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Dynamic output feedback adaptive integral sliding mode predictive control based on a dual-loop Kalman filter

Journal Title
XX(X):2–30
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sagepub.co.uk/journalsPermissions.nav
DOI: 10.1177/ToBeAssigned
www.sagepub.com/

SAGE

Huili Xiao¹, Fei Li¹, Dongya Zhao^{*1}, Zidong Liu², Xing-Gang Yan³, Sarah K. Spurgeon⁴

Abstract

This paper investigates a dynamic output feedback adaptive integral sliding mode predictive control (AISMPC) approach based on a dual-loop Kalman filter (DLKF) for linear discrete-time systems. The considered system class is subjected to bounded disturbances and noise, with constraints on both the inputs and states. The DLKF introduces one loop to estimate the states and another loop to estimate the disturbances. The dynamic output feedback AISMPC consists of two parts: one component is an output feedback adaptive integral sliding mode control (AISMPC) to handle disturbances, and the other component is an output feedback robust model predictive control (RMPC) to achieve optimal control with constraints. It is proved that the closed-loop system is stabilized in a neighborhood of the origin. Finally, numerical simulation tests are carried out to validate the effectiveness of the proposed approach.

Keywords

Adaptive integral sliding mode control; robust model predictive control; integral sliding mode predictive control; dual-loop Kalman filter; output feedback control.

Introduction

In industrial process control applications, many control systems are typically required to achieve both robust stability and optimal performance under various constraints. In order to achieve robust stability and optimal performance in the presence of uncertainties and constraints, scholars have developed many control algorithms, such as, robust control (Postlethwaite et al. (2007); Kangzhi and Y. (2016)), optimal control (Xiaojing et al. (2017); Athans and Falb (2003)) and adaptive control (Anderson and Dehghani (2008); Joshi et al. (2011)). These methods can be used for design and implementation of control systems when processes are affected by dynamic properties, constraints and external disturbances. Such existing control methods often require robustness-performance tradeoffs when dealing with disturbances and constraints, especially for dynamic environments and complex systems. It is of interest to explore more flexible and efficient control strategies to overcome such limitations.

Sliding mode control (SMC) (Edwards and Spurgeon (1998); Incremona et al. (2017)) is a classical robust control method, whose main advantage is strong robustness to model uncertainty and external disturbances. With an appropriately designed sliding mode control, the system can achieve stability within a quasi-sliding mode domain even in the presence of model uncertainty and external disturbances. With a classical sliding mode control, this robustness is achieved once the so called reaching phase is complete. Integral SMC (ISMC) (Khalid et al. (2007); Fridman et al. (2014)) guarantees that the system trajectories will start on the sliding manifold from the initial time instant. It eliminates the reaching phase of classical SMC and provides better robustness. However, the drawback of ISMC is that constraint handling in general is nontrivial. Robust model predictive control (RMPC) (Ding et al. (2008); He et al. (2022)) is an optimization approach that has become one of the most widely used control techniques due to its ability to explicitly handle constraints. To satisfy the constraints and accommodate disturbances, solutions have been proposed that embed RMPC into ISMC to form a new integral sliding mode predictive control (ISMPC) (Rubagotti et al. (2011b); Yiwen and Hedrick (2015)). This approach inherits the strong robustness properties of ISMC while incorporating the advantages of RMPC in handling constrained optimization problems (Xiao et al. (2022)).

In recent years, ISMPC has been widely studied (Rubagotti et al. (2021); Farbood et al. (2022); Mingdong et al. (2023); Héctor et al. (2024)) and successfully

¹College of New Energy, China University of Petroleum (East China), Qingdao, China

²The Department of Mechanical Engineering, City University of Hong Kong, Hong Kong, China.

³School of Engineering, University of Kent, Canterbury, United Kingdom

⁴Department of Electronic & Electrical Engineering, University College London, London, United Kingdom

Corresponding author:

Dongya Zhao, College of New Energy, China University of Petroleum (East China), Qingdao, China

Email: dyzhao@upc.edu.cn

applied to practical systems (Nicolis et al. (2020); Hongjian et al. (2023); Hongqi et al. (2024)). However, most existing studies assume that the system state can be directly measured, a premise that is difficult to achieve in many practical engineering applications. In reality, many systems only provide output signals, while state variables need to be obtained through indirect estimation. Common estimation methods include the Luenberger observer and the Kalman filter (KF), both of which play a critical role in addressing the lack of direct state measurements. Given the importance of state estimation in practical systems, researchers have proposed an observer-based output feedback ISMPC strategy (Huili and Dongya (2020); Steinberger et al. (2019)) to handle cases where direct state measurements are unavailable. This strategy uses a state estimator to reconstruct the system state and enables effective feedback control based only on output measurements. Although the ISMPC based on the Luenberger observer has a simple structure Huili and Dongya (2020), the paradigm does not consider external disturbances and noise in its design, which limits its robustness in the presence of system uncertainties. In contrast, the KF is widely recognized as an effective approach for achieving optimal state estimation under noisy conditions. It minimizes error covariance effectively and is particularly well-suited for handling stochastic noise. However, the traditional KF adopts a single-loop architecture, making it difficult to handle state estimation and disturbance compensation simultaneously. This limitation is especially evident in the presence of strong disturbances, where estimation performance tends to degrade. Incorporating disturbances as augmented states can partially compensate for their effects, but it also increases system dimensionality, adds computational complexity, and may compromise the observability of the augmented system. The traditional KF exhibits degraded performance in the presence of disturbances, mainly due to its single-loop structure, which hinders effective separation of state and disturbance components. Moreover, its reliance on accurate system models reduces its robustness to uncertainties in practical applications. To overcome these limitations, the Dual-Loop Kalman Filter (DLKF) (Bae and Oh (2017); Jingchao et al. (2022)) has been proposed. The DLKF uses two independent filtering loops: one for estimating the state and the other for estimating disturbances. The disturbance estimates from the disturbance loop are fed back into the state estimation loop, effectively compensating for disturbances and significantly enhancing state estimation accuracy. Unlike the augmented Kalman Filter (AKF) (Vettori et al. (2023)), the DLKF uses two independent loops for state and disturbance estimation. This improves estimation flexibility and allows more accurate disturbance estimation. In the AKF, disturbances are estimated as part of the system state, which can cause confusion between the state and disturbances. The dual-loop structure of the DLKF resolves this issue. Although the DLKF improves estimation performance, it still inevitably results in the presence of some estimation error. Furthermore, in industrial processes, state and input variables are often subject to physical or operational constraints. These constraints restrict the degrees of freedom of the estimator and can reduce estimation accuracy, thus amplifying errors. In the presence of significant disturbances or model uncertainties, accumulated estimation errors can severely impact the overall control performance of the system. In summary, when states cannot be directly measured and the system simultaneously encounters disturbances, noise, and constraints,

effectively managing estimation errors and enhancing control system robustness remain critical research challenges. To address these issues, this paper proposes a dynamic output-feedback adaptive integral sliding mode predictive control (AISMPC) method based on the DLKF, aiming to improve control accuracy and robustness under complex operating conditions. This method leverages the DLKF framework to independently estimate system states and disturbances. In the control loop, a novel adaptive integral sliding mode control (AISM) is developed by embedding an adaptive law into the ISMC. The controller can dynamically adjust the control gain and estimate the state and disturbance error boundaries in real time, enhancing robustness and stability while ensuring system states remain within the quasi-sliding mode domain. With this foundation, an RMPC can be designed on a system where disturbance effects have been reduced, achieving optimal control performance while satisfying any system constraints.

Based on the above analysis, the main contributions of the proposed dynamic output feedback AISMPC are summarized as follows:

(i) The proposed control strategy is based on the AISM concept and incorporates an adaptive law that enables real-time estimation of the state and disturbance error bounds. Compared to the traditional ISMC-based approach in [Xiao et al. \(2024\)](#), the proposed method relaxes prior assumptions on error bounds, thereby enhancing robustness in uncertain environments.

(ii) Compared to the method in [Roy et al. \(2020\)](#); [Haifeng et al. \(2021\)](#), this study introduces several important extensions and enhancements. Firstly, the proposed controller is designed for systems with unmeasurable states, whereas the method in [Roy et al. \(2020\)](#); [Haifeng et al. \(2021\)](#) assumes full state availability. Secondly, the DLKF is employed to achieve independent estimate disturbances, which significantly improves estimation accuracy and enhances the controller's disturbance compensation capability. Finally, the proposed controller can ensure optimal performance while satisfying state and input constraints, thereby extending the applicability of the method.

(iii) Compared to existing dynamic output feedback RMPC methods ([Xubin \(2017\)](#); [Xubin et al. \(2020\)](#)), the proposed dynamic output feedback AISMPC exhibits enhanced robustness and control performance in systems affected by both external disturbances and noise.

This paper is arranged as follows: Section II describes the ISMPC with state estimation problem, and gives the basic assumptions. In Section III, a DLKF is used to estimate the states and disturbances, followed by the design of the dynamic output feedback AISMPC. Section IV proves the recursive feasibility of the optimization problem and the stability of the closed-loop system. Section V gives simulation and is followed by conclusions in Section VI.

Notation 1. Let $\mathbf{R}$ denote the set of real numbers. For any vector x and positive-definite matrix W , $\|x\|_W^2 \triangleq x^T W x$. $x(i|k)$ is denoted as the value of x at time $k+i$ predicted at time k , and $x(0|k)$ is the current value of x at time k . $\Upsilon(M, \pi) \triangleq \{\zeta | \zeta^T M \zeta \leq \pi\}$ is denoted as the associated ellipsoidal set, where M is a positive-definite matrix and scalar $\pi > 0$. For brevity, $\Upsilon(M, 1)$ is also denoted by $\Upsilon(M)$.

Table 1
ABBREVIATION LIST.

Full name	Abbreviation
Dynamic output-feedback adaptive integral sliding mode predictive control	AISMPC
Integral sliding mode predictive control	ISMPC
Kalman filter	KF
Dual-loop Kalman filter	DLKF
Robust model predictive control	RMPC
Sliding mode control	SMC
Integral sliding mode control	ISM
Adaptive integral sliding mode control	AISM
Robust positively invariant	RPI
Luenberger observer-based ISMPC	L-ISMPC
Observer-based RMPC	O-RMPC
Linear matrix inequality	LMI
Integral of time and squared error	ITSE

Problem formulation

Consider a linear discrete-time system subject to bounded disturbances and noise described by

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) + d(k) + w(k) \\ y(k) &= Cx(k) + v(k) \end{aligned} \quad (1)$$

where $x(k) \in \mathbf{X} \subseteq \mathbf{R}^n$, $u(k) \in \mathbf{U} \subseteq \mathbf{R}^m$, $y(k) \in \mathbf{Y} \subseteq \mathbf{R}^p$ and $d(k) \in \Upsilon(P_d) \subseteq \mathbf{R}^n$ ($m \leq p < n$) respectively represent the state vector, control input, system output and external disturbance. $w(k) \in \Upsilon(P_w) \subseteq \mathbf{R}^n$ and $v(k) \in \Upsilon(P_v) \subseteq \mathbf{R}^p$ are the process noise and measurement noise with zero mean and covariance matrices $E(w(k)w^T(k)) = Q$ and $E(v(k)v^T(k)) = R$, respectively. Compared with $d(k)$, $w(k)$ and $v(k)$ are random noise that cannot be observed. $A \in \mathbf{R}^{n \times n}$, $B \in \mathbf{R}^{n \times m}$ and $C \in \mathbf{R}^{p \times n}$ are the system matrices, where $\text{rank}(B) = m$ and $\text{rank}(C) = p$. It is assumed that the system disturbances are unknown but bounded in magnitude such that

$$\|d(k)\| \leq \rho, \rho \geq 0$$

where ρ denotes the upper bound on the disturbances.

The following assumptions are imposed on the system (1).

Assumption 1. (*Edwards and Spurgeon (1998)*) The pair (A, B) is controllable and the pair (A, C) is observable.

Assumption 2. (*Edwards and Spurgeon (1998)*) The triple (A, B, C) is minimum phase.

The dynamic output feedback AISMPC takes the following form:

$$u(k) = u_{AISM}(k) + u_{RMPC}(k) \quad (2)$$

where $u_{AISM}(k) \in \mathbf{U}^I$ represents the AISMPC employed to deal with disturbances. $u_{RMPC}(k) \in \mathbf{U}^M$ denotes the RMPC utilized to obtain optimal control while satisfying constraints. The detailed descriptions of $\mathbf{U}^I$ and $\mathbf{U}^M$ can be found in Section 3.3.

When the states are unmeasured, the first component from (2) is proposed as

$$u_{AISM}(k) = -T_\varepsilon(k)\text{sgn}(s(k)) - T_\alpha s(k) - T_d \hat{d}(k) \quad (3)$$

where $T_\varepsilon(k) \in \mathbf{R}^{m \times m}$, $T_\alpha \in \mathbf{R}^{m \times m}$ and $T_d \in \mathbf{R}^{m \times n}$ are matrices to enforce the sliding motion and $s(k)$ is the integral sliding surface, $\hat{d}(k)$ is the estimation disturbances.

The second component from (2) is assumed to have the form

$$u_{RMPC}(k) = F_1 \hat{x}(k) \quad (4)$$

where $F_1 \in \mathbf{R}^{m \times n}$ is the output feedback gain matrix, $\hat{x}(k)$ is the estimated state. The detailed design of the dynamic output feedback AISMPC is given in the next sections.

The main objective of this work is to design a dynamic output feedback AISMPC for system (1), which can effectively enhance the robustness of the system under bounded disturbance $d(k)$, process noise $w(k)$ and measurement noise $v(k)$. Meanwhile, the robust stability of the closed-loop system is guaranteed while ensuring input constraints and state constraints are satisfied.

Design of the dynamic output feedback AISMPC

In this section, a DLKF is introduced in which one loop is used to estimate the state and the other loop is used to estimate the disturbances. It is shown that both state and disturbance estimation errors are bounded. Subsequently, an output feedback AISMPC is developed to deal with the disturbances, efficiently estimate the state and disturbance errors and guarantee that the system state stays within the quasi-sliding mode domain. Finally, the output feedback RMPC is derived based on the S-procedure and the robust positively invariant (RPI) set. **The overall structure of the proposed DLKF-based AISMPC method is illustrated in the block diagram of Fig. 1.**

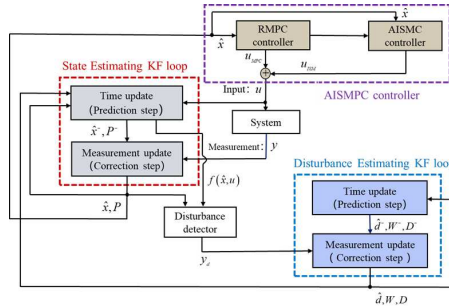


Fig. 1. Block diagram of AISMPC method based on DLKF.

In Fig. 1, $\hat{d}(k)$ is first obtained using the disturbance estimate from the KF. This estimate is fed back to the state estimation KF, thus correcting the error caused by

disturbance and noise in the prediction step and improving the accuracy of the state estimate. Next, the state estimation KF is used to obtain the final state estimate $\hat{x}(k)$ and the system model-based prediction value $f(\hat{x}, u) = A\hat{x}(k) + Bu(k) + \hat{d}(k)$. These two values are combined in the disturbance detector to produce the disturbance measurement y_d , which will be used in the correction step of the disturbance estimation KF. Meanwhile, $\hat{x}(k)$ is introduced into the AISMPC strategy to design the AISMPC controller based on the adopted RMPC and AISMC paradigm. This controller is applied to the system to obtain the output measurement y . The value y is then fed back to the state estimation KF to correct the estimated state, ultimately ensuring the desired closed-loop operation of the system is achieved.

Estimation of states and disturbances

For system (1), when the state information is unavailable and the bounded disturbance is unknown, a DLKF of the following form is designed:

- State-estimating Kalman filter (KF) loop

Prediction step:

$$\hat{x}^-(k+1) = A\hat{x}(k) + Bu(k) + \hat{d}(k) \quad (5)$$

$$P^-(k+1) = AP(k)A^T + AD(k) + D^T(k)A^T + W(k) \quad (6)$$

Update step:

$$K(k+1) = P^-(k+1)C^T(CP^-(k+1)C^T + R)^{-1} \quad (7)$$

$$\hat{x}(k+1) = \hat{x}^-(k+1) + K(k+1)(y(k+1) - C\hat{x}^-(k+1)) \quad (8)$$

$$P(k+1) = (I - K(k+1)C)P^-(k+1) \quad (9)$$

- Disturbance-estimating KF loop

Prediction step:

$$\hat{d}^-(k+1) = A_d\hat{d}(k) \quad (10)$$

$$W^-(k+1) = A_dW(k)A_d^T + Q_d \quad (11)$$

$$D^-(k+1) = AD(k)A_d^T + W(k)A_d^T \quad (12)$$

Update step:

$$Z(k+1) = (D^-(k+1))^T C^T K^T(k+1) (K(k+1)CP^-(k+1)C^T K^T(k+1) + K(k+1)RK^T(k+1) + R_d)^{-1} \quad (13)$$

$$\hat{d}(k+1) = \hat{d}^-(k+1) + Z(k+1)(y_d(k+1) - \hat{d}^-(k+1)) \quad (14)$$

$$\begin{aligned} W(k+1) = & W^-(k+1) - Z(k+1)K(k+1)CD^-(k+1) \\ & - (D^-(k+1))^T C^T K^T(k+1)Z^T(k+1) \\ & + Z(k+1)P^-(k+1)C^T K^T(k+1)Z^T(k+1) \\ & + Z(k+1)R_dZ^T(k+1) \end{aligned} \quad (15)$$

$$\begin{aligned}
D(k+1) = & (I - K(k+1)C)D^-(k+1) \\
& - (I - K(k+1)C)P^-(k+1)C^TK^T(k+1)Z^T(k+1) \\
& + K(k+1)RK^T(k+1)Z^T(k+1)
\end{aligned} \quad (16)$$

where $P(k)$ is the state estimation error covariance matrix, $K(k)$ is the state Kalman gain, $W(k)$ is the disturbance estimation error covariance matrix, $D(k)$ is the estimation error cross-covariance matrix, $Z(k)$ is the disturbance Kalman gain, $y_d(k+1) = \hat{x}(k+1) - A\hat{x}(k) - Bu(k) - \hat{d}(k)$ is the disturbance measurement, A_d is the transition matrix used for model prediction in disturbance estimating KF, Q_d is the disturbance estimating KF process noise covariance matrix and R_d is the disturbance estimating KF measurement noise covariance matrix.

Let $e(k)$ and $e_d(k)$ denote the state estimation error and the disturbance estimation error respectively. They are defined as follows

$$\begin{aligned}
e(k) & \triangleq x(k) - \hat{x}(k) \\
e_d(k) & \triangleq d(k) - \hat{d}(k)
\end{aligned} \quad (17)$$

From (17), it can be obtained that:

$$e(k+1) = (A - K(k+1)CA)e(k) + (I - K(k+1)C)e_d(k) + w(k) \quad (18)$$

To derive the disturbance estimation error system, the process model and measurement model of the disturbance are defined as follows:

$$\begin{aligned}
d(k+1) & = A_d d(k) + w_d(k) \\
d_y(k) & = d(k) + v_d(k)
\end{aligned} \quad (19)$$

where $A_d \in \mathbf{R}^{n \times n}$ and $d_y(k) \in \mathbf{R}^n$ are respectively the transition matrix and output of the disturbance system, $w_d(k) \in \Upsilon(P_{w_d})$ is the process noise of the disturbance model, and $v_d(k) \in \Upsilon(P_{v_d})$ is the measurement noise of the disturbance estimation KF.

In the process of disturbance estimation, it is assumed that there is no prior information available about the disturbance, so A_d is chosen to be an identity matrix (Bae and Oh (2017); Jingchao et al. (2022)). According to (17) and (19), it follows that

$$\begin{aligned}
e_d(k+1) = & (I - Z(k+1)K(k+1)C)e_d(k) \\
& - Z(k+1)K(k+1)CAe(k) + w_d(k)
\end{aligned} \quad (20)$$

Based on (18) and (20), the following augmented error system can be established:

$$\begin{bmatrix} e(k+1) \\ e_d(k+1) \end{bmatrix} = \begin{bmatrix} \Xi_a & \Xi_c \\ -\Xi_b & \Xi_d \end{bmatrix} \begin{bmatrix} e(k) \\ e_d(k) \end{bmatrix} + \begin{bmatrix} w(k) \\ w_d(k) \end{bmatrix}$$

where $\Xi_a = A - K(k+1)CA$, $\Xi_b = Z(k+1)K(k+1)CA$, $\Xi_c = I - K(k+1)C$ and $\Xi_d = I - Z(k+1)K(k+1)C$.

Let the augmented error $z(k) = [e^T(k) \ e_d^T(k)]^T$ and the augmented noise $h(k) = [w^T(k) \ w_d^T(k)]^T$. It follows that

$$z(k+1) = \Xi_z z(k) + h(k) \quad (21)$$

where $\Xi_z = \begin{bmatrix} \Xi_a & \Xi_c \\ -\Xi_b & \Xi_d \end{bmatrix}$. Considering that $w(k) \in \Upsilon(P_w)$ and $w_d(k) \in \Upsilon(P_{w_d})$, the vector $h(k) \in \Upsilon(P_h, 2)$ is obtained, in which $P_h = \text{diag}\{P_w, P_{w_d}\}$.

The following lemma 1 establishes the boundedness of the estimation errors, a result that is critical for the subsequent controller design. Although the stability of the KF is a well-established result (Jazwinski (1970)), it does not provide an explicit upper bound for the estimation error. Such an explicit bound is essential for ensuring sliding mode switching conditions, determining adaptive gain parameters, and performing Lyapunov stability analysis within the AISMC framework.

Lemma 1. For system (21), suppose that at time $k \geq 0$, $h(k) \in \Upsilon(P_h, 2)$ and Assumptions 1 and 2 are satisfied. If there exists a symmetric positive-definite Lyapunov matrix P_z^{-1} and a positive scalar $\tau \in (0, 1]$ satisfying the following inequality

$$\begin{bmatrix} (\tau - 1)P_z & 0 & P_z \Xi_z^T \\ 0 & -\tau P_h & I \\ \Xi_z P_z & I & -P_z \end{bmatrix} \leq 0 \quad (22)$$

then (21) is quadratically bounded, and the norm of the augmented error has an upper bound expressed by

$$\|z(k)\| \leq [\varsigma_k(\tau) \lambda_{\max}(P_z)]^{1/2}$$

where the sequence $\varsigma_k(\tau)$ is defined as

$$\varsigma_k(\tau) = (1 - \tau)^k z^T(0) P_z^{-1} z(0) + 2 - 2(1 - \tau)^k \quad (23)$$

Moreover, $\varsigma_k(\tau) \rightarrow 2$ for $k \rightarrow \infty$. Therefore, the $e(k)$ and $e_d(k)$ are bounded.

Proof: See Appendix A.

Design of the output feedback AISMC

The integral sliding surface for system (1) is defined based on the DLKF estimator of (5)-(16):

$$\begin{aligned} s(k) &= G\hat{x}(k) + \sigma(k) \\ \sigma(k+1) &= \sigma(k) + G(\hat{x}(k) - A\hat{x}(k) - Bu_{MPC}(k)) \\ \sigma(0) &= -G\hat{x}(0) \end{aligned} \quad (24)$$

where $s(k) = [s_1(k), s_2(k), \dots, s_m(k)]^T$, $G \in R^{m \times n}$ is a matrix and $GB > 0$.

From (5), (8) and (24), it follows that

$$\begin{aligned} s(k+1) - s(k) &= G\hat{x}(k+1) + \sigma(k+1) - G\hat{x}(k) - \sigma(k) \\ &= GBu_{AISMC}(k) + G\hat{d}(k|k) + f_e(k) \end{aligned} \quad (25)$$

where $f_e(k) = GK(k+1)CAe(k) + GK(k+1)Ce_d(k)$.

According to Lemma 1, both $e(k)$ and $e_d(k)$ are bounded, but their exact bounds remain unknown. Here, e^* and e_d^* denote their upper bounds, respectively, and the following inequality holds (Haifeng et al. (2021)):

$$\begin{aligned} \|f_e(k)\| &\leq \|GK(k+1)CA\| \|e(k)\| + \|GK(k+1)C\| \|e_d(k)\| \\ &\leq g_0 e^* + g_1 e_d^* = h_0^* + h_1^* \end{aligned} \quad (26)$$

where $g_0 = \|GK(k+1)CA\|$, $g_1 = \|GK(k+1)C\|$, $h_0^* \geq 0$ and $h_1^* \geq 0$. Since e^* and e_d^* are unknown, h_0^* and h_1^* are also unknown.

Based on (25) and (26), an adaptive reaching law is proposed:

$$s(k+1) - s(k) = -TK_a(k) \operatorname{sgn}(s(k)) - \alpha s(k) + Tf_e(k) \quad (27)$$

where the parameters $0 < \alpha < 1$, T is the sampling time, and the adaptive adjustment parameter $K_a(k)$ is defined as

$$K_a(k) = \hat{h}_0(k) + \hat{h}_1(k) \quad (28)$$

Let, $\Delta \hat{h}_0(k) = \hat{h}_0(k+1) - \hat{h}_0(k)$, $\Delta \hat{h}_1(k) = \hat{h}_1(k+1) - \hat{h}_1(k)$. Then, the adaptive update laws are as follows:

$$\begin{aligned} \Delta \hat{h}_0(k) &= \begin{cases} Tm_0 \|s(k)\|, & \text{if } \|s(k)\| > \|s(k-1)\| \\ Tm_0 (\|s(k)\| - \beta_0 \hat{h}_0(k)), & \text{if } \|s(k)\| \leq \|s(k-1)\| \end{cases} \\ \Delta \hat{h}_1(k) &= \begin{cases} Tm_1 \|s(k)\|, & \text{if } \|s(k)\| > \|s(k-1)\| \\ Tm_1 (\|s(k)\| - \beta_1 \hat{h}_1(k)), & \text{if } \|s(k)\| \leq \|s(k-1)\| \end{cases} \end{aligned} \quad (29)$$

where the gains m_0 , m_1 , β_0 and β_1 are positive definite constants.

Based on (25) and (27), an AISMC controller is designed:

$$u_{AISM}(k) = -(GB)^{-1} [TK_a(k) \operatorname{sgn}(s(k)) + \alpha s(k) + G\hat{d}(k)] \quad (30)$$

Theorem 1. Consider system (1). Suppose that Assumptions 1 and 2 are satisfied. If the adaptive reaching law (27) and the controller (30) are adopted, then the system trajectories will enter into the quasi-sliding mode domain:

$$\Gamma_s = \left\{ s(k) \mid \|s(k)\| \leq \sqrt{\frac{T(\beta_0(h_0^*)^2 + \beta_1(h_1^*)^2)}{4\alpha(1-\alpha)}} \right\}$$

and once the system state enters this domain, it will remain within the domain and will not escape.

Proof: See Appendix B.

This design (30) enhances system robustness against external disturbances while maintaining control accuracy. It is crucial to distinguish this approach from traditional adaptive control methods. Traditional adaptive control typically reconstructs the control model through online identification of system parameters (e.g., model reference adaptive control). In contrast, the proposed design focuses on the dynamic estimation of the error bound, making its core goal the assurance of the system's robustness rather than adaptation to system dynamics via online model parameter estimation.

From (24), $s(k+1) - s(k) = 0$, and from (25), the equivalent control is

$$u_{eq}(k) = -(GB)^{-1} \left[GK(k+1)C(Ae(k) + e_d(k)) + G\hat{d}(k) \right] \quad (31)$$

Substituting (31) into (8) yields:

$$\begin{aligned} \hat{x}(k+1) &= A\hat{x}(k) + Bu_{RMPC}(k) + EK(k+1)CAe(k) \\ &\quad + EK(k+1)Ce_d(k) + E\hat{d}(k) \end{aligned}$$

where $E = (I - B(GB)^{-1}G)$.

To minimize the norm of the matrix E , define $G = B^+$ (Rubagotti et al. (2011a)). With this, the dynamical system can be rewritten as:

$$\begin{aligned} \hat{x}(k+1) &= A\hat{x}(k) + Bu_{RMPC}(k) + E_m K(k+1)CAe(k) \\ &\quad + E_m K(k+1)Ce_d(k) + E_m \hat{d}(k) \end{aligned} \quad (32)$$

where $E_m = (I - BB^+)$.

The dynamical system (32) is an equivalent system for the system (8). Note that the robustness to disturbances is achieved from the initial time. This implies that the AISMC can effectively handle external disturbances. Therefore, the RMPC will be designed based on a system in which the impact of disturbances has been ameliorated.

Design of the output feedback RMPC

In order to achieve optimal performance for the controller (2) under the constraints, an estimated state-based RMPC (4) is obtained for the dynamical system (32).

Define $\chi(k) = \bar{\Psi}x(k)$, where $\bar{\Psi} \in \mathbf{R}^{f \times n}$. Then, the constraints on the input and state are bounded in the sets $\mathbf{U}$ and $\mathbf{X}$ respectively, which can be expressed as follows:

$$\begin{aligned} \mathbf{U} &\triangleq \{u(k) \mid -\bar{u}_i \leq u_i(k) \leq \bar{u}_i\}, \\ \mathbf{X} &\triangleq \{x(k) \mid -\bar{\psi}_j \leq \chi_j(k) \leq \bar{\psi}_j\} \end{aligned} \quad (33)$$

where $u(k) = [u_1(k), u_2(k), \dots, u_m(k)]^T$, $i \in \{1, \dots, m\}$, $\bar{u}_i > 0$, $[\bar{u}_1, \bar{u}_2, \dots, \bar{u}_m]^T = \bar{u}$, $\chi(k) = [\chi_1(k), \chi_2(k), \dots, \chi_f(k)]^T$, $\bar{\psi}_j > 0$, $j \in \{1, \dots, f\}$, $[\bar{\psi}_1, \bar{\psi}_2, \dots, \bar{\psi}_f]^T = \bar{\psi}$, and $\mathbf{U}$ and $\mathbf{X}$ are polytopic sets as detailed in Nikolaos et al. (2014).

By Lemma 1, given that $\|e_d(k)\| \leq \rho_d$, $\rho_d > 0$, it follows that $\|\hat{d}(k)\| \leq \rho_d + \rho = \rho_{\hat{d}}$. According to Theorem 1, $\|s(k)\| \leq \sqrt{\frac{T(\beta_0(h_0^*)^2 + \beta_1(h_1^*)^2)}{4\alpha(1-\alpha)}} = \rho_s$. Therefore, combining with (30), $\mathbf{U}^I$ can be defined as follows:

$$\mathbf{U}^I = \{u_{AISM}(k) \mid -T_{ti}(k) \leq u_{AISMi}(k) \leq T_{ti}(k)\}$$

where $[T_{t1}(k), T_{t2}(k), \dots, T_{tm}(k)]^T = T_t(k) = T_a(k) + T_s + T_{\hat{d}}$, $T_a(k) = -(GB)^{-1}TK_a(k)$, $T_s = -(GB)^{-1}\alpha\rho_s$, $T_{\hat{d}} = -(GB)^{-1}G\rho_{\hat{d}}$.

It follows that

$$\mathbf{U}^M = \mathbf{U} \oplus \mathbf{U}^I$$

Define the augmented state $\tilde{x}(k) = [\hat{x}^T(k) \ e^T(k)]^T$ and the augmented disturbance $\tilde{d}(k) = [e_d^T(k) \ \tilde{d}^T(k)]^T$. Based on (4), (18) and (32), at time $k+i$, the augmented closed-loop system is given by:

$$\begin{aligned} \tilde{x}(i+1|k) &= \tilde{A}\tilde{x}(i|k) + \tilde{D}\tilde{d}(i|k) + \tilde{W}w(i|k) \\ &= \tilde{A}\tilde{x}(i|k) + \tilde{E}\omega(k+i) \end{aligned} \quad (34)$$

where $\tilde{A} = \begin{bmatrix} A + BF_1 & E_m K(k+1)CA \\ 0 & A - K(k+1)CA \end{bmatrix}$, $\tilde{E} = [\tilde{D} \ \tilde{W}]$, $\tilde{W} = \begin{bmatrix} 0 \\ I \end{bmatrix}$, $\tilde{D} = \begin{bmatrix} E_m K(k+1)C & E_m \\ I - K(k+1)C & 0 \end{bmatrix}$ and $\omega(k) = [\tilde{d}(k) \ w(k)]^T$. Considering that $\tilde{d}(k) \in \Upsilon(P_{\tilde{d}})$ and $w(k) \in \Upsilon(P_w)$, then $\omega(k+i) \in \Upsilon(P_\omega, 2) \subseteq R^{2n}$, where $P_\omega = \text{diag}\{P_{\tilde{d}}, P_w\}$.

The system output and input can be rewritten as:

$$\begin{aligned} y(i|k) &= \tilde{C}\tilde{x}(i|k) + v(k+i) \\ u_{RMPC}(i|k) &= F\tilde{x}(i|k) \end{aligned} \quad (35)$$

where $\tilde{C} = [C \ 0]$, $F = [F_1 \ 0]$.

OF-RMPC is derived in the following theorem.

Theorem 2. For system (34), assume that at time $k \geq 0$, $e(k) \in \Upsilon(P_e^{-1})$, where P_e is known. If there exists a scalar $\eta(k) \in (0, 1)$ such that (36) and (37) hold, then the augmented state $\tilde{x}(k) \in \Upsilon(\tilde{Q}^{-1}, 2)$. For all possible $\tilde{x}(i|k) \in \Upsilon(\tilde{Q}^{-1}, 2)$ and $\omega(k+i) \in \Upsilon(P_\omega, 2)$, $i \geq 0$, if there exist non-negative scalars $\{\beta_1, \beta_2\}$, a positive scalar $\gamma > 0$, positive definite matrices $Y > 0$, Y_1 , F_1 and weighting matrices H_1 , H_2 , then the feedback gain $F_1 = Y_1\tilde{Q}_1^{-1}$ in (4) can be obtained from the solution of the following (LMIs):

$$\begin{bmatrix} -2 + \eta(k) & \hat{x}^T(k) \\ \hat{x}(k) & -\tilde{Q}_1 \end{bmatrix} \leq 0 \quad (36)$$

$$\begin{bmatrix} -\eta(k)P_e & P_e \\ P_e & -\tilde{Q}_2 \end{bmatrix} \leq 0 \quad (37)$$

$$\begin{bmatrix} -\beta_1 \tilde{Q} & 0 & 0 & \tilde{Q} \tilde{A}^T & \tilde{Q} \tilde{C}^T H_1^{\frac{1}{2}} & Y^T H_2^{\frac{1}{2}} \\ 0 & -\beta_2 P_\omega & 0 & \tilde{E}^T & 0 & 0 \\ 0 & 0 & 0 & 0 & H_1^{\frac{1}{2}} & 0 \\ \tilde{A} \tilde{Q} & \tilde{E} & 0 & -\tilde{Q} & 0 & 0 \\ H_1^{\frac{1}{2}} \tilde{C} \tilde{Q} & 0 & H_1^{\frac{1}{2}} & 0 & -\gamma I & 0 \\ H_2^{\frac{1}{2}} Y & 0 & 0 & 0 & 0 & -\gamma I \end{bmatrix} \leq 0 \quad (38)$$

$$2\beta_2 + \beta_1 - 1 \leq 0, 0 \leq \beta_1 \leq 1, \beta_2 \geq 0 \quad (39)$$

$$\begin{bmatrix} -\tilde{Q}_1 & 0 & \sqrt{2} Y_1^T \\ 0 & -\tilde{Q}_2 & 0 \\ \sqrt{2} Y_1 & 0 & -\tilde{Z} \end{bmatrix} \leq 0, \quad \tilde{Z}_{ss} \leq \tilde{u}_s^2, s \in \{1, \dots, m\} \quad (40)$$

$$\begin{bmatrix} -\tilde{Q}_1 & 0 & 0 & 2(Y_1^T B^T + \tilde{Q}_1 A^T) \bar{\Psi}^T \\ 0 & -\tilde{Q}_2 & 0 & \tilde{\Delta}^T \\ 0 & 0 & -P_\omega & 2(E_m^T - I) \bar{\Psi}^T \\ 2\bar{\Psi}(A\tilde{Q}_1 + BY_1) & \tilde{\Delta} & 2\bar{\Psi}(E_m - I) & -\tilde{\Xi} \end{bmatrix} \leq 0, \quad \tilde{\Xi}_{gg} \leq \bar{\psi}_g^2, g \in \{1, \dots, n\} \quad (41)$$

where $Y = \begin{bmatrix} Y_1 & 0 \end{bmatrix}$, $\tilde{Q} = \text{diag} \{ \tilde{Q}_1, \tilde{Q}_2 \}$, $\tilde{Q}_1$ and $\tilde{Q}_2$ are positive-definite matrices, $\tilde{\Delta} = 2\bar{\Psi}A\tilde{Q}_2 - 2\bar{\Psi}(E_m - I)K(k+1)CA\tilde{Q}_2$.

Proof: See Appendix C.

According to Theorem 2, the following optimization problem can be solved to minimize the scalar γ :

$$\min_{\gamma, \eta, \beta_1, \beta_2, \tilde{Q}_1, \tilde{Q}_2, Y_1} \gamma \quad \text{s.t. (36) - (41)} \quad (42)$$

Remark 1. In Theorem 2, $\tilde{x}(k) \in \Upsilon(\tilde{Q}^{-1}, 2)$ is denoted as the augmented state $\tilde{x}(k)$ can be steered by the controller (78) into the RPI set $\Upsilon(\tilde{Q}^{-1}, 2)$, at each time k , and it also enables future augmented states to evolve within the RPI set $\Upsilon(\tilde{Q}^{-1}, 2)$. This is crucial to ensure the recursive feasibility of the optimization problem and the robust stability of the closed-loop system. Lemma 1 shows that $e(k)$ is bounded, but it does not converge to zero because of external disturbances and noise. Therefore, as k tends to infinity, $e(k)$ is assumed to converges to a RPI set $\Upsilon(P_e^{-1})$. Recursive feasibility is guaranteed as long as $e(k)$ remains bounded, since the bounded error ensures consistency between successive steps.

Recursive feasibility and robust stability

To prove the feasibility of the optimization problem (42) and the robust stability of the closed-loop system, the following Lemma and Theorem are given.

Lemma 2. *For Theorem 2, if any feasible solution obtained by solving the optimization problem (42) at time k is feasible at time $k + i$ ($i \geq 1$), then it is also feasible for all time $t > k$.*

Proof.

Assuming that the optimization problem (42) is feasible at k , the optimal solution $\{\gamma, \eta, \beta_1, \beta_2, \tilde{Q}_1, \tilde{Q}_2, Y_1\}^*(k)$ is obtained. Next, the optimal solution of the optimization problem (42) at k is used to construct the feasible solution at $k + 1$.

At time $k + 1$, $\{\tilde{Q}_1(k + 1), \tilde{Q}_2(k + 1)\} = \{\tilde{Q}_1^*(k), \tilde{Q}_2^*(k)\}$ is chosen, where $\{\tilde{Q}_1(k + 1), \tilde{Q}_2(k + 1)\}$ represents the feasible solution of the optimization problem (42) at $k + 1$. According to Theorem 2, the inequality (36) - (39) guarantees $\hat{x}(1|k) \in \Upsilon\left(\left(\tilde{Q}^*(k)\right)^{-1}, 2\right)$. Since the augmented system in the optimization problem (42) considers all possible realizations of bounded disturbances and noise, this can guarantee that $\hat{x}(k + 1) \in \Upsilon\left(\left(\tilde{Q}^*(k)\right)^{-1}, 2\right)$, i.e.

$$\begin{aligned} & \hat{x}^T(k + 1) \left(\tilde{Q}(k + 1)\right)^{-1} \hat{x}(k + 1) \\ &= \hat{x}^T(k + 1) \left(\tilde{Q}_1^*(k)\right)^{-1} \hat{x}(k + 1) + e^T(k + 1) \left(\tilde{Q}_2^*(k)\right)^{-1} e(k + 1) \\ &\leq 2 \end{aligned} \quad (43)$$

From (70), it follows that $\hat{x}(k) \in \Upsilon\left(\left[(2 - \eta^*(k)) \tilde{Q}_1^*(k)\right]^{-1}\right)$. By selecting $\{\tilde{Q}_2, \eta\}(k + 1) = \{\tilde{Q}_2, \eta\}^*(k)$ at $k + 1$, it follows that (37) is satisfied, it can be ensured that

$$e^T(k + 1) \left(\tilde{Q}_2^*(k)\right)^{-1} e(k + 1) \leq \eta^*(k)$$

which implies $e(k + 1) \in \Upsilon(P_e^{-1})$.

From (36) and (43)

$$\hat{x}^T(k + 1) \left(\tilde{Q}_1^*(k)\right)^{-1} \hat{x}(k + 1) \leq 2 - \eta^*(k)$$

Then $\hat{x}(k + 1) \in \Upsilon\left(\left[(2 - \eta^*(k)) \tilde{Q}_1^*(k)\right]^{-1}\right)$ holds. From the above description, $\Upsilon(P_e^{-1})$ and $\Upsilon\left(\left[(2 - \eta^*(k)) \tilde{Q}_1^*(k)\right]^{-1}\right)$ are RPI sets of the state estimation error and estimated state at $k + 1$, respectively. Therefore, both the estimated state and the state estimation error converge to the corresponding RPI sets.

Finally, since only inequalities (36) and (37) are related to the state information, the optimization problem (42) is feasible by selecting $\{\gamma, \beta_1, \beta_2, Y_1\}(k + 1) =$

$\{\gamma, \beta_1, \beta_2, Y_1\}^*(k)$ at $k+1$. That is, a feasible solution for the optimization problem (42) at k is also feasible at $k+1$. By analogy, it is also possible at $k+i$ ($i \geq 2$). Q.E.D.

Since the state $x(k)$ is unmeasurable, the observed state $\hat{x}(k)$ must be used in the controller design process. The stability proof must use the actual state rather than the observed state. Therefore, before proving the stability of the closed-loop system, it is necessary to present the augmented closed-loop system in terms of the actual state.

Rewrite the dynamic system (32) as follows:

$$\begin{aligned} x(k+1) = & Ax(k) + Bu_{RMPC}(k) + (E_m - I)K(k+1)CAe(k) + w(k) \\ & + (I - K(k+1)C + E_mK(k+1)C)e_d(k) + E_md(k) \end{aligned} \quad (44)$$

Define the equivalent augmented state $\hat{x}(k) = [x^T(k) \ e^T(k) \ e_d^T(k)]^T$. According to (4), (18), (20) and (44), a corresponding equivalent system of the augmented closed-loop system (34) can be obtained (Baocang and Tao (2007); Baocang and Hongguang (2014)):

$$\hat{x}(k+1) = \hat{A}\hat{x}(k) + \varpi(k) \quad (45)$$

$$\begin{aligned} \text{where } \hat{A} = & \begin{bmatrix} A + BF_1 & \Delta' & \Delta'' \\ 0 & A - K(k+1)CA & I - K(k+1)C \\ 0 & -Z(k+1)K(k+1)CA & I - Z(k+1)K(k+1)C \end{bmatrix}, \\ \Delta' = & E_mK(k+1)CA - BF_1 - K(k+1)CA, \quad \Delta'' = I - K(k+1)C + E_m \\ K(k+1)C \text{ and } \varpi(k) = & \begin{bmatrix} E_md(k) + w(k) \\ w(k) \\ w_d(k) \end{bmatrix}. \end{aligned}$$

The system output is rewritten as follows:

$$y(k) = \hat{C}\hat{x}(k) + v(k) \quad (46)$$

where $\hat{C} = [C \ -C \ 0]$.

Theorem 3. Assume that the optimization problem (42) is feasible. The augmented closed-loop system (45) with the controller $u_{RMPC}(i|k) = F_1\hat{x}(i|k)$, will converge to a neighborhood of the origin, where the feedback gain F_1 is obtained by Theorem 2. That is, when the constraints (33) are satisfied, as $k \rightarrow \infty$, there exists $\mu > 0$ such that $x(k)$ will converge to the neighborhood $\Gamma_r = \{x(k) \in R^n : \|x(k)\| \leq \mu\}$ of the origin and will remain there thereafter.

Proof.

From Lemma 2, if the optimization problem (42) is feasible at $k=0$, then feasible solutions exist for all $k \geq 0$. At k , suppose that the performance cost of the optimization problem (42) is $\gamma^*(k)$. Then, at $k+1$, $\gamma(k+1) = \gamma^*(k)$ is an upper bound on the performance cost. By solving (42) at the $k+1$, $\gamma^*(k+1) \leq \gamma^*(k)$ can be obtained. This means that $\gamma^*(k)$ will tend to a constant value as time evolves for all $k \geq 0$.

For (45) and (46), the following nominal augmented closed-loop system is considered:

$$\begin{aligned}\widehat{x}_n(i+1|k) &= \widehat{A}\widehat{x}_n(i|k), \widehat{x}_n(0|k) = \widehat{x}(k) \\ y_n(i|k) &= \widehat{C}\widehat{x}_n(i|k) \\ u_{RMPCn}(i|k) &= \widehat{F}\widehat{x}_n(i|k), \forall i \geq 0\end{aligned}\quad (47)$$

where $\widehat{x}_n(k) = [x_n^T(k), e_n^T(k), e_{dn}^T(k)]^T$ and $\widehat{F} = [F_1, -F_1, 0]$.

In Theorem 2, if (73) is satisfied, then (72) holds. For (47), the following inequality is valid:

$$\begin{aligned}2 - \left\| \widehat{x}_n(i+1|k) \right\|_{(\tilde{Q}^*(k))^{-1}}^2 - \beta_1 \left(2 - \left\| \widehat{x}_n(i|k) \right\|_{(\tilde{Q}^*(k))^{-1}}^2 \right) \\ \geq \frac{1}{\gamma^*(k)} \left[\|y_n(i|k)\|_{H_1}^2 + \|u_{RMPCn}(i|k)\|_{H_2}^2 \right]\end{aligned}$$

Since $\beta_1 \in (0, 1)$, it follows that

$$\begin{aligned}\left\| \widehat{x}_n(i+1|k) \right\|_{(\tilde{Q}^*(k))^{-1}}^2 - \left\| \widehat{x}_n(i|k) \right\|_{(\tilde{Q}^*(k))^{-1}}^2 \\ < -\frac{1}{\gamma^*(k)} \left[\|y_n(i|k)\|_{H_1}^2 + \|u_{RMPCn}(i|k)\|_{H_2}^2 \right]\end{aligned}\quad (48)$$

which can guarantee the stability of the system (47). Therefore, (48) enables the nominal augmented states to converge within the RPI set $\Upsilon \left(\left(\tilde{Q}^* \right)^{-1}, 2 \right)$.

By summation of (48) from $i = 0$ to $i = \infty$

$$J_\infty(k) = \sum_{i=0}^{\infty} \left[\|y_n(i|k)\|_{H_1}^2 + \|u_{RMPCn}(i|k)\|_{H_2}^2 \right] < \gamma^*(k) \left\| \widehat{x}_n(i|k) \right\|_{(\tilde{Q}^*(k))^{-1}}^2$$

Considering $\widehat{x}_n(k) \in \Upsilon \left(\left(\tilde{Q}^* \right)^{-1}, 2 \right)$, it follows that

$$J_\infty(k) = \sum_{i=0}^{\infty} \left[\|y_n(i|k)\|_{H_1}^2 + \|u_{RMPCn}(i|k)\|_{H_2}^2 \right] < 2\gamma^*(k) \quad (49)$$

which shows that $2\gamma^*(k)$ is an upper bound on the performance cost of the system (47). Thus, condition (49) results in $\lim_{i \rightarrow \infty} y_n(i|k) = 0$ and $\lim_{i \rightarrow \infty} u_{RMPCn}(i|k) = 0$, which indicates that the future nominal outputs and the nominal inputs will converge to the origin.

Based on the above analysis, the equivalent augmented state $\widehat{x}(k)$ converges to a neighborhood of the origin when the system is subjected to bounded disturbances and noise. Therefore, the system state $x(k)$ enters and remains inside the neighborhood Γ_r of the origin. Q.E.D.

In summary, the control strategy guarantees ultimately bounded stability for the closed-loop system. After applying the AISMC, system (1) is transformed into dynamic system (32). According to Theorem 1, the system's trajectory remains stable within the quasi-sliding mode domain Γ_s . Based on Theorem 3, the system state $x(k)$ will converge to the neighbourhood Γ_r of the origin. Therefore, the proposed dynamic output feedback AISMPC guarantees that the closed-loop system will ultimately stabilize the system within the domain $\Gamma = \Gamma_s \cup \Gamma_r$.

Simulation testion

In this subsection, in order to validate the effectiveness of the proposed method, comparisons are made between the proposed method and the Luenberger observer-based ISMPC (L-ISMPC) (Huili and Dongya (2020)) and the observer-based RMPC (O-RMPC) (Xubin (2017)).

Example 1.

The following discretized heat exchanger system with shell and tube configuration (Shah et al. (2021)) is considered:

$$\begin{aligned} x(k+1) &= \begin{bmatrix} 0.9976 & 0.0464 \\ -0.0928 & 0.8585 \end{bmatrix} x(k) + \begin{bmatrix} 0.0024 \\ 0.0928 \end{bmatrix} u(k) + d(k) + w(k) \\ y(k) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(k) + v(k) \end{aligned} \quad (50)$$

where the disturbance $d(k) = \begin{bmatrix} 1 \\ -0.6 \end{bmatrix} \sin(2k\pi)$ and the noise $w(k)$ and $v(k)$ are randomly generated from the interval $[-0.1 \ 0.1]$.

The initial state values are set as $x(0) = [-3 \ 1]^T$, $\hat{x}(0) = [1 \ 0.8]^T$. For the proposed method, the covariance matrices of the noise and disturbances are chosen as $E(w(k)w^T(k)) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $E(v(k)v^T(k)) = 10$, $Q_d = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}$ and $R_d = 1$, respectively. The matrix $G = B^+ = [0.2785 \ 10.7687]$. The parameters of the adaptive reaching law (27) are $T = 0.05$, $\alpha = 0.5$, $m_0 = 1$, $m_1 = 0.5$, $\beta_0 = \beta_1 = 0.1$, $\hat{h}_0(0) = \hat{h}_1(0) = 0$, respectively. $u_{RMPC}(k) = [-0.3850 \ -0.3857]^T \hat{x}(k)$ can be obtained by solving LMIs (36) - (41).

For the L-ISMPC, the observer gains $L = [2.0561 \ 19.7072]^T$ and the corresponding positive definite weighting matrices are chosen as $Q_L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $R_L = 1$. By selecting the appropriate switching gain and parameters, the controller is given as follows:

$$u_{L-ISMPC}(k) = 0.2\text{sgn}(s(k)) + [-0.0935 \ -0.0703]^T \hat{x}(k)$$

For the O-RMPC, the controller is set as $u_{O-RMPC}(k) = [-0.4401 \ -0.4431]^T \hat{x}(k)$. The same disturbances and noise are applied for

all the simulation testing. The simulation results are shown in Fig. 2. The Integral of Time and Squared Error (ITSE) for the three methods are listed in Table 2.

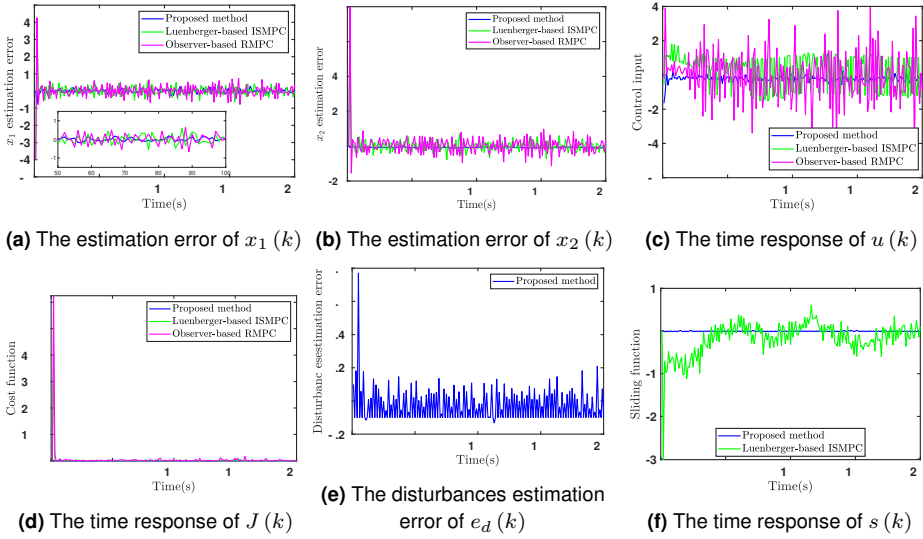


Fig. 2. Simulation results of the second-system (50) with the proposed method, L-ISMP, and O-RMPC, respectively

Table 2
ITSE FOR THE THREE METHODS.

Method	$x_1(k)$	$x_2(k)$
The proposed method	0.2168×10^3	0.1778×10^3
L-ISMP	0.2708×10^3	0.8521×10^3
O-RMPC	0.3421×10^3	1.4031×10^3

The estimation performance of the three methods for the system is depicted in Fig. 2, where the blue line is the proposed method, the green line is the L-ISMP and the purple line is the O-RMPC. As shown in Figs. 2a and 2b, the proposed method exhibits smaller fluctuations in the state estimation error than the other two methods, which means that the proposed method has better estimation accuracy. Figs. 2a-2c show that the proposed method has stronger robustness than L-RMPC in the presence of disturbances. Let the cost function $J(k) = \sum_{k=0}^{\infty} [(y(k))^2 + (u(k))^2]$. For Fig. 2d, at $k = 200$, the performance costs of the three methods are $J_{\text{proposed method}}(200) = 116.3970$, $J_{L-ISMP}(200) = 1765.4000$ and $J_{O-RMPC}(200) = 4077.2000$, respectively. It can be seen that the proposed method achieves lower cost consumption when the system has the same disturbances and noise. From Fig. 2e, it can be seen that the disturbance estimation error is bounded by using the disturbance-estimating KF used by the proposed method. This further illustrates that the proposed method can effectively exploit the capability of the output feedback controller, and then improve the control performance.

Fig. 2f shows the trajectory of the integral sliding variable $s(k)$. Additionally, this method has a smaller ITSE compared to the other two methods. In summary, the proposed method is superior to the other two methods in the simulation results.

This example uses a second-order system to preliminarily verify the effectiveness and estimation accuracy of the proposed method. To further investigate the method's performance in higher-order systems, especially under incompletely measurable states, a fourth-order continuous stirred tank reactor (CSTR) system is introduced in the following example.

Example 2.

To further evaluate the performance of the proposed method in higher-order systems where observability may become challenging, a fourth-order CSTR system is considered (Zhang et al. (2021)). The system matrices are given as follows:

$$A = \begin{bmatrix} 0.9600 & 0.0039 & 0.0722 & 0.0002 \\ -0.2488 & 0.8902 & -0.0134 & 0.0773 \\ 0.8312 & 0.0024 & 0.0657 & 0.0002 \\ -0.0235 & 0.5627 & -0.0201 & 0.0645 \end{bmatrix}, \quad B = \begin{bmatrix} 0.0072 & 0.0001 \\ -0.0009 & 0.0265 \\ 0.0247 & 0.0003 \\ -0.0200 & 0.0824 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \quad d(k) = [1 \quad -0.6 \quad 0.1 \quad -0.2]^T \sin(\pi k), \quad w(k) \quad \text{and} \quad v(k)$$

are randomly selected within the interval $[-0.2 \quad 0.2]$. The initial state $x(0) = [1 \quad 0 \quad 0.5 \quad -0.5]^T$, $\hat{x}(0) = [0 \quad 0 \quad 0 \quad 0]^T$.

For the parameters of the AISMPC (30), they are respectively: $G = \begin{bmatrix} 10.4173 & 7.1899 & 35.7233 & -2.4550 \\ 2.3266 & 5.1337 & 7.9729 & 10.4531 \end{bmatrix}$, $T = 0.1, \alpha = 0.45, m_0 = 0.5, m_1 = 0.4, \beta_0 = 0.15, \beta_1 = 0.2, \hat{h}_0(0) = \hat{h}_1(0) = 0$. The feedback gain in the RMPC (4) is: $F_1 = \begin{bmatrix} -0.6432 & 1.0026 & -0.3255 & 0.6891 \\ -0.5218 & 0.8934 & -0.2876 & 0.7321 \end{bmatrix}$. After selecting appropriate parameters, the simulation results for the three methods are shown in Fig. 3, and the ITSE results are presented in Table 3.

Table 3
ITSE FOR THE THREE METHODS.

Method	$x_1(k)$	$x_2(k)$	$x_3(k)$	$x_4(k)$
The proposed method	0.3693×10^2	0.3435×10^2	0.2750×10^2	0.8848×10^2
L-ISMP	0.7511×10^3	0.7832×10^3	0.4639×10^3	0.7996×10^3
O-RMPC	0.2021×10^4	0.1460×10^4	0.5478×10^4	0.1432×10^4

As shown in Fig. 3 and Table 3, the proposed method significantly outperforms L-ISMP and O-RMPC in the fourth-order CSTR system, a domain where state observation and estimation are particularly challenging. Its marked advantage in state estimation accuracy and disturbance rejection validates the robustness and applicability of the approach to higher-order systems.

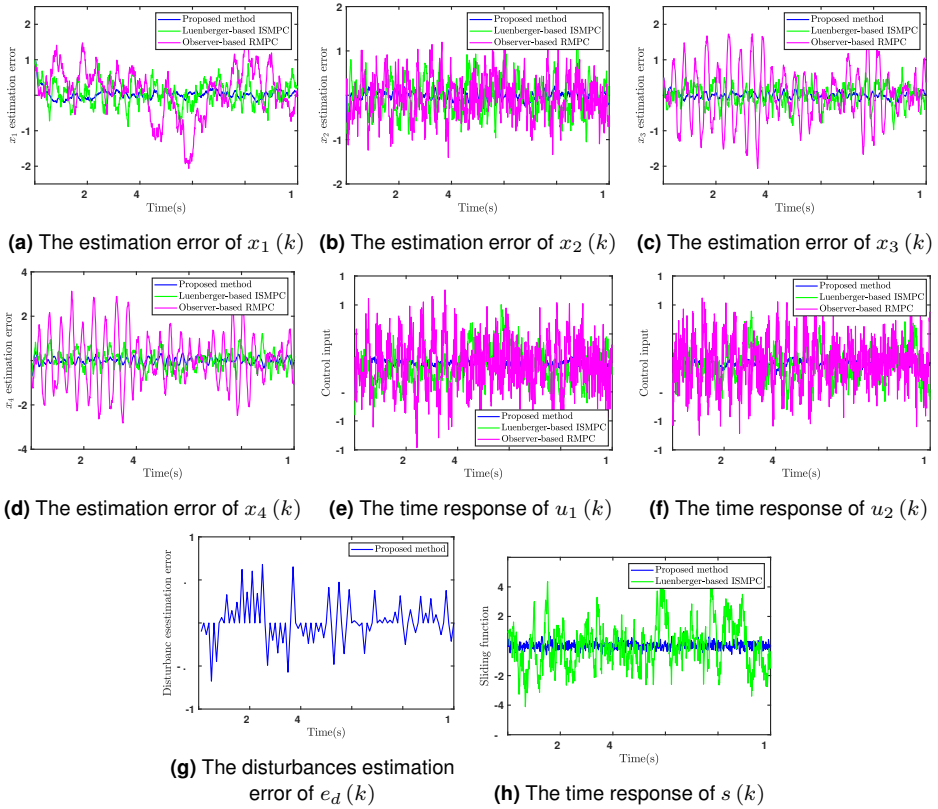


Fig. 3. Simulation results of the fourth-order system with the proposed method, L-ISMPC and O-RMPC, respectively

CONCLUSION

In this paper, where the system state is assumed unmeasured, a dynamic output feedback AISMPC is developed for constrained discrete-time systems in the presence of external disturbances and noise. The output feedback AISMC not only enables real-time estimation of state and disturbance error bounds, but also exhibits stronger robustness against disturbances. Then, the output feedback RMPC is developed to ensure optimal control in the presence of constraints. The proposed algorithm can guarantee recursive feasibility of the optimization problem and the robust stability of closed-loop system. Finally, according to the simulation results, dynamic output feedback AISMPC based on DLKF is effective in the case of unavailable state measurements and in the presence of disturbances and noise.

Although the DLKF-based control strategy can effectively address the issue of unmeasurable states, it also introduces new challenges. The need to solve online optimization problems together with dual-loop estimation imposes a heavy

computational burden, which may limit its applicability in systems with very fast dynamics or constrained computational resources. To alleviate this computational pressure, future work will investigate a static output feedback integral sliding mode predictive control approach. This approach will aim to design the controller directly based on output information without introducing observer-augmented states, thereby simplifying the overall control architecture.

Appendix A

Proof of Lemma 1

Based on (21), the following Lyapunov function is defined

$$V_e(k) = z^T(k) P_z^{-1} z(k) \quad (51)$$

Substituting from (21) into equation (51)

$$\Delta V_e(k) = \begin{bmatrix} z^T(k) & h^T(k) \end{bmatrix} \Xi_p \begin{bmatrix} z(k) \\ h(k) \end{bmatrix}$$

$$\text{where } \Xi_p = \begin{bmatrix} \Xi_z^T P_z^{-1} \Xi_z - P_z^{-1} & \Xi_z^T P_z^{-1} \\ P_z^{-1} \Xi_z & P_z^{-1} \end{bmatrix}.$$

According to the definition of quadratic boundedness (Alessandri et al. (2004)), it can be obtained that if $z^T(k) P_z^{-1} z(k) \geq 2$ then $\Delta V_e(k) \leq 0$, i.e.

$$\begin{bmatrix} z^T(k) & h^T(k) \end{bmatrix} \Xi_p \begin{bmatrix} z(k) \\ h(k) \end{bmatrix} \leq 0 \quad (52)$$

Since $h(k) \in \Upsilon(P_h, 2)$, $z^T(k) P_z^{-1} z(k) \geq 2$ is equivalent to $h^T(k) P_h h(k) \leq z^T(k) P_z^{-1} z(k)$. It follows that

$$\begin{bmatrix} z^T(k) & h^T(k) \end{bmatrix} \begin{bmatrix} -P_z^{-1} & 0 \\ 0 & P_h \end{bmatrix} \begin{bmatrix} z(k) \\ h(k) \end{bmatrix} \leq 0 \quad (53)$$

According to the S-procedure, (52) and (53) are satisfied for any possible $z(k)$ and $h(k)$ if and only if there exists a scalar $0 < \tau \leq 1$ such that

$$\begin{bmatrix} \Xi_z^T P_z^{-1} \Xi_z - P_z^{-1} & \Xi_z^T P_z^{-1} \\ P_z^{-1} \Xi_z & P_z^{-1} \end{bmatrix} - \tau \begin{bmatrix} -P_z^{-1} & 0 \\ 0 & P_h \end{bmatrix} \leq 0 \quad (54)$$

Applying the Schur Complement Lemma, (54) is equivalent to

$$\begin{bmatrix} -P_z^{-1} + \tau P_z^{-1} & 0 & \Xi_z^T \\ 0 & -\tau P_h & I \\ \Xi_z & I & -P_z \end{bmatrix} \leq 0 \quad (55)$$

Multiplying both sides of (55) by the diagonal matrix $\text{diag}\{P_z, I, I\}$, (22) can be obtained.

Inequality (54) can be written as

$$V_e(k+1) - V_e(k) \leq \tau (h^T(k) P_h h(k) - V_e(k))$$

Because $h^T(k) P_h h(k) \leq 2$, it can be easily seen that

$$V_e(k+1) \leq 2\tau + (1-\tau) V_e(k) \quad (56)$$

Assume the Lyapunov function at the initial time is $V_e(0) = z^T(0) P_z^{-1} z(0)$. Based on (56), it can be obtained recursively with the evolution of time k that:

$$V_e(k) \leq (1-\tau)^k z^T(0) P_z^{-1} z(0) + 2 - 2(1-\tau)^k$$

then (23) can be deduced. It is evident that when $0 < \tau \leq 1$, as $k \rightarrow \infty$, $\varsigma_k(\tau) \rightarrow 2$ holds.

As $V_e(k) \leq \varsigma_k(\tau)$, it is sufficient to note that $\|z(k)\|^2 \leq V_e(k) / \lambda_{\min}(P_z^{-1})$, so $\|z(k)\| \leq [\varsigma_k(\tau) \lambda_{\max}(P_z)]^{1/2}$.

From the above analysis, (21) is quadratically bounded and the upper bound of the augmented error norm can be obtained. This shows that both $e(k)$ and $e_d(k)$ are bounded. Q.E.D.

Appendix B

Proof of Theorem 1

The following Lyapunov function is selected:

$$V(k) = \frac{1}{2} s^T(k) s(k) + \frac{1}{2m_0} \tilde{h}_0^T(k) \tilde{h}_0(k) + \frac{1}{2m_1} \tilde{h}_1^T(k) \tilde{h}_1(k) \quad (57)$$

where $\tilde{h}_0(k)$ and $\tilde{h}_1(k)$ are tracking errors, which can be expressed as follows

$$\begin{aligned} \tilde{h}_0(k) &= h_0^* - \hat{h}_0(k) \\ \tilde{h}_1(k) &= h_1^* - \hat{h}_1(k) \end{aligned} \quad (58)$$

Considering the Taylor expansion of (57):

$$\begin{aligned} V(k+1) &= \frac{1}{2} s^T(k) s(k) + \frac{\partial V(k)}{\partial s(k)} \Delta s(k) + \frac{1}{2} \frac{\partial^2 V(k)}{\partial^2 s(k)} (\Delta s(k))^2 \\ &\quad + \frac{1}{2m_0} \tilde{h}_0^T(k) \tilde{h}_0(k) + \frac{\partial V(k)}{\partial \tilde{h}_0(k)} \Delta \tilde{h}_0(k) + \frac{1}{2} \frac{\partial^2 V(k)}{\partial^2 \tilde{h}_0(k)} (\Delta \tilde{h}_0(k))^2 \\ &\quad + \frac{1}{2m_1} \tilde{h}_1^T(k) \tilde{h}_1(k) + \frac{\partial V(k)}{\partial \tilde{h}_1(k)} \Delta \tilde{h}_1(k) + \frac{1}{2} \frac{\partial^2 V(k)}{\partial^2 \tilde{h}_1(k)} (\Delta \tilde{h}_1(k))^2 \end{aligned} \quad (59)$$

where

$$\begin{aligned}\Delta s(k) &= s(k+1) - s(k) \\ \Delta \tilde{h}_0(k) &= \tilde{h}_0(k+1) - \tilde{h}_0(k) \\ \Delta \tilde{h}_1(k) &= \tilde{h}_1(k+1) - \tilde{h}_1(k)\end{aligned}\quad (60)$$

It is noted that all higher order ($n > 2$) terms of (59) go to zero. The second-order difference of the Lypunov function can then be derived:

$$\begin{aligned}V(k+1) &= V(k) + s(k) \Delta s(k) + \frac{1}{2}(\Delta s(k))^2 + \frac{1}{m_0} \tilde{h}_0(k) \Delta \tilde{h}_0(k) \\ &\quad + \frac{1}{2m_0} (\Delta \tilde{h}_0(k))^2 + \frac{1}{m_1} \tilde{h}_1(k) \Delta \tilde{h}_1(k) + \frac{1}{2m_1} (\Delta \tilde{h}_1(k))^2\end{aligned}$$

According to (58) and (60), it follows that

$$\begin{aligned}\Delta \tilde{h}_0(k) &= -\Delta \hat{h}_0(k) \\ \Delta \tilde{h}_1(k) &= -\Delta \hat{h}_1(k)\end{aligned}\quad (61)$$

Then,

$$\begin{aligned}\Delta V(k+1) &= s(k) \Delta s(k) - \frac{1}{m_0} \tilde{h}_0(k) \Delta \hat{h}_0(k) - \frac{1}{m_1} \tilde{h}_1(k) \Delta \hat{h}_1(k) \\ &\quad + o\left([\Delta s(k)]^2, [\Delta \hat{h}_0(k)]^2, [\Delta \hat{h}_1(k)]^2\right)\end{aligned}\quad (62)$$

where

$$\begin{aligned}& o\left([\Delta s(k)]^2, [\Delta \hat{h}_0(k)]^2, [\Delta \hat{h}_1(k)]^2\right) \\ &= \frac{1}{2}(\Delta s(k))^2 + \frac{1}{2m_0} (\Delta \hat{h}_0(k))^2 + \frac{1}{2m_1} (\Delta \hat{h}_1(k))^2\end{aligned}\quad (63)$$

Substituting (27) into (62) yields:

$$\begin{aligned}\Delta V(k+1) &= -\alpha s^2(k) - Ts(k) K_a(k) \operatorname{sgn}(s(k)) + Ts(k) f_e(k) \\ &\quad - \frac{1}{m_0} \tilde{h}_0(k) \Delta \hat{h}_0(k) - \frac{1}{m_1} \tilde{h}_1(k) \Delta \hat{h}_1(k) \\ &\quad + o\left([\Delta s(k)]^2, [\Delta \hat{h}_0(k)]^2, [\Delta \hat{h}_1(k)]^2\right)\end{aligned}\quad (64)$$

For (64), two cases $\|s(k)\| > \|s(k-1)\|$ and $\|s(k)\| \leq \|s(k-1)\|$ are required to be considered.

(1) When $\|s(k)\| > \|s(k-1)\|$ for any $k \geq 0$.

Based on (26), (28), (29) and (64), it can be established that

$$\Delta V(k+1) \leq -\alpha \|s(k)\|^2 + o\left([\Delta s(k)]^2, [\Delta \hat{h}_0(k)]^2, [\Delta \hat{h}_1(k)]^2\right)$$

Consider the second order term $o\left([\Delta s(k)]^2, [\Delta \hat{h}_0(k)]^2, [\Delta \hat{h}_1(k)]^2\right)$ in (63):

$$\begin{aligned} & o\left([\Delta s(k)]^2, [\Delta \hat{h}_0(k)]^2, [\Delta \hat{h}_1(k)]^2\right) \\ & \leq \alpha^2 \|s(k)\|^2 + \frac{1}{2} T^2 m_0 \|s(k)\|^2 + \frac{1}{2} T^2 m_1 \|s(k)\|^2 \\ & + T^2 \left[f_e^2(k) - (K_a(k) \operatorname{sgn}(s(k)))^2 \right] \end{aligned} \quad (65)$$

From (65), the term $T^2 \left[f_e^2(k) - (K_a(k) \operatorname{sgn}(s(k)))^2 \right]$ can be neglected when T is sufficiently small. Then,

$$o\left([\Delta s(k)]^2, [\Delta \hat{h}_0(k)]^2, [\Delta \hat{h}_1(k)]^2\right) \leq \left(\alpha^2 + \frac{T^2}{2} (m_0 + m_1)\right) \|s(k)\|^2 \quad (66)$$

It follows that:

$$\Delta V(k+1) \leq \left(-\alpha + \alpha^2 + \frac{T^2}{2} (m_0 + m_1)\right) \|s(k)\|^2 \quad (67)$$

For small enough T , $\frac{T^2}{2} (m_0 + m_1) < \alpha - \alpha^2$ is satisfied. Thus, $\Delta V(k+1) < 0$ for $s(k) \neq 0$.

(2) When $\|s(k)\| \leq \|s(k-1)\|$ for any $k \geq 0$.

$$\begin{aligned} \Delta V(k+1) & \leq -\alpha \|s(k)\|^2 + T \beta_0 \tilde{h}_0(k) \hat{h}_0(k) + T \beta_1 \tilde{h}_1(k) \hat{h}_1(k) \\ & + o\left([\Delta s(k)]^2, [\Delta \hat{h}_0(k)]^2, [\Delta \hat{h}_1(k)]^2\right) \end{aligned} \quad (68)$$

Substituting (58) into (68) yields:

$$\begin{aligned} \Delta V(k+1) & \leq -\alpha \|s(k)\|^2 + \frac{T}{4} (\beta_0 (h_0^*)^2 + \beta_1 (h_1^*)^2) \\ & + o\left([\Delta s(k)]^2, [\Delta \hat{h}_0(k)]^2, [\Delta \hat{h}_1(k)]^2\right) \end{aligned}$$

According to (66), it follows that

$$\Delta V(k+1) \leq -\alpha (1 - \alpha) \|s(k)\|^2 + \frac{T}{4} (\beta_0 (h_0^*)^2 + \beta_1 (h_1^*)^2) \quad (69)$$

For (69), if $\|s(k)\| > \sqrt{\frac{T(\beta_0 (h_0^*)^2 + \beta_1 (h_1^*)^2)}{4\alpha(1-\alpha)}}$, then $\Delta V(k+1) < 0$ is established. That is, when the integral sliding mode function $s(k)$ lies outside the domain Γ_s , the system trajectory will move toward the integral sliding surface.

For the above case, it can be obtained that the adaptive reaching law (27) and the AISMC (30) ensure that the system trajectory remains within the quasi-sliding mode domain Γ_s . Q.E.D.

Appendix C

Proof of Theorem 2

For the augmented state $\tilde{x}(k) \in \Upsilon(\tilde{Q}^{-1}, 2)$, the following inequality holds

$$\tilde{x}^T(k) \tilde{Q}^{-1} \tilde{x}(k) = \hat{x}^T(k) \tilde{Q}_1^{-1} \hat{x}(k) + e^T(k) \tilde{Q}_2^{-1} e(k) \leq 2$$

Let $e^T(k) \tilde{Q}_2^{-1} e(k) \leq \eta(k) e^T(k) P_e^{-1} e(k) \leq \eta(k)$, then it follows that

$$\hat{x}^T(k) \tilde{Q}_1^{-1} \hat{x}(k) \leq 2 - \eta(k) \quad (70)$$

For (70), using the Schur Complement Lemma, (36) is obtained.

Applying the Schur Complement Lemma to $e^T(k) \tilde{Q}_2^{-1} e(k) - \eta(k) e^T(k) P_e^{-1} e(k) \leq 0$ yields:

$$\begin{bmatrix} -\eta(k) P_e^{-1} & I \\ I & -\tilde{Q}_2 \end{bmatrix} \leq 0 \quad (71)$$

Multiplying both sides of (71) by the diagonal matrix $\text{diag}\{P_e, I\}$, (37) is obtained.

For $\tilde{x}(i|k) \in \Upsilon(\tilde{Q}^{-1}, 2)$ and $\omega(k+i) \in \Upsilon(P_\omega, 2)$, $i \geq 0$, by applying the RPI set and the S-procedure (Xubin et al. (2020)), $\tilde{x}(i+1|k) \in \Upsilon(\tilde{Q}^{-1}, 2)$ is ensured if there exist non-negative scalars $\{\beta_1, \beta_2\}$ and a positive scalar $\gamma > 0$ such that

$$\begin{aligned} & 2 - \|\tilde{x}(i+1|k)\|_{\tilde{Q}^{-1}}^2 - \beta_1 \left(2 - \|\tilde{x}(i|k)\|_{\tilde{Q}^{-1}}^2 \right) - \beta_2 \left(2 - \|\omega(k+i)\|_{P_\omega}^2 \right) \\ & \geq \frac{1}{\gamma} \left[\|y(i|k)\|_{H_1}^2 + \|u_{RMPC}(i|k)\|_{H_2}^2 \right] \end{aligned} \quad (72)$$

If the sufficient conditions $\beta_2 + \beta_1 - 1 \leq 0$ and (73) are satisfied, where $0 \leq \beta_1 \leq 1, \beta_2 \geq 0$, then (72) holds.

$$\begin{aligned} & \|\tilde{x}(i+1|k)\|_{\tilde{Q}^{-1}}^2 - \beta_1 \|\tilde{x}(i|k)\|_{\tilde{Q}^{-1}}^2 - \beta_2 \|\omega(k+i)\|_{P_\omega}^2 \\ & + \frac{1}{\gamma} \left[\|y(i|k)\|_{H_1}^2 + \|u_{RMPC}(i|k)\|_{H_2}^2 \right] \leq 0 \end{aligned} \quad (73)$$

Substituting (34) and (35) into (73) yields:

$$\begin{aligned} & \left[\tilde{A}\tilde{x}(i|k) + \tilde{E}\omega(k+i) \right]^T \tilde{Q}^{-1} \left[\tilde{A}\tilde{x}(i|k) + \tilde{E}\omega(k+i) \right] - \beta_1 \tilde{x}^T(i|k) \tilde{Q}^{-1} \tilde{x}(i|k) \\ & - \beta_2 \omega^T(k+i) P_\omega \omega(k+i) + \frac{1}{\gamma} \left[\tilde{C}\tilde{x}(i|k) + v(k+i) \right]^T H_1 \left[\tilde{C}\tilde{x}(i|k) + v(k+i) \right] \\ & + \frac{1}{\gamma} [F\tilde{x}(i|k)]^T H_2 [F\tilde{x}(i|k)] \\ & = \begin{bmatrix} \tilde{x}^T(i|k) & \omega^T(k+i) & v^T(k+i) \end{bmatrix} \text{O} \begin{bmatrix} \tilde{x}(i|k) \\ \omega(k+i) \\ v(k+i) \end{bmatrix} \leq 0 \end{aligned} \quad (74)$$

where

$$O = \begin{bmatrix} \tilde{A}^T \gamma \tilde{Q}^{-1} \tilde{A} - \beta_1 \gamma \tilde{Q}^{-1} + \tilde{C}^T H_1 \tilde{C} + F^T H_2 F & & & & \\ & \tilde{E}^T \gamma \tilde{Q}^{-1} \tilde{A} & & & \\ & & H_1 \tilde{C} & & \\ & & & \tilde{A}^T \gamma \tilde{Q}^{-1} \tilde{E} & \tilde{C}^T H_1 \\ & & & \tilde{E}^T \gamma \tilde{Q}^{-1} \tilde{E} - \beta_2 \gamma P_\omega & 0 \\ & & & 0 & H_1 \end{bmatrix}.$$

Then, (74) is equivalent to

$$O \leq 0 \quad (75)$$

Multiplying (75) by γ^{-1} , and then applying the Schur Complement Lemma, (75) can be converted into the following

$$\begin{bmatrix} -\beta_1 \tilde{Q}^{-1} & 0 & 0 & \tilde{A}^T & \tilde{C}^T H_1^{\frac{1}{2}} & F^T H_2^{\frac{1}{2}} \\ 0 & -\beta_2 P_\omega & 0 & \tilde{E}^T & 0 & 0 \\ 0 & 0 & 0 & 0 & H_1^{\frac{1}{2}} & 0 \\ \tilde{A} & \tilde{E} & 0 & -\tilde{Q} & 0 & 0 \\ H_1^{\frac{1}{2}} \tilde{C} & 0 & H_1^{\frac{1}{2}} & 0 & -\gamma I & 0 \\ H_2^{\frac{1}{2}} F & 0 & 0 & 0 & 0 & -\gamma I \end{bmatrix} \leq 0 \quad (76)$$

Multiplying both sides of (76) by the diagonal matrix $\text{diag} \{ \tilde{Q} \ I \ I \ I \ I \ I \}$, it follows that:

$$\begin{bmatrix} -\beta_1 \tilde{Q} & 0 & 0 & \tilde{Q} \tilde{A}^T & \tilde{Q} \tilde{C}^T H_1^{\frac{1}{2}} & \tilde{Q} F^T H_2^{\frac{1}{2}} \\ 0 & -\beta_2 P_\omega & 0 & \tilde{E}^T & 0 & 0 \\ 0 & 0 & 0 & 0 & H_1^{\frac{1}{2}} & 0 \\ \tilde{A} \tilde{Q} & \tilde{E} & 0 & -\tilde{Q} & 0 & 0 \\ H_1^{\frac{1}{2}} \tilde{C} \tilde{Q} & 0 & H_1^{\frac{1}{2}} & 0 & -\gamma I & 0 \\ H_2^{\frac{1}{2}} F \tilde{Q} & 0 & 0 & 0 & 0 & -\gamma I \end{bmatrix} \leq 0 \quad (77)$$

Since $\tilde{Q} = \begin{bmatrix} \tilde{Q}_1 & 0 \\ 0 & \tilde{Q}_2 \end{bmatrix}$ and $F = \begin{bmatrix} F_1 & 0 \end{bmatrix}$,

$$\begin{aligned} F \tilde{Q} &= \begin{bmatrix} F_1 & 0 \end{bmatrix} \begin{bmatrix} \tilde{Q}_1 & 0 \\ 0 & \tilde{Q}_2 \end{bmatrix} = \begin{bmatrix} F_1 \tilde{Q}_1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} Y_1 & 0 \end{bmatrix} = Y \end{aligned}$$

Substituting $F \tilde{Q} = Y$ into (77), (38) is obtained.

Hence, it can be obtained that $F_1 = Y_1 \tilde{Q}_1^{-1}$ and

$$u_{RMPC}(k) = Y_1 \tilde{Q}_1^{-1} \hat{x}(k) \quad (78)$$

For input constraints, z_s is the s -th row of the m -order identity matrix. It follows that

$$\begin{aligned} \max_{i \geq 0} |z_s u(i|k)|^2 &\leq \max_{i \geq 0} \left\| z_s F \tilde{Q}^{\frac{1}{2}} \right\|^2 \left\| \tilde{Q}^{-\frac{1}{2}} \tilde{x}(i|k) \right\|^2 \\ &\leq 2 \max_{i \geq 0} \left\| z_s \begin{bmatrix} F_1 & 0 \end{bmatrix} \tilde{Q}^{\frac{1}{2}} \right\|^2 \end{aligned}$$

If there exists a symmetric matrix $\tilde{Z}$ satisfying $\tilde{Z}_{ss} \leq \bar{u}_s^2$, $s \in \{1, \dots, m\}$ such that

$$-\tilde{Z} + 2 \begin{bmatrix} F_1 & 0 \end{bmatrix} \tilde{Q} \begin{bmatrix} F_1 & 0 \end{bmatrix}^T \leq 0 \quad (79)$$

then $|u_s(i|k)| \leq \bar{u}_s$ holds.

Substituting $\tilde{Q} = \text{diag} \{ \tilde{Q}_1, \tilde{Q}_2 \}$ into (79) and applying the Schur Complement Lemma:

$$\begin{bmatrix} -\tilde{Q}_1^{-1} & 0 & \sqrt{2}F_1^T \\ 0 & -\tilde{Q}_2^{-1} & 0 \\ \sqrt{2}F_1 & 0 & -\tilde{Z} \end{bmatrix} \leq 0 \quad (80)$$

Multiplying both sides of (80) by the diagonal matrix $\text{diag} \{ \tilde{Q}_1^{-1} \quad \tilde{Q}_2^{-1} \quad I \}$, and then since $F_1 \tilde{Q}_1 = Y_1$, it can be obtained that

$$\begin{bmatrix} -\tilde{Q}_1 & 0 & \sqrt{2}Y_1^T \\ 0 & -\tilde{Q}_2 & 0 \\ \sqrt{2}Y_1 & 0 & -\tilde{Z} \end{bmatrix} \leq 0$$

which is equivalent to (40).

For state constraints, define z_g as the g -th row of the f -order identity matrix. It follows that

$$\begin{aligned} \max_{i \geq 0} |z_g \bar{\Psi} x(i+1|k)|^2 &= \max_{i \geq 0} \left| z_g \bar{\Psi} \begin{bmatrix} I & I \end{bmatrix} \begin{bmatrix} \tilde{A} & \tilde{E} \end{bmatrix} \begin{bmatrix} \tilde{x}(i|k) \\ \omega(k+i) \end{bmatrix} \right|^2 \\ &\leq 4 \max_{i \geq 0} \left\| z_g \bar{\Psi} \begin{bmatrix} I & I \end{bmatrix} \begin{bmatrix} \tilde{A} & \tilde{E} \end{bmatrix} \begin{bmatrix} \tilde{Q} & 0 \\ 0 & P_\omega^{-1} \end{bmatrix}^{\frac{1}{2}} \right\|^2 \end{aligned}$$

If there exists a symmetric matrix $\tilde{\Xi}$ satisfying $\tilde{\Xi}_{gg} \leq \bar{\psi}_g^2$, $g \in \{1, \dots, f\}$ such that

$$-\tilde{\Xi} + 4 \bar{\Psi} \begin{bmatrix} I & I \end{bmatrix} \begin{bmatrix} \tilde{A} & \tilde{E} \end{bmatrix} \begin{bmatrix} \tilde{Q}_1 & 0 & 0 \\ 0 & \tilde{Q}_2 & 0 \\ 0 & 0 & P_\omega^{-1} \end{bmatrix} \begin{bmatrix} \tilde{A}^T \\ \tilde{E}^T \end{bmatrix} \begin{bmatrix} I \\ I \end{bmatrix} \bar{\Psi}^T \leq 0$$

then $|x_g(i+1|k)| \leq \bar{\psi}_g$ holds. By applying the Schur Complement Lemma

$$\begin{bmatrix} -\tilde{Q}_1^{-1} & 0 \\ 0 & -\tilde{Q}_2^{-1} \\ 0 & 0 \\ 2\bar{\Psi}(A+BF_1) & 2\bar{\Psi}(A-(E_m-I)K(k+1)CA) \\ 0 & 2(A+BF_1)^T\bar{\Psi}^T \\ 0 & 2A^T-2A^TC^TK^T(k+1)(E_m-I)^T\bar{\Psi}^T \\ -P_\omega & 2(E_m^T-I)\bar{\Psi}^T \\ 2\bar{\Psi}(E_m-I) & -\tilde{\Xi} \end{bmatrix} \leq 0 \quad (81)$$

Multiplying both sides of (81) by the diagonal matrix $\text{diag}\{\tilde{Q}_1 \quad \tilde{Q}_2 \quad I \quad I\}$ and $F_1\tilde{Q}_1 = Y_1$, hence, (41) can be obtained. Q.E.D.

FUNDING

This work is partially supported by the National Nature Science Foundation of China (Grant No 61973315) and the Science and Technology Program of Qingdao, China (Grant No 22-3-7-cspz-15-nsh).

References

- Alessandri, A., Baglietto, M., and Battistelli, G. (2004). On estimation error bounds for receding-horizon filters using quadratic boundedness. *IEEE Transactions on Automatic Control*, 49(8):1350–1355.
- Anderson, B. D. O. and Dehghani, A. (2008). Challenges of adaptive control-past, permanent and future. *Annual Reviews in Control*, 32(2):123–135.
- Athans, M. and Falb, P. L. (2003). Optimal control: an introduction to the theory and its applications. *IEEE Transactions on Automatic Control*, 12(3):345–347.
- Bae, H. and Oh, J. H. (2017). Novel state estimation framework for humanoid robot. *Robotics and Autonomous Systems*, 98:258–275.
- Baocang, D. and Hongguang, P. (2014). Output feedback robust mpc with one free control move for the linear polytopic uncertain system with bounded disturbance. *Automatica*, 50:2929–2935.
- Baocang, D. and Tao, Z. (2007). Synthesizing output feedback predictive control for constrained uncertain time-varying discrete systems. *Acta Automatica Sinica*, 33(1):78–83.
- Ding, B., Xi, Y., Cychowski, M. T., and O’Mahony, T. (2008). A synthesis approach for output feedback robust constrained model predictive control. *Automatica*, 44(1):258–264.
- Edwards, C. and Spurgeon, S. K. (1998). Sliding mode control: Theory and applications. *London, United Kingdom: Taylor and Francis*.
- Farbood, M., Shasadeghi, M., Niknam, T., and Safarinejadian, B. (2022). Fuzzy lyapunov-based model predictive sliding mode control of nonlinear systems: An ellipsoid recursive feasibility approach. *IEEE Transactions on Fuzzy Systems*, 30(6):1929–1938.
- Fridman, L., Poznyak, A., and Bejarano, F. J. (2014). *Robust Output LQ Optimal Control via Integral Sliding Modes*. Springer New York.

- Haifeng, M., Zhenhua, X., Yangmin, L., and Zhanqiang, L. (2021). Sliding mode control for uncertain discrete-time systems using an adaptive reaching law. *IEEE Transactions on Circuits and Systems II: Express Briefs*, 68(2):722–726.
- He, X., Jiang, W., and Jiang, C. (2022). Memory nonfragile output feedback robust mpc for polytopic time-delayed systems with constraints. *International Journal of Control, Automation and Systems*, 20(2):375–391.
- Hongjian, W., Jingfei, R., Mengxue, H., Zhao, W., Kai, Z., and Xiaoning, W. (2023). Robust adaptive three-dimensional trajectory tracking control for unmanned underwater vehicles with disturbances and uncertain dynamics. *Ocean Engineering*, 289:116184.
- Hongqi, L., Luzheng, B., Xiaoya, L., and Hongping, G. (2024). Robust predictive control for eeg-based brain-robot teleoperation. *IEEE Transactions on Intelligent Transportation Systems*, pages 1–10.
- Huili, X. and Dongya, Z. (2020). Discrete-time dynamic output feedback integral sliding model predictive control. In *IEEE 2020 39th Chinese Control Conference*, pages 2357–2365.
- Héctor, R., Manuel, M., Raïssi, T., and Efimov, D. (2024). Robust interval predictive tracking control for constrained and perturbed unicycle mobile robots. *International Journal of Robust and Nonlinear Control*, 34(9):6303–6320.
- Incremona, G. P., Rubagotti, M., and Ferrara, A. (2017). Sliding mode control of constrained nonlinear systems. *IEEE Transactions on Automatic Control*, 62(6):2965–2972.
- Jazwinski, A. H. (1970). *Stochastic Processes and Filtering Theory*. New York Academic Press.
- Jingchao, L., Zhaohui, Y., Sheng, D., Jingqin, Z., and Jianrui, Z. (2022). A nonlinear state estimation framework for humanoid robots. *Robotics and Autonomous Systems*, 153:104100.
- Joshi, S. M., Tao, G., and Patre, P. (2011). Direct adaptive control using an adaptive reference model. *International Journal of Control*, 84(1):180–196.
- Kangzhi, L. and Y., Y. (2016). Robust control: Theory and applications. *John Wiley and Sons*.
- Khalid, A., Jianxin, X., and Xinghuo, Y. (2007). On the discrete-time integral sliding-mode control. *IEEE Transactions on Automatic Control*, 52(4):709–715.
- Mingdong, H., Weigan, P., and Yaozhen, H. (2023). A discrete-time integral sliding mode predictive control using iterative dynamic linearization data model. *International Journal of Control, Automation and Systems*, 21:2293 – 2302.
- Nicolis, D., Allevi, F., and Rocco, P. (2020). Operational space model predictive sliding mode control for redundant manipulators. *IEEE Transactions on Robotics*, 36(4):1348–1355.
- Nikolaos, A., George, B., and Mircea, L. (2014). Construction of invariant polytopic sets with specified complexity. *International Journal of Control*, 87(8):1681–1693.
- Postlethwaite, I., Turner, M. C., and Herrmann, G. (2007). Robust control applications. *Annual Reviews in Control*, 31(1):27–39.
- Roy, S., Baldi, S., and Fridman, L. M. (2020). On adaptive sliding mode control without a priori bounded uncertainty. *Automatica*, 111:108650.
- Rubagotti, M., Estrada, A., Castaños, F., Ferrara, A., and Fridman, L. (2011a). Integral sliding mode control for nonlinear systems with matched and unmatched perturbations. *IEEE Transactions on Automatic Control*, 56(11):2699–2704.

- Rubagotti, M., Incremona, G. P., Raimondo, D. M., and Ferrara, A. (2021). Constrained nonlinear discrete-time sliding mode control based on a receding horizon approach. *IEEE Transactions on Automatic Control*, 66(8):3802–3809.
- Rubagotti, M., Raimondo, D. M., Ferrara, A., and Magni, L. (2011b). Robust model predictive control with integral sliding mode in continuous-time sampled-data nonlinear systems. *IEEE Transactions on Automatic Control*, 56(3):556–570.
- Shah, D., Shah, A., and Mehta, A. (2021). Higher order networked sliding mode controller for heat exchanger connected via data communication network. *European Journal of Control*, 58:301–314.
- Steinberger, M., Castillo, I., Horn, M., and Fridman, L. (2019). Robust output tracking of constrained perturbed linear systems via model predictive sliding mode control. *International Journal of Robust and Nonlinear Control*, 30(3):1258–1274.
- Vettori, S., Di Lorenzo, E., Peeters, B., Luczak, M., and Chatzi, E. (2023). An adaptive-noise augmented kalman filter approach for input-state estimation in structural dynamics. *Mechanical Systems and Signal Processing*, 184:109654.
- Xiao, H., Zhang, S., Zhao, D., Yan, X., and Spurgeon, S. K. (2024). Output feedback integral sliding mode predictive control for linear discrete time systems. *European Journal of Control*, 80:101131.
- Xiao, H., Zhao, D., Gao, S., and Spurgeon, S. K. (2022). Sliding mode predictive control: A survey. *Annual Reviews in Control.*, 54:148–166.
- Xiaojing, Z., Maryam, K., Angelos, G., Paul, G., and John, L. (2017). Robust optimal control with adjustable uncertainty sets. *Automatica*, 75:249–259.
- Xubin, P. (2017). Output feedback robust mpc based on off-line observer for lpv systems via quadratic boundedness. *Asian Journal of Control*, 19(4):1641–1653.
- Xubin, P., Sen, Y., Baocang, D., Tarek, R., and Zhiwu, L. (2020). A convexity approach to dynamic output feedback robust mpc for lpv systems with bounded disturbances. *International Journal of Control, Automation and Systems*, 18(6):1378–1391.
- Yiwen, L. and Hedrick, J. K. (2015). Robust model predictive control with discrete-time integral sliding surface. In *IEEE 2015 American Control Conference*, pages 1641–1646.
- Zhang, S., Zhao, D., and Spurgeon, S. K. (2021). Output feedback robust distributed model predictive control for parallel systems in process networks with competitive characteristics. *Control Engineering Practice*, 113:104842.