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A robust approach to the capacitated hub location problem: a budget uncertainty perspective

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ABSTRACT

Formulations for the robust capacitated single and multiple allocation hub location problems under polyhedral demand uncertainty are proposed and corresponding solutions with budget uncertainty are discussed. We first develop a generic framework and show that a problem with data variability only in the objective a) can be modelled directly as a minimisation problem and b) has an equivalent nonlinear MIP formulation if deterministic and robust optimal solutions happen to be frequently the same. The proposed framework is then extended to formulate the robust counterparts of the two capacitated hub location problems where data uncertainty occurs in both the objective and constraints. Based on an extensive empirical experiment, we show that the robust formulations can be efficiently linearised and solved by a commercial optimiser like CPLEX for instance. To validate the proposed modelling approach and to demonstrate its wider applications, we have solved 256 instances of the robust uncapacitated variations of the two-hub location and the robust sorting problems and favourably compared our results with those obtained by the formulations based on the Bertsimas and Sim approach.

PRACTITIONER SUMMARY

Research on robust Hub Location Problem (HLP) has largely focussed on uncapacitated variations of the problem as existing robust optimisation (RO) approaches often struggle to deal with cases where the same type of uncertain data appears both in the objective and constraints. Assuming unlimited capacity however, is often unrealistic in practical settings like freight logistics and supply chains where congestion in hub facilities strictly govern operations. To address this challenge, in this research first we propose and employ a generic framework based on the well-known budget uncertainty approach to formulate robust counterparts of the capacitated HLPs with polyhedral demand uncertainty. Then, we solve the resulting capacitated formulations for a large number of benchmark problems, analyse the results, and discuss interesting practical and managerial insights. Among others we show, large improvement in robustness of a solution with loose capacity can be achieved with minor adjustment to the (deterministic) solution's configuration. From practical point of view this means budget uncertainty in these problems can be selected more confidently as it has limited impact on nominal cost. In problems with tight capacity however, our results show it must be selected cautiously. We also show that solutions to deterministic capacitated problems perform quite poorly under demand uncertainty as they quickly turn into infeasible solutions if demand is subject to even minor variation.

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1. Introduction

Considering parameter uncertainty in developing mathematical programming formulations also known as optimisation with/under uncertain data has been the subject of many studies in the past few decades. In particular, in recent years and perhaps due to its importance in practical settings there has been a growing interest in this subject area and its related fields. In the literature, the two main streams of research that attempt to address data uncertainty in discrete optimisation problems are (*i*) stochastic programming and (*ii*) robust optimisation (RO).

To introduce stochastic programming, Dantzig (1955) modelled data uncertainty by considering scenarios that occur with certain and different probabilities. From the practical point of view, identifying the underlying distribution and generating adequate scenarios to capture the distribution is often a very difficult task to accomplish. In addition, with a large number of scenarios the problem becomes computationally demanding and, in some cases, even intractable. RO is an approach with primary goal of optimising against worst cases that might arise in the future by utilising a min-max

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objective. This method has been widely employed to deal with data uncertainty in many application domains such as supply chain management (Bandi et al., 2019) inventory theory (Ardestani-Jaafari & Delage, 2016), scheduling (Balouka & Cohen, 2021), healthcare (Iancu et al., 2021), Markov decision processes (Wiesemann et al., 2013), location problem (Baron et al., 2011), and transit network design (Laporte et al., 2010) among others. Soyster (1973) as a pioneering researcher in this area proposed a modelling approach to find a solution that is feasible under all possible data variations. This method is likely to produce highly conservative solutions. Ben-Tal and Nemirovski (1998, 1999), El-Ghaoui and Lebret (1997), and El-Ghaoui et al. (1998) proposed a RO technique that produces less conservative solutions. According to this method, the robust counterpart of a nominal problem must be solved in the form of a conic quadratic problem. A few years later, Bertsimas and Sim (2003, 2004) proposed an efficient approach to RO based on budget of uncertainty. In the method proposed by Bertsimas and Sim the degree of conservatism could be fully controlled by allowing the value of a prespecified number of uncertain parameters (Γ_i) to change.

In the past two decades, the literature on RO has further evolved and more approaches to address various forms of uncertainty have been developed. Examples of new methodologies that have gained significant attention in the literature are data-driven RO, distributionally RO, adaptive RO, and light robustness.

A common and key step within RO methodologies is to identify an appropriate model for uncertainty. Data-driven RO is concerned with utilising historical data to design suitable uncertainty sets by means, e.g., statistical hypothesis tests (Bertsimas et al., 2018), machine learning techniques (Goerigk & Jannis Kurtz, 2023), and so on. Distributionally Robust Optimisation (DRO) is an effective method to address data uncertainty by bridging the gap between stochastic programming (where uncertainty is assumed to follow a single, perfectly known probability distribution) and RO (where the focus is exclusively on worst-case bounds). DRO search for decisions that perform best under the worst distribution among a collection of probability distributions known as ambiguity set (Kuhn et al., 2025). Adaptive Robust Optimisation (ARO) also called adjustable RO is a framework to model uncertain problems using adaptable policies. ARO models problems where decisions can be divided into "here-and-now" decisions that must be made immediately, and "wait-and-see" decisions that can be made at some point in the future (Ben-Tal et al., 2004). To address data uncertainty, Light Robustness (Fischetti & Monaci, 2009) offers a relatively simple modelling framework that combines features of the two well-known methodologies, i.e., RO and stochastic programming. It has been shown that in some cases solutions produced by LR are comparable with those obtained by RO models or stochastic programming formulations.

1.1. Hub location under uncertainty

Logistics and supply chain networks are the backbone of today's global economy as they facilitate the movement of enormous number of shipments around the globe on daily bases. In order to transport commodities and appliances companies operating in air, freight and/or truck transport sectors use special types of paradigms such as Hub and Spoke. A hub and spoke system connect (often) all origin-destination pairs with just a few numbers of links and exploits economies of scale from flow consolidation, commonly through inter-hub links. The problem that is concerned with the design of such network is called Hub Location Problem (HLP). The objective of a HLP is to find the optimal location of hubs, to allocate demand nodes to these hubs and to direct the flow through the network in such way to optimise an objective like profit, cost, and/or service level. Classical approaches to HLP are largely deterministic with no or very limited uncertain data and with no possible hub failure consideration. However, research have shown that in practice these models often produce sub-optimal and sometimes even infeasible networks if for instance, values of one or more parameters are subject to change. The two popular methodological approaches to handle data uncertainty in mathematical programmes (i.e., RO and stochastic programming) have been widely used in the hub location domain as well. The former is used when no probability law could be fitted to the uncertainty. The latter is applicable when uncertain data could be described by known probability distributions. Examples of hub location studies in which data uncertainty is managed by stochastic programming approaches are Contreras et al. (2011), Alumur et al. (2012), Taherkhani et al. (2021), Li et al. (2025), and Tran et al. (2026). In addition to dealing with the impact of data variation the importance of addressing another aspect of uncertainty that is possibility of hub failure has also been recently highlighted in the literature. Models that incorporate the chance of hub failure in design stage is often referred to as reliable systems. The dominant approach to design reliable HLPs is the use of backup hub(s) which has been studied, e.g., An et al. (2015) and Azizi and Salhi (2022) to name a few. Stochastic programming and reliability approaches to HLPs are out of scope of this research therefore, in the rest of the review we merely focus on robust HLPs.

Shahabi and Avinash (2014) studied the uncapacitated single and multiple allocation HLPs under demand uncertainty. Using the Ben-Tal and Nemirovski approach to RO, the authors modelled the two problems as mixed integer conic quadratic programmes. Meraklı and Yaman (2016, 2017) adopted hose and hybrid models to deal with demand uncertainty in the uncapacitated and later in the capacitated multiple allocation HLP. In hose model instead of considering pairwise flow variations, the total flow at each demand node is bounded. In the capacitated problem this property allows to minimise the worst-case total cost over all possible demand realisation while ensuring each open hub has enough capacity to operate under the worst-case demand realisation in the uncertainty set. Merakli and Yaman reported an important observation that in the absence of hub capacity, location of hubs in the optimal solutions obtained with and without demand uncertainty consideration are quite identical. Zetina et al. (2017) studied robust uncapacitated multi-commodity HLPs considering three cases when (1) demand (2) transportation cost and (3) both demand and transportation cost are subject to uncertainty. They adopted the Bertsimas and Sim (2003) modelling technique in which the level of conservatism in robust solutions is controlled by so called budget of uncertainty. Ghaffarinasab (2018) considered three types of demand uncertainty namely the hose, hybrid, and budget in uncapacitated multiple allocation p-HLP. Also using the Bertsimas and Sim (2003) approach, De Sa et al. (2018) proposed robust formulation for uncapacitaed multiple allocation HLP with incomplete hub network assuming uncertain demand and setup cost.

Our literature review on the robust HLPs indicates that with the exception of Meraklı and Yaman's (2017) work that consider hose demand uncertainty the rest of the research has one of the following characteristics:

- uncertain data only affect the objective,
- uncertain data are only in the constraints,
- two different types of uncertain data are subject to change in the objective and constraints.

To model robust counterpart of a combinatorial optimisation problem, e.g., a HLP using Bertsimas and Sim (2003, 2004) approach, it is crucial to ensure that the uncertain parameter, e.g., demand or cost only appears in the objective function and not in the constraints defining the feasible region (or only in the constraints and not in the objective). This is because

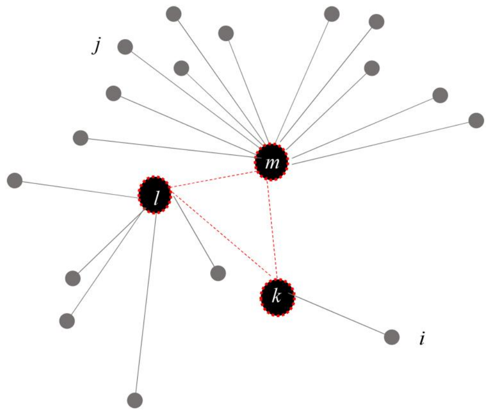


Figure 1. Network topology of a generic HLP with three hubs and 20 nodes.

where the "same" source of uncertainty appears both in the constraints and objective, variation of the uncertain parameter in the objective (e.g., demand) if not captured in the constraints, will affect feasibility of the robust problem (see also: Zetina et al. (2017)). This is perhaps the primary reason why research on robust hub location is almost entirely limited to the uncapacitated problems where either demand or cost (in the objective) or both (in the objective and constraints) are subject to uncertainty.

To model robust counterpart of a problem with common source of uncertainty in the objective and constraints (e.g., the HLPs under demand budget uncertainty) literature suggest to consider worst case of the objective and each constraint separately. Using this approach may lead to the generation of a very conservative solution as every affected constraint would require to be satisfied for all cases pertaining to uncertain parameters realisation including the worst case (Zhu & Fukushima, 2009). Perhaps more importantly in some applications (e.g., the HLP) if worst case of objective and each constraint are modelled separately the same uncertain variable will be solved with different values in the objective and constraints which lead to the generation of an infeasible robust solution. To show this limitation, we present the following small illustrative example.

1.1.1. An illustrative example

Consider a robust solution to a generic capacitated HLP with 20 nodes, 3 hubs and with flow uncertainty (Figure 1). According to the Bertsimas and Sim (2003, 2004) approach, in this (assumed)robust solution the sum of transportation cost (i.e., the nominal cost) and robustness cost is minimum. Robustness cost is the maximum additional cost that may incur if part or entire flow originated from each node i to a certain number of destinations (js) is subject to change. Let us assume for each origin i the number of destinations (js) with flow variation (i.e., uncertainty budget) is "4". This means the cost of transporting flow between each origin i and the four destinations j that cause maximum additional cost have been added to the nominal cost in the objective. Also assume that to ensure feasibility of the solution capacity of each hub facility (e.g., hub k) is decided such that the worst case of flow increase entering the hub is taken into consideration. If the worst case of the objective and each constraint is modelled separately as described above then the robust solution is feasible if and only if the(maximum) flow variation from each origin i to the four destinations causes both maximum additional cost (in the objective) and maximum additional flow to be processed at each hub (in the capacity constraints). This is unlikely to be true as the maximum impact of flow variation on the objective is a function of both volume of flow (between each origin-destination pair) and its unit transportation cost while on the total flow entering each hub depends only on the volume of flow. In other words, the maximum impact of flow variation on the objective value and on the left-hand side of the capacity constraints is measured differently. Therefore, if modelled separately it is likely that uncertain variable will be solved with different values in the objective and constraints. This is an important issue that we aim to highlight and hence to investigate in this work.

1.2. Main contributions and structure of the article

In this study, a) we address uncertainty in an important locational problem with a wide range of real-world applications, b) develop a new methodological approach to model the uncertain problem, c) transform the (uncertain) problem into a fixed non-uncertain and solvable MIP formulation, d) employ the proposed modelling approach to develop robust counterparts of two capacitated HLPs, and e) verify the effectiveness of the proposed approach and its solution technique using a large number of benchmark problems taken from two well-known datasets. This research specifically aims to

1. develop a modified variation of the well-known RO approach due to the work of Bertsimas and Sim (2003, 2004) that is applicable to problems with uncertain data only in the objective and to those (with the same uncertain data) both in the objective and constraints.
2. demonstrate that the proposed approach can be used to formulate robust counterpart of problems with uncertain data in both objective and constraints by developing formulations for the robust capacitated HLPs.
3. solve the robust capacitated single and multiple allocation HLP formulations for a large number of benchmark problems by CPLEX, report, to the best of our knowledge, the first results for this class of optimisation problem with budget uncertainty, and derive managerial insights from our extensive computational analysis.
4. show that by solving two sets of formulations, each for single and multiple allocation uncapacitated HLP, based on our proposed method and the Bertsimas and Sim (2003, 2004) approach, (i) deterministic and robust solutions to a large number of the uncapacitated HLPs we have tested are identical, (ii) for those problems with identical deterministic and robust solutions, the formulations based on our proposed method and the Bertsimas and Sim approach return the same optimal solutions and (iii) near optimal (robust) solutions are obtained by our method for the remainder problems with different deterministic and robust solutions and
5. demonstrate the proposed method's wider applications by formulating robust counterpart of the classical sorting problem and favourably compare our results with the robust solutions obtained by the formulation based on the Bertsimas and Sim (2003, 2004) approach.

It is important to note that as will be demonstrated in Section 5.2, solutions to the deterministic capacitated HLPs and their robust counterparts may differ significantly even though their uncapacitated deterministic and (uncapacitated) robust solutions might be the same. This observation is important as on one hand, it suggests that for a given HLP a deterministic capacitated solution cannot replace its (capacitated) robust counterpart and on the other hand, it highlights the importance of capacity consideration in designing robust HLPs in particular.

This article is organised into six sections. In Section 2, we present and discuss our modelling approach followed by formulations for the robust capacitated HLPs in Section 3. In Section 4, validation and wider applications of the proposed modelling approach are presented and discussed. Computational results for the capacitated robust problems and managerial insights are given in Section 5. Concluding remarks and a summary of our findings are presented in the last section.

2. An extension of the Bertsimas and Sim robust optimisation approach

2.1. Optimistic robust solutions

To control the degree of robustness of a solution, Bertsimas and Sim (2003, 2004) introduced a parameter

(Γ_0) known as *uncertainty budget*. Γ_0 could also be regarded as the maximum number of parameters, e.g., c_j in an objective function whose values could be different from their nominal. In the Bertsimas and Sim RO approach, the intention is to protect against variation of up to Γ_0 coefficients. That means a solution produced by this method is optimal when exactly Γ_0 coefficient with maximum impact on the objective are subject to change and it is feasible for other cases, i.e., when less than Γ_0 coefficients change. Therefore, a robust solution generated by this approach could be regarded as a "*pessimistic*" robust solution as its objective value reflects the worst-case for a given uncertainty budget.

Building upon the above definition of a "pessimistic" robust solution, in this section, we first introduce an "*optimistic*" robust solution to a combinatorial optimisation problem with uncertain data in the objective function and briefly explain how it can be generated. Then we argue that such optimistic robust solution can be used to derive a "pessimistic" (robust) solution to a problem "if deterministic and robust solutions to the problem are frequently the same".

An optimistic robust solution is defined as a solution in which exactly $\overline{\Gamma}_0$ coefficients with "*minimum*" impact on the objective are subject to change. Let us further clarify the definition using the following simple example. In a nominal problem with a $\boldsymbol{c}$ vector (e.g., representing an objective cost coefficients) assume $n = 10$, i.e.,

$$\boldsymbol{c}\text{: } [c_1 \ \ c_2 \ \ c_3 \ldots c_{10}]$$

In this problem for a budget of uncertainty $\Gamma_0 = 3$, assume an optimal solution to the robust counterpart of the nominal problem (e.g., model (1)) based on the Bertsimas and Sim (2003, 2004) approach marks c_1, c_2, and c_3 as the coefficients that would maximally influence the objective if they vary. Intuitively, if deterministic and robust solutions to instances of this problem (i.e., x^*) are the same then the complement set of coefficients (i.e., c_4, c_5, c_6, …, c_{10}) are the coefficients that would minimally influence the objective and vice vera. Therefore, we can model the robust counterpart of the nominal problem directly as a minimisation problem with the objective of finding a solution with minimum sum of the nominal and (optimistic) robustness costs. In an optimal solution to such problem, we can identify $\overline{\Gamma}_0$ coefficients with *minimal* influence on the objective. Identifying the coefficients with minimal impact enable us to find the rest of the coefficients which are likely to have "maximum" impact on the objective. This will lead to the calculation of the (pessimistic) robustness cost. Ultimately, the (minimised) nominal solution and the cost of (pessimistic) robustness forms the solution to the (pessimistic) RO. One particular advantage of indirect development of a pessimistic robust solution (as described above) is that we know exactly in every solution including the optimal solutions which coefficients are subject to variation.

We would like to stress that this technique is useful in cases for which robust pessimistic solutions are not available like in the capacitated HLP under budget demand uncertainty. However, it is less attractive in other cases such as the uncapacitated problem where exiting powerful methods such as the well-known method of Bertsimas and Sim do exist.

In the following a) we discuss details and formally present a modelling approach to generate optimistic robust solutions; b) clarify how an optimistic robust solution can be used to find (uncertain) coefficients that maximally impact the objective and finally; c) discuss how information on these coefficients is used to ensure feasibility of a robust solution to the same problem in which uncertain data is shared by the objective and constraints, e.g., a capacitated HLP.

2.2. Data uncertainty only in the objective

A generic combinatorial optimisation problem with linear objective function and binary variable $\boldsymbol{X} \in \{0,1\}^{\boldsymbol{n}}$ can be presented as

$$\min\left\{\sum_{j\in\overline{N}} c_j x_j | \boldsymbol{X} \in \Pi\right\} \tag{1}$$

where $\boldsymbol{c} \geq 0, \overline{N} = \{1, 2, 3, \ldots n\}$, and Π is a generic polyhedral region. Assume that in this problem the only uncertain data are the elements of vector $\boldsymbol{c}$. Each element $c_j, j \in \overline{N}$ takes values from the interval of uncertainty $[c_j, c_j + d_j]$ where d_j is the deviation from the nominal coefficient, c_j. As stated above, $\overline{\Gamma}_0$ is defined as the number of coefficients with "*minimum*" impact on the objective that are subject to change. Given the above description, "optimistic" robust counterpart of the nominal problem presented in (1) is proposed as

$$\min\left\{\sum_{j\in\overline{N}} c_j x_j + \overline{\beta}_0(\boldsymbol{X}, \overline{\Gamma}_0) | \boldsymbol{X} \in \Pi\right\} \tag{2}$$

where $\overline{\beta}_0(\boldsymbol{X}, \overline{\Gamma}_0)$ which is also known as "protection function" is defined as

$$\overline{\beta}_0(\boldsymbol{X}, \overline{\Gamma}_0) = \min\left\{\sum_{j\in\overline{N}} d_j x_j\right\}$$

It can be shown that an equivalent nonlinear MIP formulation of (2) can be presented as

$$\min\left(\sum_{j\in\overline{N}} c_j x_j + \sum_{j\in\overline{N}} d_j x_j v_{0j}\right)$$

subject to

$$\begin{aligned} \sum_{j \in \overline{N}} v_{0j} &= \overline{\Gamma}_0 \\ \boldsymbol{X} &\in \Pi \\ v_{0j} \in \{0,1\} & \quad \forall j \in \overline{N} \end{aligned} \tag{3}$$

Proposition. Model (3) is the equivalent nonlinear MIP formulation of (2)

Proof. For a given vector $\boldsymbol{X}^*$, the second minimisation term in model (2) is defined as

$$\overline{\beta}_i(\boldsymbol{X}^*, \overline{\Gamma}_0) = \min \sum_{j \in \overline{N}} d_j x_j^* v_{0j}$$

This equals to

$$\min \sum_{j \in \overline{N}} d_j x_j^* v_{0j}$$

subject to

$$\begin{aligned} \sum_{j \in \overline{N}} v_{0j} &= \overline{\Gamma}_0 \\ \boldsymbol{X} &\in \Pi \\ v_{0j} \in \{0,1\} & \quad \forall j \in \overline{N} \end{aligned} \tag{4}$$

The optimal solution to this problem consists of $\overline{\Gamma}_0$ variables at "1". This is the same as selecting a subset of coefficients with corresponding cost function $\sum_{j \in \overline{N}} d_j x_j v_{0j}$. Given the direction of optimisation in model (2), we can directly substitute for the inner minimisation problem and obtain the formulation of (3).

In model (3), $\sum_{j \in \overline{N}} d_j x_j v_{0j}$ is the "*optimistic uncertainty (robustness) cost*" for an assumed budget of (optimistic) uncertainty $\overline{\Gamma}_0$. For a given problem, the objective of model (3) is to find a solution that works even if $\overline{\Gamma}_0$ "least" impactful coefficients vary. Note that the least impactful coefficients are those coefficients that if altered by d_j will have minimal effect on the objective cost.

Remark 1. If deterministic and robust solutions to a problem are frequently identical then the relationship between Γ_0 and $\overline{\Gamma}_0$ (respectively, the number of uncertain coefficients with "maximum" and "minimum" impact on the objective) can be defined as follows. Let $\bar{J}_0$ ($\bar{J}_0 = \{j | d_j > 0\}$) be the set of uncertain c_j coefficients, $j \in \bar{J}_0$. In every feasible solution of model (3) including the optimal solution, given the direction of the optimisation (e.g., minimisation) there are $\overline{\Gamma}_0$ $\left(\overline{\Gamma}_0 \in \left[0, |\bar{J}_0|\right]\right)$ variables (i.e., v_{0j}) at 1 and $n - \overline{\Gamma}_0$ variables at 0. These v_{0j} with value at $1 (v_{0j} = 1 \forall j \in \bar{J}_0)$ determine the "least impactful" coefficients, while the variables with value at 0 determine the "most/high impactful" coefficients (Γ_0 variables in total). Therefore, the relationship between $\overline{\Gamma}_0$ and Γ_0 could be presented as

$$\Gamma_0 = n - \overline{\Gamma}_0 \tag{5}$$

where n is the total number of coefficients, e.g., in a vector/or row. From now on we call Γ_0 simply "budget of uncertainty" as defined by Bertsimas and Sim (2003, 2004). It is worth mentioning that similar to Bertsimas and Sim (2003, 2004) approach by setting $\overline{\Gamma}_0 = 0$ and n ($\overline{\Gamma}_0 = n$) in model (3), we, respectively, have Soyster's Soyster (1973) method (i.e., produces the most conservative solution) and the nominal (deterministic) problem where data uncertainty is ignored.

Remark 2. In problems with the same deterministic and robust optimal solutions (i.e., x*), the pessimistic robustness cost could be calculated by subtracting the optimistic robustness cost from the cost of robustness pertaining to the case when all coefficients are subject to change (i.e., the worst case).

To verify and further clarify this remark, consider a combinatorial optimisation problem, e.g., (1) in a generic form. Model (3) provides a framework to generate "optimistic" robust solutions to this problem. Given a budget of uncertainty Γ_0 ($\Gamma_0 < n$) and using remark 1, in every solution to this problem including the optimal solution, sum of the pessimistic and optimistic robustness costs is constant and equals the cost of robustness in the worst case when all coefficients in the objective are subject to change. Note that in an optimal solution the former cost captures the impact of changes in the values of coefficients that maximally influence the objective while the latter captures the impact of the complement set of the coefficients with minimal influence. Therefore, for every solution including the optimal solution of (3) the pessimistic robustness cost could be calculated by subtracting the optimistic robustness cost from the cost of robustness pertaining to the case when all coefficients are subject to change (i.e., the worst case). To formally derive an expression for calculating this robustness cost let us begin with the optimistic robustness cost. For any feasible solution of (3) and a given $\overline{\Gamma}_0$ the optimistic the robustness cost is calculated by

$$\sum_{j \in \overline{N}} d_j x_j v_{0j} \tag{6}$$

In the same solution, the cost of (pessimistic) robustness is the contribution of those coefficients with $v_{0j} = 0$ (see remark 1). Therefore, this cost can be calculated using the following expression

$$\sum_{j\in\overline{N}} d_j x_j (1 - v_{0j}) \tag{7}$$

Rewriting (7), we have

$$\theta = \sum_{j\in\overline{N}} d_j x_j - \sum_{j\in\overline{N}} d_j x_j v_{0j} \tag{8}$$

The first term in (8) is the robustness cost when all coefficients vary and the second term is the solution's optimistic robustness cost, i.e., when only those coefficients with minimal influence on the objective vary. The solution's (pessimistic) robustness cost is represented by θ. To include equality (8), model (3) is re-written as

$$\min\left(\sum_{j\in\overline{N}} c_j x_j + \sum_{j\in\overline{N}} d_j x_j v_{0j}\right)$$

subject to

$$\begin{aligned} \theta = \sum_{j\in\overline{N}} d_j x_j - \sum_{j\in\overline{N}} d_j x_j v_{0j} \\ \sum_{j\in\overline{N}} v_{0j} = \overline{\Gamma}_0 \\ \boldsymbol{X} \in \Pi \\ v_{0j} \in \{0,1\} \qquad \forall j \in \overline{N} \\ \theta \geq 0 \end{aligned} \tag{9}$$

It is worth mentioning that the variable θ is not optimised in (9) but it is used to derive the solution's (pessimistic) robustness cost. That means the optimal (pessimistic) robust solution to (9) is $\boldsymbol{x}^*$ with the total cost of $(\sum_{j\in\overline{N}} c_j x_j + \theta)$.

Remark 3. Model (9) is a nonlinear formulation whose objective sums the nominal (i.e., $\sum_{j\in\overline{N}} c_j x_j$) and (optimistic) robustness (i.e., $\sum_{j\in\overline{N}} d_j x_j v_{0j}$) costs. We can transform the above two objectives in (9) into an aggregated weighted objective function. The aim of this transformation is twofold. First, for problems with (frequently) identical/or similar deterministic and robust optimal solutions, the primary objective would be to minimise the nominal part of the objective function. Therefore, setting the value of w (i.e., the objective weight) relatively high (e.g., w greater than or equal 0.85) would significantly reduce the computational effort needed to solve instances of the problem. Second, with an unweighted objective function model (9) could return a local optimum, i.e., a solution with low total (sum of nominal and optimistic) costs but a high pessimistic (complement) cost. This situation is further clarified as follows.

Problems with the same deterministic and robust optimal solutions share the same nominal solution. Our analysis shows that due to the impact of the objective function's second term (in the absence of an objective weight) the nominal cost in an optimal solution to (9) can be slightly higher than its optimum value, i.e., obtaining a local nominal solution. Such an optimistic robust solution to (9) is likely to lead to a sub-optimal (pessimistic) robust solution. In this study, we suggest using a weighted objective function to avoid local optima like the one described above. Model (9) with a revised objective function is presented as

$$\min\left(w\sum_{j\in\overline{N}} c_j x_j + (1-w)\sum_{j\in\overline{N}} d_j x_j v_{0j}\right)$$

subject to

$$\begin{aligned} \theta = \sum_{j\in\overline{N}} d_j x_j - \sum_{j\in\overline{N}} d_j x_j v_{0j} \\ \sum_{j\in\overline{N}} v_{0j} = \overline{\Gamma}_0 \\ \boldsymbol{X} \in \Pi \\ v_{0j} \in \{0,1\} \qquad \forall j \in \overline{N} \\ \theta \geq 0 \end{aligned} \tag{10}$$

where $0 \leq w \leq 1$.

Remark 4. Optimal solution to the robust counterpart of (1) based on the Bertsimas and Sim approach and optimal (pessimistic robust) solution to model (10) are identical. This can be easily proven by contradiction and using remarks 1 and 2 as follows. The objective of the robust counterpart of (1) based on the Bertsimas and Sim approach minimises the sum of nominal and (pessimistic) robustness costs. For a given budget of uncertainty Γ_0, the cost of robustness captures the (maximum) impact of variation in uncertain parameter c on the total cost. Similarly, the objective of (10) minimises the (weighted) sum of the nominal and (optimistic) robustness costs (where optimistic robustness cost captures the (minimum) impact of variation in uncertain parameter c). Therefore, in problems with the same deterministic and robust solutions (i.e., the same nominal solution), optimal solutions obtained by the formulations based on Bertsimas and Sim approach and model (10) are different if and only if solution to (10) is sub-optimal, i.e., in the optimal solution to (10) there is at least one coefficient that changes in its value will not minimally impact the objective.

2.3. The same type of uncertain data in both objective and constraints

In many practical settings, the *same type of data* which is prone to uncertainty may appear in both objective function and in one or more constraint sets of a mathematical program. A typical example problem with such data uncertainty is a capacitated HLP with uncertain flow/or demand coefficients both in its objective function (to calculate transportation cost) and its constraints (to account for hub capacities). As stated in Section 1, robust counterpart of such problem can be developed using any of the existing RO approaches (e.g., budget of uncertainty) if flow variation in the constraints is ignored. Obviously, ignoring flow variations in the capacity constraints is likely to result in an infeasible robust solution as the right-hand side of the capacity constraints will account only for regular flow entering each hub centre. In Section 5.2.2, we provide a numerical example to illustrate how ignoring variation of uncertain parameter in the constraints could lead to an infeasible robust solution.

Alternatively, the robust counterpart can be represented by a formulation in which the worst case of the objective and each constraint are modelled separately. However, as we have shown in Section 1.1, the resulting robust solution is likely to be (also) infeasible as uncertain variable will be solved with different values in the objective and constraints. In the following, we elaborate on how the proposed MIP formulation in (10) can be extended to develop robust counterpart of problems with the same uncertain data in the objective and constraints.

Let us consider the following generic optimisation problem with the same source of uncertainty (i.e., c_j) in the objective and constraints

$$\min \left\{ \sum_{j \in \overline{N}} c_j x_j \middle| \ \boldsymbol{X} \in \Pi \right\} \tag{11}$$

subject to

$$g_i(x, c) \geq 0 \ \ i = 1, .., k$$

where $\boldsymbol{c} \geq 0, \overline{N} = \{1, 2, 3, \ldots n\}$, and Π is a generic polyhedral region. $g_i(x, c)$ is a generic constraint set with variable x and uncertain parameter c. If uncertain data are the elements of vector $\boldsymbol{c}$ then each element $c_j, j \in \overline{N}$ takes values from the interval of uncertainty $\left[c_j, c_j + d_j\right]$ where d_j is the deviation from the nominal coefficients, c_j. Using the framework (10), the robust counterpart of (11) could be formulated as

$$\min \left(w \sum_{j \in \overline{N}} c_j x_j + (1 - w) \sum_{j \in \overline{N}} d_j x_j v_{0j} \right)$$

subject to

$$\begin{aligned} &\theta = \sum_{j \in \overline{N}} d_j x_j - \sum_{j \in \overline{N}} d_j x_j v_{0j} \\ &\sum_{j \in \overline{N}} v_{0j} = \overline{\Gamma}_0 \\ &g_i(x, c, v) \geq 0 \ \ i = 1, .., k \\ &\boldsymbol{X} \in \Pi \\ &v_{0j} \in \{0, 1\} \qquad \forall j \in \overline{N} \\ &\theta \geq 0 \end{aligned} \tag{12}$$

As discussed earlier, the exact location of the coefficients affected by data uncertainty (i.e., the column numbers) is an output of the proposed MIP formulation (see binary variables v_{0j} in (10)). This is an important property of the proposed method that does not exist in other RO approaches. Therefore, robust counterpart of the generic model (11) can be constructed by using model (10) and by reformulating capacity constraints $g_i(x, c)$ with binary variables v_{0j} to capture variations of the parameters that are subject to uncertainty in the objective. Here, the idea is to use the complement set of binary variables v (i.e., the coefficients that if vary would maximally influence the objective) to ensure all constraints that share the same uncertain parameters with the objective remain feasible in an optimal robust solution.

To demonstrate the application of the proposed approach in modelling robust counterpart of the problems with uncertain data in both objective and constraints, in the following section, we develop and present (robust) formulations for the capacitated single and multiple allocation HLPs.

3. Robust capacitated hub location problem

3.1. The capacitated multiple allocation p-hub location problem

3.1.1. Problem description and deterministic model

Let $G = (N, A)$ be a network, where N is the set of nodes and A is the set of arcs such that $A = \{(i, j): i, j \in N, i \neq j\}$. Let H $(H \subseteq N)$ and L be the subset of potential locations to stablish hubs and the set of capacity levels, respectively. Each origin-destination (i, j) path passes through one or (maximum) two hubs of the form of $i - k - m - j$ where indices k and m are used for potential hub locations. The volume of the flow that must be transported from node i to node j is denoted by λ_{ij}. $\boldsymbol{\lambda}$ is an $n \times n$ matrix $(\boldsymbol{\lambda} \in \mathbb{R}^{n \times n})$. The elements of $\boldsymbol{\lambda}$ is known in the deterministic problem. d_{ij}, the unit cost of transhipment from node i to node j, are elements of the cost or distance matrix $\boldsymbol{D}$ $(\boldsymbol{D} \in \mathbb{R}^{n \times n})$. χ $(\chi = 1)$, α $(0 \leq \alpha \leq 1)$, and δ $(\delta = 1)$ are cost coefficients associated with collection, transfer, and distribution links on each route, respectively. The cost of transporting one unit of flow from origin node i to destination node j through hubs k and m is then presented as

$$c_{ijkm} = \chi d_{ik} + \alpha d_{km} + \delta d_{mj}$$

Let x_{ijkm} determine the fraction of flow or demand λ_{ij} that is transported\transmitted from node i to node j through the inter-hub link k-m; let $y_k \in \{0, 1\}$ be 1 if node k is selected as a hub, and 0 otherwise; and let $d_k^l \in \{0, 1\}$ be 1 if hub k operates at capacity level l, and 0 otherwise. f_k^l is the fixed cost of establishing hubs k at capacity level l. Ω_k^l is the capacity level l $(l \in L)$ of an installed hub at node k. In the p-hub location problem (p-HLP) it is required to open exactly p number of hubs. For convenience, in the following we present all the notations including those that have been already given in text.

Sets

$\bar{N}$	Set of (generic) items\products etc
$\bar{J}_0$	Set of (generic) uncertain coefficients with minimal impact on objective
N	Set of nodes
H	Set of candidate hub nodes
L	Set of capacity levels

Parameters

c_j	Generic cost component
d_j	Generic cost deviation
Γ_0	Generic uncertainty budget, the number of uncertain coefficients with "maximum" impact on the objective
$\bar{\Gamma}_0$	Generic uncertainty budget, the number of uncertain coefficients with "minimum" impact on the objective
θ	Pessimistic robustness cost
ω	Objective weight value
λ_{ij}	The volume of the flow that must be transported from node $i \in N$ to node $j \in N$
α	Fixed inter-hub link $k-m$ discount factor
χ	Collection cost per unit of flow and per unit of distance
δ	Distribution cost per unit of flow and per unit of distance
c_{ijkm}	The cost of transporting one unit of flow from origin node $i \in N$ to destination node $j \in N$ through hubs $k \in H$ and $m \in H$
f_k^l	The fixed cost of establishing hub $k \in H$ at capacity level $l \in L$
Ω_k^l	The capacity level $l \in L$ of an installed hub at node $k \in H$
p	The number of hubs to open
$\hat{\lambda}_{ij}$	The flow\demand deviation also known as the radius of uncertainty
κ	A user-defined factor to calculate flow deviation (i.e., $\hat{\lambda}_{ij} = \kappa\lambda_{ij}$)
$\bar{k}$	Items out of a set of n items to select
M	Sufficiently large number

Variables

x_j	Generic binary decision variable
v_{0j}	Generic binary decision variable; 1 if coefficient j is subject to variation, and 0 otherwise
v_{ij}	1 if variation of flow from $i \in N$ to $j \in N$ minimally influence the objective
v_j	1 if item j's deviation from its cost component minimally influences the objective
y_k	1 if node $k \in H$ is selected as a hub, and 0 otherwise
d_k^l	1 if hub $k \in H$ operates at capacity level $l \in L$, and 0 otherwise
x_{ijkm}	The fraction of flow or demand λ_{ij} that is transported\transmitted from node $i \in N$ to node $j \in N$ through the inter-hub link k-m
φ_{ijkm}	Auxiliary continuous variable (linearisation)
z_{ik}	1 if node $i \in N$ is allocated to hub $k \in H$, and 0 otherwise
ζ_{ijkm}	Auxiliary continuous variable (linearisation)
μ_i	Dual variable
p_{ij}	Dual variable
τ_j	Auxiliary continuous variable (linearisation)
z	Dual variable
p_j	Dual variable

Using the above notations, the formulation for the deterministic Capacitated Multiple Allocation p-Hub Location Problem (CMApHLP) can be presented as follows. CMApHLP is the capacitated variation of the model proposed by Hamacher et al. (2004) with multiple capacity levels.

[CMApHLP]:

$$\min \sum_{i,j \in N} \sum_{k \in H} \sum_{m \in H} \lambda_{ij} c_{ijkm} x_{ijkm} + \sum_{k \in H} \sum_{l \in l_k} f_k^l d_k^l \tag{13}$$

subject to

$$\sum_{k \in H} y_k = p \tag{14}$$

$$\sum_{k \in H} \sum_{m \in H} x_{ijkm} = 1 \qquad \forall (i,j) \in N \tag{15}$$

$$\sum_{m \in H} x_{ijkm} + \sum_{m \in H \neq k} x_{ijmk} \le y_k \qquad \forall (i,j) \in N, k \in H \tag{16}$$

$$\sum_{i,j \in N} \sum_{m \in H} \lambda_{ij} x_{ijkm} \le \sum_{l \in l_k} \Omega_k^l d_k^l \qquad \forall k \in H \tag{17}$$

$$\sum_{l \in l_k} d_k^l = y_k \qquad \forall k \in H \tag{18}$$

$$x_{ijkm} \ge 0 \qquad \forall (i,j) \in N, (k,m) \in H \tag{19}$$

$$y_k \in \{0,1\} \qquad \forall k \in H \tag{20}$$

$$d_k^l \in \{0,1\} \qquad \forall k \in H, l \in L \tag{21}$$

The first and second terms in the objective function (13) account for transportation and hub installation costs, respectively. Constraint (14) ensures exactly p hubs are opened. Constraints (15) indicate that flow associated with each origin-destination pair (i, j) has to be routed *via* one or two hubs. Constraints (16) guarantee that flow is only routed through open hubs. Constraints (17) restrict the volume of incoming (unprocessed) flow at each hub k. Consistency constraints (18) guarantee the selection of at most one capacity level for each potential. The domain for the decision variables is defined by (19)–(21).

3.1.2. Robust formulation for the CMApHLP

We assume demand represented by parameter λ_{ij} is subject to uncertainty. Each element λ_{ij} is modelled as bounded random variable $\tilde{\lambda}_{ij}$ with unknown distribution. Consider the ith row of matrix $\boldsymbol{\lambda}$ in the nominal problem. Let $\left[\lambda_{ij} - \hat{\lambda}_{ij}, \lambda_{ij} + \hat{\lambda}_{ij}\right]$ be the interval of uncertainty for parameter λ_{ij} and $\tilde{\lambda}_{ij} \in \left[\lambda_{ij} - \hat{\lambda}_{ij}, \lambda_{ij} + \hat{\lambda}_{ij}\right]$; $\hat{\lambda}_{ij}$ is the flow\demand deviation or radius of uncertainty. In this study flow deviation is calculated using a user-defined factor κ $(\hat{\lambda}_{ij} = \kappa\lambda_{ij})$. The robust counterpart of the CMApHLP is developed using the generic formulations in (10) and (12) and the CMApHLP formulation, i.e., (14)–(16) and (18)–(21). The resulting formulation is called RCMApHLP and presented as follows.

[RCMApHLP]:

$$\min w\left(\sum_{i,j \in N} \sum_{k \in H} \sum_{m \in H} \lambda_{ij} c_{ijkm} x_{ijkm} + \sum_{k \in H} \sum_{l \in l_k} f_k^l d_k^l\right) + (1-w)\left(\sum_{i,j \in N} \sum_{k \in H} \sum_{m \in H} \hat{\lambda}_{ij} c_{ijkm} x_{ijkm} v_{ij}\right) \tag{22}$$

subject to

(14)–(16); (18)–(21)

$$\theta = \sum_{i,j\in N}\sum_{k\in H}\sum_{m\in H}\hat{\lambda}_{ij}c_{ijkm}x_{ijkm} - \sum_{i,j\in N}\sum_{k\in H}\sum_{m\in H}\hat{\lambda}_{ij}c_{ijkm}x_{ijkm}v_{ij} \quad (23)$$

$$\sum_{j\in N} v_{ij} = \overline{\Gamma}_i \qquad \forall i \in N \quad (24)$$

$$\hat{\lambda}_{ij}\left(\sum_{k\in H}\sum_{m\in H} c_{ijkm}x_{ijkm}v_{ij}\right) \le \lambda_{ij}\left(\sum_{k\in H}\sum_{m\in H} c_{ijkm}x_{ijkm}\right) \qquad \forall (i,j) \in N \quad (25)$$

$$\sum_{i,j\in N}\sum_{m\in H}\lambda_{ij}x_{ijkm} + \sum_{i,j\in N}\sum_{m\in H}\hat{\lambda}_{ij}x_{ijkm}(1-v_{ij}) \le \sum_{l\in l_k}\Omega^l_k d^l_k \qquad \forall k \in H \quad (26)$$

$$\theta \ge 0 \quad (27)$$

$$v_{ij} \in \{0,1\} \qquad \forall (i,j) \in N \quad (28)$$

Equality (23) calculates the cost of uncertainty. Equality constraints (24), limits the (optimistic) uncertainty budget. Constraints (25) is a valid inequality that bounds the cost of (optimistic) uncertainty. The first term in the capacity constraint (26) captures the regular flow to be processed at hub k. The second term measures the volume of possible additional flow that need to be processed at hub k given the uncertainty budget. (27) and (28) are the domain constraints.

The proposed formulation for the RCMApHLP is nonlinear. In the following, we present the linear variation of the formulation. Details of the linearisation are provided in Appendix A.

3.1.3. RCMApHLP linear formulation.

The linear robust formulation of the RCMApHLP based on the proposed method is presented as follows.

$$\min\ w\left(\sum_{i,j\in N}\sum_{k\in H}\sum_{m\in H}\lambda_{ij}c_{ijkm}x_{ijkm} + \sum_{k\in H}\sum_{l\in l_k} f^l_k d^l_k\right) + (1-w)\left(\sum_{i,j\in N}\sum_{k\in H}\sum_{m\in H}\hat{\lambda}_{ij}c_{ijkm}\varphi_{ijkm}\right) \quad (29)$$

subject to

(14)–(16); (18)–(21); (24); (27–28)

$$\varphi_{ijkm} \le M_1 v_{ij} \qquad \forall (i,j) \in N, (k,m) \in H \quad (30)$$

$$\varphi_{ijkm} \le x_{ijkm} \qquad \forall (i,j) \in N, (k,m) \in H \quad (31)$$

$$\varphi_{ijkm} \ge x_{ijkm} - M_1(1-v_{ij}) \qquad \forall (i,j) \in N, (k,m) \in H \quad (32)$$

$$\theta = \sum_{i,j\in N}\sum_{k\in H}\sum_{m\in H}\hat{\lambda}_{ij}c_{ijkm}x_{ijkm} - \sum_{i,j\in N}\sum_{k\in H}\sum_{m\in H}\hat{\lambda}_{ij}c_{ijkm}\varphi_{ijkm} \quad (33)$$

$$\hat{\lambda}_{ij}\left(\sum_{k\in H}\sum_{m\in H} c_{ijkm}\varphi_{ijkm}\right) \le \lambda_{ij}\left(\sum_{k\in H}\sum_{m\in H} c_{ijkm}x_{ijkm}\right) \qquad \forall (i,j) \in N \quad (34)$$

$$\sum_{i,j\in N}\sum_{m\in H}\left(\lambda_{ij}+\hat{\lambda}_{ij}\right)x_{ijkm} - \sum_{i,j\in N}\sum_{m}\hat{\lambda}_{ij}\varphi_{ijkm} \le \sum_{l\in l_k}\Omega^l_k d^l_k \qquad \forall k \in H \quad (35)$$

$$\varphi_{ijkm} \ge 0 \qquad \forall (i,j) \in N, (k,m) \in H \quad (36)$$

3.2. The capacitated single allocation p-hub location problem

3.2.1. Problem description and deterministic model

In the single allocation p-HLP each node must be assigned to one hub at most. The problem description and definition of variable x, the capacity variable d^l_k and the parameters c, λ, and p are the same as those presented in Section 3.1.1 for the multiple allocation case. z_{ik}, the assignment decision variable is 1 if node i is allocated to hub k, and 0 otherwise. With these notations, the Capacitated variant of the Single Allocation p-Hub Location Problem (CSApHLP) formulation proposed by Skorin-Kapov et al. (1996) is presented as follows

[CSApHLP]:

$$\min \sum_{i,j\in N}\sum_{k\in H}\sum_{m\in H}\lambda_{ij}c_{ijkm}x_{ijkm} + \sum_{k\in H}\sum_{l\in l_k} f^l_k d^l_k \quad (37)$$

subject to

$$\sum_{k\in H} z_{ik} = 1 \qquad \forall i \in N \quad (38)$$

$$z_{ik} \le z_{kk} \qquad \forall i \in N, k \in H \quad (39)$$

$$\sum_{k\in H} z_{kk} = p \quad (40)$$

$$\sum_{m\in H} x_{ijkm} = z_{ik} \qquad \forall (i,j) \in N, k \in H \quad (41)$$

$$\sum_{k\in H} x_{ijkm} = z_{jm} \qquad \forall (i,j) \in N, m \in H \quad (42)$$

$$\sum_{i,j\in N}\sum_{m\in H}\lambda_{ij}x_{ijkm} \le \sum_{l\in l_k}\Omega^l_k d^l_k \qquad \forall k \in H \quad (43)$$

$$\sum_{l\in l_k} d^l_k = z_{kk} \qquad \forall k \in H \quad (44)$$

$$x_{ijkm} \ge 0 \qquad \forall (i,j) \in N, (k,m) \in H \quad (45)$$

$$z_{ik} \in \{0,1\} \qquad \forall i \in N, k \in H \quad (46)$$

$$d^l_k \in \{0,1\} \qquad \forall k \in H, l \in L \quad (47)$$

Constraints (38) are the single assignment restrictions. Constraints (39) indicate that a node is only assigned to an open hub. Constraint (40) ensures exactly p hubs are opened. (41) and (42) are the routeing constraints. Constraints (43) and (44) are the capacity constraints. (45), (46), and (57) define the domain for the decision variables.

3.2.2. Robust formulation for the CSApHLP

Similar to the multiple allocation case, we assume demand λ_{ij} is the uncertain data. The robust counterpart of the CSApHLP could be developed using the generic formulations in (10) and (12) and the CSApHLP formulation, i.e., (38)–(42) and (44)–(47).

The resulting formulation is called RCSApHLP and presented as follows

[RCSApHLP]:

$$\min \ w\left(\sum_{i,j\in N}\sum_{k\in H}\sum_{m\in H}\lambda_{ij}c_{ijkm}x_{ijkm}+\sum_{k\in H}\sum_{l\in l_k}f_k^l d_k^l\right) + (1-w)\left(\sum_{i,j\in N}\sum_{k\in H}\sum_{m\in H}\hat{\lambda}_{ij}c_{ijkm}x_{ijkm}v_{ij}\right) \tag{48}$$

subject to

(38)–(42); (44)–(47)

$$\theta = \sum_{i,j\in N}\sum_{k\in H}\sum_{m\in H}\hat{\lambda}_{ij}c_{ijkm}x_{ijkm} - \sum_{i,j\in N}\sum_{k\in H}\sum_{m\in H}\hat{\lambda}_{ij}c_{ijkm}x_{ijkm}v_{ij} \tag{49}$$

$$\sum_{j\in N}v_{ij}=\overline{\Gamma}_i \qquad \forall i\in N \tag{50}$$

$$\hat{\lambda}_{ij}\left(\sum_{k\in H}\sum_{m\in H}c_{ijkm}x_{ijkm}v_{ij}\right)\le\lambda_{ij}\left(\sum_{k\in H}\sum_{m\in H}c_{ijkm}x_{ijkm}\right) \quad \forall (i,j)\in N \tag{51}$$

$$\sum_{i,j\in N}\sum_{m\in H}\lambda_{ij}x_{ijkm}+\sum_{i,j\in N}\sum_{m\in H}\hat{\lambda}_{ij}x_{ijkm}(1-v_{ij}) \le \sum_{l\in l_k}\Omega_k^l d_k^l \forall k\in H \tag{52}$$

$$\theta\ge 0 \tag{53}$$

$$v_{ij}\in\{0,1\}\forall(i,j)\in N \tag{54}$$

The linear formulation of the RCSApHLP is presented in the next section. Details of the linearisation are given in Appendix B.

3.2.3. RCSApHLP linear formulation.

The linear robust formulation of the RCSApHLP based on the proposed method is presented as follows.

$$\min \ w\left(\sum_{i,j\in N}\sum_{k\in H}\sum_{m\in H}\lambda_{ij}c_{ijkm}x_{ijkm}+\sum_{k\in H}\sum_{l\in l_k}f_k^l d_k^l\right) + (1-w)\left(\sum_{i,j\in N}\sum_{k\in H}\sum_{m\in H}\hat{\lambda}_{ij}c_{ijkm}\zeta_{ijkm}\right) \tag{55}$$

subject to

$(38)-(42);(44);(46)-(47);(50);(53)-(54)$

$$\theta = \sum_{i,j\in N}\sum_{k\in H}\sum_{m\in H}\hat{\lambda}_{ij}c_{ijkm}x_{ijkm} - \sum_{i,j\in N}\sum_{k\in H}\sum_{m\in H}\hat{\lambda}_{ij}c_{ijkm}\zeta_{ijkm} \tag{56}$$

$$\hat{\lambda}_{ij}\left(\sum_{k\in H}\sum_{m\in H}c_{ijkm}\zeta_{ijkm}\right)\le\lambda_{ij}\left(\sum_{k\in H}\sum_{m\in H}c_{ijkm}x_{ijkm}\right) \quad \forall (i,j)\in N \tag{57}$$

$$\sum_{i,j\in N}\sum_{m\in H}\left(\lambda_{ij}+\hat{\lambda}_{ij}\right)x_{ijkm}-\sum_{i,j\in N}\sum_{m\in H}\hat{\lambda}_{ij}\zeta_{ijkm} \le \sum_{l\in l_k}\Omega_k^l d_k^l \quad \forall k\in H \tag{58}$$

$$\sum_{k\in H}\sum_{m\in H}\zeta_{ijkm}\le v_{ij} \qquad \forall (i,j)\in N \tag{59}$$

$$\sum_{k\in H}\sum_{m\in H}\zeta_{ijkm}\le\sum_{k\in H}\sum_{m\in H}x_{ijkm} \quad \forall (i,j)\in N \tag{60}$$

$$\zeta_{ijkm}\ge x_{ijkm}-(1-v_{ij})\forall(i,j)\in N,(k,m)\in H \tag{61}$$

$$\zeta_{ijkm}\ge 0 \qquad \forall(i,j)\in N,(k,m)\in H \tag{62}$$

$$0\le x_{ijkm}\le 1 \qquad \forall(i,j)\in N,(k,m)\in H \tag{63}$$

4. Validation of the proposed modelling approach and its wider applications

In this section, we first discuss the robust counterparts of the uncapacitated variations of the above two HLPs (i.e., USApHLP and UMApHLP formulations). We model and solve these problems to:

- investigate if and to what extent configuration of the solutions to deterministic and robust (uncapacitated) HLPs are identical,
- evaluate and compare performance of the proposed method on related problems that could also be modelled and solved by the Bertsimas and Sim approach,
- compare total costs of the capacitated and uncapacitated solutions, and
- show the proposed approach finds near optimal robust solutions even to those HLPs with different deterministic and robust solutions.

To demonstrate the proposed method's wider applications, we further model and solve robust counterpart of the classical *sorting* problem. The robust sorting problem can be modelled and solved also by the Bertsimas and Sim method.

For each of the two HLP problems, the first robust formulation has been developed according to the Bertsimas and Sim approach and the second one is based on the proposed method. The robust formulations for the USApHLP and UMApHLP based on the proposed method could be constructed by removing the capacity constraints and the fixed costs from the RCSApHLP and RCMApHLP formulations. We call the resulting two formulations, USApHLP-R2 and UMApHLP-R2. For simplicity and to avoid repetition we do not (re)present these formulations; details of their linearisation could be found in Appendices C and D. As the robust uncapacitated HLP based on the Bertsimas and Sim (2003, 2004) approach has been already addressed in the literature, we present the final (robust) formulations for these problems in Appendices E and F. These two formulations are named USApHLP-R1 and UMApHLP-R1.

In the following section, we first present the classical sorting problem followed by its robust formulations based on the Bertsimas and Sim approach (SP-R1). Then, we develop and present the robust counterpart of the problem according to our proposed method (SP-R2).

4.1. The sorting problem

4.1.1. Problem description and formulation

In Sorting Problem (SP), the objective is to minimise the total cost of selecting $\overline{k}$ items out of a set of n items. The integer programming formulation of the SP could be expressed as

[SP]:

$$\min \sum_{i \in N} c_i x_i \tag{64}$$

subject to

$$\sum_{i \in N} x_i = \overline{k} \tag{65}$$

$$\boldsymbol{x} \in \{0, 1\}^n \tag{66}$$

4.1.2. Robust counterpart of the SP formulation based on the Bertsimas and Sim (2003) approach

The robust counterpart of the SP (SP-R1) is presented as follows

[SP-R1]:

$$\min \boldsymbol{c}\prime\boldsymbol{x} + \Gamma z + \sum_{j \in \overline{N}} p_j \tag{67}$$

subject to

(65) and (66)

$$z + p_j \geq d_j x_j \qquad \forall j \in \overline{N} \tag{68}$$

$$z \geq 0 \tag{69}$$

$$p_j \geq 0 \qquad \forall j \in \overline{N} \tag{70}$$

4.1.3. Robust counterpart of the SP formulation based on the proposed approach

Using the framework of the proposed method in (9), the second robust counterpart of the SP formulation (SP-R2) is presented as follows

[SP-R2]:

$$\min \left(w \ \boldsymbol{c}\prime\boldsymbol{x} + (1 - w) \sum_{j \in \overline{N}} d_j x_j v_j \right) \tag{71}$$

subject to

(65) and (66)

$$\sum_{j \in \overline{N}} v_j = \overline{\Gamma} \tag{72}$$

$$\theta = \sum_{j \in \overline{N}} d_j x_j - \sum_{j \in \overline{N}} d_j x_j v_j \tag{73}$$

$$v_j \leq d_j x_j \qquad \forall j \in \overline{N} \tag{74}$$

$$\theta \geq 0 \tag{75}$$

The linear formulation of the SP-R2 is presented in the next section. Details of the linearisation are given in Appendix G.

4.1.4. Robust SP-R2 linear formulation.

$$\min \left(w \ \boldsymbol{c}\prime\boldsymbol{x} + (1 - w) \sum_{j \in \overline{N}} \tau_j \right) \tag{76}$$

subject to

(65) and (66); (72), (74) and (75)

$$\eta_j = d_j x_j \tag{77}$$

$$\tau_j \leq M_4 v_j \qquad \forall j \in \overline{N} \tag{78}$$

$$\tau_j \leq \eta_j \qquad \forall j \in \overline{N} \tag{79}$$

$$\tau_j \geq \eta_j - M_4(1 - v_j) \qquad \forall j \in \overline{N} \tag{80}$$

$$\theta = \sum_{j \in \overline{N}} d_j x_j - \sum_{j \in \overline{N}} \tau_j \tag{81}$$

$$\tau_j \geq 0 \qquad \forall j \in \overline{N} \tag{82}$$

4.2. Model validation and new application results

4.2.1. The uncapacitated hub location problems

We have conducted a comprehensive computational analysis using the robust single and multiple allocation p-hub location formulations. We have tested a total of 224 benchmark problems. The problems are generated from two well-known hub location datasets of CAB (O'Kelly, 1987) and TR (Tan & Kara, 2007). The CAB dataset represents movements of flight passengers between 25 major US cities. The TR dataset known as Turkish Network is obtained from ground cargo delivery operations in Turkey.

The size of the test problems varies from 10 to 40 nodes and the assumed number of hubs is 3 for problems with 10, 15, 20, and 25 and it is 5 for problems with 40 nodes. The discount factor to calculate the transportation cost in inter-hub links is 0.30. From each problem size (e.g., CAB10), we have generated a total of 16 benchmark problems by changing the flow deviation factor κ from 0.10 to 0.70, i.e., $\kappa \in \{0.10, 0.70\}$ and setting four values for the budget of uncertainty Γ_i (e.g., for CAB10 $\Gamma_i = 2, 4, 6, \text{and } 9$). It is worth mentioning that with "n" columns in the coefficient matrix, and given that $\lambda_{ii} = 0 \forall i$, the relationship between $\overline{\Gamma}_i$ and Γ_i (see remark 1) is presented as

$$\Gamma_i = n - \overline{\Gamma}_i - 1$$

Table 1. Summary of the results for the robust uncapacitated single allocation p-hub location problems.

Problem	Uncertainty budget $\Gamma_i(\overline{\Gamma}_i)$	Flow deviation factor, κ	Bertsimas and Sim (2003, 2004)		The proposed approach		Total cost % difference
			Cost of robustness	Total cost	Cost of robustness	Total cost	
CAB10	2(7)	0.7	143912834	686540378	161546449	690953868	0.60
	4(5)	0.7	238975559	781603103	254586047	783993466	0.30
CAB25	14(10)	0.7	4165554418	11302899958	4168503808	11303006207	0.0009
TR25	14(10)	0.3	3688867036	18874216156	3738271446	18879978086	0.03
		0.5	6148107180	21333456300	6230448944	21372155584	0.18
		0.7	8607346569	23792695689	8722627767	23864334407	0.30
TR40	15(24)	0.7	3730491343	11694538560	3811139667	11717200890	0.19
	20(19)	0.7	4360310633	12324357850	4420540304	12326601527	0.02

The objective weight is set to 0.98 for all test problems. Constraints with Big-M formulation have been replaced by *indicator constraints* (Bonami et al., 2015) prior to solving mathematical formulations by CPLEX 22.1. The experiments are performed on a x64-based PC I9-12900 2400 Mhz with 64.0 GB RAM.

We have solved two linear robust formulations of UMApHLP-R1 (based on the Bertsimas and Sim approach) and UMApHLP-R2 (based on the proposed modelling approach) for 112 benchmark problems and compared the results. The experiments are repeated for another two sets of formulations USApHLP-R1 (based Bertsimas and Sim approach) and USApHLP-R2 (based on the proposed modelling approach). Our computational results show that the formulations based on the two methods return the same optimal (robust) solution in 216 (out of 224) benchmark problems. For each test problem, we compared the robust and deterministic optimal solutions. We found that in all 216 test problems deterministic and robust optimal solutions are identical. For the other 8 test problems the maximum gap between the objective values obtained by the two formulations based on the Bertsimas and Sim (2003, 2004) approach and the proposed method is 0.6%. These 8 problems are instances of uncapacitated HLP with different deterministic and robust optimal solutions. Very small difference between the objectives suggests that for these 8 HLPs the proposed approach returns a very near robust optimal solutions. For convenience, a summary of the results for the 8 test problems is provided in Table 1. Detailed results of the computational experiments are given in Appendix H. Table H-1 presents the results for the multiple allocation case while Table H-2 are for the results pertaining to the single allocation case.

From the computational effort point of view, the formulations based on the two RO approaches could be solved for small instances with 10 and 15 nodes very efficiently, i.e., instances of these sizes could be solved in a very short computational time (i.e., in just few seconds) by the two formulations. However, for larger instances, our linearised formulation understandably takes longer computational time to solve. For instance, benchmark problems with 25 nodes and budget of uncertainty of 20 (multiple allocation case) could be solved by our proposed robust formulation in about 60 s on average. While the average time to solve instances with 40 nodes is 1308s.

It is worth mentioning that here we do not intend to compare efficiency of the two formulations. The aim is to show 1) deterministic and robust solutions to a large number of uncapacitated HLP instances we have tested are identical and 2) optimal solutions to all these instances obtained by the formulations based on our approach and by Bertsimas and Sim (2003, 2004) approach are the same.

4.2.2. The sorting problem

For the sorting problem, we experiment with two problems of size $|\overline{N}| = 200$, and $|\overline{N}| = 150$. The number of items to select, $\overline{k}$, in both cases is 100 (Bertsimas & Sim, 2003). The cost components of the first sorting problem are uniformly distributed in $\{20{,}150\}$ and the deviation components are uniformly distributed in $\{5{,}50\}$ and $\{20{,}100\}$. For the second problem the cost components are uniformly distributed in $\{50{,}200\}$ and the deviations are in $\{5{,}50\}$ and $\{20{,}100\}$. For each pair of cost and deviation components, we examine 8 different budget uncertainty $\Gamma_0(\overline{\Gamma}_0)$ ranging from 0(100) to 70(30) to generate 8 benchmark problems for each problem size and deviation component. As a result, the two robust sorting formulations SP-R1 and SP-R2 are solved each for a total of 32 test problems. Computational results for the sorting problem are presented in Table G-1 (Appendix G).

The results presented in Table G-1 are promising. They show that for 10 instances (out of 32) the same optimal solutions have been obtained by the formulations based on our approach and Bertsimas and Sim (2003) approach. For the rest of the problems in Table G-1 (i.e., 22 problems), solutions obtained by the formulation based on our approach are very near optimal. The findings from these experiments confirm that our proposed modelling approach can be successfully applied to other optimisation problems.

5. Computational results and analysis for the robust capacitated problems

In this section, we present and discuss our computational results for the capacitated HLPs where both the objective and the constraints are affected by flow uncertainty. Here, we provide detailed results including information about the total cost, cost of robustness and nominal cost. We also present managerial insights related to budget of uncertainty and the effect of weights in the objective function.

5.1. Data preparation

We have solved the two robust formulations of RCSApHLP & RCMApHLP for a total of 360 instances generated from two different datasets of CAB (O'Kelly, 1987) and TR (Tan & Kara, 2007). The selected problems from the CAB dataset contain 10, 15, 20, and 25 nodes. The problems from the TR datasets contain 10, 25, 30, and 40 nodes. In our experiment, we have considered both tight (T) and loose (L) fixed costs and hub capacities. For each problem size, e.g., CAB 25, we consider two cases: LL (loose fixed cost and loose capacity) and TT (tight fixed cost and tight capacity). For every potential hub and capacity type (T or L), two capacity levels are available. For problem instances with 10 nodes, level one and level two of loose capacities are generated randomly from uniform distributions in the intervals $[0.2\sum_{i,j}\lambda_{ij}, 0.4\sum_{i,j}\lambda_{ij}]$ and $[0.4\sum_{i,j}\lambda_{ij}, 0.6\sum_{i,j}\lambda_{ij}]$, respectively. For tight capacities the intervals from which the two available capacity levels are generated are $[0.1\sum_{i,j}\lambda_{ij}, 0.3\sum_{i,j}\lambda_{ij}]$ and $[0.3\sum_{i,j}\lambda_{ij}, 0.5\sum_{i,j}\lambda_{ij}]$. For the rest of the problem instances the utilised intervals for loose capacities are $[0.3\sum_{i,j}\lambda_{ij}, 0.5\sum_{i,j}\lambda_{ij}]$ and $[0.5\sum_{i,j}\lambda_{ij}, 0.8\sum_{i,j}\lambda_{ij}]$ and for tight capacities they are $[0.2\sum_{i,j}\lambda_{ij}, 0.4\sum_{i,j}\lambda_{ij}]$ and $[0.4\sum_{i,j}\lambda_{ij}, 0.6\sum_{i,j}\lambda_{ij}]$.

The fixed cost for level 1 and level 2 in loose capacities is 5×10^6 and 10×10^6 respectively and in tight capacities they are 10×10^6 and 15×10^6, respectively. For each problem size n, and capacity type (e.g., CAB10TT and CAB10LL) we have examined three values for the budget of uncertainty. The values have been selected based on the problem size and cover a wide range of budgets-see Table I-1 and Table I-2 (Appendix I). The fixed discount factor α is set to 0.30 and 0.50. The flow deviation factor κ (also known as radius of uncertainty) is set to 0.10, 0.20, and 0.30. The objective weight values (w) for CAB10 and TR10 are set to 0.1 and 0.01, respectively; for CAB15, CAB20, CAB25, TR25, TR30, and TR40 it is set to 0.10, 0.20, 0.25, 0.25, 0.25, and 0.35, respectively. The formulations are run in CPLEX 22.1 and the experiments are performed on an Intel Core i9, 2400 Mhz with 64.0 GB RAM.

5.2. Numerical results

Detailed results of the computational experiments for the robust capacitated HLPs are presented in Table I-1 to I-5 (see Appendix I). In the first two tables, i.e., I-1 and I–2, for each problem instance we report the nominal cost, the cost of robustness and the total cost that sums the two aforementioned costs. These experiments are performed for a total of 288 test problems.

5.2.1. Evaluating the robustness of the solutions to the capacitated problems

We begin the analysis by simulating the results of two robust capacitated single allocation HLPs namely CAB10 and CAB20. For the CAB10, we utilise a loose capacity and assume $\alpha = 0.3$; $\kappa = 0.2$; and $\Gamma_i(\overline{\Gamma}_i) = 4(5)$. For the CAB20, we utilise a tight capacity and assume $\alpha = 0.3$; $\kappa = 0.3$; and $\Gamma_i(\overline{\Gamma}_i) = 10(9)$. We solve the CSApHLP formulation for these instances.

The CAB10 solution has a total cost of 631,367,749 units which is the sum of nominal and robustness costs. Its robustness and nominal costs (transportation and fixed costs) are, respectively, 558,437,430 and 72,930,319. To assess the robustness of the capacitated solution to CAB10 benchmark problem, in Table 2 for each origin i ($i = 1,2, 3, \ldots, 10$), we first highlight (by "×") the four destinations (uncertainty budget) with maximum impact on the objective according to the proposed robust solution. Then using the robust capacitated network topology of the CAB10 (Figure 2(b)) for each route beginning from origin i, we calculate the additional transportation cost that may incur if demand (i.e., volume of flow) increases by 20%. In Table 2, the four largest cost increases are highlighted in "**bold**" fonts. As shown in this table, the largest possible (additional) costs incur in the same four routes proposed by the model. In the proposed robust capacitated model this information (i.e., the origin-destination pairs with maximum impact) are used to measure the additional flow that need to be processed in one or more hub facilities. Therefore, ensuring feasibility of the robust capacitated solutions. We repeated the same experiment for the CAB20, a benchmark problem with tight capacity as described above. For the CAB20 results are summarised in Table J-1 (Appendix J).

Furthermore, the base models for the capacitated and uncapacitated robust formulations (based on the proposed method) differ only in terms of capacity constraints and fixed costs which are only a

Table 2. Routes with highest additional transportation cost ($\Gamma_i(\bar{\Gamma}_i) = 4(5)$): CAB10.

Node i \ Node j	1	2	3	4	5	6	7	8	9	10
1	–	1129035	**1702545** X	**2617340** X	851882	693440	**2016587** X	684810	1158542	**1571506** X
2	**1129035** X	–	**2259106** X	**1112544** X	439371	348927	477607	**819498** X	545489	702923
3	1702545	**2259106** X	–	**4563775** X	1077432	1570455	1022355	**1756764** X	**2156232** X	916611
4	**2617340** X	1112544	**4563775** X	–	973910	655968	1015606	**4926550** X	**1926837** X	1451112
5	**851882** X	439371	**1077432** X	**973910** X	–	507582	305278	361117	**635690** X	273552
6	**693440** X	348927	**1570455** X	**655968** X	507582	–	304300	**679068** X	196416	371539
7	**2016587** X	477607	**1022355** X	1015606	305278	304300	–	**1534483** X	514647	**1517228** X
8	684810	819498	**1756764** X	**4926550** X	361117	679068	**1534483** X	–	1191550	**1256238** X
9	**1158542** X	545489	**2156232** X	**1926837** X	635690	196416	514647	**1191550** X	–	550295
10	**1571506** X	702923	916611	**1451112** X	273552	371539	**1517228** X	**1256238** X	550295	–

(a) CAB10 (α=0.3; $\kappa = 0.2$; $\Gamma_i(\bar{\Gamma}_i) = 4(5)$; deterministic/robust solution. Transportation cost: 529407419.

(b) CAB10 (α=0.3; $\kappa = 0.2$; $\Gamma_i(\bar{\Gamma}_i) = 4(5)$; robust capacitated solution. Transportation cost: 533437430.

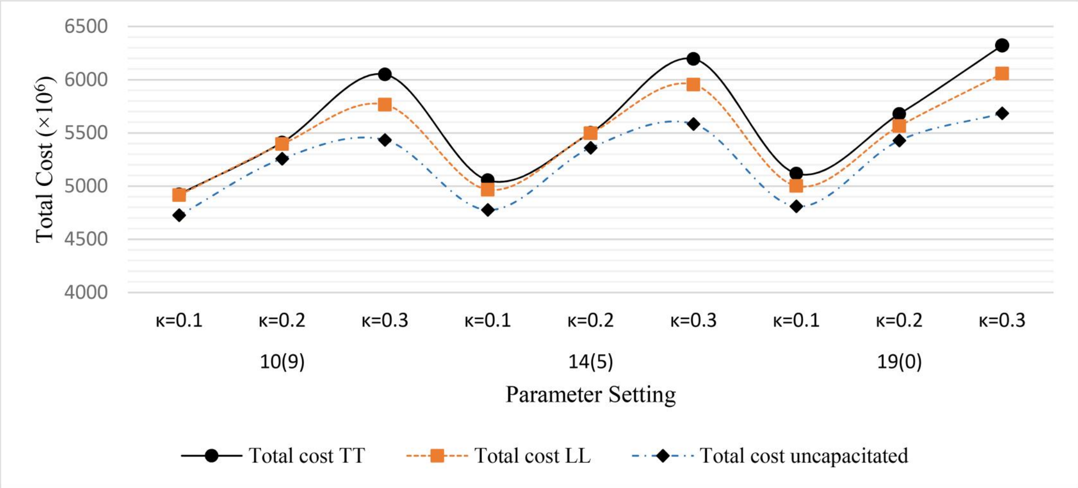
Figure 2. Network topology comparison: CAB10 single allocation problems.

part of the capacitated model. Therefore, it can be easily shown that the objective value of a robust capacitated solution must be greater than the objective value of the solution to its uncapacitated variation. We have verified this by comparing these costs in all the problems for which we have provided robust capacitated and uncapacitated solutions. In Table I-3 (Appendix I), we have compared the total and robustness costs for 16 benchmark problems.

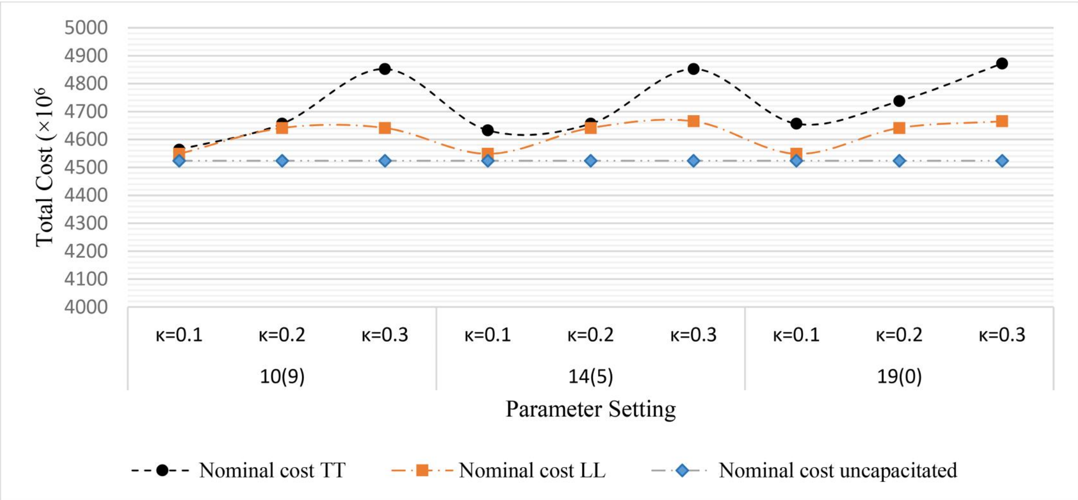
5.2.2. Capacity selection and network reconfiguration in robust solutions

In Figure 2, we have depicted and compared cost and configuration of the robust capacitated and robust uncapacitated solutions to the CAB10 problem instance introduced above. In the uncapacitated robust network (Figure 2(a)), total volume of flow that need to be processed in hubs 4, 6, and 7 are 265,779, 572,322, and 289,966 units, respectively. Now let us assume that the three selected hubs are capacitated (using loose capacities). According to the "data preparation" described in Section 5.1, the two levels of capacity available for hub 4 are 229,288 and 549,335; for hub 6, they are 265,996 and 561,233; and for hub 7 they are 357,096 and 579,781. Clearly if those hubs are indeed capacitated then the robust solution (Figure 2(a)) becomes infeasible as the total flow that potentially need to be processed in hub 6 (572,322) exceeds the largest capacity available to this hub (i.e., 561,233). To avoid infeasibility, as shown in the proposed robust capacitated network one of the nodes initially assigned to this hub (hub 6) is reallocated and assigned to another hub (hub 4) in the network (Figure 2(b)). Alteration like this ensures feasibility of a robust capacitated solution but understandably result in a new and different solution with higher nominal and robustness costs.

In Figure 3, we compare the total and nominal costs pertaining to nine problem instances with 20 nodes (i.e., CAB20) that are generated by varying flow deviation factor κ from 0.10 to 0.30 and setting three values for the budget of uncertainty Γ_i, 10, 14, and 19. The total cost is the sum of the nominal and robustness costs. As shown in this figure, the least expensive robust solutions are the uncapacitated ones. For the solutions to the uncapacitated problems the nominal cost is the same for all nine problem instances. As the hub capacities are



(a) Total cost comparison



(b) Nominal cost comparison

Figure 3. Total and nominal cost comparison between robust capacitated (Tight and Loose) and uncapacitated solutions to the CAB20.

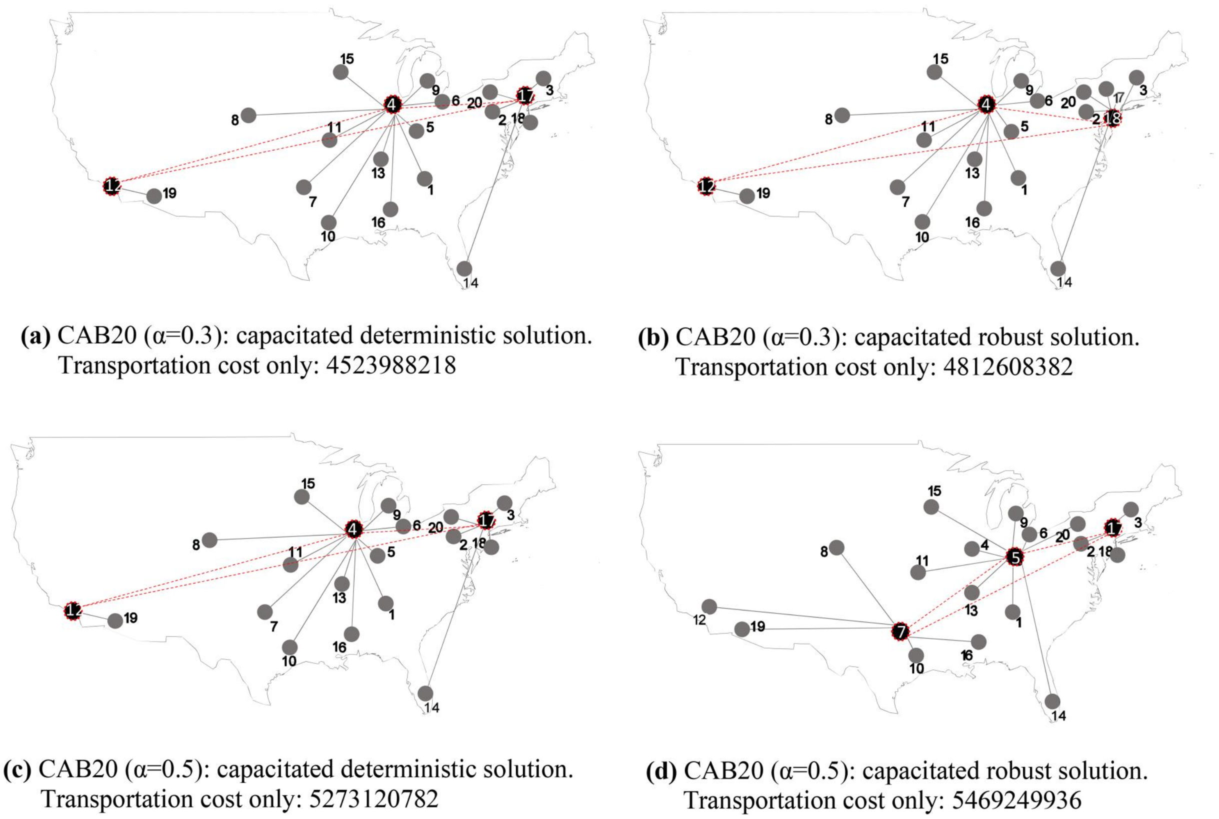
introduced (beginning with relatively loose capacities), different and more expensive robust solutions for the problems are obtained. With tighter capacities, robust solutions become even more expensive as the model attempts to further reconfigure the network to insure feasibility of the solutions.

In general, comparing the nominal costs presented in Table I-1 (Appendix I-solutions to the robust capacitated single allocation problems) and Table I-4 (solutions to their deterministic variations) clearly show that configurations of the solutions to the deterministic problems and their robust counterparts are different. The difference between solutions configuration further increases for instances with tight capacity and tight fixed cost. Our analysis shows that when capacity is scarce, the proposed robust capacitated model attempts to accommodate the additional flow caused by flow uncertainty consideration either by finding new locations for the hub facilities and/or by reallocating demand to the existing hubs. For example, in Figure 4, the cost and configuration of the capacitated deterministic and (capacitated) robust solutions to the CAB20 introduced above are depicted and compared. As illustrated in this figure, the network configurations pertaining to the deterministic and robust solutions differ both in terms of location of the hubs and allocation of non-hub nodes to these hubs.

It is important to note that in some instances where multiple allocation is allowed, the additional flow is managed by increasing the number of nodes that are allocated to multiple hubs. In Figure 5, network topologies for a number of multiple-allocation HLPs with 20 nodes are presented.

5.3. Managerial insights and sensitivity analysis

We demonstrate four items related to managerial insights. This is shown by investigating a) the impact of objective weight factors on solution total



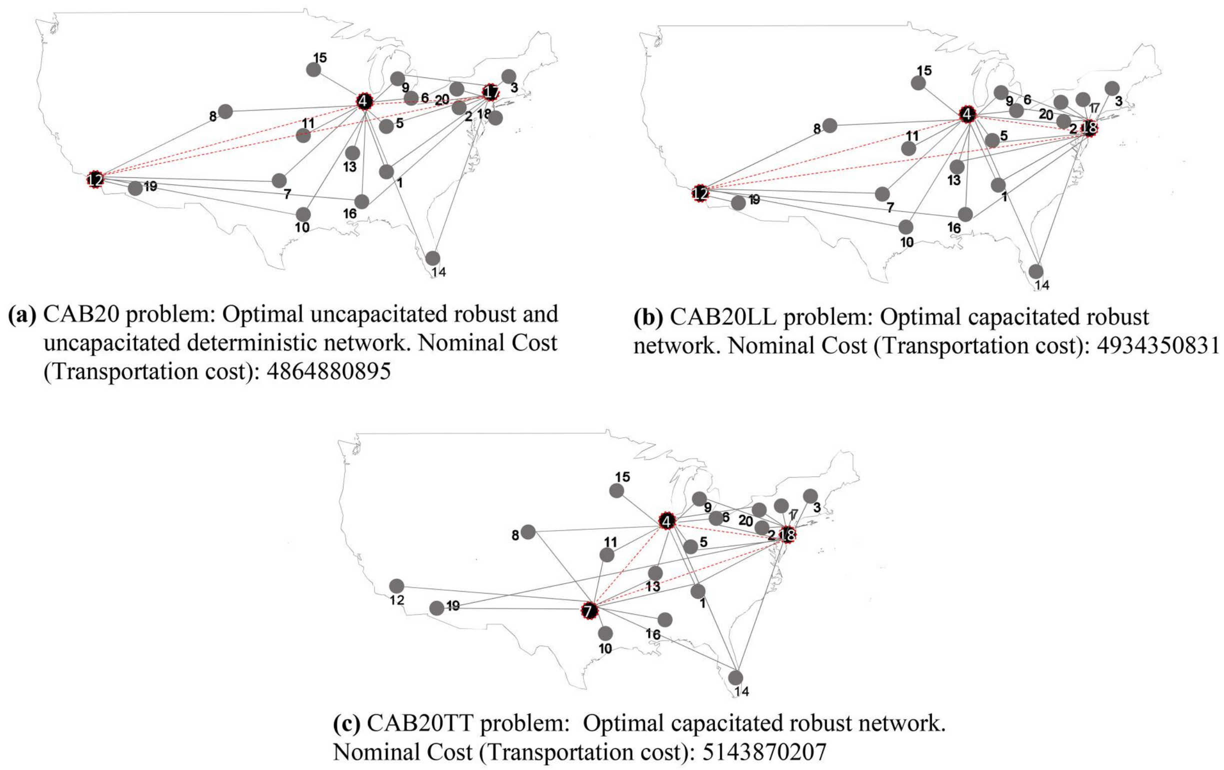
(a) CAB20 (α=0.3): capacitated deterministic solution. Transportation cost only: 4523988218

(b) CAB20 (α=0.3): capacitated robust solution. Transportation cost only: 4812608382

(c) CAB20 (α=0.5): capacitated deterministic solution. Transportation cost only: 5273120782

(d) CAB20 (α=0.5): capacitated robust solution. Transportation cost only: 5469249936

Figure 4. Network topology comparison: 20 nodes single allocation problems.



(a) CAB20 problem: Optimal uncapacitated robust and uncapacitated deterministic network. Nominal Cost (Transportation cost): 4864880895

(b) CAB20LL problem: Optimal capacitated robust network. Nominal Cost (Transportation cost): 4934350831

(c) CAB20TT problem: Optimal capacitated robust network. Nominal Cost (Transportation cost): 5143870207

Figure 5. Network topology comparisons: multiple allocation problems.

cost, on cost of uncertainty and on nominal cost, b) the impact of uncertainty budget on solution total cost, c) the impact of uncertainty budget on nominal cost, and d) infeasibility caused by demand uncertainty in capacitated deterministic solutions.

5.3.1. Impact of objective weights

Using the benchmark problems generated from the CAB dataset, in Table I-5, we report the results of a sensitivity analysis concerning the objective weight value. To perform the analysis, we have fixed the capacity type, flow deviation factor, and budget of

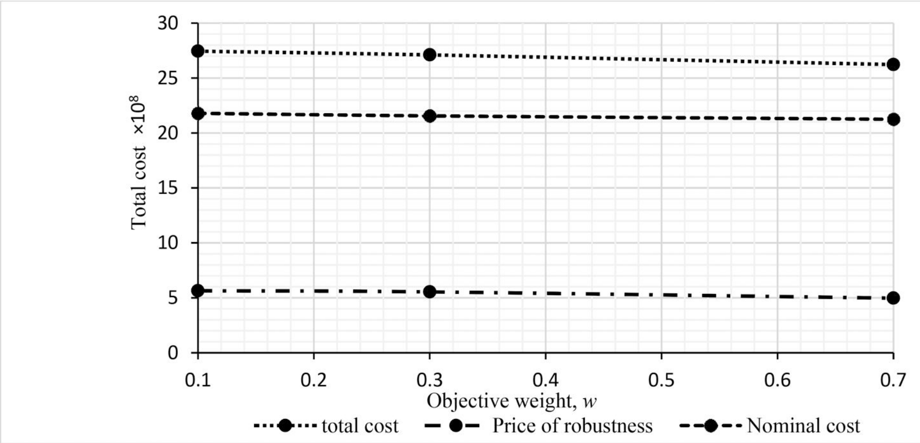


Figure 6. The impact of objective weight on total cost: CAB15TT.

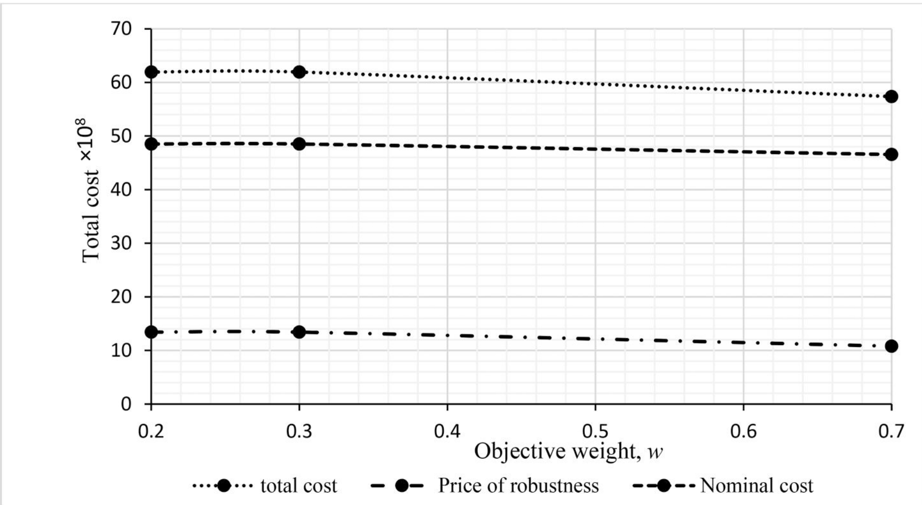


Figure 7. The impact of objective weight on total cost: CAB20TT.

uncertainty and examined a range of values for the objective weight. For instance, in this table using three objective weight values (0.05, 0.30, and 0.70) we report and compare the total costs, cost of robustness, and nominal costs for a problem instance with 10 nodes and with tight capacity and tight fixed cost, i.e., CAB10TT.

Regarding the impact of objective weight values on problems with loose capacities, in all 4 problems we found very little or no alteration in the optimal solutions as we change the weight values. In contrast, setting different weight values has resulted in different optimal solutions in 4 out of 5 test problems with tight capacity. For instance, the results for CAB15TT show that adjusting the objective weight to 0.1, 0.3, and 0.7 leads to three different robust solutions. Comparing the related costs of these four solutions reveals that the higher the values of objective weight, the cheaper the solution will be. However, the saving will come with the cost of less protection. Our analysis shows that best (robust) solutions are obtained when weight factor is set to low values, e.g., 0.05. These solutions provide maximum protection for a given budget of uncertainty. In Figures 6 and 7, we illustrate the impact of changing weight values on solution total cost, cost of uncertainty, and nominal cost for CAB15TT and CAB20TT, respectively.

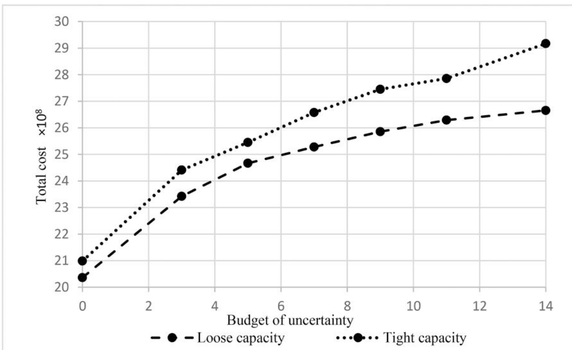


Figure 8. The impact of uncertainty budget on total cost (tight vs loose capacity): CAB15.

5.3.2. Impact of uncertainty budget on solution total cost

Further examination of the results pertaining to each problem instance in Tables I-1 and I-2

(Appendix I) show that for each test problem the overall objective value increases with increase in uncertainty budget. This is expected as more protection will be provided at higher costs. We have depicted the impact of uncertainty budget on total costs in Figure 8 for the CAB15 benchmark problem. In Figure 9 and for the same problem instance, we show that the same trend (i.e., increase in total cost with increase in uncertainty budget) could be seen regardless of the number of hubs in the network. Our computational results also show that similar to the impact of uncertainty budget on total cost, as the radius of uncertainty or flow variation factor (κ) increases the total cost of the solution will increase as well.

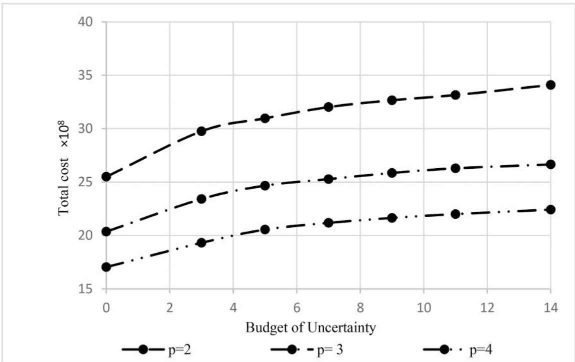


Figure 9. The impact of budget of uncertainty and number of hubs on total cost: CAB15.

5.3.3. Impact of uncertainty budget on solution nominal cost

We have performed an analysis to study trade-offs between solution robustness and nominal cost degradation in robust capacitated single and multiple allocation HLPs. For this purpose, we have considered robust solutions to CAB10, CAB15, CAB20, and CAB25 problem instances. The discount factor in CAB10&20 and CAB15&25 is 0.3 and 0.5, respectively. The flow deviation factor (i.e., κ) for all four problem instances is 0.3. For each problem, we consider solutions with loose and tight capacities. Results are depicted in Figure 10.

Analysing the results, we show in problems with loose capacity changing the budget uncertainty (or percentage of uncertain parameters) has limited impact on the nominal costs. In other words, increasing budget uncertainty in robust capacitated

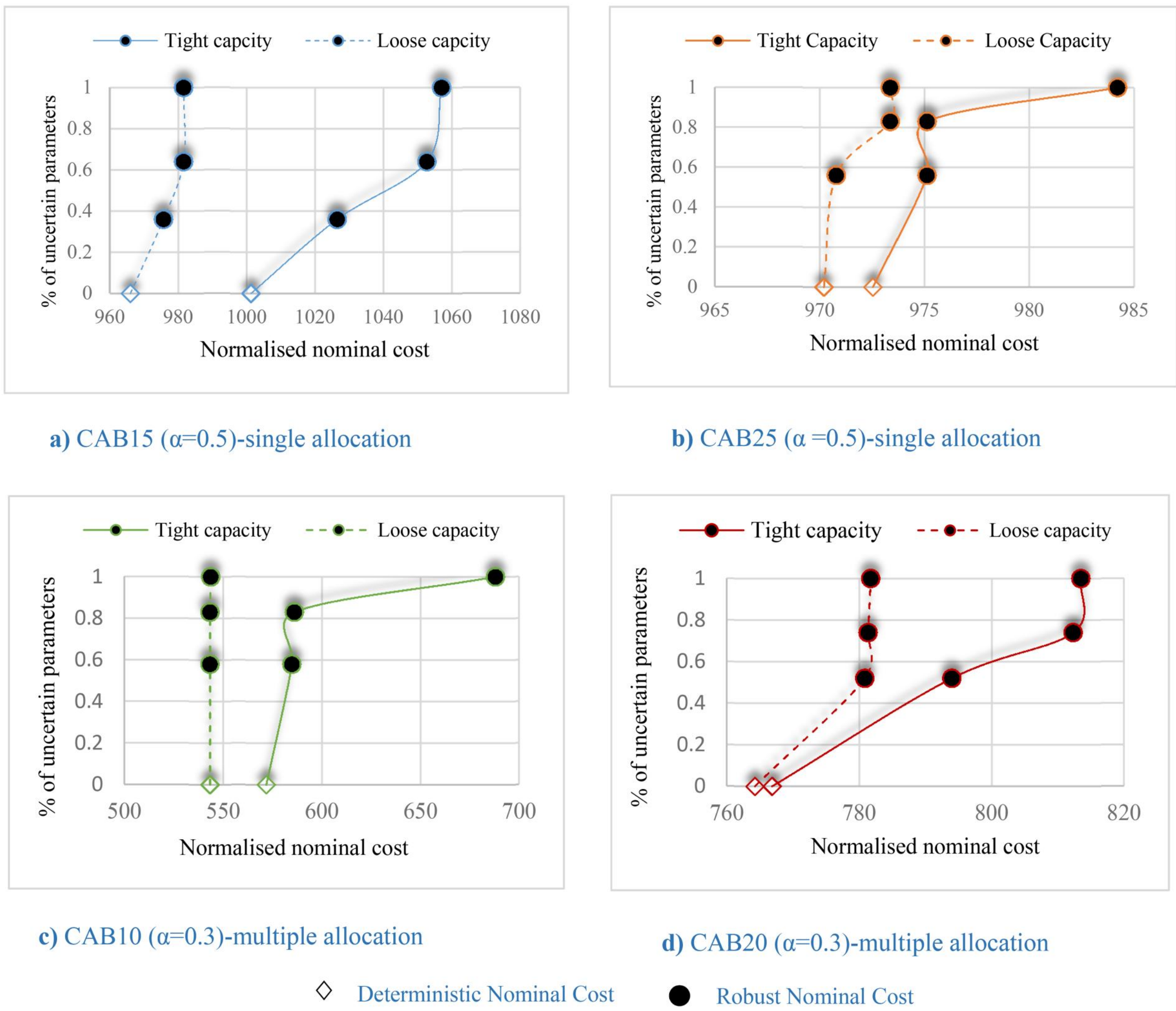


Figure 10. Impact of solution robustness on nominal cost degradation.

Table 3. Feasibility analysis of deterministic capacitated problems.

		Flow deviation factor, κ		
Problem	Deterministic	0.1	0.2	0.3
CAB15LL	2284459310	*Infeasible*	*Infeasible*	*Infeasible*
CAB15TT	2367835599	*Infeasible*	*Infeasible*	*Infeasible*
CAB20LL	5258120782	5781446120	*Infeasible*	*Infeasible*
CAB20TT	5273120782	*Infeasible*	*Infeasible*	*Infeasible*
CAB25LL	8285432329	9117008611	9943507162	*Infeasible*
CAB25TT	8305432329	9132008611	9958507162	*Infeasible*
TR25LL	17325038840	19055543301	20786049837	22516555132
TR25TT	17345038840	*Infeasible*	*Infeasible*	*Infeasible*

problems will not result in significant deviation from the nominal cost of the deterministic problem. Therefore, it can be concluded that large improvement in robustness of a solution to problems with loose capacity can be achieved with minor adjustment to the solution's configuration. This is in line with our earlier findings that if there is no or limited capacity restriction a network obtained by a deterministic model can largely handle and absorb any possible flow variations. For the same problem instances but with tight capacity however, our analysis shows as robustness of a solution increases (i.e., higher budget uncertainty used) relatively larger portion of the objective will be sacrificed. From a practical point of view, it can be said that budget uncertainty in problems with loose capacity can be selected confidently as it has limited impact on nominal cost. whereas it must be selected cautiously in problems with tight capacity.

5.3.4. Infeasibility caused by demand uncertainty

Another aspect related to managerial insights we have investigated is how solutions to deterministic capacitated problems perform under demand uncertainty. We have used the problem instances of CAB15, CAB20, CAB25, and TR25 with loose and tight capacities and fixed installation costs. The value of discount factor in all four problem instances is set to 0.5 (i.e., $\alpha = 0.5$). The experiment is conducted by calculating the (deterministic) network cost pertaining to each problem instance when flow/demand between all origin-destination nodes is subject to 10, 20, and 30% increase. Computational results are presented in Table 3. In this table instances with infeasible deterministic solutions are indicated as *infeasible*. We found that a large portion of the optimal solutions to the problems we have tested (i.e., approximately 67% of the solutions) become infeasible if there is only up to 30% variation in demand. Clearly one may expect even more increase in the number of infeasible solutions if flow deviation factor goes beyond 30%. Here, an infeasible solution means in a capacitated network possible demand uncertainty is likely to cause congestion and heavy traffic at hub facilities. This is an important practical issue with serious effects such as higher network cost and low customer satisfaction that must be avoided.

Another interesting observation obtained from the above analysis is about the impact of capacity type on the likelihood of a solution infeasibility. While discovering an infeasible solution to deterministic HLPs with tight capacity when demand is subject to uncertainty is not quite surprising, it is interesting to find that problems with loose capacities are also highly sensitive to demand variation.

6. Conclusion

We empirically show that robust and deterministic solutions to a very large instances of the uncapacitated HLP we have tested are either very similar or the same. A similar observation has been reported by Meraklı and Yaman (2017) where demand uncertainty is addressed by hose and hybrid(hose) models. We use this characteristic to develop a modified variation of the well-known RO approach proposed by Bertsimas and Sim (2003, 2004) to formulate and solve robust capacitated HLPs with polyhedral budget demand uncertainty. To the best of our knowledge, for the first time, computational results for these problems based on budget of uncertainty are reported. The proposed modelling approach has been presented in a generic form that could be easily adopted to model similar problems.

We highlight the importance of incorporating hub capacity limitation in robust HLPs while providing interesting managerial insights. We show that although in a large majority of HLP instances we have tested deterministic and robust uncapacitated solutions are frequently the same, their robust capacitated counterparts differ quite significantly in terms of location of hubs, allocations of the non-hubs and/or both. Our analysis also shows that increasing budget uncertainty in robust problems with loose capacity will not result in significant deviation from the nominal cost of the deterministic problems. Therefore, large improvement in robustness of a solution to these problems can be achieved with minor adjustment to the solution's configuration. For problems with tight capacity, we show relatively large portion of the objective need to be sacrificed to increases solution robustness.

The objective of the robust capacitated HLPs has been modelled as a weighted sum of nominal and robustness costs. Our sensitivity analysis shows that for problems with loose capacity the same optimal solutions could be obtained using a wide range of objective weight values, i.e., no or very limited sensitivity to the objective weight values. For the problems with tight capacities, we found that for a given uncertainty budget solutions with maximum protection would be obtained if very low weight values are utilised. Setting the weight value at a low level in a given problem makes the problem difficult to solve but ensures a suboptimal solution is largely avoided.

To validate the proposed modelling approach and to demonstrate its wider applications we have first formulated and, for a number of instances, solved three problems, namely robust uncapacitated single and multiple hub location and sorting problems. Our computational results for these problems are promising. We found that in a large majority of the 256 problems, we have examined the optimal solutions obtained by the two formulations based on the Bertsimas and Sim approach and our proposed method are identical and for the small number of remaining problems the gap between the objectives is at most 1.2%. In other words, the very small gaps indicate that the proposed method could effectively solve instances of those problems with different deterministic and robust optimal solutions as well.

We believe there are a number of interesting avenues for future research in this area. For instance, performance of the two pessimistic and optimistic robust approaches with uncertain data only in the objective could be studied in-depth. This is important as a pessimistic RO approach considers only the worst-case scenario therefore, its solution is (often) very costly and performs (often) poorly under different scenarios. The proposed optimistic RO approach (described by models (10) and (12)) however, offers an array of less costly solutions that provide various level of protections. This property could be attractive to practitioners who are reluctant to invest a large amount of capital on robustness to protect against uncertainty. The two proposed generic models (10) and (12) to address uncertain problems with data variation in the objective only or in both the objective and constraints, are nonlinear and therefore relatively difficult to solve for very large instances, e.g., commercial solvers. Although we are able to solve instances of the two formulations up to 40 nodes, we believe that promising extensions and adaptation of the models could be explored so that larger instances of the robust capacitated HLPs could also be solved efficiently.

Author contributions

CRediT: **Nader Azizi**: Conceptualization, Formal analysis, Investigation, Methodology, Project administration, Software, Validation, Visualization, Writing – original draft, Writing – review & editing; **Said Salhi**: Formal analysis, Methodology, Project administration, Validation, Visualization, Writing – review & editing.

Disclosure statement

The authors report there are no competing interests to declare.

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Data availability statement

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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Appendices

Appendix A. RCMApHLP linearization

To linearise the RCMApHLP, we replace the common nonlinear term, $v_{ij}x_{ijkm}$ by a new variable φ_{ijkm} i.e.,

$$\varphi_{ijkm} = v_{ij}x_{ijkm} \quad \forall i,j \in N; \forall k,m \in H \tag{A1}$$

To ensure the expression $\varphi_{ijkm} = v_{ij}x_{ijkm}$, the following three constraints need to be embedded into the RCMApHLP formulation.

$$\varphi_{ijkm} \le M_1 v_{ij} \forall i,j \in N; \quad \forall k,m \in H \tag{A2}$$

$$\varphi_{ijkm} \le x_{ijkm} \quad \forall i,j \in N; \forall k,m \in H \tag{A3}$$

$$\varphi_{ijkm} \ge x_{ijkm} - M_1(1 - v_{ij}) \quad \forall i,j \in N; \forall k,m \in H \tag{A4}$$

Appendix B. RCSApHLP linearization

To linearise the RCSApHLP, we replace the common nonlinear terms, $v_{ij}x_{ijkm}$, by a new variable ζ_{ijkm}, i.e.,

$$\zeta_{ijkm} = v_{ij}x_{ijkm} \quad \forall i,j \in N; \forall k,m \in H \tag{B1}$$

and embed the following three constraints into the RCSApHLP formulation.

$$\sum_k \sum_m \zeta_{ijkm} \le v_{ij} \quad \forall i,j \in N \tag{B2}$$

$$\sum_k \sum_m \zeta_{ijkm} \leq \sum_k \sum_m x_{ijkm} \qquad \forall i,j \in N \tag{B3}$$

$$\zeta_{ijkm} \geq x_{ijkm} - (1 - v_{ij}) \qquad \forall i,j \in N \tag{B4}$$

Appendix C. UMApHLP-R2 linearisation

To linearise the UMApHLP-R2, we first introduce an axillary variable to transfer the nonlinear terms into a product of a binary and non-binary variables.

$$h_{ij} = \sum_k \sum_m c_{ijkm} x_{ijkm} \qquad \forall i,j \in N \tag{C1}$$

The two nonlinear terms in UMApHLP-R2 are transferred to a linear expression as

$$\sum_i \sum_j \hat{\lambda}_{ij} \psi_{ij} = \sum_i \sum_k \sum_m \sum_j \hat{\lambda}_{ij} c_{ijkm} x_{ijkm} v_{ij} \tag{C2}$$

where $\psi_{ij} = h_{ij} v_{ij}$.

The nonlinear term in left side of the valid inequality constraint is replaced by a linear term as shown below

$$\hat{\lambda}_{ij} \psi_{ij} = \hat{\lambda}_{ij} \left(\sum_k \sum_m c_{ijkm} x_{ijkm} v_{ij} \right) \tag{C3}$$

and the following new elements are embedded into the problem formulation.

$$\psi_{ij} \leq M_2 v_{ij} \qquad \forall i,j \in N \tag{C4}$$

$$\psi_{ij} \leq h_{ij} \qquad \forall i,j \in N \tag{C5}$$

$$\psi_{ij} \geq h_{ij} - M_2(1 - v_{ij}) \qquad \forall i,j \in N \tag{C6}$$

$$h_{ij} = \sum_k \sum_m c_{ijkm} x_{ijkm} \qquad \forall i,j \in N \tag{C7}$$

Appendix D. USApHLP-R2 linearisation

To linearise USApHLP-R2 formulation, we first introduce an axillary variable g_{ij} to transfer the nonlinear terms into a product of a binary and non-binary variables, i.e.,

$$g_{ij} = \sum_k \sum_m c_{ijkm} x_{ijkm} \qquad \forall i,j \in N \tag{D1}$$

Then we replace the nonlinear terms in the objective and in the equality constraint by a linear expression as shown below

$$\sum_i \sum_j \hat{\lambda}_{ij} \xi_{ij} = \sum_i \sum_k \sum_m \sum_j \hat{\lambda}_{ij} c_{ijkm} x_{ijkm} v_{ij} \tag{D2}$$

where $\xi_{ij} = g_{ij} v_{ij}$.

Similarly, the nonlinear term in left side of the valid inequality constraint is replaced by a linear expression, i.e.,

$$\hat{\lambda}_{ij} \xi_{ij} = \hat{\lambda}_{ij} \left(\sum_k \sum_m c_{ijkm} x_{ijkm} v_{ij} \right) \tag{D3}$$

We need to add the following three constraints into the formulation.

$$\xi_{ij} \leq M_3 v_{ij} \qquad \forall i,j \in N \tag{D4}$$

$$\xi_{ij} \leq g_{ij} \qquad \forall i,j \in N \tag{D5}$$

$$\xi_{ij} \geq g_{ij} - M_3(1 - v_{ij}) \qquad \forall i,j \in N \tag{D6}$$

Appendix E. UMApHLP-R1 formulation

$$\min \sum_{i,j \in \overline{N}} \sum_{k \in H} \sum_{m \in H} \lambda_{ij} c_{ijkm} x_{ijkm} + \sum_{i \in N} \Gamma_i \mu_i + \sum_{i,j \in N} p_{ij} \tag{E1}$$

subject to

$$\sum_{k \in H} y_k = p \tag{E2}$$

$$\sum_{k \in H} \sum_{m \in H} x_{ijkm} = 1 \qquad \forall (i,j) \in N \tag{E3}$$

$$\sum_{m \in H} x_{ijkm} + \sum_{m \in H \neq k} x_{ijmk} \leq y_k \qquad \forall (i,j) \in N, k \in H \tag{E4}$$

$$\mu_i + p_{ij} \geq \hat{\lambda}_{ij} \left(\sum_{k \in H} \sum_{m \in H} c_{ijkm} x_{ijkm} \right) \qquad \forall (i,j) \in N \tag{E5}$$

$$x_{ijkm} \geq 0 \qquad \forall (i,j) \in N, (k,m) \in H \tag{E6}$$

$$y_k \in \{0,1\} \qquad \forall k \in H \tag{E7}$$

$$\mu_i \geq 0 \qquad \forall i \in N \tag{E8}$$

$$p_{ij} \geq 0 \qquad \forall (i,j) \in N \tag{E9}$$

Appendix F. USApHLP-R1 formulation

$$\min \sum_{i,j \in N} \sum_{k \in H} \sum_{m \in H} \lambda_{ij} c_{ijkm} x_{ijkm} + \sum_{i \in N} \Gamma_i \mu_i + \sum_{i,j \in N} p_{ij} \tag{F1}$$

subject to

$$\sum_{k \in H} z_{ik} = 1 \qquad \forall i \in N \tag{F2}$$

$$z_{ik} \leq z_{kk} \qquad \forall i \in N, k \in H \tag{F3}$$

$$\sum_{k \in H} z_{kk} = p \tag{F4}$$

$$\sum_{m \in H} x_{ijkm} = z_{ik} \qquad \forall (i,j) \in N, k \in H \tag{F5}$$

$$\sum_{k \in H} x_{ijkm} = z_{jm} \qquad \forall (i,j) \in N, m \in H \tag{F6}$$

$$\mu_i + p_{ij} \geq \hat{\lambda}_{ij} \left(\sum_{k \in H} \sum_{m \in H} c_{ijkm} x_{ijkm} \right) \qquad \forall (i,j) \in N \tag{F7}$$

$$x_{ijkm} \geq 0 \qquad \forall (i,j) \in N, (k,m) \in H \tag{F8}$$

$$z_{ik} \in \{0,1\} \qquad \forall i \in N, k \in H \tag{F9}$$

$$\mu_i \geq 0 \qquad \forall i \in N \tag{F10}$$

$$p_{ij} \geq 0 \qquad \forall (i,j) \in N \tag{F11}$$

Appendix G. SP-R2 linearisation and sorting results

To linearise the model, we first introduce an axillary variable to transfer the nonlinear terms into a product of a binary and non-binary variables.

$$\eta_j = d_j x_j \tag{G1}$$

The two nonlinear terms in the objective function and constraints are replaced by

$$\tau_j = \eta_j v_j \tag{G2}$$

To ensure the expression $\tau_j = \eta_j v_j$ the following constraints are added to the formulation.

$$\tau_j \leq M_4 v_j \qquad \forall j \in \overline{N} \tag{G3}$$

$$\tau_j \leq \eta_j \qquad \forall j \in \overline{N} \tag{G4}$$

$$\tau_j \geq \eta_j - M_4(1 - v_j) \qquad \forall j \in \overline{N} \tag{G5}$$

Table G-1. Computational results for the robust sorting problem.

Cost component, c_j	Deviation, d_j	Uncertainty budget $\Gamma_0(\bar{\Gamma}_0)$	Bertsimas and Sim (2003, 2004) Cost of robustness	Bertsimas and Sim (2003, 2004) Total cost	The proposed approach Cost of robustness	The proposed approach Total cost	Total costs % difference
U[20,150]	U[5,50]	0(100)	0	6498	0	6498	0.00
		10(90)	483	6981	483	6981	0.00
		20(80)	903	7401	903	7401	0.00
		30(70)	1274	7776	1279	7777	0.01
		40(60)	1624	8126	1631	8129	0.04
		50(50)	1922	8440	1953	8451	0.13
		60(40)	2131	8686	2223	8722	0.41
		70(30)	2333	8888	2435	8934	0.51
	U[20,100]	0(100)	0	6498	0	6498	0.00
		10(90)	962	7460	962	7460	0.00
		20(80)	1818	8320	1818	8323	0.04
		30(70)	2554	9081	2585	9095	0.15
		40(60)	3135	9738	3259	9769	0.32
		50(50)	3686	10,306	3839	10,360	0.52
		60(40)	4159	10,811	4363	10,884	0.67
		70(30)	4490	11,234	4838	11,370	1.20
U[50,200]	U[5,50]	0(100)	0	8681	0	8681	0.00
		10(90)	483	9164	483	9164	0.00
		20(80)	920	9605	926	9606	0.01
		30(70)	1320	10,010	1337	10,017	0.07
		40(60)	1641	10,352	1684	10,371	0.19
		50(50)	1868	10,618	1941	10,645	0.26
		60(40)	2071	10,833	2118	10,882	0.45
		70(30)	2230	11,011	2346	11,086	0.67
	U[20,100]	0(100)	0	8681	0	8681	0.00
		10(90)	952	9633	952	9633	0.00
		20(80)	1832	10,513	1832	10,513	0.00
		30(70)	2596	11,307	2628	11,311	0.04
		40(60)	3258	11,996	3330	12,013	0.14
		50(50)	3779	12,581	3957	12,640	0.46
		60(40)	4198	13,085	4445	13,161	0.58
		70(30)	4567	13,499	4921	13,653	1.13

Appendix H. Computational results for uncapacitated single and multiple allocation problems

Table H-1. Results for the robust uncapacitated multiple allocation p-hub location problems.

Problem	Uncertainty budget $\Gamma_i(\bar{\Gamma}_i)$	Flow deviation factor, κ	Bertsimas and Sim (2003, 2004) Cost of robustness	Bertsimas and Sim (2003, 2004) Total cost	The proposed approach Cost of robustness	The proposed approach Total cost	Total cost % difference
CAB10	2(7)	0.1	22447241	540182391	22447241	540182391	0
		0.3	67341722	585076872	67341722	585076872	0
		0.5	112236203	629971354	112236203	629971354	0
		0.7	157130685	674865835	157130685	674865835	0
	4(5)	0.1	35442181	553177331	35442181	553177331	0
		0.3	106326543	624061693	106326543	624061693	0
		0.5	177210904	694946054	177210904	694946054	0
		0.7	248095266	765830416	248095266	765830416	0
	6(3)	0.1	43976587	561711737	43976587	561711737	0
		0.3	131929761	649664911	131929761	649664911	0
		0.5	219882934	737618084	219882934	737618084	0
		0.7	307836108	825571258	307836108	825571258	0
	9(0)	0.1	51773515	569508665	51773515	569508665	0
		0.3	155320545	673055695	155320545	673055695	0
		0.5	258867575	776602725	258867575	776602725	0
		0.7	362414605	880149755	362414605	880149755	0
TR10	2(7)	0.1	15268976	409077023	15268976	409077023	0
		0.3	45806927	439614974	45806927	439614974	0
		0.5	76344877	470152924	76344877	470152924	0
		0.7	106882828	500690875	106882828	500690875	0
	4(5)	0.1	27932875	421740922	27932875	421740922	0
		0.3	83798625	477606672	83798625	477606672	0
		0.5	139664374	533472421	139664374	533472421	0
		0.7	195530124	589338171	195530124	589338171	0
	6(3)	0.1	34948909	428756956	34948909	428756956	0
		0.3	104846726	498654773	104846726	498654773	0
		0.5	174744543	568552590	174744543	568552590	0
		0.7	244642361	638450408	244642361	638450408	0
	9(0)	0.1	39380914	433188961	39380914	433188961	0
		0.3	118142743	511950790	118142743	511950790	0
		0.5	196904572	590712619	196904572	590712619	0
		0.7	275666400	669474447	275666400	669474447	0

(*continued*)

Table H-1. Continued.

Problem	Uncertainty budget $\Gamma_i(\bar{\Gamma}_i)$	Flow deviation factor, κ	Bertsimas and Sim (2003, 2004) Cost of robustness	Total cost	The proposed approach Cost of robustness	Total cost	Total cost % difference
CAB15	2(12)	0.1	69238554	2022572238	69238554	2022572238	0
		0.3	207715663	2161049347	207715663	2161049347	0
		0.5	346192772	2299526456	346192772	2299526456	0
		0.7	484669880	2438003564	484669880	2438003564	0
	5(9)	0.1	127481015	2080814699	127481015	2080814699	0
		0.3	382443044	2335776728	382443044	2335776728	0
		0.5	637405073	2590738757	637405073	2590738757	0
		0.7	892367103	2845700787	892367103	2845700787	0
	9(5)	0.1	168937752	2122271436	168937752	2122271436	0
		0.3	506813257	2460146941	506813257	2460146941	0
		0.5	844688762	2798022446	844688762	2798022446	0
		0.7	1182564267	3135897951	1182564267	3135897951	0
	14(0)	0.1	195333368	2148667052	195333368	2148667052	0
		0.3	586000105	2539333789	586000105	2539333789	0
		0.5	976666842	2930000526	976666842	2930000526	0
		0.7	1367333579	3320667263	1367333579	3320667263	0
CAB20	5(14)	0.1	252062018	4624710382	252062018	4624710382	0
		0.3	756186054	5128834418	756186054	5128834418	0
		0.5	1260310090	5632958454	1260310090	5632958454	0
		0.7	1764434126	6137082490	1764434126	6137082490	0
	10(9)	0.1	353778564	4726426928	353778564	4726426928	0
		0.3	1061335692	5433984056	1061335692	5433984056	0
		0.5	1768892821	6141541185	1768892821	6141541185	0
		0.7	2476449949	6849098313	2476449949	6849098313	0
	14(5)	0.1	403683398	4776331762	403683398	4776331762	0
		0.3	1211050194	5583698558	1211050194	5583698558	0
		0.5	2018416991	6391065355	2018416991	6391065355	0
		0.7	2825783787	7198432151	2825783787	7198432151	0
	19(0)	0.1	437264836	4809913200	437264836	4809913200	0
		0.3	1311794509	5684442873	1311794509	5684442873	0
		0.5	2186324182	6558972546	2186324182	6558972546	0
		0.7	3060853855	7433502219	3060853855	7433502219	0
CAB25	5(19)	0.1	334957961	7237422503	334957961	7237422503	0
		0.3	1004858455	7907322997	1004858455	7907322997	0
		0.5	1674791696	8577256238	1674791696	8577256238	0
		0.7	2344664653	9247129195	2344664653	9247129195	0
	14(10)	0.1	574517759	7476982301	574517759	7476982301	0
		0.3	1723553277	8626017819	1723553277	8626017819	0
		0.5	2872696383	9775160925	2872696383	9775160925	0
		0.7	4021647836	10924112378	4021647836	10924112378	0
	20(4)	0.1	660446346	7562910888	660446346	7562910888	0
		0.3	1981298612	8883763154	1981298612	8883763154	0
		0.5	3302267940	10204732482	3302267940	10204732482	0
		0.7	4622987340	11525451882	4622987340	11525451882	0
	24(0)	0.1	690275015	7592739557	690275015	7592739557	0
		0.3	2070777519	8973242061	2070777519	8973242061	0
		0.5	3451402921	10353867463	3451402921	10353867463	0
		0.7	4831742803	11734207345	4831742803	11734207345	0
TR25	5(19)	0.1	698616432	15606339632	698616432	15606339632	0
		0.3	2095849638	17003572838	2095849638	17003572838	0
		0.5	3493083710	18400806910	3493083710	18400806910	0
		0.7	4890318228	19798041428	4890318228	19798041428	0
	14(10)	0.1	1208753583	16116476783	1208753583	16116476783	0
		0.3	3626262807	18533986007	3626262807	18533986007	0
		0.5	6043766723	20951489923	6043766723	20951489923	0
		0.7	8461271882	23368995082	8461271882	23368995082	0
	20(4)	0.1	1407719617	16315442817	1407719617	16315442817	0
		0.3	4223166141	19130889341	4223166141	19130889341	0
		0.5	7038607069	21946330269	7038607069	21946330269	0
		0.7	9854045101	24761768301	9854045101	24761768301	0
	24(0)	0.1	1490772125	16398495325	1490772125	16398495325	0
		0.3	4472320445	19380043645	4472320445	19380043645	0
		0.5	7453862369	22361585569	7453862369	22361585569	0
		0.7	10435402099	25343125299	10435402099	25343125299	0
TR40	15(24)	0.1	531600173	8242443526	531600173	8242443526	0
		0.3	1594824691	9305668044	1594824691	9305668044	0
		0.5	2658044977	10368888330	2658044977	10368888330	0
		0.7	3644599738	11418732180	3644599738	11418732180	0
	20(19)	0.1	616135971	8326979324	616135971	8326979324	0
		0.3	1848421698	9559265051	1848421698	9559265051	0
		0.5	3080707597	10791550950	3080707597	10791550950	0
		0.7	4312985737	12023829090	4312985737	12023829090	0
	30(9)	0.1	722412283	8433255636	722412283	8433255636	0

(continued)

Table H-1. Continued.

Problem	Uncertainty budget $\Gamma_i(\bar{\Gamma}_i)$	Flow deviation factor, κ	Bertsimas and Sim (2003, 2004) Cost of robustness	Total cost	The proposed approach Cost of robustness	Total cost	Total cost % difference
		0.3	2167226964	9878070317	2167226964	9878070317	0
		0.5	3612044937	11322888290	3612044937	11322888290	0
		0.7	5056862917	12767706270	5056862917	12767706270	0
	39(0)	0.1	771085876	8481929229	771085876	8481929229	0
		0.3	2313257627	10024100980	2313257627	10024100980	0
		0.5	3855429377	11566272730	3855429377	11566272730	0
		0.7	5397601127	13108444480	5397601127	13108444480	0

Table H-2. Results for the robust uncapacitated single allocation p-hub location problems.

Problem	Uncertainty budget $\Gamma_i(\bar{\Gamma}_i)$	Flow deviation factor, κ	Bertsimas and Sim (2003, 2004) Cost of robustness	Total cost	The proposed approach Cost of robustness	Total cost	Total cost % difference
CAB10	2(7)	0.1	23078064	552485483	23078064	552485483	0
		0.3	69234192	598641611	69234192	598641611	0
		0.5	115390321	644797740	115390321	644797740	0
		0.7	*143912834*	*686540378*	*161546449*	*690953868*	*0.60*
	4(5)	0.1	36369435	565776854	36369435	565776854	0
		0.3	109108306	638515725	109108306	638515725	0
		0.5	181847177	711254596	181847177	711254596	0
		0.7	*238975559*	*781603103*	*254586047*	*783993466*	*0.30*
	6(3)	0.1	44962955	574370374	44962955	574370374	0
		0.3	134888865	664296284	134888865	664296284	0
		0.5	224814775	754222194	224814775	754222194	0
		0.7	314740685	844148104	314740685	844148104	0
	9(0)	0.1	52940742	582348161	52940742	582348161	0
		0.3	158822226	688229645	158822226	688229645	0
		0.5	264703710	794111129	264703710	794111129	0
		0.7	370585193	899992612	370585193	899992612	0
TR10	2(7)	0.1	15361733	411935974	15361733	411935974	0
		0.3	46085199	442659440	46085199	442659440	0
		0.5	76808665	473382906	76808665	473382906	0
		0.7	107532131	504106372	107532131	504106372	0
	4(5)	0.1	28044314	424618555	28044314	424618555	0
		0.3	84132943	480707184	84132943	480707184	0
		0.5	140221572	536795813	140221572	536795813	0
		0.7	196310200	592884441	196310200	592884441	0
	6(3)	0.1	35204163	431778404	35204163	431778404	0
		0.3	105612490	502186731	105612490	502186731	0
		0.5	176020817	572595058	176020817	572595058	0
		0.7	246429144	643003385	246429144	643003385	0
	9(0)	0.1	39657542	436231783	39657542	436231783	0
		0.3	118972627	515546868	118972627	515546868	0
		0.5	198287711	594861952	198287711	594861952	0
		0.7	277602795	674177036	277602795	674177036	0
CAB15	2(12)	0.1	70831240	2087371119	70831240	2087371119	0
		0.3	212493720	2229033599	212493720	2229033599	0
		0.5	354156200	2370696079	354156200	2370696079	0
		0.7	495818680	2512358559	495818680	2512358559	0
	5(9)	0.1	131072770	2147612649	131072770	2147612649	0
		0.3	393218309	2409758188	393218309	2409758188	0
		0.5	655363848	2671903727	655363848	2671903727	0
		0.7	917509388	2934049267	917509388	2934049267	0
	9(5)	0.1	174376796	2190916675	174376796	2190916675	0
		0.3	523130387	2539670266	523130387	2539670266	0
		0.5	871883978	2888423857	871883978	2888423857	0
		0.7	1220637569	3237177448	1220637569	3237177448	0
	14(0)	0.1	201653988	2218193867	201653988	2218193867	0
		0.3	604961964	2621501843	604961964	2621501843	0
		0.5	1008269940	3024809819	1008269940	3024809819	0
		0.7	1411577916	3428117795	1411577916	3428117795	0
CAB20	5(14)	0.1	261827267	4785815485	261827267	4785815485	0
		0.3	785481801	5309470019	785481801	5309470019	0
		0.5	1309136334	5833124552	1309136334	5833124552	0
		0.7	1832790868	6356779086	1832790868	6356779086	0
	10(9)	0.1	367063799	4891052017	367063799	4891052017	0
		0.3	1101191398	5625179616	1101191398	5625179616	0
		0.5	1835318997	6359307215	1835318997	6359307215	0
		0.7	2569446595	7093434813	2569446595	7093434813	0
	14(5)	0.1	417982063	4941970281	417982063	4941970281	0
		0.3	1253946190	5777934408	1253946190	5777934408	0
		0.5	2089910316	6613898534	2089910316	6613898534	0

(*continued*)

Table H-2. Continued.

			Bertsimas and Sim (2003, 2004)		The proposed approach		
Problem	Uncertainty budget $\Gamma_i(\bar{\Gamma}_i)$	Flow deviation factor, κ	Cost of robustness	Total cost	Cost of robustness	Total cost	Total cost % difference
		0.7	2925874443	7449862661	2925874443	7449862661	0
	19(0)	0.1	452398822	4976387040	452398822	4976387040	0
		0.3	1357196465	5881184683	1357196465	5881184683	0
		0.5	2261994109	6785982327	2261994109	6785982327	0
		0.7	3166791753	7690779971	3166791753	7690779971	0
CAB25	5(19)	0.1	347274518	7481776917	347274518	7481776917	0
		0.3	1041812480	8176314879	1041812480	8176314879	0
		0.5	1736384347	8870886746	1736384347	8870886746	0
		0.7	2430891325	9565393724	2430891325	9565393724	0
	14(10)	0.1	595515021	7730017420	595515021	7730017420	0
		0.3	1786500588	8921002987	1786500588	8921002987	0
		0.5	2977595870	10112098269	2977595870	10112098269	0
		0.7	*4165554418*	*11302899958*	*4168503808*	*11303006207*	*0.0009*
	20(4)	0.1	683133837	7817636236	683133837	7817636236	0
		0.3	2049358863	9183861262	2049358863	9183861262	0
		0.5	3415703903	10550206302	3415703903	10550206302	0
		0.7	4781791240	11916293639	4781791240	11916293639	0
	24(0)	0.1	713479595	7847981994	713479595	7847981994	0
		0.3	2140389662	9274892061	2140389662	9274892061	0
		0.5	3567426884	10701929283	3567426884	10701929283	0
		0.7	4994170330	12128672729	4994170330	12128672729	0
TR25	5(19)	0.1	706521928	15848228568	706521928	15848228568	0
		0.3	2119565334	17261271974	2119565334	17261271974	0
		0.5	3532607030	18674313670	3532607030	18674313670	0
		0.7	4945651714	20087358354	4945651714	20087358354	0
	14(10)	0.1	1246088942	16387795582	1246088942	16387795582	0
		0.3	*3688867036*	*18874216156*	*3738271446*	*18879978086*	*0.03*
		0.5	*6148107180*	*21333456300*	*6230448944*	*21372155584*	*0.18*
		0.7	*8607346569*	*23792695689*	*8722627767*	*23864334407*	*0.30*
	20(4)	0.1	1434498502	16576205142	1434498502	16576205142	0
		0.3	4303499005	19445205645	4303499005	19445205645	0
		0.5	7172493291	22314199931	7172493291	22314199931	0
		0.7	10041488680	25183195320	10041488680	25183195320	0
	24(0)	0.1	1514171250	16655877890	1514171250	16655877890	0
		0.3	4542515087	19684221727	4542515087	19684221727	0
		0.5	7570853270	22712559910	7570853270	22712559910	0
		0.7	10599191580	25740898220	10599191580	25740898220	0
TR40	15(24)	0.1	544440339	8450501562	544440339	8450501562	0
		0.3	1633345597	9539406820	1633345597	9539406820	0
		0.5	2722247487	10628308710	2722247487	10628308710	0
		0.7	*3730491343*	*11694538560*	*3811139667*	*11717200890*	0.19
	20(19)	0.1	631506984	8537568207	631506984	8537568207	0
		0.3	1894520953	9800582176	1894520953	9800582176	0
		0.5	3157532907	11063594130	3157532907	11063594130	0
		0.7	*4360310633*	*12324357850*	*4420540304*	*12326601527*	0.02
	30(9)	0.1	740495389	8646556612	740495389	8646556612	0
		0.3	2221470657	10127531880	2221470657	10127531880	0
		0.5	3702457477	11608518700	3702457477	11608518700	0
		0.7	5183442447	13089503670	5183442447	13089503670	0
	39(0)	0.1	790607698	8696668921	790607698	8696668921	0
		0.3	2371823097	10277884320	2371823097	10277884320	0
		0.5	3953038487	11859099710	3953038487	11859099710	0
		0.7	5534253887	13440315110	5534253887	13440315110	0

Appendix I. Computational results for robust capacitated single and multiple allocation problems

Table I-1. Results for the robust capacitated single allocation p-hub location problems.

Problem	Uncertainty budget $\Gamma_i(\bar{\Gamma}_i)$	α	Flow deviation factor, κ	Total cost	Cost of robustness	Nominal cost	Hubs& capacities	Solving time
CAB10TT	4(5)	0.3	0.1	629246972	37097421	592149551	4(L2); 7(L2); 9(L2)	7
			0.2	667287751	74163977	593123774	4(L2); 7(L2); 9(L2)	11
			0.3	704180619	111056845	593123774	4(L2); 7(L2); 9(L2)	14
		0.5	0.1	702461291	40810850	661650441	4(L2); 7(L2); 9(L2)	7
			0.2	743604485	81490848	662113637	4(L2); 7(L2); 9(L2)	11
			0.3	784104433	121990796	662113637	4(L2); 7(L2); 9(L2)	18
	6(3)	0.3	0.1	639481719	46357945	593123774	4(L2); 7(L2); 9(L2)	4
			0.2	685839665	92715891	593123774	4(L2); 7(L2); 9(L2)	4
			0.3	764823218	143309578	621513640	4(L2); 7(L2); 9(L2)	9
		0.5	0.1	713632423	51518786	662113637	4(L2); 7(L2); 9(L2)	4
			0.2	765151209	103037572	662113637	4(L2); 7(L2); 9(L2)	5
			0.3	850909121	158665481	692243640	4(L2); 7(L2); 9(L2)	8
	9(0)	0.3	0.1	647936151	54812377	593123774	4(L2); 7(L2); 9(L2)	0.3
			0.2	799937862	125822977	674114885	4(L2); 10(L2); 9(L2)	0.7
			0.3	1075770369	237870085	837900284	4(L2); 1(L2); 9(L2)	1.3
		0.5	0.1	723825001	61711364	662113637	4(L2); 7(L2); 9(L2)	0.4
			0.2	905621669	143436945	762184724	4(L2); 10(L2); 9(L2)	0.8
			0.3	1159919603	257289139	902630464	4(L2); 1(L2); 9(L2)	0.95
CAB10LL	4(5)	0.3	0.1	590776854	36369435	554407419	4(L2); 6(L2); 7(L1)	4
			0.2	631367749	72930319	558437430	4(L2); 6(L2); 7(L1)	8
			0.3	667832908	109395478	558437430	4(L2); 6(L2); 7(L1)	15
		0.5	0.1	671207560	40895677	630311883	4(L2); 6(L2); 7(L1)	6
			0.2	716893710	81967012	634926698	4(L2); 6(L2); 7(L1)	17
			0.3	757877215	122950517	634926698	4(L2); 6(L2); 7(L1)	62
	6(3)	0.3	0.1	599370374	44962955	554407419	4(L2); 6(L2); 7(L1)	3
			0.2	648913289	90475859	558437430	4(L2); 6(L2); 7(L1)	5
			0.3	694049395	135611965	558437430	4(L2); 6(L2); 7(L1)	7
		0.5	0.1	681373965	51062082	630311883	4(L2); 6(L2); 7(L1)	6
			0.2	738469811	102832610	635637201	4(L2); 6(L2); 7(L1)	6
			0.3	789886117	154248916	635637201	4(L2); 6(L2); 7(L1)	9
	9(0)	0.3	0.1	607348161	52940742	554407419	4(L2); 6(L2); 7(L1)	1
			0.2	665124916	106687486	558437430	4(L2); 6(L2); 7(L1)	1
			0.3	720336250	160462212	559874038	4(L2); 6(L2); 7(L1)	1
		0.5	0.1	690843071	60531188	630311883	4(L2); 6(L2); 7(L1)	0.3
			0.2	756912038	121985340	634926698	4(L2); 6(L2); 7(L1)	0.5
			0.3	818828361	183191160	635637201	4(L2); 6(L2); 7(L1)	0.7
TR10TT	4(5)	0.3	0.1	479868166	31043954	448824212	1(L2); 2(L2); 4(L1)	78
			0.2	509796389	60972177	448824212	1(L2); 2(L2); 4(L1)	140
			0.3	614621816	103350879	511270937	1(L2); 2(L2); 4(L2)	140
		0.5	0.1	583249807	38259565	544990242	1(L2); 2(L2); 8(L2)	83
			0.2	689202386	74606072	614596314	1(L2); 2(L2); 4(L1)	143
			0.3	768107411	134908946	633198465	1(L2); 2(L2); 8(L2)	148
	6(3)	0.3	0.1	485789826	36965614	448824212	1(L2); 2(L2); 4(L1)	29
			0.2	656580350	95442128	561138222	1(L2); 2(L2); 8(L2)	82
			0.3	704956478	142735141	562221337	1(L2); 2(L2); 8(L2)	32
		0.5	0.1	585385130	45394888	539990242	1(L2); 2(L2); 4(L1)	35
			0.2	740358713	108467351	631891362	1(L2); 2(L2); 8(L2)	95
			0.3	795975208	162776743	633198465	1(L2); 2(L2); 8(L2)	53
	9(0)	0.3	0.1	489706862	40882650	448824212	1(L2); 2(L2); 4(L1)	1
			0.2	604525572	93254635	511270937	1(L2); 2(L2); 4(L2)	1
			0.3	712107382	153948460	558158922	1(L2); 2(L2); 5(L2)	1
		0.5	0.1	594989539	49999297	544990242	1(L2); 2(L2); 4(L1)	1
			0.2	675423373	105904263	569519110	1(L2); 2(L2); 5(L2)	1
			0.3	788121950	171490393	616631557	1(L2); 2(L2); 5(L2)	1
TR10LL	4(5)	0.3	0.1	488173497	33540066	454633431	1(L2); 3(L2); 4(L1)	48
			0.2	521713563	67080132	454633431	1(L2); 3(L2); 4(L1)	450
			0.3	555253629	100620198	454633431	1(L2); 3(L2); 4(L1)	550
		0.5	0.1	597055289	41654864	555400425	1(L2); 3(L2); 4(L1)	334
			0.2	638710153	83309728	555400425	1(L2); 3(L2); 4(L1)	490
			0.3	680365018	124964593	555400425	1(L2); 3(L2); 4(L1)	577
	6(3)	0.3	0.1	493923305	39289874	454633431	1(L2); 3(L2); 4(L1)	38
			0.2	533213180	78579749	454633431	1(L2); 3(L2); 4(L1)	277
			0.3	572503054	117869623	454633431	1(L2); 3(L2); 4(L1)	339
		0.5	0.1	570385130	45394888	524990242	1(L2); 2(L2); 4(L1)	58
			0.2	652860111	97459686	555400425	1(L2); 3(L2); 4(L1)	345
			0.3	701589954	146189529	555400425	1(L2); 3(L2); 4(L1)	469
	9(0)	0.3	0.1	474706862	40882650	433824212	1(L2); 2(L2); 4(L1)	1
			0.2	515589513	81765301	433824212	1(L2); 2(L2); 4(L1)	1
			0.3	556472163	122647951	433824212	1(L2); 2(L2); 4(L1)	1
		0.5	0.1	574989539	49999297	524990242	1(L2); 2(L2); 4(L1)	1
			0.2	624988837	99998595	524990242	1(L2); 2(L2); 4(L1)	1
			0.3	674988134	149997892	524990242	1(L2); 2(L2); 4(L1)	1

(continued)

Table I-1. Continued.

Problem	Uncertainty budget $\Gamma_i(\bar{\Gamma}_i)$	α	Flow deviation factor, κ	Total cost	Cost of robustness	Nominal cost	Hubs& capacities	Solving time
CAB15TT	5(9)	0.3	0.1	2267078086	141429948	2125648138	9(L2); 12(L1); 13(L2)	2272
			0.2	2408508033	282859895	2125648138	9(L2); 12(L1); 13(L2)	26618
			0.3	2545364711	419716573	2125648138	9(L2); 12(L1); 13(L2)	1303
		0.5	0.1	2521259419	151178579	2370080840	9(L2); 12(L1); 13(L2)	18504
			0.2	2753159699	325801004	2427358695	9(L2); 12(L1); 13(L2)	29040
			0.3	2911979159	484620464	2427358695	9(L2); 12(L1); 13(L2)	46443
	9(5)	0.3	0.1	2308917189	183269051	2125648138	9(L2); 12(L1); 13(L2)	36
			0.2	2492186239	366538101	2125648138	9(L2); 12(L1); 13(L2)	1087
			0.3	2745282315	565456191	2179826124	9(L2); 12(L1); 13(L2)	4539
		0.5	0.1	2603003530	204504434	2398499096	4(L2); 12(L1); 13(L2)	621
			0.2	2847828588	420469893	2427358695	9(L2); 12(L1); 13(L2)	2971
			0.3	3138352051	648993381	2489358670	9(L2); 12(L1); 13(L2)	16521
	14(0)	0.3	0.1	2334212952	208564814	2125648138	9(L2); 12(L1); 13(L2)	1
			0.2	2579911928	423318655	2156593273	9(L2); 12(L1); 13(L2)	1
			0.3	2917175479	663963572	2253211907	9(L2); 12(L2); 13(L2)	1
		0.5	0.1	2634349006	235849910	2398499096	4(L2); 12(L1); 13(L2)	1
			0.2	2947557068	484592845	2462964223	4(L2); 12(L1); 13(L2)	1
			0.3	3238915437	739365101	2499550336	4(L2); 5(L1); 7(L1)	1
CAB15LL	5(9)	0.3	0.1	2183206863	132339240	2050867623	4(L2); 12(L1); 13(L1)	1368
			0.2	2388468917	282820779	2105648138	9(L2); 12(L1); 13(L1)	20778
			0.3	2466942762	403403434	2063539328	4(L2); 12(L1); 7(L1)	30922
		0.5	0.1	2458010718	150757968	2307252750	4(L2); 12(L1); 13(L1)	2310
			0.2	2608768686	301515936	2307252750	4(L2); 12(L1); 13(L1)	22666
			0.3	2759390611	452137861	2307252750	4(L2); 12(L1); 13(L1)	41305
	9(5)	0.3	0.1	2227290889	176423266	2050867623	4(L2); 12(L1); 13(L1)	173
			0.2	2403714155	352846532	2050867623	4(L2); 12(L1); 13(L1)	393
			0.3	2585879403	529639372	2056240031	4(L2); 12(L1); 13(L2)	9790
		0.5	0.1	2506255356	199002606	2307252750	4(L2); 12(L1); 13(L1)	180
			0.2	2705257961	398005211	2307252750	4(L2); 12(L1); 13(L1)	583
			0.3	2921037548	599902084	2321135464	4(L2); 12(L1); 13(L2)	1610
	14(0)	0.3	0.1	2258954385	203086762	2055867623	4(L2); 12(L1); 13(L2)	1
			0.2	2462488037	406248006	2056240031	4(L2); 12(L1); 13(L2)	1
			0.3	2665612040	609372009	2056240031	4(L2); 12(L1); 13(L2)	1
		0.5	0.1	2540978025	228725275	2312252750	4(L2); 12(L1); 13(L2)	1
			0.2	2780362557	459227093	2321135464	4(L2); 12(L1); 13(L2)	1
			0.3	3010048698	688857392	2321191306	4(L2); 12(L1); 13(L2)	1
CAB20TT	10(9)	0.3	0.1	4922480535	358492317	4563988218	5(L2); 12(L1); 18(L2)	904
			0.2	5409798481	753251724	4656546757	5(L2); 12(L1); 18(L2)	2401
			0.3	6049225620	1196617238	4852608382	5(L2); 12(L1); 18(L2)	6871
		0.5	0.1	5689621117	416500335	5273120782	4(L2); 12(L1); 17(L2)	874
			0.2	6204293308	867289086	5337004222	4(L2); 12(L1); 18(L2)	2615
			0.3	6880429855	1371179919	5509249936	5(L2); 7(L1); 17(L2)	53287
	14(5)	0.3	0.1	5055087044	421990239	4633096805	4(L2); 12(L1); 17(L2)	744
			0.2	5501195473	844648716	4656546757	4(L2); 12(L1); 18(L2)	1453
			0.3	6195157033	1342548651	4852608382	5(L2); 12(L1); 18(L2)	2353
		0.5	0.1	5828457735	491453513	5337004222	4(L2); 12(L1); 18(L2)	496
			0.2	6306081903	969077681	5337004222	4(L2); 12(L1); 18(L2)	1181
			0.3	7044177310	1534927374	5509249936	5(L2); 7(L1); 17(L2)	1703
	19(0)	0.3	0.1	5118201433	461654676	4656546757	4(L2); 12(L1); 18(L2)	38
			0.2	5677455178	939575863	4737879315	4(L2); 12(L1); 18(L2)	43
			0.3	6322030400	1449699323	4872331077	5(L2); 12(L1); 18(L2)	37
		0.5	0.1	5866704644	529700422	5337004222	4(L2); 12(L1); 18(L2)	55
			0.2	6562626658	1087104443	5475522215	4(L2); 19(L1); 18(L2)	54
			0.3	7127631972	1635607378	5492024594	4(L2); 7(L1); 18(L2)	72
CAB20LL	10(9)	0.3	0.1	4916052017	367063799	4548988218	4(L2); 12(L1); 17(L2)	375
			0.2	5394798481	753251724	4641546757	4(L2); 12(L1); 17(L2)	2490
			0.3	5765953036	1124406279	4641546757	4(L2); 12(L1); 18(L2)	7552
		0.5	0.1	5685723733	427602951	5258120782	4(L2); 12(L1); 17(L2)	1050
			0.2	6189293308	867289086	5322004222	4(L2); 12(L1); 18(L2)	2742
			0.3	6617167084	1295162862	5322004222	4(L2); 12(L1); 18(L2)	44948
	14(5)	0.3	0.1	4966970281	417982063	4548988218	4(L2); 12(L1); 17(L2)	189
			0.2	5497487483	855940726	4641546757	4(L2); 12(L1); 18(L2)	1024
			0.3	5954794172	1289306044	4665488128	4(L2); 12(L1); 18(L2)	1327
		0.5	0.1	5742936140	484815358	5258120782	4(L2); 12(L1); 17(L2)	451
			0.2	6304911248	982907026	5322004222	4(L2); 12(L1); 18(L2)	1156
			0.3	6819641067	1477639907	5342001160	4(L2); 12(L1); 18(L2)	1511
	19(0)	0.3	0.1	5001387040	452398822	4548988218	4(L2); 12(L1); 17(L2)	11
			0.2	5564856108	923309351	4641546757	4(L2); 12(L1); 18(L2)	22
			0.3	6057634566	1392146438	4665488128	4(L2); 12(L1); 18(L2)	18
		0.5	0.1	5781432860	523312078	5258120782	4(L2); 12(L1); 17(L2)	13
			0.2	6381405066	1059400844	5322004222	4(L2); 12(L1); 18(L2)	20
			0.3	6937101508	1595100348	5342001160	4(L2); 12(L1); 18(L2)	21
CAB25TT	14(10)	0.3	0.1	7773730181	598135437	7175594744	4(L2); 12(L1); 18(L2)	2783
			0.2	8371827926	1196233182	7175594744	4(L2); 12(L1); 18(L2)	7055
			0.3	9001208488	1802345401	7198863087	4(L2); 12(L1); 18(L2)	12856
		0.5	0.1	9001038552	695606223	8305432329	4(L2); 12(L1); 18(L2)	6759

(*continued*)

Table I-1. Continued.

Problem	Uncertainty budget $\Gamma_i(\bar{\Gamma}_i)$	α	Flow deviation factor, κ	Total cost	Cost of robustness	Nominal cost	Hubs& capacities	Solving time
			0.2	9696593957	1391161628	8305432329	4(L2); 12(L1); 18(L2)	9311
			0.3	10422818645	2095372192	8327446453	4(L2); 12(L1); 18(L2)	26389
	20(4)	0.3	0.1	7860115300	684520556	7175594744	4(L2); 12(L1); 18(L2)	1223
			0.2	8544580665	1368985921	7175594744	4(L2); 12(L1); 18(L2)	3966
			0.3	9250435915	2051572828	7198863087	4(L2); 12(L1); 18(L2)	2553
		0.5	0.1	9098734663	793302334	8305432329	4(L2); 12(L1); 18(L2)	3545
			0.2	9891968482	1586536153	8305432329	4(L2); 12(L1); 18(L2)	4576
			0.3	10706793966	2379347513	8327446453	4(L2); 12(L1); 18(L2)	4340
	24(0)	0.3	0.1	7889183282	713588538	7175594744	4(L2); 12(L1); 18(L2)	185
			0.2	8602705844	1427111100	7175594744	4(L2); 12(L1); 18(L2)	251
			0.3	9427007242	2166260973	7260746269	4(L2); 12(L1); 18(L2)	331
		0.5	0.1	9132008611	826576282	8305432329	4(L2); 12(L1); 18(L2)	257
			0.2	9958507162	1653074833	8305432329	4(L2); 12(L1); 18(L2)	350
			0.3	10914662745	2509569441	8405093304	4(L2); 12(L1); 18(L2)	446
CAB25LL	14(10)	0.3	0.1	7752966124	597371380	7155594744	4(L2); 12(L1); 18(L1)	4392
			0.2	8356827926	1196233182	7160594744	4(L2); 12(L1); 18(L2)	5380
			0.3	8953187953	1792593209	7160594744	4(L2); 12(L1); 18(L2)	9193
		0.5	0.1	8985182427	694750098	8290432329	4(L2); 12(L1); 18(L2)	3650
			0.2	9681593957	1391161628	8290432329	4(L2); 12(L1); 18(L2)	8504
			0.3	10375202648	2084770319	8290432329	4(L2); 12(L1); 18(L2)	15276
	20(4)	0.3	0.1	7845115300	684520556	7160594744	4(L2); 12(L1); 18(L2)	3330
			0.2	8529580665	1368985921	7160594744	4(L2); 12(L1); 18(L2)	2884
			0.3	9208861262	2049358863	7159502399	4(L2); 12(L1); 17(L2)	2782
		0.5	0.1	9083734663	793302334	8290432329	4(L2); 12(L1); 18(L2)	4437
			0.2	9876968482	1586536153	8290432329	4(L2); 12(L1); 18(L2)	4445
			0.3	10699877345	2387430892	8312446453	4(L2); 12(L1); 18(L2)	5702
	24(0)	0.3	0.1	7872981994	713479595.2	7159502399	4(L2); 12(L1); 17(L2)	190
			0.2	8586393902	1426891503	7159502399	4(L2); 12(L1); 17(L2)	194
			0.3	9299892061	2140389662	7159502399	4(L2); 12(L1); 17(L2)	196
		0.5	0.1	9117008611	826576282	8290432329	4(L2); 12(L1); 18(L2)	240
			0.2	9943507162	1653074833	8290432329	4(L2); 12(L1); 18(L2)	233
			0.3	10798722121	2486275668	8312446453	4(L2); 12(L1); 18(L2)	237
TR25TT	14(10)	0.3	0.1	16469659882	1240096852	15229563030	3(L2); 15(L2); 19(L1)	7017
			0.2	17985633287	2530052257	15455581030	3(L2); 15(L2); 19(L2)	26091
			0.3	19550558947	3869305167	15681253780	3(L2); 15(L2); 19(L2)	44822
		0.5	0.1	18804093101	1408016281	17396076820	1(L2); 3(L2); 15(L2)	10406
			0.2	20471734744	2863063054	17608671690	1(L2); 3(L2); 15(L2)	19093
			0.3	22176681883	4361782223	17814899660	1(L2); 3(L2); 15(L2)	43349
	20(4)	0.3	0.1	16715913557	1438737497	15277176060	3(L2); 15(L2); 19(L1)	4716
			0.2	18417791951	2919680281	15498111670	3(L2); 15(L2); 19(L2)	5878
			0.3	20241671250	4489192670	15752478580	1(L2); 12(L2); 17(L2)	5866
		0.5	0.1	19084850987	1644712967	17440138020	1(L2); 3(L2); 15(L2)	4820
			0.2	20973834436	3329572206	17644262230	1(L2); 3(L2); 15(L2)	5300
			0.3	22882302936	5055014766	17827288170	1(L2); 3(L2); 15(L2)	4890
	24(0)	0.3	0.1	16800894175	1523718115	15277176060	3(L2); 15(L2); 19(L1)	93
			0.2	18808506779	3127252999	15681253780	3(L2); 15(L2); 19(L2)	153
			0.3	20445641759	4707842649	15737799110	3(L2); 15(L2); 19(L2)	171
		0.5	0.1	19179652021	1739514001	17440138020	1(L2); 3(L2); 15(L2)	100
			0.2	21368882183	3553982523	17814899660	1(L2); 3(L2); 15(L2)	131
			0.3	23189317840	5340999780	17848318060	1(L2); 3(L2); 15(L2)	150
TR25LL	14(10)	0.3	0.1	16407795582	1246088942	15161706640	3(L1); 15(L2); 19(L1)	8172
			0.2	17653887086	2492180446	15161706640	3(L1); 15(L2); 19(L1)	10659
			0.3	18899978086	3738271446	15161706640	3(L1); 15(L2); 19(L1)	33358
		0.5	0.1	18738721258	1413682418	17325038840	1(L1); 3(L1); 15(L2)	6383
			0.2	20152411338	2827372498	17325038840	1(L1); 3(L1); 15(L2)	12365
			0.3	21566095153	4241056313	17325038840	1(L1); 3(L1); 15(L2)	35162
	20(4)	0.3	0.1	16596205142	1434498502	15161706640	3(L1); 15(L2); 19(L1)	1854
			0.2	18030706613	2868999973	15161706640	3(L1); 15(L2); 19(L1)	2211
			0.3	19465205644	4303499004	15161706640	3(L1); 15(L2); 19(L1)	3144
		0.5	0.1	18966417159	1641378319	17325038840	1(L1); 3(L1); 15(L2)	1920
			0.2	20607801754	3282762914	17325038840	1(L1); 3(L1); 15(L2)	2503
			0.3	22249182901	4924144061	17325038840	1(L1); 3(L1); 15(L2)	2842
	24(0)	0.3	0.1	16675877890	1514171250	15161706640	3(L1); 15(L2); 19(L1)	66
			0.2	18190050727	3028344087	15161706640	3(L1); 15(L2); 19(L1)	73
			0.3	19714221726	4542515086	15171706640	3(L2); 15(L2); 19(L2)	73
		0.5	0.1	19055543301	1730504461	17325038840	1(L1); 3(L1); 15(L2)	58
			0.2	20786049837	3461010997	17325038840	1(L1); 3(L1); 15(L2)	76
			0.3	22516555132	5191516292	17325038840	1(L1); 3(L1); 15(L2)	78
TR30LL	19(10)	0.3	0.1	6563356470	531423695.3	6031932775	3(L2); 11(L2); 15(L1)	10007
			0.2	7094794910	1062862135	6031932775	3(L2); 11(L2); 15(L1)	19159
			0.3	7626215641	1594282866	6031932775	3(L2); 11(L2); 15(L1)	35849
		0.5	0.1	7493379152	610362208	6883016944	1(L1); 3(L2); 15(L1)	13516
			0.2	8103750720	1220733776	6883016944	1(L1); 3(L2); 15(L1)	24126
			0.3	8719325510	1831182621	6888142889	1(L2); 3(L2); 15(L1)	38208
	25(4)	0.3	0.1	6614605347	582672572	6031932775	3(L2); 11(L2); 15(L1)	7650
			0.2	7197283303	1165350528	6031932775	3(L2); 11(L2); 15(L1)	24294

(*continued*)

Table I-1. Continued.

Problem	Uncertainty budget $\Gamma_i(\bar{\Gamma}_i)$	α	Flow deviation factor, κ	Total cost	Cost of robustness	Nominal cost	Hubs& capacities	Solving time
			0.3	7779943544	1748010769	6031932775	3(L2); 11(L2); 15(L1)	22436
		0.5	0.1	7549654770	666637826	6883016944	1(L1); 3(L2); 15(L1)	10961
			0.2	8216296166	1333279222	6883016944	1(L1); 3(L2); 15(L1)	15982
			0.3	8887917835	1999900891	6888016944	1(L2); 3(L2); 15(L1)	19901
	29(0)	0.3	0.1	6632627199	600694424	6031932775	1(L2); 3(L2); 15(L1)	382
			0.2	7233324415	1201391640	6031932775	11(L2); 3(L2); 15(L1)	784
			0.3	7834007394	1802074619	6031932775	11(L2); 3(L2); 15(L1)	801
		0.5	0.1	7569321416	686304472	6883016944	1(L1); 3(L2); 15(L1)	827
			0.2	8255626172	1372609228	6883016944	1(L1); 3(L2); 15(L1)	360
			0.3	8946917014	2058900070	6888016944	1(L2); 3(L2); 15(L1)	488
TR30TT	19(10)	0.3	0.1	6578356470	531423695.3	6046932775	3(L2); 11(L2); 15(L1)	14012
			0.2	7109794910	1062862135	6046932775	3(L2); 11(L2); 15(L1)	24912
			0.3	7641215641	1594282866	6046932775	3(L2); 11(L2); 15(L1)	42685
		0.5	0.1	7513379152	610362208	6903016944	1(L2); 3(L2); 15(L1)	13535
			0.2	8123750720	1220733776	6903016944	1(L2); 3(L2); 15(L1)	22360
			0.3	8735157666	1831443618	6903714048	1(L2); 3(L2); 15(L1)	49939
	25(4)	0.3	0.1	6629605347	582672572	6046932775	3(L2); 11(L2); 15(L1)	8496
			0.2	7212283303	1165350528	6046932775	3(L2); 11(L2); 15(L1)	13398
			0.3	7798922709	1748789048	6050133661	3(L2); 11(L2); 15(L1)	24104
		0.5	0.1	7569654770	666637826	6903016944	1(L2); 3(L2); 15(L1)	9888
			0.2	8236296166	1333279222	6903016944	3(L2); 11(L2); 15(L1)	15633
			0.3	8903557000	1999842952	6903714048	3(L2); 11(L2); 15(L1)	33880
	29(0)	0.3	0.1	6647627199	600694424	6046932775	3(L2); 11(L2); 15(L1)	775
			0.2	7248324415	1201391640	6046932775	3(L2); 11(L2); 15(L1)	926
			0.3	7853168416	1803034755	6050133661	3(L2); 11(L2); 15(L1)	1125
		0.5	0.1	7589321416	686304472	6903016944	1(L2); 3(L2); 15(L1)	881
			0.2	8275626172	1372609228	6903016944	1(L2); 3(L2); 15(L1)	933
			0.3	8975973609	2062143640	6913829969	1(L2); 3(L2); 15(L1)	1260
TR40LL	25(14)	0.3	0.1	8625980394	694919171.7	7931061223	1(L1),3,19,22,35	ECT
			0.2	9371258298	1385973699	7985284599	1(L1),3,19,22, 31	ECT
			0.3	10011486147	2080424924	7931061223	1(L2),3(L1),19,22,35	ECT
		0.5	0.1	10349961253	839283754.5	9510677498	1(L1),3,12,22,24	ECT
			0.2	11189254527	1678577029	9510677498	1(L1),3,12,22,24	ECT
			0.3	12028535253	2517857755	9510677498	1(L1),3,12,22,24	ECT
	30(9)	0.3	0.1	8671556612	740495389.2	7931061223	1(L1),3,19,22,35	50633
			0.2	9471849884	1486565285	7985284599	1(L1),3,19,22,31	49468
			0.3	10152531885	2221470662	7931061223	1(L1),3,19,22,35	47490
		0.5	0.1	10403593289	892915790.5	9510677498	1(L1),3,12,22,24	51682
			0.2	11296511378	1785833880	9510677498	1(L1),3,12,22,24	49468
			0.3	12199801241	2680788332	9519012909	1(L2),3(L1),12,22,24	ECT
	39(0)	0.3	0.1	8721667703	790606480.7	7931061223	1(L1),3,19,22,35	3554
			0.2	9512284072	1581222850	7931061223	1(L1),3,19,22,35	3812
			0.3	10302879964	2371818741	7931061223	1(L2),3(L1),19,22,35	4002
		0.5	0.1	10459244605	948567107	9510677498	1(L2),3(L1),12,22,24	3379
			0.2	11407826143	1897148645	9510677498	1(L2),3(L1),12,22,24	3336
			0.3	12356379443	2845701945	9510677498	1(L1),3,12,22,24	3512

Table I-2. Results for the robust capacitated multiple allocation p-hub location problems.

Problem	Uncertainty budget $\Gamma_i(\bar{\Gamma}_i)$	α	Flow deviation factor, κ	Total cost	Cost of robustness	Nominal cost	Hubs& capacities	Solving time
CAB10TT	4(5)	0.3	0.1	615846936	36814241	579032695	4(L2); 6(L2); 7(L2)	186
			0.2	654395097	72922963	581472134	4(L2); 7(L2); 9(L2)	8884
			0.3	694390053	110199861	584190192	4(L2); 7(L2); 9(L2)	34183
		0.5	0.1	661758586	38079027	623679559	4(L2); 6(L2); 7(L2)	7922
			0.2	702129655	76534621	625595034	4(L2); 7(L2); 9(L2)	25229
			0.3	746442236	116196994	630245242	4(L2); 7(L2); 9(L2)	43032
	6(3)	0.3	0.1	628787414	45926336	582861078	4(L2); 6(L2); 7(L2)	750
			0.2	673087927	90998309	582089618	4(L2); 7(L2); 9(L2)	1380
			0.3	722896535	137491953	585404582	4(L2); 7(L2); 9(L2)	12328
		0.5	0.1	672206591	48284195	623922396	4(L2); 6(L2); 7(L2)	713
			0.2	724168247	97181300	626986947	4(L2); 7(L2); 9(L2)	2859
			0.3	783115366	148094386	635020980	4(L2); 7(L2); 9(L2)	55572
	9(0)	0.3	0.1	634128508	53557137	580571371	4(L2); 9(L2); 7(L2)	1
			0.2	690851429	107641905	583209524	4(L2); 9(L2); 7(L2)	1
			0.3	879940217	192678512	687261706	2(L2); 9(L2); 10(L2)	1
		0.5	0.1	682635986	57966908	624669078	4(L2); 7(L2); 9(L2)	1
			0.2	750537647	117589608	632948040	4(L2); 9(L2); 7(L2)	1
			0.3	945886892	207896975	737989917	2(L2); 9(L2); 10(L2)	1
CAB10LL	4(5)	0.3	0.1	578177331	35442181	542735150	4(L2); 6(L2); 7(L1)	370
			0.2	613619512	70884362	542735150	4(L2); 6(L2); 7(L1)	732
			0.3	649061693	106326543	542735150	4(L2); 6(L2); 7(L1)	1462
		0.5	0.1	628140142	37549089	590591053	4(L2); 6(L2); 7(L1)	328
			0.2	665689230	75098177	590591053	4(L2); 6(L2); 7(L1)	6574
			0.3	703238319	112647266	590591053	4(L2); 6(L2); 7(L1)	2740
	6(3)	0.3	0.1	586711737	43976587	542735150	4(L2); 6(L2); 7(L1)	220
			0.2	630688324	87953174	542735150	4(L2); 6(L2); 7(L1)	438
			0.3	674664911	131929761	542735150	4(L2); 6(L2); 7(L1)	1996
		0.5	0.1	638211444	47620391	590591053	4(L2); 6(L2); 7(L1)	184
			0.2	685831834	95240781	590591053	4(L2); 6(L2); 7(L1)	2954
			0.3	733452225	142861172	590591053	4(L2); 6(L2); 7(L1)	25143
	9(0)	0.3	0.1	594508665	51773515	542735150	4(L2); 6(L2); 7(L1)	1
			0.2	646282180	103547030	542735150	4(L2); 6(L2); 7(L1)	1
			0.3	698533916	155430904	543103012	4(L2); 6(L2); 7(L1)	1
		0.5	0.1	647150158	56559105	590591053	4(L2); 6(L2); 7(L1)	1
			0.2	703709264	113118211	590591053	4(L2); 6(L2); 7(L1)	1
			0.3	760268369	169677316	590591053	4(L2); 6(L2); 7(L1)	1
CAB20TT	10(9)	0.3	0.1	4762308613	349422425	4412886188	4(L2); 12(L1); 17(L2)	483
			0.2	5236644758	728863915	4507780843	4(L2); 12(L1); 18(L2)	45557
			0.3	5680069305	1111182576	4568886729	4(L2); 12(L1); 18(L2)	47344
		0.5	0.1	5281088608	372529882	4908558726	4(L2); 12(L1); 18(L2)	559
			0.2	5773218410	801711769	4971506641	4(L2); 12(L1); 18(L2)	855
			0.3	6227068554	1217142628	5009925926	4(L2); 12(L1); 18(L2)	12367
	14(5)	0.3	0.1	4801506836	387962843	4801506836	4(L2); 12(L1); 17(L2)	300
			0.2	5333008712	823344003	4509664709	4(L2); 12(L1); 17(L2)	550
			0.3	5967706499	1293359884	4674346615	5(L2); 12(L1); 18(L2)	745
		0.5	0.1	5347891928	437265503	4910626425	4(L2); 12(L1); 17(L2)	305
			0.2	5878811126	907304485	4971506641	4(L2); 12(L1); 17(L2)	435
			0.3	6574538277	1430103850	5144434427	5(L2); 12(L1); 18(L2)	9998
	19(0)	0.3	0.1	4860955633	438268694	4422686939	4(L2); 12(L1); 17(L2)	9
			0.2	5404916741	894152790	4510763951	4(L2); 12(L1); 18(L2)	8
			0.3	6073333813	1392307803	4681026010	5(L2); 12(L1); 18(L2)	12
		0.5	0.1	5401008110	487364374	4913643736	4(L2); 12(L1); 17(L2)	7
			0.2	5957807969	986301328	4971506641	4(L2); 12(L1); 18(L2)	9
			0.3	6675031269	1531161062	5143870207	4(L2); 7(L1); 18(L2)	15
CAB20LL	10(9)	0.3	0.1	4739297839	341649475	4397648364	4(L2); 12(L1); 17(L2)	416
			0.2	5111251874	704372049	4406879825	4(L2); 12(L1); 17(L2)	498
			0.3	5586568133	1093363591	4493204542	4(L2); 12(L1); 18(L2)	15956
		0.5	0.1	5269709067	379828172	4889880895	4(L2); 12(L1); 17(L2)	344
			0.2	5664570357	764702323	4899868034	4(L2); 12(L1); 17(L2)	640
			0.3	6165976747	1209470106	4956506641	4(L2); 12(L1); 18(L2)	14190
	14(5)	0.3	0.1	4790972364	393324000	4397648364	4(L2); 12(L1); 17(L2)	307
			0.2	5229639496	789308778	4440330718	4(L2); 12(L1); 17(L2)	580
			0.3	5739033455	1242954371	4496079084	4(L2); 12(L1); 18(L2)	6533
		0.5	0.1	5838783776	900197681	4938586095	4(L2); 12(L1); 17(L2)	301
			0.2	5838783776	900197681	4938586095	4(L2); 12(L1); 17(L2)	564
			0.3	6327556724	1371050083	4956506641	4(L2); 12(L1); 18(L2)	21315
	19(0)	0.3	0.1	4834913200	437264836	4397648364	4(L2); 12(L1); 17(L2)	9
			0.2	5381755584	892792597	4488962987	4(L2); 12(L1); 17(L2)	11
			0.3	5840407018	1342017004	4498390014	4(L2); 12(L1); 18(L2)	8
		0.5	0.1	5376368985	486488090	4889880895	4(L2); 12(L1); 17(L2)	9
			0.2	5942807969	986301328	4956506641	4(L2); 12(L1); 18(L2)	9
			0.3	6439656080	1480305249	4959350831	4(L2); 12(L1); 18(L2)	8
CAB25LL	14(10)	0.3	0.1	7494853982	567389440.4	6927464542	4(L1); 12(L2); 17(L2)	2320
			0.2	8109233266	1167170902	6942062364	12(L1); 17(L2); 21(L2)	ECT
			0.3	8692687245	1750624881	6942062364	12(L1); 17(L2); 21(L2)	25932

(*continued*)

Table I-2. Continued.

Problem	Uncertainty budget $\Gamma_i(\bar{\Gamma}_i)$	α	Flow deviation factor, κ	Total cost	Cost of robustness	Nominal cost	Hubs& capacities	Solving time
		0.5	0.1	8400089398	636722679.4	7763366719	4(L1); 12(L1); 18(L2)	2616
			0.2	9064753054	1297590668	7767162386	4(L2); 12(L1); 18(L2)	47972
			0.3	9831903833	1989463043	7842440790	12(L1); 18(L2); 21(L2)	ECT
	20(4)	0.3	0.1	7569979753	642515210.7	6927464542	4(L2); 12(L1); 17(L2)	1859
			0.2	8252450742	1310388378	6942062364	12(L1); 17(L2); 21(L2)	1665
			0.3	8905331723	1977867181	6927464542	4(L2); 12(L1); 17(L2)	2572
		0.5	0.1	8511792439	728997936.4	7782794503	4(L1); 12(L1); 17(L2)	1839
			0.2	9240034564	1469707013	7770327551	4(L2); 12(L1); 17(L2)	1940
			0.3	9992892278	2222837768	7770054510	4(L2); 12(L1); 17(L2)	2542
	24(0)	0.3	0.1	7617739557	690275014.5	6927464542	4(L2); 12(L1); 17(L2)	13
			0.2	8307949688	1380485146	6927464542	4(L2); 12(L1); 17(L2)	12
			0.3	8998242061	2070777519	6927464542	4(L2); 12(L1); 17(L2)	12
		0.5	0.1	8541520730	774256891.8	7767263838	4(L2); 12(L1); 18(L2)	13
			0.2	9316333992	1548547729	7767786263	4(L2); 12(L1); 18(L2)	12
			0.3	10093612959	2323558449	7770054510	4(L2); 12(L1); 17(L2)	11

ECT: Extended Computational Time ($\geq$54,000 s.

Table I-3. Total cost comparison: robust capacitated *vs*. robust uncapacitated HLPs.

Problem	Uncertainty budget $\Gamma_i(\bar{\Gamma}_i)$	Flow deviation factor, κ	Uncapacitated		Capacitated		Total cost % difference
			Total cost	Cost of robustness	Total cost	Cost of robustness	
CAB15LL	5(9)	0.1	2147612649	131072770	2183206863	132339240	1.63
		0.3	2409758188	393218309	2466942762	403403434	2.32
	9(5)	0.1	2190916675	174376796	2227290889	176423266	1.63
		0.3	2539670266	523130387	2585879403	529639372	1.79
CAB15TT	5(9)	0.1	2147612649	131072770	2267078086	141429948	5.27
		0.3	2409758188	393218309	2545364711	419716573	5.33
	9(5)	0.1	2190916675	174376796	2308917189	183269051	5.11
		0.3	2539670266	523130387	2745282315	565456191	7.49
CAB20LL	10(9)	0.1	4891052017	367063799	4916052017	367063799	0.51
		0.3	5625179616	1101191398	5765953036	1124406279	2.44
	14(5)	0.1	4941970281	417982063	4966970281	417982063	0.50
		0.3	5777934408	1253946190	5954794172	1289306044	2.97
CAB20TT	10(9)	0.1	4891052017	367063799	4922480535	358492317	0.64
		0.3	5625179616	1101191398	6049225620	1196617238	7.01
	14(5)	0.1	4941970281	417982063	5055087044	421990239	2.24
		0.3	5777934408	1253946190	6195157033	1342548651	6.73
CAB25LL	14(10)	0.1	7730017420	595515021	7752966124	597371380	0.30
		0.3	8921002987	1786500588	8953187953	1792593209	0.36
	20(4)	0.1	7817636236	683133837	7845115300	684520556	0.35
		0.3	9183861262	2049358863	9208861262	2049358863	0.27
CAB25TT	14(10)	0.1	7730017420	595515021	7773730181	598135437	0.56
		0.3	8921002987	1786500588	9001208488	1802345401	0.89
	20(4)	0.1	7817636236	683133837	7860115300	684520556	0.54
		0.3	9183861262	2049358863	9250435915	2051572828	0.72
TR25LL	14(10)	0.1	16387795582	1246088942	16407795582	1246088942	0.12
		0.3	18874216156	3688867036	18899978086	3738271446	0.14
	20(4)	0.1	16576205142	1434498502	16596205142	1434498502	0.12
		0.3	19445205645	4303499005	19465205644	4303499004	0.10
TR25TT	14(10)	0.1	16387795582	1246088942	16469659882	1240096852	0.50
		0.3	18874216156	3688867036	19550558947	3869305167	3.46
	20(4)	0.1	16576205142	1434498502	16715913557	1438737497	0.84
		0.3	19445205645	4303499005	20241671250	4489192670	3.93

Table I-4. Solutions to the deterministic capacitated single allocation p-hub location problems.

Problem	Discount factor, α	Nominal cost	Hubs& capacities
CAB10TT	0.3	592149551	4(L2), 7(L2), 9(L2)
	0.5	661650441	4(L2), 7(L2), 9(L2)
CAB10LL	0.3	554407419	4(L2), 6(L2), 7(L1)
	0.5	630311883	4(L2), 6(L2), 7(L1)
TR10TT	0.3	448824212	1(L2), 2(L2), 4(L1)
	0.5	539990242	1(L2), 2(L2), 4(L1)
TR10LL	0.3	430250386	2(L2), 4(L1), 5(L2)
	0.5	517943149	2(L2), 4(L1), 5(L2)
CAB15TT	0.3	2098865033	4(L2), 12(L1), 13(L2)
	0.5	2367835599	4(L2), 12(L1), 13(L2)
CAB15LL	0.3	2036539879	4(L2), 7(L1), 12(L1)
	0.5	2284459310	4(L2), 7(L1), 12(L1)
CAB20TT	0.3	4563988218	4(L2), 12(L1), 17(L2)
	0.5	5273120782	4(L2), 12(L1), 17(L2)
CAB20LL	0.3	4548988218	4(L2), 12(L1), 17(L2)
	0.5	5258120782	(L2), 12(L1), 17(L2)
CAB25TT	0.3	7174502399	4(L2), 12(L1), 17(L2)
	0.5	8305432329	4(L2), 12(L1), 17(L2)
CAB25LL	0.3	7155594744	4(L2), 12(L1), 18(L1)
	0.5	8285432329	4(L2), 12(L1), 18(L1)
TR25TT	0.3	15176706640	3(L1),15(L2), 19(L1)
	0.5	17345038840	1(L2), 3(L1), 15(L2)
TR25LL	0.3	15161706640	3(L1),15(L2), 19(L1)
	0.5	17325038840	1(L1), 3(L1), 15(L2)

Table I-5. The impact of objective weight (w).

Problem	$\Gamma_i(\bar{\Gamma}_i)$	Objective weight, w	Total cost	Cost of robustness	Nominal cost	Hubs& capacities
CAB10TT	6(3)	0.05	764823218	143309578	621513640	4(L2); 7(L2);9(L2)
		0.3	720901100	127777326	593123774	4(L2); 7(L2);9(L2)
		0.7	720901100	127777326	593123774	4(L2); 7(L2);9(L2)
CAB10LL	6(3)	0.05	694049395	135611965	558437430	4(L2); 6(L2);7(L1)
		0.3	694049395	135611965	558437430	4(L2); 6(L2);7(L1)
		0.7	694049395	135611965	558437430	4(L2); 6(L2);7(L1)
CAB15TT	9(5)	0.1	2745282315	565456191	2179826124	9(L2); 12(L1);13(L2)
		0.3	2711413766	554820493	2156593273	9(L2); 12(L1);13(L2)
		0.7	2623513307	497865169	2125648138	9(L2); 12(L1);13(L2)
CAB15LL	9(5)	0.1	2585879403	529639372	2056240031	4(L2); 12(L1);13(L2)
		0.3	2585879403	529639372	2056240031	4(L2); 12(L1);13(L2)
		0.7	2585879403	529639372	2056240031	4(L2); 12(L1);13(L2)
CAB20TT	14(5)	0.2	6195157033	1342548651	4852608382	5(L2); 12(L1);18(L2)
		0.3	6195157033	1342548651	4852608382	4(L2); 12(L1);18(L2)
		0.7	5737380403	1080833646	4656546757	4(L2); 12(L1);18(L2)
CAB20LL	14(5)	0.2	5954794172	1289306044	4665488128	4(L2); 12(L1);18(L2)
		0.3	5954794172	1289306044	4665488128	4(L2); 12(L1);18(L2)
		0.7	5892293062	1250746305	4641546757	4(L2); 12(L1);18(L2)
CAB25TT	20(4)	0.25	9250435915	2051572828	7198863087	4(L2); 12(L1);18(L2)
		0.30	9250435915	2051572828	7198863087	4(L2); 12(L1);18(L2)
		0.70	9250435915	2051572828	7198863087	4(L2); 12(L1);18(L2)
CAB25LL	20(4)	0.25	9208861262	2049358863	7159502399	4(L2); 12(L1);17(L2)
		0.30	9208861262	2049358863	7159502399	4(L2); 12(L1);17(L2)
		0.70	9208861262	2049358863	7159502399	4(L2); 12(L1);17(L2)

Appendix J. Further details on robust solution to the capacitated CAB20 test problem

Table J-1. CAB20 robust capacitated solution ($\Gamma_i(\bar{\Gamma}_i) = 10(9)$): Routes with highest additional transportation cost.

Node i / Node j	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	–		1835176	3779850			4095424			2725344		2602178	2351477	12676365		3218237	8557221	1841982		
2	1210620	–	1445170	2074435			1214783	1277951		1422565		1950635		2496126		1455584	1344666			
3			–	7199791		2770222	2181997	2610141	3317126			6631795		9200395			22066259	3099901		2168976
4			7199791	–			6743122	10953787	7608340		6747742	16122251		18922379	13194078		25575618		5459197	
5	525954		764803	1460865	–		739057	506263	514685			1012012		2502395			1807055			528957
6			2770222	5066889		–	1537140	1376285	1452949			3427550		6575390	1240117		8634218	1752110		
7	4095424			6743122			–	6499241		17203174	4035344	11167259		4703763		6058100	9228428		2862813	
8			2610141	10953787			6499241	–		4171730	5205086	15003981		2325856	4481192		9205143		5264723	
9			3317126	7608340			2008069	2222391	–			5430222		10441160	2504666		11440116	2356095		2118913
10	2725344			5386830			17203174	4171730		–	1852077	7492616		5302861		9486171	9414832	1850921		
11				6747742			4035344	5205086	1175754	1852077	–	5044776		2097337	2614560		4000505		1417118	
12			6631795	16122251			11167259	15003981	5430222	7492616	5044776	–		7838666	6193165		25428496			
13	2351477			2559758			2487168		920163	1510968	865230	1585785	–	1538691		1960497	2046605			
14	10626809		9200395	18922379		6575390			10441160			7838666		–		5330742	56203257	7684456		5544764
15			2007048	13194078			1954534	4481192	2504666		2614560	6193165		2953495	–		6730251		2261435	
16	3218237	1455584		4114432			6058100	1340416		9486171		3815953	1960497	5330742		–	6139429			
17	8557221		22066259	25575618		8634218	9228428	9205143	11440116	9414832		25428496		56203257			–			
18	1841982		3099901	4574517			2146690	1910376	2356095	1850921		4322858		7684456				–		2479069
19			1169918	5459197			2862813	5264723	1498647	1153808	1417118	3416759			2261435		3882223		–	
20	1264127		2168976	5366074				1168165	2118913	1402998		3222084		5544764			6777841	2479069		–