On the structure of Foulkes modules for the symmetric group

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Abstract

This thesis concerns the structure of Foulkes modules for the symmetric group. We study 'ordinary' Foulkes modules $H^{(m^n)}$, where m and n are natural numbers, which are permutation modules arising from the action on cosets of $\mathfrak{S}_m \wr \mathfrak{S}_n \leq \mathfrak{S}_{mn}$. We also study a generalisation of these modules $H_\nu^{(m^n)}$, labelled by a partition ν of n, which we call *generalised Foulkes modules*.

Working over a field of characteristic zero, we investigate the module structure using semistandard homomorphisms. We identify several new relationships between irreducible constituents of $H^{(m^n)}$ and $H^{(m^{n+q})}$, where q is a natural number, and also apply the theory to *twisted Foulkes modules*, which are labelled by $\nu = (1^n)$, obtaining analogous results.

We make extensive use of character-theoretic techniques to study $\varphi_\nu^{(m^n)}$, the ordinary character afforded by the Foulkes module $H_\nu^{(m^n)}$, and we draw conclusions about near-minimal constituents of $\varphi_{(n)}^{(m^n)}$ in the case where m is even. Further, we prove a recursive formula for computing character multiplicities of any generalised Foulkes character $\varphi_\nu^{(m^n)}$, and we decompose completely the character $\varphi_\nu^{(2^n)}$ in the cases where ν has either two rows or two columns, or is a hook partition.

Finally, we examine the structure of twisted Foulkes modules in the modular setting. In particular, we answer questions about the structure of $H_{(1^n)}^{(2^n)}$ over fields of prime characteristic.

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Chapter 1

Introduction and Overview

The main object of study in this thesis is a permutation module $H^{(m^n)}$ for the symmetric group – called the Foulkes module – which arises from the action of $\mathfrak{S}_{mn}$ on the collection of set partitions of a set of size mn into n sets, each of size m. Equivalently, $H^{(m^n)}$ is the $k\mathfrak{S}_{mn}$-module obtained by inducing the trivial module for the imprimitive wreath product $\mathfrak{S}_m \wr \mathfrak{S}_n$. In this work, it will be convenient for us to exploit both of these descriptions of the Foulkes module.

The results in this thesis all address the task of understanding the structure of Foulkes modules, which remains an open problem in the representation theory of symmetric groups. Principally, we will work over $\mathbb{C}$ – but this can be replaced by any field of characteristic zero – since this is also the setting for *Foulkes' Conjecture*, which provides us with some additional motivation for determining the module decomposition. This conjecture, made by H. O. Foulkes in [15], states that, for all $m, n \in \mathbb{N}$ with $m < n$, there exists an injective $\mathbb{C}\mathfrak{S}_{mn}$-homomorphism $H^{(n^m)} \hookrightarrow H^{(m^n)}$.

By 1950, when Foulkes made his conjecture, Thrall had already successfully decomposed $H^{(2^r)}$ and $H^{(r^2)}$ in his paper [43] – from which the proof of Foulkes' Conjecture in the case $m = 2$ follows – and, in [32], Littlewood had given explicit descriptions of $H^{(3^n)}$ for $n \leq 6$ and $H^{(4^n)}$ for $n \leq 5$. Thus, when Foulkes decomposed $H^{(5^n)}$ and $H^{(6^n)}$ for $2 \leq n \leq 4$, and noticed that the summands of $H^{(5^4)}$ all appeared in the decomposition of $H^{(4^5)}$, he made his claim. Foulkes wrote:

> "The theorem is that for integers m, n, where $n > m$, the product $\{m\} \otimes \{n\}$ includes all terms of $\{n\} \otimes \{m\}$."

The product to which Foulkes refers is the *plethysm* of symmetric functions, a notion that was introduced by Littlewood in the 1936 paper [31]. The aforementioned decompositions due to Thrall, Littlewood and Foulkes were all in fact stated as decompositions of plethysms $\{m\} \otimes \{n\}$ rather than decompositions of the corresponding Foulkes modules $H^{(m^n)}$. Nevertheless, they are entirely equivalent statements. We shall discuss symmetric functions and plethysm multiplication, including the equivalence of the two versions of Foulkes' Conjecture, in more detail in Chapter 3. At the same time, we will also see that a statement of Foulkes' Conjecture can be given in terms of modules for the general linear group. Whilst success has been had in studying Foulkes modules using these methods, we will not adopt such approaches in this work.

Despite the fact that Foulkes' Conjecture can be tackled from a range of perspectives, it has only been proved to hold when $m \leq 4$ (work by Thrall [43], Dent and Siemons [11], and McKay [36]), $m + n \leq 19$ (work by Müller and Neunhöffer [37], and Evseev, Paget and Wildon [13]), and when n is comparatively larger than m (an asymptotic result due to Brion [4]). Similarly, the structure of the Foulkes module is only fully understood for very small m and n: in addition to the results which led to Foulkes' Conjecture, the decomposition of $H^{(m^n)}$ into irreducible modules is only known when $n = 3$ and $n = 4$ (see [11, 43] and [12, 22], respectively).

We now outline the structure of this thesis, alluding to the main results that will be proved.

In Chapter 2, we introduce all of the background material that we will require. In particular, we discuss some general results from representation theory that will be in constant use, before concentrating on the representation theory of the symmetric group. At this point, we begin to see the combinatorial nature of the topic, which will also be a feature of many of our results.

In Chapter 3, we define Foulkes modules and we aim to make transparent the relationships between the representation theory of the symmetric group, the representation theory of the general linear group and symmetric functions. We will also define generalised Foulkes modules $H_\nu^{(m^n)}$, where ν is a partition of n, of which Foulkes modules $H^{(m^n)}$ and twisted Foulkes modules $H_{(1^n)}^{(m^n)}$ are special cases. To complete the chapter, we highlight the progress that has been made to date in understanding the structure of (generalised) Foulkes modules, demonstrating the place of our results within the existing literature.

Chapters 4 and 5 feature semistandard homomorphisms as a tool for studying the structure of Foulkes modules. For both of these chapters, we are required to work over a field of characteristic zero. In Chapter 4, we illustrate the technique for Foulkes modules and show how the existing theory can be adapted to also study the structure of twisted Foulkes modules. Subsequently, we extend some existing results to the 'twisted setting', including Foulkes' Second Conjecture, which also featured in [15]. In Chapter 5, we make progress in understanding the composition factors of $H^{(m^n)}$ in the case where m is even. We prove two new relationships between irreducible constituents of $H^{(m^n)}$ and $H^{(m^{n+q})}$, where q is a natural number. We indicate when these results have analogues for twisted Foulkes modules and prove the (suitably adjusted) results.

We continue to focus on the case where m is even in Chapter 6, proving results about near-minimal constituents of $H^{(m^n)}$. The approach we take is to study the ordinary character $\varphi^{(m^n)}$ afforded by the Foulkes module – an entirely equivalent problem in the characteristic zero setting – using character-theoretic techniques. Again, where appropriate, we prove analogues of the results for twisted Foulkes characters. We subsequently continue our investigation of near-minimal constituents, proving additional results in the case where $m = 4$: we give a complete description of all constituents of $\varphi^{(4^n)}$ that are labelled by partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ of $4n$ satisfying $\lambda_1 \leq 7$ and $\lambda_2 < 7$. In the process, we identify the lexicographically smallest constituent of $\varphi^{(4^n)}$ that is labelled by a partition that has an odd part.

A key result used in Chapter 6 is a recursive formula that makes it possible to compute the multiplicity of any irreducible character in the Foulkes character $\varphi^{(m^n)}$. This formula – proved by Evseev, Paget and Wildon in [13] and used by the authors to verify Foulkes' Conjecture computationally for $m+n \leq 19$ – is in fact a special case of a new, more general formula: we prove the more general result, which enables us to calculate character multiplicities for any of the generalised Foulkes characters $\varphi_\nu^{(m^n)}$.

In the latter part of this thesis, we study generalised Foulkes modules in more detail. In Chapter 7, we obtain formulae that decompose completely the characters $\varphi_\nu^{(2^n)}$ when ν is a partition of n with two rows or two columns, or when ν is a hook partition. For any ν taking one of these forms, it turns out that the multiplicities with which the irreducible characters appear in $\varphi_\nu^{(m^n)}$ are determined by Littlewood–Richardson coefficients. We subsequently apply the formulae to obtain the explicit decomposition of $\varphi_\nu^{(2^n)}$ in a few special cases, yielding some quite elegant results. We note that the formulae that we obtain in this chapter are very different from the recursive formula in Chapter 6, which would – if given a specific n and ν – yield the same decomposition.

Finally, we turn our attention to Foulkes modules defined over a field of characteristic p. We employ the computational algebra software MAGMA to determine the structure of the twisted Foulkes module $H_{(1^n)}^{(2^n)}$ explicitly when $n = 2$, $n = 3$ and $n = 4$. Subsequently, we return to using algebraic methods from modular representation theory to investigate the module structure and we concentrate on the structure of $H_{(1^n)}^{(2^n)}$ when n is at most p.

Chapter 2

Preliminaries

In this opening chapter, we cover all of the preliminary material that will be used later. In particular, we give an introduction to the representation theory of the symmetric group, presenting the basic definitions and results upon which we will rely heavily. We do not intend to give a comprehensive exposition and so, where appropriate, we will recommend references that provide proofs and a more thorough discussion of the topic.

We will assume that the reader has a good knowledge of the following topics:

- the group algebra (of a finite group);
- indecomposable and irreducible modules;
- semisimple modules;
- homomorphisms between modules;
- direct sums;
- direct and semidirect products;
- inner and outer tensor products;
- composition factors;
- radicals and socles;
- ordinary characters.

Throughout this thesis, k denotes an algebraically closed field and $\mathfrak{S}_n$ denotes the symmetric group on a set of n symbols. Additionally, we write $k\mathfrak{S}_n$ for the group algebra of the symmetric group. We will only be dealing with right modules; with this as a standing assumption we will, from now on, simply refer to modules. Similarly, compositions of maps and products of permutations should be read from left to right.

2.1 Definitions and general representation theory

2.1.1 Induction and restriction

Let G be a finite group and let H be a subgroup of G. Given a kG-module M, we can very naturally obtain a kH-module $M\downarrow_H$ by considering only the action of kH on M. We call the resulting module the *restriction* of M to H. We can also construct a kG-module from a given kH-module, N. In particular, we define the *induced* module to be the vector space

$$N\uparrow^G = N \otimes_{kH} kG,$$

with action arising by linearly extending $(n \otimes a)g = n \otimes ag$ for $n \in N$, $a \in kG$, $g \in G$. We will also use the notation $\uparrow$ and $\downarrow$ when denoting the characters afforded by induced and restricted modules.

In [2, §8], Alperin collects together many useful facts about induction and restriction of modules, some of which we now state.

LEMMA 2.1.1
Let M, M_1, M_2 be kG-modules and let N, N_1, N_2 be kH-modules.

1. $(M_1 \oplus M_2)\downarrow_H \cong M_1\downarrow_H \oplus M_2\downarrow_H$ *and* $(N_1 \oplus N_2)\uparrow^G \cong N_1\uparrow^G \oplus N_2\uparrow^G$;
2. *If L is a subgroup of H and X is a kL-module, then* $\left(X\uparrow^H\right)\uparrow^G \cong X\uparrow^G$;
3. $M \otimes \left(N\uparrow^G\right) \cong (M\downarrow_H \otimes N)\uparrow^G$;
4. $(N\uparrow^G)^* \cong (N^*)\uparrow^G$.

Proof. See Lemma 8.5 in [2]. ■

We now state a particularly important relationship between induced and restricted modules, which we shall often need to exploit.

THEOREM 2.1.2 (FROBENIUS RECIPROCITY)
If M is a kG-module and N is a kH-module, then

1. $\mathrm{Hom}_{kG}\left(N\uparrow^G, M\right) \cong \mathrm{Hom}_{kH}\left(N, M\downarrow_H\right)$;
2. $\mathrm{Hom}_{kG}\left(M, N\uparrow^G\right) \cong \mathrm{Hom}_{kH}\left(M\downarrow_H, N\right)$.

Proof. See Lemma 8.6 in [2]. ■

Only part 1 of Theorem 2.1.2 is generally applicable to modules over a finite dimensional algebra. However, since the group algebra is a symmetric algebra, we know that it is self-dual as a module over itself. Exploiting this duality and using part 1 leads to a proof of part 2 of the theorem.

We conclude this section with a brief discussion of double cosets, so that we may state Mackey's Theorem, a result which describes the restriction of an induced module.

Let H and L be subgroups of G. Given $x \in G$, the subset $HxL := \{hx\ell \mid h \in H, \ell \in L\}$ is a *double coset* of H and L in G. The set of all double cosets is denoted by $H\backslash G/L$. If $S \subseteq G$ is a set of representatives of double cosets of H and L in G, then G can be written as the disjoint union

$$G = \bigcup_{s \in S} HsL.$$

THEOREM 2.1.3 (MACKEY'S THEOREM)
Let H and L be subgroups of G, and let S be a set of representatives of double cosets of H and L in G. Further, let N be a kH-module. Given $s \in S$, define a $k(s^{-1}Hs)$-module $N^s := \{s^{-1}ns \mid n \in N\}$, with action defined by $(s^{-1}ns)(s^{-1}hs) = s^{-1}(nh)s$ for $n \in N$ and $h \in H$, which corresponds to N in the obvious way. With this notation,

$$(N\uparrow^G)\downarrow_L \cong \bigoplus_{s \in S} (N^s \downarrow_{s^{-1}Hs\cap L})\uparrow^L.$$

Proof. See Lemma 8.7 in [2]. ■

2.1.2 Permutation modules

In this section, we introduce a special type of kG-module. A kG-module is a *permutation module* if it has a basis on which G acts as a permutation group. Given a G-set Ω, by which we mean a set with a (right) action of G, we can construct a permutation module

$$M_\Omega := \left\{ \sum_{\omega \in \Omega} c_\omega \omega \;\middle|\; c_\omega \in k \right\}$$

with action defined by linearly extending the action of G on Ω. In §2.2, we will discuss in some detail a particular example of such a module, called the Young permutation module. Of course, for us, the most important example of a permutation module is the Foulkes module, which we will introduce in Chapter 3. For now, we collect together some results concerning permutation modules, as Feit does in [14, Chapter IX: §3].

LEMMA 2.1.4
Let G be a finite group and $H \leq G$. Let M be a kG-module and let N be a kH-module.

1. *If M is a permutation module, then $M\downarrow_H$ is also a permutation module.*
2. *If N is a permutation module, then $N\uparrow^G$ is a permutation module.*

3. *The direct sum of two permutation modules is a permutation module.*

4. *The tensor product of two permutation modules is a permutation module.*

If G acts transitively on Ω, then the resulting permutation module is a *transitive permutation module*. In this case, the action of G on Ω is equivalent to the action of G on cosets of the stabiliser $G_\omega = \{g \in G \mid \omega g = \omega\}$ in G, for any $\omega \in \Omega$. Conversely, given $H \leq G$, it is possible to construct a transitive permutation module $M_{\mathcal{C}(G,H)}$ from the G-set $\mathcal{C}(G,H)$, where $\mathcal{C}(G,H)$ is the set of cosets of H in G, and G acts transitively on $\mathcal{C}(G,H)$, in the manner described above. Thus, a transitive permutation module can be viewed as an induced module $k_H \uparrow^G$, where k_H is the trivial module for $H = G_\omega \leq G$.

We will frequently need to define homomorphisms from a transitive permutation module, say $k_{G_\omega} \uparrow^G$, to another kG-module, M. To do so is strightforward: we define the homomorphism on a generator, ensuring that the image of the generator is preserved by its stabiliser in G. Indeed, it follows directly from part 1 of Theorem 2.1.2 that

$$\mathrm{Hom}_{kG}\left(k_{G_\omega} \uparrow^G, M\right) \cong \mathrm{Hom}_{k(G_\omega)}\left(k_{G_\omega}, M \downarrow_{G_\omega}\right)$$

and since G_ω acts trivially on k_{G_ω}, $f \in \mathrm{Hom}_{k(G_\omega)}\left(k_{G_\omega}, M \downarrow_{G_\omega}\right)$ must satisfy

$$f(x) \cdot g = f(xg) = f(x)$$

for $x \in k_{G_\omega}$ and $g \in G_\omega$. The resulting homomorphism from $k_{G_\omega} \uparrow^G$ to M will be well defined and any such homomorphism can be described in this way.

2.1.3 Vertices and sources

In this section, we present some background material on vertices and sources, so that we may state the Brauer correspondence. Throughout this section, we continue to let G be a finite group. We will mostly follow Alperin's exposition in [2, §9], although Benson's book [3] also gives a concise treatment of the theory.

Crucial for the definition of a vertex is the concept of a relatively projective module.

Definition 2.1.5

Let $H \leq G$. We define a kG-module X to be relatively H-projective *if, whenever U, V are kG-modules, $\alpha : X \to U$ is a kG-module homomorphism and $\beta : V \to U$ is a surjective kG-module homomorphism, then there exists a kG-module homomorphism $\gamma : X \to V$ with $\alpha = \gamma \circ \beta$, provided that there exists a kH-module homomorphism $\widehat{\gamma} : X \downarrow_H \to V \downarrow_H$ such that $\alpha = \widehat{\gamma} \circ \beta$.*

Relatively projective modules are characterised in a number of ways. In particular, Higman (see, for example, [3, Proposition 3.6.4]) establishes an important relationship between relatively projective modules and the relative trace map, which we now define.

Definition 2.1.6

Suppose that X is a kG-module. For a subgroup $L \leq G$, let the set of fixed points of X under L be denoted by $X^L := \{x \in X \mid x\ell = x \ \forall\ \ell \in L\}$. If $H \leq L \leq G$, the relative trace map *$\mathrm{tr}_H^L : X^H \to X^L$ is defined by*

$$\mathrm{tr}_H^L(x) = \sum_{\ell \in L/H} x\ell,$$

where L/H is a set of coset representatives for H in L.

If X is a kG-module, then $\mathrm{End}_k(X)$ is also a kG-module with the conjugation action $\phi g = g^{-1}\phi g$ for $\phi \in \mathrm{End}_k(X)$ and $g \in G$. Thus, noting that $\mathrm{End}_{kH}(X) = (\mathrm{End}_k(X))^H$, where $H \leq G$, it makes sense to define the relative trace map on a kH-module endomorphism. For $\phi \in \mathrm{End}_{kH}(X)$,

$$\mathrm{tr}_H^G(\phi) = \sum_{g \in G/H} g^{-1}\phi g \ \in \mathrm{End}_{kG}(X).$$

Proposition 2.1.7 (Higman's Criteria)

Let X be a kG-module. If $H \leq G$, then the following are equivalent:

(i) X is relatively H-projective;

(ii) the identity map $1_X = \mathrm{tr}_H^G(\phi)$ for some $\phi \in \mathrm{End}_{kH}(X)$;

(iii) X is a direct summand of $X\downarrow_H\uparrow^G$;

(iv) if V is a kG-module and $\psi : V \to X$ is split as a surjective kH-module homomorphism, then ψ is split as a kG-module homomorphism.

Proof. See Proposition 3.6.4 in [3]. ■

Of particular interest is the smallest subgroup of G for which a kG-module X is relatively projective. The next definition, which was introduced by Green in his 1958 paper [21], gives a name to such a subgroup.

Definition 2.1.8

We say that $Q \leq G$ is a vertex *of the indecomposable kG-module X if X is relatively Q-projective, but not relatively R-projective for any proper subgroup $R \leq Q$.*

If X has vertex Q, then a source *of X is an indecomposable kQ-module W such that X is a direct summand of $W\uparrow^G$.*

In [21], Green also details several key properties of vertices, a couple of which we now record.

Proposition 2.1.9

Let X be an indecomposable kG-module.

1. *Any two vertices of X are conjugate in G.*

2. *If k is a field of characteristic p, then a vertex of X is always a p-subgroup of G.*

2.1.4 Blocks

It will be very helpful for us to be able to decompose an algebra, so that we may study modules 'lying in' particular subalgebras of the algebra, called blocks. This idea will be made precise shortly. We will see that an arbitrary module does not necessarily lie in a block, but indecomposable modules do.

THEOREM 2.1.10
A finite dimensional algebra A has a unique decomposition

$$A = A_1 \oplus \cdots \oplus A_r$$

into a direct sum of subalgebras, each of which is indecomposable as an algebra. The subalgebras in the decomposition are called the blocks *of A.*

Proof. See Theorem 13.1 in [2]. ∎

DEFINITION 2.1.11
If M is an A-module such that $MA_i = M$ and $MA_j = 0$ for all $j \neq i$, then we say that M lies in *the block A_i.*

Remark. Submodules, quotient modules and direct sums of modules lying in a block A_i also lie in A_i. Moreover, if M_i and M_j lie in the blocks A_i and A_j, respectively, and $i \neq j$, then $\mathrm{Hom}_A(M_i, M_j) = 0$.

The notion of lying in a block is significant. The following proposition allows us to conclude that any indecomposable module lies in a block. As a consequence, we are able to study modules for the blocks of the algebra A rather than A-modules, which is often much more tractable.

PROPOSITION 2.1.12
If M is an A-module, then M has a unique direct sum decomposition

$$M = M_1 \oplus \cdots \oplus M_r,$$

where M_i lies in the block A_i.

Proof. See Proposition 13.2 in [2]. ∎

When A is a group algebra – as in our situation – it is helpful to view kG as a module for the group algebra $k(G \times G)$, with action given by $a(g_1, g_2) = g_1^{-1} a g_2$ for $a \in kG$ and $g_1, g_2 \in G$. The group algebra kG decomposes as a direct sum of indecomposable $k(G \times G)$-modules and the summands in the decomposition are the blocks of kG. We now highlight a particularly significant block of kG (see [3, p.203]).

Definition 2.1.13
The block of kG which contains the trivial kG-module k_G is called the principal block *and is denoted by $B_0 = B_0(G)$.*

The form of the vertices of blocks of kG is known. For the remainder of §2.1, we let p denote the characteristic of the field k.

Theorem 2.1.14
If B is a block of kG, then B has a vertex, as a $k(G \times G)$-module, of the form δD, where D is a p-subgroup of G and $\delta : G \to G \times G$ is defined by $\delta : g \mapsto (g, g)$.

Proof. See Theorem 13.4 in [2]. ■

Definition 2.1.15
Let B be a block of kG. The subgroups $D \leq G$, such that δD is a vertex of B, are a conjugacy class of p-subgroups of G, called the defect groups *of B. If $|D| = p^d$, then B is said to be of* defect *d.*

The following theorem shows that the defect group of a block of G is closely related to the indecomposable modules lying in the block.

Theorem 2.1.16
If B is a block of G, then any indecomposable kG-module lying in B has a vertex contained in D, the defect group of B.

Proof. See Theorem 14.5 in [2]. ■

Moreover, the defect (group) of a block is, in some sense, indicative of the complexity of the modules which lie in the block. For example, if the defect group is trivial – so that the defect of the block B is zero – then Theorem 2.1.16 tells us that the vertices of the indecomposable modules lying in the block must also be trivial. Hence, in this case, every B-module is projective. If a block has cyclic defect group, then its structure can be described by an associated Brauer tree. For more information about Brauer trees, we refer the reader to Alperin [2, §17].

2.1.5 The Brauer correspondence

The Brauer correspondence is a fundamental tool, which we will exploit later in this work to determine the vertices of summands of certain twisted Foulkes modules. The reader may wish to refer ahead to §3.3 for the definition of twisted Foulkes modules. For this section, we predominantly refer to Broué's paper [5].

Definition 2.1.17
For $L \leq G$ and a kG-module X, define

$$X^L_{<L} := \sum_{H<L} \mathrm{tr}^L_H(X^H).$$

The Brauer correspondent *of X with respect to L (or the Brauer quotient of X^L) is the $kN_G(L)$-module*

$$X(L) := X^L / X^L_{<L}$$

and the Brauer map *with respect to L is the natural surjection $\mathrm{Br}_L : X^L \to X(L)$.*

As we shall see shortly, the Brauer map is particularly helpful for studying p-permutation modules, which are defined as follows.

Definition 2.1.18
A kG-module M is said to be a p-permutation module *if, whenever P is a p-subgroup of G, there exists a basis for M that is invariant under the action of P.*

The following two theorems from Broué's paper [5] give important properties of the Brauer map, when applied to p-permutation modules.

Theorem 2.1.19
Let M be an indecomposable p-permutation kG-module. The vertices of M are the maximal p-subgroups $P \leq G$ such that $M(P) \neq 0$.

Theorem 2.1.20
The Brauer correspondence $M \to M(P)$ induces a bijection between the isomorphism classes of indecomposable p-permutation kG-modules with vertex P and the isomorphism classes of indecomposable projective $k(N_G(P)/P)$-modules.

In [20, Corollary 2.3], the authors explain that Theorem 2.1.19 has the following important corollary.

Corollary 2.1.21
Let M be a p-permutation kG-module with p-permutation basis $\mathcal{B}$ with respect to a Sylow p-subgroup P of G. Let $R \leq P$ and define $\mathcal{B}^R := \{b \in \mathcal{B} \mid br = r \ \forall\ r \in R\}$. The $kN_G(R)$-module $M(R)$ is equal to $\langle \mathcal{B}^R \rangle$, and M has an indecomposable summand with a vertex containing R if and only if $\mathcal{B}^R \neq \emptyset$.

2.2 Representation theory of symmetric groups

The material that we will present in this section will be based predominantly on chapters from [25] and so we will endeavour to be consistent with James' notation. An alternative reference for §2.2.2 is [26, Chapter 7].

2.2.1 Partitions and Young tableaux

Fundamental to the representation theory of the symmetric group is the notion of a *partition* of $n \in \mathbb{N}$, which is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ of non-negative integers such that $\lambda_i \geq \lambda_{i+1}$ for all $1 \leq i \leq \ell - 1$ and $\sum_{i=1}^{\ell} \lambda_i = n$. We note that we identify two partitions that are equal up to parts of size zero. Following standard notation, we write $\lambda \vdash n$ to indicate that λ is a partition of n and we use superscripts to denote multiple parts of the same size. For example, we may write $\left(3, 2, 1^2\right)$ instead of $(3, 2, 1, 1)$. If we do not require the parts λ_i of λ to be non-increasing, then the result is a *composition* of n, denoted by $\lambda \vDash n$.

The natural lexicographic order $<$ on the set of partitions is a total order. If λ and μ are partitions of n, then $\lambda > \mu$ if for some j we have $\lambda_j > \mu_j$ and $\lambda_i = \mu_i$ for all $i < j$. The lexicographic order will be particularly useful for us in Chapter 6. However, it is not the only order: the dominance order is a partial order, which is often preferred. We say that λ dominates μ and write $\lambda \trianglerighteq \mu$ if

$$\sum_{i=1}^{j} \lambda_i \geq \sum_{i=1}^{j} \mu_i \quad \text{for all } j. \tag{2.1}$$

If the partitions λ and μ have a different number of parts, we can still make sense of the definition of the dominance order: we add parts of size zero to the partition with fewer parts, until both partitions have the same number of parts, and then check the definition as above. Note that if λ dominates μ, then $\lambda \geq \mu$ in the lexicographic order.

Example 2.2.1

If $\lambda = (4, 2, 1)$ and $\mu = \left(3, 1^4\right)$, then we add two parts of size zero to λ so that both partitions have the same number of parts. Condition (2.1) holds, since

$$4 > 3, \quad 6 > 4, \quad 7 > 5, \quad 7 > 6, \quad 7 = 7$$

and thus, in the dominance order, $(4, 2, 1) \trianglerighteq \left(3, 1^4\right)$.

If instead we had chosen $\lambda = (3, 3)$ and $\mu = (4, 1, 1)$, then we would have found that they are incomparable in the dominance order. Indeed, $\lambda_1 < \mu_1$, but $\lambda_1 + \lambda_2 > \mu_1 + \mu_2$. However, in the lexicographic order, clearly $(4, 1, 1) > (3, 3)$.

Partitions are often represented graphically, as *Young diagrams.* The Young diagram $[\lambda]$ corresponding to $\lambda \vdash n$ consists of n boxes arranged in rows, which are left aligned. The i^{th}

row of $[\lambda]$ corresponds to the i^{th} part of λ and contains precisely λ_i boxes. Note that if we have a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$, we drop the round brackets when we want to denote the Young diagram, so that $[\lambda] = [\lambda_1, \lambda_2, \ldots, \lambda_\ell]$.

Example 2.2.2

The Young diagram of $(3, 2, 1^2)$ *is*

$$[3, 2, 1^2] = \ydiagram{3,2,1,1}\ .$$

The *conjugate partition* of $\lambda \vdash n$, denoted by λ', is the partition of n whose Young diagram $[\lambda']$ is obtained from $[\lambda]$ by interchanging rows and columns. For example, $(3, 2, 1^2)' = (4, 2, 1)$.

A λ-*tableau* is an assignment of the numbers 1 to n to the boxes of the Young diagram $[\lambda]$, using each number exactly once. If the numbers increase along the rows and down the columns of the λ-tableau t, then we describe t as *standard*. The symmetric group acts on the set of λ-tableaux in the natural way, permuting the numbers 1 to n within a tableau.

For a fixed λ-tableau t, there are two very important subgroups of $\mathfrak{S}_n$, which we shall now define. The *column stabiliser* C_t is the subgroup of $\mathfrak{S}_n$ consisting of all permutations which fix the columns of t set-wise; that is,

$$C_t := \{\pi \in \mathfrak{S}_n \mid \text{for each } 1 \leq i \leq n : i \text{ and } (i)\pi \text{ are in the same column of } t\}\,.$$

The *row stabiliser* R_t is similarly defined:

$$R_t := \{\pi \in \mathfrak{S}_n \mid \text{for each } 1 \leq i \leq n : i \text{ and } (i)\pi \text{ are in the same row of } t\}\,.$$

Given a λ-tableau, disregarding the order of the numbers within each row results in a λ-*tabloid*. Formally, we define an equivalence relation on the set of λ-tableaux, with $t_1 \sim t_2$ if $t_2 = t_1\pi$ for some $\pi \in R_{t_1}$. A λ-tabloid $\{t\}$ is the equivalence class of a λ-tableau t under this relation. Tabloids are visually different from tableaux as we only draw lines between the rows.

Example 2.2.3

If $\lambda = (3, 1)$ *then, up to equivalence, there are four* λ-*tabloids:*

$$\begin{array}{l}\hline 2\ 3\ 4 \\ \hline 1 \\ \cline{1-1}\end{array}, \quad \begin{array}{l}\hline 1\ 3\ 4 \\ \hline 2 \\ \cline{1-1}\end{array}, \quad \begin{array}{l}\hline 1\ 2\ 4 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \text{and} \quad \begin{array}{l}\hline 1\ 2\ 3 \\ \hline 4 \\ \cline{1-1}\end{array}.$$

There is a well-defined action of $\mathfrak{S}_n$ on the set of λ-tabloids, defined by

$$\{t\}\pi = \{t\pi\}.$$

Extending this transitive action linearly makes the k-vector space spanned by λ-tabloids into a $k\mathfrak{S}_n$-module, called a *Young permutation module*, which we denote by M^λ. This Young permutation module is generated by any one of the λ-tabloids. Since the stabiliser of a given λ-tabloid is the *Young subgroup*

$$\mathfrak{S}_\lambda = \mathfrak{S}_{\{1,2,\dots,\lambda_1\}} \times \mathfrak{S}_{\{\lambda_1+1,\dots,\lambda_1+\lambda_2\}} \times \cdots \times \mathfrak{S}_{\{n-\lambda_\ell+1,\dots,n\}} \cong \mathfrak{S}_{\lambda_1} \times \mathfrak{S}_{\lambda_2} \times \cdots \times \mathfrak{S}_{\lambda_\ell},$$

we may think about M^λ as the permutation module of $\mathfrak{S}_n$ on the cosets of $\mathfrak{S}_\lambda$. Equivalently,

$$M^\lambda \cong k_{\mathfrak{S}_\lambda}\big\uparrow^{\mathfrak{S}_n},$$

where $k_{\mathfrak{S}_\lambda}$ is the trivial module for $\mathfrak{S}_\lambda$.

2.2.2 Specht modules

We will see shortly that, for a given partition λ of n, the Specht module S^λ is a submodule of the corresponding Young permutation module M^λ. Before we can define it properly, we need the notion of a polytabloid.

Given a λ-tableau t, we define the *signed column sum* to be the following element of $k\mathfrak{S}_n$:

$$\kappa_t := \sum_{\pi \in C_t} \operatorname{sgn}(\pi)\pi\,.$$

The *polytabloid* arising from t is defined to be

$$e_t := \{t\}\kappa_t\,,$$

which is an element of M^λ. It follows from $\kappa_t\pi = \pi\kappa_{t\pi}$ (for $\pi \in \mathfrak{S}_n$) and the definition of e_t that

$$e_t\pi = e_{t\pi}. \tag{2.2}$$

Thus, the subspace spanned by all λ-polytabloids is a submodule of M^λ, which we call the *Specht module* S^λ corresponding to the partition λ. In [25, §8], James proves that a basis for S^λ is given by the set $\{e_t \mid t \text{ is a standard } \lambda\text{-tableau}\}$. Furthermore, (2.2) tells us that S^λ is a cyclic $k\mathfrak{S}_n$-module, generated by any one of the λ-polytabloids.

The Specht modules are particularly important, as the following theorem indicates.

THEOREM 2.2.4
The set $\left\{S^\lambda \;\middle|\; \lambda \vdash n\right\}$ is a complete set of non-isomorphic, irreducible $\mathbb{C}\mathfrak{S}_n$-modules.

Proof. See Theorem 4.12 in [25]. ■

Noteworthy are the two one-dimensional irreducible $\mathbb{C}\mathfrak{S}_n$-modules, namely the trivial module and the sign module, which are labelled by the partitions (n) and (1^n), respectively.

In later chapters, we will find it more beneficial to study characters rather than the corresponding modules. Throughout, we shall use χ^λ to denote the ordinary irreducible character of $\mathfrak{S}_n$ corresponding to the partition λ, which is precisely the character afforded by the Specht module S^λ.

If the characteristic of the field k is a prime p, then in general the Specht modules are not irreducible. However, if λ is *p-regular*, by which we mean that λ has no non-zero part repeated p times, then S^λ has a simple head $D^\lambda = S^\lambda / \operatorname{rad} S^\lambda$.

THEOREM 2.2.5
Suppose that k is a field of prime characteristic p. The set

$$\left\{ D^\lambda \mid \lambda \textit{ is a p-regular partition of } n \right\}$$

is a complete set of non-isomorphic, irreducible $k\mathfrak{S}_n$-modules.

Proof. See Theorem 11.5 in [25]. ∎

2.2.3 An alternative description of Young permutation modules

For Chapters 4 and 5, it will be very useful for us to have at our disposal information about $k\mathfrak{S}_n$-homomorphisms from S^λ to M^μ, where λ and μ are both partitions of n. In particular, we would like a basis for $\operatorname{Hom}_{k\mathfrak{S}_n}\left(S^\lambda, M^\mu\right)$. To facilitate this, we need to think more about the way we describe M^μ; the description of Young permutation modules that we saw in §2.2.1 will not suffice and so we now review a well-known alternative description.

Thus far, we have insisted that λ-tableaux contain each of the numbers 1 to n exactly once. In what follows, we require a new kind of tableau, namely one which is allowed to have repeated entries. We will keep the notation used by James in [25] and use capital letters to denote such tableaux.

Let $\lambda \vdash n$ and let $\mu \vDash n$. A λ-tableau T is said to be *of type* μ if every positive integer i occurs μ_i times in T. Define $\mathcal{T}(\lambda,\mu) := \{T \mid T \text{ is a } \lambda\text{-tableau of type } \mu\}$. Further, a tableau $T \in \mathcal{T}(\lambda,\mu)$ is called *semistandard* if the numbers are non-decreasing along rows of T and strictly increasing down the columns of T. We write $\mathcal{T}_0(\lambda,\mu)$ to denote the set of semistandard tableaux in $\mathcal{T}(\lambda,\mu)$.

As we might hope, there is a well-defined action of $\mathfrak{S}_n$ on the λ-tableaux of type μ. Fix a λ-tableau t and take $T \in \mathcal{T}(\lambda,\mu)$. Following James in [25, p.44], let $(i)T$ be the entry in T which occurs in the same position as i occurs in t. Define the action of $\mathfrak{S}_n$ on $\mathcal{T}(\lambda,\mu)$ by

$$(i)(T\pi) = (i\pi^{-1})T$$

where $T \in \mathcal{T}(\lambda,\mu)$, $\pi \in \mathfrak{S}_n$ and $1 \leq i \leq n$.

Example 2.2.6
Take $\lambda = (3, 2, 1^2)$ *and* $\mu = (5, 2)$. *If we choose*

$$t = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 6 \\ \cline{1-2} 5 \\ \cline{1-1} 7 \\ \cline{1-1} \end{array} \qquad \textit{and} \quad T = \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 1 & 1 \\ \cline{1-2} 1 \\ \cline{1-1} 2 \\ \cline{1-1} \end{array},$$

then $\pi = (2\,3\,7) \in \mathfrak{S}_7$ *acts on* T *in the following way:*

$$T\pi = \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 1 \\ \cline{1-2} 1 \\ \cline{1-1} 1 \\ \cline{1-1} \end{array}.$$

With this action, we now have all that we need to present the new description of M^μ: we take M^μ to be the $k\mathfrak{S}_n$-module spanned, as a vector space, by λ-tableaux of type μ. It is easy to see that this is equivalent to our original description of the Young permutation module: take a λ-tableau of type μ, say T, and a fixed λ-tableau t. We obtain a unique μ-tabloid $\{t_T\}$ as follows: if $(i)T = j$, then put i in row j of $\{t_T\}$. Moreover, the actions are consistent: if T corresponds to $\{t_T\}$, then $T\pi$ corresponds to $\{t_T\}\pi$.

Example 2.2.7
Let λ, μ, t, T *and* π *be as in Example 2.2.6. The* μ*-tabloid corresponding to* T *is*

$$\begin{array}{l}\hline 1\ 2\ 3\ 5\ 6 \\ \hline 4\ 7 \\ \hline \end{array}.$$

Further, the μ*-tabloid corresponding to* $T\pi$ *is*

$$\begin{array}{l}\hline 1\ 3\ 7\ 5\ 6 \\ \hline 4\ 2 \\ \hline \end{array} = \begin{array}{l}\hline 1\ 2\ 3\ 5\ 6 \\ \hline 4\ 7 \\ \hline \end{array}\ \pi.$$

2.2.4 Semistandard homomorphisms

Carter and Lusztig [7] observed that a basis for $\mathrm{Hom}_{k\mathfrak{S}_n}\left(S^\lambda, M^\mu\right)$ can be constructed from suitable homomorphisms between Young permutation modules. The theory that we have developed thus far leads us naturally to defining the maps $M^\lambda \to M^\mu$ that we will need to write down the basis. In [25, §13], James captures the essence of Carter and Lusztig's arguments when the characteristic of the ground field is not equal to two. This is sufficient for our purposes, since we will in fact require that the characteristic of k is zero when we come to use the basis.

Let t be a fixed λ-tableau. The tableaux $T_1, T_2 \in \mathcal{T}(\lambda, \mu)$ are described as being *row equivalent* if $T_2 = T_1\pi$ for some $\pi \in R_t$. If $T \in \mathcal{T}(\lambda, \mu)$, we define $\hat{\theta}_T : M^\lambda \to M^\mu$ on λ-tabloids by

$$\hat{\theta}_T : \{t\} \longmapsto \sum_{T' \sim_{\mathrm{row}} T} T', \tag{2.3}$$

where the notation $T' \sim_{\text{row}} T$ indicates that we sum over all $T' \in \mathcal{T}(\lambda,\mu)$ which are row equivalent to T. We extend this map to a homomorphism by allowing group elements to act.

The map $\hat{\theta}_T$ is clearly well-defined. Indeed, we know that the stabiliser of $\{t\}$ under the action of $\mathfrak{S}_n$ is the row stabiliser R_t of t and, using the definition of row equivalent tableaux, it is clear that R_t fixes the image $(\{t\})\hat{\theta}_T$.

Example 2.2.8

If we take $\{t\} \in M^{(3,2,1^2)}$ and $T \in \mathcal{T}\left(\left(3,2,1^2\right),(5,2)\right)$ as in Example 2.2.6, then

$$\hat{\theta}_T : \begin{array}{l}\hline 1\ 3\ 4 \\ \hline 2\ 6 \\ \hline 5 \\ \hline 7 \\ \hline \end{array} \longmapsto \ytableaushort{112,11,1,2} + \ytableaushort{121,11,1,2} + \ytableaushort{211,11,1,2}\,.$$

We are now in a position to define semistandard homomorphisms. Given a map[1] $\hat{\theta}_T : M^\lambda \to M^\mu$, let $\theta_T \in \operatorname{Hom}_{k\mathfrak{S}_n}\left(S^\lambda, M^\mu\right)$ be its restriction to the Specht module S^λ:

$$\theta_T = \hat{\theta}_T\big|_{S^\lambda}.$$

If T is a semistandard tableau, then we call θ_T a *semistandard homomorphism*. For brevity, we omit some of the details in the following calculation, but we present the image of e_t under θ_T using both of the descriptions of M^μ that we have met.

Example 2.2.9

Take $\lambda = (4,2)$ and $\mu = \left(3^2\right)$. Choosing $t = \ytableaushort{1234,56}$ and $T = \ytableaushort{1112,22}$, we have that $\kappa_t = 1 - (1\ 5) - (2\ 6) + (1\ 5)(2\ 6)$ and

$$\begin{aligned}
\theta_T : e_t \longmapsto\ & \left(\ytableaushort{1112,22} + \ytableaushort{1121,22} + \ytableaushort{1211,22} + \ytableaushort{2111,22}\right)\kappa_t \\
=\ & \ytableaushort{1112,22} + \ytableaushort{1121,22} + \ytableaushort{2212,11} + \ytableaushort{2221,11} \\
& - \ytableaushort{2112,12} - \ytableaushort{2121,12} - \ytableaushort{1212,21} - \ytableaushort{1221,21} \\
=\ & \begin{array}{l}\hline 1\ 2\ 3 \\ \hline 4\ 5\ 6 \\ \hline \end{array} + \begin{array}{l}\hline 1\ 2\ 4 \\ \hline 3\ 5\ 6 \\ \hline \end{array} + \begin{array}{l}\hline 3\ 5\ 6 \\ \hline 1\ 2\ 4 \\ \hline \end{array} + \begin{array}{l}\hline 4\ 5\ 6 \\ \hline 1\ 2\ 3 \\ \hline \end{array} \\
& - \begin{array}{l}\hline 2\ 3\ 5 \\ \hline 1\ 4\ 6 \\ \hline \end{array} - \begin{array}{l}\hline 2\ 4\ 5 \\ \hline 1\ 3\ 6 \\ \hline \end{array} - \begin{array}{l}\hline 1\ 3\ 6 \\ \hline 2\ 4\ 5 \\ \hline \end{array} - \begin{array}{l}\hline 1\ 4\ 6 \\ \hline 2\ 3\ 5 \\ \hline \end{array}\,.
\end{aligned}$$

[1] A reader familiar with the notation used by James in [25, §13] may notice that James uses θ_T to denote the map between the Young permutation modules M^λ and M^μ that is labelled by $T \in \mathcal{T}(\lambda,\mu)$. In our notation, this map is $\hat{\theta}_T : M^\lambda \to M^\mu$. James then denotes the restriction of the map to the Specht module by $\hat{\theta}_T$, whereas we do the opposite, using θ_T. This seemingly unnecessary switch will simplify our notation significantly in later chapters.

Observe that, in the setting of Example 2.2.9, $\begin{array}{|c|c|c|c|}\hline 1&2&1&1\\\hline 2&2\\\cline{1-2}\end{array}\ \kappa_t = \begin{array}{|c|c|c|c|}\hline 2&1&1&1\\\hline 2&2\\\cline{1-2}\end{array}\ \kappa_t = 0$. The following proposition from [25, p.45] describes exactly when this happens, i.e. when a tableau T is killed by the action of the signed column sum κ_t.

Proposition 2.2.10

Let λ, $\mu \vdash n$ and fix a λ-tableau t. A column of $T \in \mathcal{T}(\lambda, \mu)$ contains two identical numbers if and only if $T\kappa_t = 0$.

Convincing ourselves of the validity of Proposition 2.2.10 is not difficult. If a number is repeated in a column of T, say in the positions of t labelled by x and y, then $T\big(1-(x\ y)\big) = 0$. We can always find a set of representatives for the cosets of $\langle(x\ y)\rangle$ in C_t, say $\{\alpha_1, \ldots, \alpha_r\}$, and thus write $\kappa_t = \big(1-(x\ y)\big)\sum_{i=1}^{r}\operatorname{sgn}(\alpha_i)\alpha_i$. It follows immediately that $T\kappa_t = 0$. For the converse, proceed by contradiction.

With this result in mind, fix a λ-tableau t and consider the image of the generator $e_t \in S^\lambda$ under θ_T. Since

$$(e_t)\theta_T = (\{t\}\kappa_t)\,\theta_T = (\{t\})\theta_T\kappa_t = \left(\sum_{T' \sim_{\text{row}} T} T'\right)\kappa_t = \sum_{T' \sim_{\text{row}} T} (T'\kappa_t),$$

it is clear that sometimes θ_T is the zero map. However, by restricting our attention to semistandard tableaux, we are able to guarantee that the corresponding semistandard homomorphisms are non-zero. They are in fact the only homomorphisms that we need to write down a basis for $\operatorname{Hom}_{k\mathfrak{S}_n}\left(S^\lambda, M^\mu\right)$, as the next result states.

Theorem 2.2.11

Assume that either $\operatorname{char}(k) \neq 2$ *or that $\lambda \vdash n$ is 2-regular, that is, λ does not have two equal non-zero parts. The set*

$$\left\{\theta_T \;\middle|\; T \in \mathcal{T}_0(\lambda, \mu)\right\}$$

is a basis for $\operatorname{Hom}_{k\mathfrak{S}_n}\left(S^\lambda, M^\mu\right)$.

Proof. See [25, p.48]. ■

Corollary 2.2.12

In the setting of Theorem 2.2.11, $\dim\left(\operatorname{Hom}_{k\mathfrak{S}_n}\left(S^\lambda, M^\mu\right)\right)$ *is equal to the number of semistandard λ-tableaux of type μ.*

2.2.5 Induction and restriction of Specht modules

In this work, we will often need to induce and restrict Specht modules. There are several classical results which describe the modules that are obtained. The first such result is the Branching Rule, which gives the decomposition of a Specht module that has been induced from $\mathfrak{S}_n$ to $\mathfrak{S}_{n+1}$ or restricted from $\mathfrak{S}_n$ to $\mathfrak{S}_{n-1}$.

Theorem 2.2.13 (Branching Rule)

Let λ be a partition of n and let $[\lambda]$ be the corresponding Young diagram. Defined over a field of characteristic zero, the restriction of S^λ to $\mathfrak{S}_{n-1}$ decomposes as

$$S^\lambda \big\downarrow_{\mathfrak{S}_{n-1}} \cong \bigoplus_{\nu \in V} S^\nu,$$

where $V := \{\nu \vdash n-1 \mid [\nu]$ arises by removing a box from $[\lambda]\}$. Similarly,

$$S^\lambda \big\uparrow^{\mathfrak{S}_{n+1}} \cong \bigoplus_{\omega \in W} S^\omega,$$

where $W := \{\omega \vdash n+1 \mid [\omega]$ arises by adding a box to $[\lambda]\}$.

If the Specht module S^λ is defined over a field of prime characteristic, then $S^\lambda\big\downarrow_{\mathfrak{S}_{n-1}}$ and $S^\lambda\big\uparrow^{\mathfrak{S}_{n+1}}$ have a filtration by Specht modules, with the factors occuring being those S^ν, where $\nu \in V$, and S^ω, where $\omega \in W$, respectively.

Proof. See Theorems 9.2 and 9.3 in [25]. ■

Before continuing, we should make precise the adding and removing of boxes described in the Branching Rule. We may remove any box (sometimes called a *node*) from $[\lambda]$ that can be described as *removable*: if $(i,j) \in [\lambda]$ denotes the box in the i^{th} row and j^{th} column of $[\lambda]$, then, formally, a removable node is a node (i, λ_i) such that $\lambda_i > \lambda_{i+1}$. Similarly, there is a notion of an *addable* node, which is of the form $(1, \lambda_1 + 1)$, $(\lambda_1' + 1, 1)$, or $(i, \lambda_i + 1)$ for any $1 < i \leq \lambda_1'$ such that $\lambda_i < \lambda_{i-1}$.

The next classical result, the Littlewood–Richardson Rule, addresses the problem of inducing the outer tensor product of two Specht modules, say $S^\lambda \boxtimes S^\mu$ (where $\lambda \vdash n$ and $\mu \vdash \ell$), from the subgroup $\mathfrak{S}_n \times \mathfrak{S}_\ell \leq \mathfrak{S}_{n+\ell}$. In particular, the decomposition of the induced module is given in terms of so-called Littlewood–Richardson coefficients (which are combinatorially defined, see Theorem 2.8.13 in [26]). In Chapter 7, we will require the character-theoretic statement of the result. However, we remark that a version of the result, proved by James and Peel in [27], exists over a field of arbitrary characteristic. In [25, §16], James gives a thorough exposition of the theory needed to prove the Littlewood–Richardson Rule, including a detailed description of a method by which the Littlewood–Richardson coefficients may be computed.

Theorem 2.2.14 (Littlewood–Richardson Rule)

If λ is a partition of n and μ is a partition of ℓ, then

$$\left(\chi^\lambda \times \chi^\mu\right)\Big\uparrow_{\mathfrak{S}_n \times \mathfrak{S}_\ell}^{\mathfrak{S}_{n+\ell}} = \sum_{\nu \vdash n+\ell} c^\nu_{\lambda,\mu} \chi^\nu,$$

where $c^\nu_{\lambda,\mu}$ is the Littlewood–Richardson coefficient corresponding to the partitions λ, μ, ν.

When S^μ is the trivial $\mathfrak{S}_\ell$-module or the sign $\mathfrak{S}_\ell$-module, the result is particularly elegant. The following corollary addresses these two special cases.

Corollary 2.2.15
Let $\ell \in \mathbb{N}$ and $\lambda \vdash n$.

1. *(Young's Rule) If we define*

$$W_\ell^{\mathrm{col}} := \{\omega \vdash n+\ell \mid [\omega] \text{ arises by adding } \ell \text{ boxes to } [\lambda]\text{, no two in a column}\}$$

then

$$\left(\chi^\lambda \times 1_{\mathfrak{S}_\ell}\right)\Big\uparrow^{\mathfrak{S}_{n+\ell}} = \sum_{\omega \in W_\ell^{\mathrm{col}}} \chi^\omega.$$

2. *(Pieri's Rule) If we define*

$$W_\ell^{\mathrm{row}} := \{\omega \vdash n+\ell \mid [\omega] \text{ arises by adding } \ell \text{ boxes to } [\lambda]\text{, no two in a row}\}$$

then

$$\left(\chi^\lambda \times \mathrm{sgn}_{\mathfrak{S}_\ell}\right)\Big\uparrow^{\mathfrak{S}_{n+\ell}} = \sum_{\omega \in W_\ell^{\mathrm{row}}} \chi^\omega.$$

Remark. Given $\lambda \vdash n$ and its corresponding Young diagram $[\lambda]$, we will describe the process of adding ℓ boxes to $[\lambda]$ such that no two are added in the same column as a *Young's Rule addition of ℓ boxes.* We define a *Pieri's Rule addition of ℓ boxes* similarly.

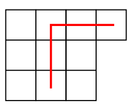
2.2.6 Hooks and the Murnaghan–Nakayama Rule

In the last section, we alluded to the fact that we may regard the Young diagram corresponding to $\lambda \vdash n$ as a set of nodes $[\lambda] = \{(i,j) \mid i \geq 1,\, 1 \leq j \leq \lambda_i\}$. For a node $(a,b) \in [\lambda]$, we define the (a,b)-*hook* to be the subset

$$h_{a,b} := \{(a,j) \in [\lambda] \mid j \geq b\} \cup \{(i,b) \in [\lambda] \mid i \geq a\}$$

and we define the *length* of the (a,b)-hook to be the number of nodes in $h_{a,b}$. We say that $h_{a,b}$ is an ℓ-hook if it has length ℓ.

Example 2.2.16
Given that $\lambda = (4,3,3)$, the hook $h_{1,2}$ is shown below.



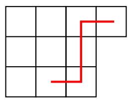
In this case, the hook length is 5.

If $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\mu = (\mu_1, \mu_2, \ldots)$ are partitions such that the Young diagram $[\lambda]$ completely contains $[\mu]$, i.e. $\mu_i \leq \lambda_i$ for all i, then the *skew-partition* λ/μ is the object corresponding to the (not necessarily connected) Young diagram which remains when the nodes in $[\mu]$ are removed from $[\lambda]$.

A skew-partition λ/μ is said to be a *rim hook* (also referred to as a *border strip*) if the Young diagram of λ/μ is a connected part of the rim of $[\lambda]$, not containing any 2×2 square, that can be removed to leave the Young diagram of a proper partition, specifically $[\mu]$. If the rim hook contains ℓ nodes, then we say that it has *length* ℓ and we describe it as a rim ℓ-hook. We define the *height* $\langle \lambda/\mu \rangle$ of the rim hook λ/μ to be one less than the number of its non-empty rows. There is a natural one-to-one correspondence between ℓ-hooks and rim ℓ-hooks: the rim hook corresponding to $h_{a,b}$ has precisely the same end nodes as $h_{a,b}$, i.e. (a, λ_a) and (λ'_b, b).

Example 2.2.17

In the setting of the Example 2.2.16, the rim 5-hook corresponding to $h_{1,2}$ is shown below.



This rim hook has height $\langle (4,3,3)/(2,2,1) \rangle = 2$.

The notion of a rim ℓ-hook is crucial for the statement of the next theorem, known as the Murnaghan–Nakayama Rule, which provides us with a way of computing entries in the character table of $\mathfrak{S}_n$. For a proof, we refer the reader to [25, §21].

Theorem 2.2.18 (Murnaghan–Nakayama Rule)

If $\rho\pi \in \mathfrak{S}_n$, where ρ is an ℓ-cycle and $\pi \in \mathfrak{S}_{n-\ell}$ permutes the remaining $n - \ell$ symbols, then

$$\chi^\lambda(\rho\pi) = \sum_{\mu} (-1)^{\langle \lambda/\mu \rangle} \chi^\mu(\pi)$$

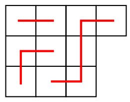
where the sum is over μ such that λ/μ is a rim ℓ-hook.

Since any character of $\mathfrak{S}_n$ is a class function, and the conjugacy classes of $\mathfrak{S}_n$ are labelled by (representatives of each of the) cycle types, it makes sense to write $\chi^\lambda(c_1, c_2, \ldots, c_r)$, where χ^λ is the irreducible character labelled by the partition λ of n, and $c_1, c_2, \ldots, c_r$ are the cycle lengths of an element of $\mathfrak{S}_n$ written in disjoint cycle notation. Formally, we define

$$\chi^\lambda(c_1, c_2, \ldots, c_r) := \chi^\lambda(\sigma) \quad \text{where } \sigma \in \mathfrak{S}_n \text{ has cycle type } (c_1, c_2, \ldots, c_r).$$

Example 2.2.19

Let $\lambda = (4,3,3)$ and choose an element of cycle type $(5,3,2)$ in $\mathfrak{S}_{10}$. There is only one way to remove a rim 5-hook from $[4,3,3]$ and subsequently a rim 3-hook, followed by a rim 2-hook, which is as shown below.



Thus, applying the Murnaghan–Nakayama Rule, we find that

$$\chi^{(4,3,3)}(5,3,2) = \chi^{(2,2,1)}(3,2) = -\chi^{(2)}(2) = -\chi^{\emptyset}(\emptyset) = -1.$$

Remark. Although it is not obvious, the character value is independent of the order in which the rim hooks are removed.

2.2.7 Blocks of symmetric groups

In the sequel, will make use of a few results concerning blocks of symmetric groups. The most important such result, which, despite having been proved, still takes the name *Nakayama's Conjecture*, is a very elegant statement describing when two Specht modules lie in the same block. Before we see this result, we need a couple of definitions.

Given a partition λ of n, the *p-core* of λ is the partition corresponding to the Young diagram that remains after as many p-rim hooks have been removed from $[\lambda]$ as possible; the number of hooks which are removed is the *p-weight* of λ. Although not immediately obvious, given any partition λ, both the p-core and p-weight of λ are well-defined (see [26, Theorem 2.7.16]). The blocks of the symmetric group $\mathfrak{S}_n$ are labelled by (γ, w), where γ is a p-core and w is the p-weight associated to γ.

Theorem 2.2.20 (Nakayama's Conjecture)
The Specht modules S^{λ} and S^{μ} lie in the p-block $B(\gamma, w)$ of $\mathfrak{S}_n$ if and only if the partitions λ and μ have the same p-core γ and the same p-weight w.

Proof. See Theorem 6.1.21 in [26]. ■

2.3 Wreath products

The main objects of study in this work are modules for the symmetric group which are induced from the trivial module for a certain imprimitive wreath product. For this reason, we should take the time to understand wreath products, which we now define as Kerber does in [28, §2]. The reader may also refer to [26, §4.1] for details of this construction.

If G is a finite group and $H \leq \mathfrak{S}_n$ acts on the set $\Omega := \{1, 2, \ldots, n\}$, then the *wreath product of G and H* is the group

$$G \wr H := \big\{(f; \pi) \mid f : \Omega \to G,\ \pi \in H\big\}$$

with multiplication defined by

$$(f; \pi) \cdot \big(f'; \pi'\big) := \big(f \cdot f'_{\pi};\ \pi\pi'\big)$$

where $f'_\pi : \Omega \to G$ is defined by $\big((\omega)\pi\big)f'_\pi := (\omega)f'$ for all $\omega \in \Omega$ and multiplication of the maps $f_1, f_2 : \Omega \to G$ is defined point-wise using the product in G, i.e. $(\omega)\big(f_1 \cdot f_2\big) = (\omega)f_1 \cdot (\omega)f_2$ for all $\omega \in \Omega$. Thus, $f \cdot f'_\pi : \omega \in \Omega \mapsto (\omega)f \cdot (\omega)f'_\pi = (\omega)f \cdot (\omega\pi^{-1})f' \in G$.

An important normal subgroup of $G \wr H$ is the *base group*

$$G^* := \{(f; 1_H) \mid f : \Omega \to G\}.$$

We should note that G^* is precisely the direct product of n copies of G, say $G_1, \dots, G_n$, where

$$G_i := \{(f; 1_H) \mid (j)f = 1_G \ \forall\ j \neq i\} \cong G.$$

A complement of G^* is the subgroup $H' := \{(e; \pi) \mid \pi \in H\} \cong H$, where $e : \Omega \to G$ is the identity map $(\omega)e = 1_G$ for all $\omega \in \Omega$. Since $G^* \cap H' = \big\{(e; 1_H)\big\}$, the identity element in $G \wr H$, it follows that $G \wr H = G^* \cdot H'$.

The case that is of interest to us is when $G = \mathfrak{S}_m$ and $H = \mathfrak{S}_n$, acting on the set $\Omega = \{1, 2, \dots, n\}$ in the natural way. In this case,

$$\mathfrak{S}_m \wr \mathfrak{S}_n = \big(\mathfrak{S}_m \times \cdots \times \mathfrak{S}_m\big) \rtimes \mathfrak{S}_n,$$

the semidirect product of the n-fold direct product of copies of $\mathfrak{S}_m$ with $\mathfrak{S}_n$, where $\mathfrak{S}_n$ acts by permuting the copies of $\mathfrak{S}_m$. For clarity, it is sometimes helpful to write $(f; \pi) \in \mathfrak{S}_m \wr \mathfrak{S}_n$ as $\big(f_1, \dots, f_n; \pi\big)$.

There is a natural embedding of $\mathfrak{S}_m \wr \mathfrak{S}_n$ into $\mathfrak{S}_{mn}$. Indeed, we let the i^{th} copy of $\mathfrak{S}_m$ permute $\{(i-1)m+1, \dots, im\} \subseteq \{1, 2, \dots, mn\}$ according to f_i, and we let $\pi \in \mathfrak{S}_n$ permute the n blocks

$$\{1, \dots, m\}, \{m+1, \dots, 2m\}, \dots, \{(n-1)m+1, \dots, mn\}.$$

For example, under this embedding,

- $\big((13), (23); 1\big) \in \mathfrak{S}_3 \wr \mathfrak{S}_2$ is mapped to $(1\,3)(5\,6) \in \mathfrak{S}_6$;
- $\big(1, (132); (12)\big) \in \mathfrak{S}_3 \wr \mathfrak{S}_2$ is mapped to $(4\,6\,5)(1\,4)(2\,5)(3\,6) = (1\,4\,3\,6\,2\,5) \in \mathfrak{S}_6$.

2.3.1 Representations of wreath products

We will benefit from having at our disposal notation that allows us to effectively study representations of wreath products. We will continue to concern ourselves with only the wreath product $\mathfrak{S}_m \wr \mathfrak{S}_n$, noting that the general theory is discussed in detail in §4.3-4.4 of James and Kerber's book [26]. However, we will not adopt James and Kerber's notation. Instead, we find the notation used by Chuang and Tan in [8] the most convenient for our purposes; it is this notation we now present.

There is a natural action of the symmetric group $\mathfrak{S}_n$ by place permutations on the n-fold tensor power $T^n(k\mathfrak{S}_m) := k\mathfrak{S}_m \otimes \cdots \otimes k\mathfrak{S}_m$. We may then define a k-algebra

$$T^n(k\mathfrak{S}_m) \otimes k\mathfrak{S}_n,$$

with multiplication

$$\big((a_1 \otimes \cdots \otimes a_n) \otimes \sigma\big)\big((b_1 \otimes \cdots \otimes b_n) \otimes \tau\big) = \big(a_1 b_{(1)\sigma^{-1}} \otimes \cdots \otimes a_n b_{(n)\sigma^{-1}}\big) \otimes \sigma\tau$$

for $(a_1 \otimes \cdots \otimes a_n)$, $(b_1 \otimes \cdots \otimes b_n) \in T^n(k\mathfrak{S}_m)$ and σ, $\tau \in k\mathfrak{S}_n$. This k-algebra is isomorphic to the group algebra of $\mathfrak{S}_m \wr \mathfrak{S}_n$. Furthermore, if $\mathbf{n} = (n_1, \ldots, n_r)$ is a composition of n, so that $\mathfrak{S}_{\mathbf{n}}$ is the Young subgroup $\mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_r}$, then

$$k(\mathfrak{S}_m \wr \mathfrak{S}_{\mathbf{n}}) := T^n(k\mathfrak{S}_m) \otimes k\mathfrak{S}_{\mathbf{n}} \cong k(\mathfrak{S}_m \wr \mathfrak{S}_{n_1}) \otimes \cdots \otimes k(\mathfrak{S}_m \wr \mathfrak{S}_{n_r})$$

is a subalgebra of $k(\mathfrak{S}_m \wr \mathfrak{S}_n)$.

Given a $k(\mathfrak{S}_m \wr \mathfrak{S}_n)$-module V and a $k\mathfrak{S}_n$-module X, we may construct from $V \otimes X$ a $k(\mathfrak{S}_m \wr \mathfrak{S}_n)$-module, denoted by $V \oslash X$, by equipping it with the action

$$(v \otimes x)(f \otimes \pi) = v(f \otimes \pi) \otimes x\pi$$

for $v \in V$, $x \in X$, $f \in T^n(k\mathfrak{S}_m)$ and $\pi \in \mathfrak{S}_n$. Note that any $k\mathfrak{S}_n$-module may be viewed as a $k(\mathfrak{S}_m \wr \mathfrak{S}_n)$-module by inflating along the canonical surjection $\mathfrak{S}_m \wr \mathfrak{S}_n \twoheadrightarrow \mathfrak{S}_n$. In particular, the inflation $\operatorname{Inf}_{\mathfrak{S}_n}^{\mathfrak{S}_m \wr \mathfrak{S}_n} X = k_{\mathfrak{S}_m \wr \mathfrak{S}_n} \oslash X$, where $k_{\mathfrak{S}_m \wr \mathfrak{S}_n}$ denotes the trivial $k(\mathfrak{S}_m \wr \mathfrak{S}_n)$-module. Furthermore, in our setting, $V \oslash X$ is the usual inner tensor product of V and $\operatorname{Inf}_{\mathfrak{S}_n}^{\mathfrak{S}_m \wr \mathfrak{S}_n} X$ over the group algebra $k(\mathfrak{S}_m \wr \mathfrak{S}_n)$.

If M is a $k\mathfrak{S}_m$-module, then the n-fold tensor power $T^n(M) = M \otimes \cdots \otimes M$ is a module for $T^n(k\mathfrak{S}_m)$, with component-wise action coming from that of $k\mathfrak{S}_m$ on M. This action may be extended to an action of $T^n(k\mathfrak{S}_m) \otimes k\mathfrak{S}_n$ by allowing any element of $\mathfrak{S}_n$ to act by place permutations. We denote the resulting $T^n(k\mathfrak{S}_m) \otimes k\mathfrak{S}_n \cong k(\mathfrak{S}_m \wr \mathfrak{S}_n)$-module by $T^{(n)}(M)$. Finally, if λ is a partition of n, we define a $k(\mathfrak{S}_m \wr \mathfrak{S}_n)$-module,

$$T^\lambda(M) := T^{(n)}(M) \oslash S^\lambda.$$

In particular, if $\lambda = (n)$, then $T^{(n)}(M) \oslash S^{(n)} = T^{(n)}(M)$ and so this definition is unambiguous.

Chuang and Tan prove an analogue of the Littlewood–Richardson Rule, which will be used extensively in Chapter 7 of this work. The statement we give here is again not the most general version of their result [8, Lemma 3.3(1)]; we remain in the setting outlined thus far.

Lemma 2.3.1

Let M be a $k\mathfrak{S}_m$-module and let $\mathbf{n} = (n_1, \dots, n_r) \vDash n$. For each $i \in \{1, 2, \dots, r\}$, let $\lambda^i \vdash n_i$. If $\lambda \vdash n$ and $c(\lambda; \lambda^1, \dots, \lambda^r)$ denotes the Littlewood–Richardson coefficient associated to the partitions $\lambda, \lambda^1, \dots, \lambda^r$, then inducing the $k(\mathfrak{S}_m \wr \mathfrak{S}_{\mathbf{n}})$-module $T^{\lambda^1}(M) \otimes \cdots \otimes T^{\lambda^r}(M)$ yields

$$\left(T^{\lambda^1}(M) \otimes \cdots \otimes T^{\lambda^r}(M)\right)\Big\uparrow_{\mathfrak{S}_m \wr \mathfrak{S}_{\mathbf{n}}}^{\mathfrak{S}_m \wr \mathfrak{S}_n} \cong \bigoplus_{\lambda \vdash n} c(\lambda; \lambda^1, \dots, \lambda^r)\, T^{\lambda}(M).$$

We conclude this chapter with a brief discussion about simple $k(\mathfrak{S}_m \wr \mathfrak{S}_n)$-modules. We will see that they may be constructed from collections of simple $k\mathfrak{S}_m$-modules and so it seems apt that they are labelled by (tuples of) partitions. Here we will detail the construction when $k = \mathbb{C}$, but a reader seeking more generality may refer to [8, Definition 3.6].

Let $\{M_i \mid i \in I\}$ be a set of irreducible $\mathbb{C}\mathfrak{S}_m$-modules. Further, suppose that $\lambda^i \vdash n_i$ and $\sum_{i \in I} n_i = n$. The module

$$M(\underline{\lambda}) = M\big(\lambda^1, \lambda^2, \dots\big) := \left(\bigotimes_{i \in I} T^{\lambda^i}(M_i)\right)\Bigg\uparrow_{\prod_{i \in I}(\mathfrak{S}_m \wr \mathfrak{S}_{n_i})}^{\mathfrak{S}_m \wr \mathfrak{S}_n}$$

is an irreducible $\mathbb{C}(\mathfrak{S}_m \wr \mathfrak{S}_n)$-module. If, further, $\{M_i \mid i \in I\}$ is a complete set of non-isomorphic simple $\mathbb{C}\mathfrak{S}_m$-modules, then $\big\{M(\underline{\lambda}) \mid \underline{\lambda} = (\lambda^i)_{i \in I} \text{ with } \lambda^i \vdash n_i \text{ and } \sum_{i \in I} n_i = n\big\}$ is a complete set of non-isomorphic simple $\mathbb{C}(\mathfrak{S}_m \wr \mathfrak{S}_n)$-modules.

Example 2.3.2

A family of non-isomorphic simple $\mathbb{C}(\mathfrak{S}_m \wr \mathfrak{S}_n)$-modules is

$$\big\{T^{\nu}\big(S^{\mu}\big) = T^{(n)}\big(S^{\mu}\big) \oslash S^{\nu} \mid \mu \vdash m,\, \nu \vdash n\big\}.$$

In particular, the trivial module for $\mathbb{C}(\mathfrak{S}_m \wr \mathfrak{S}_n)$ is $T^{(n)}\big(S^{(m)}\big)$.

Chapter 3

Foulkes modules

3.1 Foulkes modules

In this section, we introduce the main objects of study in this work. The action of the symmetric group $\mathfrak{S}_{mn}$ on the collection of set partitions of a set of size mn into n sets, each of size m, gives rise to a $k\mathfrak{S}_{mn}$-module called the *Foulkes module* $H^{(m^n)}$. As was indicated in §2.3, $H^{(m^n)}$ is the $k\mathfrak{S}_{mn}$-module induced from the trivial module of the imprimitive wreath product $\mathfrak{S}_m \wr \mathfrak{S}_n$. In particular,

$$H^{(m^n)} = \left(T^{(n)}\left(S^{(m)}\right)\right)\Big\uparrow_{\mathfrak{S}_m\wr\mathfrak{S}_n}^{\mathfrak{S}_{mn}}.$$

For most of this work, we will focus our attention on the characteristic zero setting and thus take $k = \mathbb{C}$. It will often be convenient for us to work with ordinary characters rather than modules and so we note that the permutation character of $\mathfrak{S}_{mn}$ afforded by $H^{(m^n)}$ is

$$\varphi^{(m^n)} = \left(\operatorname{Inf}_{\mathfrak{S}_n}^{\mathfrak{S}_m\wr\mathfrak{S}_n} 1_{\mathfrak{S}_n}\right)\Big\uparrow_{\mathfrak{S}_m\wr\mathfrak{S}_n}^{\mathfrak{S}_{mn}},$$

where the trivial character $1_{\mathfrak{S}_n} = \chi^{(n)}$ of $\mathfrak{S}_n$ is inflated along $\mathfrak{S}_m \wr \mathfrak{S}_n \twoheadrightarrow \mathfrak{S}_n$.

In the characteristic zero setting, Foulkes modules are the subject of a longstanding conjecture, which provides the main motivation for the study of Foulkes modules in this thesis. Foulkes' Conjecture asserts that for all natural numbers m and n with $m < n$, and for all partitions λ of mn,

$$\left\langle \varphi^{(m^n)}, \chi^\lambda \right\rangle \geq \left\langle \varphi^{(n^m)}, \chi^\lambda \right\rangle.$$

However, this is not the only formulation of the conjecture. The conjecture may also be stated in terms of plethysms or modules for the general linear group, as we will now explain.

3.2 Reformulations of Foulkes' Conjecture

Foulkes' original statement, made in 1950 in [15], was given in terms of plethysms of Schur functions. Plethysm multiplication, which may be defined for any two symmetric functions, was introduced by Littlewood in [31, §4]. We begin this section by collecting together some key facts from [33, Ch. I] about the ring of symmetric functions (in countably many independent variables), denoted by Λ, following which we will define plethysm.

The *Schur function* s_λ in the variables $x_1, x_2, \ldots, x_\ell$ is defined[1] by

$$s_\lambda(x_1, \ldots, x_\ell) = \sum x^T,$$

where the sum is over all possible semistandard λ-tableaux T whose entries – which need not be distinct – are elements of $\{1, \ldots, \ell\}$, and $x^T = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_\ell^{\alpha_\ell}$ is the monomial (of degree ℓ) such that α_i is the number of occurrences of the digit i in T.

Example 3.2.1
If $\lambda = (3,1)$ and $\ell = 2$, then the following are all the semistandard $(3,1)$-tableaux containing the entries 1 and 2:

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}\,, \qquad \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \qquad \textit{and} \qquad \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}\,.$$

Hence, $s_{(3,1)} = x_1^3 x_2 + x_1^2 x_2^2 + x_1 x_2^3$.

We highlight two Schur functions that will be of particular significance later: if $x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_\ell^{\alpha_\ell}$ for $\alpha \in \mathbb{N}^\ell$, so that the degree of x^α is $|\alpha| = \sum_j \alpha_j$, then

$$s_{(n)}(x_1, \ldots, x_\ell) = \sum_{|\alpha|=n} x^\alpha =: h_n(x_1, \ldots, x_\ell),$$

where h_n denotes the complete symmetric function corresponding to $n \in \mathbb{N}_0$, and

$$s_{(1^n)}(x_1, \ldots, x_\ell) = \sum_{1 \le i_1 < i_2 < \ldots < i_n \le \ell} x_{i_1} x_{i_2} \cdots x_{i_n} =: e_n(x_1, \ldots, x_\ell),$$

where e_n denotes the elementary symmetric function corresponding to $n \in \mathbb{N}_0$.

The ring of symmetric functions is a polynomial ring which has several generating sets. For example, Λ is freely generated by the set of elementary symmetric functions or the set of complete symmetric functions. Additionally, we can write down a variety of bases for Λ, all indexed by partitions. Indeed, if $\lambda = (\lambda_1, \lambda_2, \ldots)$ is a partition, define $e_\lambda := e_{\lambda_1} e_{\lambda_2} \cdots$ and $h_\lambda := h_{\lambda_1} h_{\lambda_2} \cdots$. With this notation, the Schur functions s_λ, the elementary symmetric functions e_λ and the complete symmetric functions h_λ are three commonly used bases.

[1]There are several definitions of Schur functions. Indeed, for an alternative, see [33, Ch. I, §3]. The definition chosen here was selected due to its simplicity to state and its combinatorial nature.

Plethysm may be defined for any two symmetric functions: let $f, g \in \Lambda$ and write g as a sum of monomials, i.e. $g = \sum_\alpha c_\alpha x^\alpha$. The plethysm[2] $f \circ g$ is defined by

$$f \circ g := f(y_1, y_2, \ldots)$$

where the new variables y_i are defined by

$$\prod_i (1 + y_i t) = \prod_\alpha (1 + x^\alpha t)^{c_\alpha}.$$

However, if the two symmetric functions are Schur functions, then $c_\alpha = 1$ for all α, and so we may avoid this formal definition by thinking about the plethysm $s_\lambda \circ s_\mu$ as the symmetric function which is obtained by substituting the monomials in s_μ for the variables of s_λ.

Example 3.2.2
If $\lambda = (3)$ and $\mu = (2)$, then

$$s_{(2)}(x, y) = x^2 + xy + y^2$$

and

$$s_{(3)} \circ s_{(2)}(x, y) = s_{(3)}\left(x^2, xy, y^2\right) = x^6 + x^5 y + 2x^4 y^2 + 2x^3 y^3 + 2x^2 y^4 + xy^5 + y^6.$$

Before we present the statement of Foulkes' Conjecture in terms of plethysms, we take the opportunity to introduce an important involution. Define the ring automorphism $\omega : \Lambda \to \Lambda$ on the polynomial generators e_r by $\omega : e_r \mapsto h_r$ for any $r \in \mathbb{N}_0$. Consequently, from the definition of the Schur functions $s_{(n)}$ and $s_{(1^n)}$ given above, it is clear that $\omega : s_{(1^n)} \mapsto s_{(n)}$. More generally,

$$\omega : s_\lambda \mapsto s_{\lambda'} \tag{3.1}$$

which is Equation (3.8) in [33, Ch. I]. We mention one more important application of this involution, which is to plethysms (see [33, p.136]). If $f \in \Lambda^m$ and $g \in \Lambda^n$, where elements of Λ^i are the homogeneous symmetric functions of degree i, then

$$\omega : f \circ g \mapsto \begin{cases} f \circ (g)\omega & n \text{ even;} \\ (f)\omega \circ (g)\omega & n \text{ odd.} \end{cases} \tag{3.2}$$

We now wish to understand how to translate information about characters of the symmetric group into the language of symmetric functions. For full details, we refer the reader to the discussion given by Macdonald in [33, Ch. I, §7]. Let $\mathcal{C}(\mathfrak{S}_n)$ denote the ring of (ordinary)

[2] In the literature, a variety of different notation is used for plethysms. For example, the plethysm $s_\lambda \circ s_\mu$ may also be denoted by $s_\lambda[s_\mu]$ or by $\{\mu\} \otimes \{\lambda\}$ – the latter was the preferred notation for Foulkes and Littlewood, c.f. Chapter 1. Further, if λ (or μ) is a partition consisting of a single part, then it is common to write, for example, s_m or $\{m\}$ instead of $s_{(m)}$ or $\{(m)\}$, respectively.

characters of $\mathfrak{S}_n$, with ring structure defined by $\chi_1\chi_2 = (\chi_1 \times \chi_2)\big\uparrow_{\mathfrak{S}_m\times\mathfrak{S}_n}^{\mathfrak{S}_{mn}}$ for $\chi_1 \in \mathcal{C}(\mathfrak{S}_m)$ and $\chi_2 \in \mathcal{C}(\mathfrak{S}_n)$. The *characteristic map*

$$\mathrm{ch} : \bigoplus_{n\geq 0} \mathcal{C}(\mathfrak{S}_n) \to \Lambda$$

is an isomorphism of rings, defined on irreducible characters of $\mathfrak{S}_n$ by $\mathrm{ch} : \chi^\lambda \to s_\lambda$. Under this map, the image of the character afforded by the $\mathbb{C}(\mathfrak{S}_{mn})$-module $\big(T^\nu\big(S^\mu\big)\big)\big\uparrow^{\mathfrak{S}_{mn}}$ is the plethysm $s_\nu \circ s_\mu$. Hence, the Foulkes character $\varphi^{(m^n)}$ corresponds to the plethysm $s_{(n)} \circ s_{(m)}$. Asking for the decomposition of the Foulkes character $\varphi^{(m^n)}$ as a sum of irreducible characters of $\mathfrak{S}_{mn}$ is therefore equivalent to decomposing $s_{(n)} \circ s_{(m)}$ as a sum of Schur functions. Moreover, stated in the language of symmetric functions, Foulkes' Conjecture asserts that, for $m < n$, $s_{(n)} \circ s_{(m)} - s_{(m)} \circ s_{(n)}$ is a sum of Schur functions with non-negative coefficients.

The notion of plethysm also arises in the representation theory of the general linear group, thus providing a third setting in which Foulkes' Conjecture can be stated. If V is a finite dimensional complex vector space, then it makes sense to consider the $GL(V)$-module $\mathrm{Sym}^m\big(\mathrm{Sym}^n(V)\big)$, where Sym^i denotes the i^{th} symmetric power. The effect on the corresponding formal characters of composing the two symmetric powers in this way is precisely the plethysm operation $s_{(m)} \circ s_{(n)}$ described above. Foulkes' Conjecture in the general linear group setting states that if $m < n$, then there is an embedding of $GL(V)$-modules $\mathrm{Sym}^n\big(\mathrm{Sym}^m(V)\big) \hookrightarrow \mathrm{Sym}^m\big(\mathrm{Sym}^n(V)\big)$.

Despite having numerous settings in which to tackle Foulkes' Conjecture, it is still an open problem. However, some progress has been made. Work by Thrall in his 1942 paper [43] gives the explicit decomposition of both $\varphi^{(2^n)}$ and $\varphi^{(m^2)}$, which we record in Theorem 3.2.3. We note that these two Foulkes characters are *multiplicity free*, which means that the multiplicity with which any irreducible summand appears in the character decomposition is at most one.

Theorem 3.2.3

The complete decomposition of the Foulkes character $\varphi^{(2^r)}$ is given by

$$\varphi^{(2^r)} = \sum_{\lambda\vdash r} \chi^{2\lambda},$$

where 2λ denotes the partition of $2r$ obtained by doubling the length of each part of λ.

Similarly, $\varphi^{(r^2)}$ decomposes as

$$\varphi^{(r^2)} = \sum_{i=0}^{\lfloor r/2\rfloor} \chi^{(2r-2i,\,2i)}.$$

Theorem 3.2.3 is sufficient to verify Foulkes' Conjecture when $m = 2$ and, together with a few small examples with m and n at most six, was known to Foulkes when he conjectured the general result. Further progress was made by Dent and Siemons, who proved the conjecture when $m = 3$ in [11]; by McKay, verifying the $m = 4$ case in [36]; and by Müller and Neunhöffer (see [37]), and Evseev, Paget and Wildon (see [13]), who computationally verified Foulkes' Conjecture for $m + n \leq 19$. We remark that the authors of [13] used an algorithm for computing character multiplicities, which will be discussed further in Chapter 6 of this work. The only other result which directly addresses Foulkes' Conjecture is an asymptotic result due to Brion (see the corollary in [4, §1.3]) proving that the conjecture holds when n is sufficiently large compared to m, although Brion does not give an explicit bound.

3.3 Generalised Foulkes modules

In this thesis, we will also consider a generalisation of the Foulkes modules discussed in §3.1. For any partition ν of n, we may construct a $k(\mathfrak{S}_m \wr \mathfrak{S}_n)$-module, $T^\nu\big(S^{(m)}\big)$, which we can induce to obtain a $k\mathfrak{S}_{mn}$-module

$$H_\nu^{(m^n)} := \Big(T^\nu\big(S^{(m)}\big)\Big)\Big\uparrow_{\mathfrak{S}_m \wr \mathfrak{S}_n}^{\mathfrak{S}_{mn}}.$$

We refer to these modules as *generalised Foulkes modules*; the ordinary characters afforded by these modules are defined by $\varphi_\nu^{(m^n)} = \big(\operatorname{Inf}_{\mathfrak{S}_n}^{\mathfrak{S}_m \wr \mathfrak{S}_n} \chi^\nu\big)\big\uparrow_{\mathfrak{S}_m \wr \mathfrak{S}_n}^{\mathfrak{S}_{mn}}$, where χ^ν is the ordinary character for the $k\mathfrak{S}_n$-module S^ν. The image of $\varphi_\nu^{(m^n)}$ under the characteristic map is $s_\nu \circ s_{(m)}$ and the corresponding $GL(V)$-module is $\Delta^\nu\big(\operatorname{Sym}^m(V)\big)$, where Δ^ν is the Schur functor corresponding to the partition ν of n. For more details about this construction and representations of the general linear group, see [17, Chapter 8] or [18, §6.1].

We remark that the Foulkes module $H^{(m^n)}$ is precisely $H_{(n)}^{(m^n)}$, but we will usually omit the subscript in this case. *Twisted Foulkes modules*, which arise by choosing $\nu = (1^n)$ for the above construction, will also feature prominently in this work. For ease of notation, we let $K^{(m^n)} := H_{(1^n)}^{(m^n)}$ and $\tau^{(m^n)} := \varphi_{(1^n)}^{(m^n)}$.

The twisted Foulkes character $\tau^{(2^r)}$ can be elegantly described (see, for example, [33, Ch. I, §8, Ex. 6]) if we introduce the following notation: let $\alpha = (\alpha_1, \ldots, \alpha_\ell) \vdash r$ have distinct parts. Define $\lambda = 2[\alpha]$ to be the partition of $2r$ which has $\lambda_i := \alpha_i + i$ for $1 \leq i \leq \ell$ and leading diagonal hook lengths $2\alpha_1, \ldots, 2\alpha_\ell$.

Figure 3.1: Young diagrams corresponding to the partitions $2[(3)] = (4, 1, 1)$ and $2[(2, 1)] = (3, 3)$, showing the leading diagonal hook lengths.

Similarly, the decomposition of the character $\tau^{(r^2)}$ is detailed in [46, Equation (5)]. The following theorem collects together these two results.

Theorem 3.3.1
The complete decomposition of the twisted Foulkes character $\tau^{(2^r)}$ is

$$\tau^{(2^r)} = \sum \chi^{2[\alpha]},$$

where the sum is over all $\alpha \vdash r$ which have distinct parts.
Similarly, $\tau^{(r^2)}$ decomposes as

$$\tau^{(r^2)} = \sum_{i=0}^{\lfloor (r-1)/2 \rfloor} \chi^{(2r-2i-1,\, 2i+1)}.$$

For example, if we take $r = 3$, then the only partitions which have distinct parts are (3) and (2, 1). Moreover, $2[(3)] = (4, 1, 1)$ and $2[(2, 1)] = (3, 3)$ (c.f. Figure 3.1) and so

$$\tau^{(2^3)} = \chi^{(4,1,1)} + \chi^{(3,3)}.$$

Also using Theorem 3.3.1, we obtain the decomposition $\tau^{(3^2)} = \chi^{(5,1)} + \chi^{(3,3)}$. This example is indicative of the fact that there is no direct analogue of Foulkes' Conjecture for the generalised Foulkes characters.

Much of the progress that has been made in understanding the structure of generalised Foulkes modules has been via symmetric functions. The plethysm $s_\nu \circ s_{(m)}$ with $\nu \vdash n$, corresponding to $\varphi_\nu^{(m^n)}$, has been fully described in the following cases:

- $s_{(2)} \circ s_{(m)}$ and $s_{(n)} \circ s_{(2)}$: work by Thrall [43], as detailed in Theorem 3.2.3 above.
- $s_{(1^2)} \circ s_{(m)}$ and $s_{(1^n)} \circ s_{(2)}$: results from [33, 46], which are detailed in Theorem 3.3.1 above.
- $s_{(3)} \circ s_{(m)}$: work originally by Thrall [43], and re-proved by Dent and Siemons [11] in a symmetric group setting as part of their proof of Foulkes' Conjecture when $m = 3$.

  The reader may have noticed that all non-zero coefficients in the plethysms $s_{(2)} \circ s_{(m)}$, $s_{(n)} \circ s_{(2)}$, $s_{(1^2)} \circ s_{(m)}$ and $s_{(1^n)} \circ s_{(2)}$ were equal to one. However, these are somewhat special cases and this multiplicity free property should not be expected in general. For example, we see coefficients greater than one appearing in $s_{(3)} \circ s_{(m)}$ for $m \geq 6$.
- $s_\nu \circ s_{(m)}$ where ν is a partition of 2, 3 or 4: the plethysms $s_{(2)} \circ s_{(m)}$, $s_{(3)} \circ s_{(m)}$ and $s_{(4)} \circ s_{(m)}$ are computed explicitly by Howe in [22, §3.5], but the author remarks in §3.6(b) that all other plethysms with $\nu \vdash 2, 3, 4$, respectively, can be obtained by an 'averaging method'. A method for calculating the plethysms $s_\nu \circ s_{(m)}$ where $\nu \vdash 4$ can also be found in Duncan's paper [12], which pre-dates Howe's work. However, if m is not small, the method appears to be computationally demanding. In [16], Foulkes gives a much simpler method for determining the coefficients in $s_\nu \circ s_{(m)}$ where $\nu \vdash 4$, which is effective for all m, no matter how large.

3.4 Existing results about constituents of generalised Foulkes characters

In this section, we collect together a variety of results that either give information about constituents of a special form, or describe relationships between constituents. The list of results is by no means exhaustive, but we aim to give an overview of the current state of the research into generalised Foulkes characters. We will often translate results into symmetric group language, since statements in this form will be most convenient for us in subsequent chapters.

A paper by Weintraub [44], detailing several observations about plethysms, is a good starting point for our discussion. In this paper, the author conjectures a result about the multiplicities of 'even partitions' in certain Foulkes modules. Two independent proofs of the conjecture have since been given: first in [6], and subsequently in [34]. We define a partition to be *even* if each of its parts is even.

Theorem 3.4.1 (Weintraub's Conjecture)
Let $m, n \in \mathbb{N}$ be such that m is even. If λ is an even partition of mn with at most n parts, then $\langle \varphi^{(m^n)}, \chi^\lambda \rangle \neq 0$.

This is not the only noteworthy result in Weintraub's paper. Constituents labelled by two-row partitions or by hook partitions can often provide a good starting point when seeking general information about a character. The following proposition includes information about the multiplicities with which hook-like constituents appear in $\varphi^{(m^n)}$ and $\tau^{(m^n)}$.

Proposition 3.4.2
If $\lambda = \big(mn - j(n-1), j^{n-1}\big)$ for $0 \leq j \leq m$, then

$$\left\langle \varphi^{(m^n)}, \chi^\lambda \right\rangle = \begin{cases} 1 & j \text{ even;} \\ 0 & j \text{ odd,} \end{cases} \qquad \text{and} \qquad \left\langle \tau^{(m^n)}, \chi^\lambda \right\rangle = \begin{cases} 1 & j \text{ odd;} \\ 0 & j \text{ even.} \end{cases}$$

If $\lambda = (mn - j, 1^j)$, i.e. a hook partition, or $\lambda = (mn - j - 2, 2, 1^j)$, then the multiplicity of the corresponding constituent χ^λ in the Foulkes character is given by

$$\left\langle \varphi^{(m^n)}, \chi^\lambda \right\rangle = \begin{cases} 1 & j = 0; \\ 0 & j > 0. \end{cases}$$

Proof. See Proposition 2.5 in [44]. ■

Example 3.4.3
If $m = 4$ and $n = 5$, then Proposition 3.4.2 tells us the following information about the Foulkes character $\varphi^{(4^5)}$:

- $\left\langle \varphi^{(4^5)}, \chi^{(20)} \right\rangle = \left\langle \varphi^{(4^5)}, \chi^{(12,2^4)} \right\rangle = \left\langle \varphi^{(4^5)}, \chi^{(4^5)} \right\rangle = 1$;
- $\left\langle \varphi^{(4^5)}, \chi^{(16,1^4)} \right\rangle = \left\langle \varphi^{(4^5)}, \chi^{(8,3^4)} \right\rangle = 0$;
- $\left\langle \varphi^{(4^5)}, \chi^{(20-j,1^j)} \right\rangle = 0$ *for all* $j \geq 1$;
- $\left\langle \varphi^{(4^5)}, \chi^{(18,2)} \right\rangle = 1$ *and* $\left\langle \varphi^{(4^5)}, \chi^{(18-j,2,1^j)} \right\rangle = 0$ *for all* $j \geq 1$.

More recently, improvements were made to the second part of Proposition 3.4.2: in [30], Langley and Remmel considered the multiplicity of the Schur functions labelled by partitions of the form $(mn-b, 1^b)$, $(mn-a-b, a, 1^b)$ and $(mn-2a-b, 2^a, 1^b)$ in the plethysms $s_\nu \circ s_\mu$ (where $\nu \vdash n$ and $\mu \vdash m$); and Giannelli [19] used symmetric group character-theoretic methods to determine the multiplicities of an even larger class of constituents, labelled by partitions whose shape is close to that of a hook, in the Foulkes character $\varphi^{(m^n)}$.

Maximal constituents of generalised Foulkes characters have been investigated by several authors. In [1], Agaoka conjectured the form of the lexicographic maximal constituent of $s_\nu \circ s_\mu$ and its associated multiplicity; a proof of the conjecture was given by Iijima in [23]. A statement of the result in the special case $\mu = (m)$ is Proposition 3.4.4, below. Yang also gives many results on the lexicographic maximal constituent of plethysms in [47].

Proposition 3.4.4

Let $m, n \in \mathbb{N}$ and let $\nu = (\nu_1, \ldots, \nu_\ell) \vdash n$. The lexicographic maximal constituent of $\varphi_\nu^{(m^n)}$ is labelled by

$$\nu^* := (\nu_1 + n(m-1), \nu_2, \ldots, \nu_\ell)$$

and $\left\langle \varphi_\nu^{(m^n)}, \chi^{\nu^*} \right\rangle = 1$.

Paget and Wildon give a complete description of the minimal constituents of generalised Foulkes modules in the dominance order: their work in [40] gives the results for $\varphi^{(m^n)}$, and for all other $\nu \vdash n$, a description of the minimal constituents of $\varphi_\nu^{(m^n)}$ can be found in [41]. We note that, in the dominance order there is, in general, not a unique minimal constituent. Paget and Wildon prove an extreme case of this fact, that all constituents of $\tau^{(2^n)}$ are minimal, and therefore also maximal, in the dominance order. However, in certain special cases, $\varphi_\nu^{(m^n)}$ does have a unique minimal constituent (see [41, Corollary 7.3]). In particular, we note the following results; the multiplicities concerned can also be deduced from Proposition 3.4.2.

Corollary 3.4.5

If m is even, then the unique minimal constituent of $\varphi^{(m^n)}$ is $\chi^{(m^n)}$ and $\langle \varphi^{(m^n)}, \chi^{(m^n)} \rangle = 1$. If m is odd, then the unique minimal constituent of $\tau^{(m^n)}$ is $\chi^{(m^n)}$ and $\langle \tau^{(m^n)}, \chi^{(m^n)} \rangle = 1$.

Remark. It is clear that the unique minimal constituent with respect to the dominance order will also be the (unique) lexicographic minimal constituent.

In addition to the results concerning constituents of a special form, there are also several results that establish relationships between constituents of generalised Foulkes modules. Foulkes' Second Conjecture – so named by Brion, who proved it in [4, §2] – is an example of such a result.

Theorem 3.4.6 (Foulkes' Second Conjecture)
Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ be a partition of mn and suppose that $\left\langle \varphi^{(m^n)}, \chi^\lambda \right\rangle = r \geq 0$. If $\widetilde{\lambda} := (\lambda_1 + n, \lambda_2, \ldots, \lambda_\ell)$, then $\left\langle \varphi^{((m+1)^n)}, \chi^{\widetilde{\lambda}} \right\rangle \geq r$.

If we restrict our attention to those constituents of the Foulkes character $\varphi^{(m^n)}$ that are labelled by partitions λ such that $\lambda_2 \leq m$, then Foulkes' Second Conjecture can be generalised in the following way.

Theorem 3.4.7 ([44, Corollary 1.8])
Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ be a partition of mn that satisfies $\lambda_2 \leq m$ and fix $\nu \vdash n$. For any $q \geq 0$, if $\widetilde{\lambda} := (\lambda_1 + nq, \lambda_2, \ldots, \lambda_\ell) \vdash (m+q)n$, then

$$\left\langle \varphi_\nu^{\left((m+q)^n\right)}, \chi^{\widetilde{\lambda}} \right\rangle = \left\langle \varphi_\nu^{(m^n)}, \chi^\lambda \right\rangle .$$

In Chapters 4 and 5, we will prove a variety of results of this type. However, we will no longer always fix ν or n.

Chapter 4

Semistandard homomorphism results for fixed n

In this chapter, we apply the general theory about semistandard homomorphisms, using it to establish some structural relationships between Foulkes modules. We also extend the theory to study the structure of twisted Foulkes modules, $K^{(m^n)}$. In the last chapter, we alluded to the fact that it will be necessary to work over a ground field of characteristic zero. So, henceforth, we fix $k = \mathbb{C}$, unless otherwise specified. The reason for this will become clear in §4.1.1.

4.1 The setting

4.1.1 Foulkes modules

One way to study the structure of the Foulkes module is to look for maps from the Specht module S^λ into $H^{(m^n)}$, where $\lambda \vdash mn$. In particular, finding such a non-zero map identifies S^λ as a composition factor of $H^{(m^n)}$. We have already acquired most of the machinery that we need to construct suitable maps: in §2.2.4, we defined semistandard homomorphisms θ_T, where T is a semistandard λ-tableau of type μ, which form a basis for $\mathrm{Hom}\left(S^\lambda, M^\mu\right)$.

Observe that there is a natural surjection ψ from the Young permutation module $M^{(m^n)}$ into $H^{(m^n)}$, which is defined on (m^n)-tabloids by

$$\psi : \begin{array}{ccc} \hline x_1 & x_2 & \dots\ x_m \\ \hline x_{m+1} & x_{m+2} & \dots\ x_{2m} \\ \hline \vdots & \vdots & \vdots \\ \hline x_{(n-1)m+1} & x_{(n-1)m+2} & \dots\ x_{nm} \\ \hline \end{array} \longmapsto \big\{X_1, X_2, \dots, X_n\big\},$$

where $X_i := \{x_{(i-1)m+1}, x_{(i-1)m+2}, \dots, x_{im}\}$ for all $1 \leq i \leq n$. So, ψ is the map which sends an (m^n)-tabloid, say $\{t\}$, to the set partition consisting of the n sets which correspond to the

n rows of $\{t\}$.

Computationally, it is often easier to work completely with the description of $M^{(m^n)}$ in terms of λ-tableaux of type (m^n), thus avoiding the need to involve (m^n)-tabloids in the calculations. Let $\overline{T}$ denote the image of the tableau $T \in M^{(m^n)}$ under ψ. Observe that two tableaux $T_1, T_2 \in \mathcal{T}\big(\lambda, (m^n)\big)$ are equivalent under ψ if there exists a relabelling permutation $\pi \in \mathfrak{S}_n$ such that $T_1 * \pi = T_2$. This says that entries in two equivalent tableaux will have the same *pattern*. For example, if

$$t = \begin{array}{|c|c|c|c|c|}\hline 1&2&3&4&5\\ \hline 6 \\ \cline{1-1}\end{array}, \quad T_1 = \begin{array}{|c|c|c|c|c|}\hline 1&1&1&2&2\\ \hline 2 \\ \cline{1-1}\end{array} \quad \text{and} \quad T_2 = \begin{array}{|c|c|c|c|c|}\hline 2&2&2&1&1\\ \hline 1 \\ \cline{1-1}\end{array},$$

so that $T_1 * (1\,2) = T_2$, then the (3^2)-tabloids corresponding to T_1 and T_2 are

$$\begin{array}{ccc}\hline 1&2&3\\ \hline 4&5&6\\ \hline\end{array} \quad \text{and} \quad \begin{array}{ccc}\hline 4&5&6\\ \hline 1&2&3\\ \hline\end{array},$$

respectively, and $\overline{T_1} = \overline{T_2}$. We should note that if two tableaux $T_1, T_2 \in \mathcal{T}\big(\lambda, (m^n)\big)$ are equivalent under ψ, then this will be true regardless of the choice of t, because t just serves as a labelling tableau.

If we take $\mu = (m^n)$, then we can construct a map $\overline{\theta}_T : S^\lambda \to H^{(m^n)}$ by composing a semistandard homomorphism with the surjection $\psi : M^{(m^n)} \twoheadrightarrow H^{(m^n)}$, as shown in Figure 4.1. Moreover, $\big\{\overline{\theta}_T \bigm| T \in \mathcal{T}_0\big(\lambda, (m^n)\big)\big\}$ is a spanning set for $\operatorname{Hom}_{\mathbb{C}\mathfrak{S}_{mn}}\big(S^\lambda, H^{(m^n)}\big)$. The Foulkes module $H^{(m^n)}$ is not only a quotient of $M^{(m^n)}$; we can also view it as a submodule of $M^{(m^n)}$. Indeed, there is a $\mathbb{C}\mathfrak{S}_{mn}$-homomorphism $\psi' : H^{(m^n)} \hookrightarrow M^{(m^n)}$ defined on set partitions $\{X_1, \ldots, X_n\}$, where $X_i = \{x_{(i-1)m+1}, x_{(i-1)m+2}, \ldots, x_{im}\}$, by

$$\psi' : \{X_1, \ldots, X_n\} \longmapsto \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \left(\begin{array}{cccc}\hline x_1 & x_2 & \cdots & x_m \\ \hline x_{m+1} & x_{m+2} & \cdots & x_{2m} \\ \hline \vdots & \vdots & & \vdots \\ \hline x_{(n-1)m+1} & x_{(n-1)m+2} & \cdots & x_{nm} \\ \hline \end{array} * \sigma \right),$$

where the effect of σ is to permute the rows of the (m^n)-tabloid. It is clear that $\psi' \circ \psi = \mathrm{id}$, where id denotes the identity map on $H^{(m^n)}$.

Example 4.1.1

The image of the set partition $\big\{\{1,2\},\{3,4\},\{5,6\}\big\}$ *under* ψ' *is*

$$\frac{1}{6}\left(\begin{array}{cc}\hline 1&2\\ \hline 3&4\\ \hline 5&6\\ \hline\end{array} + \begin{array}{cc}\hline 1&2\\ \hline 5&6\\ \hline 3&4\\ \hline\end{array} + \begin{array}{cc}\hline 3&4\\ \hline 1&2\\ \hline 5&6\\ \hline\end{array} + \begin{array}{cc}\hline 3&4\\ \hline 5&6\\ \hline 1&2\\ \hline\end{array} + \begin{array}{cc}\hline 5&6\\ \hline 1&2\\ \hline 3&4\\ \hline\end{array} + \begin{array}{cc}\hline 5&6\\ \hline 3&4\\ \hline 1&2\\ \hline\end{array} \right).$$

We should observe that the map ψ' is only defined if we are working over a field of characteristic zero, or a field of characteristic p with $p > n$. Herein lies the reason for specifying $k = \mathbb{C}$ at the start of the chapter. Maschke's Theorem applies in this setting, so

$$\begin{tikzcd}
S^\lambda \arrow[r] \arrow[rrr, bend left, "\overline{\theta}_T"] \arrow[rr, bend right, "\theta_T"'] & H^{(m^n)} \arrow[r, hook, "\psi'"] \arrow[rr, bend right, "\mathrm{id}"'] & M^{(m^n)} \arrow[r, two heads, "\psi"] & H^{(m^n)}
\end{tikzcd}$$

Figure 4.1: Maps between Specht modules, Foulkes modules and Young permutation modules

we know that the group algebra $\mathbb{C}\mathfrak{S}_{mn}$ is semisimple and thus, over $\mathbb{C}$, the submodule $H^{(m^n)}$ is a direct summand of $M^{(m^n)}$.

All $\mathbb{C}\mathfrak{S}_{mn}$-homomorphisms from S^λ to $H^{(m^n)}$ will be linear combinations of the maps $\overline{\theta}_T = \theta_T \circ \psi$, where θ_T is a basis element of $\mathrm{Hom}_{\mathbb{C}\mathfrak{S}_{mn}}\left(S^\lambda, M^{(m^n)}\right)$. Unfortunately, it is rarely obvious whether the composition of θ_T and ψ will yield a non-zero map. Examples 4.1.2 and 4.1.3 illustrate this point.

Example 4.1.2

Take $\lambda = (4,2)$, $m = 3$ and $n = 2$. Choose $t = \begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 5&6\\\cline{1-2}\end{array}$ and $T = \begin{array}{|c|c|c|c|}\hline 1&1&1&2\\\hline 2&2\\\cline{1-2}\end{array}$. In Example 2.2.9, we showed that

$$(e_t)\theta_T = \begin{array}{c}\hline 1\ 2\ 3\\\hline 4\ 5\ 6\\\hline\end{array} + \begin{array}{c}\hline 1\ 2\ 4\\\hline 3\ 5\ 6\\\hline\end{array} + \begin{array}{c}\hline 3\ 5\ 6\\\hline 1\ 2\ 4\\\hline\end{array} + \begin{array}{c}\hline 4\ 5\ 6\\\hline 1\ 2\ 3\\\hline\end{array} - \begin{array}{c}\hline 2\ 3\ 5\\\hline 1\ 4\ 6\\\hline\end{array} - \begin{array}{c}\hline 2\ 4\ 5\\\hline 1\ 3\ 6\\\hline\end{array} - \begin{array}{c}\hline 1\ 3\ 6\\\hline 2\ 4\ 5\\\hline\end{array} - \begin{array}{c}\hline 1\ 4\ 6\\\hline 2\ 3\ 5\\\hline\end{array}.$$

So, the image of $(e_t)\theta_T$ under ψ is

$$2\left(\big\{\{1,2,3\},\{4,5,6\}\big\} + \big\{\{1,2,4\},\{3,5,6\}\big\} - \big\{\{2,3,5\},\{1,4,6\}\big\} - \big\{\{2,4,5\},\{1,3,6\}\big\}\right).$$

This means that $\left(\overline{\theta}_T : S^{(4,2)} \to H^{(3^2)}\right) \neq 0$ and we conclude that $S^{(4,2)}$ appears in $H^{(3^2)}$ as a composition factor.

Example 4.1.3

Take $\lambda = (5,1)$, $m = 3$ and $n = 2$.

If we choose $t = \begin{array}{|c|c|c|c|c|}\hline 1&2&3&4&5\\\hline \multicolumn{1}{|c|}{6}\\\cline{1-1}\end{array}$ and $T = \begin{array}{|c|c|c|c|c|}\hline 1&1&1&2&2\\\hline \multicolumn{1}{|c|}{2}\\\cline{1-1}\end{array}$, then $\kappa_t = 1 - (1\ 6)$ and

$$\begin{aligned}
(e_t)\theta_T = \Bigg(&\begin{array}{|c|c|c|c|c|}\hline 1&1&1&2&2\\\hline \multicolumn{1}{|c|}{2}\\\cline{1-1}\end{array} + \begin{array}{|c|c|c|c|c|}\hline 1&1&2&1&2\\\hline \multicolumn{1}{|c|}{2}\\\cline{1-1}\end{array} + \begin{array}{|c|c|c|c|c|}\hline 1&2&1&1&2\\\hline \multicolumn{1}{|c|}{2}\\\cline{1-1}\end{array} + \begin{array}{|c|c|c|c|c|}\hline 2&1&1&1&2\\\hline \multicolumn{1}{|c|}{2}\\\cline{1-1}\end{array} + \begin{array}{|c|c|c|c|c|}\hline 1&1&2&2&1\\\hline \multicolumn{1}{|c|}{2}\\\cline{1-1}\end{array}\\
&+\begin{array}{|c|c|c|c|c|}\hline 1&2&1&2&1\\\hline \multicolumn{1}{|c|}{2}\\\cline{1-1}\end{array} + \begin{array}{|c|c|c|c|c|}\hline 2&1&1&2&1\\\hline \multicolumn{1}{|c|}{2}\\\cline{1-1}\end{array} + \begin{array}{|c|c|c|c|c|}\hline 1&2&2&1&1\\\hline \multicolumn{1}{|c|}{2}\\\cline{1-1}\end{array} + \begin{array}{|c|c|c|c|c|}\hline 2&1&2&1&1\\\hline \multicolumn{1}{|c|}{2}\\\cline{1-1}\end{array} + \begin{array}{|c|c|c|c|c|}\hline 2&2&1&1&1\\\hline \multicolumn{1}{|c|}{2}\\\cline{1-1}\end{array}\Bigg)\kappa_t\\
=\ &\begin{array}{|c|c|c|c|c|}\hline 1&1&1&2&2\\\hline \multicolumn{1}{|c|}{2}\\\cline{1-1}\end{array} + \begin{array}{|c|c|c|c|c|}\hline 1&1&2&1&2\\\hline \multicolumn{1}{|c|}{2}\\\cline{1-1}\end{array} + \begin{array}{|c|c|c|c|c|}\hline 1&2&1&1&2\\\hline \multicolumn{1}{|c|}{2}\\\cline{1-1}\end{array} + \begin{array}{|c|c|c|c|c|}\hline 1&1&2&2&1\\\hline \multicolumn{1}{|c|}{2}\\\cline{1-1}\end{array}\\
&+\begin{array}{|c|c|c|c|c|}\hline 1&2&1&2&1\\\hline \multicolumn{1}{|c|}{2}\\\cline{1-1}\end{array} + \begin{array}{|c|c|c|c|c|}\hline 1&2&2&1&1\\\hline \multicolumn{1}{|c|}{2}\\\cline{1-1}\end{array} - \begin{array}{|c|c|c|c|c|}\hline 2&1&1&2&2\\\hline \multicolumn{1}{|c|}{1}\\\cline{1-1}\end{array} - \begin{array}{|c|c|c|c|c|}\hline 2&1&2&1&2\\\hline \multicolumn{1}{|c|}{1}\\\cline{1-1}\end{array}\\
&-\begin{array}{|c|c|c|c|c|}\hline 2&2&1&1&2\\\hline \multicolumn{1}{|c|}{1}\\\cline{1-1}\end{array} - \begin{array}{|c|c|c|c|c|}\hline 2&1&2&2&1\\\hline \multicolumn{1}{|c|}{1}\\\cline{1-1}\end{array} - \begin{array}{|c|c|c|c|c|}\hline 2&2&1&2&1\\\hline \multicolumn{1}{|c|}{1}\\\cline{1-1}\end{array} - \begin{array}{|c|c|c|c|c|}\hline 2&2&2&1&1\\\hline \multicolumn{1}{|c|}{1}\\\cline{1-1}\end{array}
\end{aligned}$$

$$
\begin{aligned}
&= \begin{array}{c}\hline 1\;2\;3\\ \hline 4\;5\;6\\ \hline\end{array} + \begin{array}{c}\hline 1\;2\;4\\ \hline 3\;5\;6\\ \hline\end{array} + \begin{array}{c}\hline 1\;3\;4\\ \hline 2\;5\;6\\ \hline\end{array} + \begin{array}{c}\hline 1\;2\;5\\ \hline 3\;4\;6\\ \hline\end{array} + \begin{array}{c}\hline 1\;3\;5\\ \hline 2\;4\;6\\ \hline\end{array} + \begin{array}{c}\hline 1\;4\;5\\ \hline 2\;3\;6\\ \hline\end{array}\\
&\quad - \begin{array}{c}\hline 2\;3\;6\\ \hline 1\;4\;5\\ \hline\end{array} - \begin{array}{c}\hline 2\;4\;6\\ \hline 1\;3\;5\\ \hline\end{array} - \begin{array}{c}\hline 3\;4\;6\\ \hline 1\;2\;5\\ \hline\end{array} - \begin{array}{c}\hline 2\;5\;6\\ \hline 1\;3\;4\\ \hline\end{array} - \begin{array}{c}\hline 3\;5\;6\\ \hline 1\;2\;4\\ \hline\end{array} - \begin{array}{c}\hline 4\;5\;6\\ \hline 1\;2\;3\\ \hline\end{array}.
\end{aligned}
$$

Visibly, all terms will cancel when we compute $((e_t)\theta_T)\,\psi$ and thus,

$$\left(\overline{\theta}_T : S^{(5,1)} \longrightarrow H^{\left(3^2\right)}\right) = 0.$$

In order to conclude that a Specht module S^λ does not appear as a composition factor of $H^{(m^n)}$, it is necessary to show that all possible $\mathbb{C}\mathfrak{S}_{mn}$-homomorphisms $\overline{\theta}_T : S^\lambda \to H^{(m^n)}$ are zero maps; this means studying the homomorphisms arising from all possible semistandard λ-tableaux of type (m^n). The reader may have noticed that there is only one semistandard $(5,1)$-tableau of type $\left(3^2\right)$, namely the tableau T which was chosen in Example 4.1.3, and so we have in fact examined all $\mathbb{C}\mathfrak{S}_{mn}$-homomorphisms $\overline{\theta}_T : S^{(5,1)} \to H^{\left(3^2\right)}$. Thus, the calculation shows that $S^{(5,1)}$ does not appear as a composition factor of $H^{\left(3^2\right)}$.

The inability to easily determine whether the composition of θ_T and ψ will yield a non-zero map is perhaps indicative of the reasons why, for arbitrary m and n, there are surprisingly few results about the structure of Foulkes modules that have been proved using semistandard homomorphisms. In §4.3, we will present two existing results which can be proved using these techniques.

4.1.2 Twisted Foulkes modules

The setting for twisted Foulkes modules is very similar. Let μ be a partition of n. Recall from Fulton [17, §7.4] that $\widetilde{M}^\mu$ is the vector space spanned by all *oriented μ-column tabloids* $|t|$, corresponding to μ-tableaux t. This means that if $\sigma \in C_t$, then $|t\sigma| = \operatorname{sgn}(\sigma)|t|$.

Take $\mu = (1^n)$. The twisted Foulkes module $K^{(m^n)}$ is the vector space spanned by all oriented column (1^n)-tabloids

$$|X| = \begin{vmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{vmatrix}$$

such that the entries of the corresponding (1^n)-tableaux X are disjoint sets X_i, each of size m, and $\bigcup_{i=1}^n X_i = \{1, 2, \ldots, mn\}$. The symmetric group $\mathfrak{S}_{mn}$ acts in the obvious way, permuting $1, 2, \ldots, mn$. If a permutation in $\mathfrak{S}_{mn}$ has the effect of swapping exactly two of the sets X_i within the oriented column tabloid, then the resulting element of $K^{(m^n)}$ has the opposite orientation and so differs from the original one only by a sign.

Analogous to $\psi : M^{(m^n)} \to H^{(m^n)}$, there is a well-defined surjection $\phi : M^{(m^n)} \to K^{(m^n)}$, which is defined on (m^n)-tabloids by

$$\phi : \begin{array}{cccc} \hline x_1 & x_2 & \dots & x_m \\ \hline x_{m+1} & x_{m+2} & \dots & x_{2m} \\ \hline \vdots & \vdots & & \vdots \\ \hline x_{(n-1)m+1} & x_{(n-1)m+2} & \dots & x_{nm} \\ \hline \end{array} \longmapsto \left| \begin{array}{c} \{x_1, x_2, \dots, x_m\} \\ \{x_{m+1}, x_{m+2}, \dots, x_{2m}\} \\ \vdots \\ \{x_{(n-1)m+1}, x_{(n-1)m+2}, \dots, x_{nm}\} \end{array} \right| .$$

Just as we saw in the last section, we may use the alternative description of $M^{(m^n)}$ and thus work entirely with λ-tableaux of type (m^n). We will denote the image of such a tableau $T \in M^{(m^n)}$ under ϕ by $\underline{T}$. If t is the fixed labelling tableau, then $\underline{T} = |X|$, where $|X|$ is the oriented column tabloid whose entries are the sets $X_i = \{x \mid (x)T = i\}$.

EXAMPLE 4.1.4

If t and T are as in Example 4.1.3, then $(1)T = (2)T = (3)T = 1$ and $(4)T = (5)T = (6)T = 2$ and so

$$\phi : T \mapsto \left| \begin{array}{c} \{1, 2, 3\} \\ \{4, 5, 6\} \end{array} \right| .$$

Working with this description of $M^{(m^n)}$, we need to pay attention to more than just the pattern of the entries in the tableaux; given $T_1, T_2 \in \mathcal{T}\big(\lambda, (m^n)\big)$ such that $T_1 * \pi = T_2$ for $\pi \in \mathfrak{S}_n$, we must also record the sign of the permutation π. Indeed, swapping two rows in the (m^n)-tabloid yields, under ϕ, two elements of $K^{(m^n)}$ which differ by a sign. For example,

$$\left(\begin{array}{ccc} \hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline \end{array} \right) \phi = \left| \begin{array}{c} \{1, 2, 3\} \\ \{4, 5, 6\} \end{array} \right| = - \left| \begin{array}{c} \{4, 5, 6\} \\ \{1, 2, 3\} \end{array} \right| = - \left(\begin{array}{ccc} \hline 4 & 5 & 6 \\ \hline 1 & 2 & 3 \\ \hline \end{array} \right) \phi.$$

In general this means that if $T_1 * \pi = T_2$, then $\operatorname{sgn}(\pi)\underline{T_1} = \underline{T_2}$.

Over $\mathbb{C}$, we have $K^{(m^n)}$ appearing as a direct summand of $M^{(m^n)}$. To see this, observe that there is a $\mathbb{C}\mathfrak{S}_{mn}$-homomorphism $\phi' : K^{(m^n)} \hookrightarrow M^{(m^n)}$ defined on basis elements by

$$\phi' : \left| \begin{array}{c} \{x_1, x_2, \dots, x_m\} \\ \{x_{m+1}, \dots, x_{2m}\} \\ \vdots \\ \{x_{(n-1)m+1}, \dots, x_{nm}\} \end{array} \right| \longmapsto \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) \left(\begin{array}{cccc} \hline x_1 & x_2 & \dots & x_m \\ \hline x_{m+1} & x_{m+2} & \dots & x_{2m} \\ \hline \vdots & \vdots & & \vdots \\ \hline x_{(n-1)m+1} & x_{(n-1)m+2} & \dots & x_{nm} \\ \hline \end{array} * \sigma \right),$$

where the effect of σ is to permute the rows of the (m^n)-tabloid. Just as we saw in the Foulkes module setting, the composition $\phi' \circ \phi$ is the identity map on $K^{(m^n)}$. Moreover, all $\mathbb{C}\mathfrak{S}_{mn}$-homomorphisms $\underline{\theta}_T : S^\lambda \to K^{(m^n)}$ (which span $\operatorname{Hom}_{\mathbb{C}\mathfrak{S}_{mn}}\big(S^\lambda, K^{(m^n)}\big)$) arise as the composition of a semistandard homomorphism $\theta_T \in \operatorname{Hom}_{\mathbb{C}\mathfrak{S}_{mn}}\big(S^\lambda, M^{(m^n)}\big)$ with the surjection $\phi : M^{(m^n)} \twoheadrightarrow K^{(m^n)}$.

Example 4.1.5

Reconsider Example 4.1.3, this time composing θ_T with ϕ. We see that

$$(e_t)\underline{\theta}_T = 2\left(\begin{vmatrix}\{1,2,3\}\\\{4,5,6\}\end{vmatrix} + \begin{vmatrix}\{1,2,4\}\\\{3,5,6\}\end{vmatrix} + \begin{vmatrix}\{1,3,4\}\\\{2,5,6\}\end{vmatrix} + \begin{vmatrix}\{1,2,5\}\\\{3,4,6\}\end{vmatrix} + \begin{vmatrix}\{1,3,5\}\\\{2,4,6\}\end{vmatrix} + \begin{vmatrix}\{1,4,5\}\\\{2,3,6\}\end{vmatrix}\right)$$

and thus we can conclude that $S^{(5,1)}$ is a summand of $K^{(3^2)}$.

4.2 Notation and definitions

In this section, we introduce some definitions and notation that we will use throughout the rest of the chapter.

- If $\lambda \vdash n$ and t is any λ-tableau, then define $t_i^{(j)}$ to be the entry of t in the i^{th} row and j^{th} column. Let ℓ_j be the number of entries in column j of t.
- Similarly, denote the j^{th} column of $T \in \mathcal{T}(\lambda,\mu)$ by $T^{(j)}$ and let $T_i^{(j)}$ be the entry in the i^{th} row of the j^{th} column of T.
- If $\lambda \vdash n$ and t is any λ-tableau, then define

  $$C_t^{(j)} := \mathfrak{S}_{\left\{t_1^{(j)},t_2^{(j)},\ldots,t_{\ell_j}^{(j)}\right\}}.$$

  With this definition, the column stabiliser C_t of t is

  $$C_t = C_t^{(1)} \times C_t^{(2)} \times \cdots \times C_t^{(\lambda_1)},$$

  where λ_1 is the first part of λ.

It will be helpful later to be able to 'split' tableaux into component parts. For this reason, we introduce notation which will allow us to express a tableau T as the join of two (or more) components. We define $T = T_1 \vee T_2$ to be the tableau obtained by concatenating the rows of T_1 and T_2. In particular, when reading from left to right, row i of T consists of the entries of row i of T_1, followed by the entries of row i of T_2. For example, if

$$T_1 = \begin{array}{|c|c|c|}\hline 1&1&1\\\hline 2&3\\\cline{1-2} 4&4\\\cline{1-2}\end{array} \quad \text{and} \quad T_2 = \begin{array}{|c|c|c|c|}\hline 2&2&3&3\\\hline 4\\\cline{1-1}\end{array}$$

then

$$T = T_1 \vee T_2 = \begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&1&2&2&3&3\\\hline 2&3&4\\\cline{1-3} 4&4\\\cline{1-2}\end{array}.$$

4.3 The results

Studying Foulkes modules using semistandard homomorphisms is not unprecedented. In [10], Dent used semistandard homomorphisms to great effect, giving a new proof of a version of Foulkes' Second Conjecture (see Theorem 4.3.1). She also established a relationship between irreducible constituents of Foulkes modules by 'adding two columns' to a labelling partition (Theorem 4.3.6 (i)). In this section, we give statements of these results and prove their analogues for twisted Foulkes modules. In §4.4, we will conjecture versions of the results for generalised Foulkes modules.

4.3.1 Foulkes' Second Conjecture

Dent's version of Foulkes' Second Conjecture, proved in [10, Theorem 3.10], concerns only the partitions λ such that $\lambda_2 \leq m$; as such, it is a special case of Theorem 3.4.7. Imposing this restriction on the choice of λ actually leads to a stronger statement than the original result, since explicit multiplicities of certain constituents in the decomposition of $H^{((m+1)^n)}$ can be given.

Theorem 4.3.1 [Dent]
Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ be a partition of mn which satisfies $\lambda_2 \leq m$. Suppose that S^λ appears in $H^{(m^n)}$ with multiplicity $r \geq 0$. If $\widetilde{\lambda} := (\lambda_1 + n, \lambda_2, \ldots, \lambda_\ell)$, then $S^{\widetilde{\lambda}}$ appears in $H^{((m+1)^n)}$ with multiplicity equal to r.

It is in fact also possible to prove the original result using semistandard homomorphisms, thus giving a new proof of Theorem 3.4.6 that is similar to Dent's proof of Theorem 4.3.1. We do this now to illustrate the techniques in preparation for later results, where the proofs are more complicated.

We know that the set $\{\theta_T \mid T \in \mathcal{T}_0(\lambda, (m^n))\}$ is a basis for $\mathrm{Hom}_{\mathbb{C}\mathfrak{S}_{mn}}(S^\lambda, M^{(m^n)})$. The following lemma shows that a suitable subset of this basis gives rise to the basis elements for $\mathrm{Hom}_{\mathbb{C}\mathfrak{S}_{mn}}(S^\lambda, H^{(m^n)})$.

Lemma 4.3.2
If S^λ appears in $H^{(m^n)}$ with multiplicity $r \geq 0$, then there exist $T_1, \ldots, T_r \in \mathcal{T}_0(\lambda, (m^n))$ such that $\{\overline{\theta}_{T_1}, \ldots, \overline{\theta}_{T_r} \mid \overline{\theta}_{T_i} : S^\lambda \to H^{(m^n)}\ \forall\, 1 \leq i \leq r\}$ is a basis for $\mathrm{Hom}_{\mathbb{C}\mathfrak{S}_{mn}}(S^\lambda, H^{(m^n)})$.

Proof. By Theorem 2.2.11, $\{\theta_T \mid T \in \mathcal{T}_0(\lambda, (m^n))\}$ is a basis for $\mathrm{Hom}_{\mathbb{C}\mathfrak{S}_{mn}}(S^\lambda, M^{(m^n)})$. Therefore, $\{\overline{\theta}_T \mid T \in \mathcal{T}_0(\lambda, (m^n))\}$ spans $\mathrm{Hom}_{\mathbb{C}\mathfrak{S}_{mn}}(S^\lambda, H^{(m^n)})$ and pruning this spanning set yields a basis for $\mathrm{Hom}_{\mathbb{C}\mathfrak{S}_{mn}}(S^\lambda, H^{(m^n)})$. Since we assumed that $\dim \mathrm{Hom}_{\mathbb{C}\mathfrak{S}_{mn}}(S^\lambda, H^{(m^n)}) = r$, the basis is $\{\overline{\theta}_{T_1}, \ldots, \overline{\theta}_{T_r}\}$ for some $T_1, \ldots, T_r \in \mathcal{T}_0(\lambda, (m^n))$. ∎

Let us now set up some notation, which we will need to complete the proof of Theorem 3.4.6. Let T be a semistandard λ-tableau of type (m^n). Choose t to be the usual λ-tableau that has entries $1, 2, \ldots, mn$ in increasing order along rows. Define $\widetilde{t}$ to be the $\widetilde{\lambda}$-tableau

$$\widetilde{t} := t \vee \boxed{mn+1}\boxed{mn+2}\boxed{\ldots}\boxed{mn+n}$$

and consider the $\widetilde{\lambda}$-tableau $\widetilde{T} = T \vee N$, where $N := \boxed{1}\boxed{2}\boxed{\ldots}\boxed{n}$. It is important to notice that $\widetilde{T}$ is not necessarily semistandard. However, since T is semistandard, $\widetilde{T}$ certainly has distinct entries in each of its columns.

Lemma 4.3.3

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ be a partition of mn. If $\left(\overline{\theta}_T : S^\lambda \to H^{(m^n)}\right) \neq 0$ for some tableau $T \in \mathcal{T}_0\big(\lambda, (m^n)\big)$, then $\left(\overline{\theta}_{\widetilde{T}} : S^{\widetilde{\lambda}} \to H^{((m+1)^n)}\right) \neq 0$.

Proof. Assume that $\left(\overline{\theta}_T : S^\lambda \to H^{(m^n)}\right) \neq 0$. Since S^λ is a cyclic module with generator e_t, it follows that $(e_t)\overline{\theta}_T \neq 0$. Pick any basis element $\overline{\mathbf{R}}$ appearing in $(e_t)\overline{\theta}_T$ with non-zero coefficient. We know that there must be at least one such $\overline{\mathbf{R}}$, otherwise we contradict the fact that $(e_t)\overline{\theta}_T \neq 0$. Since ψ is surjective, there exists $\mathbf{R} \in \mathcal{T}\big(\lambda, (m^n)\big)$ such that $\psi(\mathbf{R}) = \overline{\mathbf{R}}$.

Using the definition of e_t, we may write

$$(e_t)\overline{\theta}_T = \sum_{T' \sim_{\text{row}} T} \overline{T'\kappa_t} = \sum_{\substack{T' \sim_{\text{row}} T,\\ \pi \in C_t}} \operatorname{sgn}(\pi)\overline{T'\pi}.$$

If we isolate $\overline{\mathbf{R}}$ in the sum, then

$$(e_t)\overline{\theta}_T = \sum_{\substack{T' \sim_{\text{row}} T,\\ \pi \in C_t:\\ \overline{T'\pi} = \overline{\mathbf{R}}}} \operatorname{sgn}(\pi)\overline{\mathbf{R}} + \sum_{\substack{T' \sim_{\text{row}} T,\\ \pi \in C_t:\\ \overline{T'\pi} \neq \overline{\mathbf{R}}}} \operatorname{sgn}(\pi)\overline{T'\pi}$$

and the coefficient of $\overline{\mathbf{R}}$ is

$$\mathscr{C} = \sum_{\substack{T' \sim_{\text{row}} T,\\ \pi \in C_t:\\ \overline{T'\pi} = \overline{\mathbf{R}}}} \operatorname{sgn}(\pi) \neq 0.$$

To prove the lemma, it will be sufficient to prove that the coefficient $\mathcal{C}$ of $\overline{\widetilde{\mathbf{R}}}$ in $(e_{\widetilde{t}})\overline{\theta}_{\widetilde{T}}$ is non-zero. We first determine an expression for $\mathcal{C}$ in the same way as we identified the coefficient of $\overline{\mathbf{R}}$ in $(e_t)\overline{\theta}_T$. We have that

$$(e_{\widetilde{t}})\overline{\theta}_{\widetilde{T}} = \sum_{\substack{T'' \sim_{\text{row}} \widetilde{T},\\ \rho \in C_{\widetilde{t}}:\\ \overline{T''\rho} = \overline{\widetilde{\mathbf{R}}}}} \operatorname{sgn}(\rho)\overline{\widetilde{\mathbf{R}}} + \sum_{\substack{T'' \sim_{\text{row}} \widetilde{T},\\ \rho \in C_{\widetilde{t}}:\\ \overline{T''\rho} \neq \overline{\widetilde{\mathbf{R}}}}} \operatorname{sgn}(\rho)\overline{T''\rho}$$

and hence

$$\mathcal{C} = \sum_{\substack{T'' \sim_{\text{row}} \widetilde{T}, \\ \rho \in C_{\widetilde{t}}: \\ \overline{T''\rho} = \overline{\widetilde{\mathbf{R}}}}} \operatorname{sgn}(\rho).$$

It is clear that $C_{\widetilde{t}} = C_t$ because elements of these column stabilisers only affect the first λ_2 columns of t and $\widetilde{t}$. Thus, we may as well sum over $\pi \in C_t$ and simplify the expression for $\mathcal{C}$ to

$$\mathcal{C} = \sum_{\substack{T'' \sim_{\text{row}} \widetilde{T}, \\ \pi \in C_t: \\ \overline{T''\pi} = \overline{\widetilde{\mathbf{R}}}}} \operatorname{sgn}(\pi).$$

Take $T'' \sim_{\text{row}} \widetilde{T}$, $\pi \in C_t$ such that $\overline{T''\pi} = \overline{\widetilde{\mathbf{R}}}$. Since π can only affect the first λ_2 columns of T'', $\overline{T''\pi} = \overline{\widetilde{\mathbf{R}}}$ implies that $T'' = T' \vee N\sigma$ for some $T' \sim_{\text{row}} T$ and $\sigma \in \mathfrak{S}_{\{mn+1, mn+2, \ldots, mn+n\}}$, permuting only the entries of N. So,

$$\mathcal{C} = \sum_{\substack{T' \sim_{\text{row}} T,\, \pi \in C_t, \\ \sigma \in \mathfrak{S}_{\{mn+1,\ldots,mn+n\}}: \\ \overline{(T' \vee N\sigma)\pi} = \overline{\widetilde{\mathbf{R}}}}} \operatorname{sgn}(\pi) \quad = \sum_{\substack{T' \sim_{\text{row}} T,\, \pi \in C_t, \\ \sigma \in \mathfrak{S}_{\{mn+1,\ldots,mn+n\}}: \\ \overline{T'\pi \vee N\sigma} = \overline{\widetilde{\mathbf{R}}}}} \operatorname{sgn}(\pi).$$

By definition of $\overline{\widetilde{\mathbf{R}}}$, $\overline{T'\pi \vee N\sigma} = \overline{\widetilde{\mathbf{R}}}$ if and only if $\overline{T'\pi \vee N\sigma} = \overline{\mathbf{R} \vee N}$. Comparing the component parts, it follows that $\overline{T'\pi} = \overline{\mathbf{R}}$ and $\overline{N\sigma} = \overline{N}$. Furthermore, any relabelling of the component parts must be compatible, which means that there is no choice for σ: σ must permute the entries of N in exactly the same way as the entries of $\mathbf{R}$ are relabelled to obtain $T'\pi$. Thus, we conclude that

$$\mathcal{C} = \sum_{\substack{T' \sim_{\text{row}} T, \\ \pi \in C_t: \\ \overline{T'\pi} = \overline{\mathbf{R}}}} \operatorname{sgn}(\pi) = \mathscr{C}. \qquad \blacksquare$$

Lemma 4.3.3 is sufficient to prove the existence of $S^{\widetilde{\lambda}}$ as a composition factor in $H^{((m+1)^n)}$. To complete the proof of Theorem 3.4.6, it just remains to prove that the multiplicity with which $S^{\widetilde{\lambda}}$ appears is bounded below by the multiplicity of S^λ in $H^{(m^n)}$. We first need one more piece of information.

Let $B \subseteq \{\overline{\mathbf{R}} \mid \mathbf{R} \in \mathcal{T}(\lambda, (m^n))\}$ be a basis for $H^{(m^n)}$. Let $\mathscr{B} := \left\{\overline{\widetilde{\mathbf{R}}} \;\middle|\; \overline{\mathbf{R}} \in B\right\}$ and let J be an indexing set for the elements of $\mathscr{B}$. As a consequence of the construction of $\widetilde{\mathbf{R}} \in B$ and the definition of the labelling tableau $\widetilde{t}$, if $\overline{\mathbf{R}}$ is the set partition $\{X_1, \ldots, X_n\}$, we may take $\mathbf{R}$ to be the λ-tableau of type (m^n) that has entries $(x)\mathbf{R} = i$ if $x \in X_i$ (for $1 \leq x \leq mn$) and then $\overline{\widetilde{\mathbf{R}}} \in \mathscr{B}$ is the set partition $\{X_1 \cup \{mn+1\}, \ldots, X_n \cup \{mn+n\}\}$. Since B is a basis, and therefore all of its elements are distinct, the elements of $\mathscr{B}$ must also be distinct. It follows that the formal sum $\sum_{j \in J} \beta_j \overline{\widetilde{\mathbf{R}}}_j$ is equal to zero only if $\beta_j = 0$ for all $j \in J$. In other words, $\mathscr{B}$ is a linearly independent set, which we may extend to a basis for $H^{((m+1)^n)}$.

Proof of Theorem 3.4.6. Suppose that S^λ appears in $H^{(m^n)}$ with multiplicity $r \geq 0$. By Lemma 4.3.2, we have a basis $\{\overline{\theta}_{T_1}, \ldots, \overline{\theta}_{T_r}\}$ for $\mathrm{Hom}_{\mathbb{C}\mathfrak{S}_{mn}}\left(S^\lambda, H^{(m^n)}\right)$, whose elements are labelled by $T_1, \ldots, T_r \in \mathcal{T}_0\big(\lambda, (m^n)\big)$.

For a contradiction, suppose that $\sum_{i=1}^r \alpha_i \overline{\theta}_{\widetilde{T}_i} = 0$ for some scalars α_i, which are not all zero. It follows that $(e_{\widetilde{t}})\left(\sum_{i=1}^r \alpha_i \overline{\theta}_{\widetilde{T}_i}\right) = 0$ and so the coefficient of any basis element $\overline{\widetilde{\mathbf{R}}}$ in $(e_{\widetilde{t}})\left(\sum_{i=1}^r \alpha_i \overline{\theta}_{\widetilde{T}_i}\right)$ is zero. If we let $\mathcal{C}_i$ denote the coefficient of $\overline{\widetilde{\mathbf{R}}}$ in $(e_{\widetilde{t}})\overline{\theta}_{\widetilde{T}_i}$, then

$$\sum_{i=1}^r \alpha_i \mathcal{C}_i = 0.$$

Applying Lemma 4.3.3, each $\mathcal{C}_i$ is equal to the coefficient of $\overline{\mathbf{R}}$ in $(e_t)\overline{\theta}_{T_i}$. Thus, the coefficient of $\overline{\mathbf{R}}$ in $(e_t)\left(\sum_{i=1}^r \alpha_i \overline{\theta}_{T_i}\right)$ is $\sum_{i=1}^r \alpha_i \mathcal{C}_i$ and so is also zero. We chose $\overline{\widetilde{\mathbf{R}}}$ arbitrarily and so we have that for all $\mathbf{R} \in \mathcal{T}\big(\lambda, (m^n)\big)$, $\overline{\mathbf{R}}$ has coefficient zero in $(e_t)\left(\sum_{i=1}^r \alpha_i \overline{\theta}_{T_i}\right)$. It follows that

$$(e_t)\left(\sum_{i=1}^r \alpha_i \overline{\theta}_{T_i}\right) = 0.$$

Since e_t is a generator for S^λ, this implies that $\sum_{i=1}^r \alpha_i \overline{\theta}_{T_i} = 0$, but $\{\overline{\theta}_{T_1}, \ldots, \overline{\theta}_{T_r}\}$ is a linearly independent set and so $\alpha_i = 0$ for all $1 \leq i \leq r$, which contradicts the assumption on the scalars α_i. ∎

We conclude this section by indicating how the proof of Theorem 3.4.6 may be modified to obtain a proof of the following theorem, which is the obvious analogue of Foulkes' Second Conjecture for twisted Foulkes modules. In this new result, no conditions are imposed on the partition λ and so it gives information about a larger family of constituents of $K^{((m+1)^n)}$ than those covered by Weintraub's result (Theorem 3.4.7). However, we are not able to be as precise about the multiplicities with which the constituents appear in the decomposition.

Theorem 4.3.4

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ be a partition of mn and suppose that S^λ appears in $K^{(m^n)}$ with multiplicity $r \geq 0$. If $\widetilde{\lambda} = (\lambda_1 + n, \lambda_2, \ldots, \lambda_\ell)$, then $S^{\widetilde{\lambda}}$ appears with multiplicity $\geq r$ in $K^{((m+1)^n)}$.

The following lemma, which is analogous to Lemma 4.3.2, establishes a basis for the space $\mathrm{Hom}_{\mathbb{C}\mathfrak{S}_{mn}}\left(S^\lambda, K^{(m^n)}\right)$. It is proved in the obvious analogous way.

Lemma 4.3.5

If S^λ appears in $K^{(m^n)}$ with multiplicity $r \geq 0$, then there exists $T_1, \ldots, T_r \in \mathcal{T}_0\big(\lambda, (m^n)\big)$ such that $\left\{\underline{\theta}_{T_1}, \ldots, \underline{\theta}_{T_r} \;\middle|\; \underline{\theta}_{T_i} : S^\lambda \to K^{(m^n)}\ \forall\, 1 \leq i \leq r\right\}$ is a basis for $\mathrm{Hom}_{\mathbb{C}\mathfrak{S}_{mn}}\left(S^\lambda, K^{(m^n)}\right)$.

In establishing the existence of $S^{\widetilde{\lambda}}$ as a composition factor of $K^{((m+1)^n)}$, a little more care must be taken. Indeed, whilst the arguments are mostly identical, the expression for the (non-zero) coefficient of $\underline{\mathbf{R}}$ in $(e_t)\underline{\theta}_T$ is

$$\sum_{\substack{T'\sim_{\mathrm{row}}T,\\ \pi\in C_t,\,\sigma\in\mathfrak{S}_n:\\ T'\pi=\mathbf{R}*\sigma}} \operatorname{sgn}(\pi)\operatorname{sgn}(\sigma)$$

and the initial expression for the coefficient of $\underline{\widetilde{\mathbf{R}}}$ in $(e_{\widetilde{t}})\underline{\theta}_{\widetilde{T}}$ is

$$\mathcal{C} := \sum_{\substack{T''\sim_{\mathrm{row}}\widetilde{T},\\ \rho\in C_{\widetilde{t}},\,\tau\in\mathfrak{S}_n:\\ T''\rho=\widetilde{\mathbf{R}}*\tau}} \operatorname{sgn}(\rho)\operatorname{sgn}(\tau).$$

The proof proceeds in the same manner as in the Foulkes setting, with the permutation $\tau\in\mathfrak{S}_n$ continuing to appear in the argument. When it is necessary to compare component parts, it is concluded (in the twisted Foulkes setting) that $T'\pi = R*\tau$ and $N\sigma = N*\tau$, and thus that σ is uniquely determined by τ, since σ must permute the entries of N in exactly the same way as τ relabels the entries of $\mathbf{R}$. The proof is completed by deducing that

$$\mathcal{C} = \sum_{\substack{T'\sim_{\mathrm{row}}T,\\ \pi\in C_t,\,\tau\in\mathfrak{S}_n:\\ T'\pi=\mathbf{R}*\tau}} \operatorname{sgn}(\pi)\operatorname{sgn}(\tau),$$

which is precisely equal to the non-zero coefficient of $\mathbf{R}$ in $(e_t)\underline{\theta}_T$.

The conclusion of the proof of Theorem 4.3.4, establishing a lower bound on the multiplicity with which $S^{\widetilde{\lambda}}$ appears as a composition factor in $K^{((m+1)^n)}$, is again analogous to that of Theorem 3.4.6.

4.3.2 Dent's two column result

Part (i) of the following result was proved – using semistandard homomorphisms – by Dent in [10, Theorem 3.8]. We will refer to this result as "Dent's two column result", since it establishes a relationship between irreducible constituents of Foulkes modules via the addition of two columns of length n to a labelling partition. Part (ii) of Theorem 4.3.6 may be viewed as the analogue of part (i) applicable to twisted Foulkes modules.

Theorem 4.3.6

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \vdash mn$ and define $\widehat{\lambda} := \left(\lambda_1+2, \ldots, \lambda_\ell+2, 2^{n-\ell}\right) \vdash (m+2)n$.

- (i) *(Dent) If S^λ appears in $H^{(m^n)}$ with multiplicity $r \geq 0$, then $S^{\widehat{\lambda}}$ appears in $H^{((m+2)^n)}$ with multiplicity r.*

(*ii*) *If* S^λ *appears in* $K^{(m^n)}$ *with multiplicity* $r \geq 0$*, then* $S^{\widehat{\lambda}}$ *appears in* $K^{((m+2)^n)}$ *with multiplicity* r*.*

The author cannot find any record of part (*ii*) of Theorem 4.3.6 in the literature. However, as we will see shortly, it is a straightforward corollary of a known result due to Newell [38]. In this section, we will use semistandard homomorphisms to give a new proof of Newell's result – the original proof was in terms of plethysms – thus also proving part (*ii*) of Theorem 4.3.6. Our methods are indicative of those that are required to prove Theorem 4.3.6 directly.

Theorem 4.3.7 [Newell]
Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \vdash mn$ *and define* $\widetilde{\lambda} := \left(\lambda_1 + 1, \ldots, \lambda_\ell + 1, 1^{n-\ell}\right) \vdash (m+1)n$.

(*i*) *If* S^λ *appears in* $H^{(m^n)}$ *with multiplicity* $r \geq 0$*, then* $S^{\widetilde{\lambda}}$ *appears in* $K^{((m+1)^n)}$ *with multiplicity* r*.*

(*ii*) *If* S^λ *appears in* $K^{(m^n)}$ *with multiplicity* $r \geq 0$*, then* $S^{\widetilde{\lambda}}$ *appears in* $H^{((m+1)^n)}$ *with multiplicity* r*.*

We will use notation consistent with the proof of Foulkes' Second Conjecture, redefined in the following way, so as to make it applicable to the current setting. Suppose that T is a λ-tableau of type (m^n) and define $\widetilde{T}$, a $\widetilde{\lambda}$-tableau of type $((m+1)^n)$, in the following way:

$$\widetilde{T}_i^{(j)} := \begin{cases} T_i^{(j-1)} & \text{if } j > 1; \\ i & \text{if } j = 1 \text{ and } i \in \{1, 2, \ldots, n\}. \end{cases}$$

If T is chosen to be semistandard, then the construction of $\widetilde{T}$ ensures that $\widetilde{T}$ is also semistandard. Given t, the λ-tableau that has entries $1, 2, \ldots, mn$ in increasing order along rows, define $\widetilde{t}$ to be the $\widetilde{\lambda}$-tableau

$$\widetilde{t}_i^{(j)} := \begin{cases} t_i^{(j-1)} & \text{if } j > 1; \\ mn + i & \text{if } j = 1 \text{ and } i \in \{1, 2, \ldots, n\}. \end{cases}$$

We should note that this choice of $\widetilde{t}$ is not standard. For example, if $m = 2$, $n = 3$ and we choose

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 2 \\ \hline 3 & 3 \\ \cline{1-2} \end{array} \quad \text{and} \quad t = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 \\ \cline{1-2} \end{array},$$

then

$$\widetilde{T} = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 2 \\ \hline 2 & 3 & 3 \\ \cline{1-3} 3 \\ \cline{1-1} \end{array} \quad \text{and} \quad \widetilde{t} = \begin{array}{|c|c|c|c|c|} \hline 7 & 1 & 2 & 3 & 4 \\ \hline 8 & 5 & 6 \\ \cline{1-3} 9 \\ \cline{1-1} \end{array}.$$

The first step towards proving part (*i*) of Theorem 4.3.7 is the following lemma, which establishes the existence of $S^{\widetilde{\lambda}}$ as a summand of $K^{((m+1)^n)}$.

LEMMA 4.3.8

Under the assumptions of part (i) of Theorem 4.3.7, if $\left(\overline{\theta}_T : S^\lambda \to H^{(m^n)}\right) \neq 0$ *for some semistandard* λ*-tableau* T *of type* (m^n)*, then* $\left(\underline{\theta}_{\widetilde{T}} : S^{\widetilde{\lambda}} \to K^{((m+1)^n)}\right) \neq 0$*. Moreover, the coefficient of any basis element* $\overline{\mathbf{R}}$ *in* $(e_t)\overline{\theta}_T$ *is equal to the coefficient of* $\underline{\widetilde{\mathbf{R}}}$ *in* $(e_{\widetilde{t}})\underline{\theta}_{\widetilde{T}}$*.*

Proof. Consider the image of the generator $e_{\widetilde{t}}$ of $S^{\widetilde{\lambda}}$ under $\underline{\theta}_{\widetilde{T}}$:

$$(e_{\widetilde{t}})\underline{\theta}_{\widetilde{T}} = \sum_{T'' \sim_{\mathrm{row}} \widetilde{T}} \underline{T'' \kappa_{\widetilde{t}}}. \tag{4.1}$$

In Proposition 2.2.10, it was seen that if a tableau does not have distinct entries in its columns, then multiplication by the signed column sum yields zero. Thus, when examining the sum in (4.1), we can save ourselves some unnecessary work if we restrict our attention to those T'' satisfying $T''\kappa_{\widetilde{t}} \neq 0$.

We claim that if T'' is such that $T''\kappa_{\widetilde{t}} \neq 0$, then $T'' \sim_{\mathrm{row}} \widetilde{T}$ if and only if $T' \sim_{\mathrm{row}} T$, where $T'' = \widetilde{T'}$ and $T'\kappa_t \neq 0$. Indeed, $T' \sim_{\mathrm{row}} T$ implies that $\widetilde{T'} \sim_{\mathrm{row}} \widetilde{T}$ by construction. Further, since $T'\kappa_t \neq 0$, we know that $\widetilde{T'}\kappa_{\widetilde{t}} \neq 0$ and so it follows from $T'' = \widetilde{T'}$ that $T'' \sim_{\mathrm{row}} \widetilde{T}$ and $T''\kappa_{\widetilde{t}} \neq 0$.

For the converse, assume that $T'' \sim_{\mathrm{row}} \widetilde{T}$. We need to identify those T'' which have distinct entries in columns, so that $T''\kappa_{\widetilde{t}} \neq 0$ is satisfied. Consider row i of T'' for some $i > \ell$ where, recall, ℓ is the number of parts of λ. The only entry in this row is i (occurring precisely once) and thus, row i of T'' is the same as row i of $\widetilde{T}$. Now consider row ℓ of T'': entries in this row are all greater than or equal to ℓ because $T'' \sim_{\mathrm{row}} \widetilde{T}$ and $\widetilde{T}$ is semistandard. Hence, there are two cases to consider: either ${T''}^{(1)}_\ell = \widetilde{T}^{(1)}_\ell = \ell$; or, in the rearranging of the rows of $\widetilde{T}$ to obtain T'', some digit $\alpha := \widetilde{T}^{(j)}_\ell > \ell$ $(j \geq 2)$ has been permuted with $\widetilde{T}^{(1)}_\ell$. In the second case, T'' contains a repeated digit in column 1, namely α, and thus $T''\kappa_{\widetilde{t}} = 0$. Hence, we deduce that we must have ${T''}^{(1)}_\ell = \ell$. Working inductively up the rows using similar arguments, we can conclude that we must have ${T''}^{(1)}_i = i$ for all $1 \leq i \leq \ell - 1$ and thus that ${T''}^{(1)} = \widetilde{T}^{(1)}$. It follows that $T'' \sim_{\mathrm{row}} \widetilde{T}$ with $T''\kappa_{\widetilde{t}} \neq 0$ implies that $T' \sim_{\mathrm{row}} T$, where $T'' = \widetilde{T'}$ and $T'\kappa_t = 0$, as required.

As a consequence of the claim, we can rewrite the sum in (4.1) as

$$(e_{\widetilde{t}})\underline{\theta}_{\widetilde{T}} = \sum_{T' \sim_{\mathrm{row}} T} \underline{\widetilde{T'}\, \kappa_{\widetilde{t}}}. \tag{4.2}$$

Now, assume that $\left(\overline{\theta}_T : S^\lambda \to H^{(m^n)}\right) \neq 0$. We know that $(e_t)\overline{\theta}_T \neq 0$ and therefore there exists at least one $\overline{\mathbf{R}}$ appearing with non-zero coefficient $\mathscr{C}$ in $(e_t)\overline{\theta}_T$. Pick any such $\overline{\mathbf{R}}$. Since ψ is surjective, there exists $\mathbf{R} \in \mathcal{T}\big(\lambda, (m^n)\big)$ such that $\psi : \mathbf{R} \mapsto \overline{\mathbf{R}}$. We look to determine the coefficient $\mathcal{C}$ of $\underline{\widetilde{\mathbf{R}}}$ in $(e_{\widetilde{t}})\underline{\theta}_{\widetilde{T}}$.

We first need to obtain expressions for $\mathscr{C}$ and $\mathcal{C}$, which is how we now proceed. We may write

$$(e_t)\overline{\theta}_T = \sum_{T' \sim_{\text{row}} T} \overline{T'\kappa_t} = \sum_{\substack{T' \sim_{\text{row}} T,\\ \pi \in C_t}} \operatorname{sgn}(\pi)\overline{T'\pi}$$

and subsequently, isolating $\overline{\mathbf{R}}$ in the sum, we see that

$$(e_t)\overline{\theta}_T = \sum_{\substack{T' \sim_{\text{row}} T,\\ \pi \in C_t:\\ \overline{T'\pi} = \overline{\mathbf{R}}}} \operatorname{sgn}(\pi)\overline{\mathbf{R}} + \sum_{\substack{T' \sim_{\text{row}} T,\\ \pi \in C_t:\\ \overline{T'\pi} \neq \overline{\mathbf{R}}}} \operatorname{sgn}(\pi)\overline{T'\pi}$$

and

$$\mathscr{C} = \sum_{\substack{T' \sim_{\text{row}} T,\\ \pi \in C_t:\\ \overline{T'\pi} = \overline{\mathbf{R}}}} \operatorname{sgn}(\pi) \neq 0. \tag{4.3}$$

Similarly, we may isolate $\underline{\widetilde{\mathbf{R}}}$ in the expression for $(e_{\tilde{t}})\underline{\theta}_{\widetilde{T}}$. Doing so, we obtain

$$(e_{\tilde{t}})\underline{\theta}_{\widetilde{T}} = \sum_{\substack{T' \sim_{\text{row}} T,\\ \rho \in C_{\tilde{t}}}} \operatorname{sgn}(\rho)\underline{\widetilde{T'}\rho} = \sum_{\substack{T' \sim_{\text{row}} T,\\ \rho \in C_{\tilde{t}},\, \tau \in \mathfrak{S}_n:\\ \widetilde{T'}\rho = \widetilde{\mathbf{R}} * \tau}} \operatorname{sgn}(\rho)\operatorname{sgn}(\tau)\underline{\widetilde{\mathbf{R}}} + \sum_{\substack{T' \sim_{\text{row}} T,\\ \rho \in C_{\tilde{t}}:\\ \widetilde{T'}\rho \neq \widetilde{\mathbf{R}} * \tau\\ \forall\, \tau \in \mathfrak{S}_n}} \operatorname{sgn}(\rho)\operatorname{sgn}(\tau)\underline{\widetilde{T'}\rho}$$

and thus, we have the following expression for $\mathcal{C}$:

$$\mathcal{C} = \sum_{\substack{T' \sim_{\text{row}} T,\\ \rho \in C_{\tilde{t}},\, \tau \in \mathfrak{S}_n:\\ \widetilde{T'}\rho = \widetilde{\mathbf{R}} * \tau}} \operatorname{sgn}(\rho)\operatorname{sgn}(\tau). \tag{4.4}$$

Since we can express the column stabiliser as $C_{\tilde{t}} = C_t \times C_{\tilde{t}}^{(1)}$, we may express $\rho \in C_{\tilde{t}}$ as $\rho = \pi y$, where $\pi \in C_t$ and $y \in C_{\tilde{t}}^{(1)}$. Also, since $\pi \in C_t$, it fixes all entries in column 1 of T' and thus $\widetilde{T'}\pi = \widetilde{T'\pi}$. So,

$$\mathcal{C} = \sum_{\substack{T' \sim_{\text{row}} T,\\ \pi \in C_t,\, y \in C_{\tilde{t}}^{(1)},\\ \tau \in \mathfrak{S}_n:\\ \widetilde{T'\pi}y = \widetilde{\mathbf{R}} * \tau}} \operatorname{sgn}(\pi)\operatorname{sgn}(y)\operatorname{sgn}(\tau). \tag{4.5}$$

Take $T' \sim_{\text{row}} T$, $\pi \in C_t$, $y \in C_{\tilde{t}}^{(1)}$ and $\tau \in \mathfrak{S}_n$ such that $\widetilde{T'\pi}y = \widetilde{\mathbf{R}} * \tau$. Then $\widetilde{T'\pi}y$ is a relabelling of $\widetilde{\mathbf{R}}$ by τ. Since y permutes only column 1 of $\widetilde{T'\pi}$, it must be that $T'\pi$ is a relabelling of $\mathbf{R}$. Let $\sigma \in \mathfrak{S}_n$ be the permutation which relabels $\mathbf{R}$ to give $T'\pi$. The element $y \in C_{\tilde{t}}^{(1)}$ must permute column 1 of $\widetilde{T'\pi}$ in such a way that it is a relabelling of column 1 of $\mathbf{R}$ and also consistent with the relabelling of $\mathbf{R}$ by σ. So, it must be that $\tau = \sigma$ and that y is in fact completely determined by σ, i.e. if $Y_0(T'\pi) := \left\{ y \in C_{\tilde{t}}^{(1)} \;\middle|\; \widetilde{T'\pi}y = \widetilde{\mathbf{R}} * \tau \right\}$, then necessarily $\left|Y_0(T'\pi)\right| = 1$. Furthermore, $\operatorname{sgn}(y) = \operatorname{sgn}(\sigma)$ and so $\operatorname{sgn}(y)\operatorname{sgn}(\tau) = \operatorname{sgn}(\sigma)^2 = 1$. Hence, using (4.3), we deduce that

$$\mathcal{C} = \sum_{\substack{T' \sim_{\text{row}} T,\\ \pi \in C_t, \sigma \in \mathfrak{S}_n,\\ y \in Y_0(T'\pi):\\ T'\pi = \mathbf{R} * \sigma}} \operatorname{sgn}(\pi) = \sum_{\substack{T' \sim_{\text{row}} T,\\ \pi \in C_t, \sigma \in \mathfrak{S}_n:\\ T'\pi = \mathbf{R} * \sigma}} \left|Y_0(T'\pi)\right| \operatorname{sgn}(\pi) = \sum_{\substack{T' \sim_{\text{row}} T,\\ \pi \in C_t, \sigma \in \mathfrak{S}_n:\\ T'\pi = \mathbf{R} * \sigma}} \operatorname{sgn}(\pi) = \mathscr{C}$$

and so $\mathcal{C}$ is non-zero.

We can also make deductions in the case where $\overline{\mathbf{R}}$ arises with zero coefficient in $(e_t)\overline{\theta}_T$, i.e. if

$$\sum_{\substack{T' \sim_{\text{row}} T,\\ \pi \in C_t:\\ \overline{T'\pi} = \overline{\mathbf{R}}}} \operatorname{sgn}(\pi) = 0. \tag{4.6}$$

In this case, we know that $\underline{\widetilde{\mathbf{R}}}$ also arises with coefficient zero in $(e_{\widetilde{t}})\underline{\theta}_{\widetilde{T}}$. The argument is exactly the same as in the non-zero coefficient case, except that we conclude the argument using (4.6) instead of (4.3). ■

We now make some additional observations which will be useful when we complete the proof of Theorem 4.3.7.

We saw from the proof of Lemma 4.3.8 that all basis elements of $K^{((m+1)^n)}$ that appear with non-zero coefficient in $(e_{\widetilde{t}})\underline{\theta}_{\widetilde{T}}$ are of the form $\underline{\widetilde{T'}\rho}$ for some $T' \sim_{\text{row}} T$ and $\rho \in C_{\widetilde{t}}$. In other words, $(e_{\widetilde{t}})\underline{\theta}_{\widetilde{T}}$ features exactly those oriented column tabloids

$$\underline{\widetilde{\mathbf{R}}\omega} := \begin{vmatrix} X_1 \cup \{\widetilde{t}^{(1)}_{(1)\omega}\} \\ X_2 \cup \{\widetilde{t}^{(1)}_{(2)\omega}\} \\ \vdots \\ X_n \cup \{\widetilde{t}^{(1)}_{(n)\omega}\} \end{vmatrix}$$

such that $\overline{\mathbf{R}} = \{X_1, X_2, \ldots, X_n\}$ is a set partition appearing in $(e_t)\overline{\theta}_T$ with non-zero coefficient, and $\omega \in \mathfrak{S}_n$.

Additionally, the proof of Lemma 4.3.8 showed that if the coefficient of a basis element $\overline{\mathbf{R}}$ in $(e_t)\overline{\theta}_T$ is $\mathcal{C}$, then the oriented column tabloid $\underline{\widetilde{\mathbf{R}}}$ (with $\omega = \mathrm{id}_{\mathfrak{S}_n}$) appears in $(e_{\widetilde{t}})\underline{\theta}_{\widetilde{T}}$ with coefficient $\mathscr{C} = \mathcal{C}$.

We could, quite reasonably, have proved Lemma 4.3.8 by looking at the coefficient of $\underline{\widetilde{\mathbf{R}}\omega}$ (for $\omega \neq \mathrm{id}_{\mathfrak{S}_n}$) instead and we would have found that its coefficient in $(e_{\widetilde{t}})\underline{\theta}_{\widetilde{T}}$ is $\mathscr{C} = \operatorname{sgn}(\omega)\,\mathcal{C}$. Indeed, an expression for the coefficient of $\underline{\widetilde{\mathbf{R}}\omega}$ is

$$\sum_{\substack{T' \sim_{\text{row}} T,\\ \rho \in C_{\widetilde{t}}, \tau \in \mathfrak{S}_n:\\ \widetilde{T'}\rho = \widetilde{\mathbf{R}}\omega * \tau}} \operatorname{sgn}(\rho)\operatorname{sgn}(\tau) = \sum_{\substack{T' \sim_{\text{row}} T,\\ \pi \in C_t, y \in C^{(1)}_{\widetilde{t}},\\ \tau \in \mathfrak{S}_n:\\ \widetilde{T'\pi y} = \widetilde{\mathbf{R}}\omega * \tau}} \operatorname{sgn}(\pi)\operatorname{sgn}(y)\operatorname{sgn}(\tau). \tag{4.7}$$

We now reason in the same way as we did following (4.5): y is again completely determined by σ (but this time $y = w\sigma$) and, as before, $\tau = \sigma$. So

$$\operatorname{sgn}(y)\operatorname{sgn}(\tau) = \operatorname{sgn}(\omega)\operatorname{sgn}(\sigma)^2 = \operatorname{sgn}(\omega)$$

and the result follows. Consequently, we know that if $\left(\overline{\theta}_T : S^\lambda \to H^{(m^n)}\right) = 0$, then $\left(\underline{\theta}_{\widetilde{T}} : S^{\widetilde{\lambda}} \to K^{((m+1)^n)}\right) = 0$.

Now, let $B \subseteq \{\overline{\mathbf{R}} \mid \mathbf{R} \in \mathcal{T}(\lambda, (m^n))\}$ be a basis for $H^{(m^n)}$. For each $\overline{\mathbf{R}}$, there exists $\mathbf{R}$ such that $\psi : \mathbf{R} \mapsto \overline{\mathbf{R}}$, from which we may construct $\widetilde{\mathbf{R}}$. Using reasoning similar to that given after the proof of Lemma 4.3.3, the set $\mathscr{B} := \left\{ \underline{\widetilde{\mathbf{R}}} \,\middle|\, \overline{\mathbf{R}} \in B \right\}$ is a linearly independent subset of $K^{((m+1)^n)}$ because B is linearly independent. We will use this property of $\mathscr{B}$ to complete the proof of Theorem 4.3.7.

Proof of part (i) of Theorem 4.3.7. Since the set $\{\theta_T \mid T \in \mathcal{T}_0(\widetilde{\lambda}, ((m+1)^n))\}$ is a basis for $\operatorname{Hom}_{\mathbb{C}\mathfrak{S}_{(m+1)n}}\left(S^{\widetilde{\lambda}}, M^{((m+1)^n)}\right)$, it follows that $\{\underline{\theta}_T \mid T \in \mathcal{T}_0(\widetilde{\lambda}, ((m+1)^n))\}$ is a spanning set for $\operatorname{Hom}_{\mathbb{C}\mathfrak{S}_{(m+1)n}}\left(S^{\widetilde{\lambda}}, K^{((m+1)^n)}\right)$. However, the left-most column of a semistandard $\widetilde{\lambda}$-tableau of type $((m+1)^n)$ is completely determined. Indeed, the digits $1, 2, \ldots, n$ must appear down the column in increasing order. Therefore, every semistandard $\widetilde{\lambda}$-tableau of type $((m+1)^n)$ arises as $\widetilde{T}$, where T is a semistandard λ-tableau of type (m^n). So, in fact, the spanning set for $\operatorname{Hom}_{\mathbb{C}\mathfrak{S}_{(m+1)n}}\left(S^{\widetilde{\lambda}}, K^{((m+1)^n)}\right)$ is $\{\underline{\theta}_{\widetilde{T}} \mid T \in \mathcal{T}_0(\lambda, (m^n))\}$.

Prune the spanning set to get a basis, say $\{\underline{\theta}_{\widetilde{U}_1}, \ldots, \underline{\theta}_{\widetilde{U}_s}\}$. Showing that $\{\overline{\theta}_{U_1}, \ldots, \overline{\theta}_{U_s}\}$ is a linearly independent subset of $\operatorname{Hom}_{\mathbb{C}\mathfrak{S}_{mn}}\left(S^\lambda, H^{(m^n)}\right)$ will be sufficient to prove that $s \leq \dim \operatorname{Hom}_{\mathbb{C}\mathfrak{S}_{mn}}\left(S^\lambda, H^{(m^n)}\right) = r$.

If $\{\overline{\theta}_{U_1}, \ldots, \overline{\theta}_{U_s}\}$ is not linearly independent, then $\sum_{i=1}^s \gamma_i \overline{\theta}_{U_i} = 0$ for some scalars γ_i, which are not all zero. So, $(e_t)\left(\sum_{i=1}^s \gamma_i \overline{\theta}_{U_i}\right) = 0$ and the coefficient of any basis element $\overline{\mathbf{R}}$ in $(e_t)\left(\sum_{i=1}^s \gamma_i \overline{\theta}_{U_i}\right) = \sum_{i=1}^s \gamma_i (e_t)\overline{\theta}_{U_i}$ is zero, i.e.

$$\sum_{i=1}^s \gamma_i \mathscr{C}_i^{\overline{\mathbf{R}}} = 0,$$

where $\mathscr{C}_i^{\overline{\mathbf{R}}}$ is the coefficient of $\overline{\mathbf{R}}$ in $(e_t)\overline{\theta}_{U_i}$. Thus, also, for any $\omega \in \mathfrak{S}_n$,

$$\sum_{i=1}^s \gamma_i \operatorname{sgn}(\omega) \mathscr{C}_i^{\overline{\mathbf{R}}} = \operatorname{sgn}(\omega) \sum_{i=1}^s \gamma_i \mathscr{C}_i^{\overline{\mathbf{R}}} = 0. \tag{4.8}$$

By Lemma 4.3.8 and the ensuing observation, the coefficient $\mathcal{C}_i^{\underline{\widetilde{\mathbf{R}}\omega}}$ of $\underline{\widetilde{\mathbf{R}}\omega}$ (for any choice of ω) in $(e_{\widetilde{t}})\underline{\theta}_{\widetilde{U}_i}$ is $\operatorname{sgn}(\omega)\mathscr{C}_i^{\overline{\mathbf{R}}}$ and so (4.8) says that, for all $\omega \in \mathfrak{S}_n$, the coefficient of $\underline{\widetilde{\mathbf{R}}\omega}$ in $(e_{\widetilde{t}})\left(\sum_{i=1}^s \gamma_i \underline{\theta}_{\widetilde{U}_i}\right)$ is $\sum_{i=1}^s \gamma_i \mathcal{C}_i^{\underline{\widetilde{\mathbf{R}}\omega}} = 0$. Since $\overline{\mathbf{R}}$ was chosen arbitrarily, we can conclude that all set column tabloids $\underline{\widetilde{\mathbf{R}}\omega}$ have coefficient zero in $(e_{\widetilde{t}})\left(\sum_{i=1}^s \gamma_i \underline{\theta}_{\widetilde{U}_i}\right)$; hence, it follows that

$(e_{\widetilde{t}})\left(\sum_{i=1}^{s}\gamma_i\underline{\theta}_{\widetilde{U}_i}\right)=0$. Since $e_{\widetilde{t}}$ is a generator for $S^{\widetilde{\lambda}}$, this implies that $\sum_{i=1}^{s}\gamma_i\underline{\theta}_{\widetilde{U}_i}=0$ and thus $\gamma_i=0$ for all i, which contradicts the assumptions on $\{\gamma_i \mid 1\le i\le s\}$.

By Lemma 4.3.2, we have a basis for $\mathrm{Hom}_{\mathbb{C}\mathfrak{S}_{mn}}\left(S^{\lambda},H^{(m^n)}\right)$ given by $\left\{\overline{\theta}_{T_1},\dots,\overline{\theta}_{T_r}\right\}$ with $T_1,\dots,T_r\in\mathcal{T}_0(\lambda,(m^n))$. We claim that $\left\{\underline{\theta}_{\widetilde{T}_1},\dots,\underline{\theta}_{\widetilde{T}_r}\right\}$ is a linearly independent set of homomorphisms, and therefore that $S^{\widetilde{\lambda}}$ arises as a summand of $K^{(m+1)^n}$ with multiplicity at least r. This claim will be sufficient to complete the proof of the theorem.

For a contradiction, suppose that $\sum_{i=1}^{r}\alpha_i\underline{\theta}_{\widetilde{T}_i}=0$ for some scalars α_i, which are not all zero. It follows that

$$(e_{\widetilde{t}})\left(\sum_{i=1}^{r}\alpha_i\underline{\theta}_{\widetilde{T}_i}\right)=0$$

and so the coefficient of any basis element $\underline{\widetilde{\mathbf{R}}}$ in $(e_{\widetilde{t}})\left(\sum_{i=1}^{r}\alpha_i\underline{\theta}_{\widetilde{T}_i}\right)$ is zero. We can write this coefficient of $\underline{\widetilde{\mathbf{R}}}$ as $\sum_{i=1}^{r}\alpha_i\mathcal{C}_i$, where $\mathcal{C}_i$ is the coefficient of $\underline{\widetilde{\mathbf{R}}}$ in $\underline{\theta}_{\widetilde{T}_i}$. However, from Lemma 4.3.8, we know that $\mathcal{C}_i$ is equal to the coefficient of $\overline{\mathbf{R}}$ in $(e_t)\overline{\theta}_{T_i}$ and thus we deduce that the coefficient of $\overline{\mathbf{R}}$ in $(e_t)\left(\sum_{i=1}^{r}\alpha_i\overline{\theta}_{T_i}\right)$ is $\sum_{i=1}^{r}\alpha_i\mathcal{C}_i=0$. Since $\underline{\widetilde{\mathbf{R}}}$ was chosen arbitrarily, we can conclude that for all $\mathbf{R}\in\mathcal{T}(\lambda,(m^n))$, $\overline{\mathbf{R}}$ has coefficient zero in $(e_t)\left(\sum_{i=1}^{r}\alpha_i\overline{\theta}_{T_i}\right)$. A subset of $\left\{\overline{\mathbf{R}}\;\middle|\;\mathbf{R}\in\mathcal{T}(\lambda,(m^n))\right\}$ is a basis for $H^{(m^n)}$ and thus $(e_t)\left(\sum_{i=1}^{r}\alpha_i\overline{\theta}_{T_i}\right)=0$. Since e_t is a generator for S^{λ}, we deduce that $\sum_{i=1}^{r}\alpha_i\overline{\theta}_{T_i}=0$, and the linear independence of the set $\left\{\overline{\theta}_{T_i}\;\middle|\;1\le i\le r\right\}$ tells us that $\alpha_i=0$ for all i, which is the required contradiction. ∎

Example 4.3.9

Take $\lambda=(4,2)$, $m=3$ and $n=2$. Choose $t=\ytableaushort{1234,56}$ and $T=\ytableaushort{1112,22}$.

In Example 4.1.2, we saw that $\left(\overline{\theta}_T:S^{(4,2)}\to H^{(3^2)}\right)\neq 0$. In particular,

$$\begin{aligned}(e_t)\overline{\theta}_T=2\big(&\{\{1,2,3\},\{4,5,6\}\}+\{\{1,2,4\},\{3,5,6\}\}\\&-\{\{2,3,5\},\{1,4,6\}\}-\{\{2,4,5\},\{1,3,6\}\}\big).\end{aligned}$$

Now, with $\widetilde{\lambda}=(6,2)$, $\widetilde{t}=\ytableaushort{71234,856}$ and $\widetilde{T}=\ytableaushort{11112,222}$ we have

$$\begin{aligned}(e_{\widetilde{t}})\theta_{\widetilde{T}}&=\left(\ytableaushort{11112,222}+\ytableaushort{11121,222}+\ytableaushort{11211,222}+\ytableaushort{12111,222}+\ytableaushort{21111,222}\right)\kappa_t\\
&=\ytableaushort{11112,222}+\ytableaushort{11121,222}-\ytableaushort{22212,111}-\ytableaushort{22221,111}\\
&\quad-\ytableaushort{12112,212}-\ytableaushort{12121,212}+\ytableaushort{21212,121}+\ytableaushort{21221,121}\\
&\quad-\ytableaushort{21112,122}-\ytableaushort{21121,122}+\ytableaushort{12212,211}+\ytableaushort{12221,211}\\
&\quad+\ytableaushort{22112,112}+\ytableaushort{22121,112}-\ytableaushort{11212,221}-\ytableaushort{11221,221}\end{aligned}$$

and thus, the image of $\left(e_{\widetilde{t}}\right)\theta_{\widetilde{T}}$ *under* ϕ *is*

$$\left(e_{\widetilde{t}}\right)\underline{\theta}_{\widetilde{T}} = 2\left(\begin{vmatrix} \{1,2,3,7\} \\ \{4,5,6,8\} \end{vmatrix} + \begin{vmatrix} \{1,2,4,7\} \\ \{3,5,6,8\} \end{vmatrix} - \begin{vmatrix} \{2,3,5,7\} \\ \{1,4,6,8\} \end{vmatrix} - \begin{vmatrix} \{2,4,5,7\} \\ \{1,3,6,8\} \end{vmatrix} \right.$$
$$\left. - \begin{vmatrix} \{1,2,3,8\} \\ \{4,5,6,7\} \end{vmatrix} - \begin{vmatrix} \{1,2,4,8\} \\ \{3,5,6,7\} \end{vmatrix} + \begin{vmatrix} \{2,3,5,8\} \\ \{1,4,6,7\} \end{vmatrix} + \begin{vmatrix} \{2,4,5,8\} \\ \{1,3,6,7\} \end{vmatrix} \right).$$

Proof of part (ii) *of Theorem 4.3.7.* The proof is entirely analogous to the proof of part (i), making the following changes:

- replacing occurrences of $\overline{\theta}_T$ by $\underline{\theta}_T$ and vice versa;
- replacing $\mathrm{Hom}_{\mathbb{C}\mathfrak{S}_{(m+1)n}}\left(S^{\widetilde{\lambda}}, K^{((m+1)^n)}\right)$ by $\mathrm{Hom}_{\mathbb{C}\mathfrak{S}_{(m+1)n}}\left(S^{\widetilde{\lambda}}, H^{((m+1)^n)}\right)$, and similarly $\mathrm{Hom}_{\mathbb{C}\mathfrak{S}_{mn}}\left(S^{\lambda}, H^{(m^n)}\right)$ by $\mathrm{Hom}_{\mathbb{C}\mathfrak{S}_{mn}}\left(S^{\lambda}, K^{(m^n)}\right)$;
- replacing occurrences of $\overline{\mathbf{R}}$ with $\underline{\mathbf{R}}$, and $\overline{\widetilde{\mathbf{R}}}$ with $\underline{\widetilde{\mathbf{R}}}$. ■

We mentioned at the start of §4.3.2 that Dent's two column result would follow from Theorem 4.3.7. We complete this section with a proof of this claim.

Proof of Theorem 4.3.6. Let $\lambda = (\lambda_1, \ldots, \lambda_\ell) \vdash mn$, as in the statement of the theorem, and observe that $\widetilde{\widetilde{\lambda}} = \widehat{\lambda}$.

(i) Apply part (i) of Theorem 4.3.7, followed by part (ii) of Theorem 4.3.7, taking $\lambda = \widetilde{\lambda}$ and increasing n by one when applying part (ii).

(ii) Apply part (ii) of Theorem 4.3.7, followed by part (i) of Theorem 4.3.7, taking $\lambda = \widetilde{\lambda}$ and increasing n by one when applying part (i). ■

4.4 Conjectures

We conjecture that Foulkes' Second Conjecture and Dent's two column result both have analogues for any generalised Foulkes module $H_\nu^{(m^n)}$. It is certainly reasonable to expect Foulkes' Second Conjecture to generalise, since a relationship of this type has already been established (for any ν) for labelling partitions of a particular form (recall Theorem 3.4.7). Additionally, data[1] obtained using MAGMA supports these conjectures. However, the semistandard homomorphism theory is not yet sufficiently developed to handle generalised Foulkes modules unless $\nu = (n)$ or $\nu = (1^n)$ and so we would need to find a different approach to prove the results.

[1]A selection of data can be found in Appendix C.

Conjecture 4.4.1
Let $\lambda = (\lambda_1, \lambda_2 \ldots, \lambda_\ell) \vdash mn$, $\nu \vdash n$ and suppose that S^λ appears in $H_\nu^{(m^n)}$ with multiplicity $r \geq 0$. If $\widetilde{\lambda} = \big(\lambda_1 + 2, \ldots, \lambda_\ell + 2, 2^{n-\ell}\big)$, then $S^{\widetilde{\lambda}}$ appears in $H_\nu^{((m+2)^n)}$ with multiplicity equal to r.

Conjecture 4.4.2
Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \vdash mn$, $\nu \vdash n$ and suppose that S^λ appears in $H_\nu^{(m^n)}$ with multiplicity $r \geq 0$. If $\widetilde{\lambda} = (\lambda_1 + n, \lambda_2, \ldots, \lambda_\ell)$, then $S^{\widetilde{\lambda}}$ appears in $H_\nu^{((m+1)^n)}$ with multiplicity $\geq r$.

In the same way that Dent's two column result follows from Theorem 4.3.7, we now conjecture a result that generalises Theorem 4.3.7, from which Conjecure 4.4.1 would follow as a straightforward corollary.

Conjecture 4.4.3
Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \vdash mn$ and define $\widetilde{\lambda} := \big(\lambda_1 + 1, \ldots, \lambda_\ell + 1, 1^{n-\ell}\big) \vdash (m+1)n$. If S^λ appears in $H_\nu^{(m^n)}$ with multiplicity $r \geq 0$, then $S^{\widetilde{\lambda}}$ appears in $H_{\nu'}^{((m+1)^n)}$ with multiplicity equal to r.

Chapter 5

Semistandard homomorphism results for fixed m

So far, all of the results featuring semistandard homomorphisms have been structural results for fixed n. In this section, we fix m. Given a constituent of the Foulkes module $H^{(m^n)}$ for a particular choice of natural numbers m and n, we prove the existence of related constituents in $H^{(m^{n+a})}$, where $a \in \mathbb{N}$, and determine a lower bound on the multiplicities with which these constituents appear. Where appropriate, we prove analogues of the results for twisted Foulkes modules.

In this chapter, we keep the notation that we introduced in §4.2, redefining it where necessary to suit the new setting, and we continue to work over a ground field of characteristic zero.

5.1 The results

The principal aim of this chapter is to prove the following two new theorems. We were motivated to establish a result similar to Theorem 4.3.7, but where m is fixed, and the 'new' constituent arises from the addition of a row of length m, rather than a column: the outcome is Theorem 5.1.1. In Theorem 5.1.2, we extend the idea further by adding longer rows. Specifically, we prove a relationship between irreducible constituents of (ordinary) Foulkes modules via the addition of a row, whose length is a multiple of m, above all other rows of the labelling partition.

Theorem 5.1.1

Let $\lambda = (\lambda_1, \lambda_2 \ldots, \lambda_\ell)$ be a partition of mn. Let $q \in \mathbb{N}_0$ be minimal such that $0 \leq \lambda_{q+1} \leq m$. Define $\widetilde{\lambda} := (\lambda_1, \ldots, \lambda_q, m, \lambda_{q+1}, \ldots, \lambda_\ell)$ if $q \neq 0$ and $\widetilde{\lambda} := (m, \lambda_1, \ldots, \lambda_\ell)$ if $q = 0$, so that $\widetilde{\lambda}$ is a partition of $m(n+1)$.

1. *Suppose that m is even. If S^{λ} appears in $H^{(m^n)}$ with multiplicity $r \geq 0$, then $S^{\widetilde{\lambda}}$ appears in $H^{(m^{n+1})}$ with multiplicity $\geq r$.*

2. *Suppose that m is odd. If S^{λ} appears in $K^{(m^n)}$ with multiplicity $r \geq 0$, then $S^{\widetilde{\lambda}}$ appears in $K^{(m^{n+1})}$ with multiplicity $\geq r$.*

Remark. If $q = 0$ in Theorem 5.1.1, then all parts of λ have size at most m. Additionally, we deduce from Theorem 4.13 in [25], and the fact that $H^{(m^n)}$ is a direct summand of $M^{(m^n)}$, that the Specht modules that are summands of $H^{(m^n)}$ are labelled by partitions that have at most n rows[1]. Thus, in this case, it must be that $\lambda = (m^n)$ and so $\widetilde{\lambda} = (m^{n+1})$.

Theorem 5.1.2

Let $\lambda = (\lambda_1, \lambda_2 \ldots, \lambda_\ell)$ be a partition of mn such that $0 \leq \lambda_1 < 2m$ and for any $a \geq 2$, define $\widetilde{\lambda} := (am, \lambda_1, \ldots, \lambda_\ell)$, which is a partition of $m(n+a)$. Suppose that m is even. If S^{λ} appears in $H^{(m^n)}$ with multiplicity $r \geq 0$, then $S^{\widetilde{\lambda}}$ appears in $H^{(m^{n+a})}$ with multiplicity $\geq r$.

In the case of Theorem 5.1.1, we cannot change the parity of m. For example, $S^{(7,4,4)}$ is a composition factor of $H^{(5^3)}$, appearing with multiplicity one, but $S^{(7,5,4,4)}$ does not appear as a composition factor of $H^{(5^4)}$. Similarly, the obvious analogue of Theorem 5.1.2 in the twisted Foulkes setting is false: taking $m = 3$, $n = 2$, $a = 2$ and $\lambda = (5, 1)$ provides a counterexample, since $S^{(5,1)}$ appears as a composition factor of $K^{(3^2)}$ with multiplicity one, but $S^{(6,5,1)}$ does not appear as a composition factor of $K^{(3^4)}$.

The statement of Theorem 5.1.2 is not as general as we would hope. Ideally, we would like to be able to remove the condition that $0 \leq \lambda_1 < 2m$, so that we are free to choose any constituent S^{λ} of $H^{(m^n)}$. At least for small m and n, there is data[2] to support this conjecture. However, the likelihood of proving this conjecture using semistandard homomorphisms is slim, as the tableaux involved are hard to control.

We illustrate the theorems with some examples.

Example 5.1.3

1. *$S^{(10,2)}$ appears in $H^{(4^3)}$ with multiplicity 1;*
   $S^{(10,4,2)}$ appears in $H^{(4^4)}$ with multiplicity 2;
   $S^{(10,4^2,2)}$ appears in $H^{(4^5)}$ with multiplicity 3.

2. *$S^{(7,1^2)}$ appears in $K^{(3^3)}$ with multiplicity 1;*
   $S^{(7,3,1^2)}$ appears in $K^{(3^4)}$ with multiplicity 1;
   $S^{(7,3^2,1^2)}$ appears in $K^{(3^5)}$ with multiplicity 1;
   $S^{(7,3^3,1^2)}$ appears in $K^{(3^6)}$ with multiplicity 1.

[1] An entirely similar statement holds for $K^{(m^n)}$.

[2] This data was produced using the computational algebra software MAGMA. Some data is included in Appendix C for reference.

3. $S^{(6,2)}$ *appears in* $H^{(4^2)}$ *with multiplicity 1;*
   $S^{(12,6,2)}$ *appears in* $H^{(4^5)}$ *with multiplicity 4.*

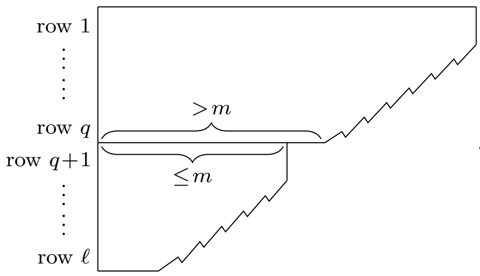
Remark. The examples illustrate that it is impractical to attempt to predict the growth of the multiplicities without deeper insight.

5.2 Tableaux

When we come to prove part 1 of Theorem 5.1.1, a key element will be showing that if $\left(\overline{\theta}_T : S^\lambda \to H^{(m^n)}\right) \neq 0$ for some λ-tableau T of type (m^n), then $\left(\overline{\theta}_{\widetilde{T}} : S^{\widetilde{\lambda}} \to H^{(m^{n+1})}\right) \neq 0$, where $\widetilde{T}$ is an appropriately chosen $\widetilde{\lambda}$-tableau of type (m^{n+1}). Thus, the proof will depend heavily on the choice of $\widetilde{T}$. The proof of part 2 will be entirely similar. We will also require an appropriate tableau for Theorem 5.1.2. In this section, we present candidates for $\widetilde{T}$.

5.2.1 Tableaux for Theorem 5.1.1

Recall from Theorem 5.1.1 that, for a given $\lambda = (\lambda_1, \dots, \lambda_\ell)$, $q \in \mathbb{N}_0$ is minimal such that $0 \leq \lambda_{q+1} \leq m$. So, any λ-tableau will have the following shape:



.

Let T be a λ-tableau of type (m^n) and define $\widetilde{T}$ in the following way:

$$\widetilde{T}_i^{(j)} := \begin{cases} T_i^{(j)} & i \leq q; \\ n+1 & i = q+1 \text{ and } j \in \{1, 2, \dots, m\}; \\ T_{i-1}^{(j)} & i > q+1. \end{cases} \tag{5.1}$$

If T is semistandard, then the construction of $\widetilde{T}$ ensures that $\widetilde{T}$ has distinct entries in columns, and that entries are non-decreasing along rows. However, $\widetilde{T}$ is certainly not semistandard in general.

Take t to be the λ-tableau which has the digits $1, 2, \dots, mn$ in increasing order along rows. Define the labelling tableau $\widetilde{t}$ by

$$\widetilde{t}_i^{(j)} := \begin{cases} t_i^{(j)} & i \leq q; \\ mn + j & i = q+1 \text{ and } j \in \{1, 2, \dots, m\}; \\ t_{i-1}^{(j)} & i > q+1. \end{cases}$$

We should note that this choice of $\widetilde{t}$ is not standard in general.

We illustrate the construction of $\widetilde{T}$ and $\widetilde{t}$ in the following example.

EXAMPLE 5.2.1

Let $\lambda = (5,1)$ and let $m = 3, n = 2$. If we take

$$T = \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & 2 & 2 \\ \hline 2 \\ \cline{1-1} \end{array} \quad \textit{and} \quad t = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & 5 \\ \hline 6 \\ \cline{1-1} \end{array},$$

then

$$\widetilde{T} = \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & 2 & 2 \\ \hline 3 & 3 & 3 \\ \cline{1-3} 2 \\ \cline{1-1} \end{array} \quad \textit{and} \quad \widetilde{t} = \begin{array}{|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & 5 \\ \hline 7 & 8 & 9 \\ \cline{1-3} 6 \\ \cline{1-1} \end{array}.$$

With these choices of tableaux, we are able to rewrite the column stabiliser $C_{\widetilde{t}}$ of $\widetilde{t}$, and subsequently the signed column sum $\kappa_{\widetilde{t}}$, in a more helpful way. Let ℓ_j be the number of entries in column j of t. Take explicit coset representatives $y_i^{(j)}$ of $C_t^{(j)}$ in $C_{\widetilde{t}}^{(j)}$: for $1 \leq j \leq m$ and $1 \leq i \leq \ell_j + 1$, define $y_i^{(j)}$ to be the transposition

$$y_i^{(j)} := \left(\widetilde{t}_i^{(j)} \ \ mn + j\right) \quad \text{if } i \neq q + 1$$

and the identity permutation if $i = q + 1$. We may write

$$C_{\widetilde{t}}^{(j)} = \bigsqcup_{i=1}^{\ell_j+1} C_t^{(j)} y_i^{(j)} \quad \text{for } 1 \leq j \leq m \qquad \text{and} \qquad C_{\widetilde{t}}^{(j)} = C_t^{(j)} \text{ for } m+1 \leq j \leq \lambda_1,$$

and then

$$Y := \left\{ \prod_{j=1}^{m} y_{x_j}^{(j)} \;\middle|\; x_j \in \{1, 2, \ldots, \ell_j + 1\} \text{ for all } 1 \leq j \leq m \right\} \tag{5.2}$$

is a set of representatives for the cosets of C_t in $C_{\widetilde{t}}$. If we define $\kappa_t^{(j)} := \sum_{\pi \in C_t^{(j)}} \operatorname{sgn}(\pi)\pi$, then we can rewrite $\kappa_{\widetilde{t}}$ as

$$\begin{aligned}
\kappa_{\widetilde{t}} &= \prod_{j=1}^{m} \left(\kappa_t^{(j)} \left(\sum_{i=1}^{\ell_j+1} \operatorname{sgn}\left(y_i^{(j)}\right) y_i^{(j)} \right) \right) \prod_{j=m+1}^{\lambda_1} \kappa_t^{(j)} \\
&= \kappa_t^{(1)} \kappa_t^{(2)} \cdots \kappa_t^{(\lambda_1)} \prod_{j=1}^{m} \left(\sum_{i=1}^{\ell_j+1} \operatorname{sgn}\left(y_i^{(j)}\right) y_i^{(j)} \right) \\
&= \kappa_t \prod_{j=1}^{m} \left(\sum_{i=1}^{\ell_j+1} \operatorname{sgn}\left(y_i^{(j)}\right) y_i^{(j)} \right).
\end{aligned}$$

5.2.2 Tableaux for Theorem 5.1.2

We also require a candidate $\widetilde{T}$ for the $\widetilde{\lambda}$-tableau of type (m^{n+a}) needed for Theorem 5.1.2. Given a λ-tableau T of type (m^n), define $\widetilde{T}$ in the following way:

$$\widetilde{T}_i^{(j)} := \begin{cases} \gamma & \text{if } i = 1 \text{ and } j \in \{(\gamma-1)m+1, (\gamma-1)m+2, \ldots, \gamma m\} \text{ with } 1 \leq \gamma \leq a; \\ T_{i-1}^{(j)} + a & \text{if } i > 1. \end{cases}$$

If T is semistandard, then the construction of $\widetilde{T}$ ensures that $\widetilde{T}$ is also semistandard. Again, take t to be the λ-tableau that has $1, 2, \ldots, mn$ in increasing order along rows. Define $\widetilde{t}$ to be the labelling tableau

$$\widetilde{t}_i^{(j)} = \begin{cases} mn + j & \text{if } i = 1 \text{ and } j \in \{1, 2, \ldots, am\}; \\ t_{i-1}^{(j)} & \text{if } i > 1. \end{cases}$$

Example 5.2.2

Let $\lambda = (3, 1)$ *and* $m = n = 2$. *Take* $T = \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}$ *and* $t = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array}$.

If $a = 3$, *then*

$$\widetilde{T} = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 2 & 2 & 3 & 3 \\ \hline 4 & 4 & 5 \\ \cline{1-3} 5 \\ \cline{1-1}\end{array} \quad \textit{and} \quad \widetilde{t} = \begin{array}{|c|c|c|c|c|c|}\hline 5 & 6 & 7 & 8 & 9 & 10 \\ \hline 1 & 2 & 3 \\ \cline{1-3} 4 \\ \cline{1-1}\end{array}.$$

Just as in §5.2.1, we are able to rewrite $C_{\widetilde{t}}$ and $\kappa_{\widetilde{t}}$. For the above choice of $\widetilde{t}$,

$$C_{\widetilde{t}}^{(j)} = \bigcup_{i=1}^{\ell_j+1} C_t^{(j)} y_i^{(j)} \quad \text{for } 1 \le j \le \lambda_1 \qquad \text{and} \qquad C_{\widetilde{t}}^{(j)} = \mathfrak{S}_{\left\{\widetilde{t}_1^{(j)}\right\}} \text{ for } \lambda_1 < j \le am,$$

where ℓ_j is the number of entries in column j of t and, for any $1 \le j \le m$ and any $1 \le i \le \ell_j + 1$, the coset representative, $y_i^{(j)}$ of $C_t^{(j)}$ in $C_{\widetilde{t}}^{(j)}$ is defined to be the transposition $y_i^{(j)} := \left(\widetilde{t}_i^{(j)} \;\; mn + j\right)$ if $i \ne 1$ and the identity permutation if $i = 1$. Therefore, since $C_{\widetilde{t}}^{(j)}$ is the symmetric group on 1 symbol for all $\lambda_1 < j \le am$, we have that

$$\begin{aligned} \kappa_{\widetilde{t}} &= \prod_{j=1}^{am} \left(\sum_{\rho \in C_{\widetilde{t}}^{(j)}} \operatorname{sgn}(\rho)\rho \right) = \prod_{j=1}^{\lambda_1} \left(\sum_{\rho \in C_{\widetilde{t}}^{(j)}} \operatorname{sgn}(\rho)\rho \right) \\ &= \prod_{j=1}^{\lambda_1} \left(\kappa_t^{(j)} \left(\sum_{i=1}^{\ell_j+1} \operatorname{sgn}\left(y_i^{(j)}\right) y_i^{(j)} \right) \right) \\ &= \kappa_t \prod_{j=1}^{\lambda_1} \left(\sum_{i=1}^{\ell_j+1} \operatorname{sgn}\left(y_i^{(j)}\right) y_i^{(j)} \right) \end{aligned}$$

and so every $\rho \in C_{\widetilde{t}}$ can be written uniquely in the form $\rho = \pi y$ for some $\pi \in C_t$ and $y \in Y$, where

$$Y := \left\{ \prod_{j=1}^{\lambda_1} y_{x_j}^{(j)} \;\middle|\; x_j \in \{1, 2, \ldots, \ell_j + 1\} \text{ for all } 1 \le j \le \lambda_1 \right\}. \tag{5.3}$$

5.3 Proof of part 1 of Theorem 5.1.1

We now have all that we need to prove part 1 of Theorem 5.1.1. The following lemma is sufficient to prove the existence of $S^{\widetilde{\lambda}}$ as a composition factor in $H^{(m^{n+1})}$. Some of the details in the proof of Lemma 5.3.1 will be important when we later address the part of Theorem 5.1.1 concerning multiplicities.

Lemma 5.3.1

In the setting of part 1 of Theorem 5.1.1, if we have a non-zero $\mathbb{C}\mathfrak{S}_{mn}$-homomorphism $\overline{\theta}_T : S^\lambda \to H^{(m^n)}$ for some tableau $T \in \mathcal{T}_0\big(\lambda, (m^n)\big)$, then $\big(\overline{\theta}_{\widetilde{T}} : S^{\widetilde{\lambda}} \to H^{(m^{n+1})}\big) \neq 0$.

Proof. Assume that $\big(\overline{\theta}_T : S^\lambda \to H^{(m^n)}\big) \neq 0$. Since S^λ is a cyclic module with generator e_t, it follows that $(e_t)\overline{\theta}_T \neq 0$. Pick any basis element $\overline{\mathbf{R}}$ appearing with non-zero coefficient in $(e_t)\overline{\theta}_T$. Since ψ is surjective, there exists $\mathbf{R} \in \mathcal{T}\big(\lambda, (m^n)\big)$ such that $\psi : \mathbf{R} \mapsto \overline{\mathbf{R}}$. We may write

$$(e_t)\overline{\theta}_T = \sum_{T' \sim_{\mathrm{row}} T} \overline{T'\kappa_t} = \sum_{\substack{T' \sim_{\mathrm{row}} T, \\ \pi \in C_t}} \mathrm{sgn}(\pi)\overline{T'\pi},$$

which allows us to identify the coefficient $\mathscr{C}$ of $\overline{\mathbf{R}}$ as

$$\mathscr{C} = \sum_{\substack{T' \sim_{\mathrm{row}} T, \\ \pi \in C_t: \\ \overline{T'\pi} = \overline{\mathbf{R}}}} \mathrm{sgn}(\pi) \neq 0. \tag{5.4}$$

We may isolate $\overline{\widetilde{\mathbf{R}}}$ in $(e_{\widetilde{t}})\overline{\theta}_{\widetilde{T}}$ in much the same way:

$$(e_{\widetilde{t}})\overline{\theta}_{\widetilde{T}} = \sum_{T'' \sim_{\mathrm{row}} \widetilde{T}} \overline{T''\kappa_{\widetilde{t}}} = \sum_{\substack{T'' \sim_{\mathrm{row}} \widetilde{T}, \\ \rho \in C_{\widetilde{t}}: \\ \overline{T''\rho} = \overline{\widetilde{\mathbf{R}}}}} \mathrm{sgn}(\rho)\overline{\widetilde{\mathbf{R}}} + \sum_{\substack{T'' \sim_{\mathrm{row}} \widetilde{T}, \\ \rho \in C_{\widetilde{t}}: \\ \overline{T''\rho} \neq \overline{\widetilde{\mathbf{R}}}}} \mathrm{sgn}(\rho)\overline{T''\rho}$$

and, visibly, its coefficient $\mathcal{C}$ is

$$\mathcal{C} = \sum_{\substack{T'' \sim_{\mathrm{row}} \widetilde{T}, \\ \rho \in C_{\widetilde{t}}: \\ \overline{T''\rho} = \overline{\widetilde{\mathbf{R}}}}} \mathrm{sgn}(\rho). \tag{5.5}$$

To prove the lemma, it will suffice to prove that $\mathcal{C}$ is non-zero.

We make an observation which allows us to write $\mathcal{C}$ in a more helpful form: that $T'' \sim_{\mathrm{row}} \widetilde{T}$ if and only if $T' \sim_{\mathrm{row}} T$, where $T' \in \mathcal{T}\big(\lambda, (m^n)\big)$ is such that $T'' = \widetilde{T'}$. To see this, observe that if $T'' \sim_{\mathrm{row}} \widetilde{T}$, then it is possible to remove row $q+1$ of T'' – the row of length m containing only $(n+1)$s – leaving a λ-tableau, say T', which is row equivalent to T. The reverse implication is clear.

Using this observation, together with the definition of Y in (5.2) and the expression of $\rho \in C_{\widetilde{t}}$ as $\rho = \pi y$ (where $\pi \in C_t$ and $y \in Y$), we have that

$$\mathcal{C} = \sum_{\substack{T' \sim_{\mathrm{row}} T, \\ \pi \in C_t,\, y \in Y: \\ \overline{\widetilde{T'}\pi y} = \overline{\widetilde{\mathbf{R}}}}} \mathrm{sgn}(\pi)\,\mathrm{sgn}(y).$$

Take $T' \sim_{\mathrm{row}} T$, $\pi \in C_t$ and $y \in Y$ such that $\overline{\widetilde{T'}\pi y} = \overline{\widetilde{\mathbf{R}}}$. Since $\pi \in C_t$, it must fix row $q+1$ of $\widetilde{T'}$. Thus, the definition of the $\sim$ construction ensures that $\widetilde{T'}\pi = \widetilde{T'\pi}$ and so

$$\overline{\widetilde{T'}\pi y} = \overline{\widetilde{\mathbf{R}}} \iff \overline{\widetilde{T'\pi}y} = \overline{\widetilde{\mathbf{R}}}. \tag{5.6}$$

The construction of $\widetilde{\mathbf{R}}$ guarantees that the entries in row $q+1$ of $\widetilde{\mathbf{R}}$ are all the same. It then follows from (5.6) that the entries in row $q+1$ of $\widetilde{T'\pi y}$ are all the same. Using a construction argument again, the entries in row $q+1$ of $\widetilde{T'\pi}$ are all identical. So, for the entries in row $q+1$ of $\widetilde{T'\pi y}$ to also be identical, it must be that $y \in Y$ either fixes row $q+1$ of $\widetilde{T'\pi}$ – in which case $y \in Y$ is the identity permutation, which we denote by id – or it must swap every identical entry, which is $n+1$, in row $q+1$ with some $\beta \in \mathrm{B}$, where

$$\mathrm{B} := \left\{ \beta \in \{1, \ldots, n\} \;\middle|\; \beta \text{ appears in precisely the columns } 1, \ldots, m \text{ of } \widetilde{T'\pi} \right\}.$$

In the latter case, if, for all $1 \leq j \leq m$, β appears in row $b_j \neq q+1$ in column j of $\widetilde{T'\pi}$, then $y = y_\beta := \prod_{j=1}^{m} y_{b_j}^{(j)}$. Define

$$Y_0\left(\widetilde{T'\pi}\right) := \{ y \in Y \mid y = \mathrm{id} \text{ or } y = y_\beta \text{ for any } \beta \in \mathrm{B} \}.$$

We have just seen that $\overline{\widetilde{T'\pi y}} = \overline{\widetilde{\mathbf{R}}}$ implies that $y \in Y_0\left(\widetilde{T'\pi}\right)$, and it is easy to see that if $y \in Y_0\left(\widetilde{T'\pi}\right)$, then $\overline{\widetilde{T'\pi y}} = \overline{\widetilde{\mathbf{R}}}$. So, we only need to sum over $y \in Y_0\left(\widetilde{T'\pi}\right)$, i.e.

$$\mathcal{C} = \sum_{\substack{T' \sim_{\text{row}} T, \\ \pi \in C_t,\, y \in Y_0\left(\widetilde{T'\pi}\right): \\ \overline{\widetilde{T'\pi y}} = \overline{\widetilde{\mathbf{R}}}}} \operatorname{sgn}(\pi) \operatorname{sgn}(y).$$

Moreover, if $y = \mathrm{id}$, then $\overline{\widetilde{T'\pi}} = \overline{\widetilde{\mathbf{R}}}$. If $y = y_\beta$, which swaps every $n+1$ in row $q+1$ with some $1 \leq \beta \leq n$, then y has the effect of relabelling $\widetilde{T'\pi}$ by the transposition $((n+1)\ \beta) \in \mathfrak{S}_{n+1}$ and so $\overline{\widetilde{T'\pi y}} = \overline{\widetilde{\mathbf{R}}}$ if and only if $\overline{\widetilde{T'\pi} * ((n+1)\ \beta)} = \overline{\widetilde{\mathbf{R}}}$. In both cases, $\widetilde{T'\pi}$ is a relabelling of $\widetilde{\mathbf{R}}$.

Let d be the number of digits in the set $\{1, 2, \ldots, n\}$ that appear in precisely columns $1, 2, \ldots, m$ of $\mathbf{R}$. By construction of $\widetilde{\mathbf{R}}$, there are necessarily $d+1$ of the digits $\{1, 2, \ldots, n+1\}$ in precisely columns $1, 2, \ldots, m$ of $\widetilde{\mathbf{R}}$. Since $\widetilde{T'\pi}$ is a relabelling of $\widetilde{\mathbf{R}}$, we conclude that $\left| Y_0\left(\widetilde{T'\pi}\right) \right| = d+1$.

We also know that if $y \in Y_0\left(\widetilde{T'\pi}\right)$, then

$$\operatorname{sgn}(y) = \begin{cases} 1 & y = \mathrm{id}; \\ (-1)^m & \text{otherwise.} \end{cases}$$

Since m is even, this says that $\operatorname{sgn}(y) = 1$ for all $y \in Y_0\left(\widetilde{T'\pi}\right)$. Hence, we may write the expression for $\mathcal{C}$ as

$$\mathcal{C} = (d+1) \sum_{\substack{T' \sim_{\text{row}} T, \\ \pi \in C_t: \\ \overline{\widetilde{T'\pi}} = \overline{\widetilde{\mathbf{R}}}}} \operatorname{sgn}(\pi).$$

Finally, we should observe that $\widetilde{T'\pi}$ is a relabelling of $\widetilde{\mathbf{R}}$ if and only if $T'\pi$ is a relabelling of $\mathbf{R}$. Indeed, this follows from the fact that row $q+1$ of $\widetilde{T'\pi}$ is identical to row $q+1$ of $\widetilde{\mathbf{R}}$; both contain m entries, all of which are $n+1$. So, using the expression for $\mathscr{C}$ given in (5.4), we are able to conclude that the coefficient of $\overline{\widetilde{\mathbf{R}}}$ in $(e_{\tilde{t}})\overline{\theta}_{\widetilde{T}}$ is a non-zero multiple of the coefficient $\mathscr{C}$ of $\overline{\mathbf{R}}$ in $(e_t)\overline{\theta}_T$: more precisely

$$\mathcal{C} = (d+1) \sum_{\substack{T' \sim_{\text{row}} T, \\ \pi \in C_t: \\ \overline{T'\pi} = \overline{\mathbf{R}}}} \operatorname{sgn}(\pi) = (d+1)\mathscr{C}. \tag{5.7}$$

■

To complete the proof of part 1 of Theorem 5.1.1, it remains to prove that the multiplicity with which $S^{\widetilde{\lambda}}$ appears as a composition factor in the decomposition of $H^{(m^{n+1})}$ is bounded below by the multiplicity of S^{λ} in the decomposition of $H^{(m^n)}$.

Let $B \subseteq \{\overline{\mathbf{R}} \mid \mathbf{R} \in \mathcal{T}(\lambda, (m^n))\}$ be a basis for $H^{(m^n)}$. Observe that there is a bijection $B \to \mathscr{B} := \left\{\overline{\widetilde{\mathbf{R}}} \mid \overline{\mathbf{R}} \in B\right\}$ defined on set partitions by

$$\overline{\mathbf{R}} = \{X_1, X_2, \ldots, X_n\} \longmapsto \big\{X_1, X_2, \ldots, X_n, \{mn+1, \ldots, mn+m\}\big\} = \overline{\widetilde{\mathbf{R}}}.$$

This is a direct consequence of the construction of $\widetilde{\mathbf{R}}$ and the definition of the labelling tableau $\tilde{t}$. So, since B is a basis, and therefore all its elements are distinct, the set partitions which are elements of $\mathscr{B} = \left\{\overline{\widetilde{\mathbf{R}}}_j \;\middle|\; 1 \leq j \leq |B|\right\}$ must also be distinct. It follows that the formal sum $\sum_{j=1}^{|B|} \beta_j \overline{\widetilde{\mathbf{R}}}_j$ is equal to zero only if $\beta_j = 0$ for all $1 \leq j \leq |B|$. In other words, $\mathscr{B}$ is a linearly independent set, which can be extended to a basis for $H^{(m^{n+1})}$.

Suppose that S^{λ} appears in $H^{(m^n)}$ with multiplicity $r \geq 0$. By Lemma 4.3.2, there is a basis $\{\overline{\theta}_{T_1}, \ldots, \overline{\theta}_{T_r}\}$ for $\operatorname{Hom}_{\mathbb{C}\mathfrak{S}_{mn}}(S^{\lambda}, H^{(m^n)})$, where $T_1, \ldots, T_r \in \mathcal{T}_0(\lambda, (m^n))$.

For a contradiction, assume that $\sum_{i=1}^{r} \alpha_i \overline{\theta}_{\widetilde{T}_i} = 0$ for some scalars α_i, which are not all zero. It follows that $(e_{\tilde{t}})\left(\sum_{i=1}^{r} \alpha_i \overline{\theta}_{\widetilde{T}_i}\right) = 0$ and so the coefficient of any basis element $\overline{\widetilde{\mathbf{R}}}$ in $(e_{\tilde{t}})\left(\sum_{i=1}^{r} \alpha_i \overline{\theta}_{\widetilde{T}_i}\right)$ is zero. If we let $\mathcal{C}_i$ denote the coefficient of $\overline{\widetilde{\mathbf{R}}}$ in $(e_{\tilde{t}})\overline{\theta}_{\widetilde{T}_i}$, then $\sum_{i=1}^{r} \alpha_i \mathcal{C}_i = 0$. Applying the result in (5.7), each $\mathcal{C}_i$ is equal to a certain non-zero multiple of the coefficient $\mathscr{C}_i$ of $\overline{\mathbf{R}}$ in $(e_t)\overline{\theta}_{T_i}$. In particular, $\mathcal{C}_i = (d+1)\mathscr{C}_i$, where d is the number of digits in the set $\{1, 2, \ldots, n\}$ that appear in precisely columns $1, 2, \ldots, m$ of $\mathbf{R}$, and so does not depend on i. Thus, the coefficient of $\overline{\mathbf{R}}$ in $(e_t)\left(\sum_{i=1}^{r} \alpha_i \overline{\theta}_{T_i}\right)$ is $\frac{1}{d+1} \sum_{i=1}^{r} \alpha_i \mathcal{C}_i$ and so is also zero. We chose $\overline{\widetilde{\mathbf{R}}}$ arbitrarily and so, for all $\mathbf{R} \in \mathcal{T}(\lambda, (m^n))$, $\overline{\mathbf{R}}$ has coefficient zero in $(e_t)\left(\sum_{i=1}^{r} \alpha_i \overline{\theta}_{T_i}\right)$. Hence, $(e_t)\left(\sum_{i=1}^{r} \alpha_i \overline{\theta}_{T_i}\right) = 0$. Since e_t is a generator for the Specht module S^{λ}, this implies that $\sum_{i=1}^{r} \alpha_i \overline{\theta}_{T_i} = 0$, which contradicts the assumptions on $\{\alpha_i \mid 1 \leq i \leq r\}$.

5.4 Proof of part 2 of Theorem 5.1.1

The proof of part 2 of Theorem 5.1.1 proceeds in much the same way as that of part 1. However, as we saw in Chapter 4, when in the twisted Foulkes setting we must pay particular attention to the relabellings of certain tableaux.

We proceed by proving the existence of $S^{\widetilde{\lambda}}$ as a composition factor in $K^{(m^{n+1})}$.

Lemma 5.4.1

Under the assumptions of part 2 of Theorem 5.1.1, if for some tableau $T \in \mathcal{T}_0\big(\lambda, (m^n)\big)$ we have a non-zero $\mathbb{C}\mathfrak{S}_{mn}$-homomorphism $\underline{\theta}_T : S^\lambda \to K^{(m^n)}$, then $\big(\underline{\theta}_{\widetilde{T}} : S^{\widetilde{\lambda}} \to K^{(m^{n+1})}\big) \neq 0$.

Proof. Assume that $\big(\underline{\theta}_T : S^\lambda \to K^{(m^n)}\big) \neq 0$. Since S^λ is a cyclic module with generator e_t, it follows that $(e_t)\underline{\theta}_T \neq 0$. Pick any basis element $\underline{\mathbf{R}}$ appearing in $(e_t)\underline{\theta}_T$ with non-zero coefficient. Since ϕ is surjective, there exists $\mathbf{R} \in \mathcal{T}\big(\lambda, (m^n)\big)$ such that $\phi : \mathbf{R} \mapsto \underline{\mathbf{R}}$. We may write

$$(e_t)\underline{\theta}_T = \sum_{T' \sim_{\mathrm{row}} T} \underline{T'\kappa_t} = \sum_{\substack{T' \sim_{\mathrm{row}} T, \\ \pi \in C_t}} \mathrm{sgn}(\pi)\underline{T'\pi}$$

and if we isolate $\underline{\mathbf{R}}$ in the sum, then we establish that the coefficient $\mathscr{C}$ of $\underline{\mathbf{R}}$ in $(e_t)\underline{\theta}_T$ is

$$\mathscr{C} = \sum_{\substack{T' \sim_{\mathrm{row}} T, \\ \pi \in C_t,\, \sigma \in \mathfrak{S}_n : \\ T'\pi = \mathbf{R} * \sigma}} \mathrm{sgn}(\pi)\,\mathrm{sgn}(\sigma) \neq 0. \tag{5.8}$$

In the same way, we may obtain an expression for the coefficient $\mathcal{C}$ of $\underline{\widetilde{\mathbf{R}}}$ in $(e_{\widetilde{t}})\underline{\theta}_{\widetilde{T}}$: we find that

$$\mathcal{C} = \sum_{\substack{T'' \sim_{\mathrm{row}} \widetilde{T}, \\ \rho \in C_{\widetilde{t}},\, \tau \in \mathfrak{S}_{n+1} : \\ T''\rho = \widetilde{\mathbf{R}} * \tau}} \mathrm{sgn}(\rho)\,\mathrm{sgn}(\tau).$$

To prove the lemma, it will suffice to prove that $\mathcal{C}$ is non-zero.

We may make the same observation as in the proof of Lemma 5.3.1: that $T'' \sim_{\mathrm{row}} \widetilde{T}$ if and only if $T' \sim_{\mathrm{row}} T$, where $T' \in \mathcal{T}\big(\lambda, (m^n)\big)$ is such that $T'' = \widetilde{T'}$. Using this observation, together with the definition of Y in (5.2) and the expression of $\rho \in C_{\widetilde{t}}$ as $\rho = \pi y$ (where $\pi \in C_t$ and $y \in Y$), we have that

$$\mathcal{C} = \sum_{\substack{T' \sim_{\mathrm{row}} T, \\ \pi \in C_t,\, y \in Y, \\ \tau \in \mathfrak{S}_{n+1} : \\ \widetilde{T'}\pi y = \widetilde{\mathbf{R}} * \tau}} \mathrm{sgn}(\pi)\,\mathrm{sgn}(y)\,\mathrm{sgn}(\tau).$$

Take $T' \sim_{\mathrm{row}} T$, $\pi \in C_t$, $y \in Y$ and $\tau \in \mathfrak{S}_{n+1}$ such that $\widetilde{T'}\pi y = \widetilde{\mathbf{R}} * \tau$. Since $\pi \in C_t$, it must fix row $q+1$ of $\widetilde{T'}$. Thus, the $\sim$ construction ensures that $\widetilde{T'}\pi = \widetilde{T'\pi}$ and so

$$\widetilde{T'}\pi y = \widetilde{\mathbf{R}} * \tau \iff \widetilde{T'\pi} y = \widetilde{\mathbf{R}} * \tau. \tag{5.9}$$

Further, we are able to conclude that $y \in Y$ either fixes row $q+1$ of $\widetilde{T'\pi}$ or it must swap every identical entry (which is $n+1$) in row $q+1$ with some digit $\beta \in \mathrm{B}$, where

$$\mathrm{B} := \Big\{ \beta \in \{1, \ldots, n\} \;\Big|\; \beta \text{ appears in precisely the columns } 1, \ldots, m \text{ of } \widetilde{T'\pi} \Big\}.$$

So, either $y \in Y$ is the identity permutation, or $y = y_\beta$ for $\beta \in \mathrm{B}$ (with y_β defined exactly as in the proof of Lemma 5.3.1). Define $Y_0\left(\widetilde{T'\pi}\right) := \{y \in Y \mid y = \mathrm{id} \text{ or } y = y_\beta \text{ for any } \beta \in \mathrm{B}\}$.

We have just seen that $\widetilde{T'\pi y} = \widetilde{\mathbf{R}} * \tau$ implies that $y \in Y_0\left(\widetilde{T'\pi}\right)$, and it is easy to see that if $y \in Y_0\left(\widetilde{T'\pi}\right)$ then $\widetilde{T'\pi y} = \widetilde{\mathbf{R}} * \tau$. So we need only sum over $y \in Y_0\left(\widetilde{T'\pi}\right)$ and therefore

$$\mathcal{C} = \sum_{\substack{T' \sim_{\mathrm{row}} T, \\ \pi \in C_t,\, y \in Y_0\left(\widetilde{T'\pi}\right), \\ \tau \in \mathfrak{S}_{n+1}: \\ \widetilde{T'\pi y} = \widetilde{\mathbf{R}} * \tau}} \operatorname{sgn}(\pi) \operatorname{sgn}(y) \operatorname{sgn}(\tau).$$

Moreover, if $y = \mathrm{id}$, then $\widetilde{T'\pi} = \widetilde{T'\pi y} = \widetilde{\mathbf{R}} * \tau$. If $y = y_\beta$, which swaps every $n+1$ in row $q+1$ with some $1 \leq \beta \leq n$, then y has the effect of relabelling $\widetilde{T'\pi}$ by the transposition $((n+1)\ \ \beta) \in \mathfrak{S}_{n+1}$ and in this case, $\widetilde{T'\pi y} = \widetilde{\mathbf{R}} * \tau$ if and only if $\widetilde{T'\pi} * ((n+1)\ \ \beta) = \widetilde{\mathbf{R}} * \tau$. At this point, we may write the expression for the coefficient $\mathcal{C}$ as

$$\mathcal{C} = \sum_{\substack{T' \sim_{\mathrm{row}} T, \\ \pi \in C_t,\, \tau \in \mathfrak{S}_{n+1}, \\ y = \mathrm{id} \in Y_0\left(\widetilde{T'\pi}\right): \\ \widetilde{T'\pi} = \widetilde{\mathbf{R}} * \tau}} \operatorname{sgn}(\pi) \operatorname{sgn}(y) \operatorname{sgn}(\tau) \quad + \sum_{\substack{T' \sim_{\mathrm{row}} T, \\ \pi \in C_t,\, \tau \in \mathfrak{S}_{n+1}, \\ y \in Y_0\left(\widetilde{T'\pi}\right) \setminus \{\mathrm{id}\}: \\ \widetilde{T'\pi} * ((n+1)\ \beta) = \widetilde{\mathbf{R}} * \tau}} \operatorname{sgn}(\pi) \operatorname{sgn}(y) \operatorname{sgn}(\tau).$$

Using the fact that m is odd, we know that if $y \in Y_0\left(\widetilde{T'\pi}\right)$, then $\operatorname{sgn}(y) = 1$ if $y = \mathrm{id}$ and $\operatorname{sgn}(y) = (-1)^m = -1$ otherwise. So, we may write $\mathcal{C}$ as

$$\mathcal{C} = \sum_{\substack{T' \sim_{\mathrm{row}} T, \\ \pi \in C_t,\, \tau \in \mathfrak{S}_{n+1}: \\ \widetilde{T'\pi} = \widetilde{\mathbf{R}} * \tau}} \operatorname{sgn}(\pi) \operatorname{sgn}(\tau) \quad - \sum_{\substack{T' \sim_{\mathrm{row}} T,\, \pi \in C_t, \\ \tau \in \mathfrak{S}_{n+1},\, \beta \in \mathrm{B}: \\ \widetilde{T'\pi} * ((n+1)\ \beta) = \widetilde{\mathbf{R}} * \tau}} \operatorname{sgn}(\pi) \operatorname{sgn}(\tau). \tag{5.10}$$

The requirement that $\widetilde{T'\pi} * ((n+1)\ \ \beta) = \widetilde{\mathbf{R}} * \tau$ says that $\widetilde{T'\pi}$ is a relabelling of $\widetilde{\mathbf{R}}$. Let d be the number of digits in the set $\{1, 2, \ldots, n\}$ that appear in precisely columns $1, 2, \ldots, m$ of $\mathbf{R}$. By construction of $\widetilde{\mathbf{R}}$, there are $d+1$ of the digits $\{1, 2, \ldots, n+1\}$ in precisely columns $1, 2, \ldots, m$ of $\widetilde{\mathbf{R}}$. So, since $\widetilde{T'\pi}$ is a relabelling of $\widetilde{\mathbf{R}}$, this forces $\left|Y_0\left(\widetilde{T'\pi}\right) \setminus \{\mathrm{id}\}\right| = |\mathrm{B}| = d$.

Consider the first sum in the right-hand side of (5.10) and observe that $\widetilde{T'\pi} = \widetilde{\mathbf{R}} * \tau$ implies that $T'\pi$ is a relabelling of $\mathbf{R}$. Indeed, if $\widetilde{T'\pi} = \widetilde{\mathbf{R}} * \tau$, then τ must not affect row $q+1$ of $\mathbf{R}$, otherwise $\widetilde{T'\pi}$ will not have $(n+1)$s in row $q+1$ (which it must do, by the definition of the $\sim$ construction). So, there exists a unique $\sigma \in \mathfrak{S}_n$ which satisfies $T'\pi = \mathbf{R} * \sigma$: take $\sigma = \tau$, from which it follows that $\operatorname{sgn}(\sigma) = \operatorname{sgn}(\tau)$.

Similarly, considering the second sum in the right-hand side of (5.10), we see that $\widetilde{T'\pi} * ((n+1)\ \ \beta) = \widetilde{\mathbf{R}} * \tau$ implies that $T'\pi$ is a relabelling of $\mathbf{R}$. In this case, if we define $\sigma \in \mathfrak{S}_n$ by $\sigma = \tau((n+1)\ \beta)$, then $T'\pi = \mathbf{R} * \sigma$ and $\operatorname{sgn}(\sigma) = -\operatorname{sgn}(\tau)$. Thus,

$$\begin{aligned}\mathcal{C} &= \sum_{\substack{T' \sim_{\text{row}} T,\\ \pi \in C_t,\, \sigma \in \mathfrak{S}_n:\\ T'\pi = \mathbf{R}*\sigma}} \operatorname{sgn}(\pi)\operatorname{sgn}(\sigma) \ - \ |\mathrm{B}| \sum_{\substack{T' \sim_{\text{row}} T,\\ \pi \in C_t,\, \sigma \in \mathfrak{S}_n:\\ T'\pi = \mathbf{R}*\sigma}} \operatorname{sgn}(\pi)\big(-\operatorname{sgn}(\sigma)\big)\\ &= (d+1) \sum_{\substack{T' \sim_{\text{row}} T,\\ \pi \in C_t,\, \sigma \in \mathfrak{S}_n:\\ T'\pi = \mathbf{R}*\sigma}} \operatorname{sgn}(\pi)\operatorname{sgn}(\sigma)\end{aligned}$$

and so, using the expression for $\mathscr{C}$ given in (5.8), we are finally able to conclude that the coefficient of $\underline{\widetilde{\mathbf{R}}}$ in $(e_{\widetilde{t}})\underline{\theta}_{\widetilde{T}}$ is a non-zero multiple of the coefficient $\mathscr{C}$ of $\underline{\mathbf{R}}$ in $(e_t)\underline{\theta}_T$: more precisely

$$\mathcal{C} = (d+1)\mathscr{C}. \tag{5.11}$$

■

To complete the proof of part 2 of Theorem 5.1.1, it remains to prove that the multiplicity with which $S^{\widetilde{\lambda}}$ appears as a composition factor in the decomposition of $K^{(m^{n+1})}$ is bounded below by the multiplicity of S^{λ} in the decomposition of $K^{(m^n)}$. The proof mirrors of part 1 of Theorem 5.1.1, using Lemma 4.3.5 instead of Lemma 4.3.2.

Remark. Our result in Theorem 5.1.1 also establishes a relationship between terms in the plethysms $s_{(n)} \circ s_{(1^m)}$ and $s_{(n+1)} \circ s_{(1^m)}$, without any restriction on the parity of m. Indeed, determining the decomposition of $H^{(m^n)}$ and $K^{(m^n)}$ into irreducible constituents corresponds to finding an expression for $s_{(n)} \circ s_{(m)}$ and $s_{(1^n)} \circ s_{(m)}$, respectively, in terms of Schur functions s_λ. Applying the involution ω defined in §3.2, we find that $\omega(s_{(n)} \circ s_{(m)}) = s_{(n)} \circ s_{(1^m)}$ when m is even and $\omega(s_{(1^n)} \circ s_{(m)}) = s_{(n)} \circ s_{(1^m)}$ when m is odd. Since $\omega(s_\lambda) = s_{\lambda'}$, (see (3.1)), the adding of a row of length m described in Theorem 5.1.1 corresponds to adding a column of length m to a partition labelling a Schur function in the plethysm $s_{(n)} \circ s_{(1^m)}$.

5.5 Proof of Theorem 5.1.2

The approach that we will take in order to prove Theorem 5.1.2 will echo the approach used in the previous two sections. So, in the setting of Theorem 5.1.2, we begin by establishing the existence of $S^{\widetilde{\lambda}}$ as a composition factor of $H^{(m^{n+a})}$ for any $a \geq 2$. Recall that, for this section, we define $\widetilde{\lambda} := (am, \lambda_1, \ldots, \lambda_\ell)$ and we redefine $\widetilde{t}$ and $\widetilde{T}$ as in §5.2.2.

Lemma 5.5.1

Under the assumptions of Theorem 5.1.2, if we have $\big(\overline{\theta}_T : S^\lambda \to H^{(m^n)}\big) \neq 0$ *for some tableau* $T \in \mathcal{T}_0\big(\lambda, (m^n)\big)$, *then* $\big(\overline{\theta}_{\widetilde{T}} : S^{\widetilde{\lambda}} \to H^{(m^{n+a})}\big) \neq 0$ *for any* $a \geq 2$.

Proof. Assume that $\big(\overline{\theta}_T : S^\lambda \to H^{(m^n)}\big) \neq 0$; it follows that $(e_t)\overline{\theta}_T \neq 0$. Pick any basis element $\overline{\mathbf{R}}$ appearing with non-zero coefficient $\mathscr{C}$ in $(e_t)\overline{\theta}_T$. Since ψ is surjective, there exists

$\mathbf{R} \in \mathcal{T}\big(\lambda, (m^n)\big)$ such that $\psi : \mathbf{R} \mapsto \overline{\mathbf{R}}$. An expression for the coefficient $\mathscr{C}$ is

$$\mathscr{C} = \sum_{\substack{T' \sim_{\text{row}} T, \\ \pi \in C_t: \\ \overline{T'\pi} = \overline{\mathbf{R}}}} \operatorname{sgn}(\pi). \tag{5.12}$$

Fix $a \geq 2$. It will suffice to show that the coefficient $\mathcal{C}$ of $\overline{\widetilde{\mathbf{R}}}$ in $(e_{\tilde{t}})\overline{\theta}_{\widetilde{T}}$ is non-zero, where

$$\mathcal{C} := \sum_{\substack{T'' \sim_{\text{row}} \widetilde{T}, \\ \rho \in C_{\tilde{t}}: \\ \overline{T''\rho} = \overline{\widetilde{\mathbf{R}}}}} \operatorname{sgn}(\rho).$$

Firstly, recall from §5.2.2 that ρ may be expressed as $\rho = \pi y$ for some unique $\pi \in C_t$ and $y \in Y$, the definition of Y being that given in (5.3). Secondly, take $T'' \sim_{\text{row}} \widetilde{T}$, $\pi \in C_t$ and $y \in Y$ such that $\overline{T''\pi y} = \overline{\widetilde{\mathbf{R}}}$. Entries in row 1 of $T''\pi y$ must have the same pattern as entries in row 1 of $\widetilde{\mathbf{R}}$, the latter being $1 \ldots 1\, 2 \ldots 2 \ldots a \ldots a$, with m copies of each digit. Also note that the first row of $\widetilde{\mathbf{R}}$ is the same as the first row of $\widetilde{T}$. Since there is only one entry in columns $\lambda_1 + 1, \ldots, am$ of T'', πy fixes these columns. Hence, entries in columns $\lambda_1 + 1, \ldots, am$ of T'' must be a relabelling (by $\omega \in \mathfrak{S}_a$, say) of the entries in columns $\lambda_1 + 1, \ldots, am$ of $\widetilde{\mathbf{R}}$. Since $\lambda_1 < 2m$, to preserve the pattern of the first row, we must have the entries in columns $m+1, \ldots, \lambda_1$ in row 1 of T'' equal to the entry in columns $\lambda_1 + 1, \ldots, 2m$, which is $(2)\omega$. The fact that $T'' \sim_{\text{row}} \widetilde{T}$ tells us that there is one remaining digit (repeated m times), which is the entry in columns $1, \ldots, m$ of row 1 of T''. We conclude that row 1 of T'' is of the form

$$\underbrace{(1)\omega\ (1)\omega\ \ldots\ (1)\omega}_{m \text{ copies}}\ \underbrace{(2)\omega\ (2)\omega\ \ldots\ (2)\omega}_{m \text{ copies}}\ \ldots\ldots\ldots\ \underbrace{(a)\omega\ (a)\omega\ \ldots\ (a)\omega}_{m \text{ copies}},$$

where $\omega \in \mathfrak{S}_a$. Entries in the remaining rows of T'' are $T''^{(j)}_{i+1} := T'^{(j)}_i + a$ (where $1 \leq i \leq \ell$) for some $T' \sim_{\text{row}} T$, that is, rows $2, 3, \ldots, \ell + 1$ of $\widetilde{T'}$. It follows that $T'' = \widetilde{T'} * \omega$, where $\omega \in \mathfrak{S}_a \subseteq \mathfrak{S}_{a+n}$, and so the expression for $\mathcal{C}$ becomes

$$\mathcal{C} = \sum_{\substack{T' \sim_{\text{row}} T,\, \omega \in \mathfrak{S}_a, \\ \pi \in C_t,\, y \in Y: \\ \overline{(\widetilde{T'} * \omega)\pi y} = \overline{\widetilde{\mathbf{R}}}}} \operatorname{sgn}(\pi) \operatorname{sgn}(y).$$

We must determine the $y \in Y$ for which $\overline{\big(\widetilde{T'} * \omega\big)\pi y} = \overline{\widetilde{\mathbf{R}}}$ holds. Recall that $\pi \in C_t$ fixes row 1 of T''. Therefore, $\overline{\big(\widetilde{T'} * \omega\big)\pi y} = \overline{\widetilde{\mathbf{R}}}$ if and only if $\overline{\big(\widetilde{T'\pi} * \omega\big)y} = \overline{\widetilde{\mathbf{R}}}$. To preserve the pattern of row 1 of $\widetilde{T'\pi} * \omega$, y must either fix row 1; or swap every $(1)\omega$ in row 1 with a digit $\beta \in \mathrm{B}$ (and fix every $(2)\omega, \ldots, (a)\omega$ because $\lambda_1 < 2m$), where

$$\mathrm{B} := \Big\{ \beta \in \{a+1, \ldots, a+n\} \;\Big|\; \beta \text{ appears in precisely columns } 1, 2 \ldots, m \text{ of } \widetilde{T'\pi} * \omega \Big\}.$$

In other words, if, for all $1 \leq j \leq m$, a β appears in row $b_j \neq 1$ in column j of $\widetilde{T'\pi} * \omega$, then

$$y \in Y_0 := \{\,\mathrm{id}\,\} \cup \Big\{ y_\beta := \prod\nolimits_{j=1}^{m} y_{b_j}^{(j)} \text{ for any } \beta \in \mathrm{B} \Big\} \subseteq Y;$$

note that Y_0 depends on $\widetilde{T'\pi} * \omega$. In fact, $\overline{\big(\widetilde{T'\pi} * \omega\big)y} = \overline{\big(\widetilde{T'\pi} * \omega\big)}$ if and only if $y \in Y_0$.

Let d be the number of digits in the set $\{1, \ldots, n\}$ that appear in precisely columns $1, \ldots, m$ of $\mathbf{R}$. By construction of $\widetilde{\mathbf{R}}$ and the requirement that $\widetilde{T'\pi} * \omega = \overline{\widetilde{\mathbf{R}}}$, we deduce that $|Y_0 \backslash \{\mathrm{id}\}| = d$. Also, since m is even, if $y \in Y_0$ then y is even. This knowledge allows us to write $\mathcal{C}$ as

$$\mathcal{C} = \sum_{\substack{T' \sim_{\mathrm{row}} T,\\ \omega \in \mathfrak{S}_a,\, \pi \in C_t:\\ \overline{\widetilde{T'\pi} * \omega} = \overline{\widetilde{\mathbf{R}}}}} |Y_0| \operatorname{sgn}(\pi) = (d+1) \sum_{\substack{T' \sim_{\mathrm{row}} T,\\ \omega \in \mathfrak{S}_a,\, \pi \in C_t:\\ \overline{\widetilde{T'\pi} * \omega} = \overline{\widetilde{\mathbf{R}}}}} \operatorname{sgn}(\pi).$$

Finally, we should observe that $\overline{\widetilde{T'\pi} * \omega} = \overline{\widetilde{\mathbf{R}}}$ implies that $\overline{T'\pi} = \overline{\mathbf{R}}$. Conversely, given any $T' \sim_{\mathrm{row}} T$ and $\pi \in C_t$ such that $\overline{T'\pi} = \overline{\mathbf{R}}$, setting $T'' = \widetilde{T'} * \omega$ for some $\omega \in \mathfrak{S}_a$ we find that $T'' \sim_{\mathrm{row}} \widetilde{T}$ and, for $y \in Y_0$, $\overline{T''\pi y} = \overline{\widetilde{T'\pi} * \omega} = \overline{\widetilde{\mathbf{R}}}$. Hence, using the expression for $\mathscr{C}$ given in (5.12), we conclude that the coefficient of $\overline{\widetilde{\mathbf{R}}}$ in $(e_{\widetilde{t}})\overline{\theta}_{\widetilde{T}}$ is a non-zero multiple of the coefficient $\mathscr{C}$ of $\overline{\mathbf{R}}$ in $(e_t)\overline{\theta}_T$: more precisely

$$\mathcal{C} = |\mathfrak{S}_a|(d+1) \sum_{\substack{T' \sim_{\mathrm{row}} T,\\ \pi \in C_t:\\ \overline{T'\pi} = \overline{\mathbf{R}}}} \operatorname{sgn}(\pi) = a!(d+1)\mathscr{C}. \tag{5.13}$$

■

It just remains to verify the bound on the multiplicity with which $S^{\widetilde{\lambda}}$ appears as a composition factor in the decomposition of $H^{(m^{n+a})}$. We will make use of the fact that, if $B \subseteq \{\overline{\mathbf{R}} \mid \mathbf{R} \in \mathcal{T}(\lambda, (m^n))\}$ is a basis for $H^{(m^n)}$, then the set $\mathscr{B} := \Big\{ \overline{\widetilde{\mathbf{R}}} \;\Big|\; \overline{\mathbf{R}} \in B \Big\} \subseteq H^{(m^{n+a})}$ is linearly independent. Indeed, for any $a \geq 2$, there exists a bijection $B \to \mathscr{B}$ defined on set partitions by

$$\{X_1, \ldots, X_n\} \mapsto \big\{X_1, \ldots, X_n, \{mn+1, \ldots, mn+m\}, \ldots, \{mn+(a-1)m+1, \ldots, mn+am\}\big\}.$$

Since elements of B are distinct, elements of $\mathscr{B}$ must also be distinct. Thus, the formal sum $\sum_{j=1}^{|B|} \beta_j \overline{\widetilde{\mathbf{R}}}_j$ is equal to zero only if $\beta_j = 0$ for all $1 \leq j \leq |B|$.

If S^λ appears in $H^{(m^n)}$ with multiplicity $r \geq 0$, then there is a basis $\{\overline{\theta}_{T_1}, \ldots, \overline{\theta}_{T_r}\}$ for $\mathrm{Hom}_{\mathbb{C}\mathfrak{S}_{mn}}\big(S^\lambda, H^{(m^n)}\big)$, where $T_i \in \mathcal{T}_0\big(\lambda, (m^n)\big)$ for all i. For a contradiction, assume that $\sum_{i=1}^{r} \alpha_i \overline{\theta}_{\widetilde{T}_i} = 0$ for some scalars α_i that are not all zero. It follows that $(e_{\widetilde{t}})\Big(\sum_{i=1}^{r} \alpha_i \overline{\theta}_{\widetilde{T}_i}\Big) = 0$ and so the coefficient of any $\overline{\widetilde{\mathbf{R}}}$ in $(e_{\widetilde{t}})\Big(\sum_{i=1}^{r} \alpha_i \overline{\theta}_{\widetilde{T}_i}\Big)$ is zero. If we let $\mathcal{C}_i$ denote the coefficient of $\overline{\widetilde{\mathbf{R}}}$ in $(e_{\widetilde{t}})\overline{\theta}_{\widetilde{T}_i}$, then $\sum_{i=1}^{r} \alpha_i \mathcal{C}_i = 0$. Applying the result in (5.13), $\mathcal{C}_i = a!(d+1)\mathscr{C}_i$, where d does not depend on i. Thus, the coefficient of $\overline{\mathbf{R}}$ in $(e_t)\big(\sum_{i=1}^{r} \alpha_i \overline{\theta}_{T_i}\big)$ is $\frac{1}{a!(d+1)} \sum_{i=1}^{r} \alpha_i \mathcal{C}_i$ and so is also zero. We chose $\overline{\widetilde{\mathbf{R}}}$ arbitrarily and so we have that for all $\mathbf{R} \in \mathcal{T}\big(\lambda, (m^n)\big)$, $\overline{\mathbf{R}}$ has coefficient zero in $(e_t)\left(\sum_{i=1}^{r}\alpha_i\overline{\theta}_{T_i}\right)$. Hence, $(e_t)\left(\sum_{i=1}^{r}\alpha_i\overline{\theta}_{T_i}\right)=0$. Since e_t is a generator for the Specht module S^λ, this implies that $\sum_{i=1}^{r}\alpha_i\overline{\theta}_{T_i}=0$, which contradicts the assumptions on $\{\alpha_i \mid 1\le i\le r\}$. This completes the proof of Theorem 5.1.2.

5.6 A prime characteristic theorem

Theorem 5.1.1 gives a description of composition factors of $H^{(m^{n+1})}$ and $K^{(m^{n+1})}$ over fields of characteristic zero. It is natural to ask whether anything can be said if instead we work over fields of prime characteristic p. As it happens, only minor changes need to be made to Theorem 5.1.1 to obtain such a result if $p>n+1$.

Theorem 5.6.1

Let k be a field of prime characteristic, $p>n+1$. Let $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_\ell)$ be a partition of mn and let $q\in\mathbb{N}_0$ be minimal such that $0\le\lambda_{q+1}\le m$. Define $\widetilde{\lambda}:=(\lambda_1,\dots,\lambda_q,m,\lambda_{q+1},\dots,\lambda_\ell)$ if $q\neq 0$ and $\widetilde{\lambda}:=(m,\lambda_1,\dots,\lambda_\ell)$ if $q=0$, so that $\widetilde{\lambda}$ is a partition of $m(n+1)$. For $T\in\mathcal{T}_0\big(\lambda,(m^n)\big)$, define $\widetilde{T}$ as in (5.1).

1. *Suppose that m is even. If $\overline{\theta}_T$ is a non-zero element of $\mathrm{Hom}_{k\mathfrak{S}_{mn}}\left(S^\lambda,H^{(m^n)}\right)$, then $\overline{\theta}_{\widetilde{T}}\in\mathrm{Hom}_{k\mathfrak{S}_{mn}}\left(S^{\widetilde{\lambda}},H^{(m^{n+1})}\right)$ is non-zero.*
2. *Suppose that m is odd. If $\underline{\theta}_T$ is a non-zero element of $\mathrm{Hom}_{k\mathfrak{S}_{mn}}\left(S^\lambda,K^{(m^n)}\right)$, then $\underline{\theta}_{\widetilde{T}}\in\mathrm{Hom}_{k\mathfrak{S}_{mn}}\left(S^{\widetilde{\lambda}},K^{(m^{n+1})}\right)$ is non-zero.*

The requirement that $p>n+1$ is far from desirable. However, since $n\in\mathbb{N}$, we avoid the often troublesome case $p=2$. Further, with this condition, the proofs given in §5.3 and §5.4 are sufficient to prove Theorem 5.6.1. Indeed, the only additional thing that we really need to check is that $d+1$ remains non-zero, but this does not pose us a problem. Recall that d is defined to be the number of digits in the set $\{1,2,\dots,n\}$ which appear in precisely columns $1,2,\dots,m$ of $\mathbf{R}$. The cardinality of the set guarantees that $d\le n$, from which we can deduce that $d+1<p$ and so cannot be equal to zero. Straightforward modular arithmetic then tells us that $(d+1)\sum\mathrm{sgn}(\pi)\neq 0$, and so the result still holds.

It is also possible to prove a result similar to Theorem 5.1.2 over fields of prime characteristic, making some minor adjustments as indicated above.

Chapter 6

The smallest non-even constituent of the Foulkes character $\varphi^{(4^n)}$

For any $m, n \in \mathbb{N}$, there is a combinatorial description of the minimal partitions (in the dominance order) that label irreducible summands of $H^{(m^n)}$ [40]. When m is even, there is a unique minimal constituent of $H^{(m^n)}$ – which is, of course, minimal with respect to the lexicographic order – and this is labelled by the partition (m^n). However, in general, these are the only 'small' constituents of $H^{(m^n)}$ that can be completely described. In this chapter, we build on the knowledge about minimal constituents and investigate lexicographically small constituents of Foulkes modules when m is even.

We will prove that no partition with first part equal to $m+1$ can label a constituent of the ordinary Foulkes character $\varphi^{(m^n)}$ if m is even. Focusing on the case $m = 4$, we will also prove that if $\lambda = (\lambda_1, \ldots, \lambda_\ell) \vdash 4n$ labels a summand of $\varphi^{(4^n)}$, then either $\lambda_1 \geq 7$ or λ has all parts even. Finally, we will give a complete description of constituents of $\varphi^{(4^n)}$ that have $\lambda_1 = 7$ and $\lambda_2 < 7$. As a consequence, we prove that when $n \geq 3$, the minimal (lexicographically ordered) summand which has an odd part is labelled by $(7, 4^{n-2}, 1)$.

In the course of this chapter, we will also prove some new results about generalised Foulkes characters. Of particular note is Theorem 6.2.6, which gives a formula for calculating multiplicities of constituents of any generalised Foulkes character $\varphi_\nu^{(m^n)}$.

6.1 Preliminaries

In this section, we present some general results, which will be useful in §6.3. We first need a preliminary lemma.

Lemma 6.1.1

There is only one double coset of $\mathfrak{S}_{mn-1}$ and $\mathfrak{S}_m \wr \mathfrak{S}_n$ in $\mathfrak{S}_{mn}$.

Proof. If we let e denote the identity element of $\mathfrak{S}_{mn}$, then the number of elements in the double coset $(\mathfrak{S}_m \wr \mathfrak{S}_n)\, e\, (\mathfrak{S}_{mn-1})$ is

$$\begin{aligned} |\,(\mathfrak{S}_m \wr \mathfrak{S}_n)\, e\, (\mathfrak{S}_{mn-1})\,| &= |\,(\mathfrak{S}_m \wr \mathfrak{S}_n)\,(\mathfrak{S}_{mn-1})\,| \\ &= \frac{|\mathfrak{S}_m \wr \mathfrak{S}_n||\mathfrak{S}_{mn-1}|}{|\,(\mathfrak{S}_m \wr \mathfrak{S}_n) \cap \mathfrak{S}_{mn-1}|}. \end{aligned}$$

Observing that $|\mathfrak{S}_m \wr \mathfrak{S}_n| = n!(m!)^n$ and

$$|\,(\mathfrak{S}_m \wr \mathfrak{S}_n) \cap \mathfrak{S}_{mn-1}| = |\,(\mathfrak{S}_m \wr \mathfrak{S}_{n-1}) \times \mathfrak{S}_{m-1}| = |\mathfrak{S}_m \wr \mathfrak{S}_{n-1}||\mathfrak{S}_{m-1}|,$$

a simple computation yields

$$|\,(\mathfrak{S}_m \wr \mathfrak{S}_n)\, e\, (\mathfrak{S}_{mn-1})\,| = \frac{n!(m!)^n(mn-1)!}{(n-1)!(m!)^{n-1}(m-1)!} = (mn)! = |\mathfrak{S}_{mn}|.$$

Thus, we conclude that $|(\mathfrak{S}_m \wr \mathfrak{S}_n)\backslash\mathfrak{S}_{mn}/\mathfrak{S}_{mn-1}| = 1$. ∎

With this lemma, we are able to prove a generalisation of the 'central observation' needed for the proof of Lemma 1 in [24], namely Proposition 6.1.2. Subsequently, we will state a corollary, detailing two important special cases of Proposition 6.1.2, which will be used repeatedly in this chapter.

Proposition 6.1.2

Let $m, n \in \mathbb{N}$ and let ν be a partition of n. Define the set

$$\mathrm{P} := \{\rho \vdash n-1 \mid [\rho] \textit{ is obtained by removing a removable node from } [\nu]\}.$$

If $\varphi_\nu^{(m^n)}$ denotes the character afforded by the generalised Foulkes module $H_\nu^{(m^n)}$, then

$$\varphi_\nu^{(m^n)}\big\downarrow_{\mathfrak{S}_{mn-1}} = \sum_{\rho \in \mathrm{P}} \left(\varphi_\rho^{(m^{n-1})} \times 1_{\mathfrak{S}_{m-1}}\right)\Big\uparrow_{\mathfrak{S}_{m(n-1)}\times\mathfrak{S}_{m-1}}^{\mathfrak{S}_{mn-1}},$$

where $1_{\mathfrak{S}_{m-1}}$ is the trivial character of the symmetric group $\mathfrak{S}_{m-1}$.

Proof. The character $\varphi_\nu^{(m^n)}\big\downarrow_{\mathfrak{S}_{mn-1}}$ is afforded by the module

$$H_\nu^{(m^n)}\big\downarrow_{\mathfrak{S}_{mn-1}} = \left(T^{(n)}(k_{\mathfrak{S}_m}) \oslash S^\nu\right)\Big\uparrow_{\mathfrak{S}_m\wr\mathfrak{S}_n}^{\mathfrak{S}_{mn}}\Big\downarrow_{\mathfrak{S}_{mn-1}}.$$

Using Mackey's Theorem, together with Lemma 6.1.1, we conclude that

$$\begin{aligned} H_\nu^{(m^n)}\big\downarrow_{\mathfrak{S}_{mn-1}} &\cong \left(T^{(n)}(k_{\mathfrak{S}_m}) \oslash S^\nu\right)\Big\downarrow_{(\mathfrak{S}_m\wr\mathfrak{S}_n)\cap\mathfrak{S}_{mn-1}}\Big\uparrow^{\mathfrak{S}_{mn-1}} \\ &= \left(T^{(n)}(k_{\mathfrak{S}_m}) \oslash S^\nu\right)\Big\downarrow_{(\mathfrak{S}_m\wr\mathfrak{S}_{n-1})\times\mathfrak{S}_{m-1}}\Big\uparrow^{\mathfrak{S}_{mn-1}} \\ &= \left(\left(T^{(n-1)}(k_{\mathfrak{S}_m}) \oslash (S^\nu)\downarrow_{\mathfrak{S}_{n-1}}\right) \boxtimes k_{\mathfrak{S}_{m-1}}\right)\Big\uparrow^{\mathfrak{S}_{mn-1}}. \end{aligned}$$

Consequently, an application of the Branching Rule yields

$$H_\nu^{(m^n)}\big\downarrow_{\mathfrak{S}_{mn-1}} \cong \left(\left(\bigoplus_{\rho\in\mathrm{P}} T^{(n-1)}(k_{\mathfrak{S}_m}) \oslash S^\rho\right) \boxtimes k_{\mathfrak{S}_{m-1}}\right)\Bigg\uparrow_{(\mathfrak{S}_m\wr\mathfrak{S}_{n-1})\times\mathfrak{S}_{m-1}}^{\mathfrak{S}_{mn-1}},$$

from which, using the properties of induction stated in Lemma 2.1.1, it follows that

$$\begin{aligned}
H_\nu^{(m^n)}\big\downarrow_{\mathfrak{S}_{mn-1}} &\cong \bigoplus_{\rho\in\mathrm{P}} \left(\left(T^{(n-1)}(k_{\mathfrak{S}_m}) \oslash S^\rho\right) \boxtimes k_{\mathfrak{S}_{m-1}}\right)\Big\uparrow_{(\mathfrak{S}_m\wr\mathfrak{S}_{n-1})\times\mathfrak{S}_{m-1}}^{\mathfrak{S}_{m(n-1)}\times\mathfrak{S}_{m-1}}\Big\uparrow_{\mathfrak{S}_{m(n-1)}\times\mathfrak{S}_{m-1}}^{\mathfrak{S}_{mn-1}} \\
&= \bigoplus_{\rho\in\mathrm{P}} \left(\left(T^{(n-1)}(k_{\mathfrak{S}_m}) \oslash S^\rho\right)\Big\uparrow_{\mathfrak{S}_m\wr\mathfrak{S}_{n-1}}^{\mathfrak{S}_{m(n-1)}} \boxtimes k_{\mathfrak{S}_{m-1}}\right)\Big\uparrow_{\mathfrak{S}_{m(n-1)}\times\mathfrak{S}_{m-1}}^{\mathfrak{S}_{mn-1}} \\
&= \bigoplus_{\rho\in\mathrm{P}} \left(H_\rho^{(m^{n-1})} \boxtimes k_{\mathfrak{S}_{m-1}}\right)\Big\uparrow_{\mathfrak{S}_{m(n-1)}\times\mathfrak{S}_{m-1}}^{\mathfrak{S}_{mn-1}}.
\end{aligned}$$

The statement of the proposition is precisely the relationship satisfied by the ordinary characters afforded by these modules. ■

Corollary 6.1.3

In the setting of Proposition 6.1.2,

1. *if $\nu = (n)$, so that $\varphi_\nu^{(m^n)}$ is the Foulkes character $\varphi^{(m^n)}$, then*

$$\varphi^{(m^n)}\big\downarrow_{\mathfrak{S}_{mn-1}} = \left(\varphi^{(m^{n-1})} \times 1_{\mathfrak{S}_{m-1}}\right)\Big\uparrow^{\mathfrak{S}_{mn-1}};$$

2. *if $\nu = (1^n)$, so that $\varphi_\nu^{(m^n)}$ is the twisted Foulkes character $\tau^{(m^n)}$, then*

$$\tau^{(m^n)}\big\downarrow_{\mathfrak{S}_{mn-1}} = \left(\tau^{(m^{n-1})} \times 1_{\mathfrak{S}_{m-1}}\right)\Big\uparrow^{\mathfrak{S}_{mn-1}}.$$

We will now prove a result which says that, if m is even, no partition with first part equal to $m+1$ can label a constituent of the Foulkes character $\varphi^{(m^n)}$. An analogous result for twisted Foulkes characters is also given. However, we will only give a detailed proof of the result in the Foulkes setting, since the twisted Foulkes result is very similar; at the end of the proof, we indicate the modifications that need to be made to prove the second result.

Proposition 6.1.4

Fix an even natural number, m. Given any $n \in \mathbb{N}$, if $\lambda \vdash mn$ such that $\lambda_1 = m+1$, then

$$\left\langle \varphi^{(m^n)}, \chi^\lambda \right\rangle = 0.$$

If instead we fix m to be an odd natural number, then, under the same assumptions on λ,

$$\left\langle \tau^{(m^n)}, \chi^\lambda \right\rangle = 0.$$

Proof. We proceed inductively to prove the first part of the proposition.

The proposition clearly holds when $n = 1$: in this case, we simply have $\varphi^{(m^1)} = \chi^{(m)}$. For some $n = r-1 \geq 1$, assume that no constituent of $\varphi^{(m^{r-1})}$ is labelled by a partition whose first part is equal to $m+1$. Consider $\varphi^{(m^r)}$ and let $\lambda = \big((m+1)^i, \lambda_{i+1}, \ldots, \lambda_\ell\big) \vdash mr$, where $i \in \mathbb{N}$ is as large as possible.

If λ labels a constituent of $\varphi^{(m^r)}$, then all constituents of $\chi^\lambda\big\downarrow_{\mathfrak{S}_{mr-1}}$ appear in $\varphi^{(m^r)}\big\downarrow_{\mathfrak{S}_{mr-1}}$. By part 1 of Corollary 6.1.3,

$$\varphi^{(m^r)}\big\downarrow_{\mathfrak{S}_{mr-1}} = \left(\varphi^{(m^{r-1})} \times 1_{\mathfrak{S}_{m-1}}\right)\Big\uparrow^{\mathfrak{S}_{mr-1}}.$$

So, our inductive assumption, together with the fact that $\left\langle \varphi^{(m^{r-1})}, \chi^{(m^{r-1})}\right\rangle = 1$, allows us to conclude that

$$\varphi^{(m^r)}\big\downarrow_{\mathfrak{S}_{mr-1}} = \left(\left(\chi^{(m^{r-1})} + \sum_{\mu \in \mathrm{M}} a_\mu \chi^\mu\right) \times 1_{\mathfrak{S}_{m-1}}\right)\Bigg\uparrow^{\mathfrak{S}_{mr-1}},$$

where M is the set of partitions which label constituents of $\varphi^{(m^{r-1})}$ and have first part at least $m+2$, and a_μ denotes the multiplicity with which χ^μ appears in $\varphi^{(m^{r-1})}$.

Young's Rule tells us how to decompose $\big(\chi^{(m^{r-1})} \times 1_{\mathfrak{S}_{m-1}}\big)\big\uparrow^{\mathfrak{S}_{mr-1}}$, yielding

$$\varphi^{(m^r)}\big\downarrow_{\mathfrak{S}_{mr-1}} = \sum_{b=0}^{m-1} \chi^{\left(m+b, m^{r-2}, m-1-b\right)} + \left(\sum_{\mu \in \mathrm{M}} a_\mu \chi^\mu \times 1_{\mathfrak{S}_{m-1}}\right)\Bigg\uparrow^{\mathfrak{S}_{mr-1}}.$$

Observe that $\varphi^{(m^r)}\big\downarrow_{\mathfrak{S}_{mr-1}}$ has a unique constituent with first part m, namely $\chi^{(m^{r-1}, m-1)}$, and a unique constituent with first part $m+1$, which is $\chi^{(m+1, m^{r-2}, m-2)}$. We have three cases to consider.

Case I ($i = 1$): In this case, $\lambda = (m+1, \lambda_2, \ldots, \lambda_\ell)$ with $\lambda_2 \leq m$. An application of the Branching Rule shows that the only constituent appearing in $\chi^\lambda\big\downarrow_{\mathfrak{S}_{mr-1}}$ that has first part equal to m is $\chi^{(m, \lambda_2, \ldots, \lambda_\ell)}$. It follows that $(m, \lambda_2, \ldots, \lambda_\ell) = (m^{r-1}, m-1)$ and therefore that $\ell = r$, $\lambda_i = m$ for all $2 \leq i \leq \ell - 1$ and $\lambda_\ell = m-1$. Thus, $\lambda = (m+1, m^{r-2}, m-1)$.

Re-examining $\chi^\lambda\big\downarrow_{\mathfrak{S}_{mr-1}}$ with $r \neq 2$, we see that

$$\chi^\lambda\big\downarrow_{\mathfrak{S}_{mr-1}} = \chi^{\left(m^{r-1}, m-1\right)} + \chi^{\left(m+1, m^{r-3}, (m-1)^2\right)} + \chi^{\left(m+1, m^{r-2}, m-2\right)},$$

but $\chi^{\left(m+1, m^{r-3}, (m-1)^2\right)}$ does not appear in $\varphi^{(m^r)}\big\downarrow_{\mathfrak{S}_{mr-1}}$. Thus, $\langle \varphi^{(m^r)}, \chi^\lambda \rangle = 0$.

If $r = 2$, then $\lambda = (m+1, m-1) \vdash 2m$. It is immediate from the known decomposition of $\varphi^{(m^2)}$ that $\langle \varphi^{(m^2)}, \chi^{(m+1, m-1)} \rangle = 0$, because all irreducible constituents in the decomposition are labelled by partitions that have two parts, which are both even.

Case II ($i = 2$): In this case, $\lambda = ((m+1)^2, \lambda_3, \ldots, \lambda_\ell)$. If $\lambda_3 \neq 0$, then, applying the Branching Rule, we see that $\chi^{((m+1)^2, \lambda_3, \ldots, \lambda_{\ell-1}, \lambda_\ell - 1)}$ is a constituent of $\chi^\lambda \big\downarrow_{\mathfrak{S}_{mr-1}}$, but not a constituent of $\varphi^{(m^r)} \big\downarrow_{\mathfrak{S}_{mr-1}}$. Thus, $\left\langle \varphi^{(m^r)}, \chi^\lambda \right\rangle = 0$.

If $\lambda_3 = 0$, then $\lambda = (m+1, m+1) \vdash mr$ for some $r \geq 2$. It follows that this is the case $m = 2$ and $r = 3$, but we know that $\left\langle \varphi^{(2^3)}, \chi^{(3,3)} \right\rangle = 0$.

Case III ($i \geq 3$): In this case, all constituents of $\chi^\lambda \big\downarrow_{\mathfrak{S}_{mr-1}}$ are labelled by partitions which have at least their first two parts equal to $m + 1$. Since no constituent of $\varphi^{(m^r)} \big\downarrow_{\mathfrak{S}_{mr-1}}$ has this property, we are forced to conclude that $\left\langle \varphi^{(m^r)}, \chi^\lambda \right\rangle = 0$.

In all three cases we saw that $\left\langle \varphi^{(m^r)}, \chi^\lambda \right\rangle = 0$. Thus, no constituent of $\varphi^{(m^r)}$ is labelled by a partition with first part equal to $m + 1$. The result follows by induction.

The proof of the second part of the proposition, concerning the twisted Foulkes character $\tau^{(m^n)}$, is proved in an entirely similar manner, taking m to be odd and using part 2 of Corollary 6.1.3 instead of part 1. ■

In §6.3, we will be turning our attention to constituents of $\varphi^{(4^n)}$ that are labelled by partitions whose first part is equal to six. In light of Proposition 6.1.4, if we are to be methodical then these are the next constituents of $\varphi^{(4^n)}$ in the lexicographic order that we would like to be able to describe. It is already possible to say something about such constituents. Indeed, recall Weintraub's Conjecture, presented in Chapter 3, which asserts that if m is even, then $\left\langle \varphi^{(m^n)}, \chi^{2\lambda} \right\rangle \neq 0$ for any partition λ of $mn/2$.

6.2 A recursive formula for generalised Foulkes characters

In §6.3, we will sharpen Weintraub's Conjecture in the case where $m = 4$ and χ^λ is labelled by a partition of $4n$ that has first part equal to six. To do this, we will rely heavily on Lemma 6.2.8, a recursion formula due to Evseev, Paget and Wildon, which arises as a corollary of the main theorem in their paper [13, Theorem 1.5]. We will, in fact, prove a more general version of this lemma, obtaining a recursive formula with which multiplicities of constituents of any generalised Foulkes character $\varphi_\nu^{(m^n)}$ can be computed. We need a few definitions, which we shall give exactly as in [13], before we can state the result.

If λ/μ is a skew-partition, then we define a *border strip tableau of shape* λ/μ to be an assignment of the elements of a set $J \subseteq \mathbb{N}$ to the boxes of the Young diagram of λ/μ so that the entries in the rows and columns are non-decreasing, and for each $j \in J$, the boxes labelled j form a border strip; if $J = \{1, \ldots, r\}$, and for each $j \in J$ the border strip formed by the boxes labelled j has length α_j, then we say that the tableau has *type* $(\alpha_1, \ldots, \alpha_r)$. Recall that the height $\langle \lambda/\mu \rangle$ of λ/μ is defined to be one less than the number of its non-empty rows.

Definition 6.2.1

Let T be a border strip tableau. The sign *of T is defined by* $\mathrm{sgn}(T) = (-1)^h$*, where h is the sum of the heights of the border strips forming T.*

Definition 6.2.2

Let λ/μ be a border strip in a partition λ. If the lowest-numbered row of λ met by λ/μ is row r then we define the first row number *of λ/μ to be r, and write $N(\lambda/\mu) = r$.*

In the next definition it is useful to note that if T is a border strip tableau of shape λ/μ and type $(\alpha_1, \ldots, \alpha_r)$, then there are partitions

$$\mu = \lambda^0 \subset \lambda^1 \subset \cdots \subset \lambda^{r-1} \subset \lambda^r = \lambda$$

such that for each $j \in \{1, \ldots, r\}$, the border strip in T labelled j is λ^j/λ^{j-1}.

Definition 6.2.3

Let $m, n \in \mathbb{N}$ and let λ/μ be a skew-partition of mn. Given a composition $\gamma = (\gamma_1, \ldots, \gamma_d)$ of n, let $\gamma^{\star m} = (\gamma_1, \ldots, \gamma_1, \gamma_2, \ldots, \gamma_2, \ldots, \gamma_d, \ldots, \gamma_d)$ denote the composition of mn obtained from γ by repeating each part m times. An m-border strip tableau of shape λ/μ and type γ *is a border strip tableau of shape λ/μ and type $\gamma^{\star m}$ such that for each $j \in \{1, \ldots, d\}$, the first row numbers of the border strips*

$$\lambda^{(j-1)m+1}/\lambda^{(j-1)m}, \ldots, \lambda^{jm}/\lambda^{jm-1},$$

corresponding to the m parts in $\gamma^{\star m}$ of length γ_j, satisfy

$$N\big(\lambda^{(j-1)m+1}/\lambda^{(j-1)m}\big) \geq \cdots \geq N\big(\lambda^{jm}/\lambda^{jm-1}\big). \tag{6.1}$$

Example 6.2.4

Let $\lambda = (5, 4, 3, 3, 1)$, $\mu = (3, 2, 1)$, $\gamma = (1, 2, 2)$, $n = 5$ and $m = 2$. There is only one 2-border strip tableau of shape $(5, 4, 3, 3, 1)/(3, 2, 1)$ and type $(1, 2, 2)$, which is

$$T = \begin{ytableau} \none & \none & \none & 1 & 2 \\ \none & \none & 4 & 4 \\ \none & 5 & 6 \\ 3 & 5 & 6 \\ 3 \end{ytableau}\ .$$

In this case, we have $N\big(\lambda^5/\lambda^4\big) = 4 = N\big(\lambda^6/\lambda^5\big)$, $N\big(\lambda^3/\lambda^2\big) = 5 \geq 2 = N\big(\lambda^4/\lambda^3\big)$ and $N\big(\lambda^1/\lambda^0\big) = 1 = N\big(\lambda^2/\lambda^1\big)$, so condition (6.1) is satisfied. The heights of the border strips labelled 1, 2, 3, 4, 5, 6 are 0, 0, 1, 0, 1, 1, respectively, and so

$$\mathrm{sgn}(T) = (-1)^3 = -1.$$

We need one final definition before we state the result.

Definition 6.2.5

Define the ℓ-sign *of* λ/μ*, denoted by* $\varepsilon_\ell(\lambda/\mu)$*, to be the sign of any* m*-border strip tableau of shape* λ/μ *and type* (ℓ)*.*

Remark. There is at most one m-border strip tableau of shape λ/μ and type (ℓ) (see [13, p.15]).

The next result is the main result of this section: a recursive formula is given, which allows character multiplicities to be computed for any generalised Foulkes character $\varphi_\nu^{(m^n)}$ that is labelled by a partition ν of n. This result is a generalisation of the recursive formula for ordinary Foulkes characters, proved by Evseev, Paget and Wildon in [13]. As such, the proof of Theorem 6.2.6 is modelled on the proof of Proposition 5.1 in [13].

Theorem 6.2.6

Let $m, n \in \mathbb{N}$ *and let* ν *be a partition of* n*. If* $\lambda \vdash mn$*, then*

$$\left\langle \varphi_\nu^{(m^n)}, \chi^\lambda \right\rangle = \frac{1}{n} \sum_{\ell=1}^{n} \sum_{\mu} \varepsilon_\ell(\lambda/\mu) \sum_{\rho} (-1)^{\langle \nu/\rho \rangle} \left\langle \varphi_\rho^{(m^{n-\ell})}, \chi^\mu \right\rangle$$

where we sum over partitions $\mu \subseteq \lambda$ *for which there exists an* m*-border strip tableau of shape* λ/μ *and type* (ℓ)*, and we sum over* ρ *such that* ν/ρ *is a rim* ℓ*-hook.*

Remark. In general, the calculations involved in the computation of a character multiplicity $\langle \varphi_\nu^{(m^n)}, \chi^\lambda \rangle$ will be lengthy. Therefore, we should not rely on such a formula for obtaining complete decompositions of $\varphi_\nu^{(m^n)}$, especially for large m and n. However, we shall see in the next section that Theorem 6.2.6 can still be very useful for determining the multiplicity of a particular irreducible character in the decomposition of $\varphi_\nu^{(m^n)}$.

Proof. If $\lambda \vdash mn$ and $\nu \vdash n$, then using Frobenius Reciprocity followed by inflation-deflation reciprocity [13, Equation (11)], we have that

$$\begin{aligned} \left\langle \varphi_\nu^{(m^n)}, \chi^\lambda \right\rangle &= \left\langle \left(\operatorname{Inf}_{\mathfrak{S}_n}^{\mathfrak{S}_m \wr \mathfrak{S}_n} \chi^\nu \right) \uparrow_{\mathfrak{S}_m \wr \mathfrak{S}_n}^{\mathfrak{S}_{mn}}, \chi^\lambda \right\rangle \\ &= \left\langle \operatorname{Inf}_{\mathfrak{S}_n}^{\mathfrak{S}_m \wr \mathfrak{S}_n} \chi^\nu, \chi^\lambda \downarrow_{\mathfrak{S}_m \wr \mathfrak{S}_n}^{\mathfrak{S}_{mn}} \right\rangle \\ &= \left\langle \chi^\nu, \operatorname{Def}_{\mathfrak{S}_n} \left(\chi^\lambda \downarrow_{\mathfrak{S}_m \wr \mathfrak{S}_n}^{\mathfrak{S}_{mn}} \right) \right\rangle. \end{aligned}$$

Since $\operatorname{Def}_{\mathfrak{S}_n} \left(\chi^\lambda \downarrow_{\mathfrak{S}_m \wr \mathfrak{S}_n}^{\mathfrak{S}_{mn}} \right) = \operatorname{Defres}_{\mathfrak{S}_n} \chi^\lambda$ (by the definition of $\operatorname{Defres}_{\mathfrak{S}_n}$ given in [13, p.3]), it follows that

$$\begin{aligned} \left\langle \varphi_\nu^{(m^n)}, \chi^\lambda \right\rangle &= \left\langle \chi^\nu, \operatorname{Defres}_{\mathfrak{S}_n} \chi^\lambda \right\rangle \\ &= \frac{1}{|\mathfrak{S}_n|} \sum_{g \in \mathfrak{S}_n} \left(\operatorname{Defres}_{\mathfrak{S}_n} \chi^\lambda \right)(g)\, \overline{\chi^\nu(g)}. \end{aligned}$$

For any $g \in \mathfrak{S}_n$, g and g^{-1} lie in the same conjugacy class and so $\overline{\chi^\nu(g)} = \chi^\nu(g^{-1}) = \chi^\nu(g)$. Furthermore, we may write $g \in \mathfrak{S}_n$ as the product of an ℓ-cycle containing the symbol 1, say $x \in \mathfrak{S}_\ell$, and some $h \in \mathfrak{S}_{n-\ell}$ which acts on the remaining $n-\ell$ symbols. Since the number of possible ℓ-cycles is $(n-1)!/(n-\ell)!$, we deduce that

$$\left\langle \varphi_\nu^{(m^n)}, \chi^\lambda \right\rangle = \frac{1}{n!} \sum_{\ell=1}^{n} \frac{(n-1)!}{(n-\ell)!} \sum_{h \in \mathfrak{S}_{n-\ell}} \left(\mathrm{Defres}_{\mathfrak{S}_n} \chi^\lambda\right)(xh)\chi^\nu(xh),$$

where $x \in \mathfrak{S}_\ell$ is an ℓ-cycle. Applying Proposition 4.5 of [13] – which says that

$$\left(\mathrm{Defres}_{\mathfrak{S}_n} \chi^\lambda\right)(xh) = \sum_{\mu} \left(\mathrm{Defres}_{\mathfrak{S}_\ell} \chi^{\lambda/\mu}\right)(x) \left(\mathrm{Defres}_{\mathfrak{S}_{n-\ell}} \chi^{\mu}\right)(h),$$

where the sum is over $\mu \subseteq \lambda$ with $|\lambda/\mu| = m\ell$ – gives

$$\left\langle \varphi_\nu^{(m^n)}, \chi^\lambda \right\rangle = \frac{1}{n} \sum_{\ell=1}^{n} \frac{1}{(n-\ell)!} \sum_{h \in \mathfrak{S}_{n-\ell}} \sum_{\mu} \left(\mathrm{Defres}_{\mathfrak{S}_\ell} \chi^{\lambda/\mu}\right)(x) \left(\mathrm{Defres}_{\mathfrak{S}_{n-\ell}} \chi^{\mu}\right)(h)\chi^\nu(xh).$$

Now, using the Murnaghan–Nakayama Rule to decompose $\chi^\nu(xh)$ as $\sum_\rho (-1)^{\langle \nu/\rho \rangle}\chi^\rho(h)$ yields

$$\left\langle \varphi_\nu^{(m^n)}, \chi^\lambda \right\rangle = \frac{1}{n} \sum_{\ell=1}^{n} \sum_{h \in \mathfrak{S}_{n-\ell}} \sum_{\mu} \sum_{\rho} \frac{(-1)^{\langle \nu/\rho \rangle}}{(n-\ell)!} \big(\mathrm{Defres}_{\mathfrak{S}_\ell} \chi^{\lambda/\mu}\big)(x) \big(\mathrm{Defres}_{\mathfrak{S}_{n-\ell}} \chi^{\mu}\big)(h)\, \chi^\rho(h),$$

where the third sum is over partitions $\mu \subseteq \lambda$ with $|\lambda/\mu| = m\ell$, and the fourth sum is over ρ such that ν/ρ is a rim ℓ-hook. Proposition 4.3 of [13] shows that

$$\big(\mathrm{Defres}_{\mathfrak{S}_\ell} \chi^{\lambda/\mu}\big)(x) = \begin{cases} \varepsilon_\ell(\lambda/\mu) & \text{if } \exists \text{ an } m\text{-border strip tableau of shape } \lambda/\mu \text{ and type } (\ell); \\ 0 & \text{otherwise,} \end{cases}$$

and thus,

$$\begin{aligned} \left\langle \varphi_\nu^{(m^n)}, \chi^\lambda \right\rangle &= \frac{1}{n} \sum_{\ell=1}^{n} \sum_{\mu} \varepsilon_\ell(\lambda/\mu) \sum_{\rho} \frac{(-1)^{\langle \nu/\rho \rangle}}{(n-\ell)!} \sum_{h \in \mathfrak{S}_{n-\ell}} \big(\mathrm{Defres}_{\mathfrak{S}_{n-\ell}} \chi^{\mu}\big)(h)\chi^\rho(h) \\ &= \frac{1}{n} \sum_{\ell=1}^{n} \sum_{\mu} \varepsilon_\ell(\lambda/\mu) \sum_{\rho} (-1)^{\langle \nu/\rho \rangle} \left\langle \varphi_\rho^{(m^{n-\ell})}, \chi^\mu \right\rangle, \end{aligned}$$

where, as above, the second sum is over $\mu \subseteq \lambda$ for which there exists an m-border strip tableau of shape λ/μ and type (ℓ), and the third sum is over ρ such that ν/ρ is a rim ℓ-hook. ■

We illustrate the application of the formula with the following example.

Example 6.2.7

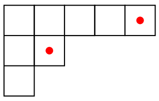
Let $m = 2$ and $n = 4$. If $\lambda = (5,2,1)$ and $\nu = (3,1)$, then

$$\left\langle \varphi_{(3,1)}^{(2^4)}, \chi^{(5,2,1)} \right\rangle = \frac{1}{4} \sum_{\ell=1}^{4} \sum_{\mu} \varepsilon_\ell\big((5,2,1)/\mu\big) \sum_{\rho} (-1)^{\langle (3,1)/\rho \rangle} \left\langle \varphi_\rho^{(2^{4-\ell})}, \chi^\mu \right\rangle,$$

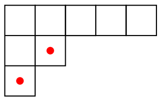
where we sum over partitions $\mu \subseteq (5,2,1)$ for which there exists a 2-border strip tableau of shape $(5,2,1)/\mu$ and type (ℓ), and we sum over ρ such that $(3,1)/\rho$ is a rim ℓ-hook.

We look to remove two ℓ-hooks from $[5,2,1]$ such that the first row numbers are weakly increasing and the remaining Young diagram has at most $4-\ell$ rows.

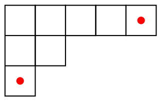
There are four ways to remove two 1-hooks.



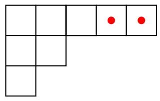
$\mu = (4,1,1)$, $\varepsilon_1(\lambda/\mu) = 1$



$\mu = (5,1)$, $\varepsilon_1(\lambda/\mu) = 1$

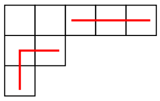


$\mu = (4,2)$, $\varepsilon_1(\lambda/\mu) = 1$



$\mu = (3,2,1)$, $\varepsilon_1(\lambda/\mu) = 1$

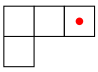
There is no way to remove two 2-hooks from $[5,2,1]$ and leave a proper partition. However, there is one way in which two 3-hooks may be removed.



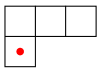
$\mu = (2)$,
$\varepsilon_3(\lambda/\mu) = -1$

There is also no way to remove two 4-hooks from $[5,2,1]$ and leave a proper partition.

We now need to determine those ρ for which $(3,1)/\rho$ is a rim ℓ-hook. We only need to concern ourselves with $\ell = 1$ and $\ell = 3$, but it is clear that there are no partitions ρ for which $(3,1)/\rho$ is a rim 3-hook. The following diagrams detail the possibilities when $\ell = 1$.



$\rho = (2,1)$, $\langle (3,1)/(2,1)\rangle = 0$



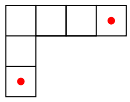
$\rho = (3)$, $\langle (3,1)/(3)\rangle = 0$

Thus,

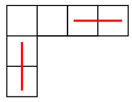
$$\left\langle \varphi^{(2^4)}_{(3,1)}, \chi^{(5,2,1)} \right\rangle = \frac{1}{4}\Big(\left\langle \varphi^{(2^3)}_{(2,1)}, \chi^{(4,1,1)} \right\rangle + \left\langle \varphi^{(2^3)}_{(2,1)}, \chi^{(5,1)} \right\rangle + \left\langle \varphi^{(2^3)}_{(2,1)}, \chi^{(4,2)} \right\rangle + \left\langle \varphi^{(2^3)}_{(2,1)}, \chi^{(3,2,1)} \right\rangle$$
$$+ \left\langle \varphi^{(2^3)}_{(3)}, \chi^{(4,1,1)} \right\rangle + \left\langle \varphi^{(2^3)}_{(3)}, \chi^{(5,1)} \right\rangle + \left\langle \varphi^{(2^3)}_{(3)}, \chi^{(4,2)} \right\rangle + \left\langle \varphi^{(2^3)}_{(3)}, \chi^{(3,2,1)} \right\rangle \Big). \quad (6.2)$$

At this point, the formula can be used repeatedly to compute character multiplicities that feature in the sum on the right-hand side of (6.2). For instance, consider $\left\langle \varphi^{(2^3)}_{(2,1)}, \chi^{(4,1,1)} \right\rangle$. We look to remove two ℓ-hooks from $[4,1,1]$ such that the first row numbers are weakly increasing and the remaining Young diagram has at most $3-\ell$ rows.

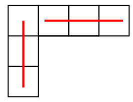
There is one way in which two ℓ-hooks may be removed for $\ell = 1, 2$ and 3.



$\mu = (3,1)$,
$\varepsilon_1\big((4,1^2)/\mu\big) = 1$

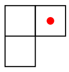


$\mu = (2)$,
$\varepsilon_2\big((4,1^2)/\mu\big) = -1$

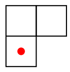


$\mu = \emptyset$,
$\varepsilon_3\big((4,1^2)/\mu\big) = 1$

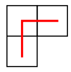
We now need to determine those ρ for which $(2,1)/\rho$ is a rim ℓ-hook. There are no partitions ρ for which $(2,1)/\rho$ is a rim 2-hook. The following diagrams detail the possibilities when $\ell = 1$ and $\ell = 3$.



$\rho = (1^2)$,
$\langle (2,1)/(1^2) \rangle = 0$



$\rho = (2)$,
$\langle (2,1)/(2) \rangle = 0$



$\rho = \emptyset$,
$\langle (2,1)/\emptyset \rangle = 1$

So, the multiplicity of $\chi^{(4,1,1)}$ in $\varphi^{(2^3)}_{(2,1)}$ is

$$\left\langle \varphi^{(2^3)}_{(3,1)}, \chi^{(4,1,1)} \right\rangle = \frac{1}{3}\left(\left\langle \varphi^{(2^2)}_{(1^2)}, \chi^{(3,1)} \right\rangle + \left\langle \varphi^{(2^2)}_{(2)}, \chi^{(3,1)} \right\rangle - \left\langle \varphi^{(2^0)}_{\emptyset}, \chi^{\emptyset} \right\rangle \right). \tag{6.3}$$

Of course, the multiplicities in the sum on the right-hand side of (6.3) can again be computed using the recursive formula. However, we know that $\varphi^{(2^2)}_{(1^2)} = \chi^{(3,1)}$ and $\varphi^{(2^2)}_{(2)} = \chi^{(4)} + \chi^{(2,2)}$. Also, by convention, $\left\langle \varphi^{(2^0)}_{\emptyset}, \chi^{\emptyset} \right\rangle = 1$ and so

$$\left\langle \varphi^{(2^3)}_{(3,1)}, \chi^{(4,1,1)} \right\rangle = \frac{1}{3}(1 + 0 - 1) = 0.$$

Computing the other multiplicities in the sum on the right-hand side of (6.2) in an entirely similar manner, we find that

$$\left\langle \varphi^{(2^4)}_{(3,1)}, \chi^{(5,2,1)} \right\rangle = \frac{1}{4}\big(0 + 0 + 1 + 0 + 1 + 1 + 1 + 0\big) = 1.$$

Setting $\nu = (n)$ in Theorem 6.2.6, we recover the recursive formula proved in [13], a statement of which is given in the following lemma.

Lemma 6.2.8

Let $m, n \in \mathbb{N}$. If λ is a partition of mn, then

$$\left\langle \varphi^{(m^n)}, \chi^{\lambda} \right\rangle = \frac{1}{n} \sum_{\ell=1}^{n} \sum_{\mu} \operatorname{sgn}(\lambda/\mu) \left\langle \varphi^{(m^{n-\ell})}, \chi^{\mu} \right\rangle, \tag{6.4}$$

where the second sum is over partitions μ of $m\ell$ such that there exists an m-border strip tableau of shape λ/μ and type (ℓ).

6.3 Constituents of $\varphi^{(4^n)}$ labelled by partitions with first part equal to six

In this section, we will focus on the case $m = 4$. We still need a few more results before we are able to say something about the smallest non-even constituent of $\varphi^{(4^n)}$. The first of these results addresses constituents of $\varphi^{(4^n)}$ that are labelled by partitions whose first part is equal to six. We can already expect constituents labelled by partitions of the form $\lambda = (6^a, 4^b, 2^c)$ (for $a, b, c \in \mathbb{N}_0$ satisfying $a + b + c \leq n$) to appear in the decomposition of $\varphi^{(4^n)}$; this is predicted by Weintraub's Conjecture. However, we prove that these are the only partitions with first part equal to six that label constituents appearing in $\varphi^{(4^n)}$ with non-zero multiplicity.

Proposition 6.3.1

Given any $n \in \mathbb{N}$, let λ be a partition of $4n$ with at most n parts and such that $\lambda_1 = 6$.

1. *If λ has an odd part, then $\left\langle \varphi^{(4^n)}, \chi^\lambda \right\rangle = 0$.*
2. *If λ has all parts even, then $\left\langle \varphi^{(4^n)}, \chi^\lambda \right\rangle = 1$.*

Proof. We proceed inductively.

When $n = 1$, the result is trivially true because the only constituent of $\varphi^{(4^1)}$ is the trivial character, $\chi^{(4)}$. For some $n = r - 1 \geq 1$, assume that the statements of the proposition hold.

We want to apply Lemma 6.2.8 to determine the multiplicity of constituents of $\varphi^{(4^r)}$ labelled by partitions of the form $\lambda = (6^a, 4^b, 2^c)$, where a, b, c are non-negative integers satisfying $a + b + c \leq r$. So, for any given triple (a, b, c) satisfying this condition, we seek the possible ways in which we can remove four rim hooks of length ℓ from $[\lambda]$ – from now on we will just write ℓ-hooks – such that the first row numbers are non-decreasing. Let $[\lambda^*]$ be the Young diagram that remains after the four ℓ-hooks have been removed.

If we remove the four ℓ-hooks from $[\lambda]$ in such a way as to leave an odd part, then our inductive assumptions tell us that $\left\langle \varphi^{(4^{r-\ell})}, \chi^{\lambda^*} \right\rangle = 0$. Similarly, if λ^* has in excess of $r - \ell$ parts, we know that $\left\langle \varphi^{(4^{r-\ell})}, \chi^{\lambda^*} \right\rangle = 0$. Hence, we only need to worry about the ways to remove four ℓ-hooks so that all parts of λ^* are even and the remaining number of parts is at most $r - \ell$, since these are the only partitions that make a non-zero contribution to the sum on the right-hand side of (6.4). Further, our inductive assumptions ensure that if λ^* has all parts even, then $\left\langle \varphi^{(4^{r-\ell})}, \chi^{\lambda^*} \right\rangle = 1$. To guarantee that all parts of λ^* are even and that the first row numbers are non-decreasing, we must always remove hooks 'in pairs'. By a pair, we mean two adjacent ℓ-hooks of the same height (see Figure 6.1).

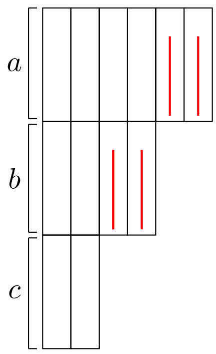
The only possible exceptions can occur when $\ell = 2$ (see the possibilities labelled (vii)–(ix) below). However, in each of these special cases, the two horizontal 2-hooks have height zero

Figure 6.1: A pair of ℓ-hooks (left) and two ℓ-hooks of the same height, which do not form a pair (right).

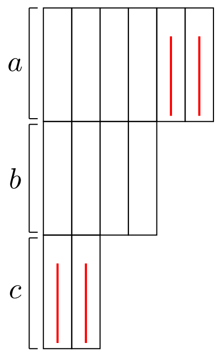
and so we may as well view them as a pair. We have just indicated the significance of 'pairing' up ℓ-hooks: both hooks in the pair have the same height. As a consequence, we always have $\mathrm{sgn}(\lambda/\lambda^*) = 1$. Thus, the task of computing $\left\langle \varphi^{(4^r)}, \chi^\lambda \right\rangle$ using Lemma 6.2.8 reduces to a straightforward combinatorial problem.

We can group the possible ways to remove four ℓ-hooks from $\left[6^a, 4^b, 2^c\right]$ so that no odd parts remain after their removal and so that the first row numbers are non-decreasing. There are ten such groups to be considered, labelled (i)–(x) below.

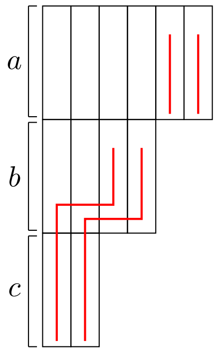
Note that we always insist that $a \neq 0$, so that λ has first part equal to six. Further, we will always be looking to satisfy the following *parts condition*: given $1 \leq \ell \leq r$, the Young diagram $[\lambda^*]$ that remains after removing the four ℓ-hooks must have at most $r - \ell$ rows. If the parts condition is not satisfied, then $\left\langle \varphi^{(4^r)}, \chi^{\lambda^*} \right\rangle = 0$.



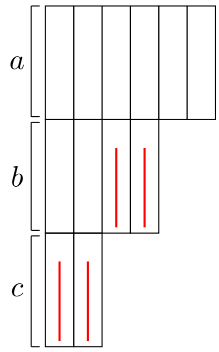
(i): This ℓ-hook removal is applicable for any $\ell \leq a$ whenever the parts condition $a + b + c \leq r - \ell$ is satisfied. The case $b < \ell$ is allowed.



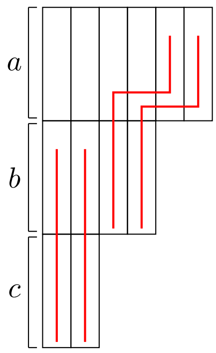
(ii): This ℓ-hook removal is applicable for any $\ell \leq c \neq 0$ whenever the parts condition $a + b + c \leq r$ is satisfied.



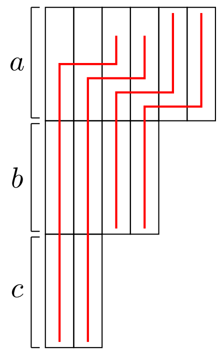
(iii): This ℓ-hook removal is applicable for any $c + 4 \leq \ell \leq a$ whenever the parts condition $a + b - 2 \leq r - \ell$ is satisfied.



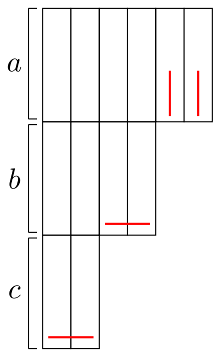
(iv): This ℓ-hook removal is applicable for any $\ell \leq b \neq 0$ whenever the parts condition $a + b + c \leq r$ is satisfied.



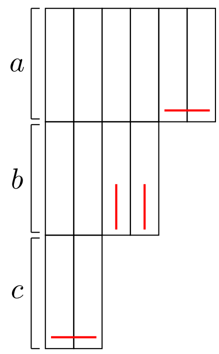
(v): This ℓ-hook removal is applicable for $b + 4 \leq \ell \leq b + c + 2$ whenever the parts condition $a + b + c \leq r$ is satisfied.



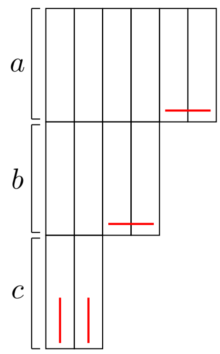
(vi): This ℓ-hook removal is applicable for $b+c+6 \leq \ell \leq a+b+2$ whenever the parts condition $a - 4 \leq r - \ell$ is satisfied.



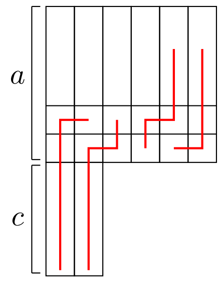
(vii): This ℓ-hook removal is applicable for $\ell = 2$ whenever $a \geq 2$ and the parts condition $a + b + c \leq r - 1$ is satisfied.



(viii): This ℓ-hook removal is applicable for $\ell = 2$ whenever $b \neq 0$ and the parts condition $a + b + c \leq r - 1$ is satisfied.



(ix): This ℓ-hook removal is applicable for $\ell = 2$ whenever $c \neq 0$ and the parts condition $a + b + c \leq r$ is satisfied.



(x): This ℓ-hook removal is applicable for $\ell = c + 3$ whenever $b = 0$ and the parts condition $a \leq r - 1 - (c + 2)$ is satisfied.

We benefit from considering the following three cases (and their sub-cases) separately. For each case, we summarise all possibilities for ℓ-hook removal in tabular form; we discuss a couple of sub-cases of Case I in full detail to illustrate the reasoning.

Case I $(a + b + c \leq r - 2)$:

In this case, a, b, c must satisfy $6a + 4b + 2c = 4r$ and $a + b + c \leq r - 2$. It follows that $a \geq c + 4$, using which we deduce that

$$\frac{1}{2}(a + c) + 2 = \frac{1}{2}(a + c + 4) \leq \frac{2a}{2} = a. \tag{6.5}$$

Table 6.1 summarises the possibilities for ℓ-hook removal when $b \neq 0$.

	$c \geq 2$	$c = 1$	$c = 0$
(i)	$\ell \leq \frac{1}{2}(a - c)$	$\ell \leq \frac{1}{2}(a - 1)$	$\ell \leq \frac{1}{2}a$
(ii)	$1 \leq \ell \leq c$	$\ell = 1$	—
(iii)	$c + 4 \leq \ell \leq \frac{1}{2}(a + c) + 2$	$5 \leq \ell \leq \frac{1}{2}(a + 1) + 2$	$4 \leq \ell \leq \frac{1}{2}a + 2$
(iv)	$1 \leq \ell \leq b$	$1 \leq \ell \leq b$	$1 \leq \ell \leq b$
(v)	$b + 4 \leq \ell \leq b + c + 2$	—	—
(vi)	$b + c + 6 \leq \ell \leq \frac{1}{2}(a + c) + b + 4$	$b + 7 \leq \ell \leq \frac{1}{2}(a - 1) + b + 4$	$b + 6 \leq \ell \leq \frac{1}{2}a + b + 4$
(vii)	$\ell = 2$	$\ell = 2$	$\ell = 2$
(viii)	$\ell = 2$	$\ell = 2$	$\ell = 2$
(ix)	$\ell = 2$	$\ell = 2$	—
(x)	—	—	—

Table 6.1: Possibilities for ℓ-hook removal when $a + b + c \leq r - 2$ and $b \neq 0$.

Let us discuss the sub-case $b \neq 0$, $c \geq 2$ in more detail. First observe that if $b \neq 0$ and $c \geq 2$, then $r = \frac{3a}{2} + b + \frac{c}{2}$. The only possibility for ℓ-hook removal that we can rule out completely is shown in diagram (x), since it requires $b = 0$.

We can remove ℓ-hooks as shown in diagram (i) for any $\ell \leq a$ provided that the parts condition holds. However, the parts condition $(a+b+c \leq r-\ell)$ is only satisfied if the hooks have length $\ell \leq \frac{1}{2}(a-c)$. By contrast, we can remove ℓ-hooks as shown in diagram (ii) for any $1 \leq \ell \leq c$ and the parts condition always holds. We can remove ℓ-hooks as shown in diagram (iii) if $c+4 \leq \ell \leq a$ provided that $a+b-2 \leq r-\ell$. This parts condition says that $a+b-2 \leq \frac{3a}{2}+b+\frac{c}{2}-\ell$, from which it follows that $\ell \leq \frac{1}{2}(a+c)+2$. Our earlier observation told us that $\min\left\{a, \frac{1}{2}(a+c)+2\right\} = \frac{1}{2}(a+c)+2$ and thus, we can remove ℓ-hooks as shown in diagram (iii) for values of ℓ satisfying $c+4 \leq \ell \leq \frac{1}{2}(a+c)+2$. The possibility shown in diagram (iv) is straightforward: we can remove ℓ-hooks for $1 \leq \ell \leq b$ and the parts condition is always satisfied. Similarly, we can remove ℓ-hooks as in diagram (v) for $b+4 \leq \ell \leq b+c+2$ without violating the parts condition. The possibility shown in diagram (vi) requires $\ell \geq b+c+6$ and $\ell \leq a+b+2$. The parts condition $a-4 \leq r-\ell$ implies that $\ell \leq \frac{a}{2}+b+\frac{c}{2}+4$, which we know by (6.5) can never be greater than $a+b+2$. Therefore, we may remove ℓ-hooks for values of ℓ satisfying $b+c+6 \leq \ell \leq \frac{1}{2}(a+c)+b+4$. The remaining three possibilities, namely (vii)–(ix), are all only possible if $\ell = 2$ and the parts condition holds automatically.

So, applying Lemma 6.2.8, we find that

$$\begin{aligned}\left\langle \varphi^{(4^r)}, \chi^{(6^a,4^b,2^c)} \right\rangle &= \frac{1}{r}\left[\frac{a-c}{2}+c+\left(\frac{a+c}{2}+2-(c+3)\right)+b+\big(b+c+2-(b+3)\big)\right.\\ &\qquad \left.+\left(\frac{a+c}{2}+b+4-(b+c+5)\right)+1+1+1\right]\\ &= \frac{1}{r}\left[\frac{3a}{2}+b+\frac{c}{2}\right] = 1.\end{aligned}$$

If instead we consider the sub-case $b \neq 0$ and $c = 1$, then we deduce that $a \geq 5$ and $r = \frac{3a}{2}+b+\frac{1}{2}$. Our reasoning proceeds almost exactly as in the previous case ($b \neq 0$ and $c \geq 2$), taking $c = 1$ where appropriate. The only exception is the possibility labelled (v) which, unlike in the previous case, must be ruled out. Indeed, to remove ℓ-hooks in the way indicated in diagram (v) we require $\ell \leq b+c+2 = b+3$ and $\ell \geq b+4$, but these cannot both be satisfied.

So, applying Lemma 6.2.8, we find that

$$\begin{aligned}\left\langle \varphi^{(4^r)}, \chi^{(6^a,4^b,2)} \right\rangle &= \frac{1}{r}\left[\frac{a-1}{2}+1+\left(\frac{a+1}{2}+2-4\right)+b\right.\\ &\qquad \left.+\left(\frac{a+1}{2}+b+4-(b+6)\right)+1+1+1\right] = \frac{1}{r}\left[\frac{3a}{2}+b+\frac{1}{2}\right] = 1.\end{aligned}$$

We omit the details for the sub-case $b \neq 0$, $c = 0$, but an application of Lemma 6.2.8

yields

$$\left\langle \varphi^{(4^r)}, \chi^{(6^a,4^b)} \right\rangle = \frac{1}{r}\left[\frac{a}{2} + \left(\frac{a}{2}+2-3\right) + b + \left(\frac{a}{2}+b+4-(b+5)\right) + 1 + 1\right]$$
$$= \frac{1}{r}\left[\frac{3a}{2}+b\right] = 1.$$

A similar table (Table 6.2) shows the possibilities for ℓ-hook removal when $b = 0$.

	$c \geq 2$	$c = 1$	$c = 0$
(i)	$\ell \leq \frac{1}{2}(a-c)$	$\ell \leq \frac{1}{2}(a-1)$	$\ell \leq \frac{1}{2}a$
(ii)	$1 \leq \ell \leq c$	$\ell = 1$	—
(iii)	$c+4 \leq \ell \leq \frac{1}{2}(a+c)+2$	$5 \leq \ell \leq \frac{1}{2}(a+1)+2$	$4 \leq \ell \leq \frac{1}{2}a+2$
(iv)	—	—	—
(v)	$4 \leq \ell \leq c+2$	—	—
(vi)	$c+6 \leq \ell \leq \frac{1}{2}(a+c)+4$	$7 \leq \ell \leq \frac{1}{2}(a+1)+4$	$6 \leq \ell \leq \frac{1}{2}a+4$
(vii)	$\ell = 2$	$\ell = 2$	$\ell = 2$
(viii)	—	—	—
(ix)	$\ell = 2$	$\ell = 2$	—
(x)	$\ell = c+3$	$\ell = 4$	$\ell = 3$

Table 6.2: Possibilities for ℓ-hook removal when $a+b+c \leq r-2$ and $b = 0$.

Applying Lemma 6.2.8 for each of the above three cases yields

$$\left\langle \varphi^{(4^r)}, \chi^{(6^a,2^c)} \right\rangle = \frac{1}{r}\left[\frac{a-c}{2} + c + \left(\frac{a+c}{2}+2-(c+3)\right) + (c+2-3) + \right.$$
$$\left. + \left(\frac{a+c}{2}+4-(c+5)\right) + 1 + 1 + 1\right] = \frac{1}{r}\left[\frac{3a}{2}+\frac{c}{2}\right] = 1,$$

$$\left\langle \varphi^{(4^r)}, \chi^{(6^a,2)} \right\rangle = \frac{1}{r}\left[\frac{a-1}{2} + 1 + \left(\frac{a+1}{2}+2-4\right) + \left(\frac{a+1}{2}+4-6\right) + 1 + 1 + 1\right]$$
$$= \frac{1}{r}\left[\frac{3a}{2}+\frac{1}{2}\right] = 1$$

and

$$\left\langle \varphi^{(4^r)}, \chi^{(6^a)} \right\rangle = \frac{1}{r}\left[\frac{a}{2} + \left(\frac{a}{2}+2-3\right) + \left(\frac{a}{2}+4-5\right) + 1 + 1\right] = \frac{1}{r}\left[\frac{3a}{2}\right] = 1,$$

when $c \geq 2$, $c = 1$ and $c = 0$, respectively.

In case I, we used on several occasions the fact that $\frac{1}{2}(a+c)+2 \leq a$. This inequality only holds if $a+b+c \leq r-2$ and thus, we need to address the case $a+b+c > r-2$ separately. We in fact consider the only two remaining cases individually as each of them requires slightly different treatment.

Case II ($a+b+c=r-1$):

In this case, a, b, c must satisfy $6a+4b+2c=4r$ and $a+b+c=r-1$. From these conditions, we deduce that $a=c+2$ (and since $c \geq 0$, this means that $a \geq 2$). The possibilities for ℓ-hook removal can be summarised as follows in Table 6.3.

	$b \neq 0$				$b = 0$		
	$c \geq 2$	$c = 1$	$c = 0$		$c \geq 2$	$c = 1$	$c = 0$
(i)	$\ell = 1$	$\ell = 1$	$\ell = 1$	(i)	$\ell = 1$	$\ell = 1$	$\ell = 1$
(ii)	$1 \leq \ell \leq c$	$\ell = 1$	—	(ii)	$1 \leq \ell \leq c$	$\ell = 1$	—
(iii)	—	—	—	(iii)	—	—	—
(iv)	$1 \leq \ell \leq b$	$1 \leq \ell \leq b$	$1 \leq \ell \leq b$	(iv)	—	—	—
(v)	$b+4 \leq \ell \leq b+c+2$	—	—	(v)	$4 \leq \ell \leq c+2$	—	—
(vi)	—	—	—	(vi)	—	—	—
(vii)	$\ell = 2$	$\ell = 2$	$\ell = 2$	(vii)	$\ell = 2$	$\ell = 2$	$\ell = 2$
(viii)	$\ell = 2$	$\ell = 2$	$\ell = 2$	(viii)	—	—	—
(ix)	$\ell = 2$	$\ell = 2$	—	(ix)	$\ell = 2$	$\ell = 2$	—
(x)	—	—	—	(x)	$\ell = c+3$	$\ell = 4$	$\ell = 3$

Table 6.3: Possibilities for ℓ-hook removal when $a+b+c=r-1$.

Thus, applying Lemma 6.2.8 in each of the above sub-cases, we conclude that if $b \neq 0$ then

$$\left\langle \varphi^{(4^r)}, \chi^{(6^{c+2},4^b,2^c)} \right\rangle = \frac{1}{r}\big[1+c+b+\big(b+c+2-(b+3)\big)+1+1+1\big]$$
$$= \frac{b+2c+3}{r} = 1,$$

$$\left\langle \varphi^{(4^r)}, \chi^{(6^3,4^b,2)} \right\rangle = \frac{1}{r}\big[1+1+b+1+1+1\big] = \frac{b+5}{r} = 1$$

and

$$\left\langle \varphi^{(4^r)}, \chi^{(6^2,4^b)} \right\rangle = \frac{1}{r}\big[1+b+1+1\big] = \frac{b+3}{r} = 1,$$

when $c \geq 2$, $c=1$ and $c=0$, respectively.

Similarly, if $b=0$, then

$$\left\langle \varphi^{(4^r)}, \chi^{(6^{c+2},2^c)} \right\rangle = \frac{1}{r}\big[1+c+(c+2-3)+1+1+1\big] = \frac{2c+3}{r} = 1$$

when $c \geq 2$,

$$\left\langle \varphi^{(4^r)}, \chi^{(6^3,2)} \right\rangle = \frac{1}{r}\big[1+1+1+1+1\big] = \frac{5}{r} = 1$$

when $c=1$, and

$$\left\langle \varphi^{(4^r)}, \chi^{(6^2)} \right\rangle = \frac{3}{r} = 1$$

when $c=0$.

Case III $(a+b+c=r)$:

In this case we have that a, b, c must satisfy $6a+4b+2c=4r$ and $a+b+c=r$. We are forced to conclude that $c=a\neq 0$. The possibilities for ℓ-hook removal are detailed in Table 6.4.

	$a=c\geq 2$, $b\neq 0$	$a=c=1$, $b=r-2$	$a=c=\frac{1}{2}r$, $b=0$
(i)	—	—	—
(ii)	$1\leq\ell\leq a$	$\ell=1$	$1\leq\ell\leq a$
(iii)	—	—	—
(iv)	$1\leq\ell\leq b$	$1\leq\ell\leq b$	—
(v)	$b+4\leq\ell\leq a+b+2$	—	$4\leq\ell\leq a+2$
(vi)	—	—	—
(vii)	—	—	—
(viii)	—	—	—
(ix)	$\ell=2$	$\ell=2$	$\ell=2$
(x)	—	—	—

Table 6.4: Possibilities for ℓ-hook removal when $a+b+c=r$.

Thus, applying Lemma 6.2.8, we find that when $a=c\geq 2$ and $b\neq 0$:

$$\left\langle\varphi^{(4^r)},\chi^{(6^a,4^b,2^a)}\right\rangle=\frac{1}{r}\Big[a+b+\big(a+b+2-(b+3)\big)+1\Big]=\frac{2a+b}{r}=1.$$

Similarly, when $a=c=1$ and $b=r-2$, we have

$$\left\langle\varphi^{(4^r)},\chi^{(6,4^{r-2},2)}\right\rangle=\frac{1}{r}\big[1+b+1\big]=\frac{b+2}{r}=1,$$

and if $a=c=\frac{r}{2}$, $b=0$ then

$$\left\langle\varphi^{(4^r)},\chi^{\left(6^{r/2},2^{r/2}\right)}\right\rangle=\frac{1}{r}\big[a+a+2-3+1\big]=\frac{2a}{r}=1.$$

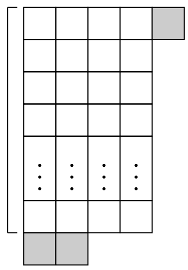
In all cases, we saw that $\langle\varphi^{(4^r)},\chi^\lambda\rangle=1$. That is, if $\lambda=(6,\lambda_2,\ldots)$ has at most n parts, all of which are even, then χ^λ is a constituent of $\varphi^{(4^r)}$ and it appears with multiplicity one.

It remains to verify that, if λ has an odd part, then $\langle\varphi^{(4^r)},\chi^\lambda\rangle=0$. We keep the same inductive assumptions and let $\lambda=(6,\lambda_2,\ldots,\lambda_\ell)\vdash 4r$ be such that λ_i is odd. Necessarily, there exists $j\neq i$ satisfying $2\leq j\leq\ell$ such that λ_j is also odd. We want to show that χ^λ is not a constituent of $\varphi^{(4^r)}$.

For a contradiction, assume that $\lambda_2\neq 6$. If χ^λ is a constituent of $\varphi^{(4^r)}$, then all constituents of $\chi^\lambda\big\downarrow_{\mathfrak{S}_{4r-1}}$ must appear as constituents of $\varphi^{(4^r)}\big\downarrow_{\mathfrak{S}_{4r-1}}$. In particular, the Branching

Rule tells us that $\chi^{(5,\lambda_2,\ldots,\lambda_\ell)}$ is a constituent of $\varphi^{(4^r)}\big\downarrow_{\mathfrak{S}_{4r-1}} = \left(\varphi^{(4^{r-1})} \times 1_{\mathfrak{S}_3}\right)\Big\uparrow^{\mathfrak{S}_{4r-1}}$. Corollary 2.2.15(1) then tells us that that we must be able to obtain $[5, \lambda_2, \ldots, \lambda_\ell]$ by a Young's Rule addition of three boxes to the Young diagram corresponding to a constituent of $\varphi^{(4^{r-1})}$.

Taking $m = 4$ in Proposition 6.1.4, we know that no partition with first part equal to five can label a constituent of $\varphi^{(4^{r-1})}$. Hence, the only way that we can obtain $[5, \lambda_2, \ldots, \lambda_\ell]$ by the method described above is if the constituent of $\varphi^{(4^{r-1})}$ is the minimal constituent, $\chi^{(4^{r-1})}$. Further, there is only one way to add the three boxes to $\left[4^{r-1}\right]$ without violating the conditions of Young's Rule; this possibility is illustrated below.

$\left[5, 4^{r-2}, 2\right] = \; r$ 

Thus, we conclude that $\lambda = (6, 4^{r-2}, 2)$, which is a contradiction because λ must have (at least) two odd parts. It follows that $\lambda_2 = 6$.

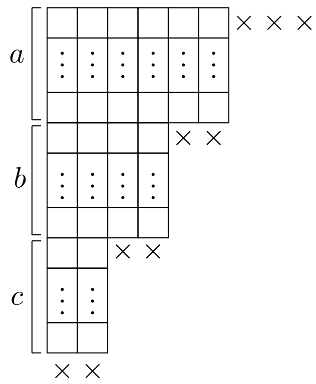
Maintaining notation, so that λ_i and λ_j are odd parts of λ, consider the partition

$$\lambda = \left(6^q, \lambda_{q+1}, \ldots, \lambda_{i-1}, \lambda_i^s, \lambda_{i+s}, \ldots, \lambda_{j-1}, \lambda_j^t, \lambda_{j+t}, \ldots, \lambda_\ell\right)$$

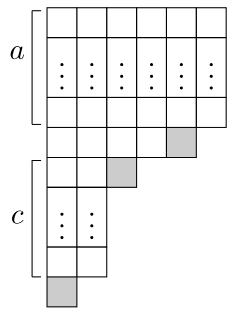
where $2 \leq q \leq i-1$ and $s + t \geq 2$. Note that q is maximal so that $\lambda_{q+1} < 6$. We may as well assume that λ_i is the first odd part of λ and that none of $\lambda_{i+s}, \ldots, \lambda_{j-1}$ are odd parts of λ.

Let $\lambda^\sharp$ be the partition obtained from λ by removing the last box from row $i-1$. Crucially, we note that $\lambda^\sharp$ has at least three odd parts: $\lambda_{i-1} - 1$, λ_i and λ_j, and the first two of these are consecutive. The Branching Rule tells us that $\chi^{\lambda^\sharp}$ is a constituent of $\chi^\lambda\big\downarrow_{\mathfrak{S}_{4r-1}}$. However, we claim that $\chi^{\lambda^\sharp}$ is not a constituent of $\left(\varphi^{(4^{r-1})} \times 1_{\mathfrak{S}_3}\right)\big\uparrow^{\mathfrak{S}_{4r-1}}$.

We want to know: from which constituents of $\varphi^{(4^{r-1})}$ could $\chi^{\lambda^\sharp}$ have arisen? Corollary 2.2.15(1) tells us that we should expect to obtain $\left[\lambda^\sharp\right]$ by a Young's Rule addition of three boxes to the Young diagram corresponding to a constituent of $\varphi^{(4^{r-1})}$. Therefore, we know that $\chi^{\lambda^\sharp}$ cannot have arisen from $\chi^{(4^{r-1})}$ because we cannot obtain more than one row of six boxes by adding three boxes to $\left[4^{r-1}\right]$. We also know that no constituent of $\varphi^{(4^{r-1})}$ is labelled by a partition whose first part is equal to five (Proposition 6.1.4). Further, by the inductive hypothesis, there are no odd parts in any partition labelling a constituent of $\varphi^{(4^{r-1})}$ that has first part equal to six. Thus, we just need to consider constituents of $\varphi^{(4^{r-1})}$ of the form $\chi^{(6^a,4^b,2^c)}$, where a, b, c are non-negative integers such that $6a + 4b + 2c = 4(r-1)$, and ask whether it is possible to obtain $\left[\lambda^\sharp\right]$ from a Young diagram corresponding to a partition of the form $(6^a, 4^b, 2^c)$ without violating the conditions imposed by Corollary 2.2.15(1). The following diagram indicates places where the boxes may be added using the symbol $\times$.



Since $\lambda^\sharp$ has first part equal to six, the boxes are not added to a row of length six in $[6^a, 4^b, 2^c]$; we must obtain $\left[\lambda^\sharp\right]$ by adding boxes in rows $a+1$, $b+1$ or $c+1$. Moreover, the conditions of Young's Rule do not permit us to add three boxes to $[6^a, 4^b, 2^c]$ in such a way as to obtain two odd parts of the same size. However, we know that $\lambda^\sharp$ has at least three odd parts and that the first two are consecutive, so the only candidate for $\lambda^\sharp$ is $(6^a, 5, 3, 2^{c-1}, 1)$, which is obtained by a Young's rule addition of three boxes to $[6^a, 4, 2^c]$.



Moreover, this means that $s = t = 1$, $i = a+2$ and $j = \ell = a+c+2$, from which we conclude that $\lambda = (6^{a+1}, 3, 2^{c-1}, 1)$, with $a, c \geq 1$.

Assume now, for a contradiction, that $c \geq 2$. In this case, the Branching Rule tells us that $\chi^{(6^{a+1},3,2^{c-2},1^2)}$ is a constituent of $\chi^\lambda\big\downarrow_{\mathfrak{S}_{4r-1}}$. However, this cannot be a constituent of $\left(\varphi^{(4^{r-1})} \times 1_{\mathfrak{S}_3}\right)\big\uparrow^{\mathfrak{S}_{4r-1}}$; if it were, then, by the same reasoning as given earlier, we must be able to obtain $\left[6^{a+1}, 3, 2^{c-2}, 1^2\right]$ by a Young's Rule addition of three boxes to a constituent labelled by a partition of $4(r-1)$ with all even parts. Clearly this cannot happen, since creating the two parts of size one would require two boxes to have been added in the same column. Hence, we are forced to conclude that $c = 1$.

We are left considering the possibility that $\lambda = \left(6^{a+1}, 3, 1\right)$ with $a \geq 1$. Recall that λ is a partition of $4r$. Certainly, λ can only take the above form if a satisfies $3(a+1) = 2(r-1)$. If such an a exists, then we claim that $\left\langle \varphi^{(4^r)}, \chi^\lambda \right\rangle = 0$.

For a contradiction, assume that $m_{(6^{a+1},3,1)} := \left\langle \varphi^{(4^r)}, \chi^{(6^{a+1},3,1)} \right\rangle \neq 0$, with $a \geq 1$. Applying the Branching Rule, it follows that $\chi^{(6^{a+1},2,1)}$ is a constituent of $\chi^{\left(6^{a+1},3,1\right)}\big\downarrow_{\mathfrak{S}_{4r-1}}$ and hence, is also a constituent of $\varphi^{(4^r)}\big\downarrow_{\mathfrak{S}_{4r-1}}$.

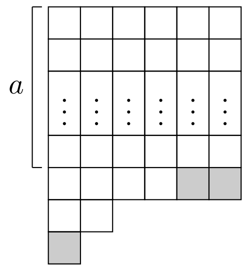
From our earlier calculations we know that $m_{(6^{a+1},2^2)} := \left\langle \varphi^{(4^r)}, \chi^{\left(6^{a+1},2^2\right)} \right\rangle = 1$. Indeed, the partition $(6^{a+1}, 2^2)$ has all parts even and there exists, by assumption, $a \geq 1$ such that $3(a+1) = 2(r-1)$. Further,

$$\chi^{\left(6^{a+1},2^2\right)}\big\downarrow_{\mathfrak{S}_{4r-1}} = \chi^{\left(6^a,5,2^2\right)} + \chi^{\left(6^{a+1},2,1\right)},$$

and thus we deduce that

$$\left\langle \varphi^{(4^r)}\big\downarrow_{\mathfrak{S}_{4r-1}}, \chi^{(6^{a+1},2,1)} \right\rangle \geq m_{(6^{a+1},3,1)} + m_{(6^{a+1},2^2)} = m_{(6^{a+1},3,1)} + 1. \tag{6.6}$$

By part 1 of Corollary 6.1.3, $\varphi^{(4^r)}\big\downarrow_{\mathfrak{S}_{4r-1}} = \left(\varphi^{(4^{r-1})} \times 1_{\mathfrak{S}_3}\right)\big\uparrow^{\mathfrak{S}_{4r-1}}$. For comparison, we need to determine the multiplicity of $\chi^{(6^{a+1},2,1)}$ in $\left(\varphi^{(4^{r-1})} \times 1_{\mathfrak{S}_3}\right)\big\uparrow^{\mathfrak{S}_{4r-1}}$. This means that we need to identify the ways in which $\left[6^{a+1}, 2, 1\right]$ may be obtained by a Young's Rule addition of three boxes to the Young diagram corresponding to a constituent of $\varphi^{(4^{r-1})}$. The only possible constituent is $\chi^{(6^a,4,2)}$, with the boxes added as shown in the diagram below.



We deduce that $\chi^{\left(6^{a+1},2,1\right)}$ appears in $\left(\varphi^{(4^{r-1})} \times 1_{\mathfrak{S}_3}\right)\big\uparrow^{\mathfrak{S}_{4r-1}}$ with multiplicity equal to the multiplicity with which $\chi^{(6^a,4,2)}$ appears in $\varphi^{(4^{r-1})}$. Since $(6^a, 4, 2)$ is an even partition, our inductive assumptions tells us that $\left\langle \varphi^{(4^{r-1})}, \chi^{(6^a,4,2)} \right\rangle = 1$ and therefore that

$$1 = \left\langle \left(\varphi^{(4^{r-1})} \times 1_{\mathfrak{S}_3}\right)\big\uparrow^{\mathfrak{S}_{4r-1}}, \chi^{(6^{a+1},2,1)} \right\rangle = \left\langle \varphi^{(4^r)}\big\downarrow_{\mathfrak{S}_{4r-1}}, \chi^{(6^{a+1},2,1)} \right\rangle. \tag{6.7}$$

Finally, (6.6) and (6.7) allow us to deduce that $m_{(6^{a+1},3,1)} = 0$, which is a contradiction. It follows immediately that the entire statement of the proposition holds for $n = r$ and thus, by induction, the proposition is proved. ■

6.4 The smallest non-even constituent of $\varphi^{(4^n)}$

We now state and prove the main theorem of this chapter.

THEOREM 6.4.1
Let n be any natural number and let $\lambda = (\lambda_1, \ldots, \lambda_\ell) \vdash 4n$. If $\left\langle \varphi^{(4^n)}, \chi^\lambda \right\rangle \neq 0$, then either $\lambda_1 \geq 7$ or λ has all parts even.

Proof. Proposition 3.4.2 tells us that $\langle \varphi^{(4^n)}, \chi^{(4^n)} \rangle = 1$ and the partition (4^n) is clearly minimal. Noting that (4^n) is the largest (lexicographically ordered) partition that has first part equal to four, the theorem follows from this fact, together with Proposition 6.1.4 and Proposition 6.3.1. ■

We now present a description of the constituents of $\varphi^{(4^n)}$ that have first part equal to seven and all subsequent parts strictly less than seven. Following this, we will be able to identify the lexicographically smallest non-even constituent of $\varphi^{(4^n)}$.

Proposition 6.4.2

Assume that $\lambda = (7, \lambda_2, \ldots, \lambda_\ell) \vdash 4n$ *with* $\lambda_2 < 7$.

- *If* $\lambda_2 < 4$, *then* $\langle \chi^\lambda, \varphi^{(4^n)} \rangle = 0$.
- *If* $\lambda_2 = 4$, *then either* $\lambda = (7, 4^{n-2}, 1)$ *with* $n \geq 2$, *in which case* $\langle \chi^\lambda, \varphi^{(4^n)} \rangle = 1$; *or otherwise* $\langle \chi^\lambda, \varphi^{(4^n)} \rangle = 0$.
- *If* $\lambda_2 = 5$, *then either* $\lambda = (7, 5, 4^{n-4}, 3, 1)$ *with* $n \geq 4$, *in which case* $\langle \chi^\lambda, \varphi^{(4^n)} \rangle = 1$; *or otherwise* $\langle \chi^\lambda, \varphi^{(4^n)} \rangle = 0$.
- *If* $\lambda_2 = 6$, *then*
  - *when* $n \geq 5$, $\left\langle \chi^{(7,6^a,5,4^b,2^c)}, \varphi^{(4^n)} \right\rangle = 1$ *for any* $a, c \in \mathbb{N}$ *and* $b \in \mathbb{N}_0$ *satisfying* $a + b + c + 2 < n$ *and* $6 + 3a + 2b + c = 2n$;
  - *when* $n \geq 4$, $\left\langle \chi^{(7,6^a,4^b,3,2^c)}, \varphi^{(4^n)} \right\rangle = 1$ *for any* $a \in \mathbb{N}$ *and* $b, c \in \mathbb{N}_0$ *satisfying* $a + b + c + 2 < n$ *and* $5 + 3a + 2b + c = 2n$;
  - *when* $n \geq 4$, $\left\langle \chi^{(7,6^a,4^b,2^c,1)}, \varphi^{(4^n)} \right\rangle = 1$ *for any* $a, b \in \mathbb{N}$ *and* $c \in \mathbb{N}_0$ *satisfying* $a + b + c + 2 \leq n$ *and* $4 + 3a + 2b + c = 2n$;
  - *when* $n \geq 6$, $\left\langle \chi^{(7,6^a,5,4^b,3,2^c,1)}, \varphi^{(4^n)} \right\rangle = 1$ *for any* $a \in \mathbb{N}$ *and* $b, c \in \mathbb{N}_0$ *satisfying* $a + b + c + 4 \leq n$ *and* $8 + 3a + 2b + c = 2n$;

  otherwise $\langle \chi^\lambda, \varphi^{(4^n)} \rangle = 0$.

When proving this proposition, will will repeatedly use the same strategy, so we outline our approach here and present some notation which will simplify the proof. For a partition $\lambda = (7, \lambda_2, \ldots, \lambda_\ell)$ of $4n$ with $\lambda_2 < 7$, we will want to determine $m_\lambda := \langle \varphi^{(4^n)}, \chi^\lambda \rangle$. To do this, we will pick a constituent of $\chi^\lambda \downarrow_{\mathfrak{S}_{4n-1}}$, say $\chi^{\lambda^\sharp}$, where $\lambda^\sharp \vdash 4n-1$, and look at its multiplicity in both $\varphi^{(4^n)} \downarrow_{\mathfrak{S}_{4n-1}}$ and $(\varphi^{(4^{n-1})} \times 1_{\mathfrak{S}_3}) \uparrow^{\mathfrak{S}_{4n-1}}$. Let

$$\mathcal{I}(\lambda^\sharp) := \left\{ \rho \vdash 4n \;\middle|\; \left\langle \chi^\rho \downarrow_{\mathfrak{S}_{4n-1}}, \chi^{\lambda^\sharp} \right\rangle = 1 \text{ and } m_\rho := \langle \varphi^{(4^n)}, \chi^\rho \rangle \neq 0 \right\} \setminus \{\lambda\}.$$

Also define

$$\mathcal{J}(\lambda^\sharp) := \left\{ \mu \vdash 4(n-1) \;\middle|\; \left\langle (\chi^\mu \times 1_{\mathfrak{S}_3}) \uparrow^{\mathfrak{S}_{4n-1}}, \chi^{\lambda^\sharp} \right\rangle = 1 \text{ and } m_\mu := \langle \varphi^{(4^{n-1})}, \chi^\mu \rangle \neq 0 \right\}.$$

By part 1 of Corollary 6.1.3, we know that

$$\left\langle \varphi^{(4^n)}\big\downarrow_{\mathfrak{S}_{4n-1}}, \chi^{\lambda^\sharp} \right\rangle = \left\langle (\varphi^{(4^{n-1})} \times 1_{\mathfrak{S}_3})\big\uparrow^{\mathfrak{S}_{4n-1}}, \chi^{\lambda^\sharp} \right\rangle,$$

from which it follows that

$$m_\lambda + \sum_{\rho \in \mathcal{I}(\lambda^\sharp)} m_\rho = \sum_{\mu \in \mathcal{J}(\lambda^\sharp)} m_\mu. \tag{6.8}$$

We will use this equation to calculate m_λ.

Proof. Fix $n \in \mathbb{N}$ and let $\lambda = (7, \lambda_2, \dots, \lambda_\ell) \vdash 4n$ with $\lambda_2 < 7$. Suppose that λ labels a constituent of $\varphi^{(4^n)}$. We know that λ can have at most one part of each odd size, otherwise the Young diagram corresponding to $\chi^{(6,\lambda_2,\dots,\lambda_\ell)}$ (which appears in the restriction of χ^λ to $\mathfrak{S}_{4n-1}$) cannot be obtained by adding three boxes to the Young diagram corresponding to a constituent of $\varphi^{(4^{n-1})}$ without violating the condition for Young's Rule that no two boxes are added in the same column. Furthermore, λ must have an even number of odd parts and so either precisely one of $\lambda_2, \dots, \lambda_\ell$ is odd, or else λ must have four odd parts: $\lambda_1 = 7$, a part of size five, a part of size three and a part of size one.

If $\lambda_2 < 4$, then $\lambda = (7, 3^a, 2^b, 1^c)$ for some $a, c \in \{0, 1\}$ and $b \in \mathbb{N}_0$ satisfying the conditions $7+3a+2b+c = 4n$ and $a+b+c+1 \leq n$. It is not hard to check that the only valid partitions of $4n$ for which these constraints hold are $\lambda = (7, 3, 2)$ with $n = 3$ and $\lambda = (7, 1)$ with $n = 2$. However, any constituent of $\varphi^{(4^2)}$ has at most two parts, both of which are even. Thus, $\left\langle \varphi^{(4^2)}, \chi^{(7,1)} \right\rangle = 0$. To rule out the possibility that $\lambda = (7, 3, 2)$ labels a constituent of $\varphi^{(4^3)}$, we appeal to the work of Dent and Siemons in [11]: a straightforward application of part (i) of their Theorem 4.1 reveals that $\left\langle \varphi^{(4^3)}, \chi^{(7,3,2)} \right\rangle = 0$.

Let us now consider the case $\lambda_2 = 4$: it is clear that λ with this property must have exactly two odd parts.

- If $\lambda_2 = 4$ and the two odd parts of λ are of size seven and size one, then $\lambda = (7, 4^a, 2^b, 1)$ for some $a \in \mathbb{N}$ and $b \in \mathbb{N}_0$ satisfying $a+b+2 \leq n$ and $4+2a+b = 2n$. These conditions on a and b are only both satisfied if $b = 0$. Hence, the only possibility is $\lambda = (7, 4^{n-2}, 1)$ for some $n \geq 3$.

  Choose $\lambda^\sharp = (6, 4^{n-2}, 1)$. It is easy to see that

  $$\mathcal{I}(\lambda^\sharp) = \left\{ (6, 4^{n-2}, 2) \right\} \quad \text{and} \quad \mathcal{J}(\lambda^\sharp) = \left\{ (6, 4^{n-3}, 2), (4^{n-1}) \right\}.$$

  Using Proposition 6.3.1 and the fact that $\left\langle \varphi^{(4^{n-1})}, \chi^{(4^{n-1})} \right\rangle = 1$, the equation (6.8) allows us to conclude that $m_\lambda = 1 + 1 - 1 = 1$.

- If $\lambda_2 = 4$ and λ has precisely two odd parts, one of size seven and one of size three, then $\lambda = (7, 4^a, 3, 2^b)$ for some $a \in \mathbb{N}$ and $b \in \mathbb{N}_0$ satisfying $a+b+2 \leq n$ and $5+2a+b = 2n$.

We conclude from these conditions on a and b that $b \leq 1$. However, it is not possible to find a such that $\lambda \vdash 4n$ when $b = 0$ and so we are left considering the possibility that $\lambda = (7, 4^{n-3}, 3, 2)$ for some $n \geq 4$.

Choosing $\lambda^\sharp = (6, 4^{n-3}, 3, 2)$, we find that

$$\mathcal{I}(\lambda^\sharp) = \{(6, 4^{n-2}, 2)\} \quad \text{and} \quad \mathcal{J}(\lambda^\sharp) = \{(6, 4^{n-3}, 2)\}.$$

Hence, by Proposition 6.3.1 and (6.8), we deduce that $m_\lambda = 1 - 1 = 0$.

Let us now consider partitions $\lambda = (7, 5, \lambda_3, \ldots, \lambda_\ell)$ of $4n$. If λ of this form has precisely two odd parts, then $\lambda = (7, 5, 4^a, 2^b)$ for some $a, b \in \mathbb{N}_0$ satisfying $6 + 2a + b = 2n$ and $a + b + 2 \leq n$. It follows from these conditions that $b \leq 2$. If $b = 1$, then it is not possible to find a such that $\lambda \vdash 4n$, so we just need to consider $b = 2$ and $b = 0$. Note that χ^λ cannot be a constituent of $\varphi^{(4^n)}$ if $n < 3$, since then $\lambda \nvdash 4n$. In fact, when $b = 2$, we actually require $n \geq 4$ to guarantee that $\lambda \vdash 4n$.

- If $b = 0$, then it must be that $\lambda = (7, 5, 4^{n-3})$ for some $n \geq 3$. In this case, we choose $\lambda^\sharp = (6, 5, 4^{n-3})$ and thus, $\mathcal{I}(\lambda^\sharp) = \{(6^2, 4^{n-3})\}$ and $\mathcal{J}(\lambda^\sharp) = \{(6, 4^{n-3}, 2)\}$. Using Proposition 6.3.1, it follows from (6.8) that $m_\lambda = 1 - 1 = 0$.
- If $b = 2$, then $\lambda = (7, 5, 4^{n-4}, 2^2)$ for $n \geq 4$. Choose $\lambda^\sharp = (6, 5, 4^{n-4}, 2^2)$ and observe that $\mathcal{I}(\lambda^\sharp) = \{(6^2, 4^{n-4}, 2^2)\}$. Partitions $\mu \in \mathcal{J}(\lambda^\sharp)$ can have at most $n - 1$ parts, otherwise $m_\mu = 0$. So, this means that there is only one possibility for μ: we have $\mathcal{J}(\lambda^\sharp) = \{(6, 4^{n-3}, 2)\}$. Using Proposition 6.3.1 and (6.8), we find that $m_\lambda = 1 - 1 = 0$.

If $\lambda_2 = 5$ and λ has four odd parts, then clearly $\lambda = (7, 5, 4^a, 3, 2^b, 1)$ for some $a, b \in \mathbb{N}_0$ satisfying $8 + 2a + b = 2n$ and $a + b + 4 \leq n$. It is not hard to show that these conditions are only both satisfied when $b = 0$ (and thus $a = n - 4$). Hence, if $n < 4$, it is immediate that $\langle \chi^\lambda, \varphi^{(4^n)} \rangle = 0$.

Assuming that $n \geq 4$, we have $\lambda = (7, 5, 4^{n-4}, 3, 1)$. Choose $\lambda^\sharp = (6, 5, 4^{n-4}, 3, 1)$. In this case, $\mathcal{I}(\lambda^\sharp) = \emptyset$ and $\mathcal{J}(\lambda^\sharp) = \{(6, 4^{n-3}, 2)\}$. Hence, Proposition 6.3.1 and (6.8) lead us to conclude that $m_\lambda = 1 - 0 = 1$.

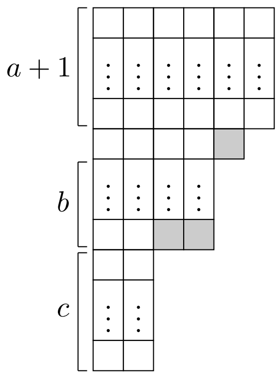
If $\lambda_2 = 6$, then we have four cases to consider.

- If $\lambda = (7, 6, \lambda_3, \ldots, \lambda_\ell)$ has two odd parts, the second of which is five, then λ takes the form $\lambda = (7, 6^a, 5, 4^b, 2^c)$ for some $a \in \mathbb{N}$ and $b, c \in \mathbb{N}_0$ satisfying $6 + 3a + 2b + c = 2n$ and $a + b + c + 2 \leq n$. It is clear that if $n < 3$, then $\lambda \nvdash 4n$. Since we require $a \neq 0$ to avoid overlapping with a case that we have already considered, we restrict our attention to $n \geq 5$. Choose $\lambda^\sharp = (6^{a+1}, 5, 4^b, 2^c)$.

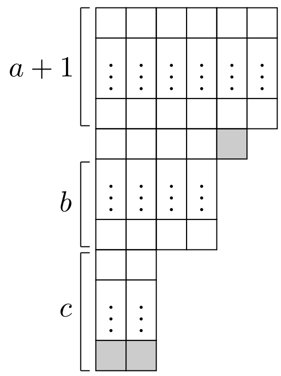
It is clear that $\mathcal{I}(\lambda^\sharp) = \{(6^{a+2}, 4^b, 2^c)\}$, but we need to take care when determining $\mathcal{J}(\lambda^\sharp)$. We find that

$$\mathcal{J}(\lambda^\sharp) = \begin{cases} \{(6^{a+1}, 4^b, 2)\} & \text{if } c = 0 \text{ and } a+b+2 < n; \\ \{(6^{a+1}, 4^b, 2^{c+1}), (6^{a+1}, 4^{b+1}, 2^{c-1})\} & \text{if } c \neq 0 \text{ and } a+b+c+2 < n; \\ \{(6^{a+1}, 4^{b+1}, 2^{c-1})\} & \text{if } c \neq 0 \text{ and } a+b+c+2 = n. \end{cases}$$

Indeed, the following diagrams indicate when and how $[6^{a+1}, 5, 4^b, 2^c]$ may be obtained by a Young's Rule addition of three boxes to $[6^{a+1}, 4^{b+1}, 2^{c-1}]$ and $[6^{a+1}, 4^b, 2^{c+1}]$.



This is possible if $a+b+c+2 < n$.



This is possible if $c \neq 0$ and $a+b+c+1 < n$.

Note that, when the conditions $c = 0$ and $a+b+2 = n$ are both satisfied, that fact that $\lambda = (7, 6^a, 5, 4^b)$ must be a partition of $4n$ leads us to deduce that $a = -2$, which is impossible.

So, applying Proposition 6.3.1, we deduce from (6.8) that

$$m_\lambda = \begin{cases} 1 - 1 = 0 & \text{if } c = 0 \text{ and } a+b+2 < n; \\ 2 - 1 = 1 & \text{if } c \neq 0 \text{ and } a+b+c+2 < n; \\ 1 - 1 = 0 & \text{if } c \neq 0 \text{ and } a+b+c+2 = n. \end{cases}$$

- If $\lambda = (7, 6, \lambda_3, \ldots, \lambda_\ell)$ has two odd parts, the second of which is three, then λ takes the form $\lambda = (7, 6^a, 4^b, 3, 2^c)$ for some $a \in \mathbb{N}$ and $b, c \in \mathbb{N}_0$ satisfying $5 + 3a + 2b + c = 2n$ and $a+b+c+2 \leq n$. We restrict our attention to $n \geq 4$ to ensure that $\lambda \vdash 4n$.

  Choose $\lambda^\sharp = (6^{a+1}, 4^b, 3, 2^c)$. It follows that $\mathcal{I}(\lambda^\sharp) = \{(6^{a+1}, 4^{b+1}, 2^c)\}$ and that

$$\mathcal{J}(\lambda^\sharp) = \begin{cases} \{(6^a, 4^{b+1}, 2), (6^{a+1}, 4^b)\} & \text{if } c = 0 \text{ and } a+b+2 < n; \\ \{(6^{a+1}, 4^b, 2^c), (6^a, 4^{b+1}, 2^{c+1})\} & \text{if } c \neq 0 \text{ and } a+b+c+2 < n; \\ \{(6^{a+1}, 4^b, 2^c)\} & \text{if } c \neq 0 \text{ and } a+b+c+2 = n. \end{cases}$$

Again, we need not worry about the case $c = 0$ and $a + b + 2 = n$, since then, the fact that $\lambda = (7, 6^a, 4^b, 3)$ must be a partition of $4n$ forces $a = -1$, which is not possible. Consequently,

$$m_\lambda = \begin{cases} 2 - 1 = 1 & \text{if } c \in \mathbb{N}_0 \text{ and } a + b + c + 2 < n; \\ 1 - 1 = 0 & \text{if } c \neq 0 \text{ and } a + b + c + 2 = n. \end{cases}$$

- If $\lambda = (7, 6, \lambda_3, \ldots, \lambda_\ell)$ has two odd parts, the second of which is one, then λ takes the form $\lambda = (7, 6^a, 4^b, 2^c, 1)$ for some $a \in \mathbb{N}$ and $b, c \in \mathbb{N}_0$ satisfying $4 + 3a + 2b + c = 2n$ and $a + b + c + 2 \leq n$. We restrict our attention to $n \geq 4$ to ensure that $\lambda \vdash 4n$.

  Choose $\lambda^\sharp = (6^{a+1}, 4^b, 2^c, 1)$. In this case, $\mathcal{I}(\lambda^\sharp) = \{(6^{a+1}, 4^b, 2^{c+1})\}$ and

$$\mathcal{J}(\lambda^\sharp) = \begin{cases} \{(6^a, 4, 2^c)\} & \text{if } b = 0 \text{ and } a + c + 1 < n; \\ \{(6^{a+1}, 4^{b-1}, 2^{c+1}), (6^a, 4^{b+1}, 2^c)\} & \text{if } b \neq 0 \text{ and } a + b + c + 1 < n. \end{cases}$$

  Thus,

$$m_\lambda = \begin{cases} 1 - 1 = 0 & \text{if } b = 0 \text{ and } a + c + 1 < n; \\ 2 - 1 = 1 & \text{if } b \neq 0 \text{ and } a + b + c + 1 < n. \end{cases}$$

- If $\lambda_2 = 6$ and λ has four odd parts, then λ is of the form $(7, 6^a, 5, 4^b, 3, 2^c, 1)$ for some $a \in \mathbb{N}$ and $b, c \in \mathbb{N}_0$ satisfying $8 + 3a + 2b + c = 2n$ and $a + b + c + 4 \leq n$. It is not hard to see that we require $n \geq 6$ to ensure that $\lambda \vdash 4n$ and thus, if $n < 6$, it is immediate that $\langle \varphi^{(4^n)}, \chi^\lambda \rangle = 0$.

  Choose $\lambda^\sharp = (6^{a+1}, 5, 4^b, 3, 2^c, 1)$. In this case, we have

$$\mathcal{I}(\lambda^\sharp) = \emptyset \quad \text{and} \quad \mathcal{J}(\lambda^\sharp) = \{(6^{a+1}, 4^{b+1}, 2^{c+1})\}.$$

  Hence, $m_\lambda = 1 - 0 = 1$. ■

Using Theorem 6.4.1 and Proposition 6.4.2, we can conclude that $S^{(7,4^{n-2},1)}$ must be the smallest non-even constituent of $H^{(4^n)}$ when $n \geq 3$. Moreover, it appears with multiplicity one.

We conclude this chapter by illustrating Proposition 6.4.2 with the following example, in which we determine explicitly the constituents χ^λ of $\varphi^{(4^6)}$ that are labelled by a partition with (necessarily) at most six parts, satisfying $\lambda_1 = 7$ and $\lambda_2 < 7$.

Example 6.4.3

If $n = 6$, then Proposition 6.4.2 tells us that $\langle \varphi^{(4^6)}, \chi^{(7,4^4,1)} \rangle = 1$ and $\langle \varphi^{(4^6)}, \chi^{(7,5,4^2,3,1)} \rangle = 1$. Moreover, these are the only constituents of $\varphi^{(4^6)}$ (appearing with non-zero multiplicity) that are labelled by a partition satisfying $\lambda_1 = 7$ and $\lambda_2 \leq 5$.

To determine all constituents χ^λ of $\varphi^{(4^6)}$ that are labelled by a partition satisfying $\lambda_1 = 7$ and $\lambda_2 = 6$, there are several cases to consider:

- *For $\lambda = (7, 6^a, 5, 4^b, 2^c)$, the only tuple $(a, b, c) \in \mathbb{N} \times \mathbb{N}_0 \times \mathbb{N}$ satisfying the conditions $a + b + c + 2 < 6$ and $6 + 3a + 2b + c = 12$ is $(1, 1, 1)$.*
- *For $\lambda = (7, 6^a, 4^b, 3, 2^c)$, the tuples $(a, b, c) \in \mathbb{N} \times \mathbb{N}_0 \times \mathbb{N}_0$ that satisfy $a + b + c + 2 < 6$ and $5 + 3a + 2b + c = 12$ are $(2, 0, 1)$ and $(1, 2, 0)$.*
- *For $\lambda = (7, 6^a, 4^b, 2^c, 1)$, the tuples $(a, b, c) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}_0$ for which $a + b + c + 2 \leq 6$ and $4 + 3a + 2b + c = 12$ hold are $(1, 2, 1)$ and $(2, 1, 0)$.*
- *Finally, for $\lambda = (7, 6^a, 5, 4^b, 3, 2^c, 1)$, the only tuple $(a, b, c) \in \mathbb{N} \times \mathbb{N}_0 \times \mathbb{N}_0$ satisfying the conditions $a + b + c + 4 \leq 6$ and $8 + 3a + 2b + c = 12$ is $(1, 0, 1)$.*

Thus, Proposition 6.4.2 allows us to conclude that

$$\left\langle \varphi^{(4^6)}, \chi^{(7,6,5,4,2)} \right\rangle = 1, \qquad \left\langle \varphi^{(4^6)}, \chi^{(7,6,4,4,3)} \right\rangle = 1, \qquad \left\langle \varphi^{(4^6)}, \chi^{(7,6,6,4,1)} \right\rangle = 1,$$
$$\left\langle \varphi^{(4^6)}, \chi^{(7,6,6,3,2)} \right\rangle = 1, \qquad \left\langle \varphi^{(4^6)}, \chi^{(7,6,4,4,2,1)} \right\rangle = 1, \qquad \left\langle \varphi^{(4^6)}, \chi^{(7,6,5,3,2,1)} \right\rangle = 1$$

and that this is an exhaustive list of constituents arising with non-zero multiplicity in $\varphi^{(4^6)}$ that are labelled by a partition satisfying $\lambda_1 = 7$ and $\lambda_2 = 6$.

6.5 Conjectures

We conjecture the following more general statements about near-minimal constituents of $\varphi^{(m^n)}$ and $\tau^{(m^n)}$, which extend the results proved in Proposition 6.3.1.

Conjecture 6.5.1
Given any $m \in \mathbb{N}$ that is even, and any $n \in \mathbb{N}$, let $\lambda \vdash mn$ with at most n parts, such that $\lambda_1 = m + 2$.

1. *If λ has an odd part, then $\left\langle \varphi^{(m^n)}, \chi^\lambda \right\rangle = 0$.*
2. *If λ has all parts even, then $\left\langle \varphi^{(m^n)}, \chi^\lambda \right\rangle = 1$.*

Choose $m \in \mathbb{N}$ that is odd, but let n and λ satisfy the same conditions as above.

1. *If λ has an even part, then $\left\langle \tau^{(m^n)}, \chi^\lambda \right\rangle = 0$.*
2. *If λ has all parts odd, then $\left\langle \tau^{(m^n)}, \chi^\lambda \right\rangle = 1$.*

We expect the proof of this conjecture to be similar to that of Proposition 6.3.1, but requiring a more extensive case-by-case analysis to cope with any even natural number m. A slightly different treatment may be required to prove the part of the conjecture concerning twisted Foulkes characters, though modelling it on the proof of Proposition 6.3.1 should still be possible.

We also make the following conjecture, concerning the lexicographically smallest constituents of $\varphi^{(m^n)}$ (with m even) and $\tau^{(m^n)}$ (with m odd) that have an odd part and an even part, respectively. The data provided in Appendix C supports the conjecture.

Conjecture 6.5.2

If $m > 3$ is even and $n \geq 3$, then the lexicographically smallest constituent of $\varphi^{(m^n)}$ that has an odd part is

$$\chi^{(m+3,\, m^{(n-2)},\, m-3)}.$$

Similarly, if $m \geq 3$ is odd and $n \geq 3$, then the lexicographically smallest constituent of $\tau^{(m^n)}$ that has an even part is

$$\chi^{(m+3,\, m^{(n-2)},\, m-3)}.$$

Chapter 7

The decomposition of $\varphi_\nu^{(2^n)}$

In Chapter 3, we presented the explicit decompositions of the Foulkes character $\varphi^{(2^n)} = \varphi_{(n)}^{(2^n)}$ (Theorem 3.2.3) and the twisted Foulkes character $\tau^{(2^n)} = \varphi_{(1^n)}^{(2^n)}$ (Theorem 3.3.1). In this chapter, we extend these known results when $m = 2$, giving formulae for the decomposition of $\varphi_\nu^{(2^n)}$ when ν is a partition that has two rows or two columns, or is a hook partition. Whilst these formulae allow us to decompose the character, information about multiplicities is not immediately visible. For this reason, we will also derive the complete explicit decomposition of $\varphi_\nu^{(2^n)}$ in a few special cases that are particularly elegant. We will see from the explicit decompositions that, in general, $\varphi_\nu^{(2^n)}$ does not exhibit the multiplicity free property possessed by the characters $\varphi_{(n)}^{(2^n)}$ and $\varphi_{(1^n)}^{(2^n)}$.

The results presented in this chapter are the outcome of joint work with Rowena Paget.

7.1 The decomposition when ν is a two-row partition

The principal result of this section is the following theorem, which provides a formula for the decomposition of the character $\varphi_\nu^{(2^n)}$ when $\nu = (n-r, r)$. The character multiplicities in the decomposition are given in terms of Littlewood–Richardson coefficients.

Theorem 7.1.1
The character $\varphi_{(n-r,r)}^{(2^n)}$ decomposes as

$$\varphi_{(n-r,r)}^{(2^n)} = \sum_{\lambda \vdash 2n} \left(\sum_{\alpha,\beta} c_{2\alpha,2\beta}^{\lambda} - \sum_{\gamma,\delta} c_{2\gamma,2\delta}^{\lambda} \right) \chi^\lambda,$$

where the second sum is over partitions α of $n-r$, β of r, γ of $n-r+1$ and δ of $r-1$.

The proof of this result and many of the subsequent results in this chapter will make use of the following lemma, which can be viewed as a corollary of Lemma 2.3.1.

LEMMA 7.1.2

If $\nu \vdash n-r$ and $\mu \vdash r$, then

$$\left(\varphi_{\nu}^{(m^{n-r})} \times \varphi_{\mu}^{(m^r)}\right)\Big\uparrow_{\mathfrak{S}_{m(n-r)}\times\mathfrak{S}_{mr}}^{\mathfrak{S}_{mn}} = \sum_{\lambda\vdash n} c_{\nu,\mu}^{\lambda}\varphi_{\lambda}^{(m^n)}.$$

Proof. An application of Lemma 2.3.1 with $M = k_{\mathfrak{S}_m}$, $\lambda^1 = \nu \vdash n-r$ and $\lambda^2 = \mu \vdash r$ yields

$$\left(T^{\nu}(k_{\mathfrak{S}_m}) \otimes T^{\mu}(k_{\mathfrak{S}_m})\right)\Big\uparrow_{(\mathfrak{S}_m\wr\mathfrak{S}_{n-r})\times(\mathfrak{S}_m\wr\mathfrak{S}_r)}^{\mathfrak{S}_m\wr\mathfrak{S}_n} \cong \bigoplus_{\lambda\vdash n} c_{\nu,\mu}^{\lambda} T^{\lambda}(k_{\mathfrak{S}_m}).$$

Inducing further, up to $\mathfrak{S}_{mn}$, and using properties of induction detailed in Lemma 2.1.1, it follows that

$$\left(T^{\nu}(k_{\mathfrak{S}_m}) \otimes T^{\mu}(k_{\mathfrak{S}_m})\right)\Big\uparrow_{(\mathfrak{S}_m\wr\mathfrak{S}_{n-r})\times(\mathfrak{S}_m\wr\mathfrak{S}_r)}^{\mathfrak{S}_{mn}} \cong \bigoplus_{\lambda\vdash n} c_{\nu,\mu}^{\lambda}\, T^{\lambda}(k_{\mathfrak{S}_m})\Big\uparrow_{\mathfrak{S}_m\wr\mathfrak{S}_n}^{\mathfrak{S}_{mn}}.$$

Exploiting the properties of induction further, we can rewrite the left-hand side of the above isomorphism as

$$\left(T^{\mu}(k_{\mathfrak{S}_m}) \otimes T^{\mu}(k_{\mathfrak{S}_m})\right)\Big\uparrow_{(\mathfrak{S}_m\wr\mathfrak{S}_{n-r})\times(\mathfrak{S}_m\wr\mathfrak{S}_r)}^{\mathfrak{S}_{m(n-r)}\times\mathfrak{S}_{mr}}\Big\uparrow_{\mathfrak{S}_{m(n-r)}\times\mathfrak{S}_{mr}}^{\mathfrak{S}_{mn}};$$

it then follows that

$$\left(H_{\nu}^{(m^{n-r})} \otimes H_{\mu}^{(m^r)}\right)\Big\uparrow_{\mathfrak{S}_{m(n-r)}\times\mathfrak{S}_{mr}}^{\mathfrak{S}_{mn}} \cong \bigoplus_{\lambda\vdash n} c_{\nu,\mu}^{\lambda} H_{\lambda}^{(m^n)}.$$

The result in the statement of the lemma is simply the relationship satisfied by the corresponding ordinary characters. ■

Proof of Theorem 7.1.1. Setting $m = 2$, $\nu = (n-r)$ and $\mu = (r)$ in Lemma 7.1.2 yields

$$\begin{aligned}\left(\varphi_{(n-r)}^{(2^{n-r})} \times \varphi_{(r)}^{(2^r)}\right)\Big\uparrow_{\mathfrak{S}_{2(n-r)}\times\mathfrak{S}_{2r}}^{\mathfrak{S}_{2n}} &= \sum_{\lambda\vdash n} c_{(n-r),(r)}^{\lambda}\varphi_{\lambda}^{(2^n)} \\ &= \varphi_{(n-r,r)}^{(2^n)} + \varphi_{(n-r+1,r-1)}^{(2^n)} + \cdots + \varphi_{(n-1,1)}^{(2^n)} + \varphi_{(n)}^{(2^n)},\end{aligned}$$

since $c_{(n-r),(r)}^{\lambda} = 1$ if and only if $[\lambda]$ can be obtained from $[n-r]$ by a Young's Rule addition of r boxes, and zero otherwise. From an entirely similar application of Lemma 7.1.2, we conclude that

$$\left(\varphi_{(n-r+1)}^{(2^{n-r+1})} \times \varphi_{(r-1)}^{(2^{r-1})}\right)\Big\uparrow_{\mathfrak{S}_{2(n-r+1)}\times\mathfrak{S}_{2(r-1)}}^{\mathfrak{S}_{2n}} = \varphi_{(n-r+1,r-1)}^{(2^n)} + \cdots + \varphi_{(n-1,1)}^{(2^n)} + \varphi_{(n)}^{(2^n)},$$

and thus, using the known decomposition of the Foulkes character $\varphi_{(k)}^{(2^k)}$, we obtain the following expression for $\varphi_{(n-r,r)}^{(2^n)}$:

$$\varphi_{(n-r,r)}^{(2^n)} = \left(\sum_{\alpha\vdash n-r}\chi^{2\alpha} \times \sum_{\beta\vdash r}\chi^{2\beta}\right)\Bigg\uparrow^{\mathfrak{S}_{2n}} - \left(\sum_{\gamma\vdash n-r+1}\chi^{2\gamma} \times \sum_{\delta\vdash r-1}\chi^{2\delta}\right)\Bigg\uparrow^{\mathfrak{S}_{2n}}.$$

An application of the Littlewood–Richardson Rule yields

$$\varphi^{(2^n)}_{(n-r,r)} = \sum_{\lambda \vdash 2n} \sum_{\alpha,\beta} c^{\lambda}_{2\alpha,2\beta} \chi^{\lambda} - \sum_{\lambda \vdash 2n} \sum_{\gamma,\delta} c^{\lambda}_{2\gamma,2\delta} \chi^{\lambda},$$

which is the required result. ■

The decomposition of $\varphi^{(2^n)}_{(n-1,1)}$ is particularly elegant if we define a_λ to be the number of distinct parts of a partition λ.

Corollary 7.1.3

The complete decomposition of $\varphi^{(2^n)}_{(n-1,1)}$ into irreducible constituents is

$$\varphi^{(2^n)}_{(n-1,1)} = \sum_{\gamma \vdash n} (a_{2\gamma} - 1)\chi^{2\gamma} + \sum_{\mu} \chi^{\mu},$$

where we sum over $\mu \vdash 2n$, with at most n parts, which have all but two parts even and the two odd parts distinct.

Proof. Set $r = 1$ in Theorem 7.1.1:

$$\varphi^{(2^n)}_{(n-1,1)} = \sum_{\lambda \vdash 2n} \left(\sum_{\alpha \vdash n-1} c^{\lambda}_{2\alpha,(2)} - \sum_{\gamma \vdash n} c^{\lambda}_{2\gamma,\emptyset} \right) \chi^{\lambda},$$

where $\emptyset$ denotes the empty partition. Clearly $c^{\lambda}_{2\gamma,\emptyset} = 1$ if and only if $\lambda = 2\gamma$; otherwise $c^{\lambda}_{2\gamma,\emptyset} = 0$. Now consider the first Littlewood–Richardson coefficient:

$$c^{\lambda}_{2\alpha,(2)} = \begin{cases} 1 & \text{if } [\lambda] \text{ may be obtained from } [2\alpha] \text{ by a Young's Rule addition of two boxes;} \\ 0 & \text{otherwise.} \end{cases}$$

Hence, $c^{\lambda}_{2\alpha,(2)}$ is non-zero if and only if either λ is an even partition, or λ has precisely two odd parts, which are distinct. In the latter case, χ^{λ} will appear in the decomposition of $\varphi^{(2^n)}_{(n-1,1)}$ with multiplicity one. The Young diagram corresponding to an even partition, say 2γ, is obtainable by a Young's Rule addition of 2 boxes from $a_{2\gamma}$ different partitions of the form 2α. Indeed, the position of the two added boxes could be at the end of the last row of any given size, as illustrated in Figure 7.1. Thus, we conclude that the decomposition of $\varphi^{(2^n)}_{(n-1,1)}$ into irreducibles is

$$\begin{aligned} \varphi^{(2^n)}_{(n-1,1)} &= \left(\sum_{\gamma \vdash n} a_{2\gamma}\chi^{2\gamma} + \sum_{\mu} \chi^{\mu} \right) - \sum_{\gamma \vdash n} \chi^{2\gamma} \\ &= \sum_{\gamma \vdash n} (a_{2\gamma} - 1)\chi^{2\gamma} + \sum_{\mu} \chi^{\mu}, \end{aligned}$$

where we sum over $\mu \vdash 2n$, with at most n parts, which have all but two parts even and the two odd parts distinct. ■

(a) There is only one way in which the Young diagram $\left[(2\gamma_1)^a\right]$ can have arisen by the Young's Rule addition of two boxes: from $\left[(2\gamma_1)^{a-1}, 2\gamma_1 - 2\right]$.

(b) The Young diagram $\left[(2\gamma_1)^a, (2\gamma_2)^b\right]$ can have arisen in two ways: either from $\left[(2\gamma_1)^{a-1}, 2\gamma_1 - 2, (2\gamma_2)^b\right]$ or from $\left[(2\gamma_1)^a, (2\gamma_2)^{b-1}, 2\gamma_2 - 2\right]$.

Figure 7.1: Examples of resulting partitions that have all parts even.

7.2 The decomposition when ν has two columns

The formula for the decomposition of $\varphi^{(2^n)}_{(2^r,1^{n-2r})}$ is obtained in much the same way as that for $\varphi^{(2^n)}_{(n-r,r)}$.

THEOREM 7.2.1
The character $\varphi^{(2^n)}_{(2^r,1^{n-2r})}$ decomposes as

$$\varphi^{(2^n)}_{(2^r,1^{n-2r})} = \sum_{\lambda \vdash 2n} \left(\sum_{\alpha,\beta} c^{\lambda}_{2[\alpha],2[\beta]} - \sum_{\gamma,\delta} c^{\lambda}_{2[\gamma],2[\delta]} \right) \chi^{\lambda},$$

where the second sum is over partitions α of $n-r$, β of r, γ of $n-r+1$ and δ of $r-1$, all with no repeated parts.

Proof. Setting $m = 2$, $\nu = (1^{n-r})$ and $\mu = (1^r)$ in Lemma 7.1.2 yields

$$\begin{aligned}\left(\varphi^{(2^{n-r})}_{(1^{n-r})} \times \varphi^{(2^r)}_{(1^r)}\right)\Big\uparrow^{\mathfrak{S}_{2n}}_{\mathfrak{S}_{2(n-r)}\times\mathfrak{S}_{2r}} &= \sum_{\lambda \vdash n} c^{\lambda}_{(1^{n-r}),(1^r)} \varphi^{(2^n)}_{\lambda} \\ &= \varphi^{(2^n)}_{(1^n)} + \varphi^{(2^n)}_{(2,1^{n-2})} + \cdots + \varphi^{(2^n)}_{(2^r,1^{n-2r})},\end{aligned}$$

since $c^{\lambda}_{(1^{n-r}),(1^r)} = 1$ if and only if $[\lambda]$ can be obtained from $[1^{n-r}]$ by a Pieri's Rule addition of r boxes, and zero otherwise. Applying Lemma 7.1.2 a second time, we conclude that

$$\varphi^{(2^n)}_{(2^r,1^{n-2r})} = \left(\varphi^{(2^{n-r})}_{(1^{n-r})} \times \varphi^{(2^r)}_{(1^r)}\right)\Big\uparrow^{\mathfrak{S}_{2n}}_{\mathfrak{S}_{2(n-r)}\times\mathfrak{S}_{2r}} - \left(\varphi^{(2^{n-r+1})}_{(1^{n-r+1})} \times \varphi^{(2^{r-1})}_{(1^{r-1})}\right)\Big\uparrow^{\mathfrak{S}_{2n}}_{\mathfrak{S}_{2(n-r+1)}\times\mathfrak{S}_{2(r-1)}}.$$

Using the known decomposition of the twisted Foulkes character $\varphi^{(2^k)}_{(1^k)}$, we obtain the following expression for $\varphi^{(2^n)}_{(2^r,1^{n-2r})}$:

$$\varphi^{(2^n)}_{(2^r,1^{n-2r})} = \left(\sum_{\alpha} \chi^{2[\alpha]} \times \sum_{\beta} \chi^{2[\beta]}\right)\Bigg\uparrow^{\mathfrak{S}_{2n}} - \left(\sum_{\gamma} \chi^{2[\gamma]} \times \sum_{\delta} \chi^{2[\delta]}\right)\Bigg\uparrow^{\mathfrak{S}_{2n}},$$

where we sum over all $\alpha \vdash n-r$, $\beta \vdash r$, $\gamma \vdash n-r+1$ and $\delta \vdash r-1$, which have distinct parts. The result follows from an application of the Littlewood–Richardson Rule. ∎

Setting $r=1$ in Theorem 7.2.1 results in another simple decomposition, the statement of which requires the following notation: for a partition γ, let $b_\gamma := |\{i \mid \gamma_i > \gamma_{i+1}+1\}|$.

Corollary 7.2.2
The character $\varphi^{(2^n)}_{(2,1^{n-2})}$ decomposes into irreducible constituents as

$$\varphi^{(2^n)}_{(2,1^{n-2})} = \sum_{\gamma} \left(b_{2[\gamma]}-1\right)\chi^{2[\gamma]} + \sum_{\mu} \chi^{\mu},$$

where the first sum is over all partitions γ of n that have distinct parts, and the second sum is over all partitions μ of $2n$ – with at most n parts – whose Young diagram may be obtained by the addition of two boxes to a partition of the form $2[\alpha]$ (where $\alpha \vdash n-1$) in such a way that the two added boxes do not lie in the same column and do not lie at opposite ends of any leading diagonal hook.

Proof. The result follows from Theorem 7.2.1, by setting $r=1$ and carefully analysing the Littlewood–Richardson coefficients that appear in the resulting formula for $\varphi^{(2^n)}_{(2,1^{n-2})}$. ∎

7.3 The decomposition when ν is a hook partition

In this section, we present two formulae for the decomposition of $\varphi^{(2^n)}_{(n-r,1^r)}$.

Theorem 7.3.1
The character $\varphi^{(2^n)}_{(n-r,1^r)}$ decomposes into irreducible constituents as

$$\varphi^{(2^n)}_{(n-r,1^r)} = \sum_{\lambda \vdash 2n} \left(\sum_{j=0}^{r} (-1)^j \sum_{\alpha^{(j)},\beta^{(j)}} c^{\lambda}_{2\alpha^{(j)},\,2[\beta^{(j)}]} \right) \chi^{\lambda},$$

where the third sum is over all partitions $\alpha^{(j)}$ of $n-r+j$, and over all partitions $\beta^{(j)}$ of $r-j$ with distinct parts. Alternatively,

$$\varphi^{(2^n)}_{(n-r,1^r)} = \sum_{\lambda \vdash 2n} \left(\sum_{j=1}^{n-r} (-1)^{j-1} \sum_{\gamma^{(j)},\,\delta^{(j)}} c^{\lambda}_{2\gamma^{(j)},\,2[\delta^{(j)}]} \right) \chi^{\lambda},$$

where the third sum is over all partitions $\gamma^{(j)}$ of $n-r-j$, and over all partitions $\delta^{(j)}$ of $r+j$ with distinct parts.

Proof. Setting $m=2$, $\nu=(n-r)$ and $\mu=(1^r)$ in Lemma 7.1.2, we conclude that

$$\begin{aligned}\left(\varphi^{(2^{n-r})}_{(n-r)} \times \varphi^{(2^r)}_{(1^r)}\right)\Big\uparrow^{\mathfrak{S}_{2n}}_{\mathfrak{S}_{2(n-r)}\times\mathfrak{S}_{2r}} &= \sum_{\lambda\vdash n} c^{\lambda}_{(n-r),(1^r)} \varphi^{(2^n)}_{\lambda} \\ &= \varphi^{(2^n)}_{(n-r,1^r)} + \varphi^{(2^n)}_{(n-r+1,1^{r-1})}.\end{aligned}$$

Using the known decompositions of $\varphi^{(2^k)}_{(k)}$ and $\varphi^{(2^k)}_{(1^k)}$, and applying the Littlewood–Richardson Rule, we obtain the following expression for $\varphi^{(2^n)}_{(n-r,1^r)}$:

$$\begin{aligned}
\varphi^{(2^n)}_{(n-r,1^r)} &= \left(\sum_{\alpha^{(0)}} \chi^{2\alpha^{(0)}} \times \sum_{\beta^{(0)}} \chi^{2[\beta^{(0)}]} \right) \Bigg\uparrow_{\mathfrak{S}_{2(n-r)}\times\mathfrak{S}_{2r}}^{\mathfrak{S}_{2n}} - \varphi^{(2^n)}_{(n-r+1,1^{r-1})} \\
&= \sum_{\lambda\vdash 2n} \left(\sum_{\alpha^{(0)},\beta^{(0)}} c^{\lambda}_{2\alpha^{(0)},2[\beta^{(0)}]} \right) \chi^{\lambda} - \varphi^{(2^n)}_{(n-r+1,1^{r-1})}
\end{aligned}$$

summing over partitions $\alpha^{(0)}$ of $n-r$, and partitions $\beta^{(0)}$ of r which have distinct parts. In an entirely similar manner, we may obtain an expression for $\varphi^{(2^n)}_{(n-r+1,1^{r-1})}$:

$$\varphi^{(2^n)}_{(n-r+1,1^{r-1})} = \sum_{\lambda\vdash 2n} \left(\sum_{\alpha^{(1)},\beta^{(1)}} c^{\lambda}_{2\alpha^{(1)},2[\beta^{(1)}]} \right) \chi^{\lambda} - \varphi^{(2^n)}_{(n-r+2,1^{r-2})},$$

this time summing over partitions $\alpha^{(1)}$ of $n-r+1$, and partitions $\beta^{(1)}$ of $r-1$ which have distinct parts. Thus,

$$\varphi^{(2^n)}_{(n-r,1^r)} = \sum_{\lambda\vdash 2n} \left(\sum_{\alpha^{(0)},\beta^{(0)}} c^{\lambda}_{2\alpha^{(0)},2[\beta^{(0)}]} - \sum_{\alpha^{(1)},\beta^{(1)}} c^{\lambda}_{2\alpha^{(1)},2[\beta^{(1)}]} \right) \chi^{\lambda} + \varphi^{(2^n)}_{(n-r+2,1^{r-2})}.$$

Repeating this process, we obtain the first formula stated in the theorem.

To obtain the second formula, we set $m=2$, $\nu=(n-r-1)$ and $\mu=(1^{r+1})$ in Lemma 7.1.2. This tells us that

$$\varphi^{(2^n)}_{(n-r,1^r)} = \left(\varphi^{(2^{n-r-1})}_{(n-r-1)} \times \varphi^{(2^{r+1})}_{(1^{r+1})} \right) \Big\uparrow_{\mathfrak{S}_{2(n-r-1)}\times\mathfrak{S}_{2(r+1)}}^{\mathfrak{S}_{2n}} - \varphi^{(2^n)}_{(n-r-1,1^{r+1})}.$$

We again use the known decompositions of $\varphi^{(2^k)}_{(k)}$ and $\varphi^{(2^k)}_{(1^k)}$, and appeal to the Littlewood–Richardson Rule, yielding

$$\varphi^{(2^n)}_{(n-r,1^r)} = \sum_{\lambda\vdash 2n} \left(\sum_{\gamma^{(1)},\delta^{(1)}} c^{\lambda}_{2\gamma^{(1)},2[\delta^{(1)}]} \right) \chi^{\lambda} - \varphi^{(2^n)}_{(n-r-1,1^{r+1})},$$

where the sum is over partitions $\gamma^{(1)}$ of $n-r-1$, and $\delta^{(1)}$ of $r+1$ with distinct parts. A recursive process, obtaining expressions for $\varphi^{(2^n)}_{(n-r-1,1^{r+1})}$, $\varphi^{(2^n)}_{(n-r-2,1^{r+2})}$ and so on, leads to the second formula stated in the theorem. ∎

In the language of symmetric functions, the results in Theorems 7.1.1, 7.2.1 and 7.3.1 decompose the plethysms $s_\nu \circ s_{(2)}$ when ν has either two rows or two columns, or ν is a hook partition. However, we note that by applying the ω involution defined in Chapter 3, the plethysms $s_\nu \circ s_{(1^2)}$ (and the corresponding characters of $\mathfrak{S}_{2n}$) are also determined for those ν specified above. In particular, if $s_\nu \circ s_{(2)} = \sum_{\lambda\vdash 2n} m_\lambda s_\lambda$, with $m_\lambda \in \mathbb{N}_0$ for all λ, then

$$s_\nu \circ s_{(1^2)} = \sum_{\lambda\vdash 2n} m_\lambda s_{\lambda'},$$

where λ' denotes the conjugate partition of λ.

Example 7.3.2

Setting $n = 3$ in Corollary 7.1.3, we find that $\varphi^{(2^3)}_{(2,1)} = \chi^{(5,1)} + \chi^{(4,2)} + \chi^{(3,2,1)}$. Hence, the plethysm $s_{(2,1)} \circ s_{(1^2)}$ decomposes as

$$\begin{aligned} s_{(2,1)} \circ s_{(1^2)} &= \big(s_{(2,1)} \circ s_{(2)}\big)\,\omega \\ &= s_{(5,1)'} + s_{(4,2)'} + s_{(3,2,1)'} \\ &= s_{(2,1^4)} + s_{(2^2,1^2)} + s_{(3,2,1)}. \end{aligned}$$

7.4 Explicit decompositions

We have already seen the explicit decompositions of $\varphi^{(2^n)}_{(n-1,1)}$ and $\varphi^{(2^n)}_{(2,1^{n-2})}$. These concise decompositions were relatively easy to obtain and far more enlightening than the corresponding formulae which gave rise to them. However, in general, without an easy way to determine Littlewood–Richardson coefficients, obtaining the explicit decomposition of $\varphi^{(2^n)}_\nu$ may be quite involved, even with the formulae that we have presented thus far. Indeed, in this section, we see that more work is required to determine the explicit decompositions of $\varphi^{(2^n)}_{(n-2,2)}$ and $\varphi^{(2^n)}_{(n-2,1^2)}$.

Proposition 7.4.1

Define $\{\lambda\} := \{\lambda_i \mid \lambda_i > 0\}$ and $r_\lambda := |\{j \mid j$ is a repeated part of $\lambda\}|$. Further, for $X, Y \subseteq \mathbb{N}$, define $N_\lambda(X \mid Y) := |\{j \geq 0 \mid 2j+x \in \{\lambda\}\,\forall\, x \in X,\ 2j+y \notin \{\lambda\}\,\forall\, y \in Y\}|$. The multiplicities of the constituents of $\varphi^{(2^n)}_{(n-2,1^2)}$ and $\varphi^{(2^n)}_{(n-2,2)}$ are given in Table 7.1.

Remark. A small amount of calculation is required to determine the multiplicities of constituents labelled by even partitions and partitions which have two odd parts. Consequently, such constituents may appear in $\varphi^{(2^n)}_{(n-2,1^2)}$ or $\varphi^{(2^n)}_{(n-2,2)}$ with zero multiplicity.

Proof. Let us consider first the multiplicities of constituents in $\varphi^{(2^n)}_{(n-2,1^2)}$. Theorem 7.3.1 tells us that

$$\varphi^{(2^n)}_{(n-2,1^2)} = \left(\sum_{\alpha^{(0)}\vdash n-2} c^\lambda_{2\alpha^{(0)},\,(3,1)} - \sum_{\alpha^{(1)}\vdash n-1} c^\lambda_{2\alpha^{(1)},\,(2)} + \sum_{\alpha^{(2)}\vdash n} c^\lambda_{2\alpha^{(2)},\,\emptyset} \right) \chi^\lambda.$$

Now, clearly $c^\lambda_{2\alpha^{(2)},\emptyset} = 1$ if and only if $\lambda = 2\alpha^{(2)}$, and zero otherwise. Hence, the third sum in the above equation will only contribute to the multiplicity of constituents labelled by even

λ	Multiplicity of χ^λ as a constituent of $\varphi_\nu^{(2^n)}$	
	$\nu = (n-2, 1^2)$	$\nu = (n-2, 2)$
λ has all parts even	$\binom{a_\lambda}{2} - a_\lambda + 1$	$a_\lambda(a_\lambda - 2) + N_\lambda(4 \mid 2) + r_\lambda$
λ has 2 odd parts that are distinct, and all other parts even	$N_\lambda(3 \mid 2) + 2N_\lambda(2 \mid 1)$ $+N_\lambda(1, 2 \mid \emptyset) - 1$	$2N_\lambda(2 \mid 1) + N_\lambda(1, 2 \mid \emptyset)$ $+N_\lambda(3 \mid 1, 2) - 1$
λ has 2 equal odd parts and all other parts even	$N_\lambda(3 \mid 2) + N_\lambda(2 \mid 1)$	0
λ has 4 odd parts that are distinct and all other parts even	3	3
λ has 4 odd parts, one repeated and two distinct, and all other parts even	1	1
λ has 4 odd parts, forming two pairs of equal odd parts, and all other parts even	0	1
λ not of the above form	0	0

Table 7.1: Multiplicities of constituents of $\varphi^{(2^n)}_{(n-2,1^2)}$ and $\varphi^{(2^n)}_{(n-2,2)}$.

partitions. Similar analysis shows that the second sum makes a contribution to multiplicities of constituents labelled by even partitions and constituents labelled by partitions which have two distinct odd parts and all other parts even.

To determine the form that partitions labelling constituents which appear in $\varphi^{(2^n)}_{(n-2,1^2)}$ with non-zero multiplicity may take, it remains to consider the first sum. The Littlewood–Richardson Rule says that $c^\lambda_{2\alpha^{(0)},(3,1)}$ is non-zero if $[\lambda]$ may be obtained from the Young diagram $\left[2\alpha^{(0)}\right]$ by adding (four boxes containing) three 1s and one 2 in such a way that no two of the 1s appear in the same column, and, when reading from right to left in successive rows, each i is preceded by more $(i-1)$s than is. In particular, this means that, of the four numbers added, the top right-most number should be a 1. Additionally, the Littlewood–Richardson Rule requires any $(\lambda/2\alpha^{(0)})$-tableau to be semistandard. Ensuring that these conditions are satisfied leads to the following possibilities.

- If we add all three 1s in the same row (and the 2 in a suitable position), then we necessarily obtain a partition which has two distinct odd parts.

- If we add precisely two of the three 1s in the same row and the third 1 in a different row (with the 2 in any suitable position) then the resulting partition may have either all parts even; two equal odd parts; or two distinct odd parts.

- If we add the three 1s in three different rows (and the 2 in any suitable position), then the resulting partition may have either four distinct odd parts; four odd parts, of which precisely two are equal; or two distinct odd parts.

We note that we cannot add all four boxes to $\left[2\alpha^{(0)}\right]$ in the same row and satisfy the conditions of the Littlewood–Richardson Rule. This is because the tableaux $\boxed{2}\boxed{1}\boxed{1}\boxed{1}$, $\boxed{1}\boxed{2}\boxed{1}\boxed{1}$ and $\boxed{1}\boxed{1}\boxed{2}\boxed{1}$ are not semistandard and in the tableau $\boxed{1}\boxed{1}\boxed{1}\boxed{2}$, the right-most number is not 1.

Thus, only constituents labelled by partitions of the form described in the statement of the proposition may appear in $\varphi^{(2^n)}_{(n-2,1^2)}$ with non-zero multiplicity. In particular, we can already rule out the possibility that λ has four odd parts (forming two pairs of equal odd parts) and all other parts even. It remains to determine the non-zero multiplicities in the other cases.

Case I: Assume that λ is an even partition, say $\lambda = 2\gamma$ for some $\gamma \vdash n$. The argument used in the proof of Corollary 7.1.3 shows that the multiplicity of $\chi^{2\gamma}$ is

$$\sum_{\alpha^{(0)} \vdash n-2} c^{2\gamma}_{2\alpha^{(0)},(3,1)} - \left(a_{2\gamma} - 1\right),$$

where, recall, $a_{2\gamma}$ denotes the number of distinct parts of 2γ. Now consider $c^{2\gamma}_{2\alpha^{(0)},(3,1)}$. If $a_{2\gamma} = 1$, i.e. all parts of 2γ are equal, then $[2\gamma]$ cannot have arisen from any $\left[2\alpha^{(0)}\right]$.

If $a_{2\gamma} \geq 2$, then the Young diagram $[2\gamma]$ can only have arisen by the addition of two of the three 1s in one row, and the addition of the remaining 1, followed by the 2, in another row, as indicated by Figure 7.2.

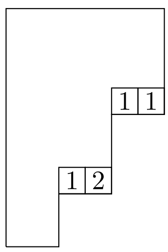


Figure 7.2: Configuration from which $[2\gamma]$ can arise.

Moreover, the pair of 1s must be higher than the $\boxed{1\,|\,2}$ combination in the diagram (although not directly above), so as not to violate the Littlewood–Richardson Rule. Thus, the number of ways in which $[2\gamma]$ can have arisen is given by the binomial coefficient $\binom{a_{2\gamma}}{2}$.

We conclude that the multiplicity with which $\chi^{2\gamma}$ appears is $\binom{a_{2\gamma}}{2} - a_{2\gamma} + 1$, noting that that there is no ambiguity in the case when $a_{2\gamma} = 1$, since $\binom{1}{2} - 1 + 1 = 0 - 1 + 1 = 0$.

Case II: Assume that λ has two distinct odd parts, say λ_i and λ_j with $i < j$. The multiplicity of a constituent labelled by a partition of this form is

$$\sum_{\alpha^{(0)} \vdash n-2} c^{\lambda}_{2\alpha^{(0)},\,(3,1)} - \sum_{\alpha^{(1)} \vdash n-1} c^{\lambda}_{2\alpha^{(1)},\,(2)} = \sum_{\alpha^{(0)} \vdash n-2} c^{\lambda}_{2\alpha^{(0)},\,(3,1)} - 1.$$

Again, we must determine when $[\lambda]$ may be obtained from a Young diagram corresponding to $2\alpha^{(0)}$ for some partition $\alpha^{(0)}$ of $n-2$ by the addition of four boxes containing three 1s and one 2, without violating the Littlewood–Richardson Rule. The possible configurations are shown in Figure 7.3.

We conclude that, in this case,

$$\sum_{\alpha^{(0)} \vdash n-2} c^{\lambda}_{2\alpha^{(0)},\,(3,1)} = N_\lambda(1,3 \mid 2) + N_\lambda(3 \mid 1,2) + N_\lambda(1,2 \mid \emptyset) + 2N_\lambda(2 \mid 1).$$

Observing that

$$\begin{aligned}\big\{j \geq 0 \mid 2j{+}1,\, 2j{+}3 \in \{\lambda\},\, 2j{+}2 \notin \{\lambda\}\big\} \cup \big\{j \geq 0 \mid 2j{+}3 \in \{\lambda\},\, 2j{+}1,\, 2j{+}2 \notin \{\lambda\}\big\} \\ = \big\{j \geq 0 \mid 2j + 3 \in \{\lambda\},\, 2j + 2 \notin \{\lambda\}\big\}\end{aligned}$$

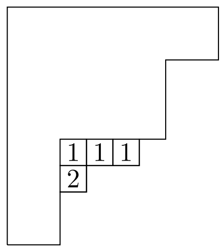
and therefore that $N_\lambda(1,3 \mid 2) + N_\lambda(3 \mid 1,2) = N_\lambda(3 \mid 2)$, we deduce that the multiplicity of χ^λ is $N_\lambda(3 \mid 2) + N_\lambda(1,2 \mid \emptyset) + 2N_\lambda(2 \mid 1) - 1$.

Case III: Assume that λ has precisely two equal odd parts, denoted by λ_i and λ_{i+1}. In this case the multiplicity of χ^λ is $\sum_{\alpha^{(0)} \vdash n-2} c^{\lambda}_{2\alpha^{(0)},(3,1)}$. Now, the Young diagram $[\lambda]$ can arise from one of two types of configurations, as shown in Figure 7.4.

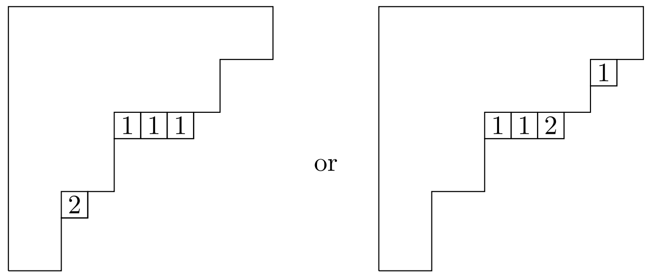
Thus, the multiplicity of χ^λ is $N_\lambda(3 \mid 2) + N_\lambda(2 \mid 1)$.

Case IV: Assume that λ has four distinct odd parts, say $\lambda_{i_1}, \lambda_{i_2}, \lambda_{i_3}, \lambda_{i_4}$. As in the previous case, the multiplicity of χ^λ is precisely $\sum_{\alpha^{(0)} \vdash n-2} c^{\lambda}_{2\alpha^{(0)},(3,1)}$. The Young diagram $[\lambda]$ can have arisen from some $\left[2\alpha^{(0)}\right]$ in precisely three ways:

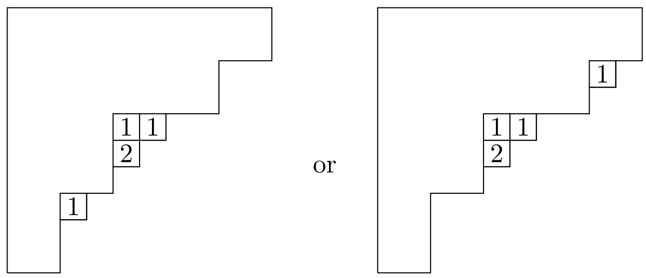
- the three 1s having been added in rows i_1, i_2 and i_3, and the 2 added in row i_4;
- the three 1s having been added in rows i_1, i_2 and i_4, and the 2 added in row i_3; or



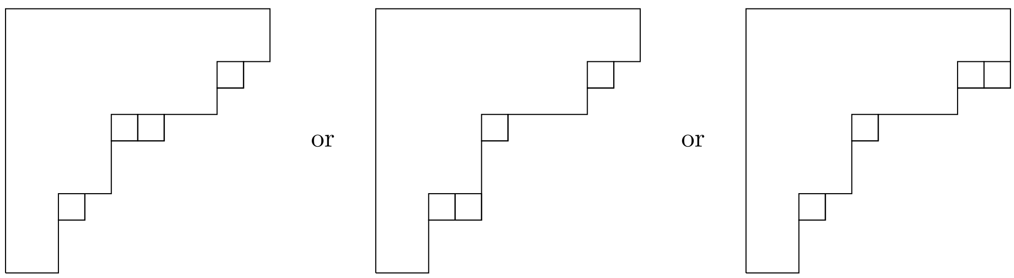
(a) This configuration can occur for all $j \geq 0$ such that $2j+3 \in \{\lambda\}$, $2j+1 \in \{\lambda\}$ and $2j+2 \notin \{\lambda\}$.



(b) This type of configuration can occur for all $j \geq 0$ such that $2j+3 \in \{\lambda\}$, $2j+2 \notin \{\lambda\}$ and $2j+1 \notin \{\lambda\}$.

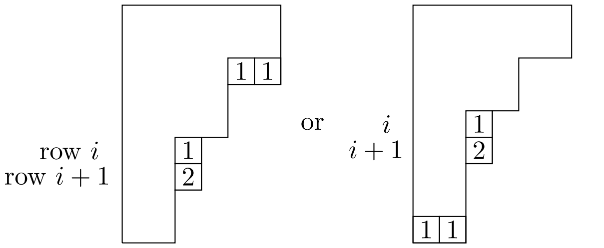


(c) This type of configuration can occur for all $j \geq 0$ such that $2j+2 \in \{\lambda\}$, $2j+1 \in \{\lambda\}$.

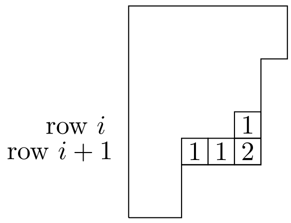


(d) This type of configuration occurs for all $j \geq 0$ such that $2j+2 \in \{\lambda\}$, $2j+1 \notin \{\lambda\}$, always with two ways to fill the four added boxes with three 1s and one 2 (not violating the Littlewood–Richardson Rule). Reading from right to left, the boxes may be filled in the following ways: for the left diagram, either 1112 or 1211; for the middle diagram, either 1121 or 1211; for the right diagram, either 1112 or 1121.

Figure 7.3: Possible configurations from which $[\lambda]$ can arise.



(a) This type of configuration can occur whenever there exists $j \geq 0$ such that $2j+2 \in \{\lambda\}$ and $2j+1 \notin \{\lambda\}$.



(b) This configuration can occur for all $j \geq 0$ such that $2j+3 \in \{\lambda\}$ and $2j+2 \notin \{\lambda\}$.

Figure 7.4: Possible configurations from which $[\lambda]$ can arise.

- the three 1s having been added in rows i_1, i_3 and i_4, and the 2 added in row i_2.

Of course, we must exclude the possibility that the 2 is added in row i_1, since in this configuration the top right-most digit is a 2. Therefore, the multiplicity of χ^λ is three.

Case V: Assume that λ has four odd parts, precisely two being equal. Let $\lambda_{i_1} = \lambda_{i_1+1}$, λ_{i_2} and λ_{i_3} denote the odd parts of λ, making no assumption about their ordering other than that λ_{i_1} and λ_{i_1+1} are consecutive. Once again, the multiplicity of χ^λ is precisely $\sum_{\alpha^{(0)} \vdash n-2} c^\lambda_{2\alpha^{(0)},(3,1)}$. There is a unique way in which $[\lambda]$ may arise (not violating the Littlewood–Richardson Rule): the three 1s having been added in rows i_1, i_2 and i_3, and the 2 added in row i_1+1. This is true regardless of the ordering of $\lambda_{i_1}, \lambda_{i_2}$ and λ_{i_3} and therefore the position of the two equal odd parts among the four odd parts of λ. Thus, in this case, the coefficient of any χ^λ is one.

This verifies the multiplicities given in Table 7.1 of the constituents of $\varphi^{(2^n)}_{(n-2,1^2)}$.

To determine the multiplicities of constituents in the decomposition of $\varphi^{(2^n)}_{(n-2,2)}$, we use Theorem 7.1.1, which tells us that

$$\varphi^{(2^n)}_{(n-2,2)} = \sum_{\lambda \vdash 2n} \left(\sum_{\alpha \vdash n-2} \left(c^\lambda_{2\alpha,(4)} + c^\lambda_{2\alpha,(2,2)} \right) - \sum_{\gamma \vdash n-1} c^\lambda_{2\gamma,(2)} \right) \chi^\lambda.$$

A careful and lengthy analysis of the Littlewood–Richardson coefficients, similar to the analysis used to determine multiplicities in $\varphi^{(2^n)}_{(n-2,1^2)}$, yields the result in the proposition. ∎

Chapter 8

Twisted Foulkes modules in prime characteristic

Thus far, our investigations into the structure of generalised Foulkes modules have all been conducted in the characteristic zero setting. In this chapter, we investigate the modular structure of twisted Foulkes modules, focusing our attention on the case $m = 2$.

The structure of the 'ordinary' Foulkes module $H^{(2^n)}$ (when defined over fields of odd prime characteristic) has already been studied. In his D.Phil thesis [45], Wildon studies summands of $H^{(2^n)}$ in blocks of arbitrary weight, with much success. For example, he shows that, under certain conditions, the only summands of $H^{(2^n)}$ in a given block are projective. In blocks of small weight, more is known: when $2n < 3p$, where p is the (odd prime) characteristic of the ground field, Wildon determines the structure of the unique indecomposable summand of $H^{(2^n)}$ lying in the principal block of $\mathfrak{S}_{2n}$, which is found to be a Scott module.

There are strong similarities between Wildon's findings and the results in this chapter. For example, when $n = p$ (an odd prime), we find that there is a unique indecomposable summand of $K^{(2^p)}$ lying in the principal block. In §8.4, we describe the Loewy layers of this summand; this is, in some sense, an analogue of Wildon's Scott module result.

When $p = 2$, the structure of $K^{(2^n)}$ is very different and, with the exception of a few small examples that we shall see in §8.1, we will not address this case at all. However, we refer an interested reader to [9], in which Collings determines both the fixed point sets and the vertices of the indecomposable summands of $H^{(2^n)}$. Collings' work also addresses the modular structure of $K^{(2^n)}$, since $K^{(2^n)} \cong H^{(2^n)}$ in characteristic two.

In this chapter, the following lemma, which provides us with some initial information about the structure of $K^{(2^n)}$ and its indecomposable summands, will be very useful.

Lemma 8.0.1

The twisted Foulkes module $K^{(2^n)}$ is self-dual.

Proof. Since $K^{(2^n)*} = \left[T^{(1^n)}(S^{(2)})\big\uparrow_{\mathfrak{S}_2\wr\mathfrak{S}_n}^{\mathfrak{S}_{2n}}\right]^*$, the result follows from Lemma 2.1.1(4); the fact that 1-dimensional modules are self-dual; and the fact that $(M_1 \otimes M_2)^* \cong M_1^* \otimes M_2^*$ for any kG-modules M_1 and M_2 (see [2, p.39]). ■

8.1 The structure of $K^{(2^n)}$ in prime characteristic for small n

To begin our investigation into the structure of $K^{(2^n)}$, we use the computational algebra software MAGMA to compute explicitly the structure of the twisted Foulkes modules $K^{(2^2)}$, $K^{(2^3)}$ and $K^{(2^4)}$ over fields of prime characteristic.

We refer the reader to §B.2 – see the function `Socle(n,p)` – for details of the MAGMA code that is used to obtain the information about dimensions of indecomposable summands and the socle series of the modules.

8.1.1 The structure of $K^{(2^2)}$

Over a field of characteristic two, $K^{(2^2)}$ has two simple indecomposable summands, whose dimensions are 1 and 2. Using knowledge about the ordinary character $\tau^{(2^2)} = \chi^{(3,1)}$ and the decomposition matrix of $\mathfrak{S}_4$ in characteristic two (see James and Kerber [26]), we conclude that

$$K^{(2^2)} \cong D^{(4)} \oplus D^{(3,1)}.$$

Over a field of characteristic $p \geq 3$,

$$K^{(2^2)} \cong S^{(3,1)} \cong D^{(3,1)}.$$

Indeed, MAGMA calculations show that, over a field of characteristic three, $K^{(2^2)}$ has a unique indecomposable summand of dimension three, which is simple; the decomposition matrix of $\mathfrak{S}_4$ in characteristic three confirms the structure. If $p \geq 5$, then in particular $p > 2n = 4$ and so, just as in the characteristic zero case, $K^{(2^2)} \cong S^{(3,1)}$.

8.1.2 The structure of $K^{(2^3)}$

We need only concern ourselves with primes $p < 6$ because when $p > 2n = 6$, the structure of $K^{(2^3)}$ over a field of characteristic p is the same as in the characteristic zero setting, i.e.

$$K^{(2^3)} \cong S^{(4,1,1)} \oplus S^{(3,3)}.$$

Henceforth, for ease of notation, we will write $M \sim X_1 + X_2 + \cdots + X_t$ if the $k\mathfrak{S}_{2n}$-module M has composition factors $X_1, X_2, \ldots, X_t$. If M_1 and M_2 are $k\mathfrak{S}_{2n}$-modules that have the same composition factors, then we may write $M \sim N$.

Over a field of characteristic two, the decomposition matrix of $\mathfrak{S}_6$ tells us that

$$S^{(4,1,1)} \sim 2D^{(6)} + D^{(5,1)} + D^{(4,2)} \quad \text{and} \quad S^{(3,3)} \sim D^{(6)} + D^{(4,2)}.$$

MAGMA calculations show that $K^{(2^3)}$ has two indecomposable summands: a summand of dimension one – we know this must be the trivial $k\mathfrak{S}_6$-module $D^{(6)}$ – and a summand of dimension fourteen. MAGMA calculations also tell us the dimensions of the modules in the socle series of the second of these summands, which are $4, 5, 9, 10, 14$. Thus, comparing dimensions with the dimensions of the composition factors known to be present and using the fact that $K^{(2^3)}$ is self-dual, we conclude that, over a field of characteristic two,

$$K^{(2^3)} \cong D^{(6)} \oplus \begin{pmatrix} D^{(4,2)} \\ D^{(6)} \\ D^{(5,1)} \\ D^{(6)} \\ D^{(4,2)} \end{pmatrix}.$$

A similar analysis over a field of characteristic three tells us that $S^{(4,1,1)} \sim D^{(5,1)} + D^{(4,1,1)}$ and $S^{(3,3)} \sim D^{(5,1)} + D^{(3,3)}$. Using MAGMA, we find that $K^{(2^3)}$ is indecomposable and the modules in the socle series have dimensions 4, 11 and 15. Thus,

$$K^{(2^3)} \cong \begin{array}{c} D^{(5,1)} \\ D^{(3,3)} \oplus D^{(4,1,1)} \\ D^{(5,1)} \end{array}.$$

Over a field of characteristic five, MAGMA calculations show that

$$K^{(2^3)} \cong S^{(4,1,1)} \oplus S^{(3,3)}.$$

Indeed, MAGMA tells us that $K^{(2^3)}$ has two summands, whose dimensions are 5 and 10. Further, the partitions $(4, 1, 1)$ and $(3, 3)$ are visibly 5-cores – hence the corresponding Specht modules lie in two different blocks: $B\big((4, 1, 1), 0\big)$ and $B\big((3, 3), 0\big)$ – and it is easy to check using the hook formula that $\dim S^{(4,1,1)} = 10$ and $\dim S^{(3,3)} = 5$.

8.1.3 The structure of $K^{(2^4)}$

For $p \geq 3$, the structure of $K^{(2^4)}$ is obtained in much the same way as that of $K^{(2^2)}$ and $K^{(2^3)}$. We find that, over a field of characteristic three,

$$K^{(2^4)} \cong \begin{array}{c} D^{(5,2,1)} \\ D^{(5,3)} \oplus D^{(4,3,1)} \\ D^{(5,2,1)} \end{array}$$

and, over a field of characteristic $p \geq 5$,

$$K^{(2^4)} \cong S^{(5,1,1,1)} \oplus S^{(4,3,1)}.$$

Over a field of characteristic two, the structure of $K^{(2^4)}$ is more complicated. The following table shows the dimensions of the three indecomposable summands and the modules in their corresponding socle series, obtained using MAGMA.

Dimension of indecomposable summands	Dimension of modules in the socle series
1	1
28	8, 14, 20, 28
76	14, 21, 61, 62, 76

Table 8.1: The dimensions of indecomposable summands of $K^{(2^4)}$.

The decomposition matrix of $\mathfrak{S}_8$ in characteristic two (see [26]) tells us that

$$\begin{aligned} S^{(5,1,1,1)} &\sim D^{(8)} + 2D^{(7,1)} + D^{(6,2)} + D^{(5,3)}, \\ S^{(4,3,1)} &\sim 2D^{(8)} + D^{(7,1)} + D^{(6,2)} + D^{(5,3)} + D^{(4,3,1)}. \end{aligned}$$

Hence, $K^{(2^4)} \sim 3D^{(8)} + 3D^{(7,1)} + 2D^{(6,2)} + 2D^{(5,3)} + D^{(4,3,1)}$. However, in order to confirm the structure of the module, we need some additional information; we require information about the heart of the indecomposable summand M of dimension 76. Using MAGMA again – making use of the function `K24decomp(p)` (see §B.2) – we find that the heart of M is decomposable:

Dimension of indecomposable summand	Dimension of modules in the socle series
6	6
42	1, 41, 42

Table 8.2: The dimensions of indecomposable summands of the heart of M.

Thus, comparing dimensions, we conclude that

$$K^{(2^4)} \cong D^{(8)} \oplus \begin{pmatrix} D^{(5,3)} \\ D^{(7,1)} \\ D^{(7,1)} \\ D^{(5,3)} \end{pmatrix} \oplus \left(\begin{array}{c} D^{(6,2)} \\ D^{(7,1)} \oplus \begin{pmatrix} D^{(8)} \\ D^{(4,3,1)} \\ D^{(8)} \end{pmatrix} \\ D^{(6,2)} \end{array} \right).$$

8.2 The abacus

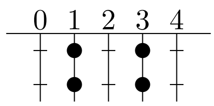
A partition λ may be displayed on an abacus consisting of p runners that are labelled (from left to right) by $0, 1, \ldots, p-1$. We choose p to be the characteristic of the field k. The positions on the runners are labelled numerically, as follows.

$$\begin{array}{cccccc}
0 & 1 & 2 & 3 & \cdots & p-1 \\
p & p+1 & p+2 & p+3 & \cdots & 2p-1 \\
2p & 2p+1 & 2p+2 & \cdots & \cdots &
\end{array}$$

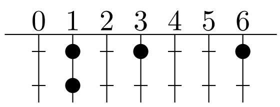
Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ of n, we obtain an abacus configuration for λ by placing beads, $\bullet$, in positions $\beta_i = \lambda_i - i + r$ for $1 \le i \le r$. Where there is no bead, we write $-$.

On a p-runner abacus, removing a rim p-hook from $[\lambda]$ corresponds to sliding a bead on the abacus up one position into a gap directly above it on the runner. In this way, the abacus configuration for a p-core of a partition λ can be obtained from that of λ by sliding all beads on the abacus up as far as possible.

Example 8.2.1
Let $\lambda = (5,4,2,1)$. We calculate that $\beta_1 = 8$, $\beta_2 = 6$, $\beta_3 = 3$ and $\beta_4 = 1$. Thus, if $p = 5$, we obtain the following abacus configuration for λ.



If instead we had chosen $p = 7$, then the abacus configuration that we would obtain is



Observe that $(5,4,2,1)$ is both a 5-core and a 7-core.

8.2.1 Abacus configurations

Let $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ be a partition of n with distinct parts. What does $2[\alpha]$ look like on a p-runner abacus? To construct the abacus:

Step 1 Put α_1 beads on the abacus, in positions $0, 1, 2, \ldots, \alpha_1 - 1$.

Step 2 Counting backwards from the last bead, take the $\alpha_\ell{}^{\text{th}}$ bead and move it to the right, across α_ℓ gaps.

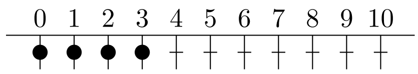
Step 3 Again, counting backwards from the last bead, take the ${\alpha_{\ell-1}}^{\text{th}}$ bead and move it to the right, across $\alpha_{\ell-1}$ gaps. The bead should be placed it in the next available gap (which is possibly after a series of beads that immediately follow the $\alpha_{\ell-1}$ gaps).

Step 4 Continue, as above, until the first bead (which is the ${\alpha_1}^{\text{th}}$ bead when counting backwards from the last bead) has been moved; the partition represented on the abacus is now $2[\alpha]$.

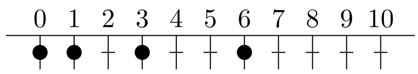
Furthermore, the beads on the abacus representing $2[\alpha]$ will be in positions $\alpha_1 + \alpha_i$ for all $1 \leq i \leq \ell$, and positions $\alpha_1 - j$ for all $1 \leq j \leq \alpha_1$ such that $j \notin \{\alpha_1, \alpha_2, \ldots, \alpha_\ell\}$. There are no other beads on the abacus and so we note that there will be gaps in positions $\alpha_1 - \alpha_i$ for all $1 \leq i \leq \ell$.

Example 8.2.2
Let $\alpha = (4, 3, 2)$ and choose $p = 11$.

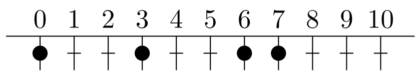
Step 1 *Place four beads on the abacus, in positions 0-3.*



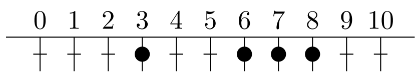
Step 2 *Move the second bead from the end – this is the bead in position 2 – to the right, across two gaps.*



Step 3 *Move the third bead from the end, which is in position 1, to the right, across three gaps.*



Step 4 *Finally, move the fourth bead from the end, which is the bead in position 0, to the right, across four gaps.*



This is the abacus configuration representing the partition $2[(4, 3, 2)] = (5, 5, 5, 3)$.

8.3 The structure of $K^{(2^n)}$ in characteristic $p > n$

THEOREM 8.3.1
If $n < p$, then $K^{(2^n)}$ is semisimple. In particular,

$$K^{(2^n)} = \bigoplus_{\alpha} S^{2[\alpha]},$$

where the sum is over all partitions α of n that have distinct parts.

Proof. If $p = 2$, then the statement of the theorem is trivial. In this case, we have that $K^{(2^1)} = S^{(2)} = D^{(2)}$.

Consider p odd. The ordinary character afforded by $K^{(2^n)}$ is

$$\tau^{(2^n)} = \sum \chi^{2[\alpha]},$$

where the sum is over all partitions α of n that have distinct parts. Consider any $2[\alpha]$ labelling a constituent of $\tau^{(2^n)}$. It will be sufficient to show that $2[\alpha]$ is a p-core, since then the Specht module $S^{2[\alpha]}$ is the unique indecomposable module lying in $B(2[\alpha], 0)$, the weight zero p-block labelled by the p-core $2[\alpha]$. The result will then follow from the (unique) block decomposition of $K^{(2^n)}$.

Now, since $n < p$, we certainly cannot remove two (or more) p-hooks and so the p-weight of $2[\alpha]$ is at most one. To rule out the possibility that the p-weight is one, let $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ and look at the abacus configuration for $2[\alpha]$.

If $\alpha_1 \leq \lfloor p/2 \rfloor$, then all beads are on row one of the abacus and so the p-weight of $2[\alpha]$ is certainly zero. If $\alpha_1 \geq \lceil p/2 \rceil$, then there are beads on exactly two rows of the abacus. Moreover, the beads on the second row are in positions $\alpha_1 + \alpha_i > p - 1$. We will show that, whenever $\alpha_1 + \alpha_i > p - 1$ for some $1 \leq i \leq \ell$, there exists $1 \leq j < \alpha_1$ with $j \notin \{\alpha_2, \ldots, \alpha_\ell\}$ such that

$$\alpha_1 + \alpha_i = \alpha_1 - j + p. \tag{8.1}$$

This means that, on runner i, we have the configuration shown in Figure 8.1 and hence the

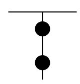


Figure 8.1: A bead configuration

lower bead on the runner cannot be raised. Again, in this case, the p-weight of $2[\alpha]$ is zero. We note that the configuration shown in Figure 8.1 cannot arise from a situation in which $\alpha_1 + \alpha_{i_1} = \alpha_1 + \alpha_{i_2} + p$ for some $1 \leq i_1, i_2 \leq \ell$, since then it follows that $\alpha_{i_1} = \alpha_{i_2} + p$ and hence $\sum_{i=1}^{\ell} \alpha_i > n$.

Let $\alpha_1 + \alpha_i > p - 1$ for some $1 \leq i \leq \ell$, so that there is a bead on the second row of the abacus. For a contradiction, assume that there is a gap directly above this bead on the runner, in position $\alpha_1 + \alpha_i - p$. Clearly $\alpha_1 + \alpha_i - p \leq \alpha_1$, which means that $\alpha_1 + \alpha_i - p = \alpha_1 - \alpha_q$ for some $1 \leq q \leq \ell$, or equivalently, that $p - \alpha_i = \alpha_q$ for $1 \leq q \leq \ell$.

Certainly, $p - \alpha_i \neq \alpha_i$, since otherwise, it follows that p is even, which is a contradiction. So, $p - \alpha_i = \alpha_q \in \{\alpha_1, \ldots, \alpha_\ell\} \backslash \{\alpha_i\}$. However, in this case,

$$n = \sum_{j=1}^{\ell} \alpha_j \geq \alpha_i + \alpha_q = p > n,$$

which is again a contradiction. Therefore, we conclude that $p - \alpha_i \neq \alpha_q$ for any $1 \leq q \leq \ell$. Now, setting $j = p - \alpha_i$, we see that $1 \leq j \leq \alpha_1$ and $j \notin \{\alpha_1, \alpha_2, \ldots, \alpha_\ell\}$. Moreover, $\alpha_1 + \alpha_i = \alpha_1 - j + p$, which says that condition (8.1) holds. ■

8.4 The structure of $K^{(2^p)}$ in characteristic p

Throughout this section, let p be an odd prime. We will investigate the structure of $K^{(2^p)}$, defined over a field of characteristic p. The structure of $K^{(2^2)}$ over a field of characteristic two, which was described explicitly in §8.1.1, is somewhat different. For instance, all indecomposable summands of $K^{(2^2)}$ lie in the principal block of $\mathfrak{S}_4$. The next proposition shows that this is not necessarily the case when p is odd.

Proposition 8.4.1

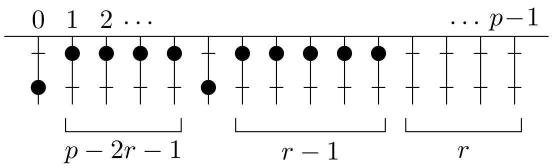
Let p be an odd prime and let $K^{(2^p)}$ be defined over a field of characteristic p. Every summand of $K^{(2^p)}$ lies in either a weight zero p-block of $\mathfrak{S}_{2p}$ or in the weight two principal block $B_0(\mathfrak{S}_{2p})$.

Proof. We examine the p-weight and p-core of each partition $2[\alpha]$, where $\alpha \vdash p$ has distinct parts; there are several cases to consider.

Case I: If $\alpha = (p)$, then $2[\alpha] = \big(p+1, 1^{p-1}\big)$, which has p-weight 2 and empty p-core.

Case II: If $\alpha = (p-r, r)$ for $1 \leq r \leq \lfloor (p-1)/2 \rfloor$, then $2[\alpha] = \big(p-r+1, r+2, 2^{r-1}, 1^{p-2r-1}\big)$.

So, on a p-runner abacus, the partition $2[\alpha]$ has configuration



and therefore, the beads on runners 0 and $p - 2r$ can both be raised. It follows that, when α is a two-row partition, $2[\alpha]$ has p-weight 2 and empty p-core.

Case III: If $\alpha \vdash p$ has $\ell > 2$ distinct parts, then the abacus configuration representing $2[\alpha]$ has beads on at most two rows of the abacus. Indeed, for any $1 \leq i \leq \ell$, $\alpha_1 + \alpha_i > 2p-1$ is impossible. To see this, observe that $\alpha_1 \leq p-2$ and thus $2\alpha_1 < 2p-1$. Additionally, $\alpha_1 + \alpha_i < 2\alpha_1$ for all $2 \leq i \leq \ell$.

Since $2[\alpha]$ is a partition of $2p$, it can have p-weight at most two. However, we will see that whenever there is a bead in the second row of the abacus configuration, there is also a bead in the first row, directly above it on the runner, and thus $2[\alpha]$ must be a p-core.

Suppose that there is a bead in position $\alpha_1 + \alpha_i > p - 1$ and, for a contradiction, suppose that there is a gap in position $\alpha_1 + \alpha_i - p$. Since α has at least three distinct parts, $\alpha_i \leq p - 2$ and hence $\alpha_1 + \alpha_i - p \leq \alpha_1 - 2$. This means that there is a gap in position $\alpha_1 + \alpha_i - p$ if and only if $\alpha_1 + \alpha_i - p = \alpha_1 - \alpha_q$ for some $1 \leq q \leq \ell$, which holds if and only if $\alpha_i + \alpha_q = p$. However, if $q = i$, then this says that p is even, which is a contradiction; if $q \neq i$, then it follows that $\alpha \vdash p$ has exactly two parts, which is another contradiction. ■

Corollary 8.4.2

The Specht modules $S^{2[(p-r,r)]}$ *with* $0 \leq r \leq \lfloor (p-1)/2 \rfloor$ *lie in* $B_0(\mathfrak{S}_{2p})$.

8.4.1 Weight zero blocks

In Proposition 8.4.1 we saw that summands of $K^{(2^p)}$ either lie in a weight zero p-block or in the principal block of $\mathfrak{S}_{2p}$. Further, we saw that if α is a partition of p that has at least three parts, necessarily all distinct, then $2[\alpha]$ has p-weight zero and these are the only partitions $2[\alpha]$ with p-weight zero.

The unique indecomposable module lying in a weight zero p-block, say $B(\gamma, 0)$, of $\mathfrak{S}_{2p}$ is the simple Specht module S^γ. Since weight zero p-blocks are semisimple, we conclude that the (direct) summands of $K^{(2^p)}$ which are in weight zero p-blocks are those Specht modules $S^{2[\alpha]}$ for which $\alpha \vdash p$ has at least three parts, all distinct.

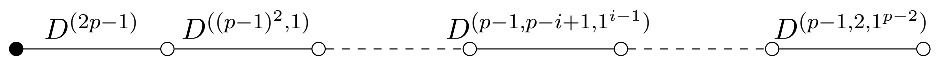
It just remains to consider the principal block of $\mathfrak{S}_{2p}$.

8.4.2 Weight two blocks

In this section, we will show that there is a unique summand of $K^{(2^p)}$ lying in the principal block of $\mathfrak{S}_{2p}$.

The first step towards determining the structure of summands of $K^{(2^p)}$ lying in the principal block of $\mathfrak{S}_{2p}$ is identifying the composition factors of each $S^{2[(p-r,r)]}$. To do this, the approach that we will take is to apply the Branching Rule to obtain a filtration by Specht modules for each $S^{2[(p-r,r)]}\downarrow_{\mathfrak{S}_{2p-1}}$. For any $0 \leq r \leq \lfloor (p-1)/2 \rfloor$, $S^{2[(p-r,r)]}$ lies in $B_0(\mathfrak{S}_{2p})$ and so its restriction will lie in a weight one p-block of $\mathfrak{S}_{2p-1}$. The weight one p-blocks of $\mathfrak{S}_{2p-1}$ are Brauer tree algebras, with the following Brauer trees (see Martin [35, Theorem 4.2.2]).

Let B_k be the weight one p-block of $\mathfrak{S}_{2p-1}$ with p-core $(k, 1^{p-k-1})$, where $1 \leq k \leq p-1$. The Brauer tree of B_{p-1} is



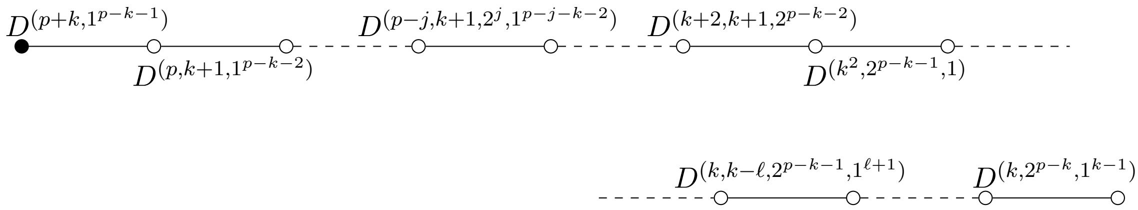
and the Specht modules $S^{(2p-1)}$ and $S^{(p-1,p-i+1,1^{i-1})}$ have the following structure:

$$S^{(2p-1)} \cong D^{(2p-1)}, \qquad S^{((p-1)^2,1)} \cong \begin{array}{c} D^{((p-1)^2,1)} \\ D^{(2p-1)} \end{array} \tag{8.2}$$

and

$$S^{(p-1,p-i+1,1^{i-1})} \cong \begin{array}{c} D^{(p-1,p-i+1,1^{i-1})} \\ D^{(p-1,p-i+2,1^{i-2})} \end{array} \qquad (3 \leq i \leq p-1). \tag{8.3}$$

For $1 \leq k \leq p-2$, the Brauer tree of B_k is



for $0 \leq j \leq p-k-2$ and $0 \leq \ell \leq k-2$. The Specht modules $S^{(p+k,1^{p-k-1})}$, $S^{(p,k+1,1^{p-k-2})}$, $S^{(p-j,k+1,2^j,1^{p-j-k-2})}$, $S^{(k^2,2^{p-k-1},1)}$ and $S^{(k,k-\ell,2^{p-k-1},1^{\ell+1})}$ have the following structure:

$$S^{(p+k,1^{p-k-1})} \cong D^{(p+k,1^{p-k-1})}, \qquad S^{(p,k+1,1^{p-k-2})} \cong \begin{array}{c} D^{(p,k+1,1^{p-k-2})} \\ D^{(p+k,1^{p-k-1})} \end{array}, \tag{8.4}$$

$$S^{(p-j,k+1,2^j,1^{p-j-k-2})} \cong \begin{array}{c} D^{(p-j,k+1,2^j,1^{p-j-k-2})} \\ D^{(p-j+1,k+1,2^{j-1},1^{p-j-k-1})} \end{array} \qquad (1 \leq j \leq p-k-2), \tag{8.5}$$

$$S^{(k^2,2^{p-k-1},1)} \cong \begin{array}{c} D^{(k^2,2^{p-k-1},1)} \\ D^{(k+2,k+1,2^{p-k-2})} \end{array} \tag{8.6}$$

and

$$S^{(k,k-\ell,2^{p-k-1},1^{\ell+1})} \cong \begin{array}{c} D^{(k,k-\ell,2^{p-k-1},1^{\ell+1})} \\ D^{(k,k-\ell+1,2^{p-k-1},1^{\ell})} \end{array} \qquad (1 \leq \ell \leq k-2). \tag{8.7}$$

We will make extensive use of the following lemma from [39, Lemma 4.1.1], which describes the restrictions to $\mathfrak{S}_{2p-1}$ of the simple modules lying in the principal block of $\mathfrak{S}_{2p}$.

Lemma 8.4.3

The simple modules lying in the principal block of $\mathfrak{S}_{2p}$ *are*

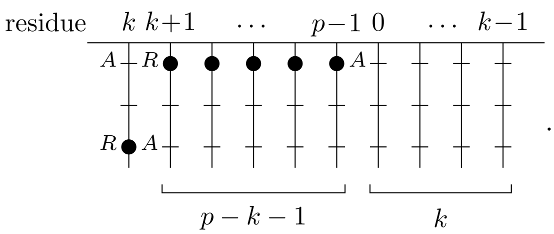
$$\begin{aligned}
u_k &:= D^{(p+k+1,1^{p-k-1})} && \text{for } 0 \le k \le p-1,\\
v_{j,k} &:= D^{(p-j,k+2,2^j,1^{p-j-k-2})} && \text{for } 0 \le j \le p-2,\ 1 \le k \le p-j-3,\\
w_k &:= D^{((k+3)^2,2^{p-k-3})} && \text{for } 0 \le k \le p-4,\\
t_j &:= D^{(p-j,2^{j+1},1^{p-j-2})} && \text{for } 0 \le j \le p-3.
\end{aligned}$$

Restricting these modules to $\mathfrak{S}_{2p-1}$*,*

$$\begin{aligned}
D^{(p+k+1,1^{p-k-1})}\big\downarrow_{\mathfrak{S}_{2p-1}} &= D^{(p+k,1^{p-k-1})},\\
D^{(p-j,k+2,2^j,1^{p-j-k-2})}\big\downarrow_{\mathfrak{S}_{2p-1}} &= D^{(p-j-1,k+2,2^j,1^{p-j-k-2})} \oplus D^{(p-j,k+1,2^j,1^{p-j-k-2})},\\
D^{((k+3)^2,2^{p-k-3})}\big\downarrow_{\mathfrak{S}_{2p-1}} &= D^{(k+3,k+2,2^{p-k-3})},\\
D^{(p-j,2^{j+1},1^{p-j-2})}\big\downarrow_{\mathfrak{S}_{2p-1}} &= D^{(p-j-1,2^{j+1},1^{p-j-2})}.
\end{aligned}$$

Proof. The proof of the first part of the lemma, which details the simple modules lying in $B_0(\mathfrak{S}_{2p})$, is straightforward: the simples are the p-regular partitions that result from the addition of two rim p-hooks to an empty p-core. To determine the restrictions of the simple modules, we apply Kleshchev's Theorem [29, Theorem 1.4]. We illustrate this idea for the simple module u_k, where $0 \le k \le p-1$.

Consider the abacus configuration for the partition $(p+k+1, 1^{p-k-1})$, which is



The nodes marked R are removable nodes and the gaps marked A indicate addable nodes. Only the removable node of residue k is normal[1]; it is also the highest normal node on runner 0 and is therefore good. Let $\mu = (p+k, 1^{p-k-1})$ be the partition that is obtained by removing this node from $(p+k+1, 1^{p-k-1})$.

By Kleshchev's Theorem, $u_k \downarrow_{\mathfrak{S}_{2p-1}} \cong M$, where M is an indecomposable module such that $\operatorname{soc}(M) \cong D^\mu$. Additionally, u_k is self-dual, from which it follows that M is also self-dual and so $M/\operatorname{rad}(M) \cong \operatorname{soc}(M)$. However, since there is only one removable node of residue k, we know (by Kleshchev's Theorem) that M has only one composition factor isomorphic to D^μ and, consequently, $M \cong D^\mu$. ■

[1]For an explanation of terminology related to Kleshchev's Theorem, we refer the reader to [29].

We now have all the information that we need to determine the composition factors of the Specht modules $S^{2[(p-r,r)]}$, where $0 \le r \le \lfloor p-1/2 \rfloor$. It will be convenient to handle some small primes separately, so the next lemma only deals with the case $p \ge 7$.

Lemma 8.4.4
Let $p \ge 7$ be prime.

1. *If $\alpha = (p)$, then $2[\alpha] = (p+1, 1^{p-1})$ and*

$$S^{2[\alpha]} \sim D^{(p+1,1^{p-1})} + D^{(p+2,1^{p-2})} = u_0 + u_1;$$

2. *If $\alpha = (p-1, 1)$, then $2[\alpha] = (p, 3, 1^{p-3})$ and*

$$\begin{aligned} S^{2[\alpha]} &\sim D^{(p+2,1^{p-2})} + D^{(p+3,1^{p-3})} + D^{(p,3,1^{p-3})} + D^{(p,4,1^{p-4})} \\ &= u_1 + u_2 + v_{0,1} + v_{0,2}; \end{aligned}$$

3. *If $\alpha = (p-r, r)$, where $2 \le r \le \frac{p-3}{2}$, then $2[\alpha] = (p-r+1, r+2, 2^{r-1}, 1^{p-2r-1})$ and*

$$\begin{aligned} S^{2[\alpha]} &\sim D^{(p-r+1,r+2,2^{r-1},1^{p-2r-1})} + D^{(p-r+1,r+3,2^{r-1},1^{p-2r-2})} \\ &\qquad + D^{(p-r+2,r+2,2^{r-2},1^{p-2r})} + D^{(p-r+2,r+3,2^{r-2},1^{p-2r-1})} \\ &= v_{r-1,r} + v_{r-1,r+1} + v_{r-2,r} + v_{r-2,r+1}; \end{aligned}$$

4. *If $\alpha = \left(\frac{p+1}{2}, \frac{p-1}{2}\right)$, then $2[\alpha] = \left(\left(\frac{p+3}{2}\right)^2, 2^{(p-3)/2}\right)$ and*

$$\begin{aligned} S^{2[\alpha]} &\sim D^{(((p+7)/2)^2,2^{(p-7)/2})} + D^{(((p+3)/2)^2,2^{(p-3)/2})} + D^{((p+5)/2,(p+3)/2,2^{(p-5)/2},1)} \\ &= w_{(p+1)/2} + w_{(p-3)/2} + v_{(p-5)/2,(p-1)/2}. \end{aligned}$$

Proof. Let $p \ge 7$.

1. By the Branching Rule, $S^{(p+1,1^{p-1})}\big\downarrow_{\mathfrak{S}_{2p-1}} \sim S^{(p,1^{p-1})} + S^{(p+1,1^{p-2})}$. Both Specht modules in the filtration are labelled by hook partitions and so, by a result due to Peel [25, Theorem 24.1], we know that $S^{(p,1^{p-1})} \cong D^{(p,1^{p-1})}$ and $S^{(p+1,1^{p-2})} \cong D^{(p+1,1^{p-2})}$. Using Lemma 8.4.3, observe that $D^{(p,1^{p-1})}$ is a composition factor of $u_0 \downarrow_{\mathfrak{S}_{2p-1}}$ and not a composition factor of any other restricted simple $k\mathfrak{S}_{2p}$-module. Similarly, $D^{(p+1,1^{p-2})}$ is only a composition factor of $u_1 \downarrow_{\mathfrak{S}_{2p-1}}$. Consequently, we deduce that

$$S^{(p+1,1^{p-1})} \sim D^{(p+1,1^{p-1})} + D^{(p+2,1^{p-2})}.$$

2. Applying the Branching Rule, $S^{(p,3,1^{p-3})}\big\downarrow_{\mathfrak{S}_{2p-1}} \sim S^{(p-1,3,1^{p-3})} + S^{(p,2,1^{p-3})} + S^{(p,3,1^{p-4})}$. The p-cores of the partitions labelling these Specht modules are $(p-1)$, (1^{p-1}) and

$(2,1^{p-3})$, respectively, and therefore, the weight one p-blocks of $\mathfrak{S}_{2p-1}$ in which these modules lie are B_{p-1}, B_1 and B_2.

By (8.3), we know that $S^{(p-1,3,1^{p-3})} \sim D^{(p-1,3,1^{p-3})} + D^{(p-1,4,1^{p-4})}$. Now, taking $k=1$ and $k=2$ in turn, (8.4) tells us that $S^{(p,2,1^{p-3})} \sim D^{(p,2,1^{p-3})} + D^{(p+1,1^{p-2})}$ and $S^{(p,3,1^{p-4})} \sim D^{(p,3,1^{p-4})} + D^{(p+2,1^{p-3})}$. We need to examine the modules in Lemma 8.4.3 to determine which of them have the above simple $k\mathfrak{S}_{2p-1}$-modules among their composition factors. We find that the simple modules can only arise as composition factors in the following restrictions:

$$\begin{aligned} v_{0,1}\downarrow_{\mathfrak{S}_{2p-1}} &= D^{(p-1,3,1^{p-3})} \oplus D^{(p,2,1^{p-3})},\\ v_{0,2}\downarrow_{\mathfrak{S}_{2p-1}} &= D^{(p-1,4,1^{p-4})} \oplus D^{(p,3,1^{p-4})},\\ u_1\downarrow_{\mathfrak{S}_{2p-1}} &= D^{(p+1,1^{p-2})},\\ \text{and}\quad u_2\downarrow_{\mathfrak{S}_{2p-1}} &= D^{(p+2,1^{p-3})}. \end{aligned}$$

Thus,

$$S^{(p,3,1^{p-3})} \sim D^{(p,3,1^{p-3})} + D^{(p,4,1^{p-4})} + D^{(p+2,1^{p-2})} + D^{(p+3,1^{p-3})}.$$

3. If $2 \le r \le \frac{p-3}{2}$, then

$$\begin{aligned} S^{(p-r+1,r+2,2^{r-1},1^{p-2r-1})}\big\downarrow_{\mathfrak{S}_{2p-1}} \sim\ & S^{(p-r,r+2,2^{r-1},1^{p-2r-1})} + S^{(p-r+1,r+1,2^{r-1},1^{p-2r-1})}\\ &+ S^{(p-r+1,r+2,2^{r-2},1^{p-2r})} + S^{(p-r+1,r+2,2^{r-1},1^{p-2r-2})}. \end{aligned}$$

Let us consider each Specht module in the filtration in turn.

- The partition $(p-r, r+2, 2^{r-1}, 1^{p-2r-1})$ has p-core $(p-r, 1^{r-1})$ and therefore the corresponding Specht module lies in the p-block B_{p-r}. Examining (8.7) with $k = p-r$ and $\ell = p-2r-2$ tells us that

$$S^{(p-r,r+2,2^{r-1},1^{p-2r-1})} \sim D^{(p-r,r+2,2^{r-1},1^{p-2r-1})} + D^{(p-r,r+3,2^{r-1},1^{p-2r-2})}.$$

- The p-core of $(p-r+1, r+1, 2^{r-1}, 1^{p-2r-1})$ is $(r, 1^{p-r-1})$ and so, by (8.5) with $j = r-1$ and $k = r$,

$$S^{(p-r+1,r+1,2^{r-1},1^{p-2r-1})} \sim D^{(p-r+1,r+1,2^{r-1},1^{p-2r-1})} + D^{(p-r+2,r+1,2^{r-2},1^{p-2r})}.$$

- The Specht module $S^{(p-r+1,r+2,2^{r-2},1^{p-2r})}$ lies in the block B_{p-r+1}. So, setting $k = p-r+1$, $\ell = p-2r-1$ and using (8.7), we conclude that

$$S^{(p-r+1,r+2,2^{r-2},1^{p-2r})} \sim D^{(p-r+1,r+2,2^{r-2},1^{p-2r})} + D^{(p-r+1,r+3,2^{r-2},1^{p-2r-1})}.$$

- The p-core of the partition $(p-r+1, r+2, 2^{r-1}, 1^{p-2r-2})$ is $(r+1, 1^{p-r-2})$ and hence, setting $j = r-1$ and $k = r+1$ in (8.5), we see that

$$S^{(p-r+1,r+2,2^{r-1},1^{p-2r-2})} \sim D^{(p-r+1,r+2,2^{r-1},1^{p-2r-2})} + D^{(p-r+2,r+2,2^{r-2},1^{p-2r-1})}.$$

Again, we look to determine which simple $k\mathfrak{S}_{2p}$-modules have these composition factors in their restrictions to $\mathfrak{S}_{2p-1}$. It is not hard to check that the following restricted modules, which feature the correct composition factors, are the only possibility. We have

$$\begin{aligned} v_{r-1,r} \downarrow_{\mathfrak{S}_{2p-1}} &= D^{(p-r,r+2,2^{r-1},1^{p-2r-1})} \oplus D^{(p-r+1,r+1,2^{r-1},1^{p-2r-1})}, \\ v_{r-1,r+1} \downarrow_{\mathfrak{S}_{2p-1}} &= D^{(p-r,r+3,2^{r-1},1^{p-2r-2})} \oplus D^{(p-r+1,r+2,2^{r-1},1^{p-2r-2})}, \\ v_{r-2,r} \downarrow_{\mathfrak{S}_{2p-1}} &= D^{(p-r+2,r+1,2^{r-1},1^{p-2r})} \oplus D^{(p-r+1,r+2,2^{r-2},1^{p-2r})} \end{aligned}$$

and

$$v_{r-2,r+1} \downarrow_{\mathfrak{S}_{2p-1}} = D^{(p-r+1,r+3,2^{r-2},1^{p-2r-1})} \oplus D^{(p-r+2,r+2,2^{r-2},1^{p-2r-1})}.$$

Thus,

$$\begin{aligned} S^{(p-r+1,r+2,2^{r-1},1^{p-2r-1})} \sim D^{(p-r+1,r+2,2^{r-1},1^{p-2r-1})} &+ D^{(p-r+1,r+3,2^{r-1},1^{p-2r-2})} \\ + D^{(p-r+2,r+2,2^{r-2},1^{p-2r})} &+ D^{(p-r+2,r+3,2^{r-2},1^{p-2r-1})}. \end{aligned}$$

4. By the Branching Rule,

$$S^{\left(\left(\frac{p+3}{2}\right)^2,2^{(p-3)/2}\right)}\Big\downarrow_{\mathfrak{S}_{2p-1}} \sim S^{\left(\frac{p+3}{2},\frac{p+1}{2},2^{(p-3)/2}\right)} + S^{\left(\left(\frac{p+3}{2}\right)^2,2^{(p-5)/2},1\right)}.$$

Since the partition $\left(\frac{p+3}{2}, \frac{p+1}{2}, 2^{(p-3)/2}\right)$ has p-core $\left(\frac{p-1}{2}, 1^{(p-1)/2}\right)$, composition factors of the corresponding Specht module are

$$S^{\left(\frac{p+3}{2},\frac{p+1}{2},2^{(p-3)/2}\right)} \sim D^{\left(\frac{p+3}{2},\frac{p+1}{2},2^{(p-3)/2}\right)} + D^{\left(\frac{p+5}{2},\frac{p+1}{2},2^{(p-5)/2},1\right)}.$$

A similar analysis of the partition $\left(\left(\frac{p+3}{2}\right)^2, 2^{(p-5)/2}, 1\right)$ shows that $\left(\frac{p+3}{2}, 1^{(p-5)/2}\right)$ is its p-core and

$$S^{\left(\left(\frac{p+3}{2}\right)^2,2^{(p-5)/2},1\right)} \sim D^{\left(\left(\frac{p+3}{2}\right)^2,2^{(p-5)/2},1\right)} + D^{\left(\frac{p+7}{2},\frac{p+5}{2},2^{(p-7)/2}\right)}.$$

Using Lemma 8.4.3, we observe that the composition factors of these Specht modules can only appear as composition factors in the following restricted simple $k\mathfrak{S}_{2p}$-modules:

$$\begin{aligned} v_{(p-5)/2,(p-1)/2} \downarrow_{\mathfrak{S}_{2p-1}} &= D^{\left(\frac{p+5}{2},\frac{p+1}{2},2^{(p-5)/2},1\right)} \oplus D^{\left(\left(\frac{p+3}{2}\right)^2,2^{(p-5)/2},1\right)}, \\ w_{(p+1)/2} \downarrow_{\mathfrak{S}_{2p-1}} &= D^{\left(\frac{p+7}{2},\frac{p+5}{2},2^{(p-7)/2}\right)} \end{aligned}$$

and

$$w_{(p-3)/2}\downarrow_{\mathfrak{S}_{2p-1}} = D^{\left(\frac{p+3}{2},\frac{p+1}{2},2^{(p-3)/2}\right)}.$$

Hence, we conclude that

$$S^{\left(\left(\frac{p+3}{2}\right)^2,2^{(p-3)/2}\right)} \sim D^{\left(\left(\frac{p+7}{2}\right)^2,2^{(p-5)/2}\right)} + D^{\left(\left(\frac{p+3}{2}\right)^2,2^{(p-3)/2}\right)} + D^{\left(\frac{p+5}{2},\frac{p+3}{2},2^{(p-5)/2},1\right)}. \quad \blacksquare$$

To conclude this section, we address the small odd primes not covered by the last result.

Lemma 8.4.5

1. *If $p = 3$, then*

$$S^{2[(3)]} = S^{(4,1^2)} \sim D^{(4,1^2)} + D^{(5,1)} = u_0 + u_1,$$
$$S^{2[(2,1)]} = S^{(3,3)} \sim D^{(3,3)} + D^{(5,1)} = w_0 + u_1.$$

2. *If $p = 5$, then*

$$S^{2[(5)]} = S^{(6,1^4)} \sim D^{(6,1^4)} + D^{(7,1^3)} = u_0 + u_1,$$
$$S^{2[(4,1)]} = S^{(5,3,1^2)} \sim D^{(5,3,1^2)} + D^{(5,4,1)} + D^{(7,1^3)} + D^{(8,1^2)} = v_{0,1} + v_{0,2} + u_1 + u_2,$$
$$S^{2[(3,2)]} = S^{(4,4,2)} \sim D^{(10)} + D^{(5,4,1)} + D^{(4,4,2)} = u_4 + v_{0,2} + w_1.$$

Proof. The proof is entirely similar to the proof of Lemma 8.4.4. ■

Remark. The composition factors of $K^{(2^3)}$ and $K^{(2^5)}$ that lie in the principal block are consistent with the more general result given in Lemma 8.4.4. However, when $p = 5$, we see a difference in notation due to the fact that $w_{(p+1)/2} = w_3$ is not defined (cf. Lemma 8.4.3): the simple module $u_4 = D^{(10)}$ is a composition factor of $K^{(2^5)}$, whereas, for $p \geq 7$, the simple module $w_{(p+1)/2}$ appears as a composition factor of $K^{(2^p)}$.

8.4.3 Ext-quivers of the principal block of $\mathfrak{S}_{2p}$

We now know the composition factors of $K^{(2^p)}$ and in particular, which of those lie in the principal block. For those composition factors lying in $B_0(\mathfrak{S}_{2p})$, we illustrate their position on the Ext-quiver, also indicating their multiplicities. First, we briefly explain what we mean by the Ext-quiver, so that we may later interpret the information it provides.

The Ext-*quiver* of an algebra A has vertices that correspond to the isomorphism classes of irreducible A-modules. An arrow is drawn from the vertex corresponding to the module M_i to the vertex corresponding to M_j if $\mathrm{Ext}^1_A(M_i, M_j) \neq 0$, or equivalently, if there is a (non-split[2]) short exact sequence

$$0 \to M_j \to X \to M_i \to 0.$$

[2]A split short exact sequence represents the zero element of $\mathrm{Ext}^1_A(M_i, M_j)$.

Moreover, the number of arrows $M_i \to M_j$ is $\dim_k \operatorname{Ext}^1_A(M_i, M_j)$. If A is a symmetric algebra, for example a group algebra, then $\dim_k \operatorname{Ext}^1_A(M_i, M_j) = \dim_k \operatorname{Ext}^1_A(M_j, M_i)$ and so we simply draw edges between the vertices rather than bidirectional arrows.

Figures 8.2-8.4 show, for a variety of primes p, the part of the Ext-quiver of the principal block of $\mathfrak{S}_{2p}$ (produced by Martin in Chapter 4 of his thesis [35]) that features the composition factors of $K^{(2^p)}$ lying in $B_0(\mathfrak{S}_{2p})$. Unlabelled vertices in the quiver correspond to (isomorphism classes of) irreducible modules that are not composition factors of $K^{(2^p)}$; black vertices correspond to composition factors of $K^{(2^p)}$ that appear with multiplicity one; and red vertices correspond to composition factors of $K^{(2^p)}$ that appear with multiplicity two.

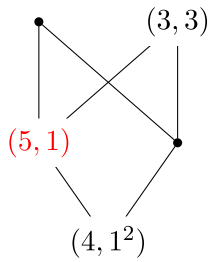


Figure 8.2: Summands of $K^{(2^3)}$ in the principal block of $\mathfrak{S}_6$ in characteristic 3.

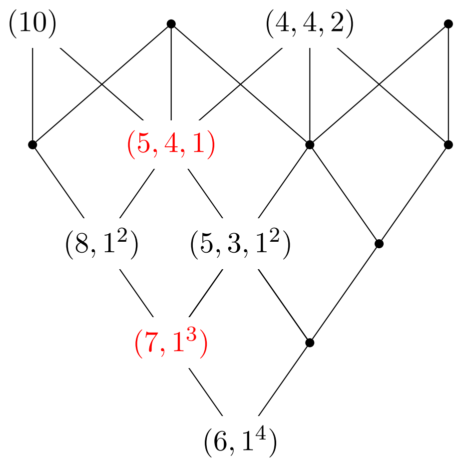


Figure 8.3: Summands of $K^{(2^5)}$ in the principal block of $\mathfrak{S}_{10}$ in characteristic 5.

8.4.4 Summands of $K^{(2^p)}$ in the principal block of $\mathfrak{S}_{2p}$

To proceed, we prove the following result, which describes the summands of $K^{(2^p)}$ that lie in the principal block. We continue to assume that p is an odd prime and take inspiration from Wildon's work in [45], modelling our approach on the methods that he used to determine the structure of the unique summand of $H^{(2^p)}$ lying in the principal block of $\mathfrak{S}_{2p}$.

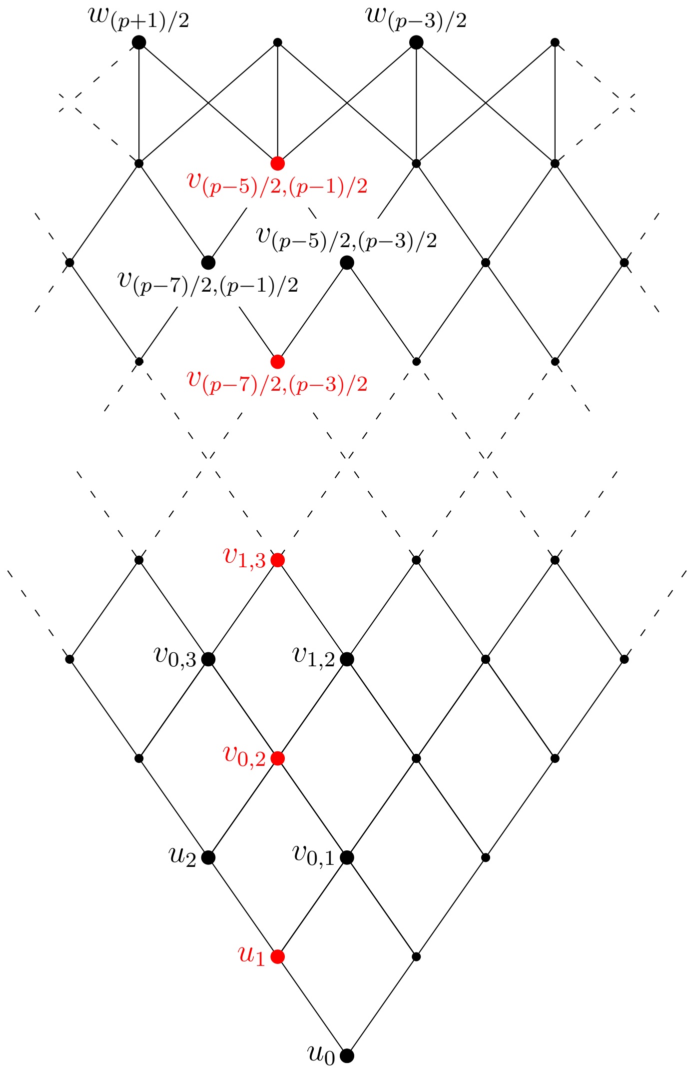


Figure 8.4: Summands of $K^{(2^p)}$ in the principal block of $\mathfrak{S}_{2p}$ in characteristic p, for $p \geq 7$.

Proposition 8.4.6

Let p be an odd prime. There is a unique, indecomposable non-projective summand of $K^{(2^p)}$ lying in $B_0(\mathfrak{S}_{2p})$.

The first step towards proving this result is establishing that $K^{(2^p)}$ is a p-permutation module, which can be done in several ways.

Our first – and simplest – method exploits the fact that the restriction or induction of a p-permutation module is also a p-permutation module [5, Proposition (0.2)]. For this reason, we may consider

$$L := T^{(1^p)}\big(S^{(2)}\big)\big\downarrow^{\mathfrak{S}_2 \wr \mathfrak{S}_p}_{\mathrm{Syl}_p(\mathfrak{S}_2 \wr \mathfrak{S}_p)}$$

and since $K^{(2^p)} = L\big\uparrow^{\mathfrak{S}_{2p}}$, it will be sufficient to show that L is a p-permutation module. Since the order of the wreath product $\mathfrak{S}_2 \wr \mathfrak{S}_p$ is $p!\,2^p$, we easily deduce that $|\,\mathrm{Syl}_p(\mathfrak{S}_2 \wr \mathfrak{S}_p)| = p$

and so $\mathrm{Syl}_p(\mathfrak{S}_2 \wr \mathfrak{S}_p)$ is isomorphic to the cyclic group C_p. Further, $L \cong \mathrm{sgn}_{\mathrm{Syl}_p(\mathfrak{S}_2 \wr \mathfrak{S}_p)}$, the sign $k\,\mathrm{Syl}_p(\mathfrak{S}_2 \wr \mathfrak{S}_p)$-module. However, since the sign of any permutation which generates $\mathrm{Syl}_p(\mathfrak{S}_2 \wr \mathfrak{S}_p)$ is 1, we in fact see that $L \cong 1_{\mathrm{Syl}_p(\mathfrak{S}_2 \wr \mathfrak{S}_p)}$, and thus is a p-permutation module.

Our second approach is to construct a suitable p-permutation basis. For this, we will need to introduce some notation. Let $X^{(2^p)}$ be the set of all (2^p)-tabloids and recall that, as a vector space, the Young permutation module $M^{(2^p)}$ is spanned by $X^{(2^p)}$. Given $\mathbf{x} \in X^{(2^p)}$, which is of the form

$$\mathbf{x} = \begin{array}{cc} \hline x_1 & x_{p+1} \\ \hline x_2 & x_{p+2} \\ \hline \vdots & \vdots \\ \hline x_p & x_{2p} \\ \hline \end{array}, \tag{8.8}$$

where $\{x_1, x_2, \ldots, x_{2p}\} = \{1, 2, \ldots, 2p\}$, define $H(\mathbf{x})$ by

$$H(\mathbf{x}) := \big\{\{x_1, x_{p+1}\}, \{x_2, x_{p+2}\}, \ldots, \{x_p, x_{2p}\}\big\}.$$

There is a natural action of $\sigma \in \mathfrak{S}_{H(\mathbf{x})} \cong \mathfrak{S}_p$, which permutes the rows of $\mathbf{x}$ according to σ. Let R be the subspace of $M^{(2^p)}$ spanned by

$$\left\{\mathbf{x} - \mathrm{sgn}(\sigma)\mathbf{x}\sigma \;\middle|\; \mathbf{x} \in X^{(2^p)}, \sigma \in \mathfrak{S}_{H(\mathbf{x})}\right\}$$

and let $\overline{\mathbf{x}}$ be the image $\mathbf{x} + R$ of $\mathbf{x}$ under the quotient map $M^{(2^p)} \to M^{(2^p)}/R \cong K^{(2^p)}$. To prove that $K^{(2^p)}$ is a p-permutation module, it will be sufficient to construct a p-permutation basis for $K^{(2^p)}$ with respect to $P := \mathrm{Syl}_p(\mathfrak{S}_{2p})$.

To construct a basis for $K^{(2^p)}$, choose any $\mathbf{x}_1 \in X^{(2p)}$ and construct $\mathcal{O}_{\mathbf{x}_1} = \{\mathbf{x}_1 g \mid g \in P\}$, the orbit of $\mathbf{x}_1$ under P. For $i \geq 2$, choose $\mathbf{x}_i \notin \bigcup_{j=1}^{i-1} \mathcal{O}_{\mathbf{x}_j}$ and construct $\mathcal{O}_{\mathbf{x}_i}$. Continue in this manner, until $\bigcup_j \mathcal{O}_{\mathbf{x}_j} = X^{(2^p)}$. Define $\mathcal{O}_{\overline{\mathbf{x}_j}} := \{\overline{\mathbf{x}_j g} \mid \mathbf{x}_j g \in \mathcal{O}_{\mathbf{x}_j}\}$ and let

$$\mathcal{B} = \bigcup_j \mathcal{O}_{\overline{\mathbf{x}_j}},$$

where $\mathcal{B}$ is viewed as a set of elements of $K^{(2^p)}$. We will see that $\mathcal{B}$ is a basis for $K^{(2^p)}$ which is invariant under $P = \mathrm{Syl}_p(\mathfrak{S}_{2p})$. More precisely, we will show that either no two elements of an orbit $\mathcal{O}_x$ are equal under the quotient map, and thus the orbit is clearly P-invariant, or that whenever $\mathbf{x}g_1, \mathbf{x}g_2 \in \mathcal{O}_{\mathbf{x}}$ are such that they differ only by a permutation of their rows, i.e. $\mathbf{x}g_1 = \mathbf{x}g_2\sigma$ for some $\sigma \in \mathfrak{S}_{H(\mathbf{x}g_2)}$, then $\mathrm{sgn}(\sigma) = 1$ and hence $\mathbf{x}g_2 - \mathbf{x}g_1 \in R$.

Let $\mathbf{x} \in X^{(2^p)}$, which is of the form given in (8.8), and let $P = \langle q, r\rangle \cong C_p \times C_p$, where $q := (1\,2\,\ldots\,p)$ and $r := (p+1\,\ldots\,2p)$. Consider $\mathcal{O}_{\mathbf{x}} = \{\mathbf{x}g \mid g \in P\}$. Define the type of a row of $\mathbf{x}$ as follows: the row of $\mathbf{x}$ with entries x_a and x_{p+a} is

- *of type 1* if $x_a \in \{1, 2, \ldots, p\}$ and $x_{p+a} \in \{p+1, \ldots, 2p\}$;

- *of type 2* if $\{x_a, x_{p+a}\} \subseteq \{1, 2, \dots, p\}$; and
- *of type 3* if $\{x_a, x_{p+a}\} \subseteq \{p+1, \dots, 2p\}$.

Since p is odd, there must exist at least one type 1 row of $\mathbf{x}$; such a row will not be fixed by the action of q^i or r^j on $\mathbf{x}$ for any $1 \leq i, j \leq p-1$. Moreover, the only way that all entries in rows of type 1, which are also elements of $\{1, 2, \dots, p\}$ (or $\{p+1, \dots, 2p\}$), will also be in a row of type 1 in $\mathbf{x}q^i$ (or $\mathbf{x}r^j$, respectively) is if all rows of $\mathbf{x}$ are of type 1.

Suppose that not all rows of $\mathbf{x}$ are of type 1. Let row a be a type 1 row of $\mathbf{x}$, whose entry $x_a \leq p$ is not in a row of type 1 in $\mathbf{x}q^i$ for some $1 \leq i \leq p-1$. For any $1 \leq i \leq p-1$, q^i fixes those entries in $\mathbf{x}$ that are elements of $\{p+1, \dots, 2p\}$. In particular, q^i fixes x_{p+a}. Thus, we have

$$\mathbf{x}q^i = \begin{array}{cc} \vdots & \vdots \\ \hline y & x_{p+a} \\ \hline \vdots & \vdots \\ \hline x_a & z \\ \hline \vdots & \vdots \end{array} \begin{array}{l} \\ \leftarrow \text{row } a \\ \\ \\ \\ \end{array},$$

where $y, z \leq p$. Similarly, for any $1 \leq j \leq p-1$, r^j fixes those entries in $\mathbf{x}$ that are elements of $\{1, \dots, p\}$ and so we know that

$$\mathbf{x}r^j = \begin{array}{cc} \vdots & \vdots \\ \hline x_a & u \\ \hline \vdots & \vdots \\ \hline v & x_{p+a} \\ \hline \vdots & \vdots \end{array} \begin{array}{l} \\ \leftarrow \text{row } a \\ \\ \\ \\ \end{array},$$

where $u > p$ and $v \in \{1, \dots, 2p\} \backslash \{x_a, x_{p+a}, u\}$. Comparing the rows of $\mathbf{x}q^i$ and $\mathbf{x}r^j$ that contain x_a, and noting that $z \leq p$ and $u > p$, it is clear that there cannot exist a permutation $\sigma \in \mathfrak{S}_{H(\mathbf{x}r^j)}$, for which $\mathbf{x}q^i = (\mathbf{x}r^j)\sigma$. Thus, under the quotient map $M^{(2^p)} \to K^{(2^p)}$, no two elements $\mathbf{x}g_1, \mathbf{x}g_2 \in \mathcal{O}_x$ are equal and hence $\mathcal{O}_{\overline{\mathbf{x}}} := \{\overline{\mathbf{x}g} \mid \mathbf{x}g \in \mathcal{O}_{\mathbf{x}}\}$ contains p^2 linearly independent elements of $K^{(2^p)}$.

If $\mathbf{x}$ is such that all of its rows are of type 1, and $\sigma \in \mathfrak{S}_{H(\mathbf{x}r^j)}$, then $\mathbf{x}q^i = (\mathbf{x}r^j)\sigma$ if and only if $j = p - i$ and $\sigma = (1\,2\,\dots,\,p)^i$. Further, since p is odd, $\mathrm{sgn}(\sigma) = 1$. Clearly, $\mathbf{x}q^i - (\mathbf{x}r^{p-i})(1\,2\,\dots,\,p)^i \in R$ and hence, $\overline{\mathbf{x}q^i} = \overline{(\mathbf{x}r^{p-i})(1\,2\,\dots,\,p)^i}$ in $K^{(2^p)}$. It follows that $\mathcal{O}_{\overline{\mathbf{x}}} = \{\overline{\mathbf{x}g} \mid \mathbf{x}g \in \mathcal{O}_{\mathbf{x}}\}$ contains p linearly independent elements of $K^{(2^p)}$.

Now that we have established that $K^{(2^p)}$ is a p-permutation module, we look to apply the Brauer correspondence. Recall that

$$K^{(2^p)} = \left(T^{(1^p)}\left(S^{(2)}\right)\right)\Big\uparrow_{\mathfrak{S}_2 \wr \mathfrak{S}_p}^{\mathfrak{S}_{2p}}.$$

By Higman's Criteria (Proposition 2.1.7), $K^{(2^p)}$ is relatively $k(\mathfrak{S}_2 \wr \mathfrak{S}_p)$-projective. It follows from Theorem 9.2 in [2] that $K^{(2^p)}\downarrow_{\mathfrak{S}_2 \wr \mathfrak{S}_p}$ is relatively $k\big(\operatorname{Syl}_p(\mathfrak{S}_2 \wr \mathfrak{S}_p)\big)$-projective and therefore, using Higman's Criteria again, $K^{(2^p)}$ is relatively $k\big(\operatorname{Syl}_p(\mathfrak{S}_2 \wr \mathfrak{S}_p)\big)$-projective. Now, by the definition of a vertex, we know that a vertex of any indecomposable summand of $K^{(2^p)}$ is a subgroup of $\operatorname{Syl}_p(\mathfrak{S}_2 \wr \mathfrak{S}_p)$. We have already established that $|\operatorname{Syl}_p(\mathfrak{S}_2 \wr \mathfrak{S}_p)| = p$ and that $\operatorname{Syl}_p(\mathfrak{S}_2 \wr \mathfrak{S}_p) \cong Q := \langle (1\,2\,\ldots\,p)(p{+}1\,\ldots\,2p)\rangle$. It follows that the vertices of indecomposable summands of $K^{(2^p)}$ are either trivial (in which case the summand is projective) or conjugate to Q.

If M is a non-projective indecomposable summand of $K^{(2^p)}$, then, under the Brauer correspondence, M corresponds to the projective indecomposable $k(N_{\mathfrak{S}_{2p}}(Q)/Q)$-module $M(Q)$. Moreover, if $K^{(2^p)} = M_1 \oplus \cdots \oplus M_s$ (into indecomposable summands), then

$$K^{(2^p)}(Q) = M_1(Q) \oplus \cdots \oplus M_s(Q).$$

To verify that not all indecomposable summands of $K^{(2^p)}$ are projective, we need to compute the Brauer quotient $K^{(2^p)}(Q)$ of $K^{(2^p)}$ with respect to $Q = \langle qr\rangle$, where $q = (1\,2\,\ldots\,p)$ and $r = (p{+}1\,\ldots\,2p)$. By Corollary 2.1.21, $K^{(2^p)}(Q) = \langle \mathcal{B}^Q\rangle_k$, where $\mathcal{B}^Q$ denotes the subspace of Q-fixed points in the p-permutation basis $\mathcal{B}$ of $K^{(2^p)}$. A basis for $\mathcal{B}^Q$ is given by the set of oriented column tabloids

$$\left\{ b_i = \left| \begin{matrix} \{1\,,\,p+1\} \\ \{2\,,\,p+2\} \\ \vdots \\ \{\,p\,,\,2p\,\} \end{matrix} \right| q^i \;\middle|\; 0 \le i \le p-1 \right\}.$$

Indeed, computation shows that acting by $(qr)^j$ for some $0 \le j \le p-1$ permutes the rows of a basis element b_i. In particular, the rows are permuted by a p-cycle, which is an even permutation, and so it follows that b_i is fixed by $(qr)^j$. Further, if $b \in \mathcal{B}$ is not one of the basis elements b_i, then $b(qr)^j \neq b$ for all $0 \le j \le p-1$.

Since $\dim K^{(2^p)}(Q) = p$ and thus, $K^{(2^p)}(Q) \neq 0$, it must be that $M_i(Q) \neq 0$ for some i. In other words, there is at least one indecomposable summand of $K^{(2^p)}$ with vertex Q.

Lemma 8.4.7

There is only one summand of $K^{(2^p)}$ lying in $B_0(\mathfrak{S}_{2p})$ with vertex Q.

Proof. Since $N_{\mathfrak{S}_{2p}}(Q)$ is a subgroup of $\mathfrak{S}_{2p}$, it is clear that $|N_{\mathfrak{S}_{2p}}(Q)| \le (2p)!$ and thus, $|N_{\mathfrak{S}_{2p}}(Q)/Q| \le (2p)!/p$. Further, $N_{\mathfrak{S}_{2p}}(Q)/Q$ contains an element of order p, namely πQ, where $\pi := (1\;\,p+1)(2\;\,p+2)\cdots(p\;\,2p)$. Therefore, $|\operatorname{Syl}_p\big(N_{\mathfrak{S}_{2p}}(Q)/Q\big)| = p$, from which it follows (by Corollary 5.7 in [2]) that every projective $k(N_{\mathfrak{S}_{2p}}(Q)/Q)$-module has dimension divisible by p. In particular, p divides $\dim M_i(Q)$, which means that $K^{(2^p)}(Q) = M_i(Q)$ because $\dim K^{(2^p)}(Q) = p$. ■

We are yet to exclude the possibility that there exist projective summands of $K^{(2^p)}$ lying in $B_0(\mathfrak{S}_{2p})$. This will complete the proof of Proposition 8.4.6.

Proof of Proposition 8.4.6. In Theorem I of [42], Scopes proves that, for a defect two block of a symmetric group in prime characteristic, all diagonal entries in the Cartan matrix are greater than or equal to three. That is, the multiplicity of a simple module D^λ as a composition factor of its projective cover $P(\lambda)$ is at least three. In §8.4.2, we saw that all simple modules lying in $B_0(\mathfrak{S}_{2p})$ arise with multiplicity at most two in $K^{(2^p)}$. Hence Scopes' result ensures there cannot be any projective summands of $K^{(2^p)}$ lying in the principal block. ∎

To summarise, we have seen that all projective summands of $K^{(2^p)}$ lie in blocks of weight zero and that there is one indecomposable non-projective summand of $K^{(2^p)}$ lying in the principal block of $\mathfrak{S}_{2p}$. We will now describe the Loewy layers of the non-projective summand.

8.4.5 Loewy layers of non-projective summands

In this section we will describe the Loewy layers of the indecomposable non-projective summand of $K^{(2^p)}$ lying in $B_0(\mathfrak{S}_{2p})$. We already know the composition factors that lie in $B_0(\mathfrak{S}_{2p})$, so we just need to determine their position within the layers.

We continue to assume that p is an odd prime. Additionally, we will not address the case $p = 3$, since the structure of $K^{(2^3)}$ is detailed in §8.1.2. However, we remark that we saw that $K^{(2^3)}$ has a single summand; this summand is non-projective and lies in the principal block.

Theorem 8.4.8
If $p = 5$, then the Loewy layers of the non-projective indecomposable summand of $K^{(2^5)}$ are

$$\begin{array}{c} D^{(5,4,1)} \oplus D^{(7,1^3)} \\ \\ D^{(10)} \oplus D^{(8,1^2)} \oplus D^{(5,3,1^2)} \oplus D^{(6,1^4)} \oplus D^{(4,4,2)} \\ \\ D^{(5,4,1)} \oplus D^{(7,1^3)} \end{array}.$$

If $p \geq 7$, then the Loewy layers of the non-projective indecomposable summand of $K^{(2^p)}$ are

$$\begin{array}{c} u_1 \oplus v_{0,2} \oplus v_{1,3} \oplus \cdots \oplus v_{(p-7)/2,(p-3)/2} \oplus v_{(p-5)/2,(p-1)/2} \\ \\ u_0 \oplus u_2 \oplus v_{0,1} \oplus \cdots \oplus v_{(p-5)/2,(p-3)/2} \oplus v_{0,3} \oplus \cdots \oplus v_{(p-7)/2,(p-1)/2} \oplus w_{(p-3)/2} \oplus w_{(p+1)/2} \\ \\ u_1 \oplus v_{0,2} \oplus v_{1,3} \oplus \cdots \oplus v_{(p-7)/2,(p-3)/2} \oplus v_{(p-5)/2,(p-1)/2} \end{array}.$$

As we observed at the end of §8.4.2, the only reason for isolating the case $p = 5$ in the statement of Theorem 8.4.8 is to account for the fact that the simple module $w_{(p+1)/2}$ is not defined when $p = 5$. In the proof of the theorem, we do not need to identify particular composition factors and therefore we will not distinguish between the case $p = 5$ and the case $p \geq 7$.

Proof. Let $p \geq 5$ and let M denote the unique non-projective indecomposable summand of $K^{(2^p)}$. For ease of notation, let $Y = \{Y_i \mid 1 \leq i \leq p\}$ be the set of composition factors of M that arise with multiplicity one, and let $Z = \{Z_j \mid 1 \leq j \leq (p-1)/2\}$ be the set of composition factors of M that arise with multiplicity two.

For a contradiction, suppose that there exists $1 \leq i \leq p$ such that $Y_i \in Y$ is a composition factor of $\operatorname{soc}(M)$. Since $K^{(2^p)}$ is self-dual, it follows that M is self-dual and therefore Y_i must also be a composition factor of $\operatorname{hd}(M) = M/\operatorname{rad}(M) \cong \operatorname{soc}(M)$. Since Y_i arises with multiplicity one, this is only possible if $\operatorname{hd}(M) = \operatorname{soc}(M)$. However, this means that the Loewy length of M is one and Y_i is a direct summand of M, which is a contradiction. Hence, the composition factors of $\operatorname{soc}(M)$ must be a subset of Z. Moreover, these composition factors must occur with multiplicity one in $\operatorname{soc}(M)$ (and multiplicity one in $\operatorname{hd}(M)$), else either M would not be self-dual or there would exist $Z_j \in Z$ with multiplicity greater than two in M, both of which are false.

For a contradiction, assume that there exists at least one $Z_j \in Z$, which is not a composition factor of $\operatorname{soc}(M)$ or $\operatorname{hd}(M)$. Consider $\operatorname{heart}(M) = \operatorname{rad}(M)/\operatorname{soc}(M)$, which we know must have Z_j and Y_i (for all $1 \leq i \leq p$) among its composition factors. It is clear from the Ext-quiver of $B_0(\mathfrak{S}_{2p})$ that $\operatorname{Ext}^1(Z_{j_1}, Z_{j_2}) = 0$ for $j_1 \neq j_2$ and therefore, Z_j cannot be a composition factor of $\operatorname{soc}^2(M)/\operatorname{soc}(M)$. It follows that M must have Loewy length at least five and the set of composition factors of $\operatorname{soc}\big(\operatorname{heart}(M)\big)$ is a subset of Y. But, since $\operatorname{heart}(M)$ is also self-dual, this means that the same subset of Y must also be the set of composition factors of $\operatorname{heart}(M)/\operatorname{rad}\big(\operatorname{heart}(M)\big)$. This is only possible if $\operatorname{heart}(M)/\operatorname{rad}\big(\operatorname{heart}(M)\big) = \operatorname{soc}\big(\operatorname{heart}(M)\big)$, but then the Loewy length of M is at most three, which is a contradiction. Hence, we conclude that $\operatorname{soc}(M) = \operatorname{hd}(M) = \bigoplus_{j=1}^{(p-1)/2} Z_j$ and $\operatorname{heart}(M) = \bigoplus_{i=1}^{p} Y_i$. ■

Appendix A

The Ext-quiver of the principal block of $\mathfrak{S}_{2p}$ in characteristic p

The following quiver was obtained by Martin in his D.Phil thesis [35].

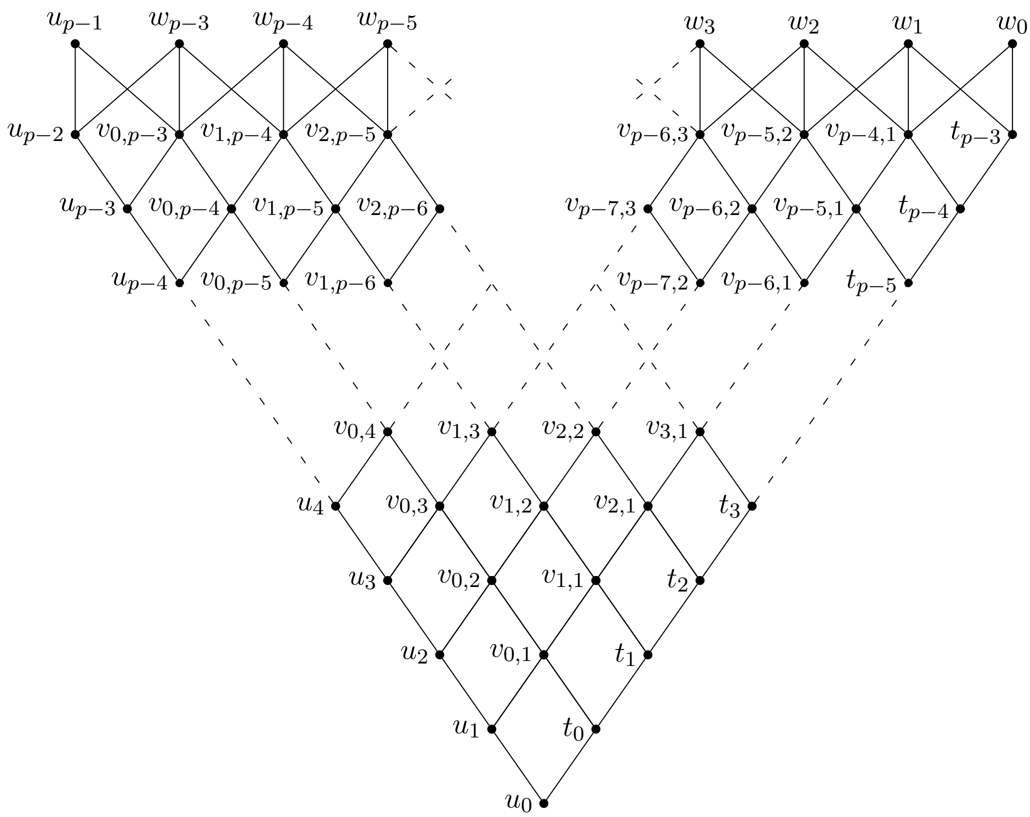


Figure A.1: The Ext-quiver of the principal block of $\mathfrak{S}_{2p}$ in characteristic p.

Appendix B

MAGMA Code

B.1 Decomposition of generalised Foulkes characters

The following MAGMA code enables the decomposition of the generalised Foulkes character $\varphi_z^{(m^n)}$ to be computed, with z used here (instead of ν) to denote the partition of n which labels the character. We use this code to generate the data given in §C.4-C.5.

```
function GenInf(m,n,z);
   S:=SymmetricGroup(m*n);
   ptn:=[m : i in [1..n]];
   ptnforchar:=z;
   H:=YoungSubgroup(ptn);
   G:=Normalizer(S,H);
   sym:=SymmetricGroup(n);
   W:=WreathProduct(Sym(m),Sym(n));
   x1:=W!([2,1]cat[i : i in [3..m*n]]);
   y1:=W!([i : i in [2..m]]cat[1]cat[j : j in [m+1..m*n]]);
   a1:=W!([m+i : i in [1..m]]cat[j : j in [1..m]]cat[k : k in [2*m+1..m*
      n]]);
   b1:=W!([i : i in [m+1..n*m]]cat[j: j in [1..m]]);
   c1:=Sym(n)!([i : i in [2..n]]cat[1]);
   q:=hom<G->sym|[G!x1->Id(sym),G!y1->Id(sym),G!a1->sym!(1,2),G!b1->sym!
      c1]>;
   symchar:=SymmetricCharacter(ptnforchar);
   sgnSmwrSn:=LiftCharacter(symchar,q,G);
   phi:=Induction(sgnSmwrSn,S);
   return [<p,InnerProduct(phi,SymmetricCharacter(p))> : p in Partitions
      (m*n)|InnerProduct(phi,SymmetricCharacter(p)) ge 1];
end function;
```

Given $m, n \in \mathbb{N}$, the following code decomposes $\varphi_\nu^{(m^n)}$ for all partitions ν of n. Of course, decomposing a single character can be computationally demanding and so this function is only really of benefit for small m and n. We use this code to obtain the data given in §C.1-C.3.

```
function All(m,n);
    "Decomposition of varphi^(m^n)_nu with <m,n> =", <m,n>;
    for z in Partitions(n)
        do "nu = ", z;
        GenInf(m,n,z);
    end for;
    return "end";
end function;
```

B.2 The structure of $K^{(2^n)}$ in prime characteristic

The following function computes the dimension of the indecomposable summands of $K^{(2^n)}$ in prime characteristic p and also returns (the dimension of) the modules appearing in the socle series of $K^{(2^n)}$.

```
function Socle(n,p);
    S:=SymmetricGroup(2*n);
    ptn:=[2 : i in [1..n]];
    H:=YoungSubgroup(ptn);
    N:=Normalizer(S,H);
    function varx(i);
        return S!(2*i-1,2*i);
    end function;
    function vary(j);
        return S!(2*j-1,2*j+1)(2*j,2*j+2);
    end function;
    lst:=[varx(i) : i in [1..n]]cat[vary(j) : j in [1..n-1]];
    G:=sub<S|lst>;
    function matrixeltx(p);
        return Matrix(GF(p),1,1,[1]);
    end function;
    function matrixelty(p);
        return Matrix(GF(p),1,1,[-1]);
    end function;
    L:=[matrixeltx(p) : i in [1..n]]cat[matrixelty(p) : i in [1..n-1]];
    A:=MatrixAlgebra<GF(p),1|L>;
    repres:=GModule(G,A);
```

```
    K:=Induction(repres,SymmetricGroup(2*n));
    IndS:=IndecomposableSummands(K);
    "K2",n;
    "characteristic", p;
    return K,[<i,SocleSeries(i)> : i in IndS];
end function;
```

To fully determine the structure of $K^{(2^4)}$ in characteristic 2, we needed to obtain more information about the heart of the indecomposable summand with the largest dimension.

```
function K24decomp(p);
    S:=SymmetricGroup(2*4);
    ptn:=[2 : i in [1..4]];
    H:=YoungSubgroup(ptn);
    N:=Normalizer(S,H);
    G:=PermutationGroup<2*4|(1,2),(3,4),(5,6),(7,8),(1,3)(2,4),(3,5)(4,6)
        ,(5,7)(6,8)>;
    A:=MatrixAlgebra<GF(p),1|[1],[1],[1],[1],[-1],[-1],[-1]>;
    repres:=GModule(G,A);
    K:=Induction(repres,SymmetricGroup(2*4));
    IndS:=IndecomposableSummands(K);
    I:=IndS[3];
    J:=JacobsonRadical(I);
    sc:=Socle(I);
    Q:=quo<J|sc>;
    IQ:=IndecomposableSummands(Q);
    return [<i,SocleSeries(i)> : i in IQ];
end function;
```

Appendix C

The decomposition of generalised Foulkes characters

The following data was obtained using the MAGMA Code from Appendix B. For each generalised Foulkes character $\varphi_\nu^{(m^n)}$, we record the partitions λ labelling the irreducible characters that appear in its decomposition with non-zero multiplicity, together with the corresponding character multiplicities $m_\lambda := \left\langle \varphi_\nu^{(m^n)}, \chi^\lambda \right\rangle$, as $<\lambda,\ m_\lambda>$.

C.1 Decompositions of $\varphi_\nu^{(2^n)}$ for $2 \leq n \leq 5$

```
Decomposition of varphi^(m^n)_nu with <m,n> = <2, 2>
```

```
nu = [ 2 ]

<[ 4 ], 1>
<[ 2, 2 ], 1>
```

```
nu = [ 1, 1 ]

<[ 3, 1 ], 1>
```

```
Decomposition of varphi^(m^n)_nu with <m,n> = <2, 3>
```

```
nu = [ 3 ]

<[ 6 ], 1>
<[ 4, 2 ], 1>
<[ 2, 2, 2 ], 1>
```

```
nu = [ 2, 1 ]

<[ 5, 1 ], 1>
<[ 4, 2 ], 1>
<[ 3, 2, 1 ], 1>
```

```
nu = [ 1, 1, 1 ]

<[ 4, 1, 1 ], 1>
<[ 3, 3 ], 1>
```

```
Decomposition of varphi^(m^n)_nu with <m,n> = <2, 4>

nu = [ 4 ]
<[ 8 ], 1>
<[ 6, 2 ], 1>
<[ 4, 4 ], 1>
<[ 4, 2, 2 ], 1>
<[ 2, 2, 2, 2 ], 1>

nu = [ 3, 1 ]
<[ 7, 1 ], 1>
<[ 6, 2 ], 1>
<[ 5, 3 ], 1>
<[ 5, 2, 1 ], 1>
<[ 4, 3, 1 ], 1>
<[ 4, 2, 2 ], 1>
<[ 3, 2, 2, 1 ], 1>

nu = [ 2, 2 ]
<[ 6, 2 ], 1>
<[ 5, 2, 1 ], 1>
<[ 4, 4 ], 1>
<[ 4, 2, 2 ], 1>
<[ 3, 3, 1, 1 ], 1>

nu = [ 2, 1, 1 ]
<[ 6, 1, 1 ], 1>
<[ 5, 3 ], 1>
<[ 5, 2, 1 ], 1>
<[ 4, 3, 1 ], 1>
<[ 4, 2, 1, 1 ], 1>
<[ 3, 3, 2 ], 1>

nu = [ 1, 1, 1, 1 ]
<[ 5, 1, 1, 1 ], 1>
<[ 4, 3, 1 ], 1>
```

```
Decomposition of varphi^(m^n)_nu with <m,n> = <2, 5>

nu = [ 5 ]
<[ 10 ], 1>
<[ 8, 2 ], 1>
<[ 6, 4 ], 1>
<[ 6, 2, 2 ], 1>
<[ 4, 4, 2 ], 1>
<[ 4, 2, 2, 2 ], 1>
<[ 2, 2, 2, 2, 2 ], 1>

nu = [ 4, 1 ]
<[ 9, 1 ], 1>
<[ 8, 2 ], 1>
<[ 7, 3 ], 1>
<[ 7, 2, 1 ], 1>
<[ 6, 4 ], 1>
<[ 6, 3, 1 ], 1>
<[ 6, 2, 2 ], 1>
<[ 5, 4, 1 ], 1>
<[ 5, 3, 2 ], 1>
<[ 5, 2, 2, 1 ], 1>
<[ 4, 4, 2 ], 1>
<[ 4, 3, 2, 1 ], 1>
<[ 4, 2, 2, 2 ], 1>
<[ 3, 2, 2, 2, 1 ], 1>

nu = [ 3, 2 ]
<[ 8, 2 ], 1>
<[ 7, 3 ], 1>
<[ 7, 2, 1 ], 1>
<[ 6, 4 ], 1>
<[ 6, 3, 1 ], 1>
<[ 6, 2, 2 ], 2>
<[ 5, 4, 1 ], 1>
<[ 5, 3, 2 ], 1>
<[ 5, 3, 1, 1 ], 1>
<[ 5, 2, 2, 1 ], 1>
<[ 4, 4, 2 ], 1>
<[ 4, 3, 2, 1 ], 1>
<[ 4, 2, 2, 2 ], 1>
<[ 3, 3, 2, 1, 1 ], 1>

nu = [ 3, 1, 1 ]
<[ 8, 1, 1 ], 1>
<[ 7, 3 ], 1>
<[ 7, 2, 1 ], 1>
<[ 6, 3, 1 ], 2>
<[ 6, 2, 1, 1 ], 1>
<[ 5, 5 ], 1>
<[ 5, 4, 1 ], 1>
<[ 5, 3, 2 ], 2>
<[ 5, 3, 1, 1 ], 1>
<[ 5, 2, 2, 1 ], 1>
<[ 4, 4, 1, 1 ], 1>
<[ 4, 3, 3 ], 1>
<[ 4, 3, 2, 1 ], 1>
<[ 4, 2, 2, 1, 1 ], 1>
<[ 3, 3, 2, 2 ], 1>

nu = [ 2, 2, 1 ]
<[ 7, 2, 1 ], 1>
<[ 6, 4 ], 1>
<[ 6, 3, 1 ], 1>
<[ 6, 2, 2 ], 1>
<[ 6, 2, 1, 1 ], 1>
<[ 5, 4, 1 ], 1>
<[ 5, 3, 2 ], 1>
<[ 5, 3, 1, 1 ], 1>
<[ 5, 2, 2, 1 ], 1>
<[ 4, 4, 2 ], 1>
<[ 4, 3, 2, 1 ], 1>
<[ 4, 3, 1, 1, 1 ], 1>
<[ 3, 3, 3, 1 ], 1>
```

```
nu = [ 2, 1, 1, 1 ]

<[ 7, 1, 1, 1 ], 1>
<[ 6, 3, 1 ], 1>
<[ 6, 2, 1, 1 ], 1>
<[ 5, 4, 1 ], 1>
<[ 5, 3, 2 ], 1>
<[ 5, 3, 1, 1 ], 1>
<[ 5, 2, 1, 1, 1 ], 1>
<[ 4, 4, 1, 1 ], 1>
<[ 4, 3, 3 ], 1>
<[ 4, 3, 2, 1 ], 1>

nu = [ 1, 1, 1, 1, 1 ]

<[ 6, 1, 1, 1, 1 ], 1>
<[ 5, 3, 1, 1 ], 1>
<[ 4, 4, 2 ], 1>
```

C.2 Decompositions of $\varphi_\nu^{(3^n)}$ for $2 \le n \le 4$

```
Decomposition of varphi^(m^n)_nu with <m,n> = <3, 2>

nu = [ 2 ]

<[ 6 ], 1>
<[ 4, 2 ], 1>

nu = [ 1, 1 ]

<[ 5, 1 ], 1>
<[ 3, 3 ], 1>
```

```
Decomposition of varphi^(m^n)_nu with <m,n> = <3, 3>

nu = [ 3 ]

<[ 9 ], 1>
<[ 7, 2 ], 1>
<[ 6, 3 ], 1>
<[ 5, 2, 2 ], 1>
<[ 4, 4, 1 ], 1>

nu = [ 2, 1 ]

<[ 8, 1 ], 1>
<[ 7, 2 ], 1>
<[ 6, 3 ], 1>
<[ 6, 2, 1 ], 1>
<[ 5, 4 ], 1>
<[ 5, 3, 1 ], 1>
<[ 4, 3, 2 ], 1>

nu = [ 1, 1, 1 ]

<[ 7, 1, 1 ], 1>
<[ 6, 3 ], 1>
<[ 5, 3, 1 ], 1>
<[ 3, 3, 3 ], 1>
```

```
Decomposition of varphi^(m^n)_nu with <m,n> = <3, 4>

nu = [ 4 ]

<[ 12 ], 1>
<[ 10, 2 ], 1>
<[ 9, 3 ], 1>
<[ 8, 4 ], 1>
<[ 8, 2, 2 ], 1>
<[ 7, 4, 1 ], 1>
<[ 7, 3, 2 ], 1>
<[ 6, 6 ], 1>
<[ 6, 4, 2 ], 1>
<[ 6, 2, 2, 2 ], 1>
<[ 5, 4, 2, 1 ], 1>
<[ 4, 4, 4 ], 1>

nu = [ 3, 1 ]

<[ 11, 1 ], 1>
<[ 10, 2 ], 1>
<[ 9, 3 ], 2>
<[ 9, 2, 1 ], 1>
<[ 8, 4 ], 1>
<[ 8, 3, 1 ], 2>
<[ 8, 2, 2 ], 1>
<[ 7, 5 ], 2>
<[ 7, 4, 1 ], 2>
<[ 7, 3, 2 ], 2>
<[ 7, 2, 2, 1 ], 1>
<[ 6, 5, 1 ], 1>
<[ 6, 4, 2 ], 2>
<[ 6, 4, 1, 1 ], 1>
<[ 6, 3, 3 ], 1>
<[ 6, 3, 2, 1 ], 1>
<[ 5, 5, 2 ], 1>
<[ 5, 4, 3 ], 1>
<[ 5, 4, 2, 1 ], 1>
<[ 5, 3, 2, 2 ], 1>
<[ 4, 4, 3, 1 ], 1>

nu = [ 2, 2 ]

<[ 10, 2 ], 1>
<[ 9, 2, 1 ], 1>
```

```
<[ 8, 4 ], 2>
<[ 8, 3, 1 ], 1>
<[ 8, 2, 2 ], 1>
<[ 7, 4, 1 ], 1>
<[ 7, 3, 2 ], 1>
<[ 7, 3, 1, 1 ], 1>
<[ 6, 6 ], 1>
<[ 6, 5, 1 ], 1>
<[ 6, 4, 2 ], 2>
<[ 6, 3, 2, 1 ], 1>
<[ 5, 5, 1, 1 ], 1>
<[ 5, 4, 3 ], 1>
<[ 5, 3, 3, 1 ], 1>
<[ 4, 4, 2, 2 ], 1>

nu = [ 2, 1, 1 ]

<[ 10, 1, 1 ], 1>
<[ 9, 3 ], 1>
<[ 9, 2, 1 ], 1>
<[ 8, 4 ], 1>
<[ 8, 3, 1 ], 2>
<[ 8, 2, 1, 1 ], 1>
<[ 7, 5 ], 1>
<[ 7, 4, 1 ], 2>
<[ 7, 3, 2 ], 2>
<[ 7, 3, 1, 1 ], 1>
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<[ 6, 4, 2 ], 1>
<[ 6, 4, 1, 1 ], 1>
<[ 6, 3, 3 ], 2>
<[ 6, 3, 2, 1 ], 1>
<[ 5, 5, 2 ], 1>
<[ 5, 4, 3 ], 1>
<[ 5, 4, 2, 1 ], 1>
<[ 5, 3, 3, 1 ], 1>
<[ 4, 3, 3, 2 ], 1>

nu = [ 1, 1, 1, 1 ]

<[ 9, 1, 1, 1 ], 1>
<[ 8, 3, 1 ], 1>
<[ 7, 4, 1 ], 1>
<[ 7, 3, 1, 1 ], 1>
<[ 6, 6 ], 1>
<[ 6, 4, 2 ], 1>
<[ 6, 3, 3 ], 1>
<[ 5, 5, 1, 1 ], 1>
<[ 5, 3, 3, 1 ], 1>
<[ 3, 3, 3, 3 ], 1>
```

C.3 Decompositions of $\varphi_\nu^{(4^n)}$ for $2 \leq n \leq 4$

```
Decomposition of varphi^(m^n)_nu with <m,n> = <4, 2>

nu = [ 2 ]

<[ 8 ], 1>
<[ 6, 2 ], 1>
<[ 4, 4 ], 1>

nu = [ 1, 1 ]

<[ 7, 1 ], 1>
<[ 5, 3 ], 1>
```

```
Decomposition of varphi^(m^n)_nu with <m,n> = <4, 3>

nu = [ 3 ]

<[ 12 ], 1>
<[ 10, 2 ], 1>
<[ 9, 3 ], 1>
<[ 8, 4 ], 1>
<[ 8, 2, 2 ], 1>
<[ 7, 4, 1 ], 1>
<[ 6, 6 ], 1>
<[ 6, 4, 2 ], 1>
<[ 4, 4, 4 ], 1>

nu = [ 2, 1 ]

<[ 11, 1 ], 1>
<[ 10, 2 ], 1>
<[ 9, 3 ], 1>
<[ 9, 2, 1 ], 1>
<[ 8, 4 ], 2>
<[ 8, 3, 1 ], 1>
<[ 7, 5 ], 1>
<[ 7, 4, 1 ], 1>
<[ 7, 3, 2 ], 1>
<[ 6, 5, 1 ], 1>
<[ 6, 4, 2 ], 1>
<[ 5, 4, 3 ], 1>
```

```
nu = [ 1, 1, 1 ]

<[ 10, 1, 1 ], 1>
<[ 9, 3 ], 1>
<[ 8, 3, 1 ], 1>
<[ 7, 5 ], 1>
<[ 7, 4, 1 ], 1>
<[ 6, 3, 3 ], 1>
<[ 5, 5, 2 ], 1>
```

```
Decomposition of varphi^(m^n)_nu with <m,n> = <4, 4>
```

```
nu = [ 4 ]

<[ 16 ], 1>
<[ 14, 2 ], 1>
<[ 13, 3 ], 1>
<[ 12, 4 ], 2>
<[ 12, 2, 2 ], 1>
<[ 11, 4, 1 ], 1>
<[ 11, 3, 2 ], 1>
<[ 10, 6 ], 2>
<[ 10, 5, 1 ], 1>
<[ 10, 4, 2 ], 2>
<[ 10, 2, 2, 2 ], 1>
<[ 9, 6, 1 ], 1>
<[ 9, 5, 2 ], 1>
<[ 9, 4, 3 ], 1>
<[ 9, 4, 2, 1 ], 1>
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<[ 8, 4, 4 ], 2>
<[ 8, 4, 2, 2 ], 1>
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<[ 7, 5, 3, 1 ], 1>
<[ 7, 4, 4, 1 ], 1>
<[ 6, 6, 4 ], 1>
<[ 6, 6, 2, 2 ], 1>
<[ 6, 4, 4, 2 ], 1>
<[ 4, 4, 4, 4 ], 1>
```

```
nu = [ 3, 1 ]

<[ 15, 1 ], 1>
<[ 14, 2 ], 1>
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<[ 13, 2, 1 ], 1>
<[ 12, 4 ], 2>
<[ 12, 3, 1 ], 2>
<[ 12, 2, 2 ], 1>
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<[ 11, 4, 1 ], 3>
<[ 11, 3, 2 ], 2>
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<[ 10, 4, 2 ], 4>
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<[ 10, 3, 3 ], 1>
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<[ 9, 5, 2 ], 4>
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<[ 9, 4, 2, 1 ], 2>
<[ 9, 3, 2, 2 ], 1>
<[ 8, 7, 1 ], 2>
<[ 8, 6, 2 ], 3>
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<[ 8, 4, 4 ], 2>
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<[ 8, 4, 2, 2 ], 1>
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<[ 7, 6, 2, 1 ], 2>
<[ 7, 5, 4 ], 2>
<[ 7, 5, 3, 1 ], 1>
<[ 7, 5, 2, 2 ], 1>
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<[ 6, 5, 4, 1 ], 1>
<[ 6, 5, 3, 2 ], 1>
<[ 6, 4, 4, 2 ], 1>
<[ 5, 4, 4, 3 ], 1>
```

```
nu = [ 2, 2 ]

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<[ 13, 2, 1 ], 1>
<[ 12, 4 ], 2>
<[ 12, 3, 1 ], 1>
<[ 12, 2, 2 ], 1>
<[ 11, 5 ], 1>
<[ 11, 4, 1 ], 2>
<[ 11, 3, 2 ], 1>
<[ 11, 3, 1, 1 ], 1>
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<[ 10, 5, 1 ], 2>
<[ 10, 4, 2 ], 3>
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<[ 9, 6, 1 ], 2>
<[ 9, 5, 2 ], 2>
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<[ 9, 4, 2, 1 ], 1>
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<[ 8, 5, 2, 1 ], 1>
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<[ 7, 4, 3, 2 ], 1>
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<[ 6, 6, 2, 2 ], 1>
<[ 6, 5, 4, 1 ], 1>
<[ 6, 4, 4, 2 ], 1>
<[ 5, 5, 3, 3 ], 1>

nu = [ 2, 1, 1 ]

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<[ 13, 2, 1 ], 1>
<[ 12, 4 ], 1>
<[ 12, 3, 1 ], 2>
<[ 12, 2, 1, 1 ], 1>
<[ 11, 5 ], 2>
<[ 11, 4, 1 ], 3>
<[ 11, 3, 2 ], 2>
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<[ 10, 4, 2 ], 2>
<[ 10, 4, 1, 1 ], 2>
<[ 10, 3, 3 ], 2>
<[ 10, 3, 2, 1 ], 1>
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<[ 9, 6, 1 ], 3>
<[ 9, 5, 2 ], 4>
<[ 9, 5, 1, 1 ], 1>
<[ 9, 4, 3 ], 3>
<[ 9, 4, 2, 1 ], 2>
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<[ 8, 6, 2 ], 2>
<[ 8, 6, 1, 1 ], 2>
<[ 8, 5, 3 ], 4>
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<[ 8, 4, 4 ], 1>
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<[ 8, 3, 3, 2 ], 1>
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<[ 7, 6, 3 ], 2>
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<[ 7, 5, 3, 1 ], 2>
<[ 7, 5, 2, 2 ], 1>
<[ 7, 4, 4, 1 ], 1>
<[ 7, 4, 3, 2 ], 1>
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<[ 6, 5, 4, 1 ], 1>
<[ 6, 5, 3, 2 ], 1>
<[ 6, 4, 3, 3 ], 1>
<[ 5, 5, 4, 2 ], 1>

nu = [ 1, 1, 1, 1 ]

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<[ 11, 4, 1 ], 1>
<[ 11, 3, 1, 1 ], 1>
<[ 10, 6 ], 1>
<[ 10, 5, 1 ], 1>
<[ 10, 4, 2 ], 1>
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<[ 9, 5, 2 ], 1>
<[ 9, 5, 1, 1 ], 1>
<[ 9, 4, 3 ], 1>
<[ 9, 3, 3, 1 ], 1>
<[ 8, 7, 1 ], 1>
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<[ 8, 5, 3 ], 1>
<[ 8, 5, 2, 1 ], 1>
<[ 8, 4, 3, 1 ], 1>
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<[ 7, 5, 3, 1 ], 1>
<[ 7, 3, 3, 3 ], 1>
<[ 6, 5, 3, 2 ], 1>
<[ 5, 5, 5, 1 ], 1>
```

C.4 Further decompositions of $\varphi^{(m^n)}$

```
Decomposition of varphi^(3^5)

<[ 15 ], 1>
<[ 13, 2 ], 1>
<[ 12, 3 ], 1>
<[ 11, 4 ], 1>
<[ 11, 2, 2 ], 1>
<[ 10, 5 ], 1>
<[ 10, 4, 1 ], 1>
<[ 10, 3, 2 ], 1>
<[ 9, 6 ], 1>
<[ 9, 4, 2 ], 2>
<[ 9, 2, 2, 2 ], 1>
<[ 8, 6, 1 ], 1>
<[ 8, 5, 2 ], 1>
<[ 8, 4, 3 ], 1>
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<[ 7, 4, 4 ], 1>
<[ 7, 4, 3, 1 ], 1>
<[ 7, 4, 2, 2 ], 1>
<[ 7, 2, 2, 2, 2 ], 1>
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<[ 6, 5, 2, 2 ], 1>
<[ 6, 4, 4, 1 ], 1>
<[ 6, 4, 2, 2, 1 ], 1>
<[ 5, 5, 3, 1, 1 ], 1>
<[ 5, 4, 4, 2 ], 1>
```

```
Decomposition of varphi^(4^5)

<[ 20 ], 1>
<[ 18, 2 ], 1>
<[ 17, 3 ], 1>
<[ 16, 4 ], 2>
<[ 16, 2, 2 ], 1>
<[ 15, 5 ], 1>
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<[ 10, 7, 2, 1 ], 2>
<[ 10, 6, 4 ], 4>
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<[ 9, 5, 4, 2 ], 2>
```

```
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<[ 8, 4, 4, 4 ], 2>
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<[ 7, 6, 4, 3 ], 1>
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<[ 7, 5, 4, 3, 1 ], 1>
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<[ 6, 6, 4, 2, 2 ], 1>
<[ 6, 4, 4, 4, 2 ], 1>
<[ 4, 4, 4, 4, 4 ], 1>
```

```
Decomposition of varphi^(5^2)

<[ 10 ], 1>
<[ 8, 2 ], 1>
<[ 6, 4 ], 1>
```

```
Decomposition of varphi^(5^3)

<[ 15 ], 1>
<[ 13, 2 ], 1>
<[ 12, 3 ], 1>
<[ 11, 4 ], 1>
<[ 11, 2, 2 ], 1>
<[ 10, 5 ], 1>
<[ 10, 4, 1 ], 1>
<[ 9, 6 ], 1>
<[ 9, 4, 2 ], 1>
<[ 8, 6, 1 ], 1>
<[ 8, 5, 2 ], 1>
<[ 7, 4, 4 ], 1>
<[ 6, 6, 3 ], 1>
```

```
Decomposition of varphi^(5^4)

<[ 20 ], 1>
<[ 18, 2 ], 1>
<[ 17, 3 ], 1>
<[ 16, 4 ], 2>
<[ 16, 2, 2 ], 1>
<[ 15, 5 ], 1>
<[ 15, 4, 1 ], 1>
<[ 15, 3, 2 ], 1>
<[ 14, 6 ], 2>
<[ 14, 5, 1 ], 1>
<[ 14, 4, 2 ], 2>
<[ 14, 2, 2, 2 ], 1>
<[ 13, 7 ], 1>
<[ 13, 6, 1 ], 2>
<[ 13, 5, 2 ], 2>
<[ 13, 4, 3 ], 1>
<[ 13, 4, 2, 1 ], 1>
<[ 12, 8 ], 2>
<[ 12, 7, 1 ], 1>
<[ 12, 6, 2 ], 3>
<[ 12, 5, 3 ], 1>
<[ 12, 5, 2, 1 ], 1>
<[ 12, 4, 4 ], 2>
<[ 12, 4, 2, 2 ], 1>
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<[ 11, 5, 4 ], 1>
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<[ 11, 5, 2, 2 ], 1>
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<[ 10, 8, 2 ], 2>
<[ 10, 7, 3 ], 1>
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<[ 8, 4, 4, 4 ], 1>
<[ 7, 6, 4, 3 ], 1>
<[ 6, 6, 6, 2 ], 1>
```

```
Decomposition of varphi^(6^2)

<[ 12 ], 1>
<[ 10, 2 ], 1>
<[ 8, 4 ], 1>
<[ 6, 6 ], 1>

Decomposition of varphi^(6^3)

<[ 18 ], 1>
<[ 16, 2 ], 1>
<[ 15, 3 ], 1>
<[ 14, 4 ], 1>
<[ 14, 2, 2 ], 1>
<[ 13, 5 ], 1>
<[ 13, 4, 1 ], 1>
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<[ 9, 6, 3 ], 1>
<[ 8, 8, 2 ], 1>
<[ 8, 6, 4 ], 1>
<[ 6, 6, 6 ], 1>
```

C.5 Further decompositions of $\varphi^{(m^n)}_{(1^n)}$

```
Decomposition of varphi^(3^5)_(1^5)

<[ 11, 1, 1, 1, 1 ], 1>
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<[ 9, 4, 2 ], 1>
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<[ 5, 5, 3, 1, 1 ], 1>
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<[ 3, 3, 3, 3, 3 ], 1>

Decomposition of varphi^(4^5)_(1^5)

<[ 16, 1, 1, 1, 1 ], 1>
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<[ 14, 4, 1, 1 ], 1>
<[ 14, 3, 1, 1, 1 ], 1>
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```

```
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<[ 8, 5, 3, 3, 1 ], 1>
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<[ 7, 7, 3, 3 ], 1>
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<[ 7, 6, 4, 3 ], 1>
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<[ 7, 5, 3, 3, 2 ], 1>
<[ 6, 6, 4, 2, 2 ], 1>
<[ 6, 5, 5, 3, 1 ], 1>
<[ 5, 5, 5, 5 ], 1>
```

```
Decomposition of varphi^(5^2)_(1^2)
```

```
<[ 9, 1 ], 1>
<[ 7, 3 ], 1>
<[ 5, 5 ], 1>
```

```
Decomposition of varphi^(5^3)_(1^3)
```

```
<[ 13, 1, 1 ], 1>
<[ 12, 3 ], 1>
<[ 11, 3, 1 ], 1>
<[ 10, 5 ], 1>
<[ 10, 4, 1 ], 1>
<[ 9, 6 ], 1>
<[ 9, 5, 1 ], 1>
<[ 9, 3, 3 ], 1>
<[ 8, 5, 2 ], 1>
<[ 7, 7, 1 ], 1>
<[ 7, 5, 3 ], 1>
<[ 5, 5, 5 ], 1>
```

```
Decomposition of varphi^(5^4)_(1^4)

<[ 17, 1, 1, 1 ], 1>
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<[ 14, 5, 1 ], 1>
<[ 14, 4, 2 ], 1>
<[ 14, 4, 1, 1 ], 1>
<[ 14, 3, 3 ], 1>
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<[ 13, 5, 2 ], 1>
<[ 13, 5, 1, 1 ], 2>
<[ 13, 4, 3 ], 1>
<[ 13, 3, 3, 1 ], 1>
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<[ 12, 7, 1 ], 2>
<[ 12, 6, 2 ], 2>
<[ 12, 5, 3 ], 2>
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<[ 12, 4, 3, 1 ], 1>
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<[ 11, 7, 1, 1 ], 2>
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<[ 11, 6, 2, 1 ], 1>
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<[ 11, 3, 3, 3 ], 1>
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<[ 10, 6, 4 ], 2>
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<[ 10, 5, 4, 1 ], 1>
<[ 10, 5, 3, 2 ], 1>
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<[ 9, 8, 3 ], 1>
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<[ 9, 7, 3, 1 ], 2>
<[ 9, 6, 5 ], 1>
<[ 9, 6, 3, 2 ], 1>
<[ 9, 5, 5, 1 ], 2>
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<[ 8, 6, 4, 2 ], 1>
<[ 8, 5, 5, 2 ], 1>
<[ 7, 7, 5, 1 ], 1>
<[ 7, 7, 3, 3 ], 1>
<[ 7, 5, 5, 3 ], 1>
<[ 5, 5, 5, 5 ], 1>

Decomposition of varphi^(6^2)_(1^2)

<[ 11, 1 ], 1>
<[ 9, 3 ], 1>
<[ 7, 5 ]

Decomposition of varphi^(6^3)_(1^3)

<[ 16, 1, 1 ], 1>
<[ 15, 3 ], 1>
<[ 14, 3, 1 ], 1>
<[ 13, 5 ], 1>
<[ 13, 4, 1 ], 1>
<[ 12, 6 ], 1>
<[ 12, 5, 1 ], 1>
<[ 12, 3, 3 ], 1>
<[ 11, 7 ], 1>
<[ 11, 6, 1 ], 1>
<[ 11, 5, 2 ], 1>
<[ 10, 7, 1 ], 1>
<[ 10, 5, 3 ], 1>
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<[ 9, 7, 2 ], 1>
<[ 9, 6, 3 ], 1>
<[ 8, 5, 5 ], 1>
<[ 7, 7, 4 ], 1>
```

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