We have

$$\begin{aligned}
e_1^*(u_1) &= \frac{\langle e^h \dot{w}_P \bar{u}_2 \cdot v^+_{\omega_1}, e_1 \cdot v^-_{\omega_1} \rangle}{\langle e^h \dot{w}_P \bar{u}_2 \cdot v^+_{\omega_1}, v^-_{\omega_1} \rangle} \\
&= (\omega_1 + \alpha_1 - \omega_1)(e^h) \frac{\langle \bar{u}_2 \cdot v^+_{\omega_1}, \dot{w}_P^{-1} e_1 \cdot v^-_{\omega_1} \rangle}{\langle \bar{u}_2 \cdot v^+_{\omega_1}, \dot{w}_P v^-_{\omega_1} \rangle} \\
&= e^t \frac{\langle \bar{u}_2 \cdot v^+_{\omega_1}, \dot{w}_P^{-1} e_1 \cdot v^-_{\omega_1} \rangle}{\langle \bar{u}_2 \cdot v^+_{\omega_1}, v^-_{\omega_1} \rangle}.
\end{aligned}$$

First look at the denominator. The only way to go from the highest weight vector $v^+_{\omega_1}$ of the first fundamental representation to the lowest $v^-_{\omega_1}$ is to apply $g \in B_+ w B_+$ for $w \geq (w^P)^{-1}$. Since $\bar{u}_2 \in B_+(\dot{w}^P)^{-1}B_+$, it follows that we need to take all factors of $\bar{u}_2$, and normalising $v^-_{\omega_1}$ appropriately, we get

$$\langle \bar{u}_2 \cdot v^+_{\omega_1}, v^-_{\omega_1} \rangle = a_1 \ldots a_{m-1} c b_{m-1} \ldots b_1.$$

Finally, we look at the numerator $\langle \bar{u}_2 \cdot v^+_{\omega_1}, \dot{w}_P^{-1} e_1 \cdot v^-_{\omega_1} \rangle$. The vector $\dot{w}_P^{-1} e_1 \cdot v^-_{\omega_1}$ has weight

$$\mu' = \dot{w}_P^{-1} s_1(-\omega_1) = \dot{w}_P^{-1}(-\epsilon_2) = \epsilon_2.$$

Write $w_P^{-1}s_1$ as a prefix $w' = s_1 s_2 \ldots s_{m-1} s_m s_{m-1} \ldots s_2$ of $(w^P)^{-1}$. We have $w' s_1 = (w^P)^{-1}$, hence the way from $v^+_{\omega_1}$ to $w' \cdot v^-_{\omega_1}$ is through s_1. From the shape of $\bar{u}_2$, it follows that $\langle \bar{u}_2 \cdot v^+_{\omega_1}, \dot{w}_P^{-1} e_1 \cdot v^-_{\omega_1} \rangle = a_1 + b_1$. □

Using the expression (4) of the superpotential from [Rie08], we immediately deduce from Lem. 7 a intermediate expression for the Landau-Ginzburg model W_q of the odd-dimensional quadric as a Laurent polynomial :

Proposition 8.

$$W_q = a_1 + \cdots + a_{m-1} + c + b_{m-1} + \cdots + b_1 + q \frac{a_1 + b_1}{a_1 \ldots a_{m-1} c b_{m-1} \ldots b_1} \quad \square \tag{8}$$

Now with the help of Lem. 6 and Prop. 8, we prove the second expression of W_q :

Proof of Prop. 5. From Lem. 6, it follows that for $\bar{u}_2$ as in (5)

$$p_{l+1} p_{2m-1-l}(\bar{u}_2) = \begin{cases} (a_{l+1} + b_{l+1})(a_1 \ldots a_l)^2 a_{l+1} \ldots a_{m-1} c b_{m-1} \ldots b_{l+1} & \text{if } l \leq m-2, \\ (a_1 \ldots a_{m-1} c)^2 & \text{if } l = m-1. \end{cases}$$

and

$$p_k p_{2m-1-k}(\bar{u}_2) = (a_k + b_k)(a_1 \ldots a_{k-1})^2 a_k \ldots a_{m-1} c b_{m-1} \ldots b_{k+1}.$$

Hence most terms in $\sum_{k=0}^{l} (-1)^k p_{l-k} p_{2m-1+k-l}(\bar{u}_2)$ cancel, and

$$\sum_{k=0}^{l} (-1)^k p_{l-k} p_{2m-1+k-l}(\bar{u}_2) = (a_1 \ldots a_l)^2 a_{l+1} \ldots a_{m-1} c b_{m-1} \ldots b_{l+1}.$$

This proves that

$$\frac{p_{l+1}p_{2m-1-l}}{p_l p_{2m-1-l} - p_{l-1}p_{2m-l} + \cdots + (-1)^l p_0 p_{2m-1}}(\bar{u}_2) = \begin{cases} a_{l+1} + b_{l+1} & \text{if } l \leq m-2, \\ c & \text{if } l = m-1. \end{cases}$$

For the first and last terms, we obtain

$$\frac{p_1}{p_0}(\bar{u}_2) = a_1 + b_1$$

and

$$\frac{p_1}{p_{2m-1}}(\bar{u}_2) = \frac{a_1 + b_1}{a_1 \ldots a_{m-1} c b_{m-1} \ldots b_1}$$

as easy consequences of Lem. 6. □

5 The open Richardson variety

We now prove that the affine subvariety $\check{X}^\circ$ defined in Equation (1) is isomorphic to the open Richardson variety $\mathcal{R}$.

Recall that $\check{X}^\circ = \check{X} \setminus D$, where

$$D := D_0 + D_1 + \ldots + D_{m-1} + D_m$$

and

$$\begin{aligned} D_0 &:= \{p_0 = 0\}, \\ D_l &:= \left\{p_l p_{2m-1-l} - p_{l-1}p_{2m-l} + \cdots + (-1)^l p_0 p_{2m-1} = 0\right\} \text{ for } 1 \leq l \leq m-1, \\ D_m &:= \{p_{2m-1} = 0\}. \end{aligned}$$

By definition, $\check{X}^\circ$ is the locus where W_q is regular. Since p_0 is non-zero on $\check{X}^\circ$, we may assume that $p_0 = 1$. Hence we have affine coordinates $(p_1, \ldots, p_{2m-1})$ on $\check{X}^\circ$. We also set, for $1 \leq j \leq 2m-1$:

$$r_j := \sum_{k=0}^{j} (-1)^k p_{j-k} p_{2m-1+k-j}.$$

Proposition 9. *The map $\pi_L \circ \pi_R^{-1} : \mathcal{R} \to \check{X}^\circ$ is an isomorphism.*

We will prove the result by constructing the inverse map. But first, let us check that the image of this map is indeed inside $\check{X}^\circ$ (and not just inside $\check{X}$). Clearly, $\mathcal{F}_d$ equals $W_q \circ \pi_L \circ \pi_R^{-1}$ as a rational map. Since $\mathcal{F}_d$ is regular on $\mathcal{R}$, it means that $W_q \circ \pi_L \circ \pi_R^{-1}$ also is, hence that W_q is regular on the image of $\pi_L \circ \pi_R^{-1}$. This proves that this image is contained in $\check{X}^\circ$.

We now define a map $\Phi : \check{X}^\circ \to B_-^{\mathrm{PSL}} \dot{w}_0$, where B_-^{PSL} is the Borel of lower triangular matrices in PSL_{2m}, so that $\Phi(p_1, \ldots, p_{2m-1}) \cdot v_j$ is equal to

$$\begin{cases} p_{2m-1} v_{2m} \text{ if } j = 1, \\ (-1)^j \frac{r_{j-1}}{r_{j-2}} + p_{2m-j} \left(\sum_{k=1}^{j-2} (-1)^{l+1} \frac{p_l}{r_{l-1}} v_{2m-l} + v_{2m} \right) \text{ if } 2 \leq j \leq m, \\ (-1)^j \frac{r_{2m-1-j}}{r_{2m-j}} v_{2m+1-j} + p_{2m-j} \left(\sum_{k=m+1}^{j-1} (-1)^k \frac{p_{k-1}}{r_{2m-k}} v_{2m+1-k} + \sum_{k=1}^{m-1} (-1)^{k-1} \frac{p_k}{r_{k-1}} v_{2m-k} \right) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } m+1 \leq j \leq 2m-1, \\ \frac{-1}{r_0} v_1 + \sum_{k=1}^{m-1} (-1)^k \frac{p_{2m-1-k}}{r_k} v_{k+1} + \sum_{k=1}^{m-1} (-1)^{k+1} \frac{p_k}{r_{k-1}} v_{2m-k} + v_{2m} \text{ if } j = 2m. \end{cases}$$

Let Ω be the open dense subset of $\check{X}^\circ$ where the coordinates $p_m, p_{m-1}, \ldots, p_{2m-2}$ do not vanish and define coordinates on Ω (as follows from Lem. 10) by

$$\begin{aligned} a_i &= \frac{p_{2m-1} r_i}{p_{2m-1-i} r_{i-1}} \text{ for all } 1 \leq i \leq m-1 \ ; \\ b_i &= \frac{p_{2m-1}}{p_{2m-1-i}} \text{ for all } 1 \leq i \leq m-1 \ ; \\ c &= \frac{p_m^2}{r_{m-1}}. \end{aligned}$$

Lemma 10. *For all $(p_1, \ldots, p_{2m-1}) \in \Omega$, $\Phi(p_1, \ldots, p_{2m-1})$ factorizes as $u_1 \dot{w}_P \bar{u}_2$, where*

$$\bar{u}_2 = y_1(a_1) \ldots y_{m-1}(a_{m-1}) y_m(c) y_{m-1}(b_{m-1}) \ldots y_1(b_1)$$

and u_1 equals

$$\begin{pmatrix} 1 & \frac{a_1+b_1}{a_1 \ldots a_{m-1} c b_{m-1} \ldots b_1} & \cdots & \frac{a_{m-1}+b_{m-1}}{a_1 \ldots a_{m-1} c b_{m-1}} & \frac{1}{a_1 \ldots a_{m-1}} & \cdots & \frac{1}{a_1} & \frac{-1}{a_1 \ldots a_{m-1} c b_{m-1} \ldots b_1} \\ & 1 & & & & & & \frac{-1}{a_1} \\ & & \ddots & & & & & \vdots \\ & & & 1 & & & & \frac{(-1)^m}{a_1 \ldots a_{m-1}} \\ & & & & 1 & & & (-1)^{m-1} \frac{a_{m-1}+b_{m-1}}{a_1 \ldots a_{m-1} c b_{m-1}} \\ & & & & & \ddots & & \vdots \\ & & & & & & 1 & \frac{a_1+b_1}{a_1 \ldots a_{m-1} c b_{m-1} \ldots b_1} \\ & & & & & & & 1 \end{pmatrix}$$

Proof. Using the definition of the y_i's, it is easy to check that $\bar{u}_2 \cdot v_j$ is equal to

$$v_j + \sum_{k=0}^{m-1-j} (a_{j+k} + b_{j+k}) b_{j+k-1} \ldots b_{j+1} b_j v_{j+k+1} + \sum_{k=0}^{m-1} a_{m-k} \ldots a_{m-1} c b_{m-1} \ldots b_j v_{m+1+k} \quad \text{if } 1 \leq j \leq m-1,$$

$$v_m + \sum_{k=0}^{m-1} a_{m-k} \ldots a_{m-1} c \text{ if } j = m,$$

$$v_j + (a_{2m-j} + b_{2m-j}) \sum_{k=0}^{2m-1-j} a_{2m-k-j} \ldots a_{2m-2-j} a_{2m-1-j} v_{j+1+k} \text{ if } m+1 \leq j \leq 2m,$$

Now a straightforward, if slightly tedious, computation shows that $\Phi(p_1, \ldots, p_{2m-1}) = u_1 \dot{w}_P \bar{u}_2$. ☐

We now need to prove that the entire image of Φ is in fact contained in $B_- \dot{w}_0 \cap U_+ \dot{w}_P U_-$ inside PSp_{2m}:

Lemma 11.

$$\Phi(\check{X}^\circ) \subset B_- \dot{w}_0 \cap U_+ \dot{w}_P U_-.$$

Proof. We first prove that $\Phi(\Omega) \subset B_- \dot{w}_0 \cap U_+ \dot{w}_P U_-$ inside PSp_{2m}. Indeed, from Lem. 10, we know that for all $(p_1, \ldots, p_{2m-1}) \in \Omega$, $\Phi(p_1, \ldots, p_{2m-1})$ factorises as $u_1 \dot{w}_P \bar{u}_2$, where u_1 and $\bar{u}_2$ are defined in the statement of the lemma. The factorisation of $\bar{u}_2$ means that

$\bar{u}_2$ is in U_- (hence in particular in PSp_{2m}). Now we prove that u_1 is also in PSp_{2m}, by showing directly that ${}^t u_1 J u_1 = J$ using the formula from Lem. 10. This is the result of a straightforward computation. It follows that $u_1 \in U_+$, hence $\Phi(p_1, \ldots, p_{2m-1}) \in U_+ \dot{w}_P U_- \subset \mathrm{PSp}_{2m}$ in this case. Now also $\Phi(p_1, \ldots, p_{2m-1}) \in B_-^{\mathrm{PSL}} \dot{w}_0 \cap \mathrm{PSp}_{2m} = B_- \dot{w}_0$. Therefore $\Phi(\Omega) \subset B_- \dot{w}_0 \cap U_+ \dot{w}_P U_-$.

Since Ω is open dense in $\check{X}^\circ$ we now have that $\Phi(\check{X}^\circ) \subset B_- \dot{w}_0 \cap \overline{U_+ \dot{w}_P U_-}$. Suppose there exists $(p_1, \ldots, p_{2m-1})$ in $\check{X}^\circ$ such that $\Phi(p_1, \ldots, p_{2m-1}) \notin U_+ \dot{w}_P U_-$. Then from Bruhat decomposition, we get $\Phi(p_1, \ldots, p_{2m-1}) \dot{w}_0^{-1} \in U_+ \dot{w} U_+$ with $w < w_P w_0$. It follows that we must have

$$\langle \Phi(p_1, \ldots, p_{2m-1}) \dot{w}_0^{-1} v_{\omega_1}^+, v_{\omega_1}^- \rangle = \langle \Phi(p_1, \ldots, p_{2m-1}) v_{\omega_1}^-, v_{\omega_1}^- \rangle = 0,$$

hence the lower-right corner of the matrix $\Phi(p_1, \ldots, p_{2m-1})$ has to be zero. But this coefficient is always 1, hence the result. □

We can now prove Prop. 9 :

Proof of Prop. 9. We have showed that the image of π_L is contained inside $\check{X}^\circ$. Moreover, we have defined a map $\Phi : \check{X}^\circ \to Z_1$, and a straightforward computation shows that it is the inverse of π_L. Hence π_L is an isomorphism. Since we saw in Sec. 2 that π_R is also an isomorphism, the proposition follows. □

The proof of Thm. 1 then follows from Prop. 5 and 9.

6 Comparison with the LG model of [GS13]

We now want to prove that our Landau-Ginzburg model (2) is isomorphic to the one stated in [GS13], which goes as follows

$$g = \sum_{i=1}^{m-1} y_i(1 + z_i) + q \frac{x^2}{(xy_1 y_2 \ldots y_{m-1} - 1) z_1 z_2 \ldots z_{m-1}}. \tag{9}$$

Note that as for Q_3, in [GS13] the superpotential denoted $\tilde{f}$ is g where the z_i are replaced by $z_i + 1$.

Assume $p_0 = 1$ and consider the change of variables :

$$\begin{aligned}
&y_1 = p_1; \qquad y_i = \frac{p_i}{p_{i-1}} \qquad \forall\, 2 \le i \le m-1; \\
&z_1 = \frac{q}{p_5}; \qquad z_i = \frac{\sum_{k=0}^{i-2} (-1)^k p_{i-2-k} p_{2m+1+k-i}}{\sum_{k=0}^{i-1} (-1)^k p_{i-1-k} p_{2m+k-i}} \qquad \forall\, 2 \le i \le m-1; \\
&x = \frac{p_m}{\sum_{k=0}^{m-1} (-1)^k p_{m-1-k} p_{m+k}}.
\end{aligned}$$

Proposition 12. *The above change of coordinates $\{x, y_i, z_i\} \mapsto \{p_i\}$ defines an isomorphism between the Landau-Ginzburg model* (9) *and ours* (2).

Proof. We have $y_1(1 + z_1) = p_1 + \frac{q}{p_{2m-1}}$, and

$$y_i(1 + z_i) = \frac{p_i p_{2m-i}}{\sum_{k=0}^{i-1} (-1)^k p_{i-1-k} p_{2m+k-i}}.$$

Moreover

$$xy_1 \dots y_{m-1} = \frac{\sum_{k=0}^{m-2}(-1)^k p_{m-2-k}p_{m+1+k}}{\sum_{k=0}^{m-1}(-1)^k p_{m-1-k}p_{m+k}},$$

$$z_1 \dots z_{m-1} = \frac{q}{\sum_{k=0}^{m-2}(-1)^k p_{m-2-k}p_{m+1+k}},$$

and

$$x^2 = \frac{p_m^2}{\left(\sum_{k=0}^{m-1}(-1)^k p_{m-1-k}p_{m+k}\right)^2},$$

hence

$$q\frac{x^2}{(xy_1y_2\dots y_{m-1}-1)z_1z_2\dots z_{m-1}} = \frac{p_m^2}{\sum_{k=0}^{m-1}(-1)^k p_{m-1-k}p_{m+k}}.$$

Hence the change of variables maps (9) to (2). Finally, it is clear that both domains of definition are the same. □

This proves Cor. 4.

7 Consequences

Let $\mathcal{H}_A$ be the sheaf of regular functions of the trivial vector bundle with fiber $H^*(X,\mathbb{C})$ over $\mathbb{C}_\hbar^* \times \mathbb{C}_q^*$ the two-dimensional complex torus with coordinates $\hbar$ and q. The *A-model connection* is defined on $\mathcal{H}_A$ by

$$\begin{aligned} {}^A\nabla_{q\partial_q} &= q\frac{\partial}{\partial q} + \frac{1}{\hbar}p_1 \star_q \bullet \\ {}^A\nabla_{\hbar\partial_\hbar} &= \hbar\frac{\partial}{\partial \hbar} + \mathrm{gr} - \frac{1}{\hbar}c_1(TX) \star_q \bullet, \end{aligned}$$

where gr is a diagonal operator on $H^*(X)$ given by $\mathrm{gr}(\alpha) = k$ for $\alpha \in H^{2k}(X)$. Here we are using the conventions of [Iri09]. Let $\mathcal{H}_A^\vee$ be the vector bundle on $\mathbb{C}_\hbar^* \times \mathbb{C}_q^*$ defined by $\mathcal{H}_A^\vee = j^*\mathcal{H}_A$ for $j : (\hbar, q) \mapsto (-\hbar, q)$. This vector bundle with the pulled back connection ${}^A\nabla^\vee = j^*({}^A\nabla)$ is dual to $(\mathcal{H}_A, {}^A\nabla)$ via the flat non-degenerate pairing,

$$\langle \sigma_i, \sigma_j\rangle = (2\pi i\hbar)^N \int_{[X]} \sigma_i \cup \sigma_j = (2\pi i \hbar)^N \delta_{i+j,N},$$

where $N = 2m-1$ is the dimension of $\check{X}^\circ$. The dual A-model connection ${}^A\nabla^\vee$ defines a system of differential equations called the (small) quantum differential equations

$${}^A\nabla^\vee_{q\partial_q} S = 0. \tag{10}$$

Define the $\mathbb{C}[\hbar^{\pm 1}, q^{\pm 1}]$-module

$$G = \Omega^N(\check{X}^\circ)[\hbar^{\pm1}, q^{\pm1}]/(d - \frac{1}{\hbar}dW_q \wedge \bullet)\Omega^{N-1}(\check{X}^\circ)[\hbar^{\pm1}, q^{\pm1}],$$

where $\Omega^k(\check{X}^\circ)$ is the space of holomorphic k-forms on $\check{X}^\circ$. We denote by $\mathcal{H}_B$ the sheaf with global sections G. Because W_q is cohomologically tame [GS13], G is a free $\mathbb{C}[\hbar^{\pm1}, q^{\pm1}]$-module of rank $2m$ (cf. [Sab99]), and $\mathcal{H}_B$ a trivial vector bundle of that dimension. It has a (Gauss-Manin) connection given by

$$
\begin{aligned}
{}^B\nabla_{q\partial_q}[\eta] &= q\frac{\partial}{\partial q}[\eta] + \frac{1}{\hbar}\left[q\frac{\partial W_q}{\partial q}\eta\right] \\
{}^B\nabla_{\hbar\partial_\hbar}[\eta] &= \hbar\frac{\partial}{\partial \hbar}[\eta] - \frac{1}{\hbar}\left[W_q\eta\right].
\end{aligned}
$$

Let ω be the canonical N-form on $\check{X}^\circ$.

Corollary 13. *The two bundles with connection* $(\mathcal{H}_A, {}^A\nabla)$ *and* $(\mathcal{H}_B, {}^B\nabla)$ *are isomorphic via* $\sigma_i \mapsto [p_i\omega]$*.*

Proof. The corollary is a consequence of the isomorphism of our LG model W_q and the one of [GS13] (see Cor. 4) together with the results of Gorbounov and Smirnov. □

Let Γ_0 be a compact oriented real N-dimensional submanifold of $\check{X}^\circ$ representing a cycle in $H^N(\check{X}^\circ, \mathbb{Z})$ dual to ω, in the sense that $\frac{1}{(2\imath\pi)^N}\int_{\Gamma_0}\omega = 1$. Then :

Corollary 14. *The integral*

$$
S_0(z,q) = \frac{1}{(2i\pi z)^N}\int_{\Gamma_0} e^{\frac{W_q}{\hbar}}\omega
$$

is a solution to the quantum differential equation (10)*.*

This implies part of [Rie08, Conj. 8.1] for odd-dimensional quadrics.

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