Representations of $\mathfrak{sl}(2,\mathbb{C})$ in category $\mathcal{O}$ and master symmetries

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Abstract

In this paper, we first give a short account on the indecomposable $\mathfrak{sl}(2,\mathbb{C})$ modules in the Bernstein-Gelfand-Gelfand (BGG) category $\mathcal{O}$. We show these modules naturally arise for homogeneous integrable nonlinear evolutionary systems. We then develop an approach to construct master symmetries for such integrable systems. This method naturally enables us to compute the hierarchy of time-dependent symmetries. We finally illustrate the method using both classical and new examples. We compare our approach to the known existing methods used to construct master symmetries. For the new integrable equations such as a Benjamin-Ono type equation, a new integrable Davey-Stewartson type equation and two different versions of (2+1)-dimensional generalised Volterra Chains, we generate their conserved densities using their master symmetries.

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1 Introduction

One of important hidden properties of integrable nonlinear evolution equations is the existence of infinitely many commuting symmetries, which has been used as a criterion to tackle the classification problems of integrable systems. It has produced fruitful results (see for example review papers [2, 45, 33, 32, 44, 30, 50] and the references therein). One way to generate these commuting symmetries is to use the well-known recursion operators [1, 38], which map a symmetry to a new symmetry. The Nijenhuis property of recursion operators enables us to construct an abelian Lie algebra of symmetries [15, 23]. This paper is devoted to the construction of master symmetries, which is an alternative method to produce this hierarchy of mutually commuting symmetries.

A master symmetry is an evolutionary vector field τ whose adjoint action ad_τ maps a symmetry to a new symmetry. The concept of master symmetries was first introduced in [10], where the authors constructed a master symmetry for the Benjamin-Ono equation and showed the equation has no recursion operator of polynomial type. This concept was further developed in [5, 8]. A recent review on it can be found in [2]. For example, the Burgers equation

$$u_t = u_{xx} + uu_x$$

possesses a master symmetry

$$\tau = x(u_{xx} + uu_x) + \frac{1}{2}u^2.$$

Starting from $a_0 = u_x$, the recurrence procedure $a_{n+1} = \mathrm{ad}_\tau a_n$ generates the hierarchy of commuting symmetries for the Burgers equation. This procedure is also called τ-scheme [8].

There are several methods to construct master symmetries. For homogeneous equations, master symmetries arise from applying their recursion operators to their corresponding scaling symmetry (see Definition 4). They can also be viewed as non-isospectral flows obtained from the Lax representation [40, 22]. Without having these extra structures at hand, master symmetries can be constructed based on the existence of time dependent higher order symmetries [16, 37]. The Burgers equation possesses infinite many t-dependent symmetries denoted by $S_n, n = 1, 2, \cdots$ and S_n is polynomial with respect to t of degree n. We list a few of them:

$$
\begin{aligned}
&S_1 = tu_x + 1;\\
&S_2 = t^2(u_{xx} + uu_x) + t(xu_x + u) + x;\\
&S_3 = t^3\left(u_{xxx} + \frac{3}{2}uu_{xx} + \frac{3}{2}u_x^2 + \frac{3}{4}u^2u_x\right) + \frac{3}{2}t^2\left(xu_{xx} + xuu_x + 2u_x + \frac{1}{2}u^2\right)\\
&\quad + \frac{3}{4}t(x^2u_x + 2xu + 2) + \frac{3}{4}x^2;\\
&S_4 = t^4\left(u_{xxxx} + 2uu_{xxx} + 5u_xu_{xx} + \frac{3}{2}u^2u_{xx} + 3uu_x^2 + \frac{1}{2}u^3u_x\right)\\
&\quad + t^3\left(x(2u_{xxx} + 3uu_{xx} + 3u_x^2 + \frac{3}{2}u^2u_x) + 5u_{xx} + 6uu_x + \frac{1}{2}u^3\right)\\
&\quad + \frac{3}{2}t^2\left(x^2u_{xx} + x^2uu_x + xu^2 + 4xu_x + 2u\right) + \frac{1}{2}t(x^3u_x + 3x^2u + 6x) + \frac{1}{2}x^3.
\end{aligned}
$$

The master symmetry τ can be obtained from the coefficient of t^2 in S_3.

In order to systematically compute t-dependent symmetries of nonlinear evolution equation, Sanders and the author observed the existence of $\mathfrak{sl}(2,\mathbb{C})$ in [48] and applied the idea to some well-known equations such as the Burgers and Kadomtsev–Petviashvili equations. For the Burgers equation, we let $e = u_x$, $h = 2(xu_x + u)$ and $f = -(x^2u_x + 2xu + 2)$, which are the t-coefficients in the above list. Then we have

$$[e,\ f] = h, \quad [h,\ e] = 2e, \quad [h,\ f] = -2f, \tag{1}$$

where the Lie bracket defined by (4), that is, these three elements form an $\mathfrak{sl}(2,\mathbb{C})$. Notice that

$$[u_t, f] = [u_{xx} + uu_x, f] = 4\tau + 8u_x$$

is a master symmetry since u_x commutes with all a_n. Later in [11], Finkel and Fokas adapted it (without explicitly mentioning $\mathfrak{sl}(2,\mathbb{C})$) for equations with nonlocal master symmetries such as the Sawada-Kotera equation. In this paper, we give justification of the method explaining where element f is from and give rigorous conditions when $[f, u_t]$ is a master symmetries. We then apply the method to various examples including some new $(2+1)$-dimensional equations presented in [18, 14].

Notice that elements e, h and f are all related to linear terms of t in S_i, $i = 1, 2, 3$. Let $w = \frac{1}{3}x^3u_x + x^2u + 2x$ the linear term in S_4. Then we have $[e, w] = f$. We show that the vector space spanned by all linear terms of S_n, $n = 1, 2, \cdots$ is an infinite dimensional module of $\mathfrak{sl}(2,\mathbb{C})$ in the Bernstein-Gelfand-Gelfand (BGG) category $\mathcal{O}$. Thus, we are able to present symmetries, master symmetries and their generalisations as elements of Lie modules. We shall refer this structure as the $\mathcal{O}$-scheme for the corresponding integrable equation. All vectors in this structure can be used to generate its (time dependent) symmetries.

The arrangement of the paper is as follows: In section 2 we give a short account on representations of $\mathfrak{sl}(2,\mathbb{C})$ in the BGG category $\mathcal{O}$. We then give definitions of symmetries and master symmetries for an evolutionary equation and recall how to generate a hierarchy of conserved densities using master symmetries in section 3. The main theoretical results are presented in section 4. We first show that for a homogeneous evolutionary equation, no matter it is integrable or not, there is a natural $\mathfrak{sl}(2,\mathbb{C})$ and further an infinite dimensional

module in the BGG category $\mathcal{O}$[1]. We then give an approach to construct master symmetries for such integrable equations and further obtain its $\mathcal{O}$-scheme. Finally, we show that the elements in this infinite dimensional module enables us to generate their time-dependent symmetries. In section 5, we use examples to demonstrate how the proposed approach apply to various examples of both $(1+1)$-dimensional and $(2+1)$-dimensional evolutionary equations. In the end, we give a short discussion on further research in this direction.

2 The BBG category of $\mathfrak{sl}(2,\mathbb{C})$ modules

In this section we give a short account on the Bernstein-Gelfand-Gelfand (BGG) category $\mathcal{O}$ of $\mathfrak{sl}(2,\mathbb{C})$ modules related to this paper. We refer the reader to the book [19, 24] for details.

The $\mathfrak{sl}(2,\mathbb{C})$ algebra, or simply written as $\mathfrak{sl}(2)$, is generated by three generators e, f, h satisfying the relations

$$[e,\ f] = h, \quad [h,\ e] = 2e, \quad [h,\ f] = -2f. \tag{2}$$

Let M be a $\mathfrak{sl}(2)$–module. For any $\lambda \in \mathbb{C}$, we define a subspace M_λ as

$$M_\lambda = \left\{ v \in M \middle| h \cdot v = \lambda v \right\}.$$

If λ is not an eigenvalue of the representation $h\cdot$, then $M_\lambda = \{0\}$. Whenever $M_\lambda \neq \{0\}$, we call λ a weight of h in M and M_λ its associated weight space.

The objects of the BGG category $\mathcal{O}$ of $\mathfrak{sl}(2)$–modules are M such that

- M is finitely generated;
- M is a weight module: M is the direct sum of its weight spaces;
- All weight spaces of M are finite dimensional.

It is clear that all finite dimensional modules lie in this category $\mathcal{O}$. In fact, all modules in this category are the direct sum of the indecomposable modules. The indecomposable modules inside the category $\mathcal{O}$ of $\mathfrak{sl}(2)$ are classified [19]. Here we list out some basic results for these nonisomorphic indecomposable modules.

2.1 The finite dimensional simple module $L(\lambda)$

The finite dimensional simple module $L(\lambda)$ with $\lambda \in \mathbb{N}$ has dimension $\lambda + 1$. It has 1-dimensional weight spaces, with weights $\lambda, \lambda-2, ..., -(\lambda-2)$ and $-\lambda$. One can choose basis vectors $v_i (0 \leq i \leq \lambda)$ so that

$$\begin{aligned} &f \cdot v_i = v_{i+1};\\ &h \cdot v_i = (\lambda - 2i) v_i;\\ &e \cdot v_i = i(\lambda - i + 1) v_{i-1};\\ &e \cdot v_0 = f \cdot v_\lambda = 0. \end{aligned}$$

Here v_0 is a highest weight vector with highest weight λ. It is (up to scaling) the unique vector vanishing under the action of e.

2.2 The Verma module $M(\lambda)$

For any highest weight $\lambda \in \mathbb{C}$, the Verma module $M(\lambda)$ is defined as the module generated by the universal enveloping algebra acting on a highest weight vector. The weights of the

[1] In recent paper [51], the author showed all indecomposable $\mathfrak{sl}(2,\mathbb{C})$ modules in the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ arises as quantised phase space of physical models.

Verma module $M(\lambda)$ are $\lambda, \lambda-2, \lambda-4, \cdots$, each with multiplicity one. One can choose basis vectors $v_i (i \geq 0)$ so that

$$\begin{aligned} f \cdot v_i &= v_{i+1}; \\ h \cdot v_i &= (\lambda - 2i) v_i; \\ e \cdot v_i &= i(\lambda - i + 1) v_{i-1}. \end{aligned}$$

When $\lambda \in \mathbb{N}$, we have $e \cdot v_{\lambda+1} = 0$. In this case the maximal submodule of $M(\lambda)$ is $M(-\lambda-2)$, and the finite dimension simple module $L(\lambda)$ is the quotient space $M(\lambda)/M(-\lambda-2)$.

When λ is not a positive integer, the Verma module $M(\lambda)$ is simple.

2.3 The dual Verma module $M^\vee(\lambda)$

Duality action within the category $\mathcal{O}$ maps the Verma module $M(\lambda)$ to its dual $M^\vee(\lambda)$. This module has the property that one can reach the highest weight vector from any vector by action of e, which is dual to the property of a Verma module $M(\lambda)$ that one can reach any vector from the highest weight vector by action of f. For any highest weight $\lambda \in \mathbb{C}$, the weights of the dual Verma module are the same as the Verma module. One can choose basis vectors $v_i (i \geq 0)$ so that

$$\begin{aligned} e \cdot v_i &= v_{i-1}; \\ h \cdot v_i &= (\lambda - 2i) v_i; \\ f \cdot v_i &= (i+1)(\lambda - i) v_{i+1}; \\ e \cdot v_0 &= 0. \end{aligned}$$

When $\lambda \in \mathbb{N}$, we have $f \cdot v_\lambda = 0$. In this case the maximal submodule of $M^\vee(\lambda)$ is $L(\lambda)$. The quotient space $M^\vee(\lambda)/L(\lambda)$ is isomorphic to the Verma module $M(-\lambda-2)$).

2.4 The projective module $P(-\lambda-2)$

For $\lambda \in \mathbb{N}$, there are also non-trivial projective modules $P(-\lambda-2)$. It has weights $\lambda, \lambda-2$, $\cdots, -\lambda$ with multiplicity one and the weights $-\lambda-2, -\lambda-4, \cdots$ with multiplicity two. Thus we can choose the basis vectors $v_i (i \geq 0)$ and $w_j (j \geq 0)$ such that

$$\begin{aligned} f \cdot v_i &= v_{i+1}; \\ h \cdot v_i &= (\lambda - 2i) v_i; \\ e \cdot v_i &= i(\lambda - i + 1) v_{i-1}; \\ f \cdot w_i &= w_{i+1}; \\ h \cdot w_i &= (-\lambda - 2 - 2i) w_i; \\ e \cdot w_i &= -i(\lambda + i + 1) w_{i-1}; \\ e \cdot w_0 &= v_\lambda. \end{aligned}$$

Notice that this indecomposable module $P(-\lambda-2)$ has a Verma submodule $M(\lambda)$, as well as a Verma submodule $M(-\lambda-2)$. The quotient space $P(-\lambda-2)/M(\lambda)$ is isomorphic to $M(-\lambda-2)$ and the quotient space $P(-\lambda-2)/M(-\lambda-2)$ is isomorphic to $M^\vee(\lambda)$.

3 Master symmetries of evolution equations

In this section we give the definitions of symmetries and master symmetries for scalar $1+1$-dimensional evolutionary partial differential equations meanwhile we also fix some notations. These definitions can be easily extend to multicomponent and $2+1$-dimensional evolutionary equations.

Let u be a scalar smooth function of the independent variables x and t. We consider an evolutionary differential equation of dependent variable u of the form

$$u_t = K[u], \tag{3}$$

where $[u]$ means that the smooth function K depends on u and x-derivatives of u up to some finite order.

All smooth functions depending on x, t, u and x-derivatives of u form a differential ring $\mathcal{F}$ with total x-derivation

$$D_x = \frac{\partial}{\partial x} + \sum_{k=0}^{\infty} u_{(k+1)x} \frac{\partial}{\partial u_{kx}},$$

where $u_{kx} = \partial_x^k u$. We simply write as $u, u_x, u_{xx}, \cdots$ instead of $u_{0x}, u_{1x}, u_{2x}, \cdots$ when k is small. The highest order of x-derivative we refer to the order of a given function. For any element $g \in \mathcal{F}$, we define an equivalence class (or a functional) $\int g$ by saying that g and h are equivalent if and only if $g - h \in \operatorname{Im} D_x$. The space of functionals, denoted by $\mathcal{F}'$, does not inherit the ring structure from $\mathcal{F}$.

A vector field (derivation) is said to be evolutionary if it commutes with the operator D_x. In the scalar case, such vector filed is completely determined by a smooth function $P \in \mathcal{F}$. We call it the characteristic of the vector field

$$\mathbf{v}_P = \sum_{k=0}^{\infty} D_x^k(P) \frac{\partial}{\partial u_{kx}}.$$

For any two evolutionary vector fields with characteristics P and Q, we define a Lie bracket as follows

$$[P,\ Q] = Q_\star(P) - P_\star(Q), \tag{4}$$

where $P_\star = \sum_j \frac{\partial P}{\partial u_{jx}} D_x^j$ is the Fréchet derivative of P. The evolutionary vector fields form a Lie algebra denoted by $\mathfrak{h}$. We simply say $P \in \mathfrak{h}$.

Definition 1. *An evolutionary vector field with characteristic P is a symmetry of system (3) if and only if*

$$\frac{\partial P}{\partial t} + \mathrm{ad}_K P = \frac{\partial P}{\partial t} + [K,\ P] = 0 \tag{5}$$

This equation is said to be integrable if it possesses infinitely many higher order symmetries.

Notice that if P is not explicitly dependent on t, the symmetry condition will reduce to $\mathrm{ad}_K P = 0$. Since the right hand side of equation (3) does not explicitly depend on time t, we can formally find t-dependent symmetry P as (cf. [16])

$$P = \exp(-t\mathrm{ad}_K)\phi$$

for any $\phi \in \mathcal{F}$ not explicitly depending on t. This expression makes sense when it reduces to a finite sum. In particular, if for some $m \in \mathbb{N}$ one has $ad_K^m \phi = 0$, then

$$P = \sum_{l=0}^{m-1} \frac{(-t)^l}{l!} \mathrm{ad}_K^l \phi$$

is a symmetry of equation (3). Moreover, the partial derivative of P with respect to t of order $r \in \mathbb{N}$, that is, $\frac{\partial^r P}{\partial t^r}$ is also a symmetry of K:

$$\frac{\partial^{r+1} P}{\partial^{r+1} t} + ad_K \frac{\partial^r P}{\partial t^r} = \frac{\partial^r}{\partial t^r}\left(\frac{\partial P}{\partial t} + \mathrm{ad}_K P\right) = 0.$$

In particular, this implies that if P is a polynomial of degree p in t, then $\frac{\partial^p P}{\partial t^p}$ is a time independent symmetry of K. Notice that it is essential for this argument to work that both K and ψ have no explicit time-dependence.

This observation was made by Fuchssteiner [16], where he defined the concept of K-generators: we say $\phi[x,u]$ is a K-generator of degree m if $\mathrm{ad}_K^{m+1}\phi = 0$ and $\mathrm{ad}_K^m\phi \neq 0$. The well-known master symmetries of equation (3) are K-generator of degree 1.

Definition 2. *Let a_0 be a time independent symmetry of equation (3), that is, $[a_0, K] = 0$, We say a t-independent evolutionary vector field τ is a master symmetry for (3) if a_n obtained by the recurrence procedure*

$$a_{n+1} = [\tau, a_n], \qquad n = 0, 1, 2, \cdots$$

mutually commute, i.e. $[a_i, a_j] = 0$, and there exists $k \in \mathbb{N}$ such that $a_k = K$.

Obviously, it follows from the definition that the existence of a master symmetry implies integrability. It also implies that

1. $\mathrm{ad}_K^2\tau = 0$, that is, τ is a K-generator of degree 1;
2. $t\mathrm{ad}_K\tau + \tau$ is a time dependent symmetry of equation (3).

These two implications have been used in the construction of a master symmetry for a given equation. We refer to [13] and its references for concrete examples.

The master symmetries can be used to construct the hierarchy of conserved densities for equation (3) if existing.

Definition 3. *We say $\rho \in \mathcal{F}'$ is a conserved density of equation (3) if*

$$\int \left(\frac{\partial \rho}{\partial t} + \rho_*(K)\right) = 0.$$

If ρ does not explicitly depend on t, the definition becomes to

$$0 = \int \rho_*(K) = \int \mathbf{v}_K(\rho) = \int K \cdot \frac{\delta \rho}{\delta u},$$

where $\frac{\delta \rho}{\delta u} = \sum_j (-D_x)^j \left(\frac{\partial \rho}{\partial u_{jx}}\right)$ is the variational derivative of ρ. For any $Q \in \mathfrak{h}$, we have (it can be easily obtained by Leibnitz rule of Lie derivatives [8])

$$\int \mathbf{v}_Q(\rho_*(K)) = \int [Q, K]\frac{\delta \rho}{\delta u} + \int K \frac{\delta \rho_*(Q)}{\delta u}. \tag{6}$$

Proposition 1. *Let ρ be a conserved density and Q be a symmetry for equation (3). If $\rho_*(Q) \not\equiv 0$, then it is also a conserved density of equation (3).*

Proof. According to Definition 3, we compute

$$\begin{aligned}
&\int \left(\frac{\partial \rho_*(Q)}{\partial t} + (\rho_*(Q))_*(K)\right) = \int \left(\left(\frac{\partial \rho}{\partial t}\right)_*(Q) + \frac{\partial Q}{\partial t}\frac{\delta \rho}{\delta u} + K\frac{\delta \rho_*(Q)}{\delta u}\right) \\
&= \int \mathbf{v}_Q(\frac{\partial \rho}{\partial t} + \rho_*(K)) + \int \left(\frac{\partial Q}{\partial t} + [K, Q]\right)\frac{\delta \rho}{\delta u} = 0,
\end{aligned}$$

where we used identity (6) and Definition 1. Thus we proved the statement. □

Taking $Q = \tau$ in (6) and using Definition 3, we can formulate the following statement:

Proposition 2. *Let ρ be a t-independent conserved density for equation (3). Assume that there is a t-independent evolutionary vector field τ such $\rho_*(\mathrm{ad}_\tau K) \equiv 0$. If $\rho_*(\tau) \not\equiv 0$, it is a conserved density of equation (3).*

Combining Proposition 1 and 2, If we know τ is a master symmetry, for any a_i we can start from a conserved density ρ_0 of $u_t = a_i$ and define

$$\rho_1 \equiv \begin{cases} \rho_{0*}(\mathrm{ad}_\tau a_i); & \text{when } \rho_{0*}(\mathrm{ad}_\tau a_j) \not\equiv 0 \\ \rho_{0*}(\tau); & \text{when } \rho_{0*}(\mathrm{ad}_\tau a_j) \equiv 0 \end{cases}$$

If $\rho_1 \neq 0$, we repeat the above procedure to define ρ_2. In this way we generate infinitely many conserved densities $\rho_i \neq 0$. If there exists $k \in \mathbb{N}$ such that $\rho_k = 0$, equation $u_t = a_i$ likely possesses only finite number of conserved densities.

For Hamiltonian systems, master symmetries can also be used to generate Hamiltonian vector fields [8].

Proposition 3. *Let $\mathcal{H}$ be a Hamiltonian operator. Assume that a_0 is a Hamiltonian vector field, that is, there exists $f \in \mathcal{F}'$ such that $a_0 = \mathcal{H}\frac{\delta f_0}{\delta u}$. So is the master symmetry τ. Then for all $n \in \mathbb{N}$, $a_n = \mathcal{H}\frac{\delta f_n}{\delta u}$, where $f_n \equiv f_{n-1*}(\tau)$.*

Proof. Since τ is a Hamiltonian vector field, there exists $g \in \mathcal{F}'$ such that $\tau = \mathcal{H}\frac{\delta g}{\delta u}$. Notice that

$$a_1 = [\tau, a_0] = [\mathcal{H}\frac{\delta g}{\delta u}, \mathcal{H}\frac{\delta f_0}{\delta u}] = \mathcal{H}\frac{\delta\{g, f_0\}}{\delta u} = \mathcal{H}\frac{\delta f_{0*}(\tau)}{\delta u},$$

where Poisson bracket $\{g, f_0\}$ is defined by Hamiltonian operator $\mathcal{H}$. So a_1 is a Hamiltonian vector field with Hamiltonian $f_1 \equiv f_{0*}(\tau)$. By induction, we can prove the statement. □

Finally, we define the homogeneity for equation (3), which is a technical requirement for the approach to constructing master symmetries presented in the next section.

Definition 4. *We say equation (3) is homogeneous (or α-homogeneous) if there exist constant α and κ such that $[xu_x + \alpha u,\ K] = \kappa K$. For this fixed α, we call $xu_x + \alpha u$ a scaling symmetry for the equation.*

In next section, we show that we are able to construct $\mathfrak{sl}(2)$ around the scaling symmetry. Thus, in other word, when equation (3) is homogeneous, the right hand of the equation is a basis for the weight space $M_{2\kappa}$.

4 Construction of master symmetries

In this section, we'll present a method to construct master symmetries based on the $\mathfrak{sl}(2, \mathbb{C})$ module. In fact, the concept of K-generators also reminds one of the representation theory of $\mathfrak{sl}(2, \mathbb{C})$ [48]. In what follows, we first present a natural $\mathfrak{sl}(2)$ for homogeneous equations and further construct an infinite dimensional module in the BGG category $\mathcal{O}$.

Lemma 1. *Given a scaling $h = 2(xu_x + \alpha u)$, where α is constant, the elements $e = u_x$ and $f = -(x^2u_x + 2\alpha xu)$ form an $\mathfrak{sl}(2, \mathbb{C})$.*

Proof. It is easy to check the three given elements h, e and f satisfy the relations (2). □

Remark 1. *Note that for given h and e in the above lemma, the element f is not unique. In fact they form $\mathfrak{sl}(2, \mathbb{C})$ with any $f' = f + f_0$ when f_0 satisfies $[e, f_0] = 0$ and $[h, f_0] = -2f_0$. For example, in (1), instead of $-(x^2u_x + 2xu)$ we take $f = -(x^2u_x + 2xu + 2)$.*

For convenience, we sometimes denote this $\mathfrak{sl}(2, \mathbb{C})$ as $\mathfrak{sl}_d(2)$ indicating its three elements lie in the diagonal direction (see Diagram 1). Notice that $\mathrm{ad}_e w = \mathrm{ad}_{u_x} w = -\frac{\partial w}{\partial x}$ for $w \in \mathfrak{h}$. So we can compute the inverse action of ad_e. This means that for any $g \in \mathfrak{h}$ we are able to find $w \in \mathfrak{h}$ such that $\mathrm{ad}_e w = g$.

Lemma 2. *Let $w = \frac{1}{3}\left(x^3u_x + 3\alpha x^2u\right)$. We have $\mathrm{ad}_e w = f$. Moreover,*

$$\mathrm{ad}_f^n w = \frac{n!}{3}\left(x^{n+3}u_x + (n+3)\alpha x^{n+2}u\right), \quad n = 0, 1, 2, \cdots \tag{7}$$

and $\mathrm{ad}_h \mathrm{ad}_f^n w = -2(n+2)\mathrm{ad}_f^n w$.

Proof. By direct computation we have $\mathrm{ad}_e w = -\frac{\partial w}{\partial x} = f$ and $\mathrm{ad}_h \mathrm{ad}_f^n w = -2(n+2)\mathrm{ad}_f^n w$.

We now prove (7) by induction. It is obviously true for $n = 0$. Assume that the formula is valid for n. Then for $n+1$, we have

$$\begin{aligned}
\mathrm{ad}_f^{n+1} w &= [f,\ \mathrm{ad}_f^n w] = [-(x^2 u_x + 2\alpha x u),\ \frac{n!}{3}\left(x^{n+3}u_x + (n+3)\alpha x^{n+2}u\right)] \\
&= \frac{n!}{3}\left(x^2 D_x\left(x^{n+3}u_x + (n+3)\alpha x^{n+2}u\right) + 2\alpha x\left(x^{n+3}u_x + (n+3)\alpha x^{n+2}u\right)\right) \\
&\quad -\frac{n!}{3}\left(x^{n+3}D_x(x^2u_x + 2\alpha x u) + (n+3)\alpha x^{n+2}(x^2u_x + 2\alpha x u)\right) \\
&= \frac{n!}{3}\left((n+1)x^{n+4}u_x + (n+1)(n+4)\alpha x^{n+3}u\right) = \frac{(n+1)!}{3}\left(x^{n+4}u_x + (n+4)\alpha x^{n+3}u\right),
\end{aligned}$$

which is the right hand side of (7) for $n+1$. So we proved the statement. □

Remark 2. *In Lemma 2, the element w satisfying $\mathrm{ad}_e w = f$ and $\mathrm{ad}_h w = -4w$ is not unique. We can take $w + w_0$ instead of w as long as $[e, w_0] = 0$ and $[h, w_0] = -4w_0$.*

From the above lemma, we know that all $\mathrm{ad}_f^n w$ are independent over $\mathbb{C}$. They generate an infinite dimensional space over $\mathbb{C}$. Together with Lemma 1, we get the following $\mathfrak{sl}_d(2)$ module:

$$e \underset{e}{\overset{f}{\rightleftarrows}} h \underset{e}{\overset{f}{\rightleftarrows}} f \underset{e}{\leftarrow} w \underset{e}{\overset{f}{\rightleftarrows}} \mathrm{ad}_f w \underset{e}{\overset{f}{\rightleftarrows}} \mathrm{ad}_f^2 w \cdots\cdots, \tag{8}$$

where we used the following notation:

Notation 1. *In a diagram, $f \xrightarrow{N} g$ means $\mathrm{ad}_N f = \gamma g$, where γ is a nonzero constant.*

This module is the quotient space $P(-4)/M(-4)$. This module and its freedom stated in both Remark 1 and Remark 2 play an important role in constructing master symmetries for homogeneous evolution equations and computing their time-dependent symmetries.

For a homogeneous evolutionary equation (3), no matter it is integrable or not, there exist the $\mathfrak{sl}_d(2)$ and further its infinite dimensional module shown (8). In the following theorem, we claim that $\mathrm{ad}_f K$ (or $\mathrm{ad}_{f'} K$ according to Remark 1) is a master symmetry for integrable equation $u_t = K[u]$ under certain conditions.

Theorem 1. *Take $a_0 = e$ and $a_1 \in \mathfrak{h}$ satisfying*

$$[a_0, a_1] = 0 \quad \textit{and} \quad [h, a_1] = [2(xu_x + \alpha u), a_1] = \lambda\ a_1,\ \textit{where } \lambda > 2$$

for a constant α. Let $f = -(x^2u_x + 2\alpha x u)$. Define $\tau = -\frac{1}{\lambda}\mathrm{ad}_f a_1$ and $a_{n+1} = [\tau, a_n]$ for $n = 0, 1, 2, \cdots$. Suppose that there exists an integer $s \geq 2$ such that

(i) $[a_{n-1}, a_n] = 0, \quad n = 1, 2, \cdots, s.$

(ii) *For any $n \in \mathbb{N}$, from $[[a_n, a_{n+1}], a_k] = 0$ for $0 \leq k \leq s-1$ it follows that there exists $m \leq 2s \in \mathbb{N}$ such that $[a_n, a_{n+1}] = \sum_{j=0}^{m} \gamma_j a_j$, where γ_j are constant.*

Then all a_n mutually commute, that is, $[a_i, a_j] = 0, \quad i, j = 0, 1, 2, \cdots$.

Before we give the proof of this theorem, we first check the definition for $a_1 = [\tau, a_0]$ is consistent with the definition of τ. Indeed,

$$[\tau, a_0] = [\tau, e] = -\frac{1}{\lambda}[[f, a_1], e] = \frac{1}{\lambda}\left([[e, f], a_1] + [[a_1, a_0], f]\right) = \frac{1}{\lambda}[h, a_1] = a_1.$$

The proof of this theorem is very similar to the one for τ-scheme in Dorfman's book [8]. However, to make the paper readable, we write it in details and begin with two lemmas.

Lemma 3. *Assume that a_n are defined as in Theorem 1. Then $[h, a_n] = (n\lambda - 2n + 2)a_n$ for $n = 0, 1, 2, \cdots$.*

Proof. We prove the statement by induction. When $n = 0$, it is valid by assumption. Assume it works for $n = l$. When $n = l + 1$, we get by the Jacobi identity

$$\begin{aligned}[h, a_{n+1}] &= [h, [\tau, a_n]] = -[\tau, [a_n, h]] - [a_n, [h, \tau]] \\ &= (n\lambda - 2n + 2)[\tau, a_n] - (\lambda - 2)[a_n, \tau] = ((n + 1)\lambda - 2n)a_{n+1}.\end{aligned}$$

Here we used the fact that $\tau = \frac{1}{\lambda}[a_1, f]$ has the weight $\lambda - 2$. □

Lemma 4. *Assume that a_n are defined as in Theorem 1 and $[a_{n-1}, a_n] = 0$ for $n \leq l$. Then $[a_m, a_n] = 0$ for all $m + n \leq 2l$.*

Proof. We again prove it by induction. When $l = 0$, this statement is trivially valid. Suppose that the statement is valid for some l. We now prove this statement for $l + 1$.

For all pairs of m and n, we assume $m \leq n$ without losing any generality. There are three cases: (i). $m + n \leq 2l$; (ii). $m + n = 2l + 1$ and (iii). $m + n = 2l + 2$. When $m + n \leq 2l$, this is covered by the induction assumption and thus there is no need to prove. For other two cases, without losing generality, we assume $m < n$. By the Jacobi identity, we have

$$\begin{aligned}[a_m, a_n] &= [a_m, [\tau, a_{n-1}]] = -[\tau, [a_{n-1}, a_m]] - [a_{n-1}, [a_m, \tau]] \\ &= -[\tau, [a_{n-1}, a_m]] + [a_{n-1}, a_{m+1}]. \end{aligned} \tag{9}$$

For case (ii), we know $[a_{n-1}, a_m] = 0$ since $m + n - 1 \leq 2l$. So we get

$$[a_m, a_n] = [a_{n-1}, a_{m+1}] = -[a_{m+1}, a_{n-1}].$$

By repeating applying this identity, we obtain $[a_m, a_n] = (-1)^{\frac{n-m-1}{2}}[a_l, a_{l+1}]$, which vanishes as an assumption.
For case (iii), we have $[a_{n-1}, a_m] = 0$ since $m + n - 1 \leq 2l + 1$. In the same way as we did for case (ii), we get

$$[a_n, a_m] = [a_{l+1}, a_{l+1}] = 0$$

and thus we complete the proof. □

We are now ready to complete the proof of Theorem 1.

Proof. To prove Theorem 1, we only need to show that $[a_{n-1}, a_n] = 0$ for all $n \in \mathbb{N}$. We prove it by induction.

From condition (i), this identity is valid for $n = 1, 2, \cdots, s$. Assume that it is valid for all $n \leq l$ and $l \geq s$. For $n = l + 1$, we have by the Jacobi identity

$$[[a_l, a_{l+1}], a_k] = -[[a_{l+1}, a_k], a_l] - [[a_k, a_l], a_{l+1}]$$

It follows from Lemma 4 that $[a_{l+1}, a_k] = [a_k, a_l] = 0$ since $l + 1 + k \leq 2l$ for $0 \leq k \leq s - 1$. This leads to

$$[[a_l, a_{l+1}], a_k] = 0, \qquad 0 \leq k \leq s - 1.$$

This implies that $[a_l, a_{l+1}] = \sum_{j=0}^{m} \gamma_j a_j$, where $m \leq 2s$ from condition (ii). We use ad_h acting on both sides of it and it follows that

$$\begin{aligned} 0 &= [h, [a_l, a_{l+1}] - \sum_{j=0}^{m} \gamma_j [h, a_j] \\ &= [[h, a_l], a_{l+1}] + [a_l, [h, a_{l+1}]] - \sum_{j=0}^{m} \gamma_j (j\lambda - 2j + 2) \\ &= \sum_{j=0}^{m} \left(\lambda + (2l - j)(\lambda - 2)\right) \gamma_j a_j. \end{aligned}$$

Since $l \geq s$ and $\lambda > 2$, we have $\lambda + (2l - j)(\lambda - 2) > 2$ for all $0 \leq j \leq m \leq 2s$. Thus all $\gamma_j = 0$ since a_j are independent over $\mathbb{C}$. This leads to $[a_l, a_{l+1}] = 0$ and we complete the proof. □

Following from this theorem we know $[e, a_n] = 0$ and a_n with weight $(\lambda-2)n+2$, where $\lambda > 2$. Therefore, starting from the highest weight vector a_n the space generated by $\mathrm{ad}_f^j a_n$, where $j = 0, 1, 2, \cdots$, is an $\mathfrak{sl}(2, \mathbb{C}) = \{e, h, f\}$ module in the BGG category $\mathcal{O}$. For a fixed n, if there exists $k \in \mathbb{N}$ such that $\mathrm{ad}_f^{k+1} a_n = 0$, then $\{\mathrm{ad}_f^j a_n, j = 0, 1, 2, \cdots, k\}$ is a finite dimensional simple module $L((\lambda-2)n+2) = L(k)$. In this case, we can find $\bar{w}$ such that $\mathrm{ad}_e \bar{w} = \mathrm{ad}_f^k a_n$ as in Lemma 2. Thus we have

$$\{\mathrm{ad}_f^j a_n, \mathrm{ad}_f^l \bar{w}, j = 0, 1, 2, \cdots, k; l = 0, 1, 2, \cdots\}$$

is a $\mathfrak{sl}(2)$ module $P(-(\lambda-2)n-4)/M(-(\lambda-2)n-4)$.

The immediate application of Theorem 1 lies in constructing a master symmetry for equation (3):

Theorem 2. *For homogeneous evolutionary equation $u_t = K[u]$ satisfying*

$$[u_x, K] = 0 \quad \textit{and} \quad [xu_x + \alpha u, K] = \kappa K, \quad \kappa > 1$$

for a certain constant α. Define

$$\tau = \frac{1}{2\kappa}[x^2 u_x + 2\alpha x u, K]$$

and $a_{n+1} = [\tau, a_n]$ with $a_0 = e = u_x$. Assume that

(i) $[[\tau, K], K] = 0$;

(ii) *If there is a Lie subalgebra $\mathfrak{h}'$ such that $a_n \in \mathfrak{h}'$ for all $n = 0, 1, 2, \cdots$. Moreover, for any $P, Q \in \mathfrak{h}'$ satisfying $[P, e] = [Q, e] = [P, K] = [Q, K] = 0$, it follows that $[P, Q] = 0$.*

Then τ is a master symmetry of equation $u_t = K[u]$, that is, all a_n mutually commute.

Proof. Following the definition, we have $a_1 = [\tau, a_0] = \frac{1}{2\kappa}[[-f, K], e] = K$. Thus the given conditions satisfy Theorem 1 for $s = 2$. This leads to τ is a master symmetry of the equation. □

This theorem provide us a method to construct master symmetries of homogeneous evolutionary equations. Moreover, starting from each a_n obtained from this theorem, we can construct an infinite dimensional $\mathfrak{sl}(2)$ module. We shall refer it as the $\mathcal{O}$-scheme and present it in the following diagram:

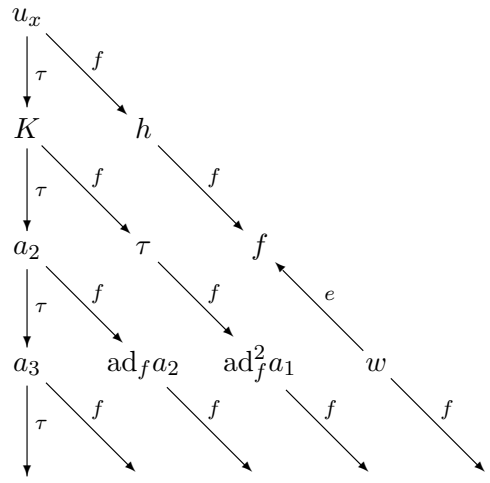


Diagram 1: The $\mathcal{O}$-scheme for $u_t = K[u]$

We are going to show that the elements in the horizontal directions are related to the t-coefficients of t-dependent symmetries for equation $u_t = K$. First we make a few remarks on Theorem 2.

Remark 3. *1. The condition $\kappa > 1$ is also trivially valid for all scalar homogeneous equations with linear terms of order greater than* 1.

2. Often we ignore the constant factor in τ (since it doesn't affect the commutativity of a_j) and simply say $\tau = [x^2 u_x + 2\alpha x u, K]$.

3. Note that condition (ii) *in Theorem 1 is a technical condition, which only used once in the proof. The corresponding condition in Theorem 2 is a special case.*

It is valid for scalar homogeneous polynomial equations with linear terms. From the construction, we know all a_n belong to the Lie subalgebra of homogeneous polynomial vector fields with nonzero linear terms. For such equations, all x, t-independent polynomial symmetries start with linear terms [47]. The Lie bracket of two symmetries, which is also a symmetry, has no linear terms. So it must be zero.

This argument can be generalised to multi-component homogeneous polynomial integrable systems with nonzero linear terms [3]. The examples in next section all belong to this class.

Lemma 5. *Under the conditions of Theorem 2, for any $n \in \mathbb{N}$ we have*

$$\mathrm{ad}_e \mathrm{ad}_K^{n-1} \mathrm{ad}_f^{n-1} a_j = \mathrm{ad}_K^n \mathrm{ad}_f^{n-1} a_j = 0, \quad j = 0, 1, 2, \cdots .$$

Proof. We prove the statement by induction. For $n = 1$, it is the result of Theorem 2. Assume it has been proved for all value less than n. We will prove the same statement for n. First notice that for all $m \in \mathbb{N}$ we have

$$\mathrm{ad}_K^m \mathrm{ad}_f = \sum_{i=0}^{m} \binom{m}{i} \mathrm{ad}_{(\mathrm{ad}_K^i f)} \mathrm{ad}_K^{m-i} = \sum_{i=0}^{2} \binom{m}{i} \mathrm{ad}_{(\mathrm{ad}_K^i f)} \mathrm{ad}_K^{m-i} \tag{10}$$

since $\mathrm{ad}_K^i f = 0$ for $i \geq 3$. Thus

$$\mathrm{ad}_K^n \mathrm{ad}_f^{n-1} a_j = \sum_{i=0}^{2} \binom{n}{i} \mathrm{ad}_{(\mathrm{ad}_K^i f)} \mathrm{ad}_K^{n-i} \mathrm{ad}_f^{n-2} a_j = \binom{n}{2} \mathrm{ad}_{(\mathrm{ad}_K^2 f)} \mathrm{ad}_K^{n-2} \mathrm{ad}_f^{n-2} a_j.$$

Here we used the induction assumption $\mathrm{ad}_K^{n-1} \mathrm{ad}_f^{n-2} a_j = 0$. Moreover, we know both $\mathrm{ad}_K^2 f$ and $\mathrm{ad}_K^{n-2} \mathrm{ad}_f^{n-2} a_j$ commute with K and $e = u_x$. Thus we have $\mathrm{ad}_{(\mathrm{ad}_K^2 f)} \mathrm{ad}_K^{n-2} \mathrm{ad}_f^{n-2} a_j = 0$ by the condition (ii) in Theorem 2.

We now show that $\mathrm{ad}_e \mathrm{ad}_K^{n-1} \mathrm{ad}_f^{n-1} a_j = 0$.

$$\begin{aligned} &\mathrm{ad}_e \mathrm{ad}_K^{n-1} \mathrm{ad}_f^{n-1} a_j = \mathrm{ad}_K^{n-1} \mathrm{ad}_e \mathrm{ad}_f^{n-1} a_j = \mathrm{ad}_K^{n-1} \mathrm{ad}_h \mathrm{ad}_f^{n-2} a_j + \mathrm{ad}_K^{n-1} ad_f \mathrm{ad}_e \mathrm{ad}_f^{n-2} a_j \\ &= \mathrm{ad}_K^{n-1} ad_f^2 \mathrm{ad}_e \mathrm{ad}_f^{n-3} a_j = \mathrm{ad}_K^{n-1} ad_f^{n-1} \mathrm{ad}_e a_j = 0. \end{aligned}$$

Here we used $\mathrm{ad}_K^{n-1} \mathrm{ad}_h \mathrm{ad}_f^{n-2} a_j = \mathrm{ad}_K^{n-2} \mathrm{ad}_h \mathrm{ad}_K \mathrm{ad}_f^{n-2} a_j = 0$ and the induction assumption $\mathrm{ad}_K^{n-1} \mathrm{ad}_f^{n-2} a_j = 0$. Thus we complete the proof. □

Notice that $\mathrm{ad}_f^n a_j$ may be zero for fixed j when n is bigger. For a given example, we can determine the values of n for fixed j. If $\mathrm{ad}_f^n a_j \neq 0$, they are K-generators of degree n following from Lemma 5. So the t-dependent symmetries for equation (3) are

$$\exp(-t\mathrm{ad}_K) \left(\mathrm{ad}_f^n a_j\right).$$

Note that there is a unique independent symmetry given the degree in t and the order of polynomial. E.g., $\frac{\partial}{\partial t} \left(\exp(-t\mathrm{ad}_K)\mathrm{ad}_f^2 a_1\right)$ is dependent with $\exp(-t\mathrm{ad}_K)\mathrm{ad}_f a_2$ in the sense that the difference between these two can be expressed as the sum of low order symmetries. Therefore we only need the elements on the contour of the diagram to generate the independent symmetries. The following theorem will answer when $\mathrm{ad}_f^j w$ for $j = 0, 1, 2, \cdots$, are K-generators.

Lemma 6. *Assume that there exists $r \in \mathbb{N}$ such that $\mathrm{ad}_K^r w = 0$ and $\mathrm{ad}_e \mathrm{ad}_K^{r-1} w = 0$. Under the conditions of Theorem 2, we have*

$$\mathrm{ad}_e \mathrm{ad}_K^{n+r-1} \mathrm{ad}_f^n w = \mathrm{ad}_K^{n+r} \mathrm{ad}_f^n w = 0, \quad n = 0, 1, 2, \cdots .$$

Proof. The proof of this theorem is similar to Lemma 5. We prove the statement by induction. For $n = 0$, it is assumption of the statement. Assume it has been proved for all value less than n. We will prove the same statement for n. Using (10), we get

$$\mathrm{ad}_K^{n+r} \mathrm{ad}_f^n w = \sum_{i=0}^{2} \binom{n+r}{i} \mathrm{ad}_{(\mathrm{ad}_K^i f)} \mathrm{ad}_K^{n+r-i} \mathrm{ad}_f^{n-1} w = \binom{n}{2} \mathrm{ad}_{(\mathrm{ad}_K^2 f)} \mathrm{ad}_K^{n+r-2} \mathrm{ad}_f^{n-1} w.$$

Here we used the induction assumption $\mathrm{ad}_K^{n+r-1} \mathrm{ad}_f^{n-1} w = 0$. Since both $\mathrm{ad}_K^{n+r-2} \mathrm{ad}_f^{n-1} w$ and $\mathrm{ad}_K^2 f$ commute with K and $e = u_x$, so $\mathrm{ad}_{(\mathrm{ad}_K^2 f)} \mathrm{ad}_K^{n+r-2} \mathrm{ad}_f^{n-1} w = 0$ by the condition (ii) in Theorem 2.

We now show that $\mathrm{ad}_e \mathrm{ad}_K^{n+r-1} \mathrm{ad}_f^n w = 0$.

$$\begin{aligned} \mathrm{ad}_e \mathrm{ad}_K^{n+r-1} \mathrm{ad}_f^n w &= \mathrm{ad}_K^{n+r-1} \mathrm{ad}_e \mathrm{ad}_f^n w = \mathrm{ad}_K^{n+r-1} \mathrm{ad}_h \mathrm{ad}_f^{n-1} w + \mathrm{ad}_K^{n+r-1} \mathrm{ad}_f \mathrm{ad}_e \mathrm{ad}_f^{n-1} w \\ &= \mathrm{ad}_K^{n+r-1} \mathrm{ad}_f^n \mathrm{ad}_e w = \mathrm{ad}_K^{n+r-1} \mathrm{ad}_f^n f = 0. \end{aligned}$$

Thus we proved the statement. □

Following from Diagram 1 and Lemma 5, instead of direct searching for the integer r such that $\mathrm{ad}_K^r w = 0$ we check whether there exists a constant $\gamma \neq 0$ such that

$$\mathrm{ad}_K w = \gamma \mathrm{ad}_f^2 K = \gamma \mathrm{ad}_f^2 a_1.$$

If this is the case, then w is a K-generator of degree 3, i.e., $\mathrm{ad}_K^4 w = 0$. Moreover, γ can be determine by the weight of K.

Lemma 7. *Let f h and w be defined as in Lemma 1 and 2. Assume that $[h, K] = 2\kappa K$, where $\kappa > 1$. If $\gamma \mathrm{ad}_K w = \mathrm{ad}_f^2 K$, then $\gamma = 2 - 4\kappa$.*

Proof. Notice that $\frac{\partial w}{\partial x} = -f$ and $\frac{\partial f}{\partial x} = -h$. We differentiate both sides of $\gamma \mathrm{ad}_K w = \mathrm{ad}_f^2 K$ with respect to x and it follows that

$$\gamma \mathrm{ad}_K(-f) = [-h, [f, K]] + [f, [-h, K]] = (-(2\kappa - 2) - 2\kappa)[f, K],$$

which implies that $\gamma = 2 - 4\kappa$ since $[f, K] \neq 0$. □

Therefore, we have the following result on t-dependent symmetries for equation (3):

Theorem 3. *Assume that the homogeneous equation $u_t = K[u]$ satisfies the conditions in Theorem 2. Let the evolutionary vector field w satisfy $\mathrm{ad}_e w = f$ and $\mathrm{ad}_K w = \frac{1}{2-4\kappa} \mathrm{ad}_f^2 K$. Then*

$$\frac{\partial^j}{\partial t^j} \exp(-t \mathrm{ad}_K) \left(\mathrm{ad}_f^n w \right), \quad \text{where } n = 0, 1, 2, \cdots, \text{and } j = 0, 1, 2, \cdots, n+2$$

are t-dependent symmetries of the equation.

Proof. Under the assumption we have shown that w is a K-generator of degree 3. From Lemma 2 we know that $\mathrm{ad}_f^j w \neq 0$ for all $j \in \mathbb{N}$ and we have $\mathrm{ad}_K^{n+4} \mathrm{ad}_f^n w = 0$ for all $n \in \mathbb{N}$ following from Lemma 6. Therefore, symmetry $\exp(-t \mathrm{ad}_K) \left(\mathrm{ad}_f^n w \right)$ is a polynomial in t of degree $n + 3$. Since the given equation does not depend x and t explicitly, the t partial derivatives of a symmetry is still a symmetry. □

We remark for some equations such as the Burgers equation studied next section, this theorem does not give us all time dependent symmetries.

5 Examples and applications

In this section, we'll demonstrate the approach in Section 4 construct a master symmetries for both multi-component and multi-dimensional evolutionary equations. We apply Theorem 2 to some new equations. We also include some well-known examples, e.g., the Burgers equation in order to compare our method to the existing methods.

5.1 $(1+1)$-dimensional differential equations

In this section, we use the approach described in Section 4 to construct known results on master symmetries for a few typical examples such as the Burgers equation (one component homogeneous equation), the Landau-Lifshitz equation (two-component non-homogeneous equation) and Benjamin-Ono equation (integro-differential equation). Meanwhile we also compare our approach to the existing methods.

5.1.1 The Burgers equation

Consider the Burgers equation

$$u_t = K = u_{xx} + uu_x, \tag{11}$$

which is homogeneous for $\alpha = 1$ and we have

$$[xu_x + u, K] = 2K.$$

According to Theorem 2, its master symmetry is

$$\tau = \frac{1}{4}[x^2u_x + 2xu, K] = x(u_{xx} + uu_x) + \frac{3}{2}u_x + \frac{1}{2}u^2$$

since

$$a_2 = [\tau, K] = 2u_{xxx} + 3uu_{xx} + 3u_x^2 + \frac{3}{2}u^2u_x$$

and $[[\tau, K], K] = 0$, and condition (ii) is trivially satisfied following from Remark 3.

We now discuss time dependent symmetries for equation (11). Let $w = \frac{1}{3}x^3u_x + x^2u$. Notice that

$$\mathrm{ad}_K w = -2(x^2uu_x + x^2u_{xx} + 3xu_x + xu^2 + u) = -\frac{1}{6}\mathrm{ad}_f^2 K.$$

According to Theorem 3, we can write down the time dependent symmetries for equation (11). For example,

$$\begin{aligned} S = \exp(-t\mathrm{ad}_K)w = &\frac{4}{3}t^3\left(2u_{4x} + 4uu_{3x} + 10u_xu_{xx} + 3u^2u_{xx} + 6uu_x^2 + u^3u_x\right) \\ &+t^2\left(x(4u_{3x} + 6uu_{xx} + 6u_x^2 + 3u^2u_x) + 8u_{xx} + 10uu_x + u^3\right) \\ &+2t\left(x^2(u_{xx} + uu_x) + x(3u_x + u^2) + u\right) + \frac{1}{3}x^3u_x + x^2u \end{aligned}$$

and $\frac{\partial^j}{\partial t^j}S$ for $j = 0, 1, 2$ are time dependent symmetries. Comparing to the list of t-dependent symmetries S_n, $n = 1, 2, 3, 4$, of the Burgers equation given in Introduction, we have

$$\frac{\partial S_4}{\partial t} = \frac{3}{2}S + 3S_2.$$

Following from Lemma 5, both $\mathrm{ad}_f a_2$ and $\mathrm{ad}_f^2 a_1$ are K-generators. However, they do not generate any new symmetries. Indeed, we have

$$\exp(-t\mathrm{ad}_K)\left(\mathrm{ad}_f a_2\right) = -\frac{3}{2}\frac{\partial^2 S}{\partial t^2} \quad \text{and} \quad \exp(-t\mathrm{ad}_K)\left(\mathrm{ad}_f^2 a_1\right) = 6\frac{\partial S}{\partial t}.$$

The master symmetries and time dependent symmetries for the Burgers equation are known in the literature (see, for example, [16, 49]). We now discuss how to explain other approaches under the frame of the $\mathfrak{sl}(2)$ representations.

In paper [48], the authors constructed two different $\mathfrak{sl}(2,\mathbb{C})$ around the scaling $h = xu_1+u$ with generators e_i, h_i and f_i, $i = 1, 2$ as follows:

$$\mathfrak{sl}_h(2,\mathbb{C}) : e_1 = K = u_{xx} + uu_x,\ f_1 = -x,\ h_1 = \frac{h}{2} = xu_x + u;$$
$$\mathfrak{sl}_v(2,\mathbb{C}) : e_2 = 1,\ f_2 = -2\tau = -2x(u_{xx} + uu_x) - 3u_x - u^2,\ h_2 = -h = -2(xu_x + u).$$

Here we can take any $f_2 + \beta u_x$, where β is constant, as f_2. To be consistent with τ defined previously, we choose the one in $\mathfrak{sl}_v(2,\mathbb{C})$. Together with $\mathfrak{sl}_d(2)$ given in Lemma 1, we present them in Diagram 2.

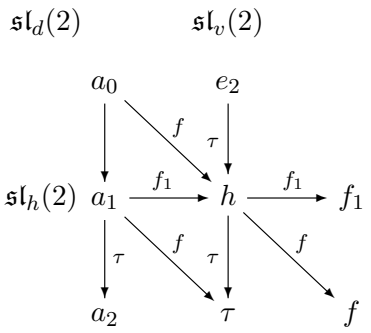


Diagram 2: Three $\mathfrak{sl}(2,\mathbb{C})$ for the Burgers equation

There three $\mathfrak{sl}(2)$ are the foundation of different approaches to constructing master symmetries. Dorfman's τ-scheme [8] is based on $\mathfrak{sl}_v(2)$. Since $[h_2, a_0] = -2a_0$ and $[e_2, a_0] = 0$, all symmetries a_j are the basis for Verma module $M(-2)$ of $\mathfrak{sl}_v(2)$. In the τ-scheme, the element τ is already known. So it can only be used to prove that the given τ is indeed a master symmetry.

Since $[a_n, K] = 0$ and $[h_1, a_n] = (n + 1)a_n$, the space spanned by

$$\{\mathrm{ad}^i_{f_1} a_n, \quad i = 0, 1, \cdots, n + 1\}$$

is an $(n + 2)$–dimensional irreducible representation of $\mathfrak{sl}_h(2)$. This is the theory behind Fuchssteiner's approach in [16]. According to him, a master symmetry for equation $u_t = K[u]$ is $[a_2, f_1]$. Indeed, for the Burgers equation,

$$[a_2, f_1] = 3x(u_{xx} + uu_x) + 6u_x + \frac{3}{2}u^2 = 3\tau + \frac{3}{2}u_x$$

is a master symmetries since $[u_x, a_n] = 0$.

The method proposed in this paper is based on $\mathfrak{sl}_d(2)$. In fact, this method was first used in [48] for Ibragimov-Shabat equation

$$u_t = u_{xxx} + 3u^2u_{xx} + 9uu_x^2 + 3u^4u_x. \tag{12}$$

and the Kadomtsev–Petviashvili equation (19) (we revisit it in section 5.2.1). Later in [11], Finkel and Fokas adapted it (without specifying $\mathfrak{sl}(2)$) to deal with nonlocal master symmetries such as the Sawada-Kotera equation. The advantage of using $\mathfrak{sl}_d(2)$ instead of other $\mathfrak{sl}(2)$ is that it is a free structure for homogeneous evolutionary equations. There are no need to search for e_2 and f_1. Indeed, for some integrable equations, they do not exist, for instance, the Ibragimov-Shabat equation. Since we do not reach out for f_1, consequently, we do not find all t-dependent symmetries.

Finally, we comment that using the quotient space $P(-4)/M(-4)$, see (8) instead of only $\mathfrak{sl}_d(2)$ enables us to find more time dependent symmetries (still may not be all). This leads to some t-dependent symmetries for equation (12) are missing in [48], which was addressed in [46].

5.1.2 The Landau-Lifshitz equation

Consider the Landau-Lifshitz equation

$$\mathbf{S}_t = K = \mathbf{S} \wedge \mathbf{S}_{xx} + \mathbf{S} \wedge J\mathbf{S}, \tag{13}$$

where $\mathbf{S}$ is a vector function of x and t in $\mathbb{R}^3$ with $|\mathbf{S}| = 1$, $\wedge$ denotes the vector product and J is 3×3 constant diagonal matrix. It is not homogeneous for only dependent variable $\mathbf{S}$. However, if we add equation $J_t = 0$, the system is homogeneous satisfying

$$[\begin{pmatrix} x\mathbf{S}_x + \alpha\mathbf{S} \\ xJ_x + 2J \end{pmatrix}, \begin{pmatrix} K \\ 0 \end{pmatrix}] = (2+\alpha)\begin{pmatrix} K \\ 0 \end{pmatrix}$$

Following from Theorem 2, we define τ as the first component (the second component can be dropped since it is always zero) of

$$\frac{1}{2(2+\alpha)}[\begin{pmatrix} x^2\mathbf{S}_x + 2x\alpha\mathbf{S} \\ x^2J_x + 4xJ \end{pmatrix}, \begin{pmatrix} K \\ 0 \end{pmatrix}] = \begin{pmatrix} x(\mathbf{S} \wedge \mathbf{S}_{xx} + \mathbf{S} \wedge J\mathbf{S}) + \frac{1+2\alpha}{2+\alpha}\mathbf{S} \wedge \mathbf{S}_x \\ 0 \end{pmatrix}.$$

We now check Condition (i) in Theorem 2. First we compute

$$\begin{aligned} a_2 = [\tau, K] = (2 + \frac{1+2\alpha}{2+\alpha}))\mathbf{S} \wedge (\mathbf{S}_x \wedge \mathbf{S}_{xx}) + 2\mathbf{S} \wedge (\mathbf{S} \wedge \mathbf{S}_{xxx}) + 2\mathbf{S} \wedge (\mathbf{S}_x \wedge J\mathbf{S}) \\ +(2 - \frac{1+2\alpha}{2+\alpha})\mathbf{S} \wedge (\mathbf{S} \wedge J\mathbf{S}_x) + \frac{1+2\alpha}{2+\alpha}\mathbf{S} \wedge J(\mathbf{S} \wedge \mathbf{S}_x) \end{aligned}$$

using $|\mathbf{S}| = 1$ and $\mathbf{S} \cdot \mathbf{S}_x = 0$. Direct computation shows that it commutes with K when $\alpha = 1$. Condition (ii) was explicitly proved in [17]. Thus we have the master symmetry

$$\tau = x(\mathbf{S} \wedge \mathbf{S}_{xx} + \mathbf{S} \wedge J\mathbf{S}) + \mathbf{S} \wedge \mathbf{S}_x, \tag{14}$$

which is the same as the one given in [17]. It generates the hierarchy of symmetries of every order starting from $a_0 = \mathbf{S}_x$.

These symmetries can also be produced by recursion operators. For the Landau-Lifshitz equation, there exists two weakly nonlocal [28] recursion operators of orders 2 and 3 and they are related by an elliptic curve equation [9]. We need to use both of them to generate all these symmetries obtained from the master symmetry (14).

We now compute the time dependent symmetries for equation (13). First we have

$$\mathrm{ad}_{\bar{K}} w = -\frac{1}{10}\mathrm{ad}_f^2 \bar{K}, \tag{15}$$

where

$$\bar{K} = \begin{pmatrix} K \\ 0 \end{pmatrix}, \quad w = \begin{pmatrix} \frac{1}{3}x^3\mathbf{S}_x + x^2\mathbf{S} \\ \frac{1}{3}x^3J_x + 2x^2J \end{pmatrix}, \quad f = -\begin{pmatrix} x^2\mathbf{S}_x + 2x\mathbf{S} \\ x^2J_x + 4xJ \end{pmatrix}.$$

Notice that the second component of $\mathrm{ad}_{\bar{K}} w$ is zero. So are all $\mathrm{ad}_{\bar{K}}\mathrm{ad}_f^n w$. We denote the first components of them by w_n and $\mathrm{ad}_K^{n+3} w_n = 0$. According to Theorem 3, the time dependent symmetries for equation (13) are

$$\frac{\partial^j}{\partial t^j}\exp(-t\mathrm{ad}_K)w_n, \quad \text{where } n = 0, 1, 2, \cdots, \text{and } j = 0, 1, 2, \cdots, n+1.$$

5.1.3 A Benjamin-Ono type equation

In [29], Mikhailov and Novikov classified the Benjamin-Ono-type equations with higher symmetries using the perturbative symmetry approach in symbolic representation. Here we construct a master symmetry for the following equation in their list:

$$u_t = K = H(u_{xx}) + D_x\left(\frac{1}{2}c_1u^2 + \frac{1}{2}c_2H(u^2) - c_2uH(u)\right), \tag{16}$$

where c_1 and c_2 are constant and H denotes the Hilbert transform

$$H(f) = \frac{1}{\mathrm{i}\pi}\int_{-\infty}^{\infty}\frac{f(y)}{y-x}dy, \qquad \mathrm{i}^2 = -1.$$

Notice that when $c_1 = 2$ and $c_2 = 0$, it reduces to the well-known Benjamin-Ono equation:

$$u_t = K = H(u_{xx}) + 2uu_x. \tag{17}$$

The higher order symmetries and conservation laws of the Benjamin-Ono type equations contain nested Hilbert transform and thus an appropriate extension of the differential ring is required. We refer [29] for the details of the extension. It is very similar to the one proposed by Mikhailov and Yamilov in [35] for $(2+1)$-dimensional equations, which we will describe in section 5.2.

We now apply Theorem 2 to construct a master symmetry of (16). First we notice that it is homogeneous and we have

$$[xu_x + u, K] = 2K.$$

In [29], it was proved that the symbolic representations of its symmetries start with linear term in u. According to Remark 1, condition (ii) in Theorem 2 is satisfied. Let

$$\tau = \frac{1}{4}[x^2u_x + 2xu, K] = xK + \frac{3}{2}H(u_x) + \frac{1}{2}c_1u^2 + \frac{1}{2}c_2H(u^2) - c_2uH(u). \tag{18}$$

We have

$$\begin{aligned} a_2 = [\tau, K] = {} & 2u_{xxx} + \frac{c_1}{2}D_x\left(3H(uu_x) + 3uH(u_x) + c_1u^3\right) \\ & + \frac{c_2}{2}D_x\left(-3H(u_xH(u)) - 3H(u)\ H(u_x) - 3c_1u^2H(u) + c_1H(u^3)\right) \\ & + \frac{c_2^2}{2}D_x\left(u^3 - 3H(u^2H(u)) + 3u(H(u))^2\right) \end{aligned}$$

and $[a_2, K] = 0$. Here we take into account the relation $H^2 = 1$ and the Hilbert-Leibnitz rule

$$H(fg) = fH(g) + gH(f) - H(H(f)H(g)).$$

Thus τ given by (18) is a master symmetry. When $c_1 = 2$ and $c_2 = 0$, we get the master symmetry of (17) in [10].

As we discussed in Section 3, we can use the master symmetry to construct conserved densities. Equation (16) is Hamiltonian with a Hamiltonian operator $\mathcal{H} = D_x$. Indeed,

$$u_t = D_x\frac{\delta\rho_1}{\delta u} = D_x\left(H(u_x) + \frac{1}{2}c_1u^2 + \frac{1}{2}c_2H(u^2) - c_2uH(u)\right),$$

where

$$\rho_1 = \frac{1}{2}\left(uH(u_x) + \frac{1}{3}c_1u^3 + c_2uH(u^2)\right).$$

Moreover, we have

$$\tau = D_x\left(x\frac{\delta\rho_1}{\delta u} + \frac{1}{2}H(u)\right) = D_x\frac{\delta(x\rho_1)}{\delta u}.$$

Let $\rho_0 = \frac{1}{2}u^2$, which corresponds to $a_0 = u_x = D_x\frac{\delta\rho_0}{\delta u}$. It follows from Proposition 3 that the symmetries $a_n = [\tau,\ a_{n-1}]$ are Hamiltonian vector fields and $a_n = D_x\frac{\delta\rho_n}{\delta u}$, where $\rho_n \equiv \rho_{n-1*}(\tau)$. We can check that $\rho_1 \equiv \rho_{0*}(\tau)$ and find

$$\begin{aligned}\rho_2 \equiv \rho_{1*}(\tau) &\equiv \frac{\delta(\rho_1)}{\delta u}D_x\left(\frac{1}{2}H(u) + x\frac{\delta(\rho_1)}{\delta u}\right) \equiv \frac{1}{2}\frac{\delta(\rho_1)}{\delta u}\left(H(u_x) + \frac{\delta(\rho_1)}{\delta u}\right)\\ &\equiv -u_x^2 + \frac{c_1}{4}\left(3u^2H(u_x) + \frac{c_1}{2}u^4\right) + \frac{c_2}{2}\left(-3uH(u)\ H(u_x) - c_1u^3H(u)\right)\\ &\quad + \frac{c_2^2}{4}\left(3u^2(H(u))^2 + \frac{u^4}{2}\right).\end{aligned}$$

When $c_1 = 2$ and $c_2 = 0$, we get the known conservation laws for the Benjamin-Ono equation (17) [10].

5.2 $(2+1)$-dimensional partial differential equations

The method of constructing master symmetries proposed in Section 4 is valid for $(2+1)$-dimensional partial differential equations. A typical example is the Kadomtsev-Petviashvili (KP) equation

$$u_t = u_{xxx} + 6uu_x + 3D_x^{-1}D_yu_y, \tag{19}$$

where the dependent variable u is a smooth function of independent variables x, y and t, and D_x^{-1} is the formal inverse of the total x-derivative. The main obstacle to directly extend the theories of $(1+1)$-dimensional nonlinear evolutionary equations to $(2+1)$-dimensional equations is the nonlocality. The concept of conservation laws and the symmetry approach for testing integrability [32] are mainly based on the locality. In 1998, Mikhailov and Yamilov addressed the nonlocality problem in [35]. They noticed that the operators D_x^{-1} and D_y^{-1} never appear alone but always in pairs like $D_x^{-1}D_y$ and $D_y^{-1}D_x$ for all known integrable equations and their hierarchies of symmetries and conservation laws. Based on this observation, they introduced the concept of quasi-local functions, which is a natural generalization of local functions. Using the symbolic representation, it was proved this observation is true for integrable equations obtained from certain scalar Lax operators [52].

We denote the derivatives of dependent variable u with respect to its independent variables x and y by $u_{ix,jy} = \partial_x^i\partial_y^ju$. For smaller i and j, we sometimes write the indexes out explicitly, that is, we write u_{xxy} and u instead of $u_{2x,1y}$ and $u_{0x,0y}$. All smooth functions depending on x, y, t, u and derivatives of u form a differential ring $\mathcal{F}$ with total x-derivation and y-derivation

$$D_x = \sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty}u_{(i+1)x,jy}\frac{\partial}{\partial u_{ix,jy}} \quad \text{and} \quad D_y = \sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty}u_{ix,(j+1)y}\frac{\partial}{\partial u_{ix,jy}}.$$

Let us denote

$$\Theta = D_x^{-1}D_y, \qquad \Theta^{-1} = D_y^{-1}D_x. \tag{20}$$

To define quasi-local functions $\mathcal{F}(\Theta)$, we consider a sequence of extensions of $\mathcal{F}$ as follows:

Let $\Theta\mathcal{F} = \{\Theta f :\ f \in \mathcal{F}\}$ and $\Theta^{-1}\mathcal{F} = \{\Theta^{-1}f :\ f \in \mathcal{F}\}$. We define $\mathcal{F}_0(\Theta) = \mathcal{F}$ and $\mathcal{F}_k(\Theta)$ is the ring closure of the union

$$\mathcal{F}_{k-1}(\Theta) \cup \Theta\mathcal{F}_{k-1}(\Theta) \cup \Theta^{-1}\mathcal{F}_{k-1}(\Theta).$$

Here the index k indicates the maximal depth of nesting the operator $\Theta^{\pm1}$ in the expression. Clearly, we have $\mathcal{F}_{k-1}(\Theta) \subset \mathcal{F}_k(\Theta)$. We now define $\mathcal{F}(\Theta) = \lim_{k\to\infty}\mathcal{F}_k(\Theta)$. However, for a given $f \in \mathcal{F}(\Theta)$, there exists k such that $f \in \mathcal{F}_k(\Theta)$. Note that $\mathcal{F}(\Theta)$ is not invariant under transformations of variables.

Since the dependent variable depends on two spatial variables x and y, we consider the scaling symmetry as $xu_x + \beta yu_y + \alpha u$, where α and β are constant. As a convention, we take the weight of x-derivative to be 1. Around h we can build two $\mathfrak{sl}(2)$ representations by choosing either $e = u_x$ or $e = u_y$. We present them in following two lemmas, which is similar to Lemma 1 and Lemma 2 for the $(1+1)$-dimensional case.

Lemma 8. *Given a scaling $h = 2(xu_x + \beta yu_y + \alpha u)$, where α and β are constant, the elements $e = u_x$ and $f = -(x^2u_x + 2\beta xyu_y + 2\alpha xu)$ form an $\mathfrak{sl}(2,\mathbb{C})$ with h. Moreover, there exists $w = \frac{1}{3}x^3u_x + \beta x^2yu_y + \alpha x^2u$. We have $\mathrm{ad}_e w = f$ and*

$$\mathrm{ad}_f^n w = \frac{n!}{3}\left(x^{n+3}u_x + (n+3)x^{n+2}(\beta yu_y + \alpha u)\right), \quad n = 0, 1, 2, \cdots \tag{21}$$

The proof of this lemma is the same as we did for Lemma 1 and Lemma 2. We won't repeat it again.

Notice that the role of x and y is equal. We can alternatively build up the $\mathfrak{sl}(2,\mathbb{C})$ by taking $e = u_y$ instead.

Lemma 9. *Assume that $\beta \neq 0$. Given a scaling $h = \frac{2}{\beta}(xu_x + \beta yu_y + \alpha u)$, where α and β are constant, when the elements $e = u_y$ and $f = -\frac{1}{\beta}(2xyu_x + \beta y^2u_y + 2\alpha yu)$ form an $\mathfrak{sl}(2,\mathbb{C})$ with h. Moreover, there exists $w = \frac{1}{3}y^3u_y + \frac{1}{\beta}y^2(xu_x + \alpha u)$. We have $\mathrm{ad}_e w = f$ and*

$$\mathrm{ad}_f^n w = \frac{n!}{3}\left(y^{n+3}u_y + \frac{n+3}{\beta}y^{n+2}(xu_x + \alpha u)\right), \quad n = 0, 1, 2, \cdots \tag{22}$$

The nonlocality can cause the invalidity of the Jacobi identity for the characteristics of nonlocal vector fields. This was first noticed in [48] when the authors systematically investigate the symmetry properties of the KP equation. To solve this problem, the notion of a ghost characteristic was introduced in [42, 39]. The ghost characteristics are the expressions in the kernels of D_x and/or of D_y. One of advantages of $\mathfrak{sl}(2)$ representations presented in Lemma 8 and 9 is that there are no such terms and thus the ghost problems will not appear.

5.2.1 The Kadomtsev–Petviashvili equation

We use our notation and rewrite the KP equation (19) as follows:

$$u_t = K = u_{xxx} + 6uu_x + 3\Theta u_y. \tag{23}$$

If ignoring the dependence of y, the equation reduces to the well-known Korteweg-de Vries (KdV) equation. As a natural generalisation of the KdV equation, its symmetry structure has been well studied in the literature. For example, its infinitely many t dependent symmetries are given in [5, 4]. Based on this, we presented its three different $\mathfrak{sl}(2)$ representations in [48], similar to the ones we discussed for the Burgers equation in section 5.1.1. Here we revisit this equation and construct its master symmetries using our approach using both $\mathfrak{sl}(2)$ representations described in Lemma 8 and 9.

The KP equation is homogeneous for scaling $h = 2yu_y + xu_x + 2u$ and

$$[h, K] = 3K.$$

It follows from Lemma 8 and 9 that there are two $\mathfrak{sl}(2,\mathbb{C})$ representations around h. When we use the $\mathfrak{sl}(2,\mathbb{C})$ in Lemma 9. According to Theorem 2, we define

$$\tau = \frac{1}{6}[xyu_x + y^2u_y + 2yu,\ K] = \frac{1}{2}y(u_{xxx} + 6uu_x + 3\Theta u_y) + xu_y + 2\Theta u,$$

which leads to the master symmetry for the KP equation in [36]. Indeed, the next symmetry, denoted by P, is

$$P = [\tau, K] = 6(u_{xxy} + \Theta^2u_y + 4uu_y + 2u_x\Theta u). \tag{24}$$

If we use the $\mathfrak{sl}(2,\mathbb{C})$ in Lemma 8, we have $\mathrm{ad}_K^3 f = \mathrm{ad}_K^3(x^2u_x + 4xyu_y + 4xu) \neq 0$. This implies that $[f, K]$ is not a master symmetry. As we mentioned in Remark 1, $f' = f + f_0$ also forms $\mathfrak{sl}(2)$ with elements e and h when $[e, f_0] = 0$ and $[h, f_0] = -2f_0$. Using the condition $\mathrm{ad}_K^3 f' = 0$, we are able to determine that

$$f_0 = y^2(u_{xxx} + 6uu_x + 3\Theta u_y) + 8y\Theta u + 2D_x^{-1}u$$

and

$$f' = f + f_0 = 2y\tau + x^2u_x + 2xyu_y + 4xu + 4y\Theta u + 2D_x^{-1}u.$$

This leads to

$$\tau' = \frac{1}{6}[f',\ K] = \frac{2}{3}yP + xK + 4u_{xx} + 8u^2 + 2u_xD_x^{-1}u + 6\Theta^2 u, \tag{25}$$

where P is defined by (24). This lies in the non-isospectral KP hierarchy given in [12] and it is a master symmetry. It is easy to see that

$$[\tau,\ u_x] = u_y \qquad \text{and} \qquad [\tau',\ u_x] = K.$$

Starting from u_x, the master symmetry τ' generates the hierarchy of symmetries with the reduction to the symmetry hierarchy of the KdV equation if u is independent of y.

In the following Diagram 3, we demonstrate the relations between these two $\mathfrak{sl}(2)$ and the corresponding master symmetries.

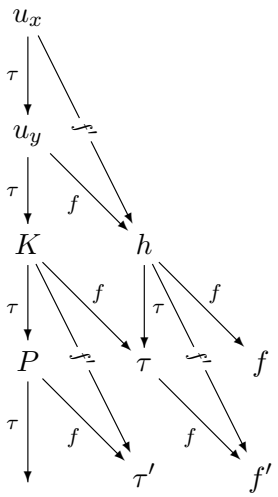


Diagram 3: Master symmetries for KP equation and two $\mathfrak{sl}(2,\mathbb{C})$

5.2.2 The noncommutative KP equation

Consider the noncommutative KP equation [6, 52]

$$u_t = K = u_{xxx} + 3uu_x + 3u_xu + 3\Theta u_y - 3C_u\Theta u, \tag{26}$$

where the dependent variable u takes its value in an associative algebra and C_u denotes the commutator in the associative algebra, that is, $C_u\Theta u = u\Theta u - (\Theta u)u$, which is zero if u takes its value in a commutative algebra. When u is independent of y, it leads to the noncommutative KdV equation (see [41, 43] for more examples and noncommutative $(1+1)$-dimensional integrable evolution equations)

$$u_t = K = u_{xxx} + 3uu_x + 3u_xu.$$

The noncommutative KP equation (26) is homogeneous with respect to the same scaling $h = 2yu_y + xu_x + 2u$ for the KP equation (23).

Notice also that the elements in $\mathfrak{sl}(2)$ presented in Lemma 8 and 9 are linear in u and its derivatives, which implies that they are valid no matter the dependent variables are commutative or noncommutative. Therefore, we can apply Theorem 2 to equation (26). We define

$$\begin{aligned}\tau &= \frac{1}{6}[xyu_x + y^2u_y + 2yu,\ K] \\ &= \frac{1}{2}y(u_{xxx} + 3uu_x + 3u_xu + 3\Theta u_y - 3C_u\Theta u) + xu_y + 2\Theta u - \frac{1}{2}C_uD_x^{-1}u. \quad (27)\end{aligned}$$

Using it, we compute

$$\begin{aligned}[\tau, K] = 6(&u_{xxy} + \Theta^2u_y + 2uu_y + 2u_yu + u_x\Theta u + (\Theta u)u_x) \\ &+6(C_uD_x^{-1}C_u\Theta u - \Theta C_u\Theta u - C_u\Theta^2u),\end{aligned}$$

which is a symmetry flow as presented in [52] implying that $[[\tau, K], K] = 0$. Condition (ii) in Theorem 2 is trivially satisfied due to Remark 3. Thus τ defined by (27) is a master symmetry for equation (26).

5.2.3 A new integrable Davey-Stewartson type equation

Recently, Huard and Novikov carried out the classification of integrable Davey-Stewartson type equations [18]. They found a few new equations. Here we use our approach to construct a master symmetries for the one (equation (3.6) in [18]), whose linear terms have constant coefficients. We further give its Hamiltonian operator and compute its conserved densities using the master symmetry.

Consider the system of the following form

$$\begin{cases} u_t = u_x\Theta^{-1}u - \frac{1}{4}c^2vv_y + v_x + \epsilon(u_{xx} + cv_{xy}) \\ v_t = D_x(v\Theta^{-1}u) + cvv_x - \epsilon v_{xx} \end{cases} \quad (28)$$

where c and ϵ are constant. It is homogeneous since we have

$$\left[\begin{pmatrix} xu_x \\ xv_x + v \end{pmatrix}, \begin{pmatrix} u_t \\ v_t \end{pmatrix}\right] = 2\begin{pmatrix} u_t \\ v_t \end{pmatrix}.$$

To shorten the expression, we use notations $p_y = u_x$ and $q_y = v_x$. According to Theorem 2, its master symmetry is

$$\tau = \frac{1}{4}\left[\begin{pmatrix} x^2u_x \\ x^2v_x + 2xv \end{pmatrix}, \begin{pmatrix} u_t \\ v_t \end{pmatrix}\right] = \begin{pmatrix} xu_t + \frac{\epsilon}{2}(u_x + cv_y) + \frac{v}{2} \\ xv_t + vp + \frac{c}{2}v^2 - \frac{3}{2}\epsilon v_x \end{pmatrix}.$$

Notice that the anti-symmetric constant operator

$$\mathcal{H} = \begin{pmatrix} -cD_y^2D_x^{-1} & D_y \\ D_y & 0 \end{pmatrix} = \begin{pmatrix} -cD_y\Theta & D_y \\ D_y & 0 \end{pmatrix}$$

is Hamiltonian, and it is a Hamiltonian operator for equation (28) since we can write it as

$$\begin{aligned}\begin{pmatrix} u_t \\ v_t \end{pmatrix} &= \mathcal{H}\begin{pmatrix} \Theta^{-1}(vp) + \frac{c}{2}\Theta^{-1}v^2 - \epsilon\Theta^{-1}v_x \\ \frac{1}{2}p^2 + cvp + \frac{3}{8}c^2v^2 + q + \epsilon p_x \end{pmatrix} \\ &= \mathcal{H}\delta_{(u,v)}\left(\frac{1}{2}(vp^2 + cv^2p + vq) + \epsilon vp_x + \frac{1}{8}c^2v^3\right). \quad (29)\end{aligned}$$

We now use the master symmetry to construct the hierarchy of conserved densities of (28). It is clear u and v are two conserved densities since we can write

$$\begin{aligned} u_t &= D_y\left(\frac{1}{2}p^2 - \frac{1}{8}c^2v^2 + q + \epsilon(p_x + cv_x)\right); \\ v_t &= D_x\left(vp + \frac{c}{2}v^2 - \epsilon v_x\right). \end{aligned}$$

However, we can't take the above two densities as starting points since we have $\int u_*(\tau) = v$ and $\int v_*(\tau) = 0$. Besides, they both generate zero Hamiltonian vector field. Notice that

$$\begin{pmatrix} u_x \\ v_x \end{pmatrix} = \mathcal{H}\begin{pmatrix} q \\ p + cv \end{pmatrix} = \mathcal{H}\delta_{(u,v)}\left(vp + \frac{c}{2}v^2\right),$$

and

$$\tau = \mathcal{H}\delta_{(u,v)}\left(\frac{x}{2}(vp^2 + cv^2p + vq) + \epsilon xvp_x + \frac{1}{8}c^2xv^3 + \frac{\epsilon}{2}uq\right).$$

Let $\rho_0 = vp + \frac{c}{2}v^2$. We apply Proposition 3 to construct Hamiltonians corresponding to symmetries, that is, $a_n = [\tau, a_{n-1}] = \mathcal{H}\delta_{(u,v)}\rho_n$ and $\rho_n = \int \rho_{n-1*}(\tau)$. In particular, we have

$$\begin{aligned}\rho_1 &= \int \rho_{0*}(\tau) = \int \left((xv_t + vp + \frac{c}{2}v^2 - \frac{3}{2}\epsilon v_x)p + v\Theta^{-1}(xu_t + \frac{\epsilon}{2}(u_x + cv_y) + \frac{v}{2})\right) \\ &+ \int cv(xv_t + vp + \frac{c}{2}v^2 - \frac{3}{2}\epsilon v_x) \\ &\equiv \frac{1}{2}\left(vp^2 + cv^2p + vq\right) + \epsilon vp_x + \frac{1}{8}c^2v^3,\end{aligned}$$

which is the Hamiltonian for this equation (see (29)), and

$$\begin{aligned}\rho_2 = \int \rho_{1*}(\tau) \equiv \frac{1}{2}\left(vp^3 + 3vpq - 3\epsilon v_x p^2 - 4\epsilon^2 p_x v_x\right) + \frac{3}{4}cv^2\left(q + \epsilon p_x + p^2\right) \\ -c\epsilon^2 v_x^2 + \frac{3}{8}c^2v^3p + \frac{1}{16}c^3v^4,\end{aligned}$$

which corresponds to the next symmetry

$$\begin{aligned}a_2 &= \mathcal{H}\delta_{(u,v)}\rho_2 \\ &= \begin{pmatrix} D_y\left(\frac{1}{2}p^3 + \frac{3}{2}pq + 3\epsilon pp_x + 2\epsilon^2 p_{xx} + \frac{3}{2}\epsilon c(p_x v + 2pv_x) - \frac{1}{8}c^2v^2(3p + cv)\right) + \\ +D_y\left(\frac{3}{2}\Theta^{-1}(pv) + \frac{3}{4}c\Theta^{-1}v^2\right) \\ \frac{3}{2}D_x\left(p^2v + vq - 2\epsilon pv_x + \frac{4}{3}\epsilon^2 v_{xx} + cpv^2 - \epsilon cvv_x + \frac{1}{4}c^2v^3\right)\end{pmatrix}.\end{aligned}$$

5.3 $(2+1)$-dimensional lattice-field equations

In this section, we apply the proposed approach in section 4 to (2+1)-dimensional lattice-field equations. Consider the differential-difference KP equation of form [7]

$$u_t = u_{yy} + 2uu_y + 2(\mathcal{S} - 1)^{-1}u_{yy}, \tag{30}$$

where the dependent variable $u = u(n, y, t)$ is a function of continuous variables y, t and discrete variable $n \in \mathbb{Z}$, and it is smooth with respect to the continuous variables. Here $\mathcal{S}$ is the shift operator mapping $u(n, y, t)$ to $u(n + 1, y, t)$. The operator $\mathcal{S} - 1$ is a discrete analogue of derivative. Notice that D_y and $(\mathcal{S} - 1)^{-1}$ appear in pair. In the same way as in the case for $(2 + 1)$-dimensional partial differential equations described in Section 5.2.1, we also introduce the concept of quasi-local functions.

We denote $\mathcal{S}^i\partial_y^j u$ by $u_{i,jy}$, where $i \in \mathbb{Z}$ and $j \in \mathbb{N}$ is the order of derivative of the dependent variable u with respect to its independent variable y. When $i = 0$ or $j = 0$, we simply write as u_i or u_{jy}. All functions depending on n, y, t, u and $u_{i,jy}$ and being smooth with respect to its variables except n form a differential ring $\mathcal{F}$ with total y-derivation D_y. Let us denote

$$\Theta = (\mathcal{S} - 1)^{-1}D_y, \qquad \Theta^{-1} = D_y^{-1}(\mathcal{S} - 1). \tag{31}$$

By considering a sequence of extensions of $\mathcal{F}$ as in Section 5.2.1, we can define the quasi-local functions $\mathcal{F}(\Theta)$.

Since y is the only continuous spatial variable for dependent variable, we consider the scaling symmetry as $yu_y + \alpha u$, where α is constant. Thus, around h we get the same $\mathfrak{sl}(2)$ representations (only changing x to y) as presented in Lemma 1 and Lemma 2 for the (1+1)-dimensional case. We are then ready for applying Theorem 2 to these type of evolutionary homogeneous equations.

5.3.1 The differential-difference KP equation

Using our notation, we rewrite equation (30) as

$$u_t = K = u_{yy} + 2uu_y + 2\Theta u_y. \tag{32}$$

Its master symmetry has recently been studied in [12, 20]. In the latter paper [20], the author also studied its time dependent symmetries using the $\mathfrak{sl}_h(2)$ (cf. Diagram 2) representation. Here we demonstrate our approach on how to construct the master symmetry. Notice that equation (32) is homogeneous with respect to $h = yu_y + u$ and

$$[yu_y + u, K] = 2K.$$

Following Remark 1 we take $f = -(y^2u_y + 2yu + f_0)$ forming $\mathfrak{sl}(2)$ with u_y and h, where f_0 satisfies $[e, f_0] = 0$ and $[h, f_0] = -2f_0$. Assume f_0 is a function of independent variable n, i.e., $f_0 = g(n)$ and define

$$\tau = \frac{1}{4}[y^2u_y + 2yu + g(n), K] = y(u_{yy} + 2uu_y + 2\Theta u_y) + \frac{1}{2}g(n)u_y + \frac{3}{2}u_y + u^2 + 3\Theta u.$$

Using the condition $[[\tau, K], K] = 0$, we can determine $g(n) = 2n$. The resulting τ is equivalent to the master symmetry presented in [12, 20] in the sense that it generates the same space of symmetries for equation (32).

5.3.2 The (2+1)-dimensional Volterra lattice

Consider the following differential-difference equation

$$\begin{aligned} u_t = K &= (\mathcal{S}+1)(\mathcal{S}-1)^{-1}u_{yy} + u_y(\mathcal{S}+1)(\mathcal{S}-1)^{-1}u_y + \exp(2u_1) - \exp(2u_{-1}) \\ &= (\mathcal{S}+1)\Theta u_y + u_y(\mathcal{S}+1)\Theta u + \exp(2u_1) - \exp(2u_{-1}). \end{aligned} \tag{33}$$

This system of periodic form is called two dimensional generalisation of the Volterra lattice [25]. Its Lax representation is invariant under dihedral reduction groups [25, 26, 27], which is generated by both inner and outer automorphisms of $\mathfrak{sl}(n, \mathbb{C})$, and it can be viewed as a discretisation of the Kadomtsev-Petviashvili equation [21]. For fixed period n, it is a bi-Hamiltonian system. When $n = 3$, its recursion operator and bi-Hamiltonian structure are explicitly constructed from its Lax representation in [53]. Its Darboux transformation for arbitrary period n has recently been constructed in [31].

If dependent variable u is independent of y, equation (33) reduces into

$$u_t = \exp(2u_1) - \exp(2u_{-1}),$$

which is the well-known Volterra chain $v_{t'} = v(v_1 - v_{-1})$ under the point transformation $v = \exp(2u)$ and $t' = 2t$.

In this section, we construct a master symmetry for (33) using our approach in Section 4 and further construct its conserved densities via master symmetries.

Notice that equation (33) is homogeneous with respect to scaling $yu_y + 1$. Indeed, by direct computation, we have $[yu_y + 1, K] = 2K$ for equation (33). The following three elements form an $\mathfrak{sl}(2)$:

$$e = u_y, \quad h = 2(yu_y + 1) \quad \text{and} \quad f = -(y^2u_y + 2y).$$

It follows from Theorem 2 that

$$\tau = \frac{1}{4}[y^2 u_y + 2y, K] = yK + nu_y + 2\Theta u \tag{34}$$

is a master symmetry for equation (33) if it satisfies the conditions. First of all, Condition (ii) of Theorem 2 is trivially satisfied according to Remark 3. We now check Condition (i) $[[\tau, K], K] = 0$. By direct calculation, we have

$$\begin{aligned} a_2 = [\tau, K] = &(\mathcal{S}^2 + \mathcal{S} + 1)\Theta^2 u_y + 3u_y \mathcal{S}^2 \Theta^2 u + 3(\Theta u)(\mathcal{S} + 1)\Theta u_y + u_y^3 + 3u_y(\Theta u)\mathcal{S}(\Theta u) \\ &+ \frac{3}{2} D_y(\exp(2u_1) + \exp(2u) + \exp(2u_{-1})) + 3\exp(2u_1)u_y \\ &+ 3\left(\exp(2u_1) - \exp(2u_{-1})\right)\Theta u, \end{aligned}$$

where we used the identity

$$(\mathcal{S} - 1)^{-1}(nu_y) = (n-1)(\mathcal{S} - 1)^{-1} u_y - (\mathcal{S} - 1)^{-2} u_y$$

and further $[a_2, K] = 0$. Here we skip the long formulas to check a_2 is a symmetry of (33), which can be carried out by organising the terms according to polynomial terms, exponential terms and mixed terms. We can also compare a_2 to the symmetry flows obtained via its Lax representation, which is given in [25, 21, 14].

We now look at conserved densities for equation (33). Equation (33) can be written as

$$u_t = K = (\mathcal{S} - 1)\left((\mathcal{S} + 1)\Theta^2 u + (\Theta u)^2 + \exp(2u) + \exp(2u_{-1})\right),$$

which implies that u is a conserved density. Since $\int u_*(\tau) = 0$, we can not use it as a start point to generate other conserved densities.

Let $\rho_0 = \frac{1}{2}(\Theta u)^2 + \exp(2u)$, which is a conserved density for (33). Indeed, we have

$$\begin{aligned} \frac{\partial \rho_0}{\partial t} &= (\Theta u)\Theta u_t + 2\exp(2u)u_t \\ &= D_y\left((\Theta u)\left((\mathcal{S} + 1)\Theta^2 u + \frac{2}{3}(\Theta u)^2 + 3\exp(2u) + \exp(2u_{-1})\right) + \exp(2u)u_y\right) \\ &\quad + (\mathcal{S} - 1)\left(\exp(2u_{-1})\Theta u_y + 2\exp(2u + 2u_{-1}) - (\Theta^2 u)^2\right). \end{aligned}$$

For fixed period $n = 3$, it reduces to the conserved density appeared in [53]. Starting from it, using the master symmetry given by (34) we get the next conserved density:

$$\rho_1 = \int \rho_{0*}(\tau) = \int \tau \cdot \frac{\delta \rho_0}{\delta u} \equiv \frac{1}{3}(\Theta u)^3 + (\Theta u)(\Theta^2 u) + 2(\Theta u)\exp(2u).$$

5.3.3 Another (2+1)-dimensional generalised Volterra Chain

In this section, we construct a master symmetry for the following equation

$$\begin{aligned} u_t &= (\mathcal{S} + 1)(\mathcal{S} - 1)^{-1} u_{yy} + u_y(\mathcal{S} + 1)(\mathcal{S} - 1)^{-1} u_y + 2\exp(u + u_1) - 2\exp(u + u_{-1}) \\ &= K = (\mathcal{S} + 1)\Theta u_y + u_y(\mathcal{S} + 1)\Theta u + 2\exp(u + u_1) - 2\exp(u + u_{-1}). \end{aligned} \tag{35}$$

This equation is appeared in [14] when the authors classified a family of equations with the non-locality of intermediate long wave type. In fact, both equations (33) and (35) are listed in the list of integrable equations of this class, and these two equations have coinciding dispersionless limits.

We can also consider the equation of periodic form. The corresponding equations of (33) and (35) for period $n = 3$ appeared in the classification of integrable systems of nonlinear Schrödinger type [34].

If dependent variable u is independent of y, equation (35) reduces into

$$u_t = 2\exp(u + u_1) - 2\exp(u + u_{-1}),$$

which is also the well-known Volterra chain $v_{t'} = v(v_1 - v_{-1})$ under the point transformation $v = \exp(u + u_1)$ and $t' = 2t$.

Notice that equation (35) is also homogeneous and shares the same scaling as equation (33). Thus we can use the same $\mathfrak{sl}(2)$ to construct its master symmetry. Here we only write out the results since the arguments of applying Theorem 2 are the same due to equations (33) and (35) having the same linear terms. We only need to check Condition (i) $[[\tau, K], K] = 0$.

We define

$$\tau = \frac{1}{4}[y^2 u_y + 2y, K] = yK + nu_y + 2\Theta u, \tag{36}$$

which is of the same form as (34), and we have

$$\begin{aligned} a_2 = [\tau, K] = {} & (\mathcal{S}^2 + \mathcal{S} + 1)\Theta^2 u_y + 3u_y \mathcal{S}^2 \Theta^2 u + 3(\Theta u)(\mathcal{S} + 1)\Theta u_y + u_y^3 + 3u_y(\Theta u)\mathcal{S}(\Theta u) \\ & + \exp(u + u_1)(3u_{1,y} + 6u_y + 6\Theta u) + \exp(u + u_{-1})(3u_{-1,y} - 6\Theta u). \end{aligned}$$

To verify $[a_2, K] = 0$, we can either compute directly or compare it to the symmetry flows obtained via its Lax representation [14].

We now look at conserved densities for equation (35) and notice that

$$\rho_0 = \frac{1}{2}(\Theta u)^2 + \exp(u + u_{-1})$$

is a conserved density. Indeed, we have

$$\begin{aligned} \frac{\partial \rho_1}{\partial t} &= (\Theta u)\Theta u_t + \exp(u + u_{-1})(u_t + u_{-1,t}) \\ &\equiv D_y \left(-\frac{1}{3}(\Theta u)^3 + 2\exp(u + u_1)\Theta u + \exp(u + u_1)u_y + \exp(u + u_{-1})u_y \right) \\ &\quad + (\mathcal{S} - 1)\left(\exp(u + u_{-1})u_y u_{-1,y} - 2\exp(u + u_{-1})u_y \Theta u - (\Theta^2 u)^2 \right). \end{aligned}$$

Starting from it, using the master symmetry given by (36) we get its next conserved density:

$$\rho_1 = \int \rho_{0*}(\tau) = \int \tau \cdot \frac{\delta \rho_0}{\delta u} \equiv \frac{1}{3}(\Theta u)^3 + (\Theta u)(\Theta^2 u) + (\exp(u + u_1) + \exp(u + u_{-1}))\Theta u.$$

6 Discussion

In this paper, we present a new structure called the $\mathcal{O}$-scheme for homogeneous evolutionary integrable equations. For an evolutionary vector field $u_t = K[u]$ the Lie algebra of symmetries is the kernel space of ad_K, the Lie algebra of master symmetries is in the kernel of ad_K^2. Symmetries, master symmetries and higher order nilpotent elements can neatly be fit as elements of modules in the BGG category $\mathcal{O}$. This is based on the observation that there is a free $\mathfrak{sl}(2)$ representation and further an infinite dimensional module in the BBG category related to such equations, whether integrable or not. We prove that under technical conditions, it is enough for us to construct master symmetries and organise time dependent symmetries using the elements in this module. It also offers us an approach to construct master symmetries.

The master symmetries for two $(2 + 1)$-dimensional Volterra Chains are of the same form, see (34) and (36), since they have the same scaling symmetry. Thus given a family of equations $u_t = K$ sharing the same scaling symmetry, condition that $\mathrm{ad}_K^2 f$ commuting K in Theorem 2 can be used as a criterion for classifying integrable equations. It is worth to explore although the calculation involved is tedious, in particular, for some $(2 + 1)$-dimensional equations whose integrability condition is challenging to formulate.

In this paper, we presented the $\mathcal{O}$-schemes for both $(1 + 1)$- and $(2 + 1)$-dimensional partial differential equations and $(2+1)$-dimensional lattice-field equations with $(\mathcal{S}-1)^{-1}D_y$ as nonlocal terms. Currently, we are working on extension of such $\mathcal{O}$-scheme to integrable differential-difference and discrete systems.

The $\mathcal{O}$-scheme in this paper is formulated in the case of $\mathfrak{sl}(2)$ modules. The extension of this construction to algebras of higher rank is a promising direction of research which would enable us to study much wider class of systems important in applications including the Boussinesq equation, the resonant wave interaction system, two dimensional Toda lattice and many others.

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