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Vector peakon equations and isospectral flows in Clifford algebras

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Abstract

Starting from a spectral problem posed in a Clifford algebra with d generators and Euclidean signature, we study an integrable, coupled system of PDEs that can be viewed as a vector perturbation of the Camassa–Holm equation with residual orthogonal symmetry. In the two-component case $d = 2$, we show that the travelling wave solutions correspond to a Liouville integrable Hamiltonian system with two degrees of freedom, making use of a reciprocal transformation linking the coupled PDEs to a symmetry of the Hirota–Satsuma system. We also present a symmetry classification of all integrable two-component perturbations of Camassa–Holm, and find that besides the $d = 2$ system analyzed here, the coupled 2CH system studied by Olver and Rosenau (as well as by Chen, Liu and Zhang, and Falqui), and equations related to either of those systems by Miura transformations, we also obtain a new system that (to the best of our knowledge) has not been reported previously. For the case of an arbitrary number of components d , we additionally investigate the short-pulse (high-frequency) regime, in which the limiting dynamics are governed by a vector-valued Hunter–Saxton-type system. Furthermore, we provide a detailed analysis of the corresponding measure-valued (weak) solutions associated with this system.

1 Introduction

In this paper we study the system of partial differential equations

$$\mathbf{m}_t = 2u_x \mathbf{m} + u \mathbf{m}_x + \mathbf{V} \mathbf{m}, \quad (1.1)$$

where $\mathbf{m} = \mathbf{m}(x, t)$ is a d -component vector and $\mathbf{V} = (v_{\mu\nu})$ is a sparse skew-symmetric $d \times d$ matrix of the form

$$\mathbf{V} = \begin{pmatrix} 0 & \mathbf{v}^T \\ -\mathbf{v} & \mathbf{0} \end{pmatrix}, \quad (1.2)$$

with $\mathbf{v} = (v_{1\nu})_{\nu=2,\dots,d}$. The scalar field u and the components of $\mathbf{v}$ are related to the components of $\mathbf{m} = (m_\mu)_{\mu=1,\dots,d}$ through the constraints

$$u_{xx} - 4\alpha u = 2m_1, \quad \text{and} \quad v_{1\mu,x} = -2m_\mu \quad \text{for } \mu = 2, \dots, d, \quad (1.3)$$

where α is an arbitrary constant. Writing (1.1) in components yields

$$\begin{aligned} m_{1,t} &= 2u_x m_1 + u m_{1,x} + \sum_{\nu=2}^n v_{1\nu} m_\nu, \\ m_{\mu,t} &= 2u_x m_\mu + u m_{\mu,x} - v_{1\mu} m_1, \quad \mu = 2, \dots, d. \end{aligned} \quad (1.4)$$

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The system (1.1) is a d -dimensional Euclidean analogue of the two-component system introduced in [4], which arose from the Euler–Bernoulli beam equation and was subsequently extended to a Clifford algebra framework in [5]. A preliminary analysis of two-component Camassa–Holm dynamics with Minkowski signature can also be found in [5]. The present paper focuses on the corresponding setting with Euclidean signature. In addition, we place (1.1) within the broader framework of the symmetry approach to integrability: in particular, for $d = 2$ we obtain a complete classification of a well-defined class of integrable perturbations of the scalar Camassa–Holm equation within this vector/Clifford setting.

In the case when $\mathbf{v} = 0$, when $m_\mu = 0$ for $2 \leq \mu \leq d$, the system (1.1) reduces to the standard (scalar) Camassa–Holm equation, which was derived as an asymptotic model of shallow water waves in [7]. As well as being completely integrable, the Camassa–Holm equation was found to combine many remarkable features, including smooth solitons, wave breaking, and an interpretation as an Euler–Poincaré flow on a diffeomorphism group, as well as a new class of weak soliton solutions known as peakons. The surprising combination of all these properties has led to many attempts to generalize from the scalar case to multi-component analogues of the Camassa–Holm equation. A complex NLS-type analogue was presented by Fokas in [12], based on the methodology of bi-Hamiltonian structures and recursion operators developed previously with Fuchssteiner, while a few different integrable two-component examples were introduced via a tri-Hamiltonian approach by Olver and Rosenau [24]. One of the latter examples, commonly referred to as 2CH, was considered in more detail by Chen, Liu and Zhang [8, 23], and also Falqui [11], who used a bi-Hamiltonian structure of hydrodynamic type as the basis for their analysis, generalizing the known structure of the original scalar CH equation. Another approach based on supersymmetry, due to Popowicz [27], produced non-integrable equations with only one Hamiltonian structure.

Since that time, multi-component analogues of Camassa–Holm have been constructed from multiple different viewpoints, usually by starting from an appropriate family of Lax operators [17, 22, 30, 31], from associated geometric flows [28], or as particular cases of a larger set of equations [6]. For a restricted class of two-component systems, there is also a classification of integrable coupled equations of Camassa–Holm type [19]. However, the literature on such equations is now unfeasibly large, so the small selection of papers cited here is necessarily incomplete.

Given the plethora of different multi-component equations of Camassa–Holm type, it is worth pointing out some of the distinctive features of the system (1.1). This system equips the scalar Camassa–Holm with internal degrees of freedom, represented by the skew-symmetric tensor $\mathbf{V}$, similar to an orbital momentum or spin. The global symmetry of the system is $O(d-1)$, under which the tensor $\mathbf{V}$ transforms into another tensor of the same type. The Clifford algebra structure creates complex interactions between internal degrees of freedom. In the simplest case of peakon solutions, this Clifford structure influences the long-time behaviour, leading to coordinated energy exchange even when the peakons are spatially separated.

To keep the presentation self-contained, we first review (and then extend) the setup of [5]. As part of this review we explain in what sense the vector PDE (1.1) (equivalently, the component system (1.4)) is integrable, by connecting it to an isospectral deformation of a linear problem formulated in the Clifford algebra $\mathcal{C}\ell(W)$, where W is a d -dimensional vector space equipped with a nondegenerate symmetric bilinear form. We show that for $d = 2$ the system (1.1) admits an infinite hierarchy of commuting symmetries. For arbitrary d , we subsequently specialize to $\alpha = 0$, corresponding to the short-pulse (Hunter–Saxton) limit [20, 21], and construct weak solutions of (1.1) arising from measure-valued elements of $\mathcal{C}\ell(W)$.

For comparison, the beam system considered in [4] is

$$m_t = (um)_x + u_x m - vm, \quad (1.5)$$

$$n_t = (un)_x + u_x n + vn, \quad (1.6)$$

with constraints

$$u_{xxx} - 4u_x = (m + n)_x, \quad v_x = (n - m), \quad (1.7)$$

corresponding to setting $\alpha = 1$ in (1.3). The underlying principle from which equations (1.5)–(1.7) were derived was that of isospectral deformations of the Euler–Bernoulli beam problem. The same system of equations also appears in the list of three interesting integrable equations connected to a 4×4 matrix Lax pair in the work of Geng and Wu [13].

Setting $m = m_1 + m_2$ and $n = m_1 - m_2$, the system consisting of (1.5) and (1.6) becomes

$$m_{1,t} = (um_1)_x + u_x m_1 - vm_2,$$

$$m_{2,t} = (um_2)_x + u_x m_2 - vm_1,$$

together with

$$u_{xxx} - 4u_x = 2m_{1,x}, \quad v_x = 2m_2.$$

This system is associated with a Clifford algebra of Minkowski signature $(+, -)$. By contrast, the Euclidean signature $(+, +)$ leads (with a minor modification in the constraint, involving the parameter α used throughout this work) to

$$\begin{cases} m_{1,t} &= (um_1)_x + u_x m_1 + v m_2, \\ m_{2,t} &= (um_2)_x + u_x m_2 - v m_1, \end{cases} \quad (1.8)$$

with constraints

$$u_{xxx} - 4\alpha u_x = 2m_{1,x}, \quad v_x = -2m_2. \quad (1.9)$$

In the second part of the paper, we concentrate on the Hunter–Saxton limit $\alpha = 0$. Moreover, we assume that the measures m_1, m_2 are compactly supported and work with the integrated form of the first constraint, namely $u_{xx} = 2m_1$. For a fixed value of $\alpha \neq 0$, the Hunter–Saxton analogue of the system corresponds to taking the short pulse, high frequency limit of (1.8). Upon rescaling the dependent and independent variables according to

$$m_1 \rightarrow \epsilon^{-2} m_1, \quad m_2 \rightarrow \epsilon^{-2} m_2, \quad u \rightarrow u, \quad v \rightarrow \epsilon^{-1} v, \quad x \rightarrow \epsilon x, \quad t \rightarrow \epsilon t,$$

and taking the limit $\epsilon \rightarrow 0$, we find that the form of the system (1.8) is preserved, and only the relation between m_1 and u is altered, so that after integrating the first relation we have:

$$2m_1 = u_{xx}, \quad 2m_2 = -v_x. \quad (1.10)$$

In section 5 below, we will be concerned with constructing weak solutions of the d -component version of the short pulse limit.

The overall structure of the paper is as follows. In the next section, we briefly review the Clifford algebra structure which provides the setting for the spectral problem and Lax pair of the d -component system (1.1), before explaining how symmetries and Hamiltonian operators appear in this context. In section 3, the main focus is on the case of two components ($d = 2$). We present a reciprocal transformation, mapping the conservation laws of the two-component PDE system to those of another system with a new set of independent variables. Under this transformation, we demonstrate that the problem of finding travelling waves for this system can be reduced to that of solving a finite-dimensional Hamiltonian system with two degrees of freedom that is Liouville integrable, which is then used to produce plots of travelling wave profiles, obtained via numerical integration. The conclusion of the third section is devoted to explaining how, when $d = 2$, the first higher symmetry of the PDE system is connected by the reciprocal transformation to another coupled pair of PDEs that admits a Miura map to the Hirota–Satsuma system. Hence, under the latter transformation, the symmetries of the original two-component system are related to the Hirota–Satsuma hierarchy. This connection prompted the subsequent investigations whose results are presented in section 4, where we present a complete classification of two-component deformations of the Camassa–Holm equation via the symmetry approach. The result of the analysis is a list of six different systems, all but one of which turn out to be related to known integrable hierarchies by a combination of reciprocal and Miura transformations, while the one remaining system appears to be new. In section 5, we consider the Hunter–Saxton limit $\alpha = 0$ of the full d -component system, and use Clifford-valued continued fractions to solve the Neumann problem for the string equation with a discrete measure M . This leads to weak soliton solutions of the vector which are piecewise linear, with moving peaks in the scalar field u at the support on the measure, hence the name peakons. For the two-component case $d = 2$, we analyze the dynamics of two peakons in more detail, and plot an explicit solution of the equations of motion of the peaks, including additional internal degrees of freedom. The paper ends with some conclusions and open problems.

Before proceeding with the rest of the paper, there are a couple of things that are worth pointing out. Firstly, unlike the original Camassa–Holm peakons, which have the shape of the Green’s function $e^{-|x|}$ for the 1D Helmholtz operator, here we are considering the Hunter–Saxton (or short pulse) limit, in which case the relevant function is $|x|$, and the peakons are piecewise linear. Secondly, the main purpose of the paper is to introduce and analyze a vector generalization of the Camassa–Holm equation, which is a well-studied nonlinear shallow-water equation with a rich mathematical and physical history; but as it stands, we do not have a specific physical interpretation of the vector system. Thus, we would like to emphasize that the work should be read as a mathematical study of an integrable vector perturbation of the Camassa–Holm equation, rather than as a direct derivation of a new physical model.

2 Lax pair and higher symmetries

We will now briefly review the setup of [5]. The system (1.1) arises from a Lax pair defined in a Clifford algebra $\mathcal{Cl}(W)$, where W is a d -dimensional vector space over a field $\mathbb{F}$ with a symmetric bilinear form $(\cdot, \cdot)$. We usually assume $\mathbb{F} = \mathbb{C}$, but sometimes we work with $\mathbb{F} = \mathbb{R}$, and consider different real forms of the underlying complex Clifford algebra. Upon choosing an orthonormal basis of generators e_μ , $\mu = 1, \dots, d$ such that $(e_\mu, e_\nu) = \varepsilon_\mu \delta_{\mu\nu}$, for signs $\varepsilon_\mu = \pm 1$, the multiplication in $\mathcal{Cl}(W)$ is then given by

$$e_\mu e_\nu + e_\nu e_\mu = 2\varepsilon_\mu \delta_{\mu\nu} 1,$$

where 1 denotes the unit $1 \in \mathbb{F} \subset \mathcal{Cl}(W)$, which will usually be omitted from formulae. The signs ε_μ determine the signature of the bilinear form $(\cdot, \cdot)$: over $\mathbb{C}$ one can always fix $\varepsilon_\mu = 1$ for all μ , which is our canonical choice for most of the paper, but on occasions when real solutions are being considered we can allow different signatures over $\mathbb{R}$. The whole algebra is a direct sum of graded components:

$$\mathcal{Cl}(W) = \mathcal{Cl}_0 \oplus \mathcal{Cl}_1 \oplus \dots \oplus \mathcal{Cl}_n,$$

where $\mathcal{Cl}_0 = \langle 1 \rangle \cong \mathbb{F}$, $\mathcal{Cl}_1 = \langle e_\mu \rangle_{\mu=1, \dots, n} \cong W$, $\mathcal{Cl}_2 = \langle e_\mu e_\nu \rangle_{\mu < \nu}$, etc. We can also equip $\mathcal{Cl}(W)$ with the trace tr , a linear functional on the Clifford algebra defined by $\text{tr}(1) = 1$ and $\text{tr}(c) = 0$ for all $c \in \mathcal{Cl}_j$ with $j \geq 1$, which then has the property that

$$\text{tr}(ab) = (a, b) \quad \forall a, b \in W \cong \mathcal{Cl}_1.$$

To construct a vector perturbation of the Camassa–Holm system, we start from a linear problem based on a string equation with a potential

$$M = \sum_{\mu=1}^n m_\mu e_\mu \in \mathcal{Cl}_1,$$

that is

$$D_x^2 \Phi = (\alpha + \lambda M) \Phi, \tag{2.1}$$

where the wave function $\Phi \in \mathcal{Cl}(V)$, with an arbitrary scalar constant $\alpha \in \mathbb{F}$ and spectral parameter λ , and couple this to a time evolution written as

$$D_t \Phi = \left(\frac{1}{2}(a - b_x) + b D_x \right) \Phi. \tag{2.2}$$

Following [4, 5], we refer to the linear equation (2.1) as a (generalized) beam problem.

Lemma 2.1 ([5]). *The compatibility conditions for the Lax pair given by (2.1) and (2.2) consist of the following two relations:*

$$\begin{aligned} a_x + \lambda[b, M] &= 0, \\ \lambda M_t &= 2\alpha b_x - \frac{1}{2}b_{xxx} + \frac{\lambda}{2} \left([a, M] + b \overset{\leftarrow}{\mathcal{L}}_M + \overset{\rightarrow}{\mathcal{L}}_M b \right); \end{aligned} \tag{2.3}$$

in the above, $\mathcal{L}_M$ is the operator $MD_x + D_x M$, and the arrow on top denotes whether it acts to the right or the left.

Remark 2.2. To clarify the notation as used in [5], note that we have $b \overset{\leftarrow}{\mathcal{L}}_M = 2b_x M + b M_x$ and $\overset{\rightarrow}{\mathcal{L}}_M b = 2M b_x + M_x b$. Thus the sum of these two terms reveals the presence of anticommutators in the compatibility conditions, as well as the commutators that are apparent in (2.3).

The equation (1.1) is an isospectral evolution, corresponding to a particular solution of the compatibility conditions (2.3), obtained by choosing

$$a = a_0 \in \mathcal{Cl}_2, \quad b = b_0 + \frac{b_{-1}}{\lambda} \quad \text{with} \quad b_0 \in \mathcal{Cl}_0, \quad b_{-1} \in \mathcal{Cl}_1,$$

and this gives a consistent result provided that b_{-1} is constant. Without loss of generality, we can always perform a gauge transformation by multiplying Φ in the linear system by a constant element (corresponding to a rotation in $SO(d)$), so that up to scale we can fix $b_{-1} = e_1$. Then we find

$$b_0 = u \in \mathbb{F}, \quad b_{-1} = e_1 \in W, \quad a_0 = \sum_{\nu=2}^d v_{1\nu} e_1 e_\nu \in \mathcal{Cl}_2, \tag{2.4}$$

where u and the coefficients appearing in a_0 are subject to the constraints (1.3).

We conclude this section with several remarks regarding the place of (1.1) in a larger context of integrable systems. We note that we can rewrite (1.1) as an evolution equation in the Clifford algebra, namely

$$M_t = \frac{1}{2}(A_M D_x + D_x A_M) u - \frac{1}{2} C_M D_x^{-1} C_M e_1, \quad (2.5)$$

with $u = b_0, e_1 = b_{-1}$ as in (2.4), where A_M, C_M respectively denote the anticommutator/commutator with respect to M .

A pair of elementary symmetries is produced by the choices

$$a = a_0 \in \mathcal{C}\ell_2, \quad b = b_0 \in \mathcal{C}\ell_0, \quad a_0, b_0 \text{ const},$$

for where the constants a_0, b_0 are arbitrary elements of degree 2 and 0, respectively, and the corresponding symmetries can be written in combination as the s -flow

$$M_s = \frac{1}{2}[a_0, M] + b_0 M_x,$$

while the first non-trivial symmetry is 3rd order, being given by the flow

$$M_\tau = \frac{1}{2}(4\alpha D_x - D_x^3) \left(|\mathbf{m}|^{-\frac{3}{2}} M \right). \quad (2.6)$$

The latter is Hamiltonian, as it can be expressed as

$$M_\tau = \mathcal{B}_1 \frac{\delta H}{\delta M},$$

with the Hamiltonian operator

$$\mathcal{B}_1 = (4\alpha D_x - D_x^3) \quad (2.7)$$

and conserved functional

$$H = \int (\text{tr } M^2)^{\frac{1}{4}} dx. \quad (2.8)$$

The conserved density associated with H is

$$\rho = (\text{tr } M^2)^{\frac{1}{4}} = \left(\sum_{\mu=1}^d m_\mu^2 \right)^{\frac{1}{4}} = |\mathbf{m}|^{\frac{1}{2}},$$

which for $d = 1$ recovers the well-known density $\sqrt{m}$ for the Camassa–Holm equation. The gradient of the Hamiltonian H is in the kernel of the second Hamiltonian structure $\mathcal{B}_2$, which is compatible with $\mathcal{B}_1$, being given by the operator

$$\mathcal{B}_2 = (A_M D_x + D_x A_M) + C_M D_x^{-1} C_M \quad (2.9)$$

(cf. (2.5) above). These operators were found in the work of Olver and Sokolov, who considered integrable Hamiltonian systems defined in associative algebras [25].

3 Two-component case: reciprocal transformation, travelling waves and Hirota–Satsuma

In this section we focus on the case $d = 2$ of (1.4), which is the two-component system

$$\begin{aligned} m_{1,t} &= 2u_x m_1 + u m_{1,x} + v m_2, \\ m_{2,t} &= 2u_x m_2 + u m_{2,x} - v m_1, \end{aligned} \quad (3.1)$$

with auxiliary scalar fields u, v being related to m_1, m_2 by the constraints

$$u_{xx} - u = 2m_1, \quad v_x = -2m_2. \quad (3.2)$$

(Note that we fixed $\alpha = 1/4$ here.) We have written (3.1) in the case of Euclidean signature. As previously mentioned in the introduction, up to a linear transformation and rescaling, this two-component system is equivalent to the beam system introduced by Beals and Szmigielski in [4].

To begin with, we present a reciprocal transformation, which changes the conservation laws in the independent variables x, t into conservation laws involving new variables X, T . We discuss how this transforms the linear system (2.1) and (2.2) for $d = 2$, thereby producing a new Lax pair in terms of the new variables X, T .

Next, we proceed to consider the most elementary solutions of (3.1), namely travelling waves: after proving an elementary result on the non-existence of such waves with vanishing boundary conditions, we present a general lemma on the correspondence between travelling waves of different PDE systems related to one another by a reciprocal transformation. We then use this result to produce a hodograph link between the travelling waves of (3.1) with a system of Hamiltonian ODEs with two degrees of freedom, and prove that this is integrable in the Liouville sense.

In a subsequent subsection, we describe the connection between (3.1) and the hierarchy of the Hirota–Satsuma equation, a well-known integrable PDE. This leads onto the following section, where we proceed to present a classification of all two-component deformations of the Camassa–Holm equation, including (3.1), which possess a higher local symmetry whose undeformed version has order 3.

3.1 Reciprocal transformation and Lax pair

The linear system consisting of (2.1) and (2.2) can be written in the first order form

$$\Psi_x = \mathbf{F}\Psi, \quad \Psi_t = \mathbf{G}\Psi, \quad (3.3)$$

where

$$\Psi = \begin{pmatrix} \Phi \\ \Phi_x \end{pmatrix}$$

and F, G are 2×2 matrices with entries in $\mathcal{C}\ell(W)$, namely

$$\mathbf{F} = \begin{pmatrix} 0 & 1 \\ \alpha 1 + \lambda M & 0 \end{pmatrix}, \quad \mathbf{G} = \begin{pmatrix} \frac{1}{2}(a - b_x) & b \\ \frac{1}{2}(a_x - b_{xx}) + \alpha b + \lambda b M & \frac{1}{2}(a + b_x) \end{pmatrix}. \quad (3.4)$$

The compatibility condition for (3.3) is the zero curvature equation

$$\mathbf{F}_t - \mathbf{G}_x + [\mathbf{F}, \mathbf{G}] = 0,$$

which produces the same set of conditions (2.3) as in Lemma 2.1.

For what follows, in the case $d = 2$ it is convenient to pick a specific 2×2 matrix representation for the Clifford algebra $\mathcal{C}\ell(W)$ with generators e_1, e_2 . This is a 4-dimensional algebra spanned by $1, e_1, e_2, e_1 e_2$, and by fixing particular representatives for the generators we may take

$$1 \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_1 \rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad e_2 \rightarrow \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad e_1 e_2 \rightarrow \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

resulting in a 4×4 representation of the linear system (3.3). Upon taking

$$M = m_1 e_1 + m_2 e_2, \quad a = v e_1 e_2, \quad b = u + \lambda^{-1} e_1,$$

in accordance with (2.4) when $d = 2$, and fixing $\alpha = \frac{1}{4}$, we obtain the linear system for (3.1) rewritten in the 1st order form (3.3), with the 4×4 matrices $\mathbf{F}, \mathbf{G}$ built from the appropriate 2×2 blocks, given by

$$\mathbf{F} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{4} & \lambda(m_1 - i m_2) & 0 & 0 \\ \lambda(m_1 + i m_2) & \frac{1}{4} & 0 & 0 \end{pmatrix}, \quad (3.5)$$

$$\mathbf{G} = \begin{pmatrix} -\frac{1}{2}(u_x - i v) & 0 & u & \lambda^{-1} \\ 0 & -\frac{1}{2}(u_x + i v) & \lambda^{-1} & u \\ -\frac{1}{4}u & \lambda u(m_1 - i m_2) + \frac{1}{4}\lambda^{-1} & \frac{1}{2}(u_x + i v) & 0 \\ \lambda u(m_1 + i m_2) + \frac{1}{4}\lambda^{-1} & -\frac{1}{4}u & 0 & \frac{1}{2}(u_x - i v) \end{pmatrix}.$$

Our main goal in this section is to transform the partial differential equations (3.1) and the associated linear system (3.3) by means of the reciprocal transformation

$$dX = \rho dx + \rho u dt, \quad dT = dt, \quad (3.6)$$

where ρ is the density associated with the functional (2.8), that is

$$\rho = |\mathbf{m}|^{\frac{1}{2}} = (m_1^2 + m_2^2)^{\frac{1}{4}} \quad (3.7)$$

when $d = 2$. This means that derivatives with respect to the independent variables transform according to

$$D_x = \rho D_X, \quad D_t = D_T + \rho u D_X.$$

The consistency of the reciprocal transformation is guaranteed by the conservation law

$$\rho_t = (\rho u)_x,$$

which transforms under (3.6) to another conservation law in the new independent variables X, T , namely

$$(\rho^{-1})_T = -u_X. \quad (3.8)$$

Then we find that it is convenient to introduce a new dependent variable ϑ , such that

$$m_1 = \rho^2 \sin \vartheta, \quad m_2 = \rho^2 \cos \vartheta,$$

and we find that the system (3.1) is transformed to a pair of equations for ρ and ϑ , namely (3.8) and the relation

$$\vartheta_T = v. \quad (3.9)$$

However, the way that the relations (3.2) between u, v and the m_j transform under the reciprocal transformation must also be taken into account.

Proposition 3.1. In terms of ϑ and another variable $\psi = \log \rho$, under the reciprocal transformation (3.6) the system (3.1) is transformed into the pair of equations

$$\begin{aligned} D_X(e^\psi \psi_{XT} - 2e^{2\psi} \sin \vartheta) + D_T(e^{-\psi}) &= 0, \\ \vartheta_{XT} + 2e^\psi \cos \vartheta &= 0. \end{aligned} \quad (3.10)$$

The latter system is the compatibility condition for the Lax pair

$$\hat{\Psi}_X = \hat{\mathbf{F}} \hat{\Psi}, \quad \hat{\Psi}_T = \hat{\mathbf{G}} \hat{\Psi}, \quad (3.11)$$

where

$$\hat{\mathbf{F}} = \begin{pmatrix} 0 & 0 & e^{-\psi} & 0 \\ 0 & 0 & 0 & e^{-\psi} \\ \frac{1}{4}e^{-\psi} & -i\lambda e^{\psi+i\vartheta} & 0 & 0 \\ i\lambda e^{\psi-i\vartheta} & \frac{1}{4}e^{-\psi} & 0 & 0 \end{pmatrix},$$

$$\hat{\mathbf{G}} = \begin{pmatrix} -\frac{1}{2}(\psi_T - i\vartheta_T) & 0 & 0 & \lambda^{-1} \\ 0 & -\frac{1}{2}(\psi_T + i\vartheta_T) & \lambda^{-1} & 0 \\ -\frac{1}{2}e^\psi \psi_{XT} + e^{2\psi} \sin \vartheta & \frac{1}{4}\lambda^{-1} & \frac{1}{2}(\psi_T + i\vartheta_T) & 0 \\ \frac{1}{4}\lambda^{-1} & -\frac{1}{2}e^\psi \psi_{XT} + e^{2\psi} \sin \vartheta & 0 & \frac{1}{2}(\psi_T - i\vartheta_T) \end{pmatrix}$$

Proof. Under the reciprocal transformation, the relation between u and m_1 produces $(\rho D_X)^2 u - u = \rho^2 \sin \vartheta$, and this can be solved for u , with u_X and u_{XX} replaced in terms of ρ from (3.8), to obtain

$$u = \rho(\log \rho)_{XT} - 2\rho^2 \sin \vartheta. \quad (3.12)$$

Similarly, the second equation in (3.2) gives the relation between v and m_2 , and this transforms to $\rho v_X = -2\rho^2 \cos \vartheta$, in which v can be replaced using (3.9). After substituting the formula (3.12) for u back into (3.8), this yields a closed system of two equations for ρ and ϑ , namely

$$\begin{aligned} D_X(\rho(\log \rho)_{XT} - 2\rho^2 \sin \vartheta) + D_T(\rho^{-1}) &= 0, \\ \vartheta_{XT} + 2\rho \cos \vartheta &= 0. \end{aligned} \quad (3.13)$$

After introducing ψ via $\rho = e^\psi$, the above system can be rewritten in the form (3.10). The same conclusion can be reached in a straightforward manner, by applying the reciprocal transformation (3.6)

directly to the linear system (3.3): indeed, if we specify the wave function as a function of the new independent variables, writing $\hat{\Psi}(X, T) = \Psi(x, t)$, then the transformed linear equations

$$\rho \hat{\Psi}_x = \mathbf{F} \hat{\Psi}, \quad \hat{\Psi}_T + \rho u \hat{\Psi}_X = \mathbf{G} \hat{\Psi}$$

immediately imply that $\hat{\mathbf{F}} = \rho^{-1} \mathbf{F}$, $\hat{\mathbf{G}} = \mathbf{G} - u \mathbf{F}$. After substituting $m_2 + i m_1 = \rho^2 e^{i\vartheta}$ into the original matrices (3.5), the compatibility conditions obtained from the new zero curvature equation $[D_X - \hat{\mathbf{F}}, D_T - \hat{\mathbf{G}}] = 0$ consist of (3.9) and (3.12), which can be rewritten in terms of ψ as

$$u = e^\psi \psi_{XT} - 2e^{2\psi} \sin \vartheta, \quad (3.14)$$

together with the system (3.13), or equivalently (3.10). In the transformed Lax pair, we can then use (3.9) and (3.12) to write all the entries purely in terms of ψ, ϑ and their derivatives. $\square$

Remark 3.2. Note that the first equation of the system (3.10) is written in conservation form, being equivalent to the equation (3.8). This system also has the conservation law

$$\hat{\mathcal{U}}_T + (2e^\psi \sin \vartheta)_X = 0, \quad (3.15)$$

where

$$\hat{\mathcal{U}} = -\frac{1}{2} \psi_{XX} - \frac{1}{4} \psi_X^2 - \frac{1}{4} e^{-2\psi} + \frac{1}{4} \vartheta_X^2. \quad (3.16)$$

When $\theta = \pi/2$, the quantity $\hat{\mathcal{U}}$ becomes the standard KdV dependent variable, and (3.10) reduces to the first negative flow of the KdV hierarchy (see e.g. [18]). In the original coordinates, this reduction corresponds precisely to the special case $v = 0 = m_2$, when the two-component system reduces to the Camassa–Holm equation.

3.2 Travelling wave reduction

In this subsection, our main goal will be to describe the properties of the travelling wave solutions in the case $d = 2$, namely for the two-component system (3.1). However, we begin by making some remarks about the case general of general d . We choose the travelling wave coordinate to be $z = x - ct$, let prime denote $\frac{d}{dz}$ and assume zero background for the fields, which depend on x and t through the combination z only, so that

$$u, \mathbf{m} \rightarrow 0 \quad \text{as} \quad |z| \rightarrow \infty.$$

Then from (1.1) we obtain

$$-(c + u) \mathbf{m}' = 2u' \mathbf{m} + V \mathbf{m}. \quad (3.17)$$

Taking the inner product with $\mathbf{m}$, and using the fact that the matrix V is skew-symmetric, we obtain

$$\frac{d}{dz} \log(|\mathbf{m}|^2) = -\frac{4u'}{c + u}, \quad (3.18)$$

and hence find the invariant

$$K = |\mathbf{m}|^2 (c + u)^4 = \text{const}. \quad (3.19)$$

Zero background gives $K = 0$, which by (1.3) forces $(u'' - 4\alpha u, \mathbf{v}') \equiv (0, 0)$ for smooth profiles; therefore nontrivial travelling waves with zero background cannot be smooth, so must be peaked/cusped, being realized when $c + u$ vanishes at a crest. So we have

Proposition 3.3. For any d , the system (1.1) has no nontrivial smooth travelling waves with zero background.

In the light of the above result, the latter sections of the paper will focus on peaked soliton (peakon) solutions of (1.1), which are weak solutions. The rest of this subsection is concerned with the smooth travelling waves for $d = 2$.

In order to analyze the ODEs for travelling waves of the two-component system (3.1), we will make use of the reciprocal transformation (3.6), which can be reduced to the level of these ODEs. Our treatment relies on a general feature of similarity reductions of reciprocal transformations (see [2], for instance), which is that under a change of independent variables, the roles of parameters and constants of motion can be switched. In particular, in the case of hodograph transformations (changing the time variable) in Hamiltonian systems, this phenomenon is referred to as “coupling constant metamorphosis” [14, 26]. For the context at hand, we can state the following

Lemma 3.4. *Under the reciprocal transformation given by*

$$dX = \rho dx + \mathcal{F} dt, \quad dT = dt, \quad (3.20)$$

the travelling wave solutions of the conservation law

$$\rho_t = \mathcal{F}_x,$$

moving with speed c , admit the conserved quantity

$$k = \mathcal{F} + c\rho = \text{const}. \quad (3.21)$$

Moreover, for a fixed value of this constant, these solutions correspond to travelling waves of the reciprocally transformed PDE

$$(\rho^{-1})_T + (\mathcal{F}\rho^{-1})_X = 0 \quad (3.22)$$

moving with speed k .

Proof. Upon reducing to travelling waves with speed c , the conservation law with density ρ and flux $\mathcal{F}$ produces the ODE $-c\rho' = \mathcal{F}'$, where the prime denotes $\frac{d}{dz}$ as before. This can be rearranged as

$$\frac{d}{dz}(\mathcal{F} + c\rho) = 0,$$

which integrates to give the conserved quantity k as in (3.21). Now consider the reduction of the reciprocal transformation (3.20) to the travelling wave solutions: by a slight abuse of notation, we have the density $\rho = \rho(z)$, being a function of the similarity variable z , and

$$\rho dz = \rho(dx - c dt) = dX - \mathcal{F} dt - c\rho dt = dX - k dT.$$

This can be rewritten as

$$dZ = \rho(z) dz, \quad (3.23)$$

where $Z = X - kT$ is a travelling wave variable in the new X, T coordinates, with wave speed k . Then, with this value of the wave speed, we see from (3.22) that the travelling waves of the reciprocally transformed PDE satisfy

$$\frac{d}{dZ}(k\rho^{-1} - \mathcal{F}\rho^{-1}) = 0,$$

which integrates and rearranges to yield (3.21), where now c is an integration constant. Thus we see that, subject to the relation (3.23) between the independent variables, there is a one-to-one correspondence between travelling wave solutions of the two reciprocally related PDEs, modulo the metamorphosis of coupling constants $c \leftrightarrow k$, exchanging the roles of these parameters. $\square$

The above result is directly applicable to the case at hand, where we have the reciprocal transformation (3.6), with the flux of the conservation law being $\mathcal{F} = \rho u$. In the light of the preceding lemma, to determine the travelling waves of the system (3.1) it will be sufficient to consider travelling waves of (3.10) moving with speed $k \neq 0$, where the original wave speed c will appear as a parameter. Thus we take

$$\psi = \psi(Z), \quad \vartheta = \theta(Z), \quad Z = X - kT. \quad (3.24)$$

In the course of doing so, for the Hamiltonian theory it will be convenient to apply imaginary rescalings to some of the variables/parameters, and consider solutions of the form

$$\psi = \psi(Z), \quad \vartheta = i\varphi(Z), \quad Z = X - iCT, \quad \text{where } k = iC, \quad (3.25)$$

with the arbitrary constant $C \neq 0$ (excluding the stationary case $C = 0$). In that case, the PDEs (3.10) reduce to a pair of ODEs for ψ, φ , given by

$$\begin{aligned} \frac{d}{dZ}(Ce^\psi \psi'' + 2e^{2\psi} \sinh \varphi + Ce^{-\psi}) &= 0, \\ C\varphi'' + 2e^\psi \cosh \varphi &= 0, \end{aligned} \quad (3.26)$$

where now the prime denotes $\frac{d}{dZ}$. Since the first equation above is a total derivative, we can integrate it once, to obtain the constant of motion

$$K = -e^\psi \psi'' - \frac{2}{C} e^{2\psi} \sinh \varphi - e^{-\psi}, \quad \text{where } K = \frac{ic}{C} \quad (3.27)$$

by the result of Lemma 3.4. Note that we will ultimately be interested in the case where c , the speed of the travelling waves of the two-component CH system, is real, and the parameter $C = -ik$ is pure imaginary, so the constant K will also be real.

After performing an integration as above, to obtain (3.27), we find that overall the system (3.26) for the reciprocally transformed travelling wave reduction can be written in canonical Hamiltonian form, as

$$\psi' = \frac{\partial h}{\partial p_\psi}, \quad \varphi' = \frac{\partial h}{\partial p_\varphi}, \quad p'_\psi = -\frac{\partial h}{\partial \psi}, \quad p'_\varphi = -\frac{\partial h}{\partial \varphi}, \quad (3.28)$$

where $p_\psi = \psi'$ and $p_\varphi = \varphi'$ are canonically conjugate momenta, and the Hamiltonian function is

$$h = \frac{1}{2}p_\psi^2 + \frac{1}{2}p_\varphi^2 + 2C^{-1}e^\psi \sinh \varphi - \frac{1}{2}e^{-2\psi} - Ke^{-\psi}. \quad (3.29)$$

In order to prove that the latter Hamiltonian system is completely integrable in the Liouville sense, we need to find a second independent constant of motion that is in involution with h , which is achieved by considering the reduction of the Lax pair to the travelling waves.

Theorem 3.5. *Via the hodograph transformation (3.23) obtained from the density ρ in (3.7), and the change of parameters and dependent variables (3.25), with $\rho = e^\psi$ and $c = -iCK$, the travelling wave solutions of the PDE system (3.1) with speed c correspond to the solutions of the Hamilton's equations (3.28) defined by (3.29). This Hamiltonian system with two degrees of freedom is the compatibility condition for the linear system*

$$\tilde{\Psi}' = \mathbf{M}\tilde{\Psi}, \quad \mathbf{L}\tilde{\Psi} = \mu\tilde{\Psi}, \quad (3.30)$$

where

$$\mathbf{L} = \begin{pmatrix} \frac{1}{2}C(p_\psi + p_\varphi) & 0 & Ce^{-\psi} & -i\lambda^{-1} \\ 0 & \frac{1}{2}C(p_\psi - p_\varphi) & -i\lambda^{-1} & Ce^{-\psi} \\ -\frac{1}{4}C(e^{-\psi} + 2K) & -\frac{1}{4}\lambda^{-1} - iCe^{\psi-\varphi}\lambda & -\frac{1}{2}C(p_\psi - p_\varphi) & 0 \\ -\frac{1}{4}\lambda^{-1} + iCe^{\psi+\varphi}\lambda & -\frac{1}{4}C(e^{-\psi} + 2K) & 0 & \frac{1}{2}C(p_\psi + p_\varphi) \end{pmatrix}$$

and

$$\mathbf{M} = \begin{pmatrix} 0 & 0 & e^{-\psi} & 0 \\ 0 & 0 & 0 & e^{-\psi} \\ \frac{1}{4}e^{-\psi} & -ie^{\psi-\varphi}\lambda & 0 & 0 \\ ie^{\psi+\varphi}\lambda & \frac{1}{4}e^{-\psi} & 0 & 0 \end{pmatrix}.$$

Moreover, it is completely integrable in the Liouville sense, having a second independent first integral, given by

$$J = p_\psi^2 p_\varphi^2 + 4C^{-1}e^\psi \cosh \varphi p_\psi p_\varphi - e^{-\psi}(e^{-\psi} + 2K)p_\varphi^2 + 4C^{-2}e^{2\psi} \cosh^2 \varphi - 4KC^{-1} \sinh \varphi. \quad (3.31)$$

Proof. The travelling wave reduction of the Lax pair (3.11) is obtained by taking the wave function in the form $\tilde{\Psi} = \exp(i\mu T)\tilde{\Psi}(Z)$, which leads to the pair of linear equations (3.30). The compatibility condition for the latter linear system is the matrix Lax equation

$$\mathbf{L}' = [\mathbf{M}, \mathbf{L}],$$

which is equivalent to Hamilton's equations (3.28) for ψ, φ and their conjugate momenta. Moreover, the spectral curve

$$\mathcal{P}(\lambda^2, \mu^2) = \lambda^4 \det(\mathbf{L}(\lambda) - \mu \mathbf{1}) = 0 \quad (3.32)$$

is invariant under the evolution defined by the Lax equation, being given by the polynomial

$$\mathcal{P} = \frac{1}{16} + \gamma_1 \lambda^2 + \gamma_2 \lambda^4 - C^4 \lambda^6 + \left(\frac{1}{2} \lambda^2 - \gamma_3 \lambda^4 \right) \mu^2 + \lambda^4 \mu^4,$$

where each of the coefficients $\gamma_1, \gamma_2, \gamma_3$ is a first integral of the system. After further calculations with computer algebra (MAPLE) we find that these coefficients are given explicitly by

$$\gamma_1 = \frac{1}{4}C^2(K^2 - h), \quad \gamma_2 = \frac{1}{4}C^4(h^2 - J), \quad \gamma_3 = C^2h,$$

where the function (3.31) is in involution with h , that is $\{h, J\} = 0$ with respect to the canonical Poisson bracket, so J is a second independent constant of motion, implying Liouville integrability of the system. $\square$

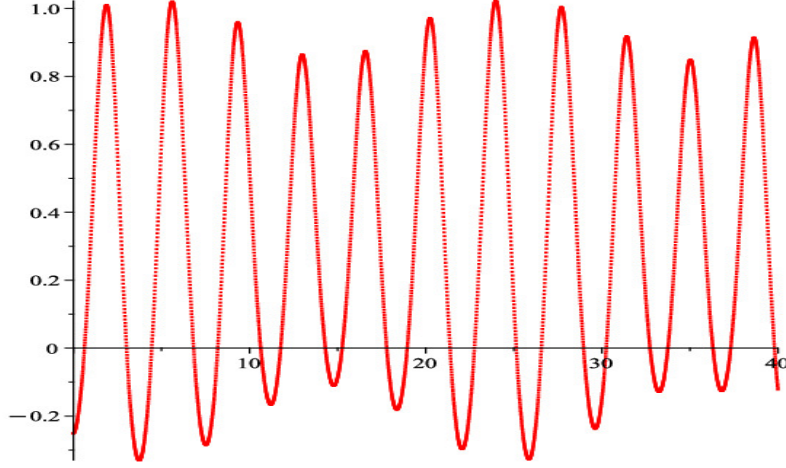


Figure 1: Travelling wave profile of $\hat{\mathcal{U}}$ given by (3.16), plotted against Z , obtained from the solution of the system (3.33) with parameters $k = 1$, $K = -3$ and the initial conditions taken as $\psi(0) = -0.3$, $\theta(0) = 1.520796327$, $\psi'(0) = -0.05$, $\theta'(0) = 0.15$.

Remark 3.6. If we let $\mathcal{C}$ denote the curve defined by the affine equation $\mathcal{P}(\lambda^2, \mu^2) = 0$ in the (λ, μ) plane, then this is an unramified double cover of the curve $\mathcal{C}'$ defined by $\mathcal{P}(\zeta, \mu^2) = 0$ in the (ζ, μ) plane, which is in turn a ramified double cover of the elliptic curve $\mathcal{E}$ defined by $\mathcal{P}(\zeta, \eta) = 0$, that is

$$\mathcal{E} : \quad \zeta^2 \eta^2 - (\gamma_3 \zeta - \tfrac{1}{2}) \zeta \eta - C^4 \zeta^3 + \gamma_2 \zeta^2 + \gamma_1 \zeta + \tfrac{1}{16} = 0,$$

so that we have a sequence of coverings:

$$\mathcal{C} \xrightarrow{2:1} \mathcal{C}' \xrightarrow{2:1} \mathcal{E}.$$

By applying the Riemann-Hurwitz theorem to each covering in succession, we see that the genus of $\mathcal{C}'$ is 3, being a double cover of the genus 1 curve $\mathcal{E}$ with 4 ramification points, while the genus of $\mathcal{C}$ is 5. General theory tells us that the flows of the Hamiltonian system linearize on $\text{Jac } \mathcal{C}$, the Jacobian variety of the spectral curve for the Lax matrix $\mathbf{L}$. However, the latter is a 5-dimensional abelian variety, whereas the invariant tori defined by the level sets $h = \text{const}$, $J = \text{const}$ are two-dimensional in this case (cf. Fig. 3); so ideally we would like to construct a separation of variables that describes straight line motion on a two-dimensional subvariety of $\text{Jac } \mathcal{C}$. There are various results on separating coordinates for natural Hamiltonians with two degrees of freedom having a second invariant that is quartic in momenta [29], including the stationary flow of the Hirota-Satsuma system [1].

The canonical coordinates and momenta $\psi, \varphi, p_\psi, p_\varphi$ and the parameter C are convenient for interpreting the travelling wave reduction as a natural Hamiltonian system with two degrees of freedom, and showing that the system is Liouville integrable, as above. However, for *real* travelling waves of (3.10) moving with real speed $k = iC \neq 0$, and corresponding real travelling waves of (3.1) with speed $c = -kK$, it is preferable to use the coordinates $(\psi, \theta, \psi', \theta') \in \mathbb{R}^4$, where $\theta = i\varphi$ and the subscripts denote derivatives with respect to the travelling wave variable $z = X - kT$. In those coordinates, the equations of motion

$$\frac{d}{dZ} \begin{pmatrix} \psi \\ \theta \\ \psi' \\ \theta' \end{pmatrix} = \begin{pmatrix} \psi' \\ \theta' \\ -2k^{-1}e^\psi \sin \theta - Ke^{-\psi} - e^{-2\psi} \\ 2k^{-1}e^\psi \cos \theta \end{pmatrix} \quad (3.33)$$

have a fixed point when

$$\psi = \log Q, \quad \theta = \pm \frac{\pi}{2}, \quad \psi' = 0, \quad \theta' = 0, \quad (3.34)$$

where Q is a root of the cubic

$$\pm 2k^{-1}Q^3 + KQ + 1 = 0. \quad (3.35)$$

Then it is instructive to consider solutions of the 4D system close to an elliptic fixed point, which should correspond to compact 2D Liouville tori. In particular, when $k = 1$, $K = -3$ (corresponding to waves in

(3.1) with speed $c = 3$) and the plus sign is chosen in (3.34) and (3.35), the cubic has a real root at $Q = 1$, and at the fixed point with $\psi = 0$, $\theta = \pi/2$ the Jacobian matrix of the 4D system (3.33) has eigenvalues $\pm\sqrt{2}i, \pm\sqrt{3}i$. A particular travelling wave solution, showing the profile of the potential $\mathcal{U}$ as in (3.16), the profile of u against $z(Z)$, and a 3D projection of the corresponding Liouville torus, is plotted in Figs. 1, 2 and 3, respectively. In these plots, the initial conditions are $\psi(0) = -0.3$, $\theta(0) = \frac{\pi}{2} - 0.05$, $\psi'(0) = -0.05$, $\theta'(0) = 0.15$ and $z(0) = 0$, with the parameter values $k = 1$, $K = -3$, so $c = 3$ is the speed of the wave in the system (3.1). Note that (3.23) gives $z' = \rho^{-1} = e^{-\psi(Z)}$, so we have fixed $z(Z) = \int_0^Z e^{-\psi(S)} dS$ to make the origin of z and Z coincide. Also, from (3.21) with $\mathcal{F} = \rho u$ in this case, we have

$$u = k\rho^{-1} - c,$$

which gives $u = e^{-\psi} - 3$ in this particular numerical example, allowing the parametric curve $(z(Z), u(Z))$ to be plotted in Fig. 2.

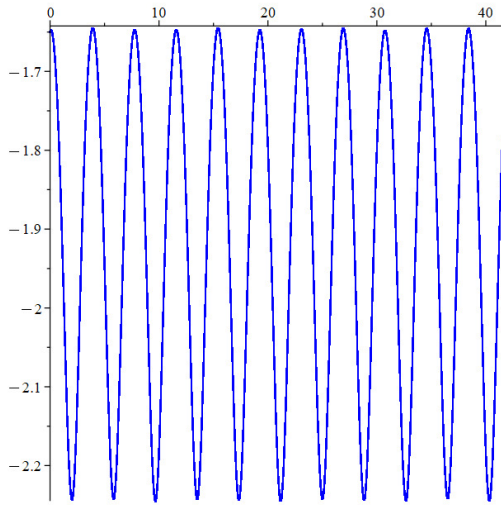


Figure 2: Travelling wave for (3.1), plotting u parametrically vs. $z = x - ct$, with speed $c = 3$, obtained from the solution of (3.33) with initial data as in Fig. 1.

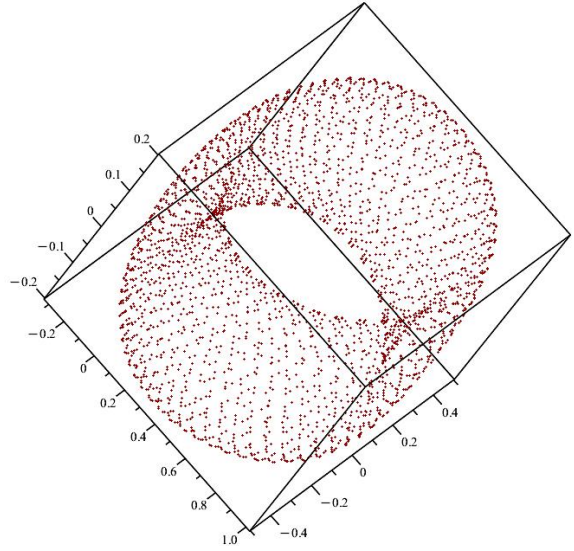


Figure 3: Liouville torus for the 4D system (3.33), projected onto the 3D space with coordinates $(\mathcal{U}, \psi', \theta')$, obtained from the same initial data as in Fig. 1.

3.3 Connection with the Hirota–Satsuma system

As mentioned above, the vector Camassa–Holm system (1.1) has a third order symmetry, given by (2.6). In the two-component case, when $d = 2$, this symmetry is given by

$$\begin{aligned} m_{1,\tau} &= \frac{1}{2}(4\alpha D_x - D_x^3) \left(\frac{m_1}{(m_1^2 + m_2^2)^{\frac{3}{4}}} \right), \\ m_{2,\tau} &= \frac{1}{2}(4\alpha D_x - D_x^3) \left(\frac{m_2}{(m_1^2 + m_2^2)^{\frac{3}{4}}} \right). \end{aligned} \quad (3.36)$$

(We have restored the arbitrary parameter α here.) The reciprocal transformation (3.6) can be extended to this symmetry, and indeed, to the whole hierarchy of higher symmetries. In order to do this, one should take

$$dX = \rho dx + \rho u dt + \sigma d\tau, \quad dT = dt, \quad d\tau' = d\tau,$$

where σ is the flux of the density ρ with respect to the τ flow, and τ' is the new time variable corresponding to the transformation of τ under the extension of (3.6). We are then able to relate the transformed symmetry to a well-known coupled third order system of PDEs, namely the Hirota–Satsuma system [15, 16]

$$\begin{aligned} \mathcal{U}_{\tau'} &= \frac{1}{2}\mathcal{U}_{XXX} + 3\mathcal{U}\mathcal{U}_X - 6\mathcal{V}\mathcal{V}_X, \\ \mathcal{V}_{\tau'} &= -\mathcal{V}_{XXX} - 3\mathcal{U}\mathcal{V}_X. \end{aligned} \quad (3.37)$$

This is described by the following result, which is verified by a direct calculation.

Proposition 3.7. The extension of the reciprocal transformation (3.6) to the third order symmetry (3.36) produces the coupled system

$$\begin{aligned}\psi_{\tau'} &= \frac{1}{2}\psi_{XXX} - \frac{1}{4}\psi_X^3 - \frac{3}{4}\vartheta_X^2\psi_X + \frac{3}{2}\vartheta_X\vartheta_{XX} - 3\alpha e^{-2\psi}\psi_X, \\ \vartheta_{\tau'} &= -\vartheta_{XXX} + \frac{3}{2}\vartheta_X\psi_{XX} + \frac{3}{4}\vartheta_X\psi_X^2 + \frac{1}{4}\vartheta_X^3 + 3\alpha e^{-2\psi}\vartheta_X,\end{aligned}\quad (3.38)$$

which is related to the Hirota–Satsuma system (3.37) via the Miura map

$$\mathcal{U} = \frac{1}{2}\psi_{XX} - \frac{1}{4}\psi_X^2 - \alpha e^{-2\psi} + 2\sqrt{\alpha}e^{-\psi}\psi_X + \frac{1}{4}\vartheta_X^2, \quad \mathcal{V} = \frac{i}{2}(\vartheta_{XX} - \vartheta_X\psi_X + 2\sqrt{\alpha}e^{-\psi}\vartheta_X). \quad (3.39)$$

Remark 3.8. The system (3.38) admits another Miura map whose first component $\tilde{\mathcal{U}}$ looks very similar to $\mathcal{U}$, being given by

$$\tilde{\mathcal{U}} = -\frac{1}{2}\psi_{XX} - \frac{1}{4}\psi_X^2 - \alpha e^{-2\psi} - \frac{1}{4}\vartheta_X^2, \quad \tilde{\mathcal{V}} = \vartheta_X^2. \quad (3.40)$$

The quantity $\tilde{\mathcal{U}}$ is related to the density for the conservation law (3.15) by taking $\tilde{\mathcal{U}} = \hat{\mathcal{U}} - \frac{1}{2}\vartheta_X^2$ (where previously $\hat{\mathcal{U}}$ in (3.16) was written in the specific case $\alpha = \frac{1}{4}$). Upon applying the second Miura map (3.40), we find a third order flow for $\tilde{\mathcal{U}}$ and $\tilde{\mathcal{V}}$ that contains powers of $\tilde{\mathcal{V}}$ as denominators on the right-hand side, that is

$$\begin{aligned}\tilde{\mathcal{U}}_{\tau'} &= \frac{1}{2}\tilde{\mathcal{U}}_{XXX} + 3\tilde{\mathcal{U}}\tilde{\mathcal{U}}_X & + \frac{3}{2}\tilde{\mathcal{V}}\tilde{\mathcal{U}}_X - \frac{3}{8}\tilde{\mathcal{V}}^{-1}\tilde{\mathcal{V}}_X\tilde{\mathcal{V}}_{XX} + \frac{3}{16}\tilde{\mathcal{V}}^{-2}\tilde{\mathcal{V}}_X^3, \\ \tilde{\mathcal{V}}_{\tau'} &= -\tilde{\mathcal{V}}_{XXX} - \frac{3}{2}\tilde{\mathcal{V}}\tilde{\mathcal{V}}_X - 3\tilde{\mathcal{V}}_X\tilde{\mathcal{U}} & - 6\tilde{\mathcal{V}}\tilde{\mathcal{U}}_X + \frac{3}{2}\tilde{\mathcal{V}}^{-1}\tilde{\mathcal{V}}_X\tilde{\mathcal{V}}_{XX} - \frac{3}{4}\tilde{\mathcal{V}}^{-2}\tilde{\mathcal{V}}_X^3.\end{aligned}\quad (3.41)$$

4 Two-component deformations of the Camassa–Holm equation

In this section we classify integrable deformations of the Camassa–Holm equation of the form

$$\begin{aligned}(1 - D_x^2)u_t &= 3uu_x - 2u_xu_{xx} - uu_{xxx} + f(u, v, u_x, v, u_{xx}, v_{xx}, u_{xxx}, v_{xxx}) \\ v_{xt} &= g(u, v, u_x, v_x, u_{xx}, v_{xx}, u_{xxx}, v_{xxx}).\end{aligned}\quad (4.1)$$

In Theorem 4.1 below, we list all integrable systems (4.1) modulo rescaling transformations $u \rightarrow \alpha u$, $v \rightarrow \beta v$, $x \rightarrow \gamma x$, $t \rightarrow \delta t$ satisfying the following conditions:

- I. Functions f, g are homogeneous quadratic polynomials in u, v and their derivatives up to order 3;
- II. System possesses a reduction $v = 0$;
- III. System (4.1) possesses a local symmetry of order 5, which upon imposing the reduction $v = 0$ becomes of order 3;
- IV. System (4.1) is non-triangular;
- V. Upon shifting $u \rightarrow u + 1$ the system gains a diagonal non-degenerate linear term.

Condition II implies that upon reduction $v = 0$ the resulting system is the scalar Camassa–Holm equation, so the system (4.1) is indeed a deformation.

As for condition III, we assume here as the definition of integrability the existence of an infinite hierarchy of local higher symmetries. We seek local symmetries in the form

$$\begin{aligned}u_\tau &= F, \\ v_\tau &= G,\end{aligned}\quad (4.2)$$

where F and G are required to be local functions of u, v , i.e. functions of u, v and their derivatives up to a finite order. By definition system (4.2) is a generator of a symmetry of system (4.1) if

$$\begin{aligned}(1 - D_x^2)D_t(F) &= D_\tau(3uu_x - 2u_xu_{xx} - uu_{xxx} + f(u, v, u_x, v, u_{xx}, v_{xx}, u_{xxx}, v_{xxx})), \\ D_xD_t(G) &= D_\tau(g(u, v, u_x, v_x, u_{xx}, v_{xx}, u_{xxx}, v_{xxx})),\end{aligned}$$

where D_t, D_τ are evolutionary derivations defined by (4.1) and (4.2). The maximal order of derivatives of u or v on which functions F or G depend is called the order of a symmetry. The scalar Camassa–Holm equation possesses local higher symmetries of orders congruent to 1 mod 2. For the class of systems being

considered here, the existence of a symmetry of order 5 is necessary for the existence of infinitely many symmetries.

We shall call the system (4.1) triangular if

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial v_x} = \frac{\partial f}{\partial v_{xx}} = \frac{\partial f}{\partial v_{xxx}} = 0$$

or

$$\frac{\partial g}{\partial v} = \frac{\partial g}{\partial v_x} = \frac{\partial g}{\partial v_{xx}} = \frac{\partial g}{\partial v_{xxx}} = 0,$$

i.e. either of the equations depends only on its own variable. According to condition IV, we shall disregard triangular systems in the classification.

Condition V is most technical of the five requirements above. It means that, upon applying the shift $u \rightarrow u + 1$, the system (4.1) becomes

$$D_t \begin{pmatrix} (1 - D_x^2)u \\ v_x \end{pmatrix} = \begin{pmatrix} 3D_x - D_x^3 & 0 \\ 0 & \mathcal{L} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 3uu_x - 2u_x u_{xx} - uu_{xxx} + f \\ g \end{pmatrix},$$

where $\mathcal{L}$ is a linear differential operator with constant coefficients. This condition is required for the explicit derivation of integrability conditions in the framework of the symmetry approach, as described in [19], for instance. Here we omit all further technical details of deriving these integrability conditions, which require extensive use of computer algebra, and proceed to state the final result of the derivation.

The following classification theorem holds:

Theorem 4.1. *Every system satisfying conditions I-V is equivalent modulo $u \rightarrow \alpha_1 u$, $v \rightarrow \alpha_2 v$, $x \rightarrow \alpha_3 x$, $t \rightarrow \alpha_4 t$, $\alpha_1, \dots, \alpha_4 \in \mathbb{C}^*$ to one of the list:*

$$\begin{aligned} (1 - D_x^2)u_t &= 3uu_x - 2u_x u_{xx} - uu_{xxx} + \frac{1}{2}D_x(v^2), \\ v_{xt} &= -(1 - D_x^2)(uv), \end{aligned} \tag{4.3}$$

$$\begin{aligned} (1 - D_x^2)u_t &= 3uu_x - 2u_x u_{xx} - uu_{xxx} + \frac{1}{2}D_x((1 + D_x)v)^2, \\ v_{xt} &= -(1 - D_x)(u(1 + D_x)v), \end{aligned} \tag{4.4}$$

$$\begin{aligned} (1 - D_x^2)u_t &= 3uu_x - 2u_x u_{xx} - uu_{xxx} + \frac{1}{2}D_x((1 - D_x^2)v)^2, \\ v_{xt} &= -u(1 - D_x^2)v, \end{aligned} \tag{4.5}$$

$$\begin{aligned} (1 - D_x^2)u_t &= 3uu_x - 2u_x u_{xx} - uu_{xxx} + \frac{1}{2}D_x(2vv_{xx} + v_x^2 - v^2), \\ v_{xt} &= vu_{xx} + u_x v_x + uv_{xx} - uv + 2vv_{xx} + v_x^2, \end{aligned} \tag{4.6}$$

$$\begin{aligned} (1 - D_x^2)u_t &= 3uu_x - 2u_x u_{xx} - uu_{xxx} + \frac{1}{2}D_x(v^2), \\ v_{xt} &= (uv)_{xx}, \end{aligned} \tag{4.7}$$

$$\begin{aligned} (1 - D_x^2)u_t &= 3uu_x - 2u_x u_{xx} - uu_{xxx} + \frac{1}{2}D_x(v_x^2), \\ v_{xt} &= (uv_x)_x. \end{aligned} \tag{4.8}$$

Remark 4.2. It is worthwhile to comment in turn on each item in the above list.

- The first system, namely (4.3), is equivalent (up to rescaling) to the $d = 2$ case of the vector Camassa-Holm system (1.1).
- System (4.4) is related to (4.3) via the Miura transformation $v \rightarrow (1 + D_x)v$.
- System (4.5) is related to (4.3) via the Miura transformation $v \rightarrow (1 - D_x^2)v$.
- System (4.7) is the 2CH system studied in [8, 11, 23] (see also equation (43) in [24]).

- System (4.8) is the potential version of the latter one, being related to (4.7) via $v \rightarrow v_x$.
- The system (4.6) appears to be new.

Upon rescaling $x \rightarrow 2\alpha^{\frac{1}{2}}x$, $u \rightarrow 2\alpha^{\frac{1}{2}}u$ and introduction of the variables

$$m_1 = \frac{1}{2}u_{xx} - 2\alpha u, \quad m_2 = -\frac{1}{2}v_x$$

the systems (4.3)-(4.8) can be rewritten as

$$\begin{aligned} m_{1,t} &= 2u_x m_1 + u m_{1,x} + v m_2, \\ m_{2,t} &= 2u_x m_2 + u m_{2,x} - v m_1, \end{aligned} \tag{4.9}$$

$$\begin{aligned} m_{1,t} &= 2u_x m_1 + u m_{1,x} + \frac{1}{2\alpha}(2\sqrt{\alpha}m_2 + m_{2,x})(\sqrt{\alpha}v - m_2), \\ m_{2,t} &= (u m_2)_x + \sqrt{\alpha}v(2\sqrt{\alpha}u - u_x), \end{aligned} \tag{4.10}$$

$$\begin{aligned} m_{1,t} &= 2u_x m_1 + u m_{1,x} + \frac{1}{8\alpha^2}(4\alpha m_2 - m_{2,xx})(m_{2,x} + 2\alpha v), \\ m_{2,t} &= u(m_{2,x} + 2\alpha v), \end{aligned} \tag{4.11}$$

$$\begin{aligned} m_{1,t} &= 2u_x m_1 + u m_{1,x} - \alpha^{-1}m_2 m_{2,x} - \frac{1}{4\alpha}v(4\alpha m_2 - m_{2,xx}), \\ m_{2,t} &= (u m_2)_x - m_1 v + \alpha^{-\frac{1}{2}}(v m_{2,x} - m_2^2), \end{aligned} \tag{4.12}$$

$$\begin{aligned} m_{1,t} &= 2u_x m_1 + u m_{1,x} + v m_2, \\ m_{2,t} &= 2u_x m_2 + u m_{2,x} - v m_1 - 2\alpha u v, \end{aligned} \tag{4.13}$$

$$\begin{aligned} m_{1,t} &= 2m_1 u_x + u m_{1,x} - \frac{1}{2\alpha}m_2 m_{2,x}, \\ m_{2,t} &= (u m_2)_x, \end{aligned} \tag{4.14}$$

With the variables m_1, m_2 as above, (4.9) is just the $d = 2$ case of the vector Camassa–Holm system (1.1). The systems (4.10) and (4.11) can also be extended to vector analogues, related to (1.1) via Miura transformations.

Example 4.3. In the limit $\alpha = 0$, the system (4.9) becomes

$$\begin{aligned} u_{xxt} &= (u u_{xx})_x + \frac{1}{2}(u_x^2)_x - \frac{1}{2}(v^2)_x, \\ v_{xt} &= (u v)_{xx}, \end{aligned} \tag{4.15}$$

which we refer to as the Hunter–Saxton limit with two components, so this is Hunter–Saxton-2, or HS2 for short. Upon integrating both equations with vanishing boundary conditions at infinity, this gives

$$\begin{aligned} u_{xt} &= u u_{xx} + \frac{1}{2}u_x^2 - \frac{1}{2}v^2, \\ v_t &= (u v)_x. \end{aligned} \tag{4.16}$$

In the remainder of the paper, we study the weak solutions for the multi-component systems of Hunter–Saxton type.

Remark 4.4. We want to comment on the similarities and differences between (4.9) and (4.13). Our first observation is that (4.9) and (4.13) share the same $\alpha = 0$ limit. In other words, the Hunter–Saxton limits for these two equations coincide. Yet the natural boundary conditions are very different in the two cases. We will elaborate on this in the next section. However, it is perhaps even more pressing to ask whether these two equations for $\alpha \neq 0$ are substantially different. We will address this question for the class of weak solutions, focusing on solutions m_1, m_2 with singular support only (peakons), consisting of sums of Dirac deltas supported on moving points $\{x_j(t)\}$. In this case, v is a primitive of m_2 and must be piecewise constant, while u is a continuous, piecewise smooth function involving weighted sums of

$e^{-2|x-x_j|}$. It is then easy to check that the left-hand sides involve a distribution of order 1 with singular support only (thus containing Dirac deltas and distributional derivatives of the Dirac deltas), so the right-hand sides in both equations must be of the same type. However, the last term in (4.13), the one involving α , is a piecewise smooth function. Thus, there are no non-trivial solutions of this type for (4.13), but, by contrast, these solutions are supported by (4.9). In [10], the system (4.16) was considered as the Hunter-Saxton limit of (4.13), with vanishing boundary conditions.

There is another important difference between (4.9) and (4.13), which is in the distinct nature of their symmetry hierarchies. All the systems in the list have a trivial symmetry of order 1, corresponding to translation in x . The lowest non-trivial symmetry of the system (4.9) is of order 3 in derivatives of m_1 and m_2 , whereas (4.13) (being of nonlinear Schrödinger type) has a symmetry of order 2.

5 The Neumann beam problem and Hunter-Saxton limit

In this section we construct a specific class of weak solutions of the generalized beam problem. For (1.1), we consider the Clifford-valued Hunter-Saxton limit ($\alpha = 0$) in Euclidean signature. Let $(\cdot, \cdot)$ be a nondegenerate symmetric bilinear form on W with Euclidean signature, and decompose

$$W = V \oplus V^\perp, \quad V = \langle e_\mu \rangle_{2 \leq \mu \leq d}, \quad V^\perp = \langle e_1 \rangle.$$

Denote by P_V the orthogonal projection onto V .

If we rewrite (1.1) in the form (2.5) as in section 2, with $b_0 = u$ and $b_{-1} = e_1$, then the measure M evolves according to

$$M_t = \mathcal{L}_M u + (M, \mathbf{v})e_1 - (M, e_1)\mathbf{v}, \quad \mathbf{v} \in V, \quad (5.1)$$

with the operator

$$\mathcal{L}_M = MD_x + D_x M$$

(which acts the same as $\frac{1}{2}(A_M D_x + D_x A_M)$ on the scalar field $u \in \mathbb{F}$), subject to

$$u_{xx} = 2(M, e_1), \quad \mathbf{v}_x = -2P_V M. \quad (5.2)$$

The right-hand side of (5.1) splits into the component parallel to M and a perpendicular rotation term.

Eliminating M yields the closed system

$$u_{xxt} = (uu_{xx})_x + \frac{1}{2}(u_x^2)_x - \frac{1}{2}(\mathbf{v}, \mathbf{v})_x, \quad (5.3)$$

$$\mathbf{v}_{xt} = (u\mathbf{v})_{xx}, \quad (5.4)$$

which corresponds to (4.15) when $d = 2$, or, after one integration of both equations, we have

$$u_{xt} = uu_{xx} + \frac{1}{2}u_x^2 - \frac{1}{2}(\mathbf{v}, \mathbf{v}), \quad (5.5)$$

$$\mathbf{v}_t = (u\mathbf{v})_x, \quad (5.6)$$

which corresponds to (4.16) when $d = 2$. The reduction $\mathbf{v} = 0$ recovers the scalar Hunter-Saxton equation [20, 21]. In the rest of the paper, we work with the unintegrated system, given by (5.3) and (5.4). The system is $O(V)$ -invariant: u is fixed and $\mathbf{v} \mapsto g\mathbf{v}$ for $g \in O(V)$.

We represent the fields in terms of M by

$$u(x) = \int_{\mathbb{R}} |x - y| (M(y), e_1) dy, \quad (5.7)$$

$$\mathbf{v}(x) = -P_V \int_{\mathbb{R}} \text{sgn}(x - y) M(y) dy, \quad (5.8)$$

which is natural for compactly supported measures. (For convenience, the implicit dependence on time t is omitted here.)

Lemma 5.1 (Asymptotics and mass conservation). *Let $\mathcal{M} = \int M(y, t) dy$ and $\mathcal{M}_1 = \int y M(y, t) dy$. Then outside the support, the fields behave as*

$$u_+(x) = (\mathcal{M}, e_1)x - (\mathcal{M}_1, e_1), \quad u_- = -u_+, \quad (5.9)$$

$$\mathbf{v}_\pm = \pm P_V \mathcal{M}, \quad (5.10)$$

where $\pm$ refers to the right (+), left (−), asymptotic regions, respectively. Moreover, we have conservation of mass, in the sense that

$$\frac{d}{dt} \mathcal{M} = 0.$$

Proof. Integrating (5.3) and (5.4) over $\mathbb{R}$ and using the asymptotics shows that all boundary contributions vanish. $\square$

5.1 Time evolution and the Weyl function

For (2.1) with $\alpha = 0$, we consider the Neumann string problem in $\mathcal{C}\ell(W)$:

Definition 5.2. For compactly supported M , the Neumann problem is

$$\Psi_{xx} = \lambda M \Psi, \quad (5.11)$$

with Ψ_x non-invertible at both spatial infinities.

We specialize to a discrete measure

$$M = \sum_{j=1}^N M_j \delta_{x_j}, \quad x_1 < \cdots < x_N, \quad (5.12)$$

with invertible $M_j \in \mathcal{C}\ell_1$. Then (5.1) reduces to

$$\dot{x}_j = -u(x_j), \quad (5.13)$$

$$\dot{M}_j = \langle u_x \rangle(x_j) M_j + \frac{1}{2} [e_1 \langle \mathbf{v} \rangle(x_j), M_j]. \quad (5.14)$$

Invertibility of each M_j for all t , provided that $M_j(0)$ is invertible, and conservation of $\mathcal{M} = \sum_j M_j$ follow directly from the above ODE system.

To incorporate time deformation into the spectral problem, we use (2.2) in the form

$$\Psi_t = \frac{1}{2} (e_1 \mathbf{v} - u_x) \Psi + \left(u + \frac{e_1}{\lambda} \right) \Psi_x. \quad (5.15)$$

Let Φ be the solution normalized by $\Phi = 1$ for $x \ll 0$. A time-dependent gauge $\Omega(t, \lambda)$ is needed to ensure compatibility. Thus, setting

$$\Psi = \Phi \Omega,$$

and analyzing the compatibility condition in the left asymptotic region, one obtains

$$\Omega_t \Omega^{-1} = \frac{1}{2} (e_1 P_V \mathcal{M} + (\mathcal{M}, e_1)), \quad (5.16)$$

which is constant in t . In the right asymptotic region, writing

$$\Phi e_1 = A(x - x_N) + B, \quad (5.17)$$

the coefficients evolve as follows.

Proposition 5.3. Let $F = \frac{1}{2} e_1 P_V \mathcal{M}$. Then A, B in (5.17) satisfy

$$A_t + [F, A] = 0, \quad (5.18)$$

$$B_t + (\mathcal{M}, e_1) B + [F, B] = \frac{e_1}{\lambda} A. \quad (5.19)$$

Proof. Using (5.16) and Lemma 5.1, we obtain

$$\dot{\Phi} + \Phi \omega + \omega \Phi = \overbrace{((\mathcal{M}, e_1)x - (\mathcal{M}_1, e_1))}^{u_+} + \frac{e_1}{\lambda} \Phi_x \quad (5.20)$$

where $\omega = \frac{e_1 P_V \mathcal{M} + (\mathcal{M}, e_1)}{2}$. The term $\Phi \omega + \omega \Phi$ can be easily simplified to

$$\Phi \omega + \omega \Phi = (\mathcal{M}, e_1) \Phi + \Phi \frac{e_1 P_V \mathcal{M}}{2} + \frac{e_1 P_V \mathcal{M}}{2} \Phi.$$

We now observe that for any $\mathbf{v} \in V$, $e_1 \mathbf{v} + \mathbf{v} e_1 = 0$, which results in the formula

$$\Phi \omega e_1 + \omega \Phi e_1 = (\mathcal{M}, e_1) \Phi e_1 + \frac{1}{2} [e_1 P_V \mathcal{M}, \Phi e_1].$$

Finally, multiply (5.20) by e_1 on the right and write $\Phi e_1 = A(x - x_N) + B$, recalling that $\dot{x}_N = -u_+(x_N)$, to get (5.18) and (5.19). $\square$

Corollary 5.4. *The time dependence of A and B is*

$$A(t) = e^{-Ft}A(0)e^{Ft}, \quad (5.21)$$

$$B(t) = e^{-(\mathcal{M}, e_1)t}e^{-Ft}B(0)e^{Ft} + \frac{e_1}{\lambda(\mathcal{M}, e_1)}(1 - e^{-(\mathcal{M}, e_1)t})A(t), \quad (5.22)$$

from which it follows that the Weyl function

$$W(\lambda) = BA^{-1} \quad (5.23)$$

has time evolution given explicitly by

$$W(\lambda, t) = e^{(\mathcal{M}, e_1)t}e^{-Ft}W(\lambda, 0)e^{Ft} + \frac{e_1}{\lambda(\mathcal{M}, e_1)}(1 - e^{-(\mathcal{M}, e_1)t}). \quad (5.24)$$

Observe that in the asymptotic region $x \gg 0$, $A(t) = \Phi_x e_1$. [Corollary 5.4](#) provides a plethora of conserved, non-local quantities in M .

Corollary 5.5. *Let x be in the right asymptotic region. Then*

$$e^{Ft}\Phi_x e_1 e^{-Ft} \quad (5.25)$$

is conserved.

5.2 Low-order λ expansion

The wave function Φ is, in general, an entire function of λ . In this paper, we will only use the first few terms in its expansion about $\lambda = 0$. We now examine the initial terms of this expansion. We construct Φ , subject to the Neumann condition for $x \ll 0$, by solving iteratively. To this end, we write the expansion around $\lambda = 0$ as

$$\Phi = 1 + \lambda\Phi^{(1)} + \lambda^2\Phi^{(2)} + O(\lambda^3), \quad (5.26)$$

where the iterates satisfy

$$\Phi_{xx}^{(n)} = M\Phi^{(n-1)}, \quad \text{with} \quad \Phi^{(n)}(x) = 0, \quad \Phi_x^{(n)}(x) = 0, \quad \text{for } x \ll 0, \text{ and all } n \geq 1. \quad (5.27)$$

In a routine way, one gets the recurrence

$$\Phi^{(n)}(x) = \int_{-\infty < z < x} (x - z)M(z)\Phi^{(n-1)}(z) dz, \quad (5.28)$$

whose solution reads

$$\Phi^{(n)}(x) = \int_{-\infty < z_1 < z_2 < \dots < z_n < x} M(z_n)M(z_{n-1}) \cdots M(z_1) dz_1 dz_2 \cdots dz_n.$$

We will only need the derivatives of $\Phi^{(1)}$ and $\Phi^{(2)}$ in the asymptotic region $x \gg 0$. The following integral formulas provide them as

$$\Phi_x^{(n)} = \int_{-\infty}^{\infty} M(z_1) dz_1 = \mathcal{M}, \quad (5.29)$$

$$\Phi_x^{(2)} = \int_{-\infty}^{\infty} M(z_2) \int_{-\infty}^{z_2} (z_2 - z_1)M(z_1) dz_1 dz_2. \quad (5.30)$$

We can now infer the time evolution of (5.30) in the asymptotic region $x \gg 0$. Indeed, by [Corollary 5.5](#),

$$\Phi_x^{(2)} = e^{-Ft}(c_1 + c_3)e^{Ft}e_1^{-1}, \quad (5.31)$$

where c_1, c_3 are constant elements of $\mathcal{C}\ell_1, \mathcal{C}\ell_3$, respectively. Of particular interest to us is, in the region $x \gg 0$, the trace of $\Phi_x^{(2)}$ which is given by

$$\text{tr}(\Phi_x^{(2)}) = \text{tr}((c_1 + c_3)e^{Ft}e_1^{-1}e^{-Ft}). \quad (5.32)$$

Proposition 5.6. When the bilinear form $(\cdot, \cdot)$ has Euclidean signature

$$e^{Ft}e_1^{-1}e^{-Ft} = \cos(\theta t)e_1^{-1} - \frac{\sin(\theta t)}{\theta}P_V\mathcal{M}, \quad (5.33)$$

where $\theta = \|(P_V\mathcal{M})\|$.

Proof. Since $e_1^{-1} = e_1$, we might as well work with $Z(t) = e^{Ft}e_1e^{-Ft}$. Recall that $F = \frac{e_1P_V\mathcal{M}}{2}$. Hence, $e^{Ft}e_1e^{-Ft}$ represents a rotation in the $(e_1, P_V\mathcal{M})$ plane. So $Z(t) = a(t)e_1 + b(t)P_V\mathcal{M}$, and

$$\dot{Z} = \dot{a}e_1 + \dot{b}P_V\mathcal{M}.$$

On the other hand

$$\dot{Z} = \left[\frac{e_1P_V\mathcal{M}}{2}, ae_1 + bP_V\mathcal{M}\right] = -a(e_1, e_1)P_V\mathcal{M} + b(P_V\mathcal{M}, P_V\mathcal{M})e_1.$$

Comparing the coefficients of $e_1, P_V\mathcal{M}$, we get a system of ODEs

$$\dot{a} = \|(P_V\mathcal{M})\|^2 b, \quad \dot{b} = -a,$$

whose solution is

$$a(t) = \cos(\|(P_V\mathcal{M})\| t), \quad b(t) = -\frac{\sin(\|(P_V\mathcal{M})\| t)}{\|(P_V\mathcal{M})\|}.$$

□

Corollary 5.7. In the region $x \gg 0$, the trace of $\Phi_x^{(2)}$ takes the form

$$\text{tr}(\Phi_x^{(2)}) = C_1 \cos(\theta t) + C_2 \sin(\theta t), \quad (5.34)$$

for some constants C_1, C_2 , and θ defined in [Proposition 5.6](#).

5.3 The forward problem

We start by solving the initial value problem

$$\Phi_{xx} = \lambda M\Phi, \quad \text{with} \quad \Phi(x) = 1, \quad \Phi_x(x) = 0 \quad \text{for} \quad x < x_1, \quad (5.35)$$

for the Clifford value measure specified in [\(5.12\)](#). The solution Φ is piecewise linear in x , while Φ_x is piecewise constant. Thus, for any λ , the solution is uniquely characterized by $\Phi_j = \Phi(x_j)$, the left derivatives $\Phi'_j = \Phi_x(x_j-)$, the jump condition

$$\Phi'_{j+1} - \Phi_j = \lambda M_j \Phi_j, \quad 1 \leq j \leq N, \quad (5.36)$$

and the relation

$$\Phi_{j+1} = \Phi_j + l_j \Phi'_{j+1}, \quad 1 \leq j \leq N-1. \quad (5.37)$$

We note that Φ'_{N+1} is just the right derivative $\Phi_x(x_{N+})$. Both recurrence relations can be written in matrix form, as follows:

$$\begin{bmatrix} \Phi_{j+1} \\ \Phi'_{j+1} \end{bmatrix} = \begin{bmatrix} 1 & l_j \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \Phi_j \\ \Phi'_{j+1} \end{bmatrix}, \quad 1 \leq j \leq N-1, \quad (5.38a)$$

$$\begin{bmatrix} \Phi_j \\ \Phi'_{j+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \lambda M_j & 1 \end{bmatrix} \begin{bmatrix} \Phi_j \\ \Phi'_j \end{bmatrix}, \quad 1 \leq j \leq N. \quad (5.38b)$$

Proposition 5.8. The quantities Φ_j and Φ'_j are polynomials in λ with degrees $\deg(\Phi_j) = \deg(\Phi'_j) = j-1$. For all $j \geq 2$, Φ_j and Φ'_j are invertible for sufficiently large λ .

Proof. The proof proceeds by induction on j . The fact that they are polynomials of the given degrees follows directly from the recurrence relations. The base case corresponds to $j = 2$. We have

$$\begin{bmatrix} \Phi_2 \\ \Phi'_2 \end{bmatrix} = \begin{bmatrix} 1 + \lambda l_1 M_1 & l_1 \\ \lambda M_1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 + \lambda l_1 M_1 \\ \lambda M_1 \end{bmatrix}, \quad (5.39)$$

where we used $\Phi_1 = 1$ and $\Phi'_1 = 0$ from the initial value problem (5.35). Both entries are invertible for large λ because M_1 is invertible. The recurrence relations give the following:

$$\begin{bmatrix} \Phi_{j+1} \\ \Phi'_{j+1} \end{bmatrix} = \begin{bmatrix} 1 + \lambda l_j M_j & l_j \\ \lambda M_j & 1 \end{bmatrix} \begin{bmatrix} \Phi_j \\ \Phi'_j \end{bmatrix} = \begin{bmatrix} \Phi_j + \lambda l_j M_j \Phi'_j \\ \Phi'_j + \lambda M_j \Phi_j \end{bmatrix}.$$

The dominant terms in λ are $\lambda l_j M_j \Phi'_j$ for the first entry, $\lambda M_j \Phi_j$ for the second entry. Both are invertible by the inductive hypothesis, so Φ_{j+1} and Φ'_{j+1} are invertible. $\square$

We can now summarize the recursive way of solving the initial value problem.

Proposition 5.9. The solution of the initial value problem (5.35) can be expressed by the recursive formula

$$\begin{bmatrix} \Phi_N \\ \Phi'_{N+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \lambda M_N & 1 \end{bmatrix} \left(\prod_{j=1}^{N-1} \begin{bmatrix} 1 + \lambda l_j M_j & l_j \\ \lambda M_j & 1 \end{bmatrix} \right) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (5.40)$$

From the definition of the Weyl function given by (5.23), we get a formula $W(\lambda, t) = B(\lambda, t)A^{-1}(\lambda, t) = \Phi_N e_1 (\Phi'_{N+1} e_1)^{-1} = \Phi_N (\Phi'_{N+1})^{-1}$.

Theorem 5.10 (Stieltjes-Clifford fraction). *For large λ , the Weyl function is given by a finite continued fraction of Stieltjes type, with Clifford algebra-valued partial quotients, that is*

$$W(\lambda) = \frac{1}{\lambda M_N + \frac{1}{l_{N-1} + \frac{1}{\lambda M_{N-1} + \frac{1}{\ddots + \frac{1}{\lambda M_1}}}}} \quad (5.41)$$

Proof. We note that by Proposition 5.8 all Φ_j and Φ'_j are invertible for sufficiently large λ . By the last iteration of each recurrence relation ($j = N$ in (5.36) and (5.37)), we have

$$\begin{aligned} \Phi'_{N+1} &= \Phi'_N + \lambda M_N \Phi_N, \\ \Phi_N &= \Phi_{N-1} + l_{N-1} \Phi'_N, \end{aligned}$$

which gives

$$\begin{aligned} \Phi_N (\Phi'_{N+1})^{-1} &= (\lambda M_N + \Phi'_N \Phi_N^{-1})^{-1} = (\lambda M_N + (\Phi_N (\Phi'_N)^{-1})^{-1})^{-1} \\ &= (\lambda M_N + (l_{N-1} + \Phi_{N-1} (\Phi'_N)^{-1})^{-1})^{-1}. \end{aligned}$$

This is the first step in the iteration; we moved from $\Phi_N (\Phi'_{N+1})^{-1}$ to $\Phi_{N-1} (\Phi'_N)^{-1}$. We proceed by iterating down until we reach $\Phi_1 (\Phi'_2)^{-1} = \lambda M_1$, since $\Phi_1 = 1$ and $\Phi'_1 = 0$ for the Neumann problem. This concludes the proof. $\square$

Finally, the time evolution of W is given by (5.24), while the continued fraction expansion of W determines the masses M_j and the spacings $l_{N-1}, \dots, l_1$. We will provide a detailed analysis of the inverse problem elsewhere, with particular emphasis on the explicit representations of the quantities M_j and l_j in terms of the Weyl matrix function W . To complete the formal solution of the inverse problem, we still need to determine the position of the last particle, x_N .

5.4 Determining the position x_N

We cannot determine the position x_N from the Weyl function W . This is similar to the scalar Hunter-Saxton equation, for which the same question was addressed in [3]. x_N satisfies (see (5.13))

$$\dot{x}_N = -u(x_N) = -u_+(x_N) = -((\mathcal{M}, e_1)x_N - (\mathcal{M}_1, e_1)),$$

or,

$$\dot{x}_N + (\mathcal{M}, e_1)x_N = (\mathcal{M}_1, e_1). \quad (5.42)$$

The latter equation can be easily integrated, provided we know the time dependence of $(\mathcal{M}_1, e_1)$.

Proposition 5.11. The inner product of the first moment $\mathcal{M}_1$ with e_1 satisfies

$$\frac{d}{dt}(\mathcal{M}_1, e_1) = -\text{tr}(\Phi_x^{(2)}). \quad (5.43)$$

Proof. We want to compute $\frac{d}{dt} \sum_j x_j (M_j, e_1)$. Since $\dot{x}_j = -u(x_j)$ and $\dot{M}_j = \langle u_x \rangle(x_j) M_j + \frac{1}{2} [e_1 \langle v \rangle(x_j), M_j]$ we obtain

$$\frac{d}{dt} \sum_j x_j (M_j, e_1) = \overbrace{\sum_j (-u(x_j) + x_j \langle u_x \rangle(x_j)) (M_j, e_1)}^{=I_1} + \overbrace{\sum_j x_j (\langle v \rangle(x_j), M_j) (e_1, e_1)}^{=I_2} \quad (5.44)$$

We recall that $u(x) = \int_{\mathbb{R}} |x - y| (M(y), e_1) dy$ which in the discrete case becomes

$$u(x_j) = \sum_k |x_j - x_k| (M_k, e_1).$$

Likewise, $\langle u_x \rangle = \int_{\mathbb{R}} \text{sgn}(x - y) (M(y), e_1) dy$. Hence,

$$\langle u_x \rangle(x_j) = \sum_k \text{sgn}(j - k) (M_k, e_1).$$

After symmetrizing, this gives

$$\sum_j x_j \langle u_x \rangle(x_j) = \frac{1}{2} \sum_{j,k} |x_j - x_k| (M_k, e_1) (M_j, e_1).$$

Thus we have

$$I_1 = -\frac{1}{2} \sum_{j,k} |x_j - x_k| (M_k, e_1) (M_j, e_1). \quad (5.45)$$

To compute I_2 we need to evaluate $(\langle v \rangle(x_j), M_j)$. Recall that

$$v(x) = -P_V \int_{\mathbb{R}} \text{sgn}(x - y) M(y) dy.$$

Since $P_V M = M - P_{V^\perp} M = M - (M, e_1) e_1$ we get

$$\langle v \rangle(x_j) = -\sum_k \text{sgn}(j - k) M_k + \sum_k \text{sgn}(j - k) (M, e_1) e_1,$$

and,

$$I_2 = -\sum_{j,k} x_j \text{sgn}(j - k) (M_k, M_j) (e_1, e_1) + \sum_{j,k} x_j \text{sgn}(j - k) (M_k, e_1) (M_j, e_1)$$

After symmetrizing, we get

$$I_2 = -\frac{1}{2} \sum_{j,k} |x_j - x_k| (M_k, M_j) (e_1, e_1) + \frac{1}{2} \sum_{j,k} |x_j - x_k| (M_k, e_1) (M_j, e_1) \quad (5.46)$$

Combining (5.45) and (5.46) we get

$$I_1 + I_2 = -\frac{1}{2} \sum_{j,k} |x_j - x_k| (M_k, M_j). \quad (5.47)$$

For comparison, let us compute $\text{tr} \Phi_x^{(2)}$ in the right asymptotic region assuming the mass is discrete. From (5.30) we get

$$\Phi_x^{(2)} = \sum_{j < k} (x_k - x_j) M_k M_j,$$

and

$$\begin{aligned} \text{tr} \Phi_x^{(2)} &= \sum_{j < k} (x_k - x_j) \text{tr} (M_k M_j) = \sum_{j < k} (x_k - x_j) (M_k, M_j) \\ &= \frac{1}{2} \left(\sum_{j < k} (x_k - x_j) (M_k, M_j) + \sum_{j > k} (x_j - x_k) (M_j, M_k) \right). \end{aligned}$$

Hence,

$$\text{tr } \Phi_x^{(2)} = \frac{1}{2} \sum_{j,k} |x_k - x_j| (M_k, M_j) \quad (5.48)$$

which implies the final claim. $\square$

This result, together with [Corollary 5.7](#), allows one to integrate (5.42). Let us set $(\mathcal{M}_1, e_1) = c_1 \cos(\theta t) + c_2 \sin(\theta t)$. Then the solution x_N reads as follows:

$$x_N(t) = c_3 e^{-(\mathcal{M}, e_1)t} + \frac{(c_1 (\mathcal{M}, e_1) - c_2 \theta) \cos(\theta t) + (c_1 \theta + c_2 (\mathcal{M}, e_1)) \sin(\theta t)}{(\mathcal{M}, e_1)^2 + \theta^2}. \quad (5.49)$$

This completes the solution of the forward and inverse problems for the generalized Neumann beam in the Hunter–Saxton regime.

5.5 The case of two peakons with two components

Here we can specialize the results above to the case $N = 2$, to understand the interaction of two peakons, and further restrict to the two-component Hunter-Saxton case $d = 2$, which we refer to as HS2, given by (4.15). For any d , in the special case of $N = 2$, the wave $u(x, t)$ defined by (5.7) takes the form

$$u(x, t) = m_{11}(t) |x - x_1(t)| + m_{21}(t) |x - x_2(t)|, \quad (5.50)$$

where the coefficient functions are defined by $m_{j\mu}(t) = (M_j(t), e_\mu)$ for $j = 1, \dots, N$, $\mu = 1, \dots, d$.

In the case of HS2, with two components ($d = 2$), the interaction of two peakons is illustrated in Fig.4 below.

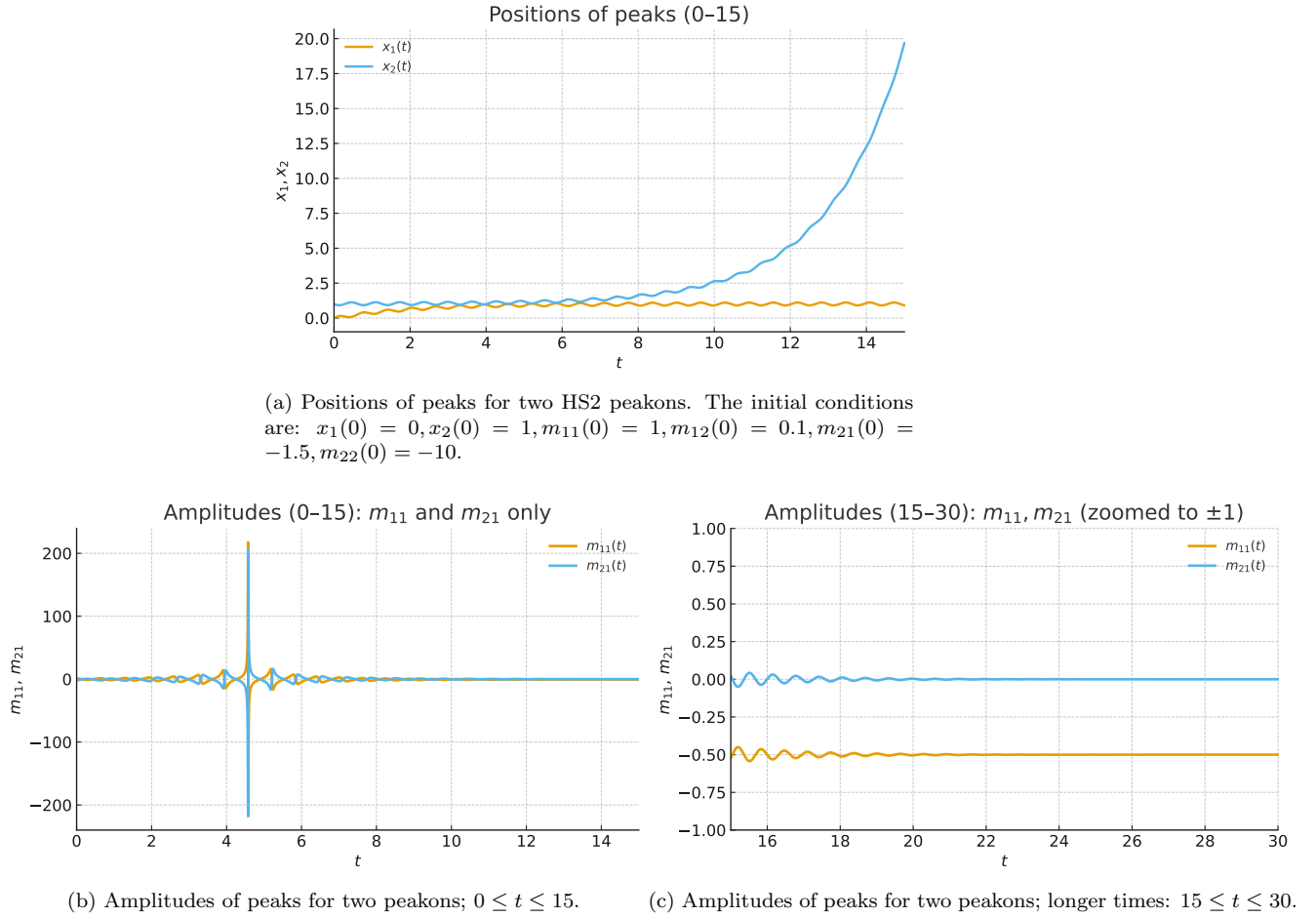


Figure 4: HS2 peakons satisfying (4.16).

Example 5.12. Let us set $(M_j, e_\mu) = m_{j\mu}$ for $j = 1, \dots, N$, $\mu = 1, \dots, d$. When $N = 2, d = 2$, the equations of motion (5.13) and (5.14) read:

$$\dot{x}_1 = -m_{21}|x_1 - x_2|, \quad (5.51)$$

$$\dot{x}_2 = -m_{11}|x_1 - x_2|, \quad (5.52)$$

$$\dot{m}_{11} = m_{11}^2 - m_{12}^2 - C_1 m_{11} + C_2 m_{12}, \quad (5.53)$$

$$\dot{m}_{12} = 2m_{11}m_{12} - C_1 m_{12} - C_2 m_{11}, \quad (5.54)$$

where $m_{11} + m_{21} = C_1$ and $m_{12} + m_{22} = C_2$ are constants of motion. The corresponding field $u(x, t)$, satisfying one half of the system (4.16), is given by the formula (5.50), where the amplitudes m_{11} and $m_{21} = C_1 - m_{11}$ and the peak positions x_1, x_2 are obtained from the solution of this system of ODEs. The plots in Fig.4 are produced from a particular solution of the latter system, with $(\mathcal{M}, e_1) = m_{11} + m_{21} = C_1 = -0.5$. We can make the following observations:

- **Wave centered at $x_1(t)$:**
 - The center of the wave $x_1(t)$ becomes *asymptotically periodic* as $t \rightarrow \infty$.
 - A longer time simulation shows that amplitude m_{11} stabilizes around -0.5 .
- **Wave centered at $x_2(t)$:**
 - The center of the wave $x_2(t)$ increases rapidly while oscillating and tends to $+\infty$ as $t \rightarrow \infty$. Since $(\mathcal{M}, e_1) < 0$, this behaviour is consistent with (5.49).
 - A longer time simulation shows that m_{21} stabilizes around 0; the wave component $m_{21}(t)|x - x_2(t)|$ becomes increasingly flat and moves away from the region of interest.
 - As a result, this component becomes *asymptotically negligible* in the spatial region near $x_1(t)$.

Remark 5.13. Constantin and Ivanov studied the short-pulse limit of the 2CH system in reference [10], noting that although 2CH has no peakon solutions, its short-pulse limit (4.16), a two-component generalization of the Hunter-Saxton equation, does. Our analysis differs in two crucial ways. We present a general approach to the d -component generalized Hunter-Saxton equation, rather than the special case $d = 2$. Even for $d = 2$, our analysis considers different boundary conditions: the coupling field v does not vanish at $\pm\infty$, and neither does u . This is consistent with our interpretation of v as an internal degree of freedom, akin to a spin variable.

6 Conclusions

We have studied a Clifford-algebraic generalization of the Camassa–Holm (CH) hierarchy in Euclidean signature, in which the scalar CH variable is coupled to internal degrees of freedom taking values in a Clifford algebra with d generators. This construction breaks the full $O(d)$ symmetry to the residual $O(d-1)$ invariance associated with rotations in the subspace orthogonal to the distinguished direction e_1 , and can be interpreted as endowing the CH dynamics with “spin”-type internal modes. For $d = 2$, we used symmetry methods to classify the resulting perturbations of CH within this framework, obtaining the system analyzed here as well as an additional integrable candidate.

A key point of the paper is that the integrable structure persists in several complementary guises. On the one hand, the Hunter–Saxton (short-pulse) reduction admits a measure-valued formulation and a Neumann-type spectral problem whose Weyl function evolves by an explicit isospectral flow. This leads to a concrete forward/inverse scheme for discrete measures and provides a Stieltjes-type continued fraction expansion for the Weyl function in the Clifford setting. On the other hand, via a reciprocal transformation, we relate our model to the Hirota–Satsuma system, and we include a travelling-wave analysis that clarifies how the internal variables modify the classical CH wave families.

To illustrate the dynamical impact of the internal degrees of freedom, we carried out computations for N -peakon solutions in the Hunter–Saxton regime, with particular emphasis on the two-atom case. In the example presented, the solution exhibits a striking asymptotic decomposition: as $t \rightarrow \infty$ the field $u(x, t)$ becomes effectively dominated, on any fixed observable spatial window, by a single spatially localized oscillatory structure centered near $x_1(t)$, while the second component is transported to $+\infty$ and simultaneously flattens so that its contribution becomes negligible near $x_1(t)$. The presence of persistent harmonic modes in the amplitudes is a genuinely new feature compared to the scalar Hunter–Saxton dynamics and appears to be a robust signature of the Clifford coupling.

Several directions emerge naturally from this work:

1. **Physical interpretation.** Identify a compelling physical setting—for instance, in fluid models with internal/charged degrees of freedom—in which the Clifford-valued generalization and the residual $O(d-1)$ symmetry arise from first principles.
2. **Signature dependence.** Determine how changing the signature of the underlying bilinear form affects qualitative behaviour (e.g., boundedness versus growth, oscillatory versus hyperbolic modes), and how this is reflected in the spectral data and time evolution.
3. **Higher-rank classification.** Extend the $d = 2$ symmetry classification to general d and clarify which members of the resulting families are integrable, including the role of reciprocal transformations and Hamiltonian structures.
4. **Peakon and spectral asymptotics.** Develop a systematic description of long-time behaviour for multi-peakon solutions (scattering, clustering, and synchronization mechanisms), and relate these to the invariants encoded by the Clifford Weyl function and its continued fraction data.

We expect that progress on these problems will sharpen the connection between the spectral theory of generalized beam/string problems and the nonlinear dynamics of CH-type equations with internal degrees of freedom. While we have only considered the case of Hunter-Saxton peakons here with $\alpha = 0$, the spectral analysis of peakons in the full vector system with $\alpha \neq 0$ is already underway. There is a wealth of other coupled CH-type models whose solutions exhibit interesting behaviours, such as “waltzing” peakons [9, 32], and we expect that pursuing an approach via spectral theory will shed further light on these as well.

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