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Bounded real lemmas for positive descriptor systems

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Abstract

A well known result in the theory of linear positive systems is the existence of positive definite diagonal matrix (PDDM) solutions to some well known linear matrix inequalities (LMIs). In this paper, based on the positivity characterization, a novel bounded real lemma for continuous positive descriptor systems in terms of strict LMI is first established by the separating hyperplane theorem. The result developed here provides a necessary and sufficient condition for systems to possess H_∞ norm less than γ and shows the existence of PDDM solution. Moreover, under certain condition, a simple model reduction method is introduced, which can preserve positivity, stability and H_∞ norm of the original systems. An advantage of such method is that systems' matrices of the reduced order systems do not involve solving of LMIs conditions. Then, the obtained results are extended to discrete case. Finally, a numerical example is given to illustrate the effectiveness of the obtained results.
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1. Introduction

Positive systems are dynamical systems whose state and output variables remain nonnegative for all future time whenever their initial conditions and inputs are nonnegative. Such systems can be found in different areas, such as economics [1], biology [2,33] physiology [3], and epidemiology [4]. In general, the variables chosen in these systems can only take nonnegative

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values, which may represent quantities of goods, densities of species, chemical concentrations. The theory of nonnegative matrices is a basic mathematical tool for analysis and synthesis of linear positive systems, which leads to many novel results in this field. During recent decades, positive systems have been of great interest to many researchers and a large number of theoretical results have been reported see [5–8], for more detail, refer to the two monographs, see [9,10]. Amongst many new results for linear positive systems, one is the existence of PPDM solutions to some well-known results for linear general systems. Another one is linear co-positive Lyapunov functions available for linear positive systems only, which has resulted in many remarkable and simpler results. For details of the literature related to PPDM solutions and linear co-positive Lyapunov functions, the reader is referred to [3,9,11–13] and the references cited therein.

However, in practice, descriptor systems are of more widespread applications than standard systems. It is necessary to guarantee positivity of systems in economics, biology and chemical reaction, which can be modeled more accurately using descriptor systems. It is well known that the Leontief model is a representative example of linear positive descriptor systems. Therefore, the study on positive descriptor systems is of great importance and far reaching practical significance. Due to singularity of the derivative matrix and nonnegativity restriction on systems' variables, compared to linear positive standard systems, analysis and synthesis for linear positive descriptor systems are more complicated and therefore much of the developed theory for such systems is still not up to a quantitative level. Although some topics of linear positive descriptor systems have been available, for instance, positivity and stability [14–17], reachability and controllability [18], positive preserving balanced truncation [19], admissibility [20], there are a lot of open problems in the study of such systems, such as stabilization, optimal control, dissipativity and systems subject to uncertainty.

It is well known that linear positive systems are defined on cones, to be more exact, convex cones, rather than linear space. Therefore, convex optimization is a powerful tool for analysis of linear positive systems. Relevant published results by convex optimization can be found in [21,22]. On the other hand, the problem of H_∞ control has been a topic of recurring interest for several decades. Great efforts have been made on H_∞ control and many results on such a topic for general standard systems as well as general descriptor systems have been reported. In recent years, increasing attention has been devoted to H_∞ analysis for linear positive standard systems by means of convex optimization. The KYP Lemma for linear continuous positive standard systems is proved using semidefinite programming duality in [23]. In [24], the alternative proofs along the line of the rank-one separable property are given to several remarkable results for linear positive standard systems and main result given in [23]. The KYP Lemma for linear discrete positive standard systems is studied by a theorem of alternatives on the feasibility of LMIs in [25]. At the same time, H_∞ model reduction has received considerable attention. Positivity-preserving H_∞ model reduction for continuous positive systems is investigated in [26]. Shen and Lam [27] study the H_∞ model reduction problems for the discrete positive systems with inhomogeneous initial conditions and the output error between the original system and the reduced-order one is bounded by a weighted sum of the magnitude of the input and that of the initial condition. Zhang et al. [19] are concerned with positivity preserving model reduction method of balanced truncation for descriptor systems. Unfortunately, to the best of authors' knowledge, little attempt has been made on H_∞ analysis for positive descriptor systems, and little attention has been paid to analysis for descriptor systems by means of convex analysis, which motivates the present work.

The objective of this paper is to study bounded real lemmas of linear positive descriptor systems. By virtue of the separating hyperplane theorem, a novel bounded real lemma is first established to check H_∞ norm less than γ for continuous positive descriptor systems. An elegant feature of this obtained result is the existence of PPDM solution, which is one of the important properties of positive systems. Moreover, this result in terms of strict LMI can be solved directly by standard LMI solvers. At the same time, the exact value of H_∞ norm can be calculated. On the other hand, under certain condition, a simple model reduction method is introduced, which can preserve positivity, stability and H_∞ norm. Such method does not involve solving of LMIs conditions and therefore is easy to deal with. Then, the obtained results are extended to discrete case.

This paper is organized as follows. In Section 2, some necessary preliminaries are introduced. Section 3 is devoted to bounded real lemma of continuous positive descriptor systems and positivity-H_∞ norm-preserving model reduction. The obtained results in Section 3 are extended to discrete case in Section 4. A numerical example is provided to show the effectiveness of the theoretical results in Section 5. Section 6 concludes this paper.

2. Preliminaries

In this section, some necessary preliminaries and results are to be presented, which are helpful for the proofs of main results and understanding of subsequent sections.

At first, the following notations will be used throughout this paper.

$\mathbb{C}$ is the set of complex number. $\mathbb{R}^n$ is the space of column vectors of dimension n. $\mathbb{R}^n_+$ is the nonnegative orthant of $\mathbb{R}^n$. S^n is the space of all real symmetric matrices with dimension $n \times n$. $\mathbb{R}^{n\times m}$ is the space of $n \times m$ matrices with real numbers. $D^{n\times n}_+$ denotes the set of all positive definite diagonal matrices. A_{ij} denotes the ijth entry of matrix A. A_{ci} and A_{ri} denote the ith column and row of A, respectively. For $A, B \in \mathbb{R}^{n\times m}$, $A \ge B$, $A > B$, $A \gg (\ll) B$ mean that $A_{ij} \ge B_{ij}$, $A_{ij} \ge B_{ij}$ but $A \ne B$, $A_{ij} > (<) B_{ij}$, $\forall i, j$, respectively. For $A \in \mathbb{R}^{n\times n}$, $A \succeq 0$ and $A \prec 0$ mean that A is a positive semidefinite matrix and a negative definite matrix, respectively. A^D denotes the Drazin inverse of A which satisfies $A^D A A^D = A^D$, $A^D A = A A^D$, $A^D A^{k+1} = A^k$. rank(A) represents the rank of A and tr(A) is the trace of A. $\rho(A)$ and $\mu(A)$ denote the spectral radius and maximal real part of eigenvalues of A, respectively. $\lambda(E, A)$ is the set of all finite eigenvalues of matrix pair (E, A). Index of (E, A) is denoted by ind(E, A). $\mu(E, A)$ and $\rho(E, A)$ represent the maximal real part and spectral radius of finite eigenvalues of (E, A), respectively. $D(A)$ is the vector which is composed of the diagonal entries of $A \in \mathbb{R}^{n\times n}$. $\langle X, Y\rangle = \mathrm{tr}(XY)$ is the inner product on S^n. For $A \in \mathbb{R}^{n\times m}$, im($A$) denotes the image space of A. We use I and $0_{m,n}$ refer to the identity matrix of appropriate dimensions and $m \times n$ zero matrix, respectively. Sometimes, for simplicity, 0 is used to refer to zero matrix with appropriate dimensions.

As known, for matrices $E, A \in \mathbb{R}^{n\times n}$, if there exists a scalar $\lambda \in \mathbb{C}$ such that $\det(\lambda E - A) \ne 0$, the matrix pair (E, A) is said to be regular. In this case, there exist two nonsingular matrices $P, Q \in \mathbb{R}^{n\times n}$ such that

$$PEQ = \begin{bmatrix} I_r & 0 \\ 0 & N \end{bmatrix}, \quad PAQ = \begin{bmatrix} J & 0 \\ 0 & I_{n-r} \end{bmatrix},$$

where J and N are matrices in the Jordan canonical form, and specifically, N is a nilpotent matrix. This transformation is called the Weierstrass canonical form transformation. Index of the matrix pair (E, A) is defined as the nilpotent index of matrix N in the Weierstrass canonical form [28]. In particular, ind$(E, A) = 0$ if E is nonsingular, ind$(E, A) = 1$ if rank$(E) = r < n$ and $N = 0$.

For a given matrix $A \in \mathbb{R}^{n \times n}$, it is said to be Hurwitz stable if $\mu(A) < 0$. It is said to be Schur stable if $\rho(A) < 1$. It is called a Metzler matrix if $A_{ij} \geq 0$ for all i, j with $i \neq j$, then $e^{At} \geq 0$, $\forall t \geq 0$. A matrix $A \in \mathbb{R}^{n \times m}$ is called nonnegative on a subset $\Omega \subset \mathbb{R}^m$ if $Ax \in \mathbb{R}_+^n$ for all $x \in \Omega \cap \mathbb{R}_+^m$ [14].

Definition 1 (*Ebihara [24]*). For a given matrix $H \in S^n$ with $H \succcurlyeq 0$, the vector $h \in \mathbb{R}_+^n$ is defined by $h_i := \sqrt{H_{ii}}$, $i = 1, 2, \ldots, n$.

Lemma 1 (*Mason [22], Ebihara [24]*). *For a given Metzler matrix $A \in \mathbb{R}^{n \times n}$ and $H \in S^n$ with $H \succcurlyeq 0$, the following statements hold*:

(i) $(hh^T)_{ii} = H_{ii}, (hh^T)_{ij} \geq H_{ij}, \ i \neq j$.
(ii) $D(hh^T A) \geq D(HA)$.

where $h \in \mathbb{R}_+^n$ is defined from H as in Definition 1.

This lemma is very useful in the proofs of the main results developed in this paper. We now recall the following lemma named separating hyperplane theorem which plays a key role in the proofs in the following sections.

Lemma 2 (*Boyd [29]*). *Suppose that Ω_1 and Ω_2 are two convex sets that do not intersect, that is, $\Omega_1 \cap \Omega_2 = \varnothing$. Then there exist $a \neq 0$ and b such that $\langle a, x \rangle \leq b$ for all $x \in \Omega_1$ and $\langle a, x \rangle \geq b$ for all $x \in \Omega_2$.*

It should be noted that, in this paper, if Ω_1 and Ω_2 are two convex subsets in $\mathbb{R}^{n \times n}$, then $a \in \mathbb{R}^{n \times n}$ 0 and $\langle a, x \rangle = \mathrm{tr}(ax)$.

3. Bounded real lemma for continuous case

3.1. Analysis for bounded real lemma

In this subsection, basic characteristics of linear continuous positive descriptor systems are introduced at first. Then some lemmas needed for later analysis are given. Finally, a necessary and sufficient condition in terms of strict LMI to check H_∞ norm less than γ is proposed.

Consider the following linear continuous descriptor system

$$E_c \dot{x}(t) = A_c x(t) + B_c u(t), \tag{1a}$$

$$y(t) = C_c x(t) + D_c u(t), \tag{1b}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$ are the state, input and output vectors, respectively. E_c, A_c, B_c, C_c, D_c are real matrices with compatible dimensions, and $\mathrm{rank}(E_c) = r \leq n$. System (1) is called a linear continuous standard system if $E_c = I$.

A scalar $\lambda \in \mathbb{C}$ is a finite eigenvalue of a matrix pair (E_c, A_c) if $\det(\lambda E_c - A_c) = 0$. System (1) is said to be stable if $\mu(E_c, A_c) < 0$; it is said to be impulse-free if $\mathrm{ind}(E_c, A_c) = 1$; it is said to be admissible if it is regular, impulse-free and stable [30].

Suppose that (E_c, A_c) is regular and $\mathrm{ind}(E_c, A_c) = \nu$, $\hat{\lambda}$ is a complex number such that $(\hat{\lambda} E_c - A_c)^{-1}$ exists, then an explicit solution to Eq. (1a) in the form of Drazin inverses [28] is given by

$$x(t)=e^{\hat{E}_c^D\hat{A}_c t}\hat{E}_c^D\hat{E}_c x(0)+\int_0^t e^{\hat{E}_c^D\hat{A}_c(t-\tau)}\hat{E}_c^D\hat{B}_c u(\tau)\,d\tau-(I-\hat{E}_c^D\hat{E}_c)\sum_{i=0}^{\nu-1}(\hat{E}_c\hat{A}_c^D)^i\hat{A}_c^D\hat{B}_c u^{(i)}(t),$$

where $\hat{E}_c=(\hat{\lambda}E_c-A_c)^{-1}E_c$, $\hat{A}_c=(\hat{\lambda}E_c-A_c)^{-1}A_c$, $\hat{B}_c=(\hat{\lambda}E_c-A_c)^{-1}B_c$, $x(0)$ is an admissible initial condition, $u^{(i)}$ is the ith derivative of u, $i=0,1,\ldots,\nu-1$. The admissible initial condition $x(0)$ satisfies

$$x(0)=\hat{E}^D\hat{E}w-(I-\hat{E}_c^D\hat{E}_c)\sum_{i=0}^{\nu-1}(\hat{E}_c\hat{A}_c^D)^i\hat{A}_c^D\hat{B}_c u^{(i)}(0),$$

where $w\in\mathbb{R}^n$.

Transforming (E_c,A_c) into the Weierstrass canonical form by $P_c,Q_c\in\mathbb{R}^{n\times n}$ leads to

$$\hat{E}_c=Q_c\begin{bmatrix}(\hat{\lambda}I-J_c)^{-1} & 0\\ 0 & (\hat{\lambda}N_c-I)^{-1}N_c\end{bmatrix}Q_c^{-1},\quad \hat{E}_c^D=Q_c\begin{bmatrix}\hat{\lambda}I-J_c & 0\\ 0 & 0\end{bmatrix}Q_c^{-1},$$

$$\hat{A}_c=Q_c\begin{bmatrix}(\hat{\lambda}I-J_c)^{-1}J_c & 0\\ 0 & (\hat{\lambda}N_c-I)^{-1}\end{bmatrix}Q_c^{-1},\quad \hat{A}_c^D=Q_c\begin{bmatrix}J_c^D(\hat{\lambda}I-J_c) & 0\\ 0 & \hat{\lambda}N_c-I\end{bmatrix}Q_c^{-1}.$$

B_c,C_c are partitioned accordingly,

$$P_cB_c=\begin{bmatrix}B_{c1}\\ B_{c2}\end{bmatrix},\quad C_cQ_c=[C_{c1}\ \ C_{c2}].$$

Then by computation,

$$\hat{E}_c\hat{E}_c^D=\hat{E}_c^D\hat{E}_c=Q_c\begin{bmatrix}I & 0\\ 0 & 0\end{bmatrix}Q_c^{-1},\quad \hat{E}_c^D\hat{A}_c=\hat{A}_c\hat{E}_c^D=Q_c\begin{bmatrix}J_c & 0\\ 0 & 0\end{bmatrix}Q_c^{-1},$$

$$\hat{A}_c^D\hat{E}_c=\hat{E}_c\hat{A}_c^D=Q_c\begin{bmatrix}J_c^D & 0\\ 0 & N_c\end{bmatrix}Q_c^{-1},\quad \hat{E}_c^D\hat{B}_c=Q_c\begin{bmatrix}B_{c1}\\ 0\end{bmatrix},$$

$$\hat{A}_c^D\hat{B}_c=Q_c\begin{bmatrix}J_c^DB_{c1}\\ B_{c2}\end{bmatrix},\quad C_c\hat{E}_c^D\hat{E}_c=[C_{c1}\ \ 0]Q_c^{-1}.$$

Definition 2 (*Virnik [14]*). System (1) is said to be positive if $x(t)\geq 0, y(t)\geq 0, t\geq 0$ for any admissible initial condition $x(0)\geq 0$ and any input satisfying $u^{(i)}(\tau)\geq 0,\ 0\leq\tau\leq t, i=0,1,\ldots,\nu-1$.

Lemma 3 (*Virnik [14]*). *Suppose that* (E_c,A_c) *is regular*, $\mathrm{ind}(E_c,A_c)=\nu$ *and*

$$\hat{E}_c^D\hat{E}_c\geq 0,\quad (I-\hat{E}_c^D\hat{E}_c)(\hat{E}_c\hat{A}_c^D)^i\hat{A}_c^D\hat{B}_c\leq 0,\ i=0,1,\ldots,\nu-1. \tag{2}$$

Then system (1) *with* $D_c=0$ *is positive if and only if the following conditions hold*:

(i) *there exists a scalar* $\alpha\geq 0$ *such that* $\overline{M}:=\hat{E}_c^D\hat{A}_c-\alpha I+\alpha\hat{E}_c^D\hat{E}_c$ *is a Metzler matrix*,
(ii) $\hat{E}_c^D\hat{B}_c\geq 0$,

(iii) C_c *is nonnegative on the subspace* χ_c *defined as*

$$\chi_c := \mathrm{im}_+ \left[\hat{E}_c^D \hat{E}_c \quad -(I-\hat{E}_c^D\hat{E}_c)\hat{A}_c^D\hat{B}_c \quad \cdots \quad -(I-\hat{E}_c^D\hat{E}_c)(\hat{E}_c\hat{A}_c^D)^{v-1}\hat{A}_c^D\hat{B}_c \right],$$

where for a matrix $A \in \mathbb{R}^{n\times m}$, $\mathrm{im}_+ A := \left\{ w_1 \in \mathbb{R}^n \mid \exists w_2 \in \mathbb{R}_+^m : Aw_2 = w_1 \right\}$.

Throughout this section, it is assumed that $D_c \geq 0$ unless explicitly stated otherwise. Obviously, such assumption does not spoil positivity since only nonnegative input is allowed.

The transfer function matrix of system (1) is given by

$$G_c(s) = C_c(sE_c - A_c)^{-1}B_c + D_c, \quad s \in \mathbb{C} \setminus \lambda(E_c, A_c), \tag{3}$$

and its H_∞ norm is defined as $\| G_c \|_\infty := \sup_{\omega \in \mathbb{R}} \overline{\sigma}(G_c(j\omega))$, where $\overline{\sigma}(G_c(j\omega))$ denotes the maximal singular value of $G_c(j\omega)$.

Taking into account the Weierstrass canonical form transformation of (E_c, A_c), the transfer function matrix (3) can be rewritten as

$$\begin{aligned} G_c(s) &= C_c(sE_c - A_c)^{-1}B_c + D_c \\ &= C_c Q_c \begin{bmatrix} sI - J_c & 0 \\ 0 & sN - I \end{bmatrix}^{-1} \begin{bmatrix} B_{c1} \\ B_{c2} \end{bmatrix} + D_c \\ &= C_{c1}(sI - J_c)^{-1}B_{c1} + C_{c2}(sN - I)^{-1}B_{c2} + D_c. \end{aligned}$$

As shown in [14], matrix (3) can be decomposed as

$$G_c(s) = G_{csp}(s) + G_{cp}(s),$$

where

$$G_{csp}(s) = C_c\hat{E}_c^D\hat{E}_c(sI - \hat{E}_c^D\hat{A}_c)^{-1}\hat{E}_c^D\hat{B}_c$$

is the strictly proper part and

$$G_{cp}(s) = C_c(I - \hat{E}_c^D\hat{E}_c)(s(I - \hat{E}_c^D\hat{E}_c)\hat{E}_c\hat{A}_c^D - I)^{-1}(I - \hat{E}_c^D\hat{E}_c)\hat{A}_c^D\hat{B}_c + D_c$$

is the polynomial part. In fact, by computation, we have

$$\begin{aligned} G_{csp}(s) &= C_c\hat{E}_c^D\hat{E}_c(sI - \hat{E}_c^D\hat{A}_c)^{-1}\hat{E}_c^D\hat{B}_c \\ &= [C_{c1} \;\; 0] \begin{bmatrix} sI - J_c & 0 \\ 0 & sI \end{bmatrix}^{-1} \begin{bmatrix} B_{c1} \\ 0 \end{bmatrix} \\ &= C_{c1}(sI - J_c)^{-1}B_{c1} \\ &= [C_{c1} \;\; 0] \begin{bmatrix} sI - J_c & 0 \\ 0 & (s+\alpha)I \end{bmatrix}^{-1} \begin{bmatrix} B_{c1} \\ 0 \end{bmatrix} \\ &= C_c\hat{E}_c^D\hat{E}_c(sI - \overline{M})^{-1}\hat{E}_c^D\hat{B}_c, \end{aligned}$$

$$\begin{aligned} G_{cp}(s) &= C_c(I - \hat{E}_c^D\hat{E}_c)(s(I - \hat{E}_c^D\hat{E}_c)\hat{E}_c\hat{A}_c^D - I)^{-1}\hat{A}_c^D\hat{B}_c + D_c \\ &= [C_{c1} \;\; 0] \begin{bmatrix} -I & 0 \\ 0 & (sN - I)^{-1} \end{bmatrix} \begin{bmatrix} J_c^D B_{c1} \\ B_{c2} \end{bmatrix} + D_c \\ &= C_{c2}(sN - I)^{-1}B_{c2} + D_c. \end{aligned}$$

Note that $(sN-I)^{-1} = -\sum_{i=0}^{\nu-1} s^i N^i$, if $\nu > 1$, which implies that system (1) has impulse, then $\lim_{s\to\infty} C_{c2}(sN-I)^{-1}B_{c2} = \infty$. Hence, when the H_∞ norm of system (1) is considered, $\nu = 1$ is a desired condition.

For convenience, the following matrix expressions are introduced:

$$\overline{A}_c = \hat{E}_c^D \hat{A}_c, \quad \tilde{B}_c = \hat{E}_c^D \hat{B}_c, \quad \overline{C}_c = C_c \hat{E}_c^D \hat{E}_c, \quad \overline{D}_c = D_c - C_c(I - \hat{E}_c^D \hat{E}_c)(\hat{A}_c^D \hat{B}_c),$$
$$\overline{B}_c' = -(I - \hat{E}_c^D \hat{E}_c)\hat{A}_c^D \hat{B}.$$

Therefore, if system (1) is impulse-free, then $G_c(s) = \overline{G}_c(s)$, where $\overline{G}_c(s) = \overline{C}_c(sI - \overline{M})^{-1}\overline{B}_c + \overline{D}_c$.

It has been pointed out in [14] that all finite eigenvalues of (E_c, A_c) are also eigenvalues of $\overline{M}$ and the eigenvalue ∞ of (E_c, A_c) is mapped to the eigenvalue $-\alpha$ of $\overline{M}$. Therefore, it is natural to assume $\alpha > 0$ such that $\overline{M}$ is a Metzler matrix due to the following two facts:

(i) If $\overline{\alpha} \geq 0$ such that $\overline{M}$ is a Metzler matrix, then for any $\alpha > \overline{\alpha} \geq 0$, $\overline{M}$ is Metzler.
(ii) For any $\alpha > 0$, $\overline{M}$ is Hurwitz stable if and only if system (1) is stable.

Throughout this section, it is always assumed that $\alpha > 0$ such that $\overline{M}$ is a Metzler matrix unless otherwise stated.

Before stating the main result, two lemmas which is helpful for the proof of main result are to be presented.

Lemma 4. *For a given $H \in S^{n+m}$ with $H \succcurlyeq 0$, if (E_c, A_c) is regular, condition (2) holds and system (1) is positive, then* $\operatorname{tr}(hh^T W) \geq \operatorname{tr}(HW)$, *where*

$$W = \begin{bmatrix} \overline{C}_c^T \overline{C}_c & \overline{C}_c^T \overline{D}_c \\ \overline{D}_c^T \overline{C}_c & \overline{D}_c^T \overline{D}_c - I \end{bmatrix},$$

$h \in \mathbb{R}_+^{n+m}$ is defined from H as in Definition 1.

Proof. Since $h \in \mathbb{R}_+^{n+m}$ is defined from H as in Definition 1, then it follows from Lemma 1 that

$$\overline{H} = hh^T - H \geq 0, \quad \overline{H}_{ii} = 0, \ \forall i = 1, 2, \ldots, n+m,$$

which yields

$$\operatorname{tr}\left(\overline{H}\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}\right) = 0.$$

Thus,

$$\operatorname{tr}\left(\overline{H}W\right) = \operatorname{tr}\left(\overline{H}\,\overline{W}\right) = \operatorname{tr}\left(\sum_{i=1}^{n+m}\sum_{j=1}^{n+m} \overline{H}_{ij} e_i e_j^T \overline{W}\right) = \sum_{i=1}^{n+m}\sum_{j=1}^{n+m} \overline{H}_{ij} e_j^T \overline{W} e_i,$$

where

$$\overline{W} = \begin{bmatrix} \overline{C}_c^T \overline{C}_c & \overline{C}_c^T \overline{D}_c \\ \overline{D}_c^T \overline{C}_c & \overline{D}_c^T \overline{D}_c \end{bmatrix},$$

$e_i, i = 1, 2, \ldots, n+m$, denotes the ith vector of the canonical basis in $\mathbb{R}^{n+m}$. Since system (1) is positive, it is easy to see that $\overline{H}_{ij} e_j^T \overline{W} e_i \geq 0$. Hence, $\operatorname{tr}(hh^T W) \geq \operatorname{tr}(HW)$. □

Lemma 5. *Suppose that* (E_c, A_c) *is regular, condition* (2) *holds and system* (1) *is positive. Consider a linear system in the form of*

$$\dot{x}(t) = \overline{M}x(t) + \overline{B}_c u(t), \tag{4}$$

under feedback $u(t) = K\overline{C}_c x(t)$, *where* $K \geq 0$ *with appropriate dimension. Then the following statements hold*:

(i) *The corresponding closed-loop system is positive.*
(ii) *The corresponding closed-loop system is Hurwitz stable if and only if there exists a PDDM* X *such that* $(\overline{M} + \overline{B}_c K\overline{C}_c)^T X + X(\overline{M} + \overline{B}_c K\overline{C}_c) \prec 0$.

Proof. (i) Since system (1) is positive, so is system (4). Under feedback $u(t) = K\overline{C}_c x(t)$, the following closed-loop system is obtained:

$$\dot{x}(t) = (\overline{M} + \overline{B}_c K\overline{C}_c)x(t).$$

Since C_c is nonnegative on the subspace χ_c,

$$x(t) = e^{(\overline{M} + \overline{B}_c K\overline{C}_c)t}x(0) = e^{\overline{M}t}(I + \overline{B}_c K\overline{C}_c t + \cdots)x(0) \geq 0, \quad \forall t \geq 0.$$

Therefore, the corresponding closed-loop system is positive.

(ii) The proof is similar to that of Theorem 2 in [24]. □

It has been proved in [23] that there exists a PDDM solution to the existed condition for $\|G\|_\infty < 1$. In [14], it has been pointed out that the existence of PDDM solution can be generalized to stability criterion for positive system (1). It is natural to ask whether a similar result is available for bounded real lemma of positive system (1). Next, we give a positive answer to this question.

Theorem 1. *Suppose that* (E_c, A_c) *is regular, condition* (2) *holds, system* (1) *is positive and* $\mathrm{ind}(E_c, A_c) = 1$. *Then the following statements are equivalent*:

(i) *System* (1) *is stable and* $\|G_c\|_\infty < \gamma$.
(ii) *There exists a PDDM* X *such that*

$$\begin{bmatrix} \overline{M}^T X + X\overline{M} + \overline{C}_c^T \overline{C}_c & X\overline{B}_c + \overline{C}_c^T \overline{D}_c \\ \overline{B}_c^T X + \overline{D}_c^T \overline{C}_c & \overline{D}_c^T \overline{D}_c - \gamma^2 I \end{bmatrix} \prec 0. \tag{5}$$

Proof. From the existed condition to check $\|G_c\|_\infty < \gamma$ for linear general systems, (ii) ⇒ (i) is immediately obtained. Now we will show that there exists $X \in \mathbb{D}_+^{n \times n}$ such that Eq. (5) holds in the case that C_c is nonnegative on the subspace χ_c. It is noted that $\|G_c\|_\infty < \gamma$ is equivalent to $\|(1/\gamma)G_c\|_\infty < 1$. Set $\overline{G}_c(s) = (1/\gamma)G_c(s)$, which is the transfer function matrix of the following system:

$$\begin{aligned} \dot{x}(t) &= \overline{M}x(t) + \overline{B}_c u(t), \\ y(t) &= \frac{1}{\gamma}\overline{C}_c x(t) + \frac{1}{\gamma}\overline{D}_c u(t). \end{aligned} \tag{6}$$

It is obvious that positivity and stability of system (6) is equivalent to these of system (1). To the contrary, suppose that there does not exist any PDDM X such that LMI (5) holds, define the

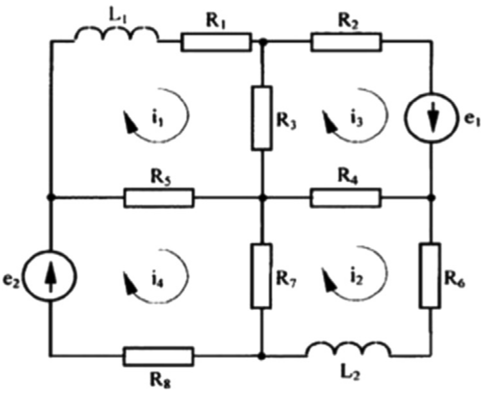

Fig. 1. A four-mesh circuit.

following two sets $\{P|P\prec 0, P\in S^{n+m}\}$ and

$$\left\{\begin{bmatrix}\overline{M}^TX+X\overline{M}+\overline{C}_c^T\overline{C}_c & X\overline{B}_c+\overline{C}_c^T\overline{D}_c\\ \overline{B}_c^TX+\overline{D}_c^T\overline{C}_c & \overline{D}_c^T\overline{D}_c-\gamma^2I\end{bmatrix}|X\in D_+^{n\times n}\right\}$$

which are both convex and do not intersect each other, then from the separating hyperplane theorem, there exists a nonzero matrix H such that

$$\mathrm{tr}(HP)\le 0,\quad \forall P\prec 0,$$
$$\mathrm{tr}\left(H\begin{bmatrix}\overline{M}^TX+X\overline{M}+\overline{C}_c^T\overline{C}_c & X\overline{B}_c+\overline{C}_c^T\overline{D}_c\\ \overline{B}_c^TX+\overline{D}_c^T\overline{C}_c & \overline{D}_c^T\overline{D}_c-\gamma^2I\end{bmatrix}\right)\ge 0,\quad \forall X\in D_+^{n\times n}.$$

Then we can conclude that there exists nonzero $H\succcurlyeq 0$ such that

$$\mathrm{tr}\left(H\begin{bmatrix}\overline{M}^TX+X\overline{M}+\overline{C}_c^T\overline{C}_c & X\overline{B}_c+\overline{C}_c^T\overline{D}_c\\ \overline{B}_c^TX+\overline{D}_c^T\overline{C}_c & \overline{D}_c^T\overline{D}_c-\gamma^2I\end{bmatrix}\right)\ge 0,\quad \forall X\in D_+^{n\times n},$$

from which it follows that

$$\mathrm{tr}\left(H\begin{bmatrix}\overline{M}^TX+X\overline{M} & X\overline{B}_c\\ \overline{B}_c^TX & 0\end{bmatrix}\right)\ge 0,\quad \forall X\in D_+^{n\times n},\tag{7}$$

$$\mathrm{tr}\left(H\begin{bmatrix}\frac{1}{\gamma^2}\overline{C}_c^T\overline{C}_c & \frac{1}{\gamma^2}\overline{C}_c^T\overline{D}_c\\ \frac{1}{\gamma^2}\overline{D}_c^T\overline{C}_c & \frac{1}{\gamma^2}\overline{D}_c^T\overline{D}_c-I\end{bmatrix}\right)\ge 0.\tag{8}$$

Since $\overline{M}$ is a Metzler matrix and $\overline{B}_c\ge 0$, it is easy to check that $X\overline{M}$ is also Metzler and $X\overline{B}_c\ge 0$. Then, according to Lemmas 1 and 4, it follows from conditions (7) and (8) that

$$\mathrm{tr}\left(hh^T\begin{bmatrix}\overline{M}^TX+X\overline{M} & X\overline{B}_c\\ \overline{B}_c^TX & 0\end{bmatrix}\right)\ge 0,\quad \forall X\in D_+^{n\times n},$$

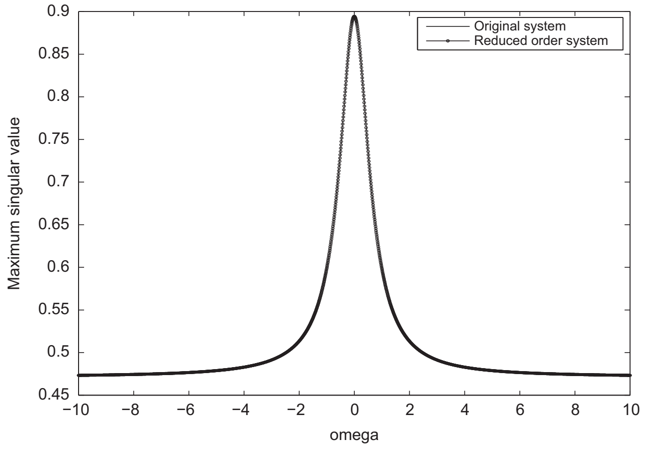

Fig. 2. The maximum singular values' plots of $G_c(s)$ and $G_{cr}(s)$.

$$\mathrm{tr}\left(hh^T \begin{bmatrix} \frac{1}{\gamma^2}\overline{C}_c^T\overline{C}_c & \frac{1}{\gamma^2}\overline{C}_c^T\overline{D}_c \\ \frac{1}{\gamma^2}\overline{D}_c^T\overline{C}_c & \frac{1}{\gamma^2}\overline{D}_c^T\overline{D}_c - I \end{bmatrix} \right) \ge 0,$$

where $h \in \mathbb{R}_+^{n+m}$ is defined as in Definition 1. Now h is partitioned as $h := [h_1^T \ h_2^T]^T$. The two above inequalities can be rewritten in the equivalent form as follows:

$$h_1^T(\overline{M}^T X + X\overline{M})h_1 + h_2^T B^T X h_1 + h_1^T X B h_2 \ge 0, \quad \forall X \in D_+^{n\times n}, \tag{9}$$

$$\left(\frac{1}{\gamma}\overline{C}_c h_1 + \frac{1}{\gamma}\overline{D}_c h_2\right)^T \left(\frac{1}{\gamma}\overline{C}_c h_1 + \frac{1}{\gamma}\overline{D}_c h_2\right) \ge h_2^T h_2. \tag{10}$$

Note that system (1) is stable. Then matrix $\overline{M}$ is Hurwitz stable. Inasmuch as h is nonzero, the following three cases are considered:

(1) $h_1 \ne 0,\ h_2 = 0$. It follows from condition (9) that $h_1^T(\overline{M}^T X + X\overline{M})h_1 \ge 0$, that is to say, there does not exist any PDDM X such that $\overline{M}^T X + X\overline{M} \prec 0$, which contradicts the fact that $\overline{M}$ is Hurwitz stable.
(2) $h_1 = 0,\ h_2 \ne 0$. By condition (10), $(1/\gamma^2)h_2^T\overline{D}_c^T\overline{D}_c h_2 \ge h_2^T h_2$ which violates the fact that $\|\overline{G_c}\|_\infty < 1$.
(3) $h_1 \ne 0,\ h_2 \ne 0$. It is easy to check that $(1/\gamma)\overline{C}_c h_1 + (1/\gamma)\overline{D}_c h_2 \ne 0$. Define

$$\Delta = h_2\left(\frac{1}{\gamma}\overline{C}_c h_1 + \frac{1}{\gamma}\overline{D}_c h_2\right)^T \Big/ \left(\frac{1}{\gamma}\overline{C}_c h_1 + \frac{1}{\gamma}\overline{D}_c h_2\right)^T \left(\frac{1}{\gamma}\overline{C}_c h_1 + \frac{1}{\gamma}\overline{D}_c h_2\right). \tag{11}$$

It can be easily seen that $\Delta \ge 0$, $\overline{\sigma}(\Delta) \le 1$ and $h_2 = (I - \Delta(1/\gamma)\overline{D}_c)^{-1}\Delta(1/\gamma)\overline{C}_c h_1 \ge 0$ since C_c is nonnegative on the subspace χ_c. It follows from Eq. (9) that

$$h_1^T\left(\left(\overline{M} + \overline{B}_c\left(I - \Delta\frac{1}{\gamma}\overline{D}_c\right)^{-1}\Delta\frac{1}{\gamma}\overline{C}_c\right)^T X\right.$$

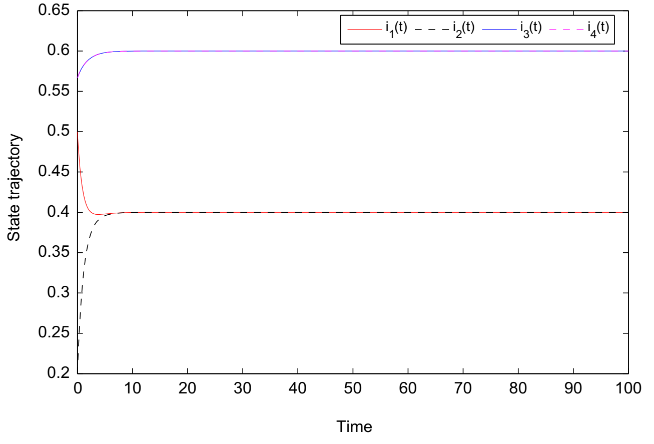

Fig. 3. The state trajectory of the original system.

$$+\left(\overline{M}+\overline{B}_c\left(I-\Delta\frac{1}{\gamma}\overline{D}_c\right)^{-1}\Delta\frac{1}{\gamma}\overline{C}_c\right)X\right)h_1\geq 0,\ \forall X\in D_+^{n\times n}.$$

According to Lemma 5, the above condition implies that $\overline{M}+\overline{B}_c(I-\Delta(1/\gamma)\overline{D}_c)^{-1}\Delta(1/\gamma)\overline{C}_c$ is unstable which contradicts the fact that $\|\overline{G}_c\|_\infty<1$. □

Remark 1. If $E_c=I$, $\gamma=1$, LMI (5) in Theorem 1 is exactly LMI (6) presented in [23]. On the other hand, it has been pointed out in [24] that $\|G_c\|_\infty=\|G_c(0)\|$ if system (1) with $E=I$ is positive and Hurwitz stable. We can obtain $\|\overline{G}_c\|_\infty=\|\overline{G}_c(0)\|$ following the same method for Corollary 2 in [24]. Therefore, for system (1) with $\mathrm{ind}(E_c,A_c)=1$, based on the positivity and stability characteristics, the exact value of H_∞ norm is given by $\|G_c\|_\infty=\|G_c(0)\|$. It is worthwhile to note that Theorem 1 can be extended to general descriptor systems. If PDDM in LMI (5) is substituted for positive definite matrix, then Theorem 1 is an H_∞ criterion for system (1) without positivity restriction.

Remark 2. It is important to mention that the magnitude of α has nothing to do with H_∞ norm of positive system (1). It is easy to see from aforementioned discussion that the transfer function matrix $G(s)$ is independent of α.

Remark 3. In general, whether a descriptor system is in normal operation largely depends on impulse. For a regular continuous descriptor system, the internal stability contains not only stability, but also impulse-free. Necessary and sufficient conditions for stability and impulse-free of positive system (1) have been established in [14,20]. However, in practice, it is not possible to be stable or impulse-free for any continuous descriptor system. Therefore, it is necessary to develop stabilization problem for system (1) with positivity preserved, which has not been well studied in the literature and is the subject of ongoing work. The greatest difficulty of this problem is checking positivity of the closed-loop system which involves the computation of Drazin inverses of the closed-loop systems' matrices.

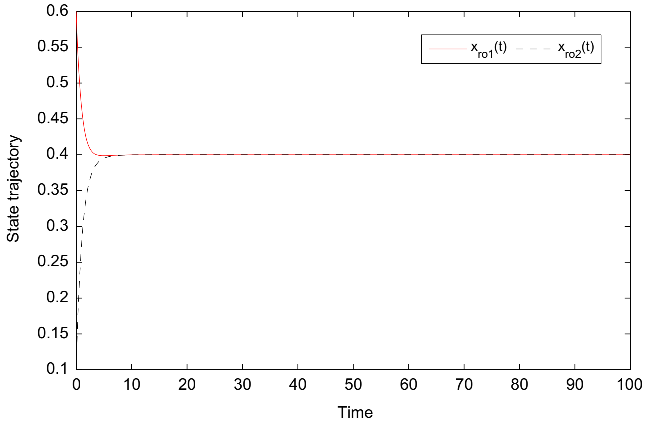

Fig. 4. The state trajectory of the reduced order system.

The following algorithm can be proposed to check H_∞ norm of positive system (1).

Step 1: Check the regularity of matrix pair (E_c, A_c). If it is regular, go to the next step, otherwise, go to end.

Step 2: Compute the index of matrix pair (E_c, A_c). If $\mathrm{ind}(E_c, A_c) = 1$, then continue, otherwise, stop.

Step 3: Compute matrices $\hat{E}_c^D \hat{E}_c$, $-(I - \hat{E}_c^D \hat{E}_c)\hat{A}_c^D \hat{B}_c$. If both are nonnegative matrices, then go to step 4, if not, go to end.

Step 4: Compute matrices $\hat{E}_c^D \hat{A}_c$, $\hat{E}_c^D \hat{B}_c$, then check the positivity of system (1) using Lemma 3. If system (1) is positive, go to step 5, otherwise, go to end.

Step 5: Choose a scalar $\alpha > 0$ arbitrarily such that $\overline{M}$ is a Metzler matrix, then solve the LMI (5) in Theorem 1. If there exists a feasible solution, positive system (1) is stable and $\| G_c \|_\infty < \gamma$.

End.

3.2. Positivity-H_∞ norm-preserving model reduction

In this subsection, a simple model reduction which can preserve the positivity, stability and H_∞ norm of positive system (1) is to be introduced.

It has been pointed out in [10] that state equation (1a) and the following equation

$$\dot{x}(t) = \overline{A}_c x(t) + \overline{B}_c u(t) - (I - \hat{E}_c^D \hat{E}_c) \sum_{i=1}^{\nu} (\hat{E}_c \hat{A}_c^D)^i \hat{A}_c^D \hat{B}_c u^{(i)}(t)$$

have the same solution (2) for any admissible initial condition $x(0)$ of system (1) and a given $u(t)$. If $\mathrm{ind}(E_c, A_c) = 1$, it is easy to verify that system (1) and the following system

$$\begin{aligned} \dot{x}(t) &= \overline{A}_c x(t) + \overline{B}_c u(t) + \overline{B}_c' \dot{u}(t), \\ y(t) &= \overline{C}_c x(t) + \overline{D}_c u(t) \end{aligned} \tag{12}$$

have the same transfer function matrix, solution and output for any admissible initial condition $x(0)$ and a given $u(t)$.

Note that $\hat{E}_c^D \hat{E}_c \overline{A}_c = \overline{A}_c \hat{E}_c^D \hat{E}_c = \overline{A}_c$. If $(\hat{E}_c^D \hat{E}_c)_{ci} = 0$, then it immediately follows by computation that $(\overline{A}_c)_{ci} = 0$, $(\overline{C}_c)_{ci} = 0$. In such case, from system (12), we can observe that the state variable $x_i(t)$ has no impact on other state variables and output. Set

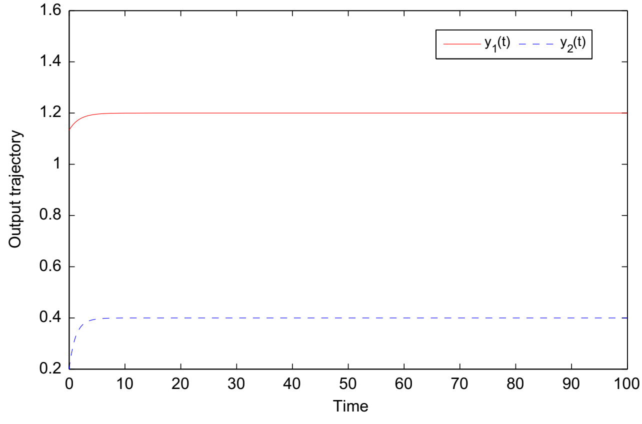

Fig. 5. The output trajectory of the original system.

$Z=\left\{i \middle| (\hat{E}_c^D\hat{E}_c)_{ci}=0\right\}$, delete the ith row and ith column of $\hat{E}_c^D\hat{E}_c$ and $\overline{A}_c$,the ith column of C_c, the ith row of $\overline{B}_c$ and $\overline{B}_c'$, then the reduced order system is given by

$$\dot{x}_{cr}(t)=(\overline{A}_c)_{ro}x_{cr}(t)+(\overline{B}_c)_{ro}u(t)+(\overline{B'}_c)_{ro}\dot{u}(t), \tag{13a}$$

$$y_{cr}(t)=(\overline{C}_c)_{ro}x_{cr}(t)+\overline{D}_c u(t), \tag{13b}$$

where $(\overline{A}_c)_{ro}$, $(\overline{B}_c)_{ro}$, $(\overline{B}_c')_{ro}$, $(\overline{C}_c)_{ro}$ are the reduced order matrices. The admissible initial condition is given by $x_{cr}(0)=(\hat{E}_c^D\hat{E}_c)_{ro}w+(\overline{B}_c')_{ro}u(0)$.

Next, we will show that the model reduction introduced above can preserve positivity, stability and H_∞ norm of positive system (1). At first, projector is introduced which is used in the proof of next theorem.

A matrix $A\in\mathbb{R}^{n\times n}$ is called a projector if $A^2=A$. If A is a projector, then there exists a nonsingular matrix T such that [14]

$$A=T^{-1}\begin{bmatrix} I & 0\\ 0 & 0\end{bmatrix}T.$$

Theorem 2. *Suppose that* (E_c,A_c) *is regular, condition* (2) *holds*, $\mathrm{ind}(E_c,A_c)=1$, *system* (1) *is positive, stable and* $\|G_c\|_\infty<\gamma$. *If the set Z is nonempty, then the reduced order system* (13) *is also positive, stable and* $\|G_{cr}\|_\infty<\gamma$, *moreover*, $\|G_c\|_\infty=\|G_{cr}\|_\infty=\|G_{cr}(0)\|$.

Proof. Since condition (2) holds and system (1) is positive, then the reduced order matrices

$$(\hat{E}_c^D\hat{E}_c)_{ro}\geq 0, \quad (\overline{B}_c)_{ro}\geq 0, \quad (\overline{B}_c')_{ro}\geq 0,$$

which leads to $x_{cr}(0)\geq 0$. If $\overline{\alpha}>0$ such that $\overline{M}$ is a Metzler matrix, it is easy to check that $\overline{M}_{ro}:=(\overline{A}_c)_{ro}+\alpha(\hat{E}_c^D\hat{E}_c)_{ro}-\alpha I$, $\forall\alpha\geq\overline{\alpha}>0$, is also a Metzler matrix. Suppose that the set Z has only one element, without loss of generality, $(\hat{E}_c^D\hat{E}_c)_{c1}=0$, which implies that $(\overline{A}_c)_{c1}=0$, $(\overline{C}_c)_{c1}=0$. Set

$$\hat{E}_c^D\hat{E}_c=\begin{bmatrix} 0_{1,1} & (\hat{E}_c^D\hat{E}_c)_{1,n-1}\\ 0_{n-1,1} & (\hat{E}_c^D\hat{E}_c)_{ro}\end{bmatrix}, \quad \overline{A}_c=\begin{bmatrix} 0_{1,1} & (\overline{A}_c)_{1,n-1}\\ 0_{n-1,1} & (\overline{A}_c)_{ro}\end{bmatrix},$$

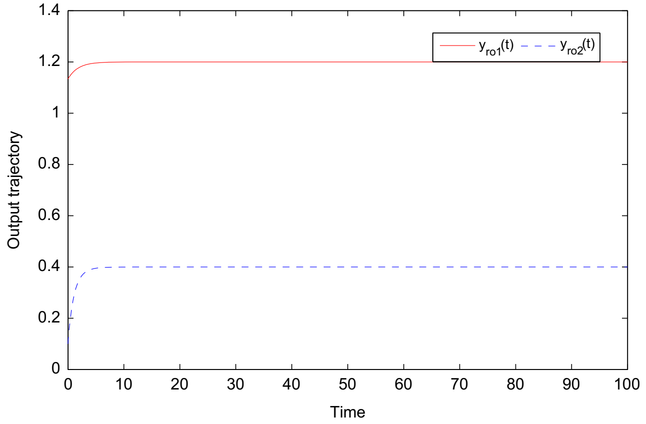

Fig. 6. The output trajectory of the reduced order system.

$$\overline{C}_c = [0_{p,1}\ \ (\overline{C}_c)_{ro}],\quad \overline{B}_c = [(\overline{B}_c)_{r1}^T\ \ (\overline{B}_c)_{ro}^T]^T,\quad \overline{B}'_c = [(\overline{B}'_c)_{r1}^T\ \ (\overline{B}'_c))_{ro}^T]^T.$$

Inasmuch as $\hat{E}_c^D\hat{E}_c\hat{E}_c^D\hat{E}_c = \hat{E}_c^D\hat{E}_c$, we have

$$\hat{E}_c^D\hat{E}_c\hat{E}_c^D\hat{E}_c = \begin{bmatrix} 0_{1,1} & (\hat{E}_c^D\hat{E}_c)_{1,n-1}(\hat{E}_c^D\hat{E}_c)_{ro} \\ 0_{n-1,1} & (\hat{E}_c^D\hat{E}_c)_{ro}(\hat{E}_c^D\hat{E}_c)_{ro} \end{bmatrix} = \begin{bmatrix} 0_{1,1} & (\hat{E}_c^D\hat{E}_c)_{1,n-1} \\ 0_{n-1,1} & (\hat{E}_c^D\hat{E}_c)_{ro} \end{bmatrix},$$

which implies $(\hat{E}_c^D\hat{E}_c)_{ro}(\hat{E}_c^D\hat{E}_c)_{ro} = (\hat{E}_c^D\hat{E}_c)_{ro}$. In a similar way, we can conclude that $(\overline{A}_c)_{ro}(\hat{E}_c^D\hat{E}_c)_{ro} = (\hat{E}_c^D\hat{E}_c)_{ro}(\overline{A}_c)_{ro} = (\overline{A}_c)_{ro}$, $(\overline{C}_c)_{ro} = (\overline{C}_c)_{ro}(\hat{E}_c^D\hat{E}_c)_{ro}$, $(\overline{B}_c)_{ro} = (\hat{E}_c^D\hat{E}_c)_{ro}(\overline{B}_c)_{ro}$, $(\overline{A}_c)_{ro}(\overline{B}'_c)_{ro} = 0$, $(\overline{C}_c)_{ro}(\overline{B}'_c)_{ro} = 0$. Then it is easy to check that

$$x(t) = e^{(\overline{A}_c)_{ro}t}(\hat{E}_c^D\hat{E}_c)_{ro}x(0) + \int_0^t e^{(\overline{A}_c)_{ro}(t-\tau)}(\overline{B}_c)_{ro}u(\tau)\,d\tau + (\overline{B}'_c)_{ro}u(t)$$

is a solution of state equation (13a). If $(\hat{E}_c^D\hat{E}_c)_{ro}$ is nonsingular, it follows from $(\hat{E}_c^D\hat{E}_c)_{ro}(\hat{E}_c^D\hat{E}_c)_{ro} = (\hat{E}_c^D\hat{E}_c)_{ro}$ that $(\hat{E}_c^D\hat{E}_c)_{ro} = I$, that is, system (13) is a standard system. Then, it is easy to see that $x_{cr}(t) \geq 0,\ t \geq 0$. On the other hand, we obtain $(\overline{C}_c)_{ro}x_{cr}(t) = \overline{C}_c x(t)$, this is due to the fact that $(\overline{C}_c)_{c1} = 0$ which implies that x_1 has no impact on output. Therefore, according to Lemma 3, $y_{cr}(t) \geq 0,\ t \geq 0$. If $(\hat{E}_c^D\hat{E}_c)_{ro}$ is singular, from Taylor expansion,

$$e^{\overline{M}_{ro}t}(\hat{E}_c^D\hat{E}_c)_{ro} = (\hat{E}_c^D\hat{E}_c)_{ro}e^{\overline{M}_{ro}t}(\hat{E}_c^D\hat{E}_c)_{ro} = e^{(\overline{A}_c)_{ro}t}(\hat{E}_c^D\hat{E}_c)_{ro} = (\hat{E}_c^D\hat{E}_c)_{ro}e^{(\overline{A}_c)_{ro}t}(\hat{E}_c^D\hat{E}_c)_{ro}.$$

Thus, it can be easily verified from Lemma 3 that $x_{cr}(t) \geq 0,\ y_{cr}(t) \geq 0,\ t \geq 0$.

Since system (1) is stable and impulse-free, from the aforementioned discussion, the number of finite eigenvalues and eigenvalue ∞ of (E_c, A_c) are r and $n-r$, respectively, which means that $\overline{A}_c$ has r finite eigenvalues with negative real parts and $n-r$ eigenvalue 0, or equivalently, $\overline{M}$ has r finite eigenvalues with negative real parts and $n-r$ eigenvalue $-\alpha$. Define

$$\overline{M} = \begin{bmatrix} 0_{1,1} & \overline{M}_{1,n-1} \\ 0_{n-1,1} & \overline{M}_{ro} \end{bmatrix}.$$

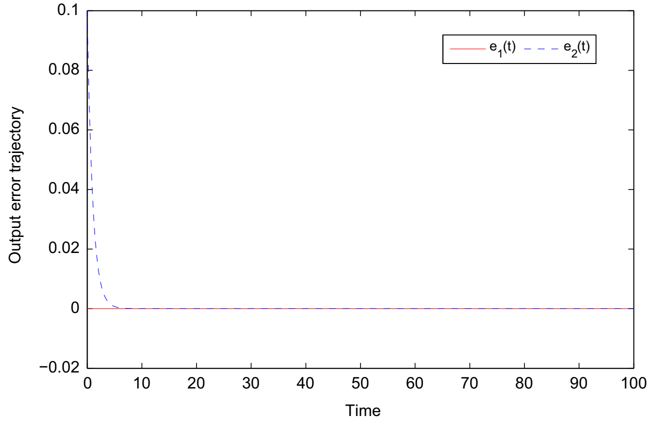

Fig. 7. The output error trajectory between the original system and the reduced order system.

Then expanding the determinant of $sI-\overline{M}$ by column, we have

$$\det(sI-\overline{M})=\det\left(\begin{bmatrix} s & \overline{M}_{1,n-1} \\ 0_{n-1,1} & sI-\overline{M}_{ro}\end{bmatrix}\right)=s\det(sI-\overline{M}_{ro}),$$

from which one can observe that the reduced order matrix $\overline{M}_{ro}$ preserves the finite eigenvalues of (E_c,A_c), in other words, $\overline{M}_{ro}$ has r finite eigenvalues with negative real parts and $n-r-1$ eigenvalue $-\alpha$. Therefore, the reduced order system (13) is stable.

According to the aforementioned analysis, the transfer function matrix of system (13) is given by

$$G_{cr}(s)=\left(\overline{C}_c\right)_{ro}\left(sI-\left(\overline{A}_c\right)_{ro}\right)^{-1}\left(\overline{B}_c\right)_{ro}+\overline{D}_c.$$

On the other hand, since $(\hat{E}_c^D\hat{E}_c)_{ro}$ is a projector, then there exists a nonsingular matrix T such that

$$(\hat{E}_c^D\hat{E}_c)_{ro}=T^{-1}\begin{bmatrix} I & 0 \\ 0 & 0\end{bmatrix}T.$$

Matrix $(\overline{A}_c)_{ro}$ is partitioned accordingly,

$$(\overline{A}_c)_{ro}=T^{-1}\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4\end{bmatrix}T,$$

from which it follows that $A_2=0$, $A_3=0$, $A_4=0$. Hence, we have

$$\begin{aligned}
G_c(s)&=\overline{C}_c(sI-\overline{M})^{-1}\overline{B}_c+\overline{D}_c\\
&=[0_{1,1}\;\;(\overline{C}_c)_{ro}]\left(sI-\begin{bmatrix} -\alpha & \overline{M}_{1,n-1} \\ 0_{n-1,1} & (\overline{A}_c)_{ro}+\alpha(\hat{E}_c^D\hat{E}_c)_{ro}-\alpha I\end{bmatrix}\right)^{-1}\begin{bmatrix}(\overline{B}_c)_{c1} \\ (\overline{B}_c)_{ro}\end{bmatrix}+\overline{D}_c\\
&=\left(\overline{C}_c\right)_{ro}\left(sI-\overline{M}_{ro}\right)^{-1}\left(\overline{B}_c\right)_{ro}+\overline{D}_c\\
&=\left(\overline{C}_c\right)_{ro}\left(sI-\left(\overline{A}_c\right)_{ro}\right)^{-1}\left(\overline{B}_c\right)_{ro}+\overline{D}_c\\
&=G_{cr}(s).
\end{aligned}$$

Therefore, $\|G_c\|_\infty = \|G_{cr}\|_\infty < \gamma$, furthermore, $\|G_{cr}\|_\infty = \|G_{cr}(0)\|$. If the set Z has two or more than two elements, in the same way, we can conclude that the reduced order system (13) is positive, stable and $\|G_{cr}\|_\infty < \gamma$. □

Remark 4. If $(\hat{E}_c^D \hat{E}_c)_{ro}$ is nonsingular, then the reduced order system (13) is a standard system, conversely, it is a descriptor system, although it is described in the form of a standard system. It is should be stressed here that this model reduction method can be applied to not only positive descriptor systems, but also general descriptor systems. However, for general descriptor systems, we cannot derive the result $\|G_{cr}\|_\infty = \|G_{cr}(0)\|$. On the other hand, this method cannot be applied to standard systems, although state variable x_i has no impact on other state variables, it may exercise a great influence on output.

Remark 5. A class of descriptor systems with some zero rows and zero columns in matrix E can be found in many practical models, such as DC motor [32], circuit network [10,30,31], and biological complex systems [32]. It is easy to check by computation that if there exists $(E_c)_{ci} = 0$, then $(\hat{E}_c^D \hat{E}_c)_{ci} = 0$. In this case, the introduced model reduction above is available. When high-order positive descriptor models are concerned, such method can be reduced order easily and efficiently, which leads to simpler analysis.

4. Bounded real lemma for discrete case

In this section, the results presented in Section 3 will be extended to discrete case.

Consider a linear discrete descriptor system of the form

$$E_d x(k+1) = A_d x(k) + B_d u(k), \tag{14a}$$

$$y(k) = C_d x(k) + D_d u(k), \tag{14b}$$

where $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$, $y(k) \in \mathbb{R}^p$ are the state, input and output vectors, respectively. E_d, A_d, B_d, C_d, D_d are real matrices with compatible dimensions, and $\operatorname{rank}(E_d) = r \le n$. System (14) is called a linear discrete standard system if $E_d = I$.

Suppose that (E_d, A_d) is regular and $\operatorname{ind}(E_d, A_d) = \nu$. An explicit solution in terms of Drazin inverses to Eq. (14a) is given by [28]

$$\begin{aligned} x(k) &= (\hat{E}_d^D \hat{A}_d)^k \hat{E}_d^D \hat{E}_d x(0) \\ &+ \sum_{i=0}^{k-1} (\hat{E}_d^D \hat{A}_d)^{k-1-i} \hat{E}_d^D \hat{B}_d u(i) - (I - \hat{E}_d^D \hat{E}_d) \sum_{i=0}^{\nu-1} (\hat{E}_d \hat{A}_d^D)^i \hat{A}_d^D \hat{B}_d u(k+i), \end{aligned} \tag{15}$$

where $\hat{E}_d = (\lambda E_d - A_d)^{-1} E_d$, $\hat{A}_d = (\lambda E_d - A_d)^{-1} A_d$, $\hat{B}_d = (\lambda E_d - A_d)^{-1} B_d$, and $x(0)$ is an admissible initial condition. For system (14), it is said to be stable if $\rho(E_d, A_d) < 1$; it is said to be causal if $\operatorname{ind}(E_d, A_d) = 1$; it is said to be admissible if it is regular, causal and stable [30].

It has been pointed out in [14] that all finite eigenvalues of (E_d, A_d) are also eigenvalues of $\overline{A}_d$ and the eigenvalue ∞ of (E_d, A_d) is mapped to the eigenvalue 0 of $\overline{A}_d$. In other words, system (14) is stable if and only if matrix $\overline{A}_d$ is Schur stable.

Definition 3 (*Virnik [14]*). System (14) is said to be positive if $x(k) \ge 0, y(k) \ge 0, k \ge 0$ for any admissible initial condition $x(0) \ge 0$ and any input $u(\tau) \ge 0, 0 \le \tau \le k + \nu - 1$.

Lemma 6 (*Virnik [14]*). *Suppose that* (E_d, A_d) *is regular,* $\operatorname{ind}(E_d, A_d) = \nu$ *and* $\hat{E}_d^D \hat{E}_d \ge 0$. *Then system* (14) *with* $D_d = 0$ *is positive if and only if* $\hat{E}_d^D \hat{A}_d \ge 0$, $\hat{E}_d^D \hat{B}_d \ge 0$,

$(I-\hat{E}_d^D\hat{E}_d)(\hat{E}_d\hat{A}_d^D)^i\hat{A}_d^D\hat{B}_d \le 0, i=0,\dots,v-1$, *and* C_d *are nonnegative on the subspace* χ_d *defined by* $\chi_d{:=}\mathrm{im}_+[\hat{E}_d^D\hat{E}_d \;\; -(I-\hat{E}_d^D\hat{E}_d)\hat{A}_d^D\hat{B}_d \;\cdots\; -(I-\hat{E}_d^D\hat{E}_d)(\hat{E}_d\hat{A}_d^D)^{v-1}\hat{A}_d^D\hat{B}_d]$.

For convenience, define

$$\overline{A}_d=\hat{E}_d^D\hat{A}_d, \quad \overline{B}_d=\hat{E}_d^D\hat{B}_d, \quad \overline{B}'_d=-(I-\hat{E}_d^D\hat{E}_d)(\hat{A}_d^D\hat{B}_d),$$
$$\overline{C}_d=C_d\hat{E}_d^D\hat{E}_d, \quad \overline{D}_d=D_d-C_d(I-\hat{E}_d^D\hat{E}_d)(\hat{A}_d^D\hat{B}_d).$$

Suppose that (E_d,A_d) is regular, $\hat{E}_d^D\hat{E}_d\ge 0$ and system (14) is positive. Consider a linear system given by

$$x(k+1)=\overline{A}_dx(k)+\overline{B}_du(k),$$

under feedback $u(k)=K\overline{C}_dx(k)$, where $K\ge 0$. Then the corresponding closed-loop system is also positive. Furthermore, it is Schur stable if and only if there exists $X\in D_+^{n\times n}$ such that $(\overline{A}_d+\overline{B}_dK\overline{C}_d)^TX(\overline{A}_d+\overline{B}_dK\overline{C}_d)-X\prec 0$.

The transfer function matrix of system (14) is given by

$$G_d(z)=C_d(zE_d-A_d)^{-1}B_d+D_d, \quad s\in\mathbb{C}/\lambda(E_d,A_d),$$

and its H_∞ norm is defined as $\|G_d\|_\infty{:=}\sup\overline{\sigma}(G_d(e^{j\theta})), \theta\in[0,2\pi)$. Similarly, if system (14) is admissible, then $G_d(s)=\overline{G}_d(s)$, where $\overline{G}_d(s)=\overline{C}_d(sI-\overline{A}_d)^{-1}\overline{B}_d+\overline{D}_d$.

Throughout this section, it is assumed that $D_d\ge 0$ unless otherwise specified. Then the following result is obtained.

Theorem 3. *Suppose that* (E_d,A_d) *is regular,* $\hat{E}_d^D\hat{E}_d\ge 0$, *system* (14) *is positive and* $\mathrm{ind}(E_d,A_d)=1$. *The following statements are equivalent*:

(i) *System* (14) *is stable and* $\|G_d\|_\infty<\gamma$.
(ii) *There exists a PDDM X such that*

$$\begin{bmatrix} \overline{A}_d^TX\overline{A}_d-X+\overline{C}_d^T\overline{C}_d & \overline{A}_d^TX\overline{B}_d+\overline{C}_d^T\overline{D}_d \\ \overline{B}_d^TX\overline{A}_d+\overline{D}_d^T\overline{C}_d & \overline{B}_d^TX\overline{B}_d+\overline{D}_d^T\overline{D}_d-\gamma^2I \end{bmatrix}\prec 0. \quad (16)$$

Proof. The proof is similar to that of Theorem 1. To the contrary, suppose that Eq. (16) does not hold for any PPDM X. Then there exists a nonzero matrix $H\succcurlyeq 0$ such that

$$\mathrm{tr}\left(hh^T\begin{bmatrix} \overline{A}_d^TX\overline{A}_d-X & \overline{A}_d^TX\overline{B}_d \\ \overline{B}_d^TX\overline{A}_d & \overline{B}_d^TX\overline{B}_d \end{bmatrix}\right)\ge \mathrm{tr}\left(H\begin{bmatrix} \overline{A}_d^TX\overline{A}_d-X & \overline{A}_d^TX\overline{B}_d \\ \overline{B}_d^TX\overline{A}_d & \overline{B}_d^TX\overline{B}_d \end{bmatrix}\right)\ge 0, \quad \forall X\in D_+^{n\times n},$$

$$\mathrm{tr}\left(hh^T\begin{bmatrix} \frac{1}{\gamma^2}\overline{C}_d^T\overline{C}_d & \frac{1}{\gamma^2}\overline{C}_d^T\overline{D}_d \\ \frac{1}{\gamma^2}\overline{D}_d^T\overline{C}_d & \frac{1}{\gamma^2}\overline{D}_d^T\overline{D}_d-I \end{bmatrix}\right)\ge \mathrm{tr}\left(H\begin{bmatrix} \frac{1}{\gamma^2}\overline{C}_d^T\overline{C}_d & \frac{1}{\gamma^2}\overline{C}_d^T\overline{D}_d \\ \frac{1}{\gamma^2}\overline{D}_d^T\overline{C}_d & \frac{1}{\gamma^2}\overline{D}_d^T\overline{D}_d-I \end{bmatrix}\right)\ge 0.$$

Partition $h{:=}[h_1^T \; h_2^T]^T$, where $h_1\in\mathbb{R}_+^n$, $h_2\in\mathbb{R}_+^m$, the two above conditions can be rewritten as follows:

$$h_1^T(\overline{A}_d^TXA_d-X)h_1+h_2^TB_d^TXA_d+h_1^T\overline{A}_d^TXB_dh_2+h_2^TB_d^TXBh_2\ge 0, \quad \forall X\in D_+^{n\times n}, \quad (17)$$

$$\left(\frac{1}{\gamma}\overline{C}_d h_1 + \frac{1}{\gamma}\overline{D}_d h_2\right)^T \left(\frac{1}{\gamma}\overline{C}_d h_1 + \frac{1}{\gamma}\overline{D}_d h_2\right) \geq h_2^T h_2. \tag{18}$$

Since H is nonzero, it only needs to consider the following three cases:

(1) $h_1 = 0, h_2 \neq 0$. Condition (18) violates $\|\overline{G}_d\|_\infty < \gamma$.
(2) $h_1 \neq 0, h_2 = 0$. From Eq. (17), $h_1^T(\overline{A}_d^T X A_d - X)h_1 \geq 0, \forall X \in D_+^{n\times n}$, which means that there is no PDDM X such that $\overline{A}_d^T X A_d - X \prec 0$. This is a contradiction.
(3) $h_1 \neq 0, h_2 \neq 0$. Define Δ as in Eq. (11). Then from Eq. (17), it follows that

$$h_1^T\left(\tilde{A}_d^T X \tilde{A}_d - X\right)h_1 \geq 0, \quad \forall X \in D_+^{n\times n},$$

where $\tilde{A}_d$:=$\overline{A}_d + \overline{B}_d(I - \Delta(1/\gamma)\overline{D}_d)^{-1}\Delta(1/\gamma)\overline{C}_d$. Then $\rho(\tilde{A}_d) \geq 1$ which contradicts $\|\overline{G}_d\|_\infty < \gamma$. □

Remark 6. If $E_d = I$, $\gamma = 1$, LMI (16) in Theorem 3 is exactly LMI condition to check $\|G_d\|_\infty < 1$ given in [25]. Similar to continuous case, the exact value of H_∞ norm of positive system (14) can be computed directly by $\|G_d(1)\|$, in other words, $\|G_d\|_\infty = \|G_d(1)\|$.

In [25], a necessary and sufficient condition in the form of linear programming to check $\|G_d\|_\infty < 1$ for positive standard systems has been presented if m=p. Such result can also be extended to positive descriptor system (14).

Corollary 1. *Suppose that* (E_d, A_d) *is regular,* $\hat{E}_d^D \hat{E}_d \geq 0$*, system* (14) *is positive and* $\text{ind}(E_d, A_d) = 1$. *If* m=p*, then the following statements are equivalent*:

(i) *System* (14) *is stable and* $\|G_d\|_\infty < \gamma$.
(ii) *There exists a vector* $\beta \gg 0$ *such that*

$$\begin{bmatrix} \overline{A}_d - I & \overline{B}_d \\ \overline{C}_d & \overline{D}_d - \gamma I \end{bmatrix}^T \beta \ll 0.$$

As shown in [31], system (14) and the following system

$$x(k+1) = \overline{A}_d x(k) + \overline{B}_d u(k) + \overline{B}_d' u(k+1),$$
$$y(k) = \overline{C}_d x(k) + \overline{D}_d u(k),$$

have the same transfer function matrix, solution (15) and output for any admissible initial condition $x(0)$ and a given $u(k)$.

If Z_d:=$\{i|(\hat{E}_c^D \hat{E}_c)_{ci} = 0\}$ is nonempty, applying the same method for continuous case, the reduced order system is described as

$$x_{dr}(k+1) = (\overline{A}_d)_{ro} x_{dr}(k) + (\overline{B}_d)_{ro} u(k) + (\overline{B}_d')_{ro} u(k+1),$$
$$y_{dr}(k) = (\overline{C}_d)_{ro} x_{dr}(k) + \overline{D}_d u(k). \tag{19}$$

The transfer function matrix of system (19) is given by

$$G_{dr}(s) = (\overline{C}_d)_{ro}\left(sI - (\overline{A}_d)_{ro}\right)^{-1}(\overline{B}_d)_{ro} + \overline{D}_d.$$

Theorem 4. *Suppose that* (E_d, A_d) *is regular,* $\hat{E}_d^D \hat{E}_d \geq 0$, *system* (14) *is positive,* $\text{ind}(E_d, A_d) = 1$, *stable and* $\| G_d \|_\infty < \gamma$. *If the set* Z_d *is nonempty, then the reduced order system* (19) *is also positive, stable and* $\| G_{dr} \|_\infty < \gamma$, *moreover,* $\| G_d \|_\infty = \| G_{dr} \|_\infty = \| G_{dr}(1) \|$.

Proof. It follows exactly the same line of Theorem 2. □

Remark 7. Compared with the classical model reduction technique [27] which tackles positive standard systems, our model reduction method is only applicable to positive descriptor systems in that the set $Z_d := \left\{ i | (\hat{E}_c^D \hat{E}_c)_{ci} = 0 \right\}$ must be nonempty. Even though we cannot give a numerical comparison between the method in [27] and that in this paper, the numerical example in the next section also shows that under the assumption that the initial condition of the original system and the reduced order system are different and the input vector is the same, the output vector of reduced order system asymptotically approaches that of the original system. It should also be mentioned that using our proposed method, the norm of the transfer function matrix of the original system equals to that of the reduced order system, which means under the same input vector, the norms of the output vector of the two systems are equal. In the classical H_∞ model reduction setting, an additional performance index $\| y(t) - y_{ro}(t) \| < \gamma \| u(t) \|$ needs to be satisfied, while using our method, when $\| y(t) \| < \frac{1}{2} \gamma \| u(t) \|$ and $\| y_{ro}(t) \| < \frac{1}{2} \gamma \| u(t) \|$ hold, it suffices to prove $\| y(t) - y_{ro}(t) \| < \gamma \| u(t) \|$. In fact, based on the norm inequality $\| y(t) - y_{ro}(t) \| \leq \| y(t) \| + \| y_{ro}(t) \|$, we have $\| y(t) - y_{ro}(t) \| \leq \| y(t) \| + \| y_{ro}(t) \| < \gamma \| u(t) \|$ which is the classical performance index.

5. Numerical example

In this section, a numerical example is given to illustrate the effectiveness of the obtained results.

Now consider a linear electrical circuit consisting of resistances, inductances and source voltages (see Fig. 1) which is introduced in [10] as an example of weakly positive descriptor systems. As pointed out in [10], using the mesh method, the following equations can be derived:

$$L_1 \frac{di_1}{dt} = -(R_1 + R_3 + R_5) i_1 + R_3 i_3 + R_5 i_4,$$

$$\begin{aligned} L_2 \frac{di_2}{dt} &= -(R_4 + R_6 + R_7) i_2 + R_4 i_3 + R_7 i_4, \\ 0 &= R_3 i_1 + R_4 i_2 - (R_2 + R_3 + R_4) i_3 + e_1, \\ 0 &= R_5 i_1 + R_7 i_2 - (R_5 + R_7 + R_8) i_3 + e_2. \end{aligned}$$

The mesh currents $x_1 = i_1,\ x_2 = i_2,\ x_3 = i_3,\ x_4 = i_4$ are chosen as the state variables, and the voltages $u_1 = e_1$, $u_2 = e_2$ and $y_1 = L di_1/dt + R_{11} i_1, y_2 = R_6 i_2$ are chosen as the input and output variables, respectively, then the system can be written in the form of system (1) with

$$E_c = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_c = \begin{bmatrix} -R_{11}/L_1 & 0 & R_{13}/L_1 & R_{14}/L_1 \\ 0 & -R_{22}/L_2 & R_{23}/L_2 & R_{24}/L_2 \\ R_{31} & R_{32} & -R_{33} & 0 \\ R_{41} & R_{42} & 0 & -R_{44} \end{bmatrix},$$

$$B_c = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C_c = \begin{bmatrix} 0 & 0 \\ 0 & R_6 \\ R_3 & 0 \\ R_5 & 0 \end{bmatrix}^T, \quad D_c = 0,$$

$$R_{11} = R_1 + R_3 + R_5, \quad R_{13} = R_{31} = R_3, \quad R_{14} = R_{41} = R_5,$$

$$R_{22} = R_4 + R_6 + R_7, \quad R_{23} = R_{32} = R_4, \quad R_{24} = R_{42} = R_7,$$

$$R_{33} = R_2 + R_3 + R_4, \quad R_{44} = R_5 + R_7 + R_8.$$

Let $R_1 = R_2 = R_3 = R_4 = R_5 = R_6 = R_7 = R_8 = 1, L_1 = L_2 = 3$. Then by computation, (E_c, A_c) is regular, $\text{ind}(E_c, A_c) = 1$ and

$$\hat{E}_c^D \hat{E}_c = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & 0 \end{bmatrix}, \quad -\left(I - \hat{E}_c^D \hat{E}_c\right)\left(\hat{A}_c^D \hat{B}_c\right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{bmatrix},$$

$$\hat{E}_c^D \hat{A}_c = \begin{bmatrix} \frac{-7}{9} & \frac{2}{9} & 0 & 0 \\ \frac{2}{9} & \frac{-7}{9} & 0 & 0 \\ \frac{-5}{27} & \frac{-5}{27} & 0 & 0 \\ \frac{-5}{27} & \frac{-5}{27} & 0 & 0 \end{bmatrix} \overline{B}_c = \begin{bmatrix} \frac{1}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{9} \\ \frac{2}{27} & \frac{2}{27} \\ \frac{2}{27} & \frac{2}{27} \end{bmatrix},$$

$$\overline{C}_c = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad -C_c\left(I - \hat{E}_c^D \hat{E}_c\right)\left(\hat{A}_c^D \hat{B}_c\right) = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ 0 & 0 \end{bmatrix}.$$

It is straightforward to see that $\overline{M} := \hat{E}_c^D \hat{A}_c + \alpha \hat{E}_c^D \hat{E}_c - \alpha I$ is a Metzler matrix for any $\alpha \geq \frac{5}{9}$. According to Lemma 3, the system is a positive descriptor system. Then choosing $\alpha = 1$ and solving condition (5) in Theorem 1, one feasible solution can be obtained as

$$\gamma = 0.8945, \quad X = \begin{bmatrix} 4.4728 & 0 & 0 & 0 \\ 0 & 5.5908 & 0 & 0 \\ 0 & 0 & 0.0021 & 0 \\ 0 & 0 & 0 & 0.0021 \end{bmatrix}.$$

By computation, $\| G_c(0) \| = \| \overline{G}_c(0) \| = 0.8944$, which shows that the exact value of $\| G_c \|_\infty$ is closed to the optimal value γ in LMI (5). Let $\alpha = 3$, the following feasible solution to LMI (6) is obtained:

$$\gamma = 0.8945, \quad X = \begin{bmatrix} 4.4738 & 0 & 0 & 0 \\ 0 & 5.5919 & 0 & 0 \\ 0 & 0 & 0.0005 & 0 \\ 0 & 0 & 0 & 0.0005 \end{bmatrix}.$$

Direct computation shows that $\| G_c(0) \| = \| \overline{G}_c(0) \| = 0.8944$. For $\alpha = 8$, one feasible solution to LMI (5) is

$$\gamma = 0.8945, \quad X = \begin{bmatrix} 4.4736 & 0 & 0 & 0 \\ 0 & 5.5914 & 0 & 0 \\ 0 & 0 & 0.0002 & 0 \\ 0 & 0 & 0 & 00002 \end{bmatrix}.$$

$\| G_c(0) \| = \| \overline{G}_c(0) \| = 0.8944$ also holds. From the obtained feasible solutions, one can see that the magnitude of α has nothing to do with H_∞ norm of this system. However, it has impact on the diagonal entries of X. It can be easily seen that the bigger the magnitude of α, the smaller the last two diagonal entries of X. Therefore, the above facts illustrate the effectiveness of the theoretical results presented in this paper.

Note that $(\hat{E}_c^D \hat{E}_c)_{c3} = 0, (\hat{E}_c^D \hat{E}_c)_{c4} = 0$, then we have

$$(\hat{E}_c^D \hat{E}_c)_{ro} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad (\overline{A}_c)_{ro} = \begin{bmatrix} \frac{-7}{9} & \frac{2}{9} \\ \frac{2}{9} & \frac{-7}{9} \end{bmatrix},$$

$$(\overline{B}_c)_{ro} = \begin{bmatrix} \frac{1}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{9} \end{bmatrix}, \quad (\overline{B}_c^{'})_{ro} = 0, \quad (\overline{C}_c)_{ro} = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} \\ 0 & 1 \end{bmatrix}.$$

The positive descriptor system can be reduced to the following system:

$$\dot{x}_{ro}(t) = \begin{bmatrix} \frac{-7}{9} & \frac{2}{9} \\ \frac{2}{9} & \frac{-7}{9} \end{bmatrix} x_{ro}(t) + \begin{bmatrix} \frac{1}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{9} \end{bmatrix} u(t),$$

$$y_{ro}(t) = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} \\ 0 & 1 \end{bmatrix} x_{ro}(t) + \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ 0 & 0 \end{bmatrix} u(t),$$

Obviously, the reduced order system is a positive standard system. Solving the LMI in Theorem 1, one feasible solution is

$$\gamma = 0.8945, \quad X = \begin{bmatrix} 4.4721 & 0 \\ 0 & 5.5902 \end{bmatrix}.$$

Direct computation also shows that $\| G_{cr} \|_\infty = \| G_{cr}(0) \| = 0.8944$. Fig. 2 shows the maximum singular values of $G_c(j\omega)$ and $G_{cr}(j\omega)$, $\omega \in [-10, 10]$, from which we can see that all the maximum singular values are smaller than 0.8945 and plots of original system and the reduced order system overlap completely. This fact shows that this model reduction method can be used to simplify analysis for high-order positive descriptor systems with some zero columns in derivative matrix E. Suppose that the initial state vector of the original system and that of the reduced order system are $x_0 = [0.5\ 0.2\ 0.57\ 0.57]^T$ and $x_0 = [0.6\ 0.1]^T$ respectively, the voltages $u_1 = u_2 = 1$, then simulation results are present in Figs. 3–7. Among them, Figs. 3 and 4 show the state trajectory of the original system and that of the reduced order system, Figs. 5 and 6 depict the output trajectory of the original system and that of the reduced order system, the output error between the original system and the reduced order system is also provided in Fig. 7.

6. Conclusion

In this paper, bounded real lemmas for linear continuous and discrete positive descriptor systems have been investigated. By using the separating hyperplane theorem, necessary and sufficient conditions to check $\|G_c\|_\infty < \gamma$ have been presented, which show that under the assumption of positivity characteristics, there exist PDDM solutions to the obtained LMIs conditions. Moreover, a simple model reduction which can preserve positivity, stability and H_∞ norm of original systems is proposed. Finally, an example is provided to demonstrate the effectiveness of the theoretical results. However, we have only restricted our attention to the analysis of H_∞ norm in this paper, an open problem is the development of stabilization of positive descriptor systems with positivity preserved, which is left for future research.

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