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Discrete Equations from Bäcklund Transformations of the Fifth Painlevé Equation

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Abstract

In this paper discrete equations are derived from Bäcklund transformations of the fifth Painlevé equation, including a new discrete equation which has ternary symmetry. There are two classes of rational solutions of the fifth Painlevé equation, one expressed in terms of the generalised Laguerre polynomials and the other in terms of the generalised Umemura polynomials, both of which can be expressed as Wronskians of Laguerre polynomials. Hierarchies of rational solutions of the discrete equations are derived in terms of the generalised Laguerre and generalised Umemura polynomials. It is known that there is non-uniqueness of some rational solutions of the fifth Painlevé equation. Pairs of non-unique rational solutions are used to derive distinct hierarchies of rational solutions which satisfy the same discrete equation.

Keywords Discrete Painlevé equations · Bäcklund transformations · Fifth Painlevé equation · Rational solutions

Mathematics Subject Classification 33E17 · 34M55 · 37J70 · 39A10

1 Introduction

In this paper we are concerned with discrete Painlevé equations associated with the fifth Painlevé equation

$$\frac{d^2 w}{dz^2} = \left(\frac{1}{2w} + \frac{1}{w-1} \right) \left(\frac{dw}{dz} \right)^2 - \frac{1}{z} \frac{dw}{dz} + \frac{(w-1)^2(\alpha w^2 + \beta)}{z^2 w} + \frac{\gamma w}{z} + \frac{\delta w(w+1)}{w-1}, \quad (1.1)$$

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with α, β, γ and δ constants. We consider the generic case of (1.1) when $\delta \neq 0$, so we set $\delta = -\frac{1}{2}$, without loss of generality (by rescaling z if necessary) and obtain

$$\frac{d^2 w}{dz^2} = \left(\frac{1}{2w} + \frac{1}{w-1} \right) \left(\frac{dw}{dz} \right)^2 - \frac{1}{z} \frac{dw}{dz} + \frac{(w-1)^2(\alpha w^2 + \beta)}{z^2 w} + \frac{\gamma w}{z} - \frac{w(w+1)}{2(w-1)}, \quad (1.2)$$

which we will refer to as P_V . In the case when $\delta = 0$ in (1.1)

$$\frac{d^2 w}{dz^2} = \left(\frac{1}{2w} + \frac{1}{w-1} \right) \left(\frac{dw}{dz} \right)^2 - \frac{1}{z} \frac{dw}{dz} + \frac{(w-1)^2(\alpha w^2 + \beta)}{z^2 w} + \frac{\gamma w}{z}, \quad (1.3)$$

which is known as the *degenerate fifth Painlevé equation* (deg- P_V).

The six Painlevé equations were discovered by Painlevé, Gambier and their colleagues in the late 19th and early 20th centuries whilst studying second order ordinary differential equations of the form

$$\frac{d^2 w}{dz^2} = F \left(z, w, \frac{dw}{dz} \right), \quad (1.4)$$

where F is rational in dw/dz and w and analytic in z . The Painlevé transcendents, i.e. the solutions of the Painlevé equations, can be thought of as nonlinear analogues of the classical special functions. The Painlevé equations are regarded as being “completely integrable” and solvable using Riemann-Hilbert techniques [27]. The general solutions of the Painlevé equations are transcendental in the sense that they cannot be expressed in terms of known elementary functions and so require the introduction of a new transcendental function to describe their solution. However, it is well known that all the Painlevé equations, except P_I , possess rational solutions, algebraic solutions and solutions expressed in terms of the classical special functions for special values of the parameters, see, e.g. [15, 39] and the references therein. These hierarchies are usually generated from “seed solutions” using the associated Bäcklund transformations and frequently can be expressed in the form of determinants.

Discrete Painlevé equations also have a long history. The discrete Painlevé I (d P_I) equation is

$$x_{n+1} + x_n + x_{n-1} = 1 + \frac{\lambda n + \rho}{x_n}, \quad (1.5)$$

with λ and ρ constants. Equation (1.5) was studied in 1939 by Shohat [77], developing ideas due to Laguerre [49] dating from 1885, in the study of orthogonal polynomials associated with the quartic weight $\exp(-\frac{1}{4}x^4)$ on the real line. Subsequently this problem was studied by Freud [30], see also [8], though it was not recognised at the time as a discrete Painlevé equation. Equation (1.5) was then rediscovered in the 1980s in models of gauge field theory by Bessis, Itzykson and Zuber [7] and in quantum gravity by Brézin and Kazakov [9], who showed that its continuum limit is the first

Painlevé equation (P_I)

$$\frac{d^2 w}{dz^2} = 6w^2 + z, \quad (1.6)$$

see also [26].

The discrete Painlevé II (dP_{II}) equation is

$$x_{n+1} + x_{n-1} = \frac{\zeta[(n + \lambda)x_n + \rho]}{1 - x_n^2}, \quad (1.7)$$

with ζ , λ and ρ constants, which was derived by Periwal and Shevitz [70] in a study of unitary matrix models in a double scaling limit using orthogonal polynomials on the unit circle. Around the same time, Nijhoff and Papageorgiou [57] independently derived dP_{II} (1.7) and showed that its continuum limit is the second Painlevé equation (P_{II})

$$\frac{d^2 w}{dz^2} = 2w^3 + zw + \alpha, \quad (1.8)$$

with α a constant; Periwal and Shevitz [70] gave a continuum limit to P_{II} (1.8) with $\alpha = 0$.

One method for deriving discrete Painlevé equations is using Bäcklund transformations of (continuous) Painlevé equations. For example, it is well known that P_{II} (1.8) has the Bäcklund transformations

$$w(z; \alpha + 1) = -w(z; \alpha) - \frac{2\alpha + 1}{2w^2(z; \alpha) + 2w'(z; \alpha) + z}, \quad (1.9a)$$

$$w(z; \alpha - 1) = -w(z; \alpha) - \frac{2\alpha - 1}{2w^2(z; \alpha) - 2w'(z; \alpha) + z}. \quad (1.9b)$$

with $' \equiv d/dz$ [24, 32]. Hence letting $x_n = w(z; n + \lambda - \frac{1}{2})$ and $\alpha = n + \lambda - \frac{1}{2}$, then by eliminating $w'(z; \alpha)$ we obtain another discrete equation, commonly known as alt-dP_I

$$\frac{n + \lambda}{x_{n+1} + x_n} + \frac{n + \lambda - 1}{x_n + x_{n-1}} + 2x_n^2 + z = 0, \quad (1.10)$$

which has P_I (1.6) as its continuum limit [25]. The alt-dP_I equation (1.10) appeared in 1981 in the work of Jimbo and Miwa [43, equation (5.3)], though it was not recognised to be a discrete Painlevé equation until the work of Fokas, Grammaticos and Ramani [25].

In some respects the discrete Painlevé equations are much richer than the Painlevé equations. For example, they can have solutions which have no continuum limit and other properties which are lost in the continuous limit. Perhaps the most fundamental difference is that, whilst the Painlevé equations are unique up to a Möbius transformation, this is not true for the discrete Painlevé equations. For each discrete Painlevé

equation there exists a number of different forms, as illustrated by dP_I (1.5) and alt-dP_I (1.10) which both have P_I (1.6) as a continuum limit yet are not equivalent as there is no transformation relating the solutions (1.5) and (1.10). Further their exact solutions are quite different since dP_I (1.5) arises from Bäcklund transformations of the fourth Painlevé equation (P_{IV})

$$\frac{d^2 w}{dz^2} = \frac{1}{2w} \left(\frac{dw}{dz} \right)^2 + \frac{3}{2} w^3 + 4zw^2 + 2(z^2 - \alpha)w + \frac{\beta}{w}, \quad (1.11)$$

with α and β constants [34], whereas alt-dP_I (1.10) arises from Bäcklund transformations of P_{II} (1.8), as shown above.

The nomenclature for discrete Painlevé equations can be misleading. The original naming of discrete Painlevé equations came from their continuum limits but this can be ambiguous. For example, there are generalisations of dP_I (1.5) and dP_{II} (1.7), see equations (3.32) and (3.7), respectively, which have continuum limits to other Painlevé equations, as discussed below. Another approach is to classify the discrete Painlevé equations by the type of affine Weyl group and associated affine Dynkin diagram they are derived from, which is now commonly used. Sakai [75], extending the earlier work of Okamoto [64] for the (continuous) Painlevé equations, approached the discrete Painlevé equations from the point of view of algebraic geometry. Sakai clarified the algebraic nature of discrete Painlevé equations and gave a classification scheme. A comprehensive review of the geometric aspects of the continuous and discrete Painlevé equations is given by Kajiwara, Noumi and Yamada [46]. There is a recent discussion of the symmetry group of dP_{II} (1.7) in [22]. Discrete Painlevé equations arise in various applications such as random matrix theory and orthogonal polynomials, cf. [82, 83].

There are three distinct types of discrete Painlevé equations: (i), those for which the discrete variable is additive, which is the case for the discrete equations we discuss; (ii), the q -discrete Painlevé equations for which the discrete variable is multiplicative; and (iii), elliptic discrete Painlevé equations whose coefficients are elliptic functions of the independent variable n .

From the works of Okamoto [65–68], it is known that the parameter spaces of P_{II} – P_{VI} all admit the action of an extended affine Weyl group; the group acts as a group of Bäcklund transformations. In a series of papers, Noumi and Yamada [58, 59, 61, 63] have implemented this idea to derive a hierarchy of dynamical systems associated to the affine Weyl group of type $\widetilde{A}_N^{(1)}$, often known as “*symmetric forms of the Painlevé equations*”. The behaviour of each dynamical system varies depending on whether N is even or odd. The first member of the $\widetilde{A}_{2n+1}^{(1)}$ hierarchy, i.e. $\widetilde{A}_3^{(1)}$, usually known as symmetric- P_V (sP_V), is equivalent to P_V (1.2) and given by

$$z \frac{df_0}{dz} = f_0 f_2 (f_1 - f_3) + \left(\frac{1}{2} - \vartheta_2\right) f_0 + \vartheta_0 f_2, \quad (1.12a)$$

$$z \frac{df_1}{dz} = f_1 f_3 (f_2 - f_0) + \left(\frac{1}{2} - \vartheta_3\right) f_1 + \vartheta_1 f_3, \quad (1.12b)$$

$$z \frac{df_2}{dz} = f_2 f_0 (f_3 - f_1) + \left(\frac{1}{2} - \vartheta_0\right) f_2 + \vartheta_2 f_0, \quad (1.12c)$$

$$z \frac{df_3}{dz} = f_3 f_1 (f_0 - f_2) + \left(\frac{1}{2} - \vartheta_1\right) f_3 + \vartheta_3 f_1, \quad (1.12d)$$

with the normalisations

$$f_0(z) + f_2(z) = \sqrt{z}, \quad f_1(z) + f_3(z) = \sqrt{z}, \quad (1.12e)$$

and $\vartheta_0, \vartheta_1, \vartheta_2$ and ϑ_3 are constants such that

$$\vartheta_0 + \vartheta_1 + \vartheta_2 + \vartheta_3 = 1. \quad (1.12f)$$

The symmetric system sP_V (1.12) was found by Adler [1] in the context of periodic chains of Bäcklund transformations and has applications in random matrix theory, cf. [29].

The outline of this paper is as follows. In §2 we describe Bäcklund transformations of P_V (1.2). In §3 we derive some discrete Painlevé equations from these Bäcklund transformations, including a new discrete equation which has a ternary symmetry. There have been earlier studies deriving discrete Painlevé equations from Bäcklund transformations of P_V (1.2), see [25, 80]. Our study explicitly gives the Bäcklund transformations which generate the discrete Painlevé equations. These then can be used to obtain exact solutions of the discrete equations in terms of known solutions of P_V (1.2). In §4 we describe the rational solutions of P_V (1.2) which are expressed in terms of Wronskians of Laguerre polynomials. In §5 we derive hierarchies of rational solutions of the discrete Painlevé equations derived in §3 in terms of the rational solutions of P_V (1.2) given in §4. Further pairs of rational solutions which although different, which satisfy P_V (1.2) for the same parameters, are used to derive distinct hierarchies of rational solutions that satisfy the same discrete equation. Finally in §6 we discuss our results.

2 Bäcklund Transformations of the Fifth Painlevé Equation

In this section we describe the Bäcklund transformations of P_V (1.2).

Definition 2.1 A *Bäcklund transformation* maps a solution of a given Painlevé equation to another solution of the same Painlevé equation, with different values of the parameters, or another equation.

Definition 2.2 Let $w = w(z; \alpha, \beta, \gamma)$ be a solution of P_V (1.2) such that

$$\Phi(w; z) = z \frac{dw}{dz} - \varepsilon_2 a w^2 + (\varepsilon_2 a - \varepsilon_3 b + \varepsilon_1 z) w + \varepsilon_3 b \neq 0, \quad (2.1)$$

and

$$\Phi(w; z) - 2\varepsilon_1 z w \neq 0, \quad (2.2)$$

where $a = \sqrt{2\alpha}$ and $b = \sqrt{-2\beta}$ and $c = \gamma$. Then the *Bäcklund transformation* $\mathcal{T}_{\varepsilon_1, \varepsilon_2, \varepsilon_3}$ is defined by

$$\mathcal{T}_{\varepsilon_1, \varepsilon_2, \varepsilon_3} : \quad W(z; A, B, C) = 1 - \frac{2\varepsilon_1 zw}{\Phi(w; z)} = 1 - \frac{2\varepsilon_1 zw}{zw' - \varepsilon_2 aw^2 + (\varepsilon_2 a - \varepsilon_3 b + \varepsilon_1 z)w + \varepsilon_3 b}, \quad (2.3)$$

with $' \equiv d/dz$, $\varepsilon_j = \pm 1$, $j = 1, 2, 3$, independently, and for the parameters

$$\mathcal{T}_{\varepsilon_1, \varepsilon_2, \varepsilon_3}(a, b, c) = \left(\frac{1}{2}[c + \varepsilon_1(1 - \varepsilon_3 b - \varepsilon_2 a)], \frac{1}{2}[c - \varepsilon_1(1 - \varepsilon_3 b - \varepsilon_2 a)], \varepsilon_1(\varepsilon_3 b - \varepsilon_2 a) \right), \quad (2.4)$$

so

$$(A, B, C) = \left(\frac{1}{8}[c + \varepsilon_1(1 - \varepsilon_3 b - \varepsilon_2 a)]^2, -\frac{1}{8}[c - \varepsilon_1(1 - \varepsilon_3 b - \varepsilon_2 a)]^2, \varepsilon_1(\varepsilon_3 b - \varepsilon_2 a) \right),$$

cf. [36–38], [39, §39] or [69, §32.7(v)].

Note that we are using the same notation as in [69, §32.7(v)] which slightly differs from that used in [36–38] or [39, §39].

Remarks 2.3

- (i) Since $\varepsilon_j = \pm 1$, $j = 1, 2, 3$, independently, there are eight distinct transformations $\mathcal{T}_{\varepsilon_1, \varepsilon_2, \varepsilon_3}$. The parameter ε_1 reflects that the transformation $(z, \gamma) \rightarrow (-z, -\gamma)$ is a discrete symmetry of P_V . The parameters ε_2 and ε_3 reflect that $\alpha = \frac{1}{2}a^2$ and $\beta = -\frac{1}{2}b^2$, respectively, so a and b can take either sign.
- (ii) The Bäcklund transformation $\mathcal{T}_{\varepsilon_1, \varepsilon_2, \varepsilon_3}$ is usually given for the parameters α , β and γ , in terms of a , b and c . However in order to apply a sequence of Bäcklund transformations, it's preferable to define the action of Bäcklund transformation on the parameters a , b and c , rather than α , β and γ , to avoid any ambiguity in taking a square root.
- (iii) We note that the Hamiltonian for P_V (1.2) in [43] is written in terms of the monodromy parameters θ_0 , θ_1 and θ_∞ , which are related to the parameters α , β and γ through

$$\alpha = \frac{1}{8}(\theta_0 - \theta_1 + \theta_\infty)^2, \quad \beta = -\frac{1}{8}(\theta_0 - \theta_1 - \theta_\infty)^2, \quad \gamma = 1 - \theta_0 - \theta_1, \quad (2.5)$$

see also [66], as are the Schlesinger transformations, which relate two solutions of the associated monodromy problem, for P_V (1.2), see [28, 56]. Further the discrete symmetries of P_V (1.2), corresponding to the extended affine Weyl group of type $A_3^{(1)}$, act naturally on the root variables, see, for example, [18]. Hence it is natural to define the Bäcklund transformations for P_V (1.2) in terms of the parameters $a = \sqrt{2\alpha}$, $b = \sqrt{-2\beta}$ and $c = \gamma$, rather than α , β and γ .

- (iv) When applying the Bäcklund transformation $\mathcal{T}_{\varepsilon_1, \varepsilon_2, \varepsilon_3}$ followed by the Bäcklund transformation $\mathcal{T}_{\sigma_1, \sigma_2, \sigma_3}$ we use the notation $\mathcal{T}_{\sigma_1, \sigma_2, \sigma_3} \circ \mathcal{T}_{\varepsilon_1, \varepsilon_2, \varepsilon_3}$, i.e.

$$\mathcal{T}_{\sigma_1, \sigma_2, \sigma_3}(\mathcal{T}_{\varepsilon_1, \varepsilon_2, \varepsilon_3}(w; a, b, c)) = \mathcal{T}_{\sigma_1, \sigma_2, \sigma_3} \circ \mathcal{T}_{\varepsilon_1, \varepsilon_2, \varepsilon_3}(w; a, b, c).$$

Further if

$$\mathcal{T}_{\varepsilon_1, \varepsilon_2, \varepsilon_3}(w; a, b, c) = \left(W(w, w', a, b, c); A(a, b, c), B(a, b, c), C(a, b, c) \right),$$

then

$$\begin{aligned} \mathcal{T}_{\sigma_1, \sigma_2, \sigma_3} \circ \mathcal{T}_{\varepsilon_1, \varepsilon_2, \varepsilon_3}(w; a, b, c) &= \mathcal{T}_{\sigma_1, \sigma_2, \sigma_3} \\ &\left(W(w, w', a, b, c); A(a, b, c), B(a, b, c), C(a, b, c) \right). \end{aligned}$$

3 Deriving discrete Painlevé equations

In this section we derive four discrete Painlevé equations which are associated with P_V (1.2) using the Bäcklund transformations described in Sect. 2.

To derive a discrete Painlevé equation from Bäcklund transformations of P_V (1.2), we use the Bäcklund transformation $\mathcal{R}_+$, which relates a solution $w(z; a, b, c)$ to another solution $w_+(z; a_+, b_+, c_+)$ and its inverse transformation $\mathcal{R}_- = \mathcal{R}_+^{-1}$, which relates $w(z; a, b, c)$ to a third solution $w_-(z; a_-, b_-, c_-)$. Eliminating the derivative between the two Bäcklund transformations gives an algebraic relation between $w(z; a, b, c)$, $w_+(z; a_+, b_+, c_+)$ and $w_-(z; a_-, b_-, c_-)$, which is the discrete equation. Then setting $w(z; a, b, c) = w_n$, $w_{\pm}(z; a_{\pm}, b_{\pm}, c_{\pm}) = w_{n\pm 1}$, with $(a, b, c) = (a_n, b_n, c_n)$ and $(a_{\pm}, b_{\pm}, c_{\pm}) = (a_{n\pm 1}, b_{n\pm 1}, c_{n\pm 1})$, gives the chain

$$\{w_{n-1}, a_{n-1}, b_{n-1}, c_{n-1}\} \begin{array}{c} \xrightarrow{\mathcal{R}_+} \\ \xleftarrow{\mathcal{R}_-} \end{array} \{w_n, a_n, b_n, c_n\} \begin{array}{c} \xrightarrow{\mathcal{R}_+} \\ \xleftarrow{\mathcal{R}_-} \end{array} \{w_{n+1}, a_{n+1}, b_{n+1}, c_{n+1}\}.$$

The parameters satisfy a linear discrete system derived from (2.4). See, for example, [19, 25] for more details of this procedure.

Remarks 3.1

- (i) It is essential that $\mathcal{R}_-$ is the inverse transformation of $\mathcal{R}_+$, see Example 3.2. It is possible to create an algebraic relation between three solutions of P_V (1.2) using two Bäcklund transformations $\mathcal{R}_1$ and $\mathcal{R}_2$ when $\mathcal{R}_2 \neq \mathcal{R}_1^{-1}$, but this won't necessarily result in a discrete Painlevé equation. For example, in [39, §39], see also [37, 38], there are algebraic relations of three (and four) solutions of P_V , but these are not discrete Painlevé equations.
- (ii) The Bessel function $J_n(z)$ satisfies the differential equation

$$z^2 \frac{d^2 J_n}{dz^2} + z \frac{d J_n}{dz} + (z^2 - n^2) J_n = 0, \quad (3.1)$$

and the recurrence relations

$$z \frac{d J_n}{dz}(z) = z J_{n-1}(z) - n J_n(z), \quad (3.2a)$$

$$z \frac{dJ_n}{dz}(z) = nJ_n(z) - zJ_{n+1}(z), \quad (3.2b)$$

$$zJ_{n+1}(z) - 2nJ_n(z) + zJ_{n-1}(z) = 0, \quad (3.2c)$$

cf. [69, §10]. Eliminating $\frac{dJ_n}{dz}(z)$ between (3.2a) and (3.2b) yields the recurrence relation (3.2c). Letting $n \rightarrow n+1$ in (3.2a) and eliminating $J_{n+1}(z)$ using (3.2b) gives the differential equation (3.1). Thus the Bessel function $J_n(z)$ satisfies both the differential equation (3.1) and the difference equation (3.2c). An analogous situation occurs for the Painlevé equations, except P_I (1.6).

In the next three subsections, we derive discrete equations using the Bäcklund transformations

$$\begin{aligned} \mathcal{R}_1 &= \mathcal{T}_{1,-1,1}, & \mathcal{R}_1^{-1} &= \mathcal{T}_{-1,1,-1}, \\ \mathcal{R}_2 &= \mathcal{T}_{-1,-1,-1} \circ \mathcal{T}_{-1,1,-1}, & \mathcal{R}_2^{-1} &= \mathcal{T}_{1,1,1} \circ \mathcal{T}_{1,-1,1}, \\ \mathcal{R}_3 &= \mathcal{T}_{-1,1,-1} \circ \mathcal{T}_{1,-1,-1} \circ \mathcal{T}_{1,-1,1}, & \mathcal{R}_3^{-1} &= \mathcal{T}_{-1,1,1}, \end{aligned}$$

as well as a discrete equation which has a ternary symmetry using the Bäcklund transformation

$$\mathcal{R}_4 = \mathcal{R}_3^2 = \mathcal{T}_{-1,1,-1} \circ \mathcal{T}_{1,-1,-1}^2 \circ \mathcal{T}_{1,-1,1}, \quad \mathcal{R}_4^{-1} = \mathcal{T}_{-1,1,1}^2.$$

These Bäcklund transformations are the only combinations which we have found that give rise to a discrete equation.

Example 3.2 Consider the Bäcklund transformations $\mathcal{T}_{-1,1,1}$ and $\mathcal{T}_{1,-1,1}$. First applying $\mathcal{T}_{-1,1,1}$ then $\mathcal{T}_{1,-1,1}$ gives

$$\mathcal{T}_{-1,1,1}(w) = 1 + \frac{2zw}{zw' - aw^2 + (a - b - z)w + b}, \quad \mathcal{T}_{1,-1,1} \circ \mathcal{T}_{-1,1,1}(w) = w,$$

with $' \equiv d/dz$, where we have used the fact that w satisfies P_V (1.2) with $(\alpha, \beta, \gamma) = (\frac{1}{2}a^2, -\frac{1}{2}b^2, c)$. However applying $\mathcal{T}_{1,-1,1}$ then $\mathcal{T}_{-1,1,1}$ gives

$$\begin{aligned} \mathcal{T}_{1,-1,1}(w) &= 1 - \frac{2zw}{zw' + aw^2 - (a + b - z)w + b}, \\ \mathcal{T}_{-1,1,1} \circ \mathcal{T}_{1,-1,1}(w) &= w + \frac{(a - b - c + 1)w(w - 1)^2}{zw' + (b + c - 1)w^2 - (2b + c - 1 + z)w + b}. \end{aligned}$$

Hence we conclude that $\mathcal{T}_{1,-1,1} \neq \mathcal{T}_{-1,1,1}^{-1}$ and so these Bäcklund transformations can not be used to derive a discrete equation. We remark that the parameters transform as follows

$$\begin{aligned} \mathcal{T}_{1,-1,1} \circ \mathcal{T}_{-1,1,1}(a, b, c) &= (a, -b, c), \\ \mathcal{T}_{-1,1,1} \circ \mathcal{T}_{1,-1,1}(a, b, c) &= \left(\frac{1}{2}(a + b + c - 1), \frac{1}{2}(a + b - c + 1), a - b + 1\right), \end{aligned}$$

confirming that $\mathcal{T}_{1,-1,1} \neq \mathcal{T}_{-1,1,1}^{-1}$.

3.1 First Discrete Equation: Asymmetric Discrete P_{II}

Consider the Bäcklund transformation $\mathcal{R}_1 = \mathcal{T}_{1,-1,1}$ which has an inverse $\mathcal{R}_1^{-1} = \mathcal{T}_{-1,1,-1}$, then

$$w_{n+1} = \mathcal{R}_1(w_n) = 1 - \frac{2zw_n}{zw'_n + a_n w_n^2 - (a_n + b_n - z)w_n + b_n}, \quad (3.3a)$$

$$w_{n-1} = \mathcal{R}_1^{-1}(w_n) = 1 + \frac{2zw_n}{zw'_n - a_n w_n^2 + (a_n + b_n - z)w_n - b_n}, \quad (3.3b)$$

with $' \equiv d/dz$, where a_n, b_n and c_n satisfy

$$\begin{aligned} (a_{n+1}, b_{n+1}, c_{n+1}) &= \mathcal{R}_1(a_n, b_n, c_n) \\ &= \left(\frac{1}{2}(a_n - b_n + c_n + 1), \frac{1}{2}(-a_n + b_n + c_n - 1), a_n + b_n\right), \\ (a_{n-1}, b_{n-1}, c_{n-1}) &= \mathcal{R}_1^{-1}(a_n, b_n, c_n) \\ &= \left(\frac{1}{2}(a_n - b_n + c_n - 1), \frac{1}{2}(-a_n + b_n + c_n + 1), a_n + b_n\right), \end{aligned}$$

which have solution

$$a_n = \frac{1}{2}[n + \lambda + \rho + (-1)^n \varphi], \quad b_n = -\frac{1}{2}[n + \lambda - \rho - (-1)^n \varphi], \quad c_n = \rho - (-1)^n \varphi, \quad (3.4)$$

with λ, ρ and φ arbitrary constants. From (3.3) we obtain

$$\frac{1}{w_{n+1} - 1} + \frac{1}{2w_n} \frac{dw_n}{dz} + \frac{a_n w_n^2 - (a_n + b_n - z)w_n + b_n}{2zw_n} = 0, \quad (3.5a)$$

$$\frac{1}{w_{n-1} - 1} - \frac{1}{2w_n} \frac{dw_n}{dz} + \frac{a_n w_n^2 - (a_n + b_n - z)w_n + b_n}{2zw_n} = 0, \quad (3.5b)$$

so adding gives

$$\frac{1}{w_{n+1} - 1} + \frac{1}{w_{n-1} - 1} + \frac{a_n w_n^2 - (a_n + b_n - z)w_n + b_n}{zw_n} = 0,$$

which is equation (3.14) in [40].

Setting $w_n = (q_n + 1)/(q_n - 1)$ and $w_{n\pm 1} = (q_{n\pm 1} + 1)/(q_{n\pm 1} - 1)$ in (3.3) gives

$$q_{n+1} + \frac{2}{1 - q_n^2} \frac{dq_n}{dz} - \frac{2}{z} \frac{(n + \lambda)q_n + \rho + (-1)^n \varphi}{1 - q_n^2} = 0, \quad (3.6a)$$

$$q_{n-1} - \frac{2}{1 - q_n^2} \frac{dq_n}{dz} - \frac{2}{z} \frac{(n + \lambda)q_n + \rho + (-1)^n \varphi}{1 - q_n^2} = 0. \quad (3.6b)$$

Hence by adding we obtain

$$q_{n+1} + q_{n-1} = \frac{4}{z} \frac{(n + \lambda)q_n + \rho + (-1)^n \varphi}{1 - q_n^2}, \quad (3.7)$$

which is known as the *asymmetric dP_{II} equation* [76] and is a generalisation of dP_{II} (1.7). Further q_n satisfies the second-order equation

$$\begin{aligned} \frac{d^2 q_n}{dz^2} = & \frac{q_n}{q_n^2 - 1} \left(\frac{dq_n}{dz} \right)^2 - \frac{1}{z} \frac{dq_n}{dz} - \frac{a_n^2(q_n + 1)^2 - b_n^2(q_n - 1)^2}{z^2(q_n^2 - 1)} - \frac{c_n(q_n^2 - 1)}{2z} \\ & + \frac{1}{4}q_n(q_n^2 - 1), \end{aligned} \quad (3.8)$$

with a_n , b_n and c_n given by (3.4).

In order to consider the alternating term in (3.7), i.e. the binary symmetry, we set $q_{2n} = x_n$ and $q_{2n+1} = y_n$ in (3.7) and obtain the discrete system

$$x_{n+1} + x_n = \frac{4}{z} \frac{(2n + \lambda + 1)y_n + \rho - \varphi}{1 - y_n^2}, \quad y_n + y_{n-1} = \frac{4}{z} \frac{(2n + \lambda)x_n + \rho + \varphi}{1 - x_n^2}, \quad (3.9)$$

which is regarded as a discrete P_{III} equation (dP_{III}) since it admits a continuous limit to the third Painlevé equation (P_{III})

$$\frac{d^2 w}{dz^2} = \frac{1}{w} \left(\frac{dw}{dz} \right)^2 - \frac{1}{z} \frac{dw}{dz} + \frac{\alpha w^2 + \beta}{w} + \gamma w^3 + \frac{\delta}{w}, \quad (3.10)$$

with α , β , γ and δ constants [33, 74].

We note that

$$\mathcal{R}_1^2(a, b, c) = (a + 1, b - 1, c),$$

and so

$$\mathcal{R}_1^{2n}(a, b, c) = (n + a, b - n, c).$$

Therefore since

$$\begin{aligned}(a_0, b_0, c_0) &= \left(\frac{1}{2}(\lambda + \rho + \varphi), -\frac{1}{2}(\lambda - \rho - \varphi), \rho - \varphi\right), \\(a_1, b_1, c_1) &= \left(\frac{1}{2}(\lambda + \rho - \varphi + 1), -\frac{1}{2}(\lambda - \rho + \varphi + 1), \rho + \varphi\right),\end{aligned}$$

then

$$\begin{aligned}(a_{2n}, b_{2n}, c_{2n}) &= \left(\frac{1}{2}(2n + \lambda + \rho + \varphi), -\frac{1}{2}(2n + \lambda - \rho - \varphi), \rho - \varphi\right), \\(a_{2n+1}, b_{2n+1}, c_{2n+1}) &= \left(\frac{1}{2}(2n + 1 + \lambda + \rho - \varphi), -\frac{1}{2}(2n + 1 + \lambda - \rho + \varphi), \rho + \varphi\right).\end{aligned}$$

If $w_0(z)$ satisfies P_V (1.2) with $\alpha = \frac{1}{8}(\lambda + \rho + \varphi)^2$, $\beta = -\frac{1}{8}(\lambda - \rho - \varphi)^2$ and $\gamma = \rho - \varphi$ then

$$x_n = q_{2n} = \frac{\mathcal{R}_1^{2n}(w_0) + 1}{\mathcal{R}_1^{2n}(w_0) - 1}, \quad y_n = q_{2n+1} = \frac{\mathcal{R}_1^{2n+1}(w_0) + 1}{\mathcal{R}_1^{2n+1}(w_0) - 1}, \quad (3.11)$$

satisfy (3.9).

Remarks 3.3

(i) In the case when $\rho = \varphi = 0$, equation (3.8) becomes

$$\frac{d^2 q_n}{dz^2} = \frac{q_n}{q_n^2 - 1} \left\{ \left(\frac{dq_n}{dz} \right)^2 - \frac{(4n + \lambda)^2}{z^2} \right\} - \frac{1}{z} \frac{dq_n}{dz} + \frac{1}{4} q_n (q_n^2 - 1), \quad (3.12)$$

which is equivalent to P_V (1.2) with $\alpha = -\beta = \frac{1}{2}(n + \lambda)^2$ and $\gamma = 0$. Making the transformation $q_n = 1/\sqrt{1 - u_n}$ with $z = 2\sqrt{x}$ in equation (3.12) gives $\text{deg-}P_V$ (1.3) with $\alpha = 0$, $\beta = -2(n + \lambda)^2$ and $\gamma = \frac{1}{2}$. Solutions of $\text{deg-}P_V$ (1.3) are related to those of P_{III} (3.10), cf. [39, Theorem 34.2]. Solutions of equation (3.12), in terms of Bessel functions, are discussed in [16, 41].

(ii) Equation (3.12) arises as a reduction of the two-dimensional complex sine-Gordon equation, also known as the Pohlmeyer-Lund-Regge model [50, 51, 71], in quantum systems [10–12], in relativity [31] and in coefficients of the three-term recurrence relation for orthogonal polynomials with respect to the weight $w(\theta) = \exp(t \cos \theta)$ on the unit circle, see [82, equation (3.13)]. The orthogonal polynomials for this weight are related to unitary random matrices [70].

3.2 Second Discrete Equation

Consider the Bäcklund transformation $\mathcal{R}_2 = \mathcal{T}_{-1,-1,-1} \circ \mathcal{T}_{-1,1,-1}$ which has the inverse $\mathcal{R}_2^{-1} = \mathcal{T}_{1,1,1} \circ \mathcal{T}_{1,-1,1}$. Then we have

$$\begin{aligned}
\mathcal{T}_{-1,1,-1}(w) &= 1 + \frac{2zw}{zw' - aw^2 + (a+b-z)w - b}, \\
\mathcal{T}_{-1,-1,-1} \circ \mathcal{T}_{-1,1,-1}(w) &= \frac{1}{w} \left\{ 1 + \frac{(a-b-c-1)(w-1)^2}{zw' - aw^2 + (2a-c-1+z)w - (a-c-1)} \right\}, \\
\mathcal{T}_{1,-1,1}(w) &= 1 - \frac{2zw}{zw' + aw^2 - (a+b-z)w + b}, \\
\mathcal{T}_{1,1,1} \circ \mathcal{T}_{1,-1,1}(w) &= \frac{1}{w} \left\{ 1 - \frac{(a-b-c+1)(w-1)^2}{zw' + aw^2 - (2a-c+1+z)w + (a-c+1)} \right\},
\end{aligned}$$

with $' \equiv d/dz$, where we have used the fact that w satisfies P_V (1.2) with $(\alpha, \beta, \gamma) = (\frac{1}{2}a^2, -\frac{1}{2}b^2, c)$ to eliminate w'' . Also for the parameters (a, b, c) we have

$$\begin{aligned}
\mathcal{T}_{-1,1,-1}(a, b, c) &= \left(\frac{1}{2}(a-b+c-1), -\frac{1}{2}(a-b-c-1), a+b \right), \\
\mathcal{T}_{-1,-1,-1} \circ \mathcal{T}_{-1,1,-1}(a, b, c) &= \left(\frac{1}{2}(a+b-c-1), \frac{1}{2}(a+b+c+1), -a+b+1 \right), \\
\mathcal{T}_{1,-1,1}(a, b, c) &= \left(\frac{1}{2}(a-b+c+1), -\frac{1}{2}(a-b-c+1), a+b \right), \\
\mathcal{T}_{1,1,1} \circ \mathcal{T}_{1,-1,1}(a, b, c) &= \left(\frac{1}{2}(a+b-c+1), \frac{1}{2}(a+b+c-1), -a+b-1 \right).
\end{aligned}$$

Consequently, if $w_n = w(z; \frac{1}{2}a_n^2, -\frac{1}{2}b_n^2, c_n)$ and $w_{n\pm 1} = w(z; \frac{1}{2}a_{n\pm 1}^2, -\frac{1}{2}b_{n\pm 1}^2, c_{n\pm 1})$ then

$$w_{n+1} = \mathcal{R}_2(w_n) = \frac{1}{w_n} \left\{ 1 + \frac{(a_n - b_n - c_n - 1)(w_n - 1)^2}{zw'_n - a_n w_n^2 + (2a_n - c_n - 1 + z)w_n - (a_n - c_n - 1)} \right\}, \quad (3.13a)$$

$$w_{n-1} = \mathcal{R}_2^{-1}(w_n) = \frac{1}{w_n} \left\{ 1 - \frac{(a_n - b_n - c_n + 1)(w_n - 1)^2}{zw'_n + a_n w_n^2 - (2a_n - c_n + 1 + z)w_n + (a_n - c_n + 1)} \right\}, \quad (3.13b)$$

where a_n, b_n and c_n satisfy

$$\begin{aligned}
(a_{n+1}, b_{n+1}, c_{n+1}) &= \mathcal{R}_2(a_n, b_n, c_n) \\
&= \left(\frac{1}{2}(a_n + b_n - c_n - 1), \frac{1}{2}(a_n + b_n + c_n + 1), -a_n + b_n + 1 \right), \\
(a_{n-1}, b_{n-1}, c_{n-1}) &= \mathcal{R}_2^{-1}(a_n, b_n, c_n) \\
&= \left(\frac{1}{2}(a_n + b_n - c_n + 1), \frac{1}{2}(a_n + b_n + c_n - 1), -a_n + b_n - 1 \right),
\end{aligned}$$

which have solution

$$\begin{aligned}
a_n &= -\frac{1}{2}[n + \lambda + \rho + (-1)^n \varphi], & b_n &= \frac{1}{2}[n + \lambda - \rho + (-1)^n \varphi], \\
c_n &= n + \lambda - (-1)^n \varphi,
\end{aligned} \quad (3.14)$$

with λ , ρ and φ arbitrary constants. From (3.13) we obtain

$$\frac{a_n - b_n - c_n - 1}{w_n w_{n+1} - 1} - \frac{z}{(w_n - 1)^2} \frac{dw_n}{dz} + a_n - \frac{z - c_n - 1}{w_n - 1} - \frac{z}{(w_n - 1)^2} = 0, \quad (3.15a)$$

$$\frac{a_n - b_n - c_n + 1}{w_n w_{n-1} - 1} + \frac{z}{(w_n - 1)^2} \frac{dw_n}{dz} + a_n - \frac{z - c_n + 1}{w_n - 1} - \frac{z}{(w_n - 1)^2} = 0, \quad (3.15b)$$

Hence adding and using (3.14) gives the difference equation

$$\begin{aligned} & \frac{2n + 2\lambda + 1}{w_n w_{n+1} - 1} + \frac{2n + 2\lambda - 1}{w_n w_{n-1} - 1} + n + \lambda + \rho + (-1)^n \varphi \\ & - \frac{2[n + \lambda - (-1)^n \varphi]}{w_n - 1} + \frac{2zw_n}{(w_n - 1)^2} = 0, \end{aligned} \quad (3.16)$$

which is equivalent to equation (2.8) in [73].

Setting $w_n = (q_n + 1)/(q_n - 1)$ and $w_{n\pm 1} = (q_{n\pm 1} + 1)/(q_{n\pm 1} - 1)$ in (3.13) gives

$$\frac{2n + 2\lambda + 1}{q_{n+1} + q_n} - \frac{z}{1 - q_n^2} \frac{dq_n}{dz} - \frac{1}{2}z + \frac{[n + \lambda + (-1)^n \varphi]q_n + \rho}{1 - q_n^2} = 0, \quad (3.17a)$$

$$\frac{2n + 2\lambda - 1}{q_n + q_{n-1}} + \frac{z}{1 - q_n^2} \frac{dq_n}{dz} - \frac{1}{2}z + \frac{[n + \lambda + (-1)^n \varphi]q_n + \rho}{1 - q_n^2} = 0. \quad (3.17b)$$

Hence by adding we obtain

$$\frac{2n + 2\lambda + 1}{q_{n+1} + q_n} + \frac{2n + 2\lambda - 1}{q_n + q_{n-1}} = z - \frac{2[n + \lambda + (-1)^n \varphi]q_n + 2\rho}{1 - q_n^2}, \quad (3.18)$$

and then letting $\xi_n = n + \lambda$ gives

$$\frac{\xi_{n+1} + \xi_n}{q_{n+1} + q_n} + \frac{\xi_n + \xi_{n-1}}{q_n + q_{n-1}} = z - \frac{2[\xi_n + (-1)^n \varphi]q_n + 2\rho}{1 - q_n^2}. \quad (3.19)$$

This is equation (2.9) in [73], equation (3.25) in [80] and equation (13) in [35]. Following [73], we introduce a new variable p_n through

$$p_n = 1 - \frac{2}{z} \frac{\xi_{n+1} + \xi_n}{q_{n+1} + q_n},$$

and then (3.19) becomes

$$p_n + p_{n-1} = \frac{4}{z} \frac{[\xi_n + (-1)^n \varphi]q_n + \rho}{1 - q_n^2}, \quad q_{n+1} + q_n = \frac{2}{z} \frac{\xi_{n+1} + \xi_n}{1 - p_n}, \quad (3.20)$$

which is system (2.10) in [73]. Ramani, Ohta and Grammaticos [73] state that “the system (3.20) describes a motion in the same parameter space as asymmetric dP_{II} which is associated with the A_3 affine Weyl group. While asymmetric dP_{II} describes a motion along the axes, the geometry of (3.20) is more complicated: the associated motion is a ‘staircase’ one”. To our knowledge¹, there is no discussion in the literature as to which (continuous) Painlevé equation arises in the continuum limit for either (3.19) or (3.20).

In order to consider the alternating term in (3.18), i.e. the binary symmetry, setting $q_{2n} = x_n$ and $q_{2n+1} = y_n$ yields the discrete system

$$\frac{4n + 2\lambda + 3}{x_{n+1} + y_n} + \frac{4n + 2\lambda + 1}{x_n + y_n} = z - \frac{2(2n + \lambda - \varphi + 1)y_n + 2\rho}{1 - y_n^2}, \quad (3.21a)$$

$$\frac{4n + 2\lambda + 1}{x_n + y_n} + \frac{4n + 2\lambda - 1}{x_n + y_{n-1}} = z - \frac{2(2n + \lambda + \varphi)x_n + 2\rho}{1 - x_n^2}, \quad (3.21b)$$

and so if $\xi_n = n + \lambda$ then

$$\begin{aligned} \frac{\xi_{2n+2} + \xi_{2n+1}}{x_{n+1} + y_n} + \frac{\xi_{2n+1} + \xi_{2n}}{x_n + y_n} &= z - \frac{2(\xi_{2n+1} - \varphi)y_n + 2\rho}{1 - y_n^2}, \\ \frac{\xi_{2n+1} + \xi_{2n}}{x_n + y_n} + \frac{\xi_{2n} + \xi_{2n-1}}{x_n + y_{n-1}} &= z - \frac{2(\xi_{2n} + \varphi)x_n + 2\rho}{1 - x_n^2}. \end{aligned}$$

Further q_n satisfies the second-order equation

$$\begin{aligned} \frac{d^2 q_n}{dz^2} &= \frac{q_n}{q_n^2 - 1} \left(\frac{dq_n}{dz} \right)^2 - \frac{1}{z} \frac{dq_n}{dz} - \frac{a_n^2(q_n + 1)^2 - b_n^2(q_n - 1)^2}{z^2(q_n^2 - 1)} \\ &\quad - \frac{c_n(q_n^2 - 1)}{2z} + \frac{1}{4}q_n(q_n^2 - 1), \end{aligned} \quad (3.22)$$

with a_n , b_n and c_n given by (3.14).

We note that

$$\mathcal{R}_2^2(a, b, c) = (a - 1, b + 1, c + 2),$$

and so

$$\mathcal{R}_2^{2n}(a, b, c) = (a - n, b + n, c + 2n).$$

Therefore since

$$\begin{aligned} (a_0, b_0, c_0) &= \left(-\frac{1}{2}(\lambda + \rho + \varphi), \frac{1}{2}(\lambda - \rho + \varphi), \lambda - \varphi \right), \\ (a_1, b_1, c_1) &= \left(-\frac{1}{2}(\lambda + \rho - \varphi + 1), \frac{1}{2}(\lambda - \rho - \varphi + 1), \lambda + \varphi + 1 \right). \end{aligned}$$

then

¹ This was confirmed in an email communication with Basil Grammaticos.

$$\begin{aligned}(a_{2n}, b_{2n}, c_{2n}) &= \left(-\frac{1}{2}(2n + \lambda + \rho + \varphi), \frac{1}{2}(2n + \lambda - \rho + \varphi), 2n + \lambda - \varphi\right), \\(a_{2n+1}, b_{2n+1}, c_{2n+1}) &= \left(-\frac{1}{2}(2n + \lambda + \rho - \varphi + 1), \frac{1}{2}(2n + \lambda - \rho - \varphi + 1), \right. \\&\quad \left.2n + \lambda + \varphi + 1\right).\end{aligned}$$

If $w_0(z)$ satisfies P_V (1.2) with $\alpha = \frac{1}{8}(\lambda + \rho + \varphi)^2$, $\beta = -\frac{1}{8}(\lambda - \rho + \varphi)^2$ and $\gamma = \lambda - \varphi$ then

$$x_n = q_{2n} = \frac{\mathcal{R}_2^{2n}(w_0) + 1}{\mathcal{R}_2^{2n}(w_0) - 1}, \quad y_n = q_{2n+1} = \frac{\mathcal{R}_2^{2n+1}(w_0) + 1}{\mathcal{R}_2^{2n+1}(w_0) - 1}, \quad (3.23)$$

satisfy (3.21)

Remarks 3.4 Since the Bäcklund transformation $\mathcal{R}_2 = \mathcal{T}_{-1,-1,-1} \circ \mathcal{T}_{-1,1,-1}$ involves a sequence of two Bäcklund transformations then it might be expected that the resulting transformation is quadratic in dw/dz . Generally if one applies the Bäcklund transformation $\mathcal{T}_{\varepsilon_1, \varepsilon_2, \varepsilon_3}$ followed by the Bäcklund transformation $\mathcal{T}_{\sigma_1, \sigma_2, \sigma_3}$ then the resulting transformation is quadratic in dw/dz , but in some cases there is a simplification, which happens in this case.

3.3 Third Discrete Equation: Ternary Discrete P_I

Consider the Bäcklund transformation $\mathcal{R}_3 = \mathcal{T}_{-1,1,-1} \circ \mathcal{T}_{1,-1,-1} \circ \mathcal{T}_{1,-1,1}$, which has an inverse $\mathcal{R}_3^{-1} = \mathcal{T}_{-1,1,1}$, then

$$\begin{aligned}\mathcal{T}_{1,-1,1}(w) &= 1 - \frac{2zw}{zw' + aw^2 - (a+b-z)w + b}, \\ \mathcal{T}_{1,-1,-1} \circ \mathcal{T}_{1,-1,1}(w) &= \frac{1}{w} \left\{ 1 - \frac{(a-b+c+1)(w-1)^2}{zw' + aw^2 - (2a+c-z+1)w + b} \right\}, \\ \mathcal{T}_{-1,1,-1} \circ \mathcal{T}_{1,-1,-1} \circ \mathcal{T}_{1,-1,1}(w) &= 1 - \frac{2zw}{zw' + aw^2 - (a+b-z)w + b}, \\ \mathcal{T}_{-1,1,1}(w) &= 1 + \frac{2zw}{zw' - aw^2 + (a-b-z)w + b},\end{aligned}$$

with $' \equiv d/dz$, where we have used the fact that w satisfies P_V (1.2) with $(\alpha, \beta, \gamma) = (\frac{1}{2}a^2, -\frac{1}{2}b^2, c)$ to eliminate w'' . Also for the parameters (a, b, c) we have

$$\begin{aligned}\mathcal{T}_{1,-1,1}(a, b, c) &= \left(\frac{1}{2}(a-b+c+1), -\frac{1}{2}(a-b-c+1), a+b\right), \\ \mathcal{T}_{1,-1,-1} \circ \mathcal{T}_{1,-1,1}(a, b, c) &= \left(\frac{1}{2}(a+b+c+1), \frac{1}{2}(a+b-c-1), a-b+1\right), \\ \mathcal{T}_{-1,1,-1} \circ \mathcal{T}_{1,-1,-1} \circ \mathcal{T}_{1,-1,1}(a, b, c) &= \left(\frac{1}{2}(a-b+c+1), \frac{1}{2}(a-b-c+1), a+b\right), \\ \mathcal{T}_{-1,1,1}(a, b, c) &= \left(\frac{1}{2}(a+b+c-1), -\frac{1}{2}(a+b-c-1), a-b\right),\end{aligned}$$

Consequently, if $w_n = w(z; \frac{1}{2}a_n^2, -\frac{1}{2}b_n^2, c_n)$ and $w_{n\pm 1} = w(z; \frac{1}{2}a_{n\pm 1}^2, -\frac{1}{2}b_{n\pm 1}^2, c_{n\pm 1})$ then

$$w_{n+1} = \mathcal{R}_3(w_n) = 1 - \frac{2zw_n}{zw'_n + a_n w_n^2 - (a_n + b_n - z)w_n + b_n}, \quad (3.24a)$$

$$w_{n-1} = \mathcal{R}_3^{-1}(w_n) = 1 + \frac{2zw_n}{zw'_n - a_n w_n^2 + (a_n - b_n - z)w_n + b_n}, \quad (3.24b)$$

with $' \equiv d/dz$, where a_n, b_n and c_n satisfy

$$\begin{aligned} (a_{n+1}, b_{n+1}, c_{n+1}) &= \mathcal{R}_3(a_n, b_n, c_n) \\ &= \left(\frac{1}{2}(a_n - b_n + c_n + 1), \frac{1}{2}(a_n - b_n - c_n + 1), a_n + b_n\right), \end{aligned} \quad (3.25a)$$

$$\begin{aligned} (a_{n-1}, b_{n-1}, c_{n-1}) &= \mathcal{R}_3^{-1}(a_n, b_n, c_n) \\ &= \left(\frac{1}{2}(a_n + b_n + c_n - 1), -\frac{1}{2}(a_n + b_n - c_n - 1), a_n - b_n\right). \end{aligned} \quad (3.25b)$$

These have solution

$$a_n = \frac{1}{3}n + \rho + \lambda \cos\left(\frac{2}{3}\pi n\right) + \frac{1}{3}\sqrt{3}\varphi \sin\left(\frac{2}{3}\pi n\right), \quad (3.26a)$$

$$b_n = \frac{1}{3} + \sqrt{3}\lambda \sin\left(\frac{2}{3}\pi n\right) - \varphi \cos\left(\frac{2}{3}\pi n\right), \quad (3.26b)$$

$$c_n = \frac{1}{3}n + \rho - 2\lambda \cos\left(\frac{2}{3}\pi n\right) - \frac{2}{3}\sqrt{3}\varphi \sin\left(\frac{2}{3}\pi n\right), \quad (3.26c)$$

with λ, ρ and φ arbitrary constants (the factor of $\sqrt{3}$ is for convenience), i.e.

$$\begin{aligned} a_{3n} &= n + \lambda + \rho, & b_{3n} &= -\varphi + \frac{1}{3}, & c_{3n} &= n - 2\lambda + \rho, \\ a_{3n+1} &= n - \frac{1}{2}\lambda + \frac{1}{2}\varphi + \rho + \frac{1}{3}, & b_{3n+1} &= \frac{3}{2}\lambda + \frac{1}{2}\varphi + \frac{1}{3}, & c_{3n+1} &= n + \lambda - \varphi + \rho + \frac{1}{3}, \\ a_{3n+2} &= n - \frac{1}{2}\lambda - \frac{1}{2}\varphi + \rho + \frac{2}{3}, & b_{3n+2} &= -\frac{3}{2}\lambda + \frac{1}{2}\varphi + \frac{1}{3}, & c_{3n+2} &= n + \lambda + \varphi + \rho + \frac{2}{3}. \end{aligned}$$

which have a ternary symmetry in contrast to the binary symmetry for the parameters discussed in §3.1 and §3.2 above.

Setting $w_n = 1 + 1/x_n$ and $w_{n\pm 1} = 1 + 1/x_{n\pm 1}$ in (3.24) gives

$$x_{n+1} - \frac{1}{2x_n(x_n + 1)} \frac{dx_n}{dz} + \frac{a_n}{2zx_n} - \frac{b_n}{2z(x_n + 1)} + \frac{1}{2} = 0, \quad (3.27a)$$

$$x_{n-1} + \frac{1}{2x_n(x_n + 1)} \frac{dx_n}{dz} + \frac{a_n}{2zx_n} + \frac{b_n}{2z(x_n + 1)} + \frac{1}{2} = 0. \quad (3.27b)$$

Hence by adding we obtain

$$x_n(x_{n+1} + x_{n-1} + 1) + \frac{a_n}{z} = 0, \quad (3.28)$$

which is known as the *ternary dP_I equation* [34, 80]. We remark that this derivation of (3.28) was given by two of us (Clarkson and Mitchell, in collaboration with Dzhamay and Hone), see [18].

Since equation (3.28) has a ternary symmetry, then it can be written as a system of three equations. Setting $x_{3n} = X_n$, $x_{3n+1} = Y_n$ and $x_{3n+2} = Z_n$ in (3.28) yields the discrete system

$$X_n(Y_n + Z_{n-1} + 1) + \frac{a_{3n}}{z} = 0, \quad (3.29a)$$

$$Y_n(Z_n + X_n + 1) + \frac{a_{3n+1}}{z} = 0, \quad (3.29b)$$

$$Z_n(X_{n+1} + Y_n + 1) + \frac{a_{3n+2}}{z} = 0. \quad (3.29c)$$

Further x_n satisfies the second-order equation

$$\begin{aligned} \frac{d^2 x_n}{dz^2} = & \frac{2x_n + 1}{2x_n(x_n + 1)} \left(\frac{dx_n}{dz} \right)^2 - \frac{1}{z} \frac{dx_n}{dz} - \frac{a_n^2(x_n + 1)^2 - b_n^2 x_n^2}{2z^2 x_n(x_n + 1)} - \frac{c_n x_n(x_n + 1)}{z} \\ & + \frac{x_n(x_n + 1)(2x_n + 1)}{2}, \end{aligned} \quad (3.30)$$

with a_n , b_n and c_n given by (3.26).

We note that

$$\mathcal{R}_3^3(a, b, c) = (a + 1, b, c + 1),$$

and so

$$\mathcal{R}_3^{3n}(a, b, c) = (a + n, b, c + n).$$

Therefore since

$$\begin{aligned} (a_0, b_0, c_0) &= \left(\lambda + \rho, -\varphi + \frac{1}{3}, -2\lambda + \rho \right), \\ (a_1, b_1, c_1) &= \left(-\frac{1}{2}\lambda + \frac{1}{2}\varphi + \rho + \frac{1}{3}, \frac{3}{2}\lambda + \frac{1}{2}\varphi + \frac{1}{3}, \lambda - \varphi + \rho + \frac{1}{3} \right), \\ (a_2, b_2, c_2) &= \left(-\frac{1}{2}\lambda - \frac{1}{2}\varphi + \rho + \frac{2}{3}, -\frac{3}{2}\lambda + \frac{1}{2}\varphi + \frac{1}{3}, \lambda + \varphi + \rho + \frac{2}{3} \right), \end{aligned}$$

then

$$\begin{aligned} (a_{3n}, b_{3n}, c_{3n}) &= \left(n + \lambda + \rho, -\varphi + \frac{1}{3}, n - 2\lambda + \rho \right), \\ (a_{3n+1}, b_{3n+1}, c_{3n+1}) &= \left(n - \frac{1}{2}\lambda + \frac{1}{2}\varphi + \rho + \frac{1}{3}, \frac{3}{2}\lambda + \frac{1}{2}\varphi + \frac{1}{3}, n + \lambda - \varphi + \rho + \frac{1}{3} \right), \\ (a_{3n+2}, b_{3n+2}, c_{3n+2}) &= \left(n - \frac{1}{2}\lambda - \frac{1}{2}\varphi + \rho + \frac{2}{3}, -\frac{3}{2}\lambda + \frac{1}{2}\varphi + \frac{1}{3}, n + \lambda + \varphi + \rho + \frac{2}{3} \right). \end{aligned}$$

If w_0 satisfies P_V (1.2) with $\alpha = \frac{1}{2}(\lambda + \rho)^2$, $\beta = -\frac{1}{2}(\varphi - \frac{1}{3})^2$ and $\gamma = -2\lambda + \rho$ then

$$X_n = x_{3n} = \frac{1}{\mathcal{R}_3^{3n}(w_0) - 1}, \quad Y_n = x_{3n+1} = \frac{1}{\mathcal{R}_3^{3n+1}(w_0) - 1},$$

$$Z_n = x_{3n+2} = \frac{1}{\mathcal{R}_3^{3n+2}(w_0) - 1},$$

satisfy (3.29).

Remarks 3.5

- (i) In terms of w , the transformation (3.24a) is the same as $\mathcal{T}_{1,-1,1}(w)$. However, the parameters (a, b, c) map differently since

$$\mathcal{T}_{1,-1,1}(a, b, c) = \left(\frac{1}{2}(a - b + c - 1), \frac{1}{2}(-a + b + c - 1), a + b\right).$$

It is straightforward to show that $\mathcal{T}_{1,-1,1} \circ \mathcal{T}_{-1,1,1}(w) = w$, whereas $\mathcal{T}_{-1,1,1} \circ \mathcal{T}_{1,-1,1}(w) \neq w$, so $\mathcal{T}_{1,-1,1}$ is *not* the inverse of $\mathcal{T}_{-1,1,1}$.

- (ii) It is interesting to note that given a solution $w_0(z)$ of P_V (1.2) with $(\alpha_0, \beta_0, \gamma_0) = (\frac{1}{2}a_0^2, -\frac{1}{2}b_0^2, c_0)$, the transformation $\mathcal{R}_1$ (3.3a) maps it to the solution $w_1(z)$ with $(\alpha_1, \beta_1, \gamma_1) = (\frac{1}{2}a_1^2, -\frac{1}{2}b_1^2, c_1)$ and the transformation $\mathcal{R}_3$ (3.24a) maps it to the solution $w_3(z)$ with $(\alpha_3, \beta_3, \gamma_3) = (\frac{1}{2}a_3^2, -\frac{1}{2}b_3^2, c_3)$ where

$$w_1(z) = w_3(z), \quad a_1 = a_3, \quad b_1 = -b_3, \quad c_1 = c_3,$$

i.e. both transformations map $w_0(z)$ to the *same* solution of P_V . However applying the transformation a second time gives different solutions since

$$\mathcal{R}_1^2(a_0, b_0, c_0) = (a_0 + 1, b_0 - 1, c_0),$$

$$\mathcal{R}_3^2(a_0, b_0, c_0) = \left(\frac{1}{2}(a_0 + b_0 + c_0 + 1), -\frac{1}{2}(a_0 + b_0 - c_0 - 1), a_0 - b_0 + 1\right),$$

so the transformations $\mathcal{R}_1$ (3.3) and $\mathcal{R}_3$ (3.24) will generate *different* hierarchies of solutions.

- (iii) The symmetric case of equation (3.28), i.e. when $\lambda = \varphi = 0$,

$$x_n(x_{n+1} + x_{n-1} + 1) + \frac{\rho + \frac{1}{3}n}{z} = 0, \quad (3.31)$$

is a discrete P_I [72, equation (3.6)], whilst the full equation, which was derived from a discrete dressing transformation [34], is a discrete form of P_{IV} (1.11).

- (iv) The ternary dP_I (3.28) is associated with P_V (1.2), though the asymmetric generalisation of dP_I (1.5), i.e.

$$x_{n+1} + x_n + x_{n-1} = 1 + \frac{\lambda n + \rho + (-1)^n \varphi}{x_n}, \quad (3.32)$$

with λ , ρ and φ constants, is associated with P_{IV} (1.11) cf. [26, 52], and can be derived from Bäcklund transformations of P_{IV} [19, 25]. We remark that the continuum limit of (3.32) with $\varphi \neq 0$ is P_{II} (1.8) rather than P_I (1.6), as shown in [20].

- (v) Recently Clarkson et al. [18] studied the special case of the ternary dP_I equation given by

$$v_n(v_{n+1} + v_{n-1} + 1) = (n+1)\varepsilon, \quad \varepsilon > 0, \quad (3.33)$$

with $v_{-1} = 0$, which arises in quantum minimal surfaces [5, 42] and found a unique solution. The expressions for v_{3n} , v_{3n+1} and v_{3n+2} were written explicitly as ratios of Wronskian determinants in terms of the modified Bessel functions $K_{1/6}(z)$ and $K_{5/6}(z)$ and were all different showing the three-fold structure of the solutions.

- (vi) Felder and Hoppe [23] show that ternary dP_I equation (3.33) also arises in connection with orthogonal polynomials in the complex plane with respect to the weight

$$w(z; t) = \exp \left\{ -t|z|^2 + \frac{1}{3}i(z^3 + \bar{z}^3) \right\}, \quad t > 0,$$

with $z = x + iy \in \mathbb{C}$, and has the *same* unique solution as in [18]. Teodorescu et al. [79], see also [78, equation (26)], show that equation (3.33) arises in the theory of random normal matrices [86], which motivated the study in [23].

- (vii) A study of the asymptotics of solutions of (3.33) is given by Aptekarev and Novokshenov [2].

3.4 Fourth Discrete Equation: New Equation with Ternary Symmetry

It is possible to derive a discrete equation relating the three solutions x_n , x_{n+2} and x_{n-2} of the ternary dP_I equation (3.28) as illustrated in the following Lemma.

Lemma 3.6 *Suppose that x_n , x_{n+2} and x_{n-2} are solutions of the ternary dP_I equation (3.28), then they satisfy the discrete equation*

$$\frac{a_{n+1}}{x_{n+2} + x_n + 1} + \frac{a_{n-1}}{x_n + x_{n-2} + 1} = z + \frac{a_n}{x_n}, \quad (3.34)$$

where a_n is given by (3.26a), i.e.

$$a_n = \frac{1}{3}n + \rho + \lambda \cos\left(\frac{2}{3}\pi n\right) + \frac{1}{3}\sqrt{3}\varphi \sin\left(\frac{2}{3}\pi n\right), \quad (3.35)$$

with λ , ρ and φ arbitrary constants.

Proof Letting $n \rightarrow n-1$ in equation (3.28) gives

$$x_{n-1}(x_n + x_{n-2} + 1) + \frac{a_{n-1}}{z} = 0.$$

Solving this equation and (3.28) for x_{n+1} and x_{n-1} gives

$$x_{n+1} = \frac{a_{n-1}}{z(x_n + x_{n-2} + 1)} - 1 - \frac{a_n}{zx_n}, \quad x_{n-1} = -\frac{a_{n-1}}{z(x_n + x_{n-2} + 1)}.$$

Letting $n \rightarrow n + 2$ in the expression for x_{n-1} gives

$$x_{n+1} = -\frac{a_{n+1}}{z(x_{n+2} + x_n + 1)}.$$

Equating the two expressions for x_{n+1} gives the result. $\square$

An alternative proof is to consider the Bäcklund transformation $\mathcal{R}_4 = \mathcal{R}_3^2$, with $\mathcal{R}_4^{-1} = \mathcal{T}_{-1,1,1}^2$, then

$$w_{n+2} = \mathcal{R}_4(w_n) = \frac{1}{w_n} \left\{ 1 + \frac{(a_n - b_n + c_n + 1)(w_n - 1)^2}{zw'_n + a_n w_n^2 - (2a_n + c_n + 1 - z)w_n + a_n + c_n + 1} \right\}, \quad (3.36a)$$

$$w_{n-2} = \mathcal{R}_4^{-1}(w_n) = \frac{1}{w_n} \left\{ 1 + \frac{(a_n + b_n + c_n - 1)(w_n - 1)^2}{zw'_n - a_n w_n^2 + (2a_n + c_n - 1 - z)w_n - a_n - c_n + 1} \right\}, \quad (3.36b)$$

with $' \equiv d/dz$ and a_n, b_n and c_n are given by (3.26). Letting $w_n = 1 + 1/x_n$ and $w_{n\pm 2} = 1 + 1/x_{n\pm 2}$ gives

$$\frac{a_n - b_n + c_n + 1}{x_{n+2} + x_n + 1} = z - \frac{z}{x_n(x_n + 1)} \frac{dx_n}{dz} + \frac{(a_n - b_n)x_n + a_n}{x_n(x_n + 1)}, \quad (3.37a)$$

$$\frac{a_n + b_n + c_n - 1}{x_n + x_{n-2} + 1} = z + \frac{z}{x_n(x_n + 1)} \frac{dx_n}{dz} + \frac{(a_n + b_n)x_n + a_n}{x_n(x_n + 1)}, \quad (3.37b)$$

and so by adding we obtain

$$\frac{a_{n+1}}{x_{n+2} + x_n + 1} + \frac{a_{n-1}}{x_n + x_{n-2} + 1} = z + \frac{a_n}{x_n},$$

since from (3.25), $a_{n+1} = \frac{1}{2}(a_n - b_n + c_n + 1)$ and $a_{n-1} = \frac{1}{2}(a_n + b_n + c_n - 1)$. The parameters $\tilde{a}_n, \tilde{b}_n$ and $\tilde{c}_n$ satisfy

$$\begin{aligned} \tilde{a}_{n+1} &= \frac{1}{2}(\tilde{a}_n + \tilde{b}_n + \tilde{c}_n + 1), & \tilde{a}_{n-1} &= \frac{1}{2}(\tilde{a}_n - \tilde{b}_n + \tilde{c}_n - 1), \\ \tilde{b}_{n+1} &= -\frac{1}{2}(\tilde{a}_n + \tilde{b}_n - \tilde{c}_n - 1), & \tilde{b}_{n-1} &= \frac{1}{2}(\tilde{a}_n - \tilde{b}_n - \tilde{c}_n + 1), \\ \tilde{c}_{n+1} &= \tilde{a}_n - \tilde{b}_n + 1, & \tilde{c}_{n-1} &= \tilde{a}_n + \tilde{b}_n - 1, \end{aligned}$$

which have solution

$$\begin{aligned} \tilde{a}_n &= \frac{2}{3}n + \rho + \lambda \cos\left(\frac{2}{3}\pi n\right) + \frac{1}{3}\sqrt{3}\varphi \sin\left(\frac{2}{3}\pi n\right), \\ \tilde{b}_n &= \frac{1}{3} - \sqrt{3}\lambda \sin\left(\frac{2}{3}\pi n\right) + \varphi \cos\left(\frac{2}{3}\pi n\right), \\ \tilde{c}_n &= \frac{2}{3}n + \rho - 2\lambda \cos\left(\frac{2}{3}\pi n\right) - \frac{2}{3}\sqrt{3}\varphi \sin\left(\frac{2}{3}\pi n\right), \end{aligned}$$

with λ , ρ and φ arbitrary constants. We remark that the Bäcklund transformation (3.36) has similar structure to the Bäcklund transformation $\mathcal{R}_2$ (3.13), except the parameters have a ternary symmetry rather than a binary symmetry. We note that since $\mathcal{R}_3 = \mathcal{T}_{-1,1,-1} \circ \mathcal{T}_{1,-1,-1} \circ \mathcal{T}_{1,-1,1}$, then

$$\begin{aligned}\mathcal{R}_4 = \mathcal{R}_3^2 &= \mathcal{T}_{-1,1,-1} \circ \mathcal{T}_{1,-1,-1} \circ \mathcal{T}_{1,-1,1} \circ \mathcal{T}_{-1,1,-1} \circ \mathcal{T}_{1,-1,-1} \circ \mathcal{T}_{1,-1,1} \\ &= \mathcal{T}_{-1,1,-1} \circ \mathcal{T}_{1,-1,-1} \circ \mathcal{T}_{1,-1,-1} \circ \mathcal{T}_{1,-1,1},\end{aligned}$$

as $\mathcal{T}_{1,-1,1} \circ \mathcal{T}_{-1,1,-1} = \mathcal{I}$, the identity.

Corollary 3.7 *If x_n satisfies ternary dP_I equation (3.28) and a_n is given by (3.35), then $q_n = x_{2n}$ and $p_n = x_{2n+1}$ satisfy*

$$\frac{a_{2n+1}}{q_{n+1} + q_n + 1} + \frac{a_{2n-1}}{q_n + q_{n-1} + 1} = z + \frac{a_{2n}}{q_n}, \quad (3.38a)$$

$$\frac{a_{2n+2}}{p_{n+1} + p_n + 1} + \frac{a_{2n}}{p_n + p_{n-1} + 1} = z + \frac{a_{2n+1}}{p_n}, \quad (3.38b)$$

respectively. Further $X_n = x_{3n}$, $Y_n = x_{3n+1}$ and $Z_n = x_{3n+2}$ satisfy the system

$$\frac{a_{3n+1}}{X_n + Z_n + 1} + \frac{a_{3n-1}}{X_n + Y_{n-1} + 1} = z + \frac{a_{3n}}{X_n}, \quad (3.39a)$$

$$\frac{a_{3n+2}}{X_{n+1} + Y_n + 1} + \frac{a_{3n}}{Y_n + Z_{n-1} + 1} = z + \frac{a_{3n+1}}{Y_n}, \quad (3.39b)$$

$$\frac{a_{3n+3}}{Y_{n+1} + Z_n + 1} + \frac{a_{3n+1}}{X_n + Z_n + 1} = z + \frac{a_{3n+2}}{Z_n}. \quad (3.39c)$$

Proof Letting $n \rightarrow 2n$ and $n \rightarrow 2n + 1$ in (3.34) and then setting $q_n = x_{2n}$ and $p_n = x_{2n+1}$, respectively, gives (3.38). Similarly for the system (3.39), which is equivalent to the system (3.29). $\square$

3.5 Differential-Difference Equations

In the previous subsections, we have obtained discrete equations by eliminating the derivative between two equations, i.e. by adding the equations in (3.5), (3.15) and (3.27). If instead we subtract the equations then we obtain differential-difference equations.

(i) Subtracting the equations in (3.6) gives the *modified Volterra lattice* [84]

$$\frac{dq_n}{dz} = \frac{1}{4}(q_n^2 - 1)(q_{n+1} - q_{n-1}).$$

which is a special case of equation (3.40) below.

(ii) Subtracting the equations in (3.17) gives

$$z \frac{dq_n}{dz} = \frac{1}{2}(1 - q_n^2) \left(\frac{2n + 2\lambda + 1}{q_{n+1} + q_n} - \frac{2n + 2\lambda - 1}{q_n + q_{n-1}} \right).$$

which is a non-autonomous special case of equation (3.41) below.

(iii) Subtracting the equations in (3.27) gives

$$z \frac{dx_n}{dz} = zx_n(x_n + 1)(x_{n+1} - x_{n-1}) - b_n x_n,$$

with b_n given by (3.26b).

(iv) Subtracting the equations in (3.37) gives

$$z \frac{dx_n}{dz} = -x_n(x_n + 1) \left(\frac{a_{n+1}}{x_{n+2} + x_n + 1} - \frac{a_{n-1}}{x_n + x_{n-2} + 1} \right) - b_n x_n,$$

with a_n and b_n given by (3.26). Consequently setting $x_{2n} = \frac{1}{2}(q_n - 1)$ and $x_{2n+1} = \frac{1}{2}(p_n - 1)$ gives

$$\begin{aligned} z \frac{dq_n}{dz} &= (1 - q_n^2) \left(\frac{a_{2n+1}}{q_{n+1} + q_n} - \frac{a_{2n-1}}{q_n + q_{n-1}} \right) - b_{2n}(q_n - 1), \\ z \frac{dp_n}{dz} &= (1 - p_n^2) \left(\frac{a_{2n+2}}{p_{n+1} + p_n} - \frac{a_{2n}}{p_n + p_{n-1}} \right) - b_{2n+1}(p_n - 1), \end{aligned}$$

which are non-autonomous special cases of equation (3.41) below.

Remarks 3.8 Yamilov [88] gives a classification of (autonomous) Volterra-type equations. The first two are:

$$\frac{du_n}{dz} = P(u_n)(u_{n+1} - u_{n-1}), \quad (3.40)$$

$$\frac{du_n}{dz} = P(u_n^2) \left(\frac{1}{u_{n+1} + u_n} - \frac{1}{u_n + u_{n-1}} \right), \quad (3.41)$$

where $P(u) = au^2 + bu + c$, with a, b and c constants.

4 Rational Solutions of P_V

In this section we describe the rational solutions of P_V (1.2), which are classified in the following theorem.

Theorem 4.1 Equation (1.2) has a rational solution if and only if one of the following holds:

- (i) $\alpha = \frac{1}{2}m^2$, $\beta = -\frac{1}{2}(m + 2n + 1 + \kappa)^2$, $\gamma = \kappa$, for $m \geq 1$;
- (ii) $\alpha = \frac{1}{2}(m + \kappa)^2$, $\beta = -\frac{1}{2}(n + \varepsilon\kappa)^2$, $\gamma = m + \varepsilon n$, with $\varepsilon = \pm 1$, provided that $m \neq 0$ or $n \neq 0$;
- (iii) $\alpha = \frac{1}{2}(m + \frac{1}{2})^2$, $\beta = -\frac{1}{2}(n + \frac{1}{2})^2$, $\gamma = \kappa$, provided that $m \neq 0$ or $n \neq 0$,

where $m, n \in \mathbb{Z}$ and κ is an arbitrary constant, together with the solutions obtained through the symmetries

$$\begin{aligned} \mathcal{S}_1 : \quad w_1(z; \alpha_1, \beta_1, \gamma_1, -\tfrac{1}{2}) &= w(-z; \alpha, \beta, \gamma, -\tfrac{1}{2}), & (\alpha_1, \beta_1, \gamma_1, -\tfrac{1}{2}) &= (\alpha, \beta, -\gamma, -\tfrac{1}{2}), \\ \mathcal{S}_2 : \quad w_2(z; \alpha_2, \beta_2, \gamma_3, -\tfrac{1}{2}) &= \frac{1}{w(z; \alpha, \beta, \gamma, -\tfrac{1}{2})}, & (\alpha_2, \beta_2, \gamma_3, -\tfrac{1}{2}) &= (-\beta, -\alpha, -\gamma, -\tfrac{1}{2}), \end{aligned} \quad (4.1)$$

where $w(z; \alpha, \beta, \gamma, -\frac{1}{2})$ is a solution of P_V (1.2).

Proof See Kitaev, Law and McLeod [48, Theorem 1.1]; also [39, Theorem 40.3]. $\square$

Remarks 4.2

- (i) Kitaev, Law and McLeod [48, Theorem 1.1] give four cases, though their cases (I) and (II) are related by the symmetry (4.2). Further Kitaev, Law and McLeod [48, Theorem 1.2] prove that if $\gamma \notin \mathbb{Z}$ in case (i), then the rational solution is unique and if $\gamma \in \mathbb{Z}$ then there are at most two rational solutions, as illustrated in [17]; see also [3, 4].
- (ii) Rational solutions in case (i) are expressed in terms of *generalised Laguerre polynomials*, which are discussed in §5.1, and in cases (ii) and (iii) in terms of *generalised Umemura polynomials*, discussed in §5.2.

4.1 Generalised Laguerre Polynomials

Rational solutions in case (i) of Theorem 4.1 are expressed in terms of *generalised Laguerre polynomials*, which are written in terms of a determinant of Laguerre polynomials, or equivalently as a Wronskian, see [17, §3]. These rational solutions are special cases of the solutions of P_V (1.2) expressible in terms of the Kummer functions $M(a, b, z)$ and $U(a, b, z)$ discussed in [53, 66] since

$$U(-n, \alpha + 1, z) = (-1)^n (\alpha + 1)_n M(-n, \alpha + 1, z) = (-1)^n n! L_n^{(\alpha)}(z), \quad (4.3)$$

with $L_n^{(\alpha)}(z)$ the Laguerre polynomial

$$L_n^{(\alpha)}(z) = \frac{z^{-\alpha} e^z}{n!} \frac{d^n}{dz^n} (z^{n+\alpha} e^{-z}), \quad n \geq 0, \quad (4.4)$$

see [69, equation (13.6.19)]. First we define the generalised Laguerre polynomial $T_{m,n}^{(\mu)}(z)$; properties of it are discussed in [6, 17].

Definition 4.3 The *generalised Laguerre polynomial* $T_{m,n}^{(\mu)}(z)$, which is a polynomial of degree $(m+1)n$, is defined by

$$T_{m,n}^{(\mu)}(z) = \det \left[\frac{d^{j+k}}{dz^{j+k}} L_{m+n}^{(\mu+1)}(z) \right]_{j,k=0}^{n-1}, \quad m \geq 0, \quad n \geq 1, \quad (4.5)$$

with $T_{m,0}^{(\mu)}(z) = 1$, where $L_n^{(\mu)}(z)$ is the Laguerre polynomial (4.4).

For case (i) in Theorem 4.1, Kitaev, Law and McLeod [48] state that rational solutions arise when

$$\alpha = \frac{1}{2}\ell^2, \quad \beta = -\frac{1}{2}(k + \varepsilon\gamma)^2, \quad \varepsilon = \pm 1,$$

where $\ell > 0$ and $k + \ell$ is odd, with $k, \ell \in \mathbb{Z}$, and $\beta \neq 0$ when $|k| < \ell$. Since the sign of γ can be changed by letting $z \rightarrow -z$, we set $\varepsilon = 1$, without loss of generality. There are three cases (a), $k > \ell$, (b), $|k| \leq \ell$ and (c), $-k > \ell$, which are given in the following theorem.

Theorem 4.4 Suppose that $T_{m,n}^{(\mu)}(z)$ is the generalised Laguerre polynomial given by (4.5) and $L_n^{(\alpha)}(z)$ is the Laguerre polynomial, then P_V (1.2) has the following three sets of rational solutions.

(a) If $\ell = m$ and $k = m + 2n + 1$, with $m \geq 1$, so $k + \ell = 2m + 2n + 1$, then for $n \geq 1$

$$w_{m,n}^{(\mu)}(z) = \frac{T_{m-1,n}^{(\mu)}(z) T_{m-1,n+1}^{(\mu)}(z)}{T_{m,n}^{(\mu)}(z) T_{m-2,n+1}^{(\mu)}(z)},$$

$$(\alpha, \beta, \gamma) = \left(\frac{1}{2}m^2, -\frac{1}{2}(m + 2n + 1 + \mu)^2, \mu\right). \quad (4.6a)$$

In the case when $n = 0$, then for $m \geq 1$

$$w_{m,0}^{(\mu)}(z) = \frac{L_m^{(\mu+1)}(z)}{L_{m-1}^{(\mu+1)}(z)}, \quad (\alpha, \beta, \gamma) = \left(\frac{1}{2}m^2, -\frac{1}{2}(m + 1 + \mu)^2, \mu\right). \quad (4.6b)$$

(b) If $\ell = m + n + 1$ and $k = m - n$, with $m \geq 0$, so $k + \ell = 2m + 1$, then for $m \geq 1$

$$v_{m,n}^{(\mu)}(z) = \frac{m - n + \mu}{m + n + 1} \frac{T_{m,n}^{(\mu-2n)}(z) T_{m-1,n+1}^{(\mu-2n-2)}(z)}{T_{m-1,n}^{(\mu-2n)}(z) T_{m,n+1}^{(\mu-2n-2)}(z)},$$

$$(\alpha, \beta, \gamma) = \left(\frac{1}{2}(m + n + 1)^2, -\frac{1}{2}(m - n + \mu)^2, \mu\right). \quad (4.7a)$$

In the case when $m = 0$ and $n \geq 1$ then

$$v_{0,n}^{(\mu)}(z) = \frac{n - \mu}{n + 1} \frac{L_n^{(-\mu-1)}(-z)}{L_{n+1}^{(-\mu-1)}(-z)}, \quad (\alpha, \beta, \gamma) = \left(\frac{1}{2}(n + 1)^2, -\frac{1}{2}(n - \mu)^2, \mu\right), \quad (4.7b)$$

$$\tilde{v}_{0,n}^{(\mu)}(z) = \frac{L_{n+1}^{(-1-\mu)}(-z)}{L_n^{(-1-\mu)}(-z)}, \quad (\alpha, \beta, \gamma) = \left(\frac{1}{2}(n + 1)^2, -\frac{1}{2}(n - \mu)^2, \mu + 2\right). \quad (4.7c)$$

In the case when $n = 0$ and $m \geq 0$ then

$$v_{m,0}^{(\mu)}(z) = \frac{m + \mu}{m + 1} \frac{L_m^{(\mu-1)}(z)}{L_{m+1}^{(\mu-1)}(z)}, \quad (\alpha, \beta, \gamma) = \left(\frac{1}{2}(m+1)^2, -\frac{1}{2}(m+\mu)^2, \mu\right), \quad (4.7d)$$

$$\tilde{v}_{m,0}^{(\mu)}(z) = \frac{L_{m+1}^{(\mu-1)}(z)}{L_m^{(\mu-1)}(z)}, \quad (\alpha, \beta, \gamma) = \left(\frac{1}{2}(m+1)^2, -\frac{1}{2}(m+\mu)^2, \mu-2\right). \quad (4.7e)$$

(c) If $\ell = m$ and $k = -m - 2n - 1$, with $m \geq 1$ and $n \geq 0$, so $k + \ell = -2m - 1$, then $u_{m,n}^{(\mu)}(z) = w_{m,n}^{(-\mu)}(-z)$, i.e.

$$u_{m,n}^{(\mu)}(z) = \frac{T_{n-1,m}^{(\mu-2m-2n)}(z) T_{n,m}^{(\mu-2m-2n-2)}(z)}{T_{n,m-1}^{(\mu-2m-2n)}(z) T_{n-1,m+1}^{(\mu-2m-2n-2)}(z)}, \quad (\alpha, \beta, \gamma) = \left(\frac{1}{2}m^2, -\frac{1}{2}(m+2n+1-\mu)^2, \mu\right). \quad (4.8a)$$

In the case when $n = 0$, then for $m \geq 1$

$$u_{m,0}^{(\mu)}(z) = \frac{L_m^{(1-\mu)}(-z)}{L_{m-1}^{(1-\mu)}(-z)}, \quad (\alpha, \beta, \gamma) = \left(\frac{1}{2}m^2, -\frac{1}{2}(m+1-\mu)^2, \mu\right). \quad (4.8b)$$

Proof For cases (a) and (c) see Clarkson and Dunning [17, Theorem 3] and [17, Corollary 1], respectively. Case (b) follows in a similar way from the special function solutions of P_V (1.2) given by Masuda [53, Theorem 2.2]. The solutions when either $m = 0$ or $n = 0$ follow from recurrence relations for the Laguerre polynomial, cf. [69, §18.9]. $\square$

It is known that rational solutions of P_{III} can be expressed either in terms of four special polynomials or in terms of the logarithmic derivative of the ratio of two special polynomials, cf. [13, Theorem 2.4]. Hence it might be expected that the rational solutions discussed in Theorem 4.4 can also be written in terms of the logarithmic derivative of the ratio of two generalised Laguerre polynomials. We note that Masuda [53, Remark 2.3] has written the special function solutions of sP_V (1.12) as logarithmic derivatives.

Remarks 4.5

- (i) The rational solutions in Theorem 4.4 can also be written as logarithmic derivatives

$$w_{m,n}^{(\mu)}(z) = \frac{z}{m} \frac{d}{dz} \ln \frac{T_{m-2,n+1}^{(\mu)}(z)}{T_{m,n}^{(\mu)}(z)} - \frac{z - m - 2n - 1 - \mu}{m}, \quad (4.9a)$$

$$v_{m,n}^{(\mu)}(z) = \frac{z}{m+n+1} \frac{d}{dz} \ln \frac{T_{m-1,n}^{(\mu-2n)}(z)}{T_{m,n+1}^{(\mu-2n-2)}(z)} + 1, \quad (4.9b)$$

$$u_{m,n}^{(\mu)}(z) = \frac{z}{m} \frac{d}{dz} \ln \frac{T_{n,m-1}^{(\mu-2m-2n)}(z)}{T_{n-1,m+1}^{(\mu-2m-2n-2)}(z)} + \frac{z+m+2n+1-\mu}{m}. \quad (4.9c)$$

Using computer algebra we have verified these alternative forms of the rational solutions in Theorem 4.4 given by (4.9) for several small values of m and n . Comparing these expressions with those in Theorem 4.4 gives the relations (A.3). We envisage these relations can be proved using the Jacobi identity [21], sometimes known as the *Lewis Carroll formula*, for the determinant $\mathcal{D}$

$$\mathcal{D} \mathcal{D} \begin{bmatrix} i, k \\ j, \ell \end{bmatrix} = \mathcal{D} \begin{bmatrix} i \\ j \end{bmatrix} \mathcal{D} \begin{bmatrix} k \\ \ell \end{bmatrix} - \mathcal{D} \begin{bmatrix} k \\ j \end{bmatrix} \mathcal{D} \begin{bmatrix} i \\ \ell \end{bmatrix} \quad (4.10)$$

where $\mathcal{D} \begin{bmatrix} i \\ j \end{bmatrix}$ is the determinant with the i^{th} row and the j^{th} column removed from $\mathcal{D}$, though we don't pursue this further here.

(ii) From the definition of $T_{m,n}^{(\mu)}(z)$ (4.5)

$$T_{m-1,n}^{(\mu)}(z) = \det \left[\frac{d^{j+k}}{dz^{j+k}} L_{m+n-1}^{(\mu+1)}(z) \right]_{j,k=0}^{n-1},$$

$$T_{m-2,n+1}^{(\mu)}(z) = \det \left[\frac{d^{j+k}}{dz^{j+k}} L_{m+n-1}^{(\mu+1)}(z) \right]_{j,k=0}^n,$$

which are well-defined for $m \geq 1$ and $n \geq 1$.

Recall that solutions of the discrete equations (3.7) and (3.18) also satisfy the second-order equation

$$\frac{d^2 q}{dz^2} = \frac{q}{q^2 - 1} \left(\frac{dq}{dz} \right)^2 - \frac{1}{z} \frac{dq}{dz} - \frac{2\alpha(q+1)^2 + 2\beta(q-1)^2}{z^2(q^2 - 1)} - \frac{\gamma(q^2 - 1)}{2z} + \frac{1}{4}q(q^2 - 1), \quad (4.11)$$

see equations (3.8) and (3.22), and solutions of the ternary dP_I (3.28) also satisfy the second-order equation

$$\frac{d^2 x}{dz^2} = \frac{2x+1}{2x(x+1)} \left(\frac{dx}{dz} \right)^2 - \frac{1}{z} \frac{dx}{dz} - \frac{\alpha(x+1)^2 + \beta x^2}{z^2 x(x+1)} - \frac{\gamma x(x+1)}{z} + \frac{x(x+1)(2x+1)}{2}, \quad (4.12)$$

see equation (3.30). In the following Lemma we derive the solutions of equations (4.11) and (4.12) in terms of generalised Laguerre polynomials.

Lemma 4.6 If $w_{m,n}^{(\mu)}(z)$, $v_{m,n}^{(\mu)}(z)$ and $u_{m,n}^{(\mu)}(z)$ are rational solutions of P_V defined in Theorem 4.4 and $T_{m,n}^{(\mu)}(z)$ is the generalised Laguerre polynomial given by (4.5), then

$$\mathcal{W}_{m,n}^{(\mu)}(z) = \frac{w_{m,n}^{(\mu)}(z) + 1}{w_{m,n}^{(\mu)}(z) - 1} = 1 + 2 \frac{d}{dz} \ln \frac{T_{m-1,n+1}^{(\mu+1)}(z)}{T_{m-1,n+1}^{(\mu-1)}(z)}, \quad (4.13a)$$

$$\mathcal{V}_{m,n}^{(\mu)}(z) = \frac{v_{m,n}^{(\mu)}(z) + 1}{v_{m,n}^{(\mu)}(z) - 1} = -1 + \frac{2\mu}{z} + 2 \frac{d}{dz} \ln \frac{T_{m-1,n+1}^{(\mu-2n-1)}(z)}{T_{m,n}^{(\mu-2n-1)}(z)}, \quad (4.13b)$$

$$\mathcal{U}_{m,n}^{(\mu)}(z) = \frac{u_{m,n}^{(\mu)}(z) + 1}{u_{m,n}^{(\mu)}(z) - 1} = 1 + 2 \frac{d}{dz} \ln \frac{T_{n,m}^{(\mu-2m-2n-1)}}{T_{n-1,m}^{(\mu-2m-2n-1)}}, \quad (4.13c)$$

are solutions of equation (4.11) and

$$\mathcal{X}_{m,n}^{(\mu)}(z) = \frac{1}{w_{m,n}^{(\mu)}(z) - 1} = \frac{d}{dz} \ln \frac{T_{m-1,n}^{(\mu+1)}(z)}{T_{m-1,n+1}^{(\mu-1)}(z)}, \quad (4.14a)$$

$$\mathcal{Y}_{m,n}^{(\mu)}(z) = \frac{1}{v_{m,n}^{(\mu)}(z) - 1} = -1 + \frac{\mu}{z} + \frac{d}{dz} \ln \frac{T_{m-1,n+1}^{(\mu-2n-1)}(z)}{T_{m,n}^{(\mu-2n-1)}(z)}, \quad (4.14b)$$

$$\mathcal{Z}_{m,n}^{(\mu)}(z) = \frac{1}{u_{m,n}^{(\mu)}(z) - 1} = \frac{d}{dz} \ln \frac{T_{n,m}^{(\mu-2m-2n-1)}}{T_{n-1,m}^{(\mu-2m-2n-1)}}, \quad (4.14c)$$

are solutions of equation (4.12).

Proof Since

$$\frac{w + 1}{w - 1} = 1 + \frac{2}{w - 1}, \quad (4.15)$$

then we prove (4.14a) first. By definition

$$w_{m,n}^{(\mu)} = \frac{T_{m-1,n}^{(\mu)} T_{m-1,n+1}^{(\mu)}}{T_{m,n}^{(\mu)} T_{m-2,n+1}^{(\mu)}},$$

and so

$$\frac{1}{w_{m,n}^{(\mu)} - 1} = \frac{T_{m,n}^{(\mu)} T_{m-2,n+1}^{(\mu)}}{T_{m-1,n}^{(\mu)} T_{m-1,n+1}^{(\mu)} - T_{m,n}^{(\mu)} T_{m-2,n+1}^{(\mu)}}.$$

Also

$$\frac{d}{dz} \ln \frac{T_{m-1,n}^{(\mu+1)}}{T_{m-1,n+1}^{(\mu-1)}} = \frac{D_z(T_{m-1,n}^{(\mu+1)} \cdot T_{m-1,n+1}^{(\mu-1)})}{T_{m-1,n}^{(\mu+1)} T_{m-1,n+1}^{(\mu-1)}} = \frac{T_{m,n}^{(\mu)} T_{m-2,n+1}^{(\mu)}}{T_{m-1,n}^{(\mu)} T_{m-1,n+1}^{(\mu)} - T_{m,n}^{(\mu)} T_{m-2,n+1}^{(\mu)}},$$

since from the identity (A.1c)

$$D_z(T_{m-1,n}^{(\mu+1)} \cdot T_{m-1,n+1}^{(\mu-1)}) = T_{m,n}^{(\mu)} T_{m-2,n+1}^{(\mu)},$$

where $D_z(f \cdot g)$ is the Hirota operator defined by

$$D_z(f \cdot g) = \frac{df}{dz}g - f\frac{dg}{dz}, \quad (4.16)$$

and from the identity (A.2c)

$$T_{m-1,n}^{(\mu+1)} T_{m-1,n+1}^{(\mu-1)} = T_{m-1,n}^{(\mu)} T_{m-1,n+1}^{(\mu)} - T_{m,n}^{(\mu)} T_{m-2,n+1}^{(\mu)},$$

which completes the proof of (4.14a). The identity (4.13a) follows from equation (4.15).

By definition

$$v_{m,n}^{(\mu+2n+1)} = \frac{m+n+\mu+1}{m+n+1} \frac{T_{m,n}^{(\mu+1)} T_{m-1,n+1}^{(\mu-1)}}{T_{m-1,n}^{(\mu+1)} T_{m,n+1}^{(\mu-1)}},$$

and so

$$\frac{1}{v_{m,n}^{(\mu+2n+1)} - 1} + 1 = \frac{(m+n+\mu+1)T_{m,n}^{(\mu+1)} T_{m-1,n+1}^{(\mu-1)}}{(m+n+\mu+1)T_{m,n}^{(\mu+1)} T_{m-1,n+1}^{(\mu-1)} - (m+n+1)T_{m-1,n}^{(\mu+1)} T_{m,n+1}^{(\mu-1)}}.$$

From (4.7a) and (4.9b) it follows

$$zD_z(T_{m-1,n}^{(\mu+1)} \cdot T_{m,n+1}^{(\mu-1)}) = (m+n+\mu+1)T_{m,n}^{(\mu+1)} T_{m-1,n+1}^{(\mu-1)} - (m+n+1)T_{m,n+1}^{(\mu-1)} T_{m-1,n}^{(\mu+1)},$$

and since from (A.1b)

$$D_z(T_{m-1,n}^{(\mu+1)} \cdot T_{m,n+1}^{(\mu-1)}) = T_{m,n}^{(\mu)} T_{m-1,n+1}^{(\mu)},$$

then we have

$$zT_{m,n}^{(\mu)} T_{m-1,n+1}^{(\mu)} = (m+n+\mu+1)T_{m,n}^{(\mu+1)} T_{m-1,n+1}^{(\mu-1)} - (m+n+1)T_{m,n+1}^{(\mu-1)} T_{m-1,n}^{(\mu+1)}.$$

Therefore

$$\frac{1}{v_{m,n}^{(\mu+2n+1)} - 1} + 1 = \frac{m+n+\mu+1}{z} \frac{T_{m,n}^{(\mu+1)} T_{m-1,n+1}^{(\mu-1)}}{T_{m-1,n+1}^{(\mu)} T_{m,n}^{(\mu)}},$$

and

$$\frac{m+n+\mu+1}{z} + \frac{d}{dz} \ln \frac{T_{m-1,n+1}^{(\mu)}}{T_{m,n}^{(\mu)}} = \frac{m+n+\mu+1}{z} \frac{T_{m,n}^{(\mu+1)} T_{m-1,n+1}^{(\mu-1)}}{T_{m-1,n+1}^{(\mu)} T_{m,n}^{(\mu)}}.$$

Hence we obtain (4.14b). The identity (4.13b) follows from equation (4.15).

By definition

$$u_{m,n}^{(\mu+2m+2n+1)}(z) = \frac{T_{n-1,m}^{(\mu+1)} T_{n,m}^{(\mu-1)}}{T_{n,m-1}^{(\mu+1)} T_{n-1,m+1}^{(\mu-1)}},$$

so

$$\frac{1}{u_{m,n}^{(\mu+2m+2n+1)}(z) - 1} = \frac{T_{n,m-1}^{(\mu+1)} T_{n-1,m+1}^{(\mu-1)}}{T_{n-1,m}^{(\mu+1)} T_{n,m}^{(\mu-1)} - T_{n,m-1}^{(\mu+1)} T_{n-1,m+1}^{(\mu-1)}}.$$

Also

$$\frac{d}{dz} \ln \frac{T_{n,m}^{(\mu)}}{T_{n-1,m}^{(\mu)}} = \frac{D_z(T_{n,m}^{(\mu)} \cdot T_{n-1,m}^{(\mu)})}{T_{n,m}^{(\mu)} T_{n-1,m}^{(\mu)}} = \frac{T_{n,m-1}^{(\mu+1)} T_{n-1,m+1}^{(\mu-1)}}{T_{n-1,m}^{(\mu+1)} T_{n,m}^{(\mu-1)} - T_{n,m-1}^{(\mu+1)} T_{n-1,m+1}^{(\mu-1)}},$$

since from the identity (A.1f)

$$D_z(T_{n,m}^{(\mu)} \cdot T_{n-1,m}^{(\mu)}) = T_{n,m-1}^{(\mu+1)} T_{n-1,m+1}^{(\mu-1)},$$

and from the identity (A.2b)

$$T_{n,m}^{(\mu)} T_{n-1,m}^{(\mu)} = T_{n-1,m}^{(\mu+1)} T_{n,m}^{(\mu-1)} - T_{n,m-1}^{(\mu+1)} T_{n-1,m+1}^{(\mu-1)},$$

which completes the proof of (4.14c). The identity (4.13c) follows from equation (4.15). $\square$

4.2 Generalised Umemura Polynomials

Rational solutions in cases (ii) and (iii) of Theorem 4.1 are expressed in terms of *generalised Umemura polynomials*. Umemura [81] defined some polynomials through a differential-difference equation to describe rational solutions of P_V (1.2); see also [14, 60, 87]. Subsequently these were generalised by Masuda, Ohta and Kajiwara [55], who defined the generalised Umemura polynomial $R_{m,n}^{(\kappa)}(z)$ through coupled

differential-difference equations and gave a representation as the determinant

$$R_{m,n}^{(\ell)}(z) = \begin{vmatrix} q_1^{(\ell)} & \cdots & q_{-m+2}^{(\ell)} & q_{-m+1}^{(\ell)} & \cdots & q_{-m-n+2}^{(\ell)} \\ q_3^{(\ell)} & \cdots & q_{-m+4}^{(\ell)} & q_{-m+3}^{(\ell)} & \cdots & q_{-m-n+4}^{(\ell)} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ q_{2m-1}^{(\ell)} & \cdots & q_m^{(\ell)} & q_{m-1}^{(\ell)} & \cdots & q_{m-n}^{(\ell)} \\ p_{n-m}^{(\ell)} & \cdots & p_{n+1}^{(\ell)} & p_n^{(\ell)} & \cdots & p_{2n-1}^{(\ell)} \\ p_{n-m-2}^{(\ell)} & \cdots & p_{n-1}^{(\ell)} & p_{n-2}^{(\ell)} & \cdots & p_{2n-3}^{(\ell)} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ p_{-n-m+2}^{(\ell)} & \cdots & p_{-n+1}^{(\ell)} & p_{-n+2}^{(\ell)} & \cdots & p_1^{(\ell)} \end{vmatrix}, \quad (4.17)$$

where $p_k^{(\ell)}(z) = L_k^{(\ell-1)}(\frac{1}{2}z)$ and $q_k^{(\ell)}(z) = L_k^{(\ell-1)}(-\frac{1}{2}z)$, with $p_k^{(\ell)} = q_k^{(\ell)} = 0$ for $k < 0$, and $L_k^{(\alpha)}(z)$ is the Laguerre polynomial (4.4). Recently Masuda [54] studied properties of zeros of the polynomial $R_{m,n}^{(\ell)}(z)$.

Rather than use (4.17) for the generalised Umemura polynomial, we prefer to use an equivalent representation given in Definition 4.7, which we find simpler to compute and useful to prove algebraic and combinatorial properties of the polynomial, though we do not pursue this further here.

Definition 4.7 The generalised Umemura polynomial $U_{m,n}^{(\kappa)}(z)$ is given by the Wronskian

$$U_{m,n}^{(\kappa)}(z) = c_{m,n} \exp(\frac{1}{2}mz) \operatorname{Wr}(\varphi_1^{(\kappa)}, \varphi_3^{(\kappa)}, \dots, \varphi_{2m-1}^{(\kappa)}; \psi_1^{(\kappa)}, \psi_3^{(\kappa)}, \dots, \psi_{2n-1}^{(\kappa)}), \quad m, n \geq 0, \quad (4.18)$$

where

$$c_{m,n} = (-1)^m 2^{(m+n)(m+n+1)/2} \prod_{j=1}^{m-1} \frac{(2j+1)!}{j!} \prod_{k=1}^{n-1} \frac{(2k+1)!}{k!}, \quad (4.19)$$

$\varphi_n^{(\kappa)}(z) = e^{-z/2} L_n^{(\kappa)}(\frac{1}{2}z)$ and $\psi_n^{(\kappa)}(z) = L_n^{(\kappa)}(-\frac{1}{2}z)$, with κ a parameter, and $L_n^{(\kappa)}(z)$ is the Laguerre polynomial (4.4) and $U_{0,0}^{(\kappa)}(z) = 1$.

Remarks 4.8

- (i) The generalised Laguerre polynomial $T_{m,n}^{(\mu)}(z)$ is a Wronskian involving one sequence of Laguerre polynomials, whereas the generalised Umemura polynomial $U_{m,n}^{(\kappa)}(z)$ involves two sequences of Laguerre polynomials, unless either $m = 0$ or $n = 0$. This is analogous to the situation for rational solutions of PIV (1.11), where one family of rational solutions involves one sequence of Hermite polynomials and a second family involves two sequences of Hermite polynomials, cf. [15, 62].

- (ii) The generalised Umemura polynomial $U_{m,n}^{(\kappa)}(z)$ given by (4.18) is a monic polynomial of degree $\frac{1}{2}m(m+1) + \frac{1}{2}n(n+1)$ and has the symmetry

$$U_{m,n}^{(\kappa)}(z) = U_{n,m}^{(-\kappa-2m-2n)}(z). \quad (4.20)$$

- (iii) The polynomials $R_{m,n}^{(\ell)}(z)$ and $U_{m,n}^{(\kappa)}(z)$ given by (4.17) and (4.18) respectively, are equivalent, which can be shown using matrix row and column operations. Further the two polynomials are related as follows

$$U_{m,n}^{(\kappa)}(z) = c_{m,n}(-1)^{m+n(n+1)/2} \left(\frac{1}{2}\right)^{(m+n)(m+n-1)/2} R_{m,n}^{(-\kappa-m-n)}(z),$$

with $c_{m,n}$ given by (4.19).

- (iv) When $m = 0$, the polynomial $U_{0,n}^{(\kappa)}(z)$ is equivalent to the polynomial defined by Umemura [81] through a differential-difference equation. Also $U_{0,n}^{(\kappa)}(z)$ is equivalent to the determinant $\tau_m^{(n)}(z)$ defined in Theorem 5.1 below; see also [60]. Due to the symmetry (4.20), the same is true for the polynomial $U_{m,0}^{(\kappa)}(z)$.

Theorem 4.9 Suppose that $U_{m,n}^{(\kappa)}(z)$ is the generalised Umemura polynomial given by (4.18).

- (a) In case (ii) of Theorem 4.1, P_V (1.2) has the rational solutions for $m \geq 1$ and $n \geq 1$

$$\begin{aligned} \widehat{w}_{m,n}^{(\kappa)}(z) &= -\frac{U_{m,n-1}^{(2\kappa)}(z) U_{m-1,n}^{(2\kappa+2)}(z)}{U_{m,n-1}^{(2\kappa+2)}(z) U_{m-1,n}^{(2\kappa)}(z)}, \\ (\alpha, \beta, \gamma) &= \left(\frac{1}{2}(m+\kappa)^2, -\frac{1}{2}(n+\kappa)^2, m+n\right), \end{aligned} \quad (4.21a)$$

$$\begin{aligned} \widehat{v}_{m,n}^{(\kappa)}(z) &= -\frac{U_{m-1,n-1}^{(2\kappa-2n+3)}(z) U_{m,n}^{(2\kappa-2n-1)}(z)}{U_{m-1,n-1}^{(2\kappa-2n+1)}(z) U_{m,n}^{(2\kappa-2n+1)}(z)}, \\ (\alpha, \beta, \gamma) &= \left(\frac{1}{2}(m+\kappa)^2, -\frac{1}{2}(n-\kappa)^2, m-n\right), \end{aligned} \quad (4.21b)$$

In the case when $n = 0$, then for $m \geq 1$

$$\begin{aligned} \widehat{w}_{m,0}^{(\kappa)}(z) &= \widehat{v}_{m,0}^{(\kappa)}(z) = -\frac{U_{m,0}^{(2\kappa-1)}(z) U_{m-1,0}^{(2\kappa+2)}(z)}{U_{m,0}^{(2\kappa+1)}(z) U_{m-1,0}^{(2\kappa)}(z)}, \\ (\alpha, \beta, \gamma) &= \left(\frac{1}{2}(m+\kappa)^2, -\frac{1}{2}\kappa^2, m\right), \end{aligned} \quad (4.21c)$$

and in the case when $m = 0$ then for $n \geq 1$

$$\begin{aligned} \widehat{w}_{0,n}^{(\kappa)}(z) &= \widehat{v}_{0,n}^{(-\kappa)}(z) = -\frac{U_{0,n}^{(2\kappa+1)}(z) U_{0,n-1}^{(2\kappa)}(z)}{U_{0,n}^{(2\kappa-1)}(z) U_{0,n-1}^{(2\kappa+2)}(z)}, \\ (\alpha, \beta, \gamma) &= \left(\frac{1}{2}\kappa^2, -\frac{1}{2}(n+\kappa)^2, n\right). \end{aligned} \quad (4.21d)$$

(b) In case (iii) of Theorem 4.1, P_V (1.2) has the rational solution for $m \geq 1$ and $n \geq 1$

$$\begin{aligned}\widehat{u}_{m,n}^{(\kappa)}(z) &= -\frac{U_{m,n-1}^{(\kappa+1)}(z) U_{m,n+1}^{(\kappa-1)}(z)}{U_{m-1,n}^{(\kappa+1)}(z) U_{m+1,n}^{(\kappa-1)}(z)}, \\ (\alpha, \beta, \gamma) &= \left(\frac{1}{2}\left(m + \frac{1}{2}\right)^2, \left(n + \frac{1}{2}\right)^2, m + n + k\right).\end{aligned}\quad (4.22a)$$

In the case when $n = 0$, then for $m \geq 1$

$$\begin{aligned}\widehat{u}_{m,0}^{(\kappa)}(z) &= -\frac{U_{m,0}^{(\kappa)}(z) U_{m,1}^{(\kappa-1)}(z)}{U_{m-1,0}^{(\kappa+1)}(z) U_{m+1,0}^{(\kappa-1)}(z)}, \\ (\alpha, \beta, \gamma) &= \left(\frac{1}{2}\left(m + \frac{1}{2}\right)^2, -\frac{1}{8}, m + \kappa\right),\end{aligned}\quad (4.22b)$$

and in the case when $m = 0$ then for $n \geq 1$

$$\begin{aligned}\widehat{u}_{0,n}^{(\kappa)}(z) &= -\frac{U_{0,n-1}^{(\kappa+1)}(z) U_{0,n+1}^{(\kappa-1)}(z)}{U_{0,n}^{(\kappa)}(z) U_{1,n}^{(\kappa-1)}(z)}, \\ (\alpha, \beta, \gamma) &= \left(\frac{1}{8}, -\frac{1}{2}\left(n + \frac{1}{2}\right)^2, n + \kappa\right).\end{aligned}\quad (4.22c)$$

Proof See Masuda, Ohta and Kajiwara [55, Theorem 1.1]. $\square$

Remarks 4.10 The rational solutions in Theorem 4.9 can also be written as logarithmic derivatives

$$\widehat{w}_{m,n}^{(\kappa)}(z) = \frac{z}{m + \kappa} \frac{d}{dz} \ln \frac{U_{m-1,n}^{(2\kappa)}(z)}{U_{m,n-1}^{(2\kappa+2)}(z)} - \frac{n + \kappa}{m + \kappa}, \quad (4.23a)$$

$$\widehat{v}_{m,n}^{(\kappa+n)}(z) = \frac{z}{m + n + \kappa} \frac{d}{dz} \ln \frac{U_{m-1,n-1}^{(2\kappa+1)}(z)}{U_{m,n}^{(2\kappa+1)}(z)} - \frac{\kappa}{m + n + \kappa}, \quad (4.23b)$$

$$\widehat{u}_{m,n}^{(\kappa)}(z) = \frac{2z}{2m + 1} \frac{d}{dz} \ln \frac{U_{m-1,n}^{(\kappa+1)}(z)}{U_{m+1,n}^{(\kappa-1)}(z)} + 1. \quad (4.23c)$$

Using computer algebra we have verified these alternative forms of the rational solutions in Theorem 4.9 given by (4.9) for several small values of m and n . Comparing these expressions with those in Theorem 4.9 gives the relations (A1.2), which we expect can be proved using the Jacobi identity (4.10). We note that Masuda [54, Appendix A] has written the rational solutions expressed in terms of generalised Umemura polynomials of sP_V (1.12) as logarithmic derivatives.

In the following Lemma we derive the solutions of equations (4.11) and (4.12) in terms of generalised Umemura polynomials.

Lemma 4.11 If $\widehat{w}_{m,n}^{(\kappa)}(z)$, $\widehat{v}_{m,n}^{(\kappa)}(z)$ and $\widehat{u}_{m,n}^{(\kappa)}(z)$ are rational solutions of P_V defined in Theorem 4.9 and $U_{m,n}^{(\mu)}(z)$ is the generalised Umemura polynomial given by (4.18), then

$$\widehat{\mathcal{W}}_{m,n}^{(\kappa)}(z) = \frac{\widehat{w}_{m,n}^{(\kappa)}(z) + 1}{\widehat{w}_{m,n}^{(\kappa)}(z) - 1} = 2 \frac{d}{dz} \ln \frac{U_{m,n}^{(2\kappa)}(z)}{U_{m-1,n-1}^{(2\kappa+2)}(z)}, \quad (4.24a)$$

$$\widehat{\mathcal{V}}_{m,n}^{(\kappa)}(z) = \frac{\widehat{v}_{m,n}^{(\kappa)}(z) + 1}{\widehat{v}_{m,n}^{(\kappa)}(z) - 1} = 2 \frac{d}{dz} \ln \frac{U_{m,n-1}^{(2\kappa-2n+1)}(z)}{U_{m-1,n}^{(2\kappa-2n+1)}(z)}, \quad (4.24b)$$

$$\widehat{\mathcal{U}}_{m,n}^{(\kappa)}(z) = \frac{\widehat{u}_{m,n}^{(\kappa)}(z) + 1}{\widehat{u}_{m,n}^{(\kappa)}(z) - 1} = \frac{2(m+n+\kappa)}{z} + 2 \frac{d}{dz} \ln \frac{U_{m,n}^{(\kappa+1)}(z)}{U_{m,n}^{(\kappa-1)}(z)}, \quad (4.24c)$$

are solutions of (4.11) and

$$\widehat{\mathcal{X}}_{m,n}^{(\kappa)}(z) = \frac{1}{\widehat{w}_{m,n}^{(\kappa)}(z) - 1} = -\frac{1}{2} + \frac{d}{dz} \ln \frac{U_{m,n}^{(2\kappa)}(z)}{U_{m-1,n-1}^{(2\kappa+2)}(z)}, \quad (4.25a)$$

$$\widehat{\mathcal{Y}}_{m,n}^{(\kappa)}(z) = \frac{1}{\widehat{v}_{m,n}^{(\kappa)}(z) - 1} = -\frac{1}{2} + \frac{d}{dz} \ln \frac{U_{m,n-1}^{(2\kappa-2n+1)}(z)}{U_{m-1,n}^{(2\kappa-2n+1)}(z)}, \quad (4.25b)$$

$$\widehat{\mathcal{Z}}_{m,n}^{(\kappa)}(z) = \frac{1}{\widehat{u}_{m,n}^{(\kappa)}(z) - 1} = -\frac{1}{2} + \frac{(m+n+\kappa)}{z} + \frac{d}{dz} \ln \frac{U_{m,n}^{(\kappa+1)}(z)}{U_{m,n}^{(\kappa-1)}(z)} \quad (4.25c)$$

are solutions of (4.12).

Proof We note that (4.25) follow from (4.24) using (4.15). By definition

$$\widehat{w}_{m,n}^{(\kappa)}(z) = -\frac{U_{m,n-1}^{(2\kappa)}(z) U_{m-1,n}^{(2\kappa+2)}(z)}{U_{m,n-1}^{(2\kappa+2)}(z) U_{m-1,n}^{(2\kappa)}(z)},$$

so that we have

$$\begin{aligned} \frac{\widehat{w}_{m,n}^{(\kappa)}(z) + 1}{\widehat{w}_{m,n}^{(\kappa)}(z) - 1} &= \frac{U_{m,n-1}^{(2\kappa)}(z) U_{m-1,n}^{(2\kappa+2)}(z) - U_{m,n-1}^{(2\kappa+2)}(z) U_{m-1,n}^{(2\kappa)}(z)}{U_{m,n-1}^{(2\kappa)}(z) U_{m-1,n}^{(2\kappa+2)}(z) + U_{m,n-1}^{(2\kappa+2)}(z) U_{m-1,n}^{(2\kappa)}(z)}, \\ \frac{d}{dz} \ln \frac{U_{m,n}^{(2\kappa)}(z)}{U_{m-1,n-1}^{(2\kappa+2)}(z)} &= \frac{D_z(U_{m,n}^{(2\kappa)}(z) \cdot U_{m-1,n-1}^{(2\kappa+2)}(z))}{U_{m,n}^{(2\kappa)}(z) U_{m-1,n-1}^{(2\kappa+2)}(z)}, \end{aligned}$$

where $D_z(f \cdot g)$ is the Hirota operator (4.16). From the identities (A.9b) and (A.10b)

$$\begin{aligned} 4D_z(U_{m,n}^{(2\kappa)}(z) \cdot U_{m-1,n-1}^{(2\kappa+2)}(z)) &= U_{m,n-1}^{(2\kappa)}(z) U_{m-1,n}^{(2\kappa+2)}(z) - U_{m,n-1}^{(2\kappa+2)}(z) U_{m-1,n}^{(2\kappa)}(z), \\ 2U_{m,n}^{(2\kappa)}(z) U_{m-1,n-1}^{(2\kappa+2)}(z) &= U_{m,n-1}^{(2\kappa)}(z) U_{m-1,n}^{(2\kappa+2)}(z) + U_{m,n-1}^{(2\kappa+2)}(z) U_{m-1,n}^{(2\kappa)}(z), \end{aligned}$$

respectively, therefore

$$\frac{2D_z(U_{m,n}^{(2\kappa)}(z) \cdot U_{m-1,n-1}^{(2\kappa+2)}(z))}{U_{m,n}^{(2\kappa)}(z) U_{m-1,n-1}^{(2\kappa+2)}(z)} = \frac{U_{m,n-1}^{(2\kappa)}(z) U_{m-1,n}^{(2\kappa+2)}(z) - U_{m,n-1}^{(2\kappa+2)}(z) U_{m-1,n}^{(2\kappa)}(z)}{U_{m,n-1}^{(2\kappa)}(z) U_{m-1,n}^{(2\kappa+2)}(z) + U_{m,n-1}^{(2\kappa+2)}(z) U_{m-1,n}^{(2\kappa)}(z)},$$

which proves (4.24a).

By definition

$$\widehat{v}_{m,n}^{(\kappa+n-1/2)}(z) = -\frac{U_{m-1,n-1}^{(2\kappa+2)}(z) U_{m,n}^{(2\kappa-2)}(z)}{U_{m-1,n-1}^{(2\kappa)}(z) U_{m,n}^{(2\kappa)}(z)},$$

so that we have

$$\begin{aligned} \frac{\widehat{v}_{m,n}^{(\kappa+n-1/2)}(z) + 1}{\widehat{v}_{m,n}^{(\kappa+n-1/2)}(z) - 1} &= \frac{U_{m,n}^{(2\kappa-2)} U_{m-1,n-1}^{(2\kappa+2)} - U_{m,n}^{(2\kappa)} U_{m-1,n-1}^{(2\kappa)}}{U_{m,n}^{(2\kappa-2)} U_{m-1,n-1}^{(2\kappa+2)} + U_{m,n}^{(2\kappa)} U_{m-1,n-1}^{(2\kappa)}}, \\ \frac{d}{dz} \ln \frac{U_{m,n-1}^{(2\kappa)}}{U_{m-1,n}^{(2\kappa)}} &= \frac{D_z(U_{m,n-1}^{(2\kappa)} \cdot U_{m-1,n}^{(2\kappa)})}{U_{m,n-1}^{(2\kappa)} U_{m-1,n}^{(2\kappa)}}. \end{aligned}$$

From the identities (A.9a) and (A.10a)

$$\begin{aligned} 4D_z(U_{m,n-1}^{(2\kappa)} \cdot U_{m-1,n}^{(2\kappa)}) &= U_{m,n}^{(2\kappa-2)} U_{m-1,n-1}^{(2\kappa+2)} - U_{m,n}^{(2\kappa)} U_{m-1,n-1}^{(2\kappa)}, \\ 2U_{m,n-1}^{(2\kappa)} U_{m-1,n}^{(2\kappa)} &= U_{m,n}^{(2\kappa-2)} U_{m-1,n-1}^{(2\kappa+2)} + U_{m,n}^{(2\kappa)} U_{m-1,n-1}^{(2\kappa)}, \end{aligned}$$

respectively, therefore

$$\frac{2D_z(U_{m,n-1}^{(2\kappa)} \cdot U_{m-1,n}^{(2\kappa)})}{U_{m,n-1}^{(2\kappa)} U_{m-1,n}^{(2\kappa)}} = \frac{U_{m,n}^{(2\kappa-2)} U_{m-1,n-1}^{(2\kappa+2)} - U_{m,n}^{(2\kappa)} U_{m-1,n-1}^{(2\kappa)}}{U_{m,n}^{(2\kappa-2)} U_{m-1,n-1}^{(2\kappa+2)} + U_{m,n}^{(2\kappa)} U_{m-1,n-1}^{(2\kappa)}},$$

which proves (4.24b).

By definition

$$\widehat{u}_{m,n}^{(\kappa)}(z) = -\frac{U_{m,n-1}^{(\kappa+1)}(z) U_{m,n+1}^{(\kappa-1)}(z)}{U_{m-1,n}^{(\kappa+1)}(z) U_{m+1,n}^{(\kappa-1)}(z)},$$

and so

$$\frac{\widehat{u}_{m,n}^{(\kappa)} + 1}{\widehat{u}_{m,n}^{(\kappa)} - 1} = \frac{U_{m,n-1}^{(\kappa+1)} U_{m,n+1}^{(\kappa-1)} - U_{m-1,n}^{(\kappa+1)} U_{m+1,n}^{(\kappa-1)}}{U_{m,n-1}^{(\kappa+1)} U_{m,n+1}^{(\kappa-1)} + U_{m-1,n}^{(\kappa+1)} U_{m+1,n}^{(\kappa-1)}}.$$

Also

$$\begin{aligned} \frac{d}{dz} \ln \frac{U_{m,n}^{(\kappa+1)}}{U_{m,n}^{(\kappa-1)}} + \frac{m+n+\kappa}{z} &= \frac{D_z(U_{m,n}^{(\kappa+1)} \cdot U_{m,n}^{(\kappa-1)})}{U_{m,n}^{(\kappa+1)} U_{m,n}^{(\kappa-1)}} + \frac{m+n+\kappa}{z} \\ &= \frac{z D_z(U_{m,n}^{(\kappa+1)} \cdot U_{m,n}^{(\kappa-1)}) + (m+n+\kappa) U_{m,n}^{(\kappa+1)} U_{m,n}^{(\kappa-1)}}{z U_{m,n}^{(\kappa+1)} U_{m,n}^{(\kappa-1)}} \end{aligned}$$

It can be shown that

$$U_{m,n-1}^{(\kappa+1)} U_{m,n+1}^{(\kappa-1)} + U_{m-1,n}^{(\kappa+1)} U_{m+1,n}^{(\kappa-1)} = 2z U_{m,n}^{(\kappa+1)} U_{m,n}^{(\kappa-1)}, \quad (4.26)$$

$$U_{m,n-1}^{(\kappa+1)} U_{m,n+1}^{(\kappa-1)} - U_{m-1,n}^{(\kappa+1)} U_{m+1,n}^{(\kappa-1)} = 4z D_z(U_{m,n}^{(\kappa+1)} \cdot U_{m,n}^{(\kappa-1)}) + 4(m+n+\kappa) U_{m,n}^{(\kappa+1)} U_{m,n}^{(\kappa-1)}, \quad (4.27)$$

which proves (4.24c). Equation (4.26) is the identity (A.13) and using computer algebra, we have verified equation (4.27) for several small values of m and n . $\square$

5 Rational Solutions of the Discrete Equations

In this section we discuss rational solutions of the discrete equations derived in §3. It is known that there are rational solutions of the dP_{II} equation (1.7), see [44, 45, 47, 60]. This is illustrated in the following theorem.

Theorem 5.1 Define $\tau_m^{(n)}(z)$ to be the $m \times m$ determinant given by

$$\tau_m^{(n)}(z) = \begin{vmatrix} p_m^{(n)} & p_{m+1}^{(n)} & \cdots & p_{2m-1}^{(n)} \\ p_{m-2}^{(n)} & p_{m-1}^{(n)} & \cdots & p_{2m-3}^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ p_{-m+2}^{(n)} & p_{-m+3}^{(n)} & \cdots & p_1^{(n)} \end{vmatrix}, \quad (5.1)$$

where $p_k^{(n)}(z) = L_k^{(n)}(\frac{1}{2}z)$, with $L_k^{(n)}(z)$ the Laguerre polynomial, for $k \geq 0$ and $p_k^{(n)}(z) = 0$ for $k < 0$, then

$$q_n(z; m) = \frac{\tau_{m+1}^{(n+1)}(z) \tau_m^{(n-1)}(z)}{\tau_{m+1}^{(n)}(z) \tau_m^{(n)}(z)} - 1, \quad (5.2)$$

satisfies

$$q_{n+1} + q_{n-1} = \frac{4(n+1)q_n - m - 1}{z(1 - q_n^2)}, \quad (5.3)$$

which is dP_{II} (1.7) with $\zeta = 4/z$, $\lambda = 1$ and $\rho = -m - 1$.

Proof See Kajiwara, Yamamoto and Ohta [47]. $\square$

Table 1 The effect on the parameters of applying the transformations $\mathcal{R}_1$ (3.3) and $\mathcal{R}_1^2$ on a solution $w(z)$ of PV with $(\alpha, \beta, \gamma) = (\frac{1}{2}a^2, -\frac{1}{2}b^2, c)$

	α_1	β_1	γ_1
$\mathcal{R}_1(a, b, c)$	$\frac{1}{8}(a - b + c + 1)^2$	$-\frac{1}{8}(a - b - c + 1)^2$	$a + b$
$\mathcal{R}_1(-a, b, c)$	$\frac{1}{8}(a + b - c - 1)^2$	$-\frac{1}{8}(a + b + c - 1)^2$	$-a + b$
$\mathcal{R}_1(a, -b, c)$	$\frac{1}{8}(a + b + c + 1)^2$	$-\frac{1}{8}(a + b - c + 1)^2$	$a - b$
$\mathcal{R}_1(-a, -b, c)$	$\frac{1}{8}(a - b - c - 1)^2$	$-\frac{1}{8}(a - b + c - 1)^2$	$-a - b$
$\mathcal{R}_1^2(a, b, c)$	$\frac{1}{2}(a + 1)^2$	$-\frac{1}{2}(b - 1)^2$	c
$\mathcal{R}_1^2(-a, b, c)$	$\frac{1}{2}(a - 1)^2$	$-\frac{1}{2}(b - 1)^2$	c
$\mathcal{R}_1^2(a, -b, c)$	$\frac{1}{2}(a + 1)^2$	$-\frac{1}{2}(b + 1)^2$	c
$\mathcal{R}_1^2(-a, -b, c)$	$\frac{1}{2}(a - 1)^2$	$-\frac{1}{2}(b + 1)^2$	c

Remarks 5.2

(i) The determinant $\tau_m^{(n)}(z)$ can be written as the Wronskian

$$\tau_m^{(n)}(z) = (-1)^{m(m-1)/2} \text{Wr} \left(p_1^{(n-m+1)}, p_3^{(n-m+1)}, \dots, p_{2m-1}^{(n-m+1)} \right), \quad (5.4)$$

with $p_k^{(n)}(z) = L_k^{(n)}(\frac{1}{2}z)$, and is related to the generalised Umemura polynomial by

$$U_{m,0}^{(n-m+1)}(z) = U_{0,m}^{(-n-m-1)}(z) = c_{0,m}(-1)^{m(m+1)/2} \left(\frac{1}{2}\right)^{m(m-1)/2} \tau_m^{(n)}(z).$$

(ii) The function $q_n(z; m)$ satisfies equation (4.11) with

$$(\alpha, \beta, \gamma) = \left(\frac{1}{8}(m-n)^2, -\frac{1}{8}(m+n+2)^2, -m-1\right).$$

5.1 Rational Solutions in Terms of Generalised Laguerre Polynomials

Each of the transformations $\mathcal{R}_1$ (3.3), $\mathcal{R}_2$ (3.13) and $\mathcal{R}_3$ (3.24) is actually four transformations since there are two independent choices of sign for $a = \pm\sqrt{2\alpha}$ and $b = \pm\sqrt{2\beta}$. This can lead to four different solutions from applying the transformation to a given solution. This is illustrated in Table 1 which shows the effect on the parameters of applying the transformations $\mathcal{R}_1$ (3.3) and $\mathcal{R}_1^2$ on a solution $w(z)$ of PV (1.2) with $(\alpha, \beta, \gamma) = (\frac{1}{2}a^2, -\frac{1}{2}b^2, c)$; see also Example 5.3.

5.1.1 First discrete equation: asymmetric discrete P_{II}

For the Bäcklund transformation $\mathcal{R}_1 = \mathcal{T}_{1,-1,1}$, we know that $\mathcal{R}_1^2(a, b, c) = (a + 1, b - 1, c)$, so that

$$\mathcal{R}_1^2(w_{m,n}^{(\mu)}; m, m + 2n + 1 + \mu, \mu) = (w_{m+1,n-1}^{(\mu)}; m + 1, m + 2n + \mu, \mu), \quad (5.5a)$$

$$\mathcal{R}_1^2(w_{m,n}^{(\mu)}; -m, m+2n+1+\mu, \mu) = (w_{m-1,n}^{(\mu)}; 1-m, m+2n+\mu, \mu), \quad (5.5b)$$

$$\mathcal{R}_1^2(w_{m,n}^{(\mu)}; m, -m-2n-1-\mu, \mu) = (w_{m+1,n}^{(\mu)}; m+1, -m-2n-2-\mu, \mu), \quad (5.5c)$$

$$\mathcal{R}_1^2(w_{m,n}^{(\mu)}; -m, -m-2n-1-\mu, \mu) = (w_{m-1,n+1}^{(\mu)}; 1-m, -m-2n-2-\mu, \mu). \quad (5.5d)$$

From (5.5) we see that except in the case (5.5c), the Bäcklund transformation $\mathcal{R}_1$ (3.3) can only be applied a finite number of times before the sequence terminates as either m or n becomes zero, as illustrated in the following example.

Example 5.3 Consider the rational solution

$$w_{1,1}^{(\mu-3)}(z) = -\frac{(z-\mu+1)[z^2-2\mu z+\mu(\mu+1)]}{z^2-2\mu z+\mu(\mu-1)},$$

which satisfies P_V (1.2) with $(\alpha, \beta, \gamma) = (\frac{1}{2}, -\frac{1}{2}(\mu+1)^2, \mu-3)$, so that

$$q_0(z) = \frac{w_{1,1}^{(\mu-3)}(z)+1}{w_{1,1}^{(\mu-3)}(z)-1} = 1 + \frac{2}{z-\mu} - \frac{4(z-\mu+1)}{z^2-2(\mu-1)z+\mu(\mu-1)}. \quad (5.6)$$

To derive a sequence of solutions we define the Bäcklund transformation

$$\mathcal{Q}_1(q; a, b, c) = \left(\frac{2}{z(q^2-1)} \left\{ z \frac{dq}{dz} - (a-b)q - a-b \right\}; \frac{1}{2}(a-b+c+1), \right. \\ \left. \frac{1}{2}(-a+b+c-1), a+b \right), \quad (5.7)$$

since

$$\mathcal{Q}_1(q) = \frac{\mathcal{R}_1(\frac{q+1}{q-1})+1}{\mathcal{R}_1(\frac{q+1}{q-1})-1} = \frac{2}{z(q^2-1)} \left\{ z \frac{dq}{dz} - (a-b)q - a-b \right\}.$$

Applying the Bäcklund transformation $\mathcal{Q}_1$ (5.7), with $(a, b, c) = (1, \mu+1, \mu-3)$, gives the sequence of rational solutions

$$q_1^{[a]}(z) = 1 - \frac{2}{z-\mu+1} + \frac{4(z-\mu)}{z^2-2\mu z+\mu(\mu-1)}, \\ q_2^{[a]}(z) = 1 - \frac{4(z-\mu+1)}{z^2-2(\mu-1)z+(\mu-1)(\mu-2)}, \quad q_3^{[a]}(z) = 1.$$

Applying $\mathcal{Q}_1$ (5.7), with $(a, b, c) = (-1, \mu+1, \mu-3)$, gives the sequence of rational solutions

$$q_1^{[b]}(z) = -1 + \frac{2\mu}{z} - \frac{2}{z-\mu+1}, \quad q_2^{[b]}(z) = 1,$$

and applying $\mathcal{Q}_1$ (5.7), with $(a, b, c) = (-1, -\mu - 1, \mu - 3)$, gives the sequence of rational solutions

$$q_1^{[d]}(z) = -1 - \frac{4(z - \mu)}{z^2 - 2\mu z + \mu(\mu + 1)}, \quad q_2^{[d]}(z) = 1.$$

In each case the Bäcklund transformation only produces a finite number of solutions.

Applying $\mathcal{Q}_1$ (5.7), with $(a, b, c) = (1, -\mu - 1, \mu - 3)$, gives an infinite sequence of solutions. The next few are

$$\begin{aligned} q_1(z) &= 1 - \frac{2\mu}{z} + 2 \frac{d}{dz} \ln \frac{z^2 - 2\mu z + \mu(\mu - 1)}{z^2 - 2\mu z + \mu(\mu + 1)}, \\ q_2(z) &= 1 + 2 \frac{d}{dz} \ln \frac{z^2 - 2(\mu + 1)z + \mu(\mu + 1)}{z^4 - 4\mu z^3 + 6\mu^2 z^2 - 4\mu(\mu^2 - 1)z + \mu^2(\mu^2 - 1)}, \\ q_3(z) &= 1 - \frac{2\mu}{z} \\ &\quad + 2 \frac{d}{dz} \ln \frac{z^3 - 3(\mu + 1)z^2 + 3\mu(\mu + 1)z - \mu(\mu^2 - 1)}{z^4 - 4(\mu + 1)z^3 + 6(\mu + 1)^2 z^2 - 4\mu(\mu + 1)(\mu + 2)z + \mu(\mu + 1)^2(\mu + 2)}. \end{aligned}$$

Generally for $n \geq 0$

$$q_{2n}(z) = 1 + 2 \frac{d}{dz} \ln \frac{T_{n,1}^{(\mu-2)}(z)}{T_{n,2}^{(\mu-4)}(z)}, \quad q_{2n+1}(z) = 1 - \frac{2\mu}{z} + 2 \frac{d}{dz} \ln \frac{T_{n+1,1}^{(\mu-3)}(z)}{T_{n,2}^{(\mu-3)}(z)},$$

satisfy

$$q_{n+1} + q_{n-1} = \frac{4}{z} \frac{(n + \mu + 2)q_n - \frac{3}{2} - (\mu - \frac{3}{2})(-1)^n}{1 - q_n^2},$$

which is asymmetric dP_{II} (3.7) with $\lambda = \mu + 2$, $\rho = -\frac{3}{2}$ and $\varphi = -\mu + \frac{3}{2}$, whilst $x_n = q_{2n}$ and $y_n = q_{2n+1}$ satisfy

$$x_{n+1} + x_n = \frac{4}{z} \frac{(2n + \mu + 3)y_n + \mu - 3}{1 - y_n^2}, \quad y_n + y_{n-1} = \frac{4}{z} \frac{(2n + \mu + 2)x_n - \mu}{1 - x_n^2}.$$

Also, applying the inverse transformation $\mathcal{Q}_1^{-1}$ we obtain

$$q_{-1}(z) = 1 - \frac{2\mu}{z} + \frac{2}{z - \mu + 1}, \quad q_{-2}(z) = 1.$$

We note that for the solution q_0 (5.6)

$$\begin{aligned} \mathcal{Q}_1^3(q_0; 1, \mu + 1, \mu - 3) &= 1, \quad \mathcal{Q}_1^2(q_0; -1, \mu + 1, \mu - 3) = 1, \\ \mathcal{Q}_1^2(q_0; -1, -\mu - 1, \mu - 3) &= 1. \end{aligned}$$

where $\mathcal{Q}_1$ is the Bäcklund transformation (5.7). Further if $\mathcal{R}_1$ (3.3) is the Bäcklund transformation given by (5.5c) then

$$w_{1,1}^{(\mu-3)} \xrightarrow{\mathcal{R}_1} \frac{1}{v_{1,1}^{(\mu)}} \xrightarrow{\mathcal{R}_1} w_{2,1}^{(\mu-3)} \xrightarrow{\mathcal{R}_1} \frac{1}{v_{2,1}^{(\mu)}} \xrightarrow{\mathcal{R}_1} w_{3,1}^{(\mu-3)}$$

or equivalently

$$\begin{aligned} \frac{w_{1,1}^{(\mu-3)} + 1}{w_{1,1}^{(\mu-3)} - 1} &\xrightarrow{\mathcal{Q}_1} \frac{1 + v_{1,1}^{(\mu)}}{1 - v_{1,1}^{(\mu)}} \xrightarrow{\mathcal{Q}_1} \frac{w_{2,1}^{(\mu-3)} + 1}{w_{1,1}^{(\mu-3)} - 1} \xrightarrow{\mathcal{Q}_1} \frac{1 + v_{2,1}^{(\mu)}}{1 - v_{2,1}^{(\mu)}} \\ &\xrightarrow{\mathcal{Q}_1} \frac{w_{3,1}^{(\mu-3)} + 1}{w_{3,1}^{(\mu-3)} - 1} \end{aligned}$$

More generally, using the solution $w_{m,n}^{(\mu)}(z)$, for fixed m and n , then we have the following Lemma.

Lemma 5.4 Consider the rational functions

$$Q_{2N}(z) = X_N(z) = \frac{w_{N+m,n}^{(\mu)}(z) + 1}{w_{N+m,n}^{(\mu)}(z) - 1}, \quad Q_{2N+1}(z) = Y_N(z) = \frac{1 + v_{N+m,n}^{(\mu+2n+1)}(z)}{1 - v_{N+m,n}^{(\mu+2n+1)}(z)},$$

with $w_{m,n}^{(\mu)}(z)$ and $v_{m,n}^{(\mu)}(z)$ as given in Theorem 4.4, then $Q_N(z)$ satisfies

$$Q_{N+1} + Q_{N-1} = \frac{4(N + 2m + 2n + \mu + 1)Q_N - n - \frac{1}{2} - (n + \mu + \frac{1}{2})(-1)^N}{1 - Q_N^2},$$

which is asymmetric dP_{II} (3.7) with $\lambda = 2m + 2n + \mu + 1$, $\rho = -n - \frac{1}{2}$ and $\varphi = -n - \mu - \frac{1}{2}$, whilst $X_N(z)$ and $Y_N(z)$ satisfy the discrete system

$$X_{N+1} + X_N = \frac{4(2N + 2m + 2n + \mu + 2)Y_N + \mu}{z(1 - Y_N^2)}, \quad (5.8a)$$

$$Y_N + Y_{N-1} = \frac{4(2N + 2m + 2n + \mu + 1)X_N - 2n - \mu - 1}{z(1 - X_N^2)}. \quad (5.8b)$$

Proof From the transformations

$$\begin{aligned} &\mathcal{R}_1(w_{m,n}^{(\mu)}; m, -m - 2n - 1 - \mu, \mu) \\ &= (1/v_{m,n}^{(\mu+2n+1)}; m + n + 1 + \mu, -m - n - 1, -\mu - 2n - 1), \\ &\mathcal{R}_1(v_{m,n}^{(\mu)}; m + n + 1, -m + n - \mu, \mu) \\ &= (1/w_{m+1,n}^{(\mu-2n-1)}; m + \mu + 1, -m - 1, -\mu + 2n + 1), \end{aligned}$$

we obtain the sequence of solutions

$$w_{m,n}^{(\mu)} \xrightarrow{\mathcal{R}_1} \frac{1}{v_{m,n}^{(\mu+2n+1)}} \xrightarrow{\mathcal{R}_1} w_{m+1,n}^{(\mu)} \xrightarrow{\mathcal{R}_1} \frac{1}{v_{m+1,n}^{(\mu+2n+1)}} \xrightarrow{\mathcal{R}_1} w_{m+2,n}^{(\mu)}$$

from which the result follows. We note that

$$\mathcal{R}_1^2(w_{m,n}^{(\mu)}) = w_{m+1,n}^{(\mu)}, \quad \mathcal{R}_1^2(1/v_{m,n}^{(\mu)}) = 1/v_{m+1,n}^{(\mu)}.$$

□

Remarks 5.5

- (i) An equivalent result can be obtained using the inverse transformation $\mathcal{R}_1^{-1} = \mathcal{T}_{-1,1,-1}$ since

$$\begin{aligned} \mathcal{R}_1^{-1}(w_{m,n}^{(\mu)}; -m, m+2n+1+\mu, \mu) \\ &= (v_{m,n}^{(\mu+2n+1)}; -m-n-1, m+n+1+\mu, 2n+1+\mu), \\ \mathcal{R}_1^{-1}(v_{m,n}^{(\mu)}; -m-n-1, m-n+\mu, \mu) \\ &= (w_{m+1,n}^{(\mu-2n-1)}; -m-1, m+1+\mu, -2n-1+\mu). \end{aligned}$$

- (ii) We note that if $\mu = -n - \frac{1}{2}$ then

$$Q_{2N}(z) = \frac{w_{N+m,n}^{(-n-1/2)}(z) + 1}{w_{N+m,n}^{(-n-1/2)}(z) - 1}, \quad Q_{2N+1}(z) = \frac{1 + v_{N+m,n}^{(n+1/2)}(z)}{1 - v_{N+m,n}^{(n+1/2)}(z)},$$

are solutions to

$$Q_{N+1} + Q_{N-1} = \frac{4(N+2m+n+\frac{1}{2})Q_N - n - \frac{1}{2}}{z(1 - Q_N^2)},$$

which is dP_{II} (1.7) with $\zeta = 4/z$, $\lambda = 2m+n+\frac{1}{2}$ and $\rho = -n - \frac{1}{2}$. Hence, although there is no alternating term, the structure of the even and odd terms is different.

- (iii) As remarked above, Kitaev, Law and McLeod [48, Theorem 1.2] proved that rational solutions expressed in terms of generalised Laguerre polynomials are unique if $\mu \notin \mathbb{Z}$. Consequently the hierarchies can be determined through the parameters using induction. To obtain the rational solution $w_{m,n}^{(\ell)}$, with $\ell \in \mathbb{Z}$, then we use $w_{m,n}^{(\ell)} = \lim_{\mu \rightarrow \ell} w_{m,n}^{(\mu)}$ and similarly for $v_{m,n}^{(\ell)}$ and $u_{m,n}^{(\ell)}$. In fact it is necessary to do this in order to evaluate the rational solution when $\mu \in \mathbb{Z}$ using Maple.

Alternatively, using the solution $v_{m,n}^{(\mu)}(z)$, for fixed m and n , then we have the following Lemma.

Lemma 5.6 Consider the rational functions

$$Q_{2N}(z) = X_N(z) = \frac{v_{m,N+n}^{(\mu)} + 1}{v_{m,N+n}^{(\mu)} - 1}, \quad Q_{2N+1}(z) = Y_N(z) = \frac{u_{N+n+1,m}^{(\mu+2m+1)} + 1}{u_{N+n+1,m}^{(\mu+2m+1)} - 1},$$

with $v_{m,n}^{(\mu)}(z)$ and $u_{m,n}^{(\mu)}(z)$ as given in Theorem 4.4, then $Q_N(z)$ satisfies

$$Q_{N+1} + Q_{N-1} = \frac{4(N + 2n + 1 - \mu)Q_N + \mu + (m + \frac{1}{2})[1 + (-1)^N]}{z(1 - Q_N^2)},$$

which is asymmetric dP_{II} (3.7) with $\lambda = 2n + 1 - \mu$, $\rho = m + \frac{1}{2} + \mu$ and $\varphi = m + \frac{1}{2}$, whilst $X_N(z)$ and $Y_N(z)$ satisfy the discrete system

$$X_{N+1} + X_N = \frac{4(2N + 2n + 2 - \mu)Y_N + \mu}{z(1 - Y_N^2)}, \quad (5.9a)$$

$$Y_N + Y_{N-1} = \frac{4(2N + 2n + 1 - \mu)X_N + \mu + 2m + 1}{z(1 - X_N^2)}. \quad (5.9b)$$

Proof From the transformations

$$\begin{aligned} \mathcal{R}_1(v_{m,n}^{(\mu)}; m+n+1, m-n+\mu, \mu) &= (u_{n+1,m}^{(\mu+2m+1)}; n+1, -n-1+\mu, 2m+1+\mu), \\ \mathcal{R}_1(u_{m,n}^{(\mu)}; m, -m-2n-1+\mu, \mu) &= (v_{n,m}^{(\mu-2n-1)}; m+n+1, -m-n-1+\mu, \\ &\quad -2n-1+\mu), \end{aligned}$$

we obtain the sequence of solutions

$$v_{m,n}^{(\mu)} \xrightarrow{\mathcal{R}_1} u_{n+1,m}^{(\mu+2m+1)} \xrightarrow{\mathcal{R}_1} v_{m,n+1}^{(\mu)} \xrightarrow{\mathcal{R}_1} u_{n+2,m}^{(\mu+2m+1)} \xrightarrow{\mathcal{R}_1} v_{m,n+2}^{(\mu)}$$

from which the result follows. We note that

$$\mathcal{R}_1^2(v_{m,n}^{(\mu)}) = v_{m,n+1}^{(\mu)}, \quad \mathcal{R}_1^2(u_{n,m}^{(\mu)}) = u_{n+1,m}^{(\mu)}.$$

□

5.1.2 Second Discrete Equation

The effect on the parameters of applying the transformations (3.13) $\mathcal{R}_2$ and $\mathcal{R}_2^2$ on a solution $w(z)$ of P_V with $(\alpha, \beta, \gamma) = (\frac{1}{2}a^2, -\frac{1}{2}b^2, c)$ is given in Table 2.

In the following example, we illustrate how using the Bäcklund transformation $\mathcal{R}_2$ (3.13) gives rise to a hierarchy of solutions of (3.18) in terms of generalised Laguerre polynomials.

Table 2 The effect on the parameters of applying the transformations $\mathcal{R}_2$ (3.13) and $\mathcal{R}_2^2$ on a solution $w(z)$ of PV with $(\alpha, \beta, \gamma) = (\frac{1}{2}a^2, -\frac{1}{2}b^2, c)$

	α_1	β_1	γ_1
$\mathcal{R}_2(a, b, c)$	$\frac{1}{8}(a+b-c-1)^2$	$-\frac{1}{8}(a+b+c+1)^2$	$-a+b+1$
$\mathcal{R}_2(-a, b, c)$	$\frac{1}{8}(a-b+c+1)^2$	$-\frac{1}{8}(a-b-c-1)^2$	$a+b+1$
$\mathcal{R}_2(a, -b, c)$	$\frac{1}{8}(a-b-c-1)^2$	$-\frac{1}{8}(a-b+c+1)^2$	$-a-b+1$
$\mathcal{R}_2(-a, -b, c)$	$\frac{1}{8}(a+b+c+1)^2$	$-\frac{1}{8}(a+b-c-1)^2$	$a-b+1$
$\mathcal{R}_2^2(a, b, c)$	$\frac{1}{2}(a-1)^2$	$-\frac{1}{2}(b+1)^2$	$c+2$
$\mathcal{R}_2^2(-a, b, c)$	$\frac{1}{2}(a+1)^2$	$-\frac{1}{2}(b+1)^2$	$c+2$
$\mathcal{R}_2^2(a, -b, c)$	$\frac{1}{2}(a-1)^2$	$-\frac{1}{2}(b-1)^2$	$c+2$
$\mathcal{R}_2^2(-a, -b, c)$	$\frac{1}{2}(a+1)^2$	$-\frac{1}{2}(b-1)^2$	$c+2$

Example 5.7 Applying the Bäcklund transformation $\mathcal{R}_2$ (3.13), with $(a, b, c) = (-3, \mu, \mu)$, to the solution

$$v_{1,1}^{(\mu)}(z) = 1 - \frac{z[z^2 - 2\mu z + \mu(\mu+1)][z^2 - 2\mu z + \mu(\mu-1)]}{(z-\mu)[z^4 - 4\mu z^3 + 6\mu^2 z^2 - 4\mu(\mu^2-1)z + \mu^2(\mu^2-1)]},$$

we obtain the sequence of rational solutions of PV (1.2)

$$v_{1,1}^{(\mu)} \xrightarrow{\mathcal{R}_2} u_{2,1}^{(\mu+4)} \xrightarrow{\mathcal{R}_2} v_{1,2}^{(\mu+2)} \xrightarrow{\mathcal{R}_2} u_{3,1}^{(\mu+6)} \xrightarrow{\mathcal{R}_2} v_{1,3}^{(\mu+4)}$$

Hence we obtain the solutions of equation (3.18)

$$\begin{aligned}
 q_0(z) &= \frac{v_{1,1}^{(\mu)} + 1}{v_{1,1}^{(\mu)} - 1} = -1 + \frac{2\mu}{z} + \frac{d}{dz} \ln \frac{z^2 - 2\mu + \mu(\mu+1)}{z^2 - 2\mu + \mu(\mu-1)}, \\
 q_1(z) &= \frac{u_{2,1}^{(\mu+4)} + 1}{u_{2,1}^{(\mu+4)} - 1} = 1 \\
 &\quad + 2 \frac{d}{dz} \ln \frac{z^4 - 4(\mu+1)z^3 + 6(\mu+1)^2 z^2 - 4\mu(\mu+1)(\mu+2)z + \mu(\mu+1)^2(\mu+2)}{z^2 - 2\mu + \mu(\mu+1)}.
 \end{aligned}$$

Consequently, if we define the rational functions

$$\begin{aligned}
 q_{2n}(z) = x_n(z) &= \frac{v_{1,n+1}^{(2n+\mu)}(z) + 1}{v_{1,n+1}^{(2n+\mu)}(z) - 1} = -1 + \frac{2(\mu+2n)}{z} + 2 \frac{d}{dz} \ln \frac{T_{0,n+2}^{(\mu-3)}}{T_{1,n+1}^{(\mu-3)}(z)}, \\
 q_{2n+1}(z) = y_n(z) &= \frac{u_{n+2,1}^{(2n+\mu+4)}(z) + 1}{u_{n+2,1}^{(2n+\mu+4)}(z) - 1} = 1 + 2 \frac{d}{dz} \ln \frac{T_{1,n+2}^{(\mu-3)}}{T_{0,n+2}^{(\mu-3)}(z)},
 \end{aligned}$$

then $q_n(z)$ satisfies

$$\frac{n + \mu + 2}{q_{n+1} + q_n} + \frac{n + \mu + 1}{q_n + q_{n-1}} = \frac{1}{2}z - \frac{\{n + \mu + \frac{3}{2}[1 + (-1)^n]\}q_n + 3 - \mu}{1 - q_n^2},$$

which is equation (3.18) with $\lambda = \mu + \frac{3}{2}$, $\rho = 3 - \mu$ and $\varphi = \frac{3}{2}$, whilst $x_n(z)$ and $y_n(z)$ satisfy the discrete system

$$\begin{aligned} \frac{2n + \mu + 3}{x_{n+1} + y_n} + \frac{2n + \mu + 2}{x_n + y_n} &= \frac{1}{2}z - \frac{(2n + \mu + 1)y_n + 3 - \mu}{1 - y_n^2}, \\ \frac{2n + \mu + 2}{x_n + y_n} + \frac{2n + \mu + 1}{x_n + y_{n-1}} &= \frac{1}{2}z - \frac{(2n + \mu + 3)x_n + 3 - \mu}{1 - x_n^2}. \end{aligned}$$

More generally, using the solution $v_{m,n}^{(\mu)}(z)$, for fixed m and n , then we have the following Lemma.

Lemma 5.8 Consider the rational functions

$$Q_{2N}(z) = X_N(z) = \frac{v_{m,N+n}^{(2N+\mu)}(z) + 1}{v_{m,N+n}^{(2N+\mu)}(z) - 1}, \quad Q_{2N+1}(z) = Y_N(z) = \frac{u_{N+n+1,m}^{(2N+\mu+2m+2)}(z) + 1}{u_{N+n+1,m}^{(2N+\mu+2m+2)}(z) - 1},$$

with $v_{m,n}^{(\mu)}(z)$ and $u_{m,n}^{(\mu)}(z)$ as given in Theorem 4.4, then $Q_N(z)$ satisfies

$$\frac{N + m + \mu + 1}{Q_{N+1} + Q_N} + \frac{N + m + \mu}{Q_N + Q_{N-1}} = \frac{1}{2}z - \frac{\{N + [1 + (-1)^N](m + \frac{1}{2}) + \mu\}Q_N + 2n + 1 - \mu}{1 - Q_N^2},$$

which is equation (3.18) with $\lambda = m + \mu - \frac{1}{2}$, $\rho = 2n - \mu + 1$ and $\varphi = m + \frac{1}{2}$, whilst $X_N(z)$ and $Y_N(z)$ satisfy the discrete system

$$\frac{2N + m + \mu + 2}{X_{N+1} + Y_N} + \frac{2N + m + \mu + 1}{X_N + Y_N} = \frac{1}{2}z - \frac{(2N + \mu + 1)Y_N + 2n + 1 - \mu}{1 - Y_N^2}, \quad (5.10a)$$

$$\frac{2N + m + \mu + 1}{X_N + Y_N} + \frac{2N + m + \mu}{X_N + Y_{N-1}} = \frac{1}{2}z - \frac{(2N + 2m + \mu + 1)X_N + 2n + 1 - \mu}{1 - X_N^2}. \quad (5.10b)$$

Proof From the transformations

$$\begin{aligned} &\mathcal{R}_2(v_{m,n}^{(\mu)}; -m - n - 1, m - n + \mu, \mu) \\ &= (u_{n+1,m}^{(\mu+2m+2)}; -n - 1, -n + \mu, 2m + 2 + \mu), \\ &\mathcal{R}_2(u_{n,m}^{(\mu+2m+2)}; -n, -n + 1 + \mu, 2m + 2 + \mu) \\ &= (v_{m,n}^{(\mu+2)}; m + n + 1, -m - n - 1 + \mu, 2m - 1 + \mu), \end{aligned}$$

we obtain the sequence of solutions

$$v_{m,n}^{(\mu)} \xrightarrow{\mathcal{R}_2} u_{n+1,m}^{(\mu+2m+2)} \xrightarrow{\mathcal{R}_2} v_{m,n+1}^{(\mu+2)} \xrightarrow{\mathcal{R}_2} u_{n+2,m}^{(\mu+2m+4)} \xrightarrow{\mathcal{R}_2} v_{m,n+2}^{(\mu+4)}$$

from which the result follows. We note that

$$\mathcal{R}_2^2(v_{m,n}^{(\mu)}) = v_{m,n+1}^{(\mu+2)}, \quad \mathcal{R}_2^2(u_{n,m}^{(\mu)}) = u_{n+1,m}^{(\mu+2)}.$$

□

Alternatively, using the solution $w_{m,n}^{(\mu)}(z)$, for fixed m and n , then we have the following Lemma.

Lemma 5.9 Consider the rational functions

$$Q_{2N}(z) = X_N(z) = \frac{w_{N+m,n}^{(\mu-2N)}(z) + 1}{w_{N+m,n}^{(\mu-2N)}(z) - 1}, \quad Q_{2N+1}(z) = Y_N(z) = \frac{v_{N+m,n}^{(\mu+2n-2N)}(z) + 1}{v_{N+m,n}^{(\mu+2n-2N)}(z) - 1},$$

with $w_{m,n}^{(\mu)}(z)$ and $v_{m,n}^{(\mu)}(z)$ as given in Theorem 4.4, then $Q_N(z)$ satisfies

$$\begin{aligned} & \frac{N-n-\mu}{Q_{N+1} + Q_N} + \frac{N-n-\mu-1}{Q_N + Q_{N-1}} + \frac{1}{2}z \\ & + \frac{\{N-\mu-(n+\frac{1}{2})[1+(-1)^N]\}Q_N + 2m+2n+1+\mu}{1-Q_N^2} = 0, \end{aligned}$$

which is equation (3.18) with $\lambda = -n-\mu-\frac{1}{2}$, $\rho = 2m+2n+\mu+1$ and $\varphi = -n-\frac{1}{2}$, whilst $X_N(z)$ and $Y_N(z)$ satisfy the discrete system

$$\frac{2N-n-\mu+1}{X_{N+1} + Y_N} + \frac{2N-n-\mu}{X_N + Y_N} + \frac{1}{2}z + \frac{(2N-\mu+1)Y_N + 2m+2n+1+\mu}{1-Y_N^2} = 0, \quad (5.11a)$$

$$\frac{2N-n-\mu}{X_N + Y_N} + \frac{2N-n-\mu-1}{X_N + Y_{N-1}} + \frac{1}{2}z + \frac{(2N-2n-\mu-1)X_N + 2m+2n+1+\mu}{1-X_N^2} = 0. \quad (5.11b)$$

Proof From the transformations

$$\begin{aligned} \mathcal{R}_2^{-1}(w_{m,n}^{(\mu)}; m, m+2n+1+\mu, \mu) &= (v_{m,n}^{(\mu+2n)}; m+n+1, m+n+\mu, 2n+\mu), \\ \mathcal{R}_2^{-1}(v_{m,n}^{(\mu+2n)}; m+n+1, m+n+\mu, 2n+\mu) &= (w_{m+1,n}^{(\mu-2)}; m, m+2n+1+\mu, \mu) \end{aligned}$$

we obtain the sequence of solutions

$$w_{m,n}^{(\mu)} \xrightarrow{\mathcal{R}_2^{-1}} v_{m,n}^{(\mu+2n)} \xrightarrow{\mathcal{R}_2^{-1}} w_{m+1,n}^{(\mu-2)} \xrightarrow{\mathcal{R}_2^{-1}} v_{m,n}^{(\mu+2n-2)} \xrightarrow{\mathcal{R}_2^{-1}} w_{m+2,n}^{(\mu-4)}$$

from which the result follows. □

Table 3 The effect on the parameters of applying the transformation $\mathcal{R}_3$ (3.24), $\mathcal{R}_3^2$ and $\mathcal{R}_3^3$ on a solution $w(z)$ of PV with $(\alpha, \beta, \gamma) = (\frac{1}{2}a^2, -\frac{1}{2}b^2, c)$

	α_1	β_1	γ_1
$\mathcal{R}_3(a, b, c)$	$\frac{1}{8}(a - b + c + 1)^2$	$-\frac{1}{8}(a - b - c + 1)^2$	$a + b$
$\mathcal{R}_3(-a, b, c)$	$\frac{1}{8}(a + b - c - 1)^2$	$-\frac{1}{8}(a + b + c - 1)^2$	$-a + b$
$\mathcal{R}_3(a, -b, c)$	$\frac{1}{8}(a + b + c + 1)^2$	$-\frac{1}{8}(a + b - c + 1)^2$	$a - b$
$\mathcal{R}_3(-a, -b, c)$	$\frac{1}{8}(a - b - c - 1)^2$	$-\frac{1}{8}(a - b + c - 1)^2$	$-a - b$
$\mathcal{R}_3^2(a, b, c)$	$\frac{1}{8}(a + b + c + 1)^2$	$-\frac{1}{8}(a + b - c - 1)^2$	$a - b + 1$
$\mathcal{R}_3^2(-a, b, c)$	$\frac{1}{8}(a - b - c - 1)^2$	$-\frac{1}{8}(a - b + c + 1)^2$	$-a - b + 1$
$\mathcal{R}_3^2(a, -b, c)$	$\frac{1}{8}(a - b + c + 1)^2$	$-\frac{1}{8}(a - b - c - 1)^2$	$a + b + 1$
$\mathcal{R}_3^2(-a, -b, c)$	$\frac{1}{8}(a + b - c - 1)^2$	$-\frac{1}{8}(a + b + c + 1)^2$	$-a + b + 1$
$\mathcal{R}_3^3(a, b, c)$	$\frac{1}{2}(a + 1)^2$	$-\frac{1}{2}b^2$	$c + 1$
$\mathcal{R}_3^3(-a, b, c)$	$\frac{1}{2}(a - 1)^2$	$-\frac{1}{2}b^2$	$c + 1$
$\mathcal{R}_3^3(a, -b, c)$	$\frac{1}{2}(a + 1)^2$	$-\frac{1}{2}b^2$	$c + 1$
$\mathcal{R}_3^3(-a, -b, c)$	$\frac{1}{2}(a - 1)^2$	$-\frac{1}{2}b^2$	$c + 1$

Remarks 5.10 From the transformations

$$\begin{aligned}
 &\mathcal{R}_2^{-1}(w_{m,n}^{(\mu)}; m, -m - 2n - 1 - \mu, \mu) \\
 &= (1/u_{n+1,m}^{(\mu+2m+2n+2)}; -n - \mu, -n - 1, -2m - 2n - 2 - \mu), \\
 &\mathcal{R}_2^{-1}(1/u_{n,m}^{(\mu)}; 2m + n + 1 - \mu, -n, -\mu) \\
 &= (w_{m+1,n}^{(\mu-2m-2n-2)}; m + 1, m - \mu, -2m - 2n - 2 + \mu),
 \end{aligned}$$

we obtain the sequence of solutions

$$\begin{aligned}
 w_{m,n}^{(\mu)} &\xrightarrow{\mathcal{R}_2^{-1}} \frac{1}{u_{n+1,m}^{(\mu+2m+2n+2)}} \xrightarrow{\mathcal{R}_2^{-1}} w_{m+1,n+1}^{(\mu-2)} \xrightarrow{\mathcal{R}_2^{-1}} \frac{1}{u_{n+2,m+1}^{(\mu+2m+2n+4)}} \\
 &\xrightarrow{\mathcal{R}_2^{-1}} w_{m+2,n+2}^{(\mu-4)}
 \end{aligned}$$

5.1.3 Third Discrete Equation: Ternary Discrete P_I

The effect on the parameters of applying the transformations $\mathcal{R}_3$ (3.24), $\mathcal{R}_3^2$ and $\mathcal{R}_3^3$ on a solution $w(z)$ of PV with $(\alpha, \beta, \gamma) = (\frac{1}{2}a^2, -\frac{1}{2}b^2, c)$ is given in Table 3.

In the following example, we illustrate how using the Bäcklund transformation $\mathcal{R}_3$ (3.24) gives rise to a hierarchy of solutions of ternary dP_I (3.28) in terms of generalised Laguerre polynomials.

Example 5.11 Applying the Bäcklund transformation $\mathcal{R}_3$ (3.24), with $(a, b, c) = (\mu + 1, -1, -\mu + 3)$, to the solution

$$\frac{1}{w_{1,1}^{(\mu-3)}(z)} = -\frac{z^2 - 2\mu z + \mu(\mu - 1)}{(z - \mu + 1)[z^2 - 2\mu z + \mu(\mu + 1)]},$$

we obtain the sequence of rational solutions of P_V (1.2)

$$\frac{1}{w_{1,1}^{(\mu-3)}} \xrightarrow{\mathcal{R}_3} v_{1,1}^{(\mu)} \xrightarrow{\mathcal{R}_3} u_{2,1}^{(\mu+3)} \xrightarrow{\mathcal{R}_3} \frac{1}{w_{1,2}^{(\mu-4)}} \xrightarrow{\mathcal{R}_3} v_{1,2}^{(\mu+1)} \xrightarrow{\mathcal{R}_3} u_{3,1}^{(\mu+4)}$$

so that the effect of applying $\mathcal{R}_3$ three times is

$$\mathcal{R}_3^3\left(1/w_{1,1}^{(\mu-3)}\right) = 1/w_{1,1}^{(\mu-4)}, \quad \mathcal{R}_3^3\left(v_{1,1}^{(\mu)}\right) = v_{1,2}^{(\mu+1)}, \quad \mathcal{R}_3^3\left(u_{2,1}^{(\mu+3)}\right) = u_{3,1}^{(\mu+4)}.$$

Hence we obtain the rational solutions of the ternary dP_I (3.28)

$$\begin{aligned} x_0(z) &= \frac{w_{1,1}^{(\mu-3)}(z)}{1 - w_{1,1}^{(\mu-3)}(z)} = -1 + \frac{d}{dz} \ln \frac{z^2 - 2(\mu - 1)z + \mu(\mu - 1)}{z - \mu}, \\ x_1(z) &= \frac{1}{v_{1,1}^{(\mu)}(z) - 1} = -1 + \frac{\mu}{z} + \frac{d}{dz} \ln \frac{z^2 - 2\mu z + \mu(\mu - 1)}{z^2 - 2\mu z + \mu(\mu + 1)}, \\ x_2(z) &= \frac{1}{u_{2,1}^{(\mu+3)}(z) - 1} = \frac{d}{dz} \ln \frac{z^4 - 4\mu z^3 + 6\mu^2 z^2 - 4\mu(\mu^2 - 1)z + \mu^2(\mu^2 - 1)}{z^2 - 2(\mu + 1)z + \mu(\mu - 1)}. \end{aligned}$$

Consequently, if we define the rational functions

$$\begin{aligned} x_{3n}(z) &= X_n(z) = \frac{w_{1,n+1}^{(\mu-n-3)}(z)}{1 - w_{1,n+1}^{(\mu-n-3)}(z)} = -1 + \frac{d}{dz} \ln \frac{T_{0,n+2}^{(\mu-n-4)}(z)}{T_{0,n+1}^{(\mu-n-2)}(z)}, \\ x_{3n+1}(z) &= Y_n(z) = \frac{1}{v_{1,n+1}^{(\mu+n)}(z) - 1} = -1 + \frac{\mu + n}{z} + \frac{d}{dz} \ln \frac{T_{0,n+2}^{(\mu-n-3)}(z)}{T_{1,n+1}^{(\mu-n-3)}(z)}, \\ x_{3n+2}(z) &= Z_n(z) = \frac{1}{u_{n+2,1}^{(\mu+n+3)}(z) - 1} = \frac{d}{dz} \ln \frac{T_{1,n+2}^{(\mu-n-4)}(z)}{T_{0,n+2}^{(\mu-n-4)}(z)}, \end{aligned}$$

then $x_n(z)$ satisfies ternary dP_I

$$x_n(x_{n+1} + x_{n-1} + 1) + \frac{a_n}{z} = 0,$$

with

$$a_n = \frac{1}{3}(n + \mu + 5) + \frac{2}{3}(\mu - 1) \cos\left(\frac{2}{3}\pi n\right) + \frac{4}{9}\sqrt{3} \sin\left(\frac{2}{3}\pi n\right),$$

whilst $X_n(z)$, $Y_n(z)$ and $Z_n(z)$ satisfy the discrete system

$$\begin{aligned} X_n(Y_n + Z_{n-1} + 1) + \frac{n + \mu + 1}{z} &= 0, \\ Y_n(Z_n + X_n + 1) + \frac{n + 3}{z} &= 0, \\ Z_n(X_{n+1} + Y_n + 1) + \frac{n + 2}{z} &= 0. \end{aligned}$$

More generally, using the solution $w_{m,n}^{(\mu)}(z)$, for fixed m and n , as the initial solution then we have the following Lemma.

Lemma 5.12 *Consider the rational functions*

$$\begin{aligned} X_N(z) &= \frac{w_{m,N+n}^{(\mu-N)}(z)}{1 - w_{m,N+n}^{(\mu-N)}(z)}, \quad Y_N(z) = \frac{1}{v_{m,N+n}^{(\mu+N+2n+1)}(z) - 1}, \\ Z_N(z) &= \frac{1}{u_{N+n+1,m}^{(\mu+N+2m+2n+2)}(z) - 1}, \end{aligned} \quad (5.12)$$

with $w_{m,n}^{(\mu)}(z)$, $v_{m,n}^{(\mu)}(z)$ and $u_{m,n}^{(\mu)}(z)$ as given in Theorem 4.4, then $X_N(z)$, $Y_N(z)$ and $Z_N(z)$ satisfy the discrete system

$$X_N(Y_N + Z_{N-1} + 1) + \frac{N + m + 2n + \mu + 1}{z} = 0, \quad (5.13a)$$

$$Y_N(Z_N + X_N + 1) + \frac{N + m + n + 1}{z} = 0, \quad (5.13b)$$

$$Z_N(X_{N+1} + Y_N + 1) + \frac{N + n + 1}{z} = 0, \quad (5.13c)$$

which is the system (3.29) with $a_{3N} = N + m + 2n + \mu + 1$, $a_{3N+1} = N + m + n + 1$ and $a_{3N+2} = N + n + 1$, i.e. (3.26a) with $\lambda = \frac{1}{3}(m + 2n + 2\mu + 1)$, $\rho = \frac{1}{3}(2m + 4n + \mu + 2)$ and $\varphi = m + \frac{1}{3}$.

Proof From the transformations

$$\begin{aligned} &\mathcal{R}_3(1/w_{m,n}^{(\mu)}; m + 2n + 1 + \mu, -m, -\mu) \\ &= (v_{m,n}^{(\mu+2n+1)}; m + n + 1, m + n + \mu + 1, 2n + 1 + \mu), \\ &\mathcal{R}_3(v_{m,n}^{(\mu+2n+1)}; m + n + 1, m + n + \mu + 1, 2n + 1 + \mu) \\ &= (u_{n+1,m}^{(\mu+2m+2n+2)}; n + 1, -n - \mu, 2m + 2n + 2 + \mu), \\ &\mathcal{R}_3(u_{n+1,m}^{(\mu+2m+2n+2)}; n + 1, -n - \mu, 2m + 2n + 2 + \mu) \\ &= (1/w_{m,n+1}^{(\mu-1)}; m + 2n + \mu + 2, -m, \mu - 1), \end{aligned}$$

we obtain the sequence of solutions

$$\frac{1}{w_{m,n}^{(\mu)}} \xrightarrow{\mathcal{R}_3} v_{m,n}^{(\mu+2n+1)} \xrightarrow{\mathcal{R}_3} u_{n+1,m}^{(\mu+2m+2n+2)} \xrightarrow{\mathcal{R}_3} \frac{1}{w_{m,n+1}^{(\mu-1)}} \xrightarrow{\mathcal{R}_3} v_{m,n+1}^{(\mu+2n+2)}$$

from which the result follows. We note that

$$\mathcal{R}_3^3\left(1/w_{m,n}^{(\mu)}\right) = 1/w_{m,n+1}^{(\mu-1)}, \quad \mathcal{R}_3^3\left(v_{m,n}^{(\mu)}\right) = v_{m,n+1}^{(\mu+1)}, \quad \mathcal{R}_3^3\left(u_{n,m}^{(\mu)}\right) = u_{n+1,m}^{(\mu+1)}$$

□

Lemma 5.13 *Consider the rational functions*

$$\begin{aligned} X_N(z) &= \frac{1}{v_{m,N+n}^{(N+\mu)}(z) - 1}, & Y_N(z) &= \frac{1}{u_{N+n+1,m}^{(N+\mu+2m+1)}(z) - 1}, \\ Z_N(z) &= \frac{w_{m,N+n+1}^{(-N+\mu-2n-2)}(z)}{1 - w_{m,N+n+1}^{(-N+\mu-2n-2)}(z)}, \end{aligned} \quad (5.14)$$

with $w_{m,n}^{(\mu)}(z)$, $v_{m,n}^{(\mu)}(z)$ and $u_{m,n}^{(\mu)}(z)$ as given in Theorem 4.4, then $X_N(z)$, $Y_N(z)$ and $Z_N(z)$ satisfy the discrete system

$$X_N(Y_N + Z_{N-1} + 1) + \frac{N + m + n + 1}{z} = 0, \quad (5.15a)$$

$$Y_N(Z_N + X_N + 1) + \frac{N + n + 1}{z} = 0, \quad (5.15b)$$

$$Z_N(X_{N+1} + Y_N + 1) + \frac{N + m + \mu + 1}{z} = 0, \quad (5.15c)$$

which is the system (3.29) with $a_{3N} = N + m + n + 1$, $a_{3N+1} = N + n + 1$ and $a_{3N+2} = N + m + \mu + 1$, i.e. (3.26a) with $\lambda = \frac{1}{3}(m+n-\mu+1)$, $\rho = \frac{1}{3}(2m+2n+\mu+2)$ and $\varphi = -m + n - \mu + \frac{1}{3}$.

Proof From the transformations

$$\begin{aligned} \mathcal{R}_3(v_{m,n}^{(\mu)}; m+n+1, m-n+\mu, \mu) &= (u_{n+1,m}^{(\mu+2m+1)}; n+1, n+1-\mu, 2m+1+\mu), \\ \mathcal{R}_3(u_{m,n}^{(\mu)}; m, m+2n+1-\mu, \mu) &= (1/w_{n,m}^{(\mu-2m-2n-1)}; -n+\mu, -n, 2m+2n+1-\mu), \\ \mathcal{R}_3(1/w_{m,n}^{(\mu)}; m+2n+1+\mu, -m, -\mu) &= (v_{m,n}^{(\mu+2n+1)}; m+n+1, m+n+\mu+1, 2n+1+\mu), \end{aligned}$$

we obtain the sequence of solutions

$$v_{m,n}^{(\mu)} \xrightarrow{\mathcal{R}_3} u_{n+1,m}^{(\mu+2m+1)} \xrightarrow{\mathcal{R}_3} \frac{1}{w_{m,n+1}^{(\mu-2n-2)}} \xrightarrow{\mathcal{R}_3} v_{m,n+1}^{(\mu+1)} \xrightarrow{\mathcal{R}_3} u_{n+2,m}^{(\mu+2m+2)}$$

from which the result follows. We note that

$$\mathcal{R}_3^3(1/w_{m,n}^{(\mu)}) = 1/w_{m,n+1}^{(\mu-1)}, \quad \mathcal{R}_3^3(v_{m,n}^{(\mu)}) = v_{m,n+1}^{(\mu+1)}, \quad \mathcal{R}_3^3(u_{n,m}^{(\mu)}) = u_{n+1,m}^{(\mu+1)}.$$

□

Remarks 5.14

- (i) There are no solutions in terms of generalised Laguerre polynomials in the symmetric case when $\lambda = \varphi = 0$.
- (ii) Lemmas 5.12 and 5.13 imply that there are hierarchies of rational solutions of the fourth discrete equation (3.34) in terms of generalised Laguerre polynomials.

5.1.4 Partial Difference Systems

In this subsection we derive some partial difference systems which have solutions in terms of generalised Laguerre polynomials.

Lemma 5.15 *If we define*

$$\mathcal{W}_{m,n}^{(\mu)} = \frac{w_{m,n}^{(\mu)} + 1}{w_{m,n}^{(\mu)} - 1}, \quad \mathcal{V}_{m,n}^{(\mu)} = \frac{v_{m,n}^{(\mu)} + 1}{v_{m,n}^{(\mu)} - 1}, \quad \mathcal{U}_{m,n}^{(\mu)} = \frac{u_{m,n}^{(\mu)} + 1}{u_{m,n}^{(\mu)} - 1}, \quad (5.16)$$

with $w_{m,n}^{(\mu)}(z)$, $v_{m,n}^{(\mu)}(z)$ and $u_{m,n}^{(\mu)}(z)$ as given in Theorem 4.4, then they satisfy the partial difference systems

$$\mathcal{W}_{m+1,n}^{(\mu)} + \mathcal{W}_{m,n}^{(\mu)} + \frac{4(2m+2n+\mu+2)\mathcal{V}_{m,n}^{(\mu+2n+1)} - \mu}{z \frac{1 - (\mathcal{V}_{m,n}^{(\mu+2n+1)})^2}{}} = 0, \quad (5.17a)$$

$$\mathcal{V}_{m,n}^{(\mu+2n+1)} + \mathcal{V}_{m-1,n}^{(\mu+2n+1)} + \frac{4(2m+2n+\mu+1)\mathcal{W}_{m,n}^{(\mu)} - 2n - \mu - 1}{z \frac{1 - (\mathcal{W}_{m,n}^{(\mu)})^2}{}} = 0, \quad (5.17b)$$

and

$$\mathcal{V}_{m,n+1}^{(\mu)} + \mathcal{V}_{m,n}^{(\mu)} = \frac{4(2n+2-\mu)\mathcal{U}_{n+1,m}^{(\mu+2m+1)} - \mu}{z \frac{1 - (\mathcal{U}_{n+1,m}^{(\mu+2m+1)})^2}{}}, \quad (5.18a)$$

$$\mathcal{U}_{n+1,m}^{(\mu+2m+1)} + \mathcal{U}_{n,m}^{(\mu+2m+1)} = \frac{4(2n+1-\mu)\mathcal{V}_{m,n}^{(\mu)} + 2m + \mu + 1}{z \frac{1 - (\mathcal{V}_{m,n}^{(\mu)})^2}{}}. \quad (5.18b)$$

Proof These follow from setting $N = 0$ in (5.8) and (5.9), respectively. □

Lemma 5.16 *If $\mathcal{W}_{m,n}^{(\mu)}$, $\mathcal{V}_{m,n}^{(\mu)}$ and $\mathcal{U}_{m,n}^{(\mu)}$ are given by (5.16), then they satisfy the partial difference systems*

$$\mathcal{W}_{m+1,n}^{(\mu)} + \mathcal{W}_{m,n}^{(\mu)} + \frac{4}{z} \frac{(2n + \mu + 1) \mathcal{V}_{m,n}^{(\mu+2n+1)} - \mu}{1 - (\mathcal{V}_{m,n}^{(\mu+2n+1)})^2} = 0, \quad (5.19a)$$

$$\mathcal{V}_{m,n}^{(\mu+2n+1)} + \mathcal{V}_{m-1,n}^{(\mu+2n+1)} + \frac{4}{z} \frac{(2m + 2n + \mu + 1) \mathcal{W}_{m,n}^{(\mu)} - (2n + \mu + 1)}{1 - (\mathcal{W}_{m,n}^{(\mu)})^2} = 0, \quad (5.19b)$$

and

$$\mathcal{W}_{m+1,n-1}^{(\mu)} + \mathcal{W}_{m,n}^{(\mu)} + \frac{4}{z} \frac{(2n + \mu) \mathcal{U}_{n,m}^{(\mu+2m+2n+1)} - \mu}{1 - (\mathcal{U}_{n,m}^{(\mu+2m+2n+1)})^2} = 0, \quad (5.20a)$$

$$\mathcal{U}_{n+1,m-1}^{(\mu+2m+2n+1)} + \mathcal{U}_{n,m}^{(\mu+2m+2n+1)} + \frac{4}{z} \frac{(2n + \mu + 1) \mathcal{W}_{m,n}^{(\mu)} - (2m + 2n + \mu + 1)}{1 - (\mathcal{W}_{m,n}^{(\mu)})^2} = 0. \quad (5.20b)$$

Proof From the transformations

$$\begin{aligned} & \mathcal{R}_1(w_{m,n}^{(\mu)}; -m, m + 2n + \mu + 1, \mu) \\ &= (v_{m-1,n}^{(\mu+2n+1)}; -m - n, m + n + \mu, \mu + 2n + 1), \\ & \mathcal{R}_1(v_{m,n}^{(\mu)}; -m - n - 1, m - n + \mu, \mu) \\ &= (w_{m,n}^{(\mu-2n-1)}; -m, m + \mu, \mu - 2n - 1), \end{aligned}$$

we obtain the sequence

$$w_{m,n}^{(\mu)} \xrightarrow{\mathcal{R}_1} v_{m-1,n}^{(\mu+2n+1)} \xrightarrow{\mathcal{R}_1} w_{m-1,n}^{(\mu)} \xrightarrow{\mathcal{R}_1} v_{m-2,n}^{(\mu+2n+1)} \xrightarrow{\mathcal{R}_1} w_{m-2,n}^{(\mu)}$$

from which the system (5.19) follows.

From the transformations

$$\begin{aligned} & \mathcal{R}_1(w_{m,n}^{(\mu)}; m, m + 2n + \mu + 1, \mu) = (u_{n,m}^{(\mu+2m+2n+1)}; -n, n + \mu, \mu + 2m + 2n + 1), \\ & \mathcal{R}_1(u_{m,n}^{(\mu)}; -m, -m - 2n - 1 + \mu, \mu) = (w_{n+1,m-1}^{(\mu-2m-2n-1)}; n + 1, -n - 1 + \mu, \mu - 2m - 2n - 1), \end{aligned}$$

we obtain the sequence

$$\begin{aligned} & w_{m,n}^{(\mu)} \xrightarrow{\mathcal{R}_1} u_{n,m}^{(\mu+2m+2n+1)} \xrightarrow{\mathcal{R}_1} w_{m+1,n-1}^{(\mu)} \xrightarrow{\mathcal{R}_1} u_{n-1,m+1}^{(\mu+2m+2n+1)} \\ & \xrightarrow{\mathcal{R}_1} w_{m+2,n-2}^{(\mu)} \end{aligned}$$

from which the system (5.20) follows. $\square$

Lemma 5.17 If $\mathcal{W}_{m,n}^{(\mu)}$, $\mathcal{V}_{m,n}^{(\mu)}$ and $\mathcal{U}_{m,n}^{(\mu)}$ are given by (5.16), then they satisfy the partial difference systems

$$\frac{m + \mu + 1}{\mathcal{V}_{m,n}^{(\mu+1)} + \mathcal{U}_{n,m}^{(\mu+2m+1)}} + \frac{m + \mu}{\mathcal{V}_{m,n-1}^{(\mu-1)} + \mathcal{U}_{n,m}^{(\mu+2m+1)}} = \frac{1}{2}z - \frac{\mu \mathcal{U}_{n,m}^{(\mu+2m+1)} + 2n - \mu}{1 - (\mathcal{U}_{n,m}^{(\mu+2m+1)})^2}, \quad (5.21a)$$

$$\frac{m + \mu}{\mathcal{V}_{m,n-1}^{(\mu-1)} + \mathcal{U}_{n,m}^{(\mu+2m+1)}} + \frac{m + \mu - 1}{\mathcal{V}_{m,n-1}^{(\mu-1)} + \mathcal{U}_{n-1,m}^{(\mu+2m-1)}} = \frac{1}{2}z - \frac{(2m + \mu)\mathcal{V}_{m,n-1}^{(\mu-1)} + 2n - \mu}{1 - (\mathcal{V}_{m,n-1}^{(\mu-1)})^2}, \quad (5.21b)$$

and

$$\frac{n + \mu - 1}{\mathcal{W}_{m+1,n}^{(\mu-2)} + \mathcal{V}_{m,n}^{(\mu+2n)}} + \frac{n + \mu}{\mathcal{W}_{m,n}^{(\mu)} + \mathcal{V}_{m,n}^{(\mu+2n)}} = \frac{1}{2}z - \frac{(\mu - 1)\mathcal{V}_{m,n}^{(\mu+2n)} - (2m + 2n + 1 + \mu)}{1 - (\mathcal{V}_{m,n}^{(\mu+2n)})^2}, \quad (5.22a)$$

$$\frac{n + \mu}{\mathcal{W}_{m,n}^{(\mu)} + \mathcal{V}_{m,n}^{(\mu+2n)}} + \frac{n + \mu + 1}{\mathcal{W}_{m,n}^{(\mu)} + \mathcal{V}_{m-1,n}^{(\mu+2n+2)}} = \frac{1}{2}z - \frac{(\mu + 2n + 1)\mathcal{W}_{m,n}^{(\mu)} - (2m + 2n + 1 + \mu)}{1 - (\mathcal{W}_{m,n}^{(\mu)})^2}. \quad (5.22b)$$

Proof These follow from setting $N = 0$ in (5.10) and (5.11), respectively. $\square$

Lemma 5.18 Consider the rational functions

$$\mathcal{X}_{m,n}^{(\mu)} = \frac{w_{m,n}^{(\mu)}}{1 - w_{m,n}^{(\mu)}}, \quad \mathcal{Y}_{m,n}^{(\mu)} = \frac{1}{v_{m,n}^{(\mu)} - 1}, \quad \mathcal{Z}_{m,n}^{(\mu)} = \frac{1}{u_{m,n}^{(\mu)} - 1},$$

with $w_{m,n}^{(\mu)}(z)$, $v_{m,n}^{(\mu)}(z)$ and $u_{m,n}^{(\mu)}(z)$ as given in Theorem 4.4, then they satisfy the partial difference system

$$\mathcal{X}_{m,n}^{(\mu)}(\mathcal{Y}_{m,n}^{(\mu+2n+1)} + \mathcal{Z}_{n,m}^{(\mu+2m+2n+1)} + 1) + \frac{m + 2n + \mu + 1}{z} = 0, \quad (5.23a)$$

$$\mathcal{Y}_{m,n}^{(\mu+2n+1)}(\mathcal{Z}_{n+1,m}^{(\mu+2m+2n+2)} + \mathcal{X}_{m,n}^{(\mu)} + 1) + \frac{m + n + 1}{z} = 0, \quad (5.23b)$$

$$\mathcal{Z}_{n+1,m}^{(\mu+2m+2n+2)}(\mathcal{X}_{m,n+1}^{(\mu-1)} + \mathcal{Y}_{m,n}^{(\mu+2n+1)} + 1) + \frac{n + 1}{z} = 0. \quad (5.23c)$$

Proof This follows from setting $N = 0$ in (5.13). $\square$

5.2 Rational Solutions in Terms of Generalised Umemura Polynomials

5.2.1 First Discrete Equation: Asymmetric Discrete P_{II}

In the following example, we illustrate how using the Bäcklund transformation $\mathcal{R}_1$ (3.3) gives rise to a hierarchy of solutions of asymmetric dP_{II} (3.7) in terms of generalised Umemura polynomials. We remark that Kitaev, Law and McLeod [48, Theorem 1.2] proved that rational solutions expressed in terms of generalised Umemura polynomials are unique.

Example 5.19 Applying the Bäcklund transformation $\mathcal{R}_1$ with $(a, b, c) = (\kappa + \frac{1}{2}, -\kappa + \frac{1}{2}, 2)$ to the solution

$$\widehat{w}_{2,1}^{(\kappa-3/2)}(z) = -\frac{[z^2 - 16\kappa(\kappa+1)][z^3 - 6(2\kappa-1)z^2 + 48\kappa(\kappa-1)z - 32\kappa(\kappa-1)(2\kappa-1)]}{[z^2 - 16\kappa(\kappa-1)][z^3 - 6(2\kappa+1)z^2 + 48\kappa(\kappa+1)z - 32\kappa(\kappa+1)(2\kappa+1)]},$$

gives the sequence of rational solutions

$$\widehat{w}_{2,1}^{(\kappa-3/2)} \xrightarrow{\mathcal{R}_1} \widehat{v}_{2,1}^{(\kappa)} \xrightarrow{\mathcal{R}_1} \widehat{w}_{2,1}^{(\kappa-1/2)} \xrightarrow{\mathcal{R}_1} \widehat{v}_{2,1}^{(\kappa+1)} \xrightarrow{\mathcal{R}_1} \widehat{w}_{2,1}^{(\kappa+1/2)}$$

Hence we obtain solutions of asymmetric dP_{II} (3.7)

$$\begin{aligned} q_0(z) &= \frac{\widehat{w}_{2,1}^{(\kappa-3/2)}(z) + 1}{\widehat{w}_{2,1}^{(\kappa-3/2)}(z) - 1} = 2 \frac{d}{dz} \ln \frac{z^4 - 8\kappa z^3 + 128\kappa(\kappa^2 - 1)z - 256\kappa^2(\kappa^2 - 1)}{z - 4\kappa}, \\ q_1(z) &= \frac{\widehat{v}_{2,1}^{(\kappa)}(z) + 1}{\widehat{v}_{2,1}^{(\kappa)}(z) - 1} = 2 \frac{d}{dz} \ln \frac{z^2 - 16\kappa(\kappa + 1)}{z^3 - 6(2\kappa + 1)z^2 + 48\kappa(\kappa + 1)z - 32\kappa(\kappa + 1)(2\kappa + 1)}, \\ q_2(z) &= \frac{\widehat{w}_{2,1}^{(\kappa-1/2)}(z) + 1}{\widehat{w}_{2,1}^{(\kappa-1/2)}(z) - 1} \\ &= 2 \frac{d}{dz} \ln \frac{z^4 - 8(\kappa + 1)z^3 + 128\kappa(\kappa + 1)(\kappa + 2)z - 256\kappa(\kappa + 1)^2(\kappa + 2)}{z - 4(\kappa + 1)}, \\ q_3(z) &= \frac{\widehat{v}_{2,1}^{(\kappa+1)}(z) + 1}{\widehat{v}_{2,1}^{(\kappa+1)}(z) - 1} \\ &= 2 \frac{d}{dz} \ln \frac{z^2 - 16(\kappa + 1)(\kappa + 2)}{z^3 - 6(2\kappa + 3)z^2 + 48(\kappa + 1)(\kappa + 2)z - 32(\kappa + 1)(\kappa + 2)(2\kappa + 3)}. \end{aligned}$$

If we define the rational functions

$$\begin{aligned} q_{2n}(z) &= x_n(z) = \frac{\widehat{w}_{2,1}^{(\kappa+n-3/2)}(z) + 1}{\widehat{w}_{2,1}^{(\kappa+n-3/2)}(z) - 1} = 2 \frac{d}{dz} \ln \frac{U_{2,1}^{(2\kappa+2n-3)}(z)}{U_{1,0}^{(2\kappa+2n+1)}(z)} \\ &= 2 \frac{d}{dz} \ln \frac{z^4 - 8(n+\kappa)z^3 + 128(n+\kappa)[(n+\kappa)^2 - 1]z - 256(n+\kappa)^2[(n+\kappa)^2 - 1]}{z - 4(n+\kappa)}, \\ q_{2n+1}(z) &= y_n(z) = \frac{\widehat{v}_{2,1}^{(\kappa+n)}(z) + 1}{\widehat{v}_{2,1}^{(\kappa+n)}(z) - 1} = 2 \frac{d}{dz} \ln \frac{U_{2,0}^{(2\kappa+2n+1/2)}(z)}{U_{1,1}^{(2\kappa+2n+1/2)}(z)} \\ &= 2 \frac{d}{dz} \ln \frac{z^3 - 6(2n+2\kappa+1)z^2 + 48(n+\kappa)(n+\kappa+1)z - 32(n+\kappa)(n+\kappa+1)(2n+2\kappa+1)}{z^2 - 16(n+\kappa)(n+\kappa+1)}, \end{aligned}$$

then $q_n(z)$ satisfies

$$q_{n+1} + q_{n-1} = \frac{4(n+2\kappa)q_n + 2 - (-1)^n}{z(1 - q_n^2)},$$

which is asymmetric dP_{II} (3.7) with $\lambda = 2\kappa$, $\rho = 2$ and $\varphi = -1$, whilst $x_n(z)$ and $y_n(z)$ satisfy the discrete system

$$x_{n+1} + x_n = \frac{4(2n + 2\kappa + 1)y_n + 3}{z(1 - y_n^2)}, \quad y_n + y_{n-1} = \frac{4(2(n + \kappa)x_n + 1)}{z(1 - x_n^2)}.$$

More generally, using the solution $\widehat{w}_{m,n}^{(\kappa)}$, for fixed m and n , then we have the following Lemma.

Lemma 5.20 *Consider the rational functions*

$$Q_{2N}(z) = X_N(z) = \frac{\widehat{w}_{m,n}^{(N+\kappa)}(z) + 1}{\widehat{w}_{m,n}^{(N+\kappa)}(z) - 1}, \quad Q_{2N+1}(z) = Y_N(z) = \frac{\widehat{v}_{m,n}^{(N+\kappa+n+1/2)}(z) + 1}{\widehat{v}_{m,n}^{(N+\kappa+n+1/2)}(z) - 1},$$

with $\widehat{w}_{m,n}^{(\kappa)}(z)$ and $\widehat{v}_{m,n}^{(\kappa)}(z)$ as given in Theorem 4.9, then $Q_N(z)$ satisfies

$$Q_{N+1} + Q_{N-1} = \frac{4(N + m + n + 2\kappa)Q_N + m - n(-1)^N}{z(1 - Q_N^2)},$$

which is asymmetric dP_{II} (3.7) with $\lambda = m + n + 2\kappa$, $\rho = m$ and $\varphi = -n$, whilst $X_N(z)$ and $Y_N(z)$ satisfy the discrete system

$$X_{N+1} + X_N = \frac{4(2N + m + n + 2\kappa + 1)Y_N + m + n}{z(1 - Y_N^2)}, \quad (5.24a)$$

$$Y_N + Y_{N-1} = \frac{4(2N + m + n + 2\kappa)X_N + m - n}{z(1 - X_N^2)}. \quad (5.24b)$$

Proof From the transformations

$$\begin{aligned} \mathcal{R}_1(\widehat{w}_{m,n}^{(\kappa)}; m + \kappa, -n - \kappa, m + n) &= (\widehat{v}_{m,n}^{(\kappa+n+1/2)}; m + n + \kappa + \tfrac{1}{2}, -\kappa - \tfrac{1}{2}, m - n), \\ \mathcal{R}_1(\widehat{v}_{m,n}^{(\kappa)}; m + \kappa, n - \kappa, m - n) &= (\widehat{w}_{m,n}^{(\kappa-n+1/2)}; m - n + \kappa + \tfrac{1}{2}, -\kappa - \tfrac{1}{2}, m + n), \end{aligned}$$

we obtain the sequence of solutions

$$\widehat{w}_{m,n}^{(\kappa)} \xrightarrow{\mathcal{R}_1} \widehat{v}_{m,n}^{(\kappa+n+1/2)} \xrightarrow{\mathcal{R}_1} \widehat{w}_{m,n}^{(\kappa+1)} \xrightarrow{\mathcal{R}_1} \widehat{v}_{m,n}^{(\kappa+n+3/2)} \xrightarrow{\mathcal{R}_1} \widehat{w}_{m,n}^{(\kappa+2)}$$

from which the result follows. We note that

$$\mathcal{R}_1^2(\widehat{w}_{m,n}^{(\kappa)}) = \widehat{w}_{m,n}^{(\kappa+1)}, \quad \mathcal{R}_1^2(\widehat{v}_{m,n}^{(\kappa)}) = \widehat{v}_{m,n}^{(\kappa+1)},$$

so that only the parameter changes. $\square$

Alternatively, using the solution $\widehat{v}_{m,n}^{(\kappa)}$, for fixed m and n , then we have the following Lemma.

Lemma 5.21 *Consider the rational functions*

$$Q_{2N}(z) = X_N(z) = \frac{\widehat{v}_{N+m,N+n}^{(\kappa)}(z) + 1}{\widehat{v}_{N+m,N+n}^{(\kappa)}(z) - 1}, \quad Q_{2N+1}(z) = Y_N(z) = \frac{\widehat{u}_{N+m,N+n}^{(2\kappa-2N-2n)}(z) + 1}{\widehat{u}_{N+m,N+n}^{(2\kappa-2N-2n)}(z) - 1},$$

with $\widehat{v}_{m,n}^{(\kappa)}(z)$ and $\widehat{u}_{m,n}^{(\kappa)}(z)$ as given in Theorem 4.9, then $Q_N(z)$ satisfies

$$Q_{N+1} + Q_{N-1} = \frac{4(N+m+n)Q_N + m - n + \kappa[1 + (-1)^N]}{z(1 - Q_N^2)},$$

which is asymmetric dP_{II} (3.7) with $\lambda = m + n$, $\rho = m - n + \kappa$ and $\varphi = \kappa$, whilst $X_N(z)$ and $Y_N(z)$ satisfy the discrete system

$$X_{N+1} + X_N = \frac{4(2N+m+n+1)Y_N + m - n}{z(1 - Y_N^2)}, \quad (5.25a)$$

$$Y_N + Y_{N-1} = \frac{4(2N+m+n)X_N + 2\kappa + m - n}{z(1 - X_N^2)}. \quad (5.25b)$$

Proof From the transformations

$$\begin{aligned} \mathcal{R}_1(\widehat{v}_{m,n}^{(\kappa)}; m + \kappa, -n + \kappa, m - n) &= (\widehat{u}_{m,n}^{(2\kappa-2n)}; m + \tfrac{1}{2}, -n - \tfrac{1}{2}, m - n + 2\kappa), \\ \mathcal{R}_1(\widehat{u}_{m,n}^{(2\kappa)}; m + \tfrac{1}{2}, -n - \tfrac{1}{2}, m + n + 2\kappa) &= (\widehat{v}_{m+1,n+1}^{(n+\kappa)}; m + n + \kappa + 1, \kappa - 1, m - n), \end{aligned}$$

we obtain the sequence of solutions

$$\widehat{v}_{m,n}^{(\kappa)} \xrightarrow{\mathcal{R}_1} \widehat{u}_{m,n}^{(2\kappa-2n)} \xrightarrow{\mathcal{R}_1} \widehat{v}_{m+1,n+1}^{(\kappa)} \xrightarrow{\mathcal{R}_1} \widehat{u}_{m+1,n+1}^{(2\kappa-2n-2)} \xrightarrow{\mathcal{R}_1} \widehat{v}_{m+2,n+2}^{(\kappa)}$$

from which the result follows. We note that

$$\mathcal{R}_1^2(\widehat{w}_{m,n}^{(\kappa)}) = \widehat{w}_{m+1,n+1}^{(\kappa)}, \quad \mathcal{R}_1^2(\widehat{u}_{m,n}^{(2\kappa-2n)}) = \widehat{u}_{m+1,n+1}^{(2\kappa-2n-2)}.$$

□

Remarks 5.22 From the transformations

$$\mathcal{R}_1(\widehat{w}_{m,n}^{(\kappa)}; m + \kappa, n + \kappa, m + n) = (\widehat{u}_{m,n-1}^{(2\kappa+1)}; m + \tfrac{1}{2}, n - \tfrac{1}{2}, m + n + 2\kappa), \quad (5.26a)$$

$$\mathcal{R}_1(\widehat{u}_{m,n-1}^{(2\kappa+1)}; m + \tfrac{1}{2}, n - \tfrac{1}{2}, m + n + 2\kappa) = (\widehat{w}_{m+1,n-1}^{(\kappa)}; m + \kappa + 1, n + \kappa - 1, m + n), \quad (5.26b)$$

we obtain the sequence

$$\widehat{w}_{m,n}^{(\kappa)} \xrightarrow{\mathcal{R}_1} \widehat{u}_{m,n-1}^{(2\kappa+1)} \xrightarrow{\mathcal{R}_1} \widehat{w}_{m+1,n-1}^{(\kappa)} \xrightarrow{\mathcal{R}_1} \widehat{u}_{m,n-2}^{(2\kappa+1)} \xrightarrow{\mathcal{R}_1} \widehat{w}_{m+2,n-2}^{(\kappa)}.$$

It might be expected that the sequence would terminate when the second index becomes zero. However it can be shown that

$$\widehat{w}_{m,1}^{(\kappa)} \xrightarrow{\mathcal{R}_1} \widehat{u}_{m,0}^{(2\kappa+1)} \xrightarrow{\mathcal{R}_1} \widehat{w}_{m+1,0}^{(\kappa)} \xrightarrow{\mathcal{R}_1} \widehat{u}_{m+1,0}^{(2\kappa)}.$$

and

$$\widehat{u}_{m+1,0}^{(2\kappa)} \xrightarrow{\mathcal{R}_1} \widehat{v}_{m+2,1}^{(\kappa)} \xrightarrow{\mathcal{R}_1} \widehat{u}_{m+2,1}^{(2\kappa-2)} \xrightarrow{\mathcal{R}_1} \widehat{v}_{m+3,2}^{(\kappa)}.$$

This is illustrated in the following example.

Example 5.23 Consider the rational solution

$$\widehat{w}_{1,1}^{(\kappa)}(z) = -\frac{(z-4\kappa-2)(z+4\kappa+6)}{(z+4\kappa+2)(z-4\kappa-6)},$$

which satisfies P_V with $(\alpha, \beta, \gamma) = (\frac{1}{2}(\kappa+1)^2, \frac{1}{2}(\kappa+1)^2, 2)$, and so using (5.26) we obtain the sequence

$$\widehat{w}_{1,1}^{(\kappa)} \xrightarrow{\mathcal{R}_1} \widehat{u}_{1,0}^{(2\kappa+1)} \xrightarrow{\mathcal{R}_1} \widehat{w}_{2,0}^{(\kappa)} \xrightarrow{\mathcal{R}_1} \widehat{u}_{2,0}^{(2\kappa)} \xrightarrow{\mathcal{R}_1} \widehat{v}_{3,1}^{(\kappa)} \xrightarrow{\mathcal{R}_1} u_{3,1}^{(2\kappa-2)}.$$

The first few solutions are

$$\begin{aligned} q_0(z) &= \frac{\widehat{w}_{1,1}^{(\kappa)}(z) + 1}{\widehat{w}_{1,1}^{(\kappa)}(z) - 1} = \frac{4z}{z^2 - 4(2\kappa+1)(2\kappa+3)}, \\ q_1(z) &= \frac{4(\kappa+1)}{z} + 2 \frac{d}{dz} \ln \frac{z-4\kappa-6}{z-4\kappa-2}, \\ q_2(z) &= \frac{6[z^2 - 8(\kappa+1)z + 4(2\kappa+1)(2\kappa+3)]}{z^3 - 12(\kappa+1)z^2 + 12(2\kappa+1)(2\kappa+3)z - 16(\kappa+1)(2\kappa+1)(2\kappa+3)} \\ &\quad - \frac{2}{z-4\kappa-4}, \\ q_3(z) &= \frac{4(\kappa+1)}{z} + 2 \frac{d}{dz} \ln \frac{z^3 - 6(2\kappa+3)z^2 + 48(\kappa+1)(\kappa+2)z - 32(\kappa+1)(\kappa+2)(2\kappa+3)}{z^3 - 6(2\kappa+1)z^2 + 48\kappa(\kappa+1)z - 32\kappa(\kappa+1)(2\kappa+1)}, \end{aligned}$$

and satisfy

$$q_{n+1} + q_{n-1} = \frac{4nq_n + 2 + [1 + (-1)^n]\kappa}{z(1 - q_n^2)}, \quad (5.27)$$

which is asymmetric dP_{II} (3.7) with $\lambda = 0$, $\rho = 2 + \kappa$ and $\varphi = \kappa$. In general

$$q_{2n} = x_n = 2 \frac{d}{dz} \ln \frac{U_{n+1,n-2}^{(2\kappa-2n+3)}}{U_{n,n-1}^{(2\kappa-2n+3)}}, \quad q_{2n+1} = y_n = \frac{4(\kappa+1)}{z} + 2 \frac{d}{dz} \ln \frac{U_{n+1,n-1}^{(2\kappa-2n+3)}}{U_{n+1,n-1}^{(2\kappa-2n+1)}},$$

satisfy (5.27) and

$$x_{n+1} + x_n = \frac{4}{z} \frac{(2n+1)y_n + 2}{1 - y_n^2}, \quad y_n + y_{n-1} = \frac{4}{z} \frac{2nx_n + 2\kappa + 2}{1 - x_n^2}.$$

Symmetric case. In the symmetric case when $\varphi = 0$ in (3.7), then

$$q_n(z) = \frac{\widehat{w}_{m,0}^{(\kappa+n/2)}(z) + 1}{\widehat{w}_{m,0}^{(\kappa+n/2)}(z) - 1}, \quad p_n(z) = \frac{\widehat{w}_{0,m}^{(\kappa-n/2)}(z) + 1}{\widehat{w}_{0,m}^{(\kappa-n/2)}(z) - 1},$$

where $\widehat{w}_{m,0}^{(\kappa)}(z)$ and $\widehat{w}_{0,m}^{(\kappa)}(z)$ are given by (4.21c) and (4.21d) respectively, which satisfy

$$q_{n+1} + q_{n-1} = \frac{4}{z} \frac{(n+m+2\kappa)q_n + m}{1 - q_n^2}, \quad p_{n+1} + p_{n-1} = \frac{4}{z} \frac{(n-m-2\kappa)p_n + m}{1 - p_n^2}.$$

5.2.2 Second Discrete Equation

In the following example, we illustrate how using the Bäcklund transformation $\mathcal{R}_2$ (3.13) gives rise to a hierarchy of solutions of (3.18) in terms of generalised Umemura polynomials.

Example 5.24 Applying the Bäcklund transformation $\mathcal{R}_2$ (3.13), with $(a, b, c) = (-\kappa - 1, -\kappa - m, m + 1)$ to the solution $\widehat{w}_{1,m}^{(\kappa)}(z)$ gives the sequence

$$\widehat{w}_{1,m}^{(\kappa)} \xrightarrow{\mathcal{R}_2} \widehat{v}_{2,m}^{(\kappa+m-1/2)} \xrightarrow{\mathcal{R}_2} \widehat{w}_{3,m}^{(\kappa-1)} \xrightarrow{\mathcal{R}_2} \widehat{v}_{4,m}^{(\kappa+m-3/2)}$$

Hence if we define the rational functions

$$q_{2n}(z) = x_n(z) = \frac{\widehat{w}_{2n+1,m}^{(\kappa-n)}(z) + 1}{\widehat{w}_{2n+1,m}^{(\kappa-n)}(z) - 1} = 2 \frac{d}{dz} \ln \frac{U_{2n+1,m}^{(2\kappa-2n)}(z)}{U_{2n,m-1}^{(2\kappa-2n+2)}(z)},$$

$$q_{2n+1}(z) = y_n(z) = \frac{\widehat{v}_{2n+2,m}^{(\kappa+m-n-1/2)}(z) + 1}{\widehat{v}_{2n+2,m}^{(\kappa+m-n-1/2)}(z) - 1} = 2 \frac{d}{dz} \ln \frac{U_{2n+2,m-1}^{(2\kappa-2n-2m+2)}(z)}{U_{2n+1,m}^{(2\kappa-2n-2m+2)}(z)},$$

then $q_n(z)$ satisfies

$$\frac{2n+3}{q_{n+1}+q_n} + \frac{2n+1}{q_n+q_{n-1}} = z - \frac{2[n+1-m(-1)^n]q_n+4\kappa+2m+2}{1-q_n^2},$$

which is equation (3.18) with $\lambda = m$, $\varphi = -n$ and $\rho = m+n+2\kappa$, whilst $x_n(z)$ and $y_n(z)$ satisfy the discrete system

$$\begin{aligned} \frac{4n+5}{x_{n+1}+y_n} + \frac{4n+3}{x_n+y_n} &= z - \frac{2(2n+m+1)y_n+4\kappa+2m+2}{1-y_n^2}, \\ \frac{4n+3}{x_n+y_n} + \frac{4n+1}{x_n+y_{n-1}} &= z - \frac{2(2n-m+1)x_n+4\kappa+2m+2}{1-x_n^2}. \end{aligned}$$

More generally, using the solution $\widehat{w}_{m,n}^{(\kappa)}$, for fixed m and n , then we have the following Lemma.

Lemma 5.25 *Consider the rational functions*

$$Q_{2N}(z) = X_N(z) = \frac{\widehat{w}_{2N+m,n}^{(\kappa-N)}(z) + 1}{\widehat{w}_{2N+m,n}^{(\kappa-N)}(z) - 1}, \quad Q_{2N+1}(z) = Y_N(z) = \frac{\widehat{v}_{2N+m+1,n}^{(\kappa+n-N-1/2)}(z) + 1}{\widehat{v}_{2N+m+1,n}^{(\kappa+n-N-1/2)}(z) - 1},$$

with $\widehat{w}_{m,n}^{(\kappa)}(z)$ and $\widehat{v}_{m,n}^{(\kappa)}(z)$ as given in Theorem 4.9, then $Q_N(z)$ satisfies

$$\begin{aligned} \frac{2N+2m+1}{Q_{N+1}+Q_N} + \frac{2N+2m-1}{Q_N+Q_{N-1}} &= z \\ &- \frac{2[N+m-n(-1)^N]Q_N+2(m+n+2\kappa)}{1-Q_N^2}, \end{aligned}$$

which is equation (3.18) with $\lambda = m$, $\varphi = -n$ and $\rho = m+n+2\kappa$, whilst $X_N(z)$ and $Y_N(z)$ satisfy the discrete system

$$\begin{aligned} \frac{4N+2m+3}{X_{N+1}+Y_N} + \frac{4N+2m+1}{X_N+Y_N} \\ = z - \frac{2[(2N+m+n+1)Y_N+m+n+2\kappa]}{1-Y_N^2}, \end{aligned} \quad (5.28a)$$

$$\begin{aligned} \frac{4N+2m+1}{X_N+Y_N} + \frac{4N+2m-1}{X_N+Y_{N-1}} \\ = z - \frac{2[(2N+m-n)X_N+m+n+2\kappa]}{1-X_N^2}. \end{aligned} \quad (5.28b)$$

Proof From the transformations

$$\mathcal{R}_2(\widehat{w}_{m,n}^{(\kappa)}; -m-\kappa, -n-\kappa, m+n)$$

$$\begin{aligned}
&= (\widehat{v}_{m+1,n}^{(\kappa+n-1/2)}; -m-n-\kappa-\frac{1}{2}, -\kappa+\frac{1}{2}, m-n+1), \\
&\mathcal{R}_2(\widehat{v}_{m,n}^{(\kappa)}; -m-\kappa, n-\kappa, m-n) \\
&= (\widehat{w}_{m+1,n}^{(\kappa-n-1/2)}; -m-\kappa-\frac{1}{2}, n-\kappa+\frac{1}{2}, m+n+1)
\end{aligned}$$

we obtain the sequence of solutions

$$\widehat{w}_{m,n}^{(\kappa)} \xrightarrow{\mathcal{R}_2} \widehat{v}_{m+1,n}^{(\kappa+n-1/2)} \xrightarrow{\mathcal{R}_2} \widehat{w}_{m+2,n}^{(\kappa-1)} \xrightarrow{\mathcal{R}_2} \widehat{v}_{m+3,n}^{(\kappa+n-3/2)} \xrightarrow{\mathcal{R}_2} \widehat{w}_{m+4,n}^{(\kappa-2)}$$

from which the result follows. We note that

$$\mathcal{R}_2^2(\widehat{w}_{m,n}^{(\kappa)}) = \widehat{w}_{m+2,n}^{(\kappa-1)}, \quad \mathcal{R}_2^2(\widehat{v}_{m,n}^{(\kappa)}) = \widehat{v}_{m+2,n}^{(\kappa-1)}.$$

□

When $n = 0$ there is no alternating term and so

$$Q_{2N}(z) = \frac{\widehat{w}_{2N+m,0}^{(\kappa-N)}(z) + 1}{\widehat{w}_{2N+m,0}^{(\kappa-N)}(z) - 1}, \quad Q_{2N+1}(z) = \frac{\widehat{v}_{2N+m+1,0}^{(\kappa-N-1/2)}(z) + 1}{\widehat{v}_{2N+m+1,0}^{(\kappa-N-1/2)}(z) - 1},$$

satisfy

$$\frac{2N+2m+1}{Q_{N+1}+Q_N} + \frac{2N+2m-1}{Q_N+Q_{N-1}} = z - \frac{2(N+m)Q_N+2m+4\kappa}{1-Q_N^2}.$$

It is also possible to obtain a sequence of solutions involving the solutions $\widehat{w}_{m,n}^{(\kappa)}$ and $\widehat{u}_{m,n}^{(2\kappa+1)}$ as illustrated in the following Lemma.

Lemma 5.26 *Consider the rational functions*

$$Q_{2N}(z) = X_N(z) = \frac{\widehat{w}_{N+m,N+n}^{(\kappa)}(z) + 1}{\widehat{w}_{N+m,N+n}^{(\kappa)}(z) - 1}, \quad Q_{2N+1}(z) = Y_N(z) = \frac{\widehat{u}_{N+m,N+n}^{(2\kappa+1)}(z) + 1}{\widehat{u}_{N+m,N+n}^{(2\kappa+1)}(z) - 1},$$

with $\widehat{w}_{m,n}^{(\kappa)}(z)$ and $\widehat{u}_{m,n}^{(\kappa)}(z)$ as given in Theorem 4.9, then $Q_N(z)$ satisfies

$$\begin{aligned}
&\frac{2N+2m+2n+2\kappa+1}{Q_{N+1}+Q_N} + \frac{2N+2m+2n+2\kappa-1}{Q_N+Q_{N-1}} = \\
&z - \frac{2[N+m+n+\kappa\{1+(-1)^N\}]Q_N+2(m-n)}{1-Q_N^2},
\end{aligned}$$

which is equation (3.18) with $\lambda = m+n+\kappa$, $\varphi = \kappa$ and $\rho = m-n$, whilst $X_N(z)$ and $Y_N(z)$ satisfy the discrete system

$$\frac{4N+2m+2n+2\kappa+3}{X_{N+1}+Y_N} + \frac{4N+2m+2n+2\kappa+1}{X_N+Y_N}$$

$$= z - \frac{2[(2N + m + n + 1)Y_N + m - n]}{1 - Y_N^2}, \quad (5.29a)$$

$$\begin{aligned} & \frac{4N + 2m + 2n + 2\kappa + 1}{X_N + Y_N} + \frac{4N + 2m + 2n + 2\kappa - 1}{X_N + Y_{N-1}} \\ &= z - \frac{2[(2N + m + n + 2\kappa)X_N + m - n]}{1 - X_N^2}. \end{aligned} \quad (5.29b)$$

Proof From the transformations

$$\begin{aligned} & \mathcal{R}_2(\widehat{w}_{m,n}^{(\kappa)}; -m - \kappa, n + \kappa, m + n) \\ &= (\widehat{u}_{m,n}^{(2\kappa+1)}; -m - \frac{1}{2}, n + \frac{1}{2}, m + n + 2\kappa + 1), \\ & \mathcal{R}_2(\widehat{u}_{m,n}^{(2\kappa+1)}; -m - \frac{1}{2}, n + \frac{1}{2}, m + n + 2\kappa + 1) \\ &= (\widehat{w}_{m+1,n+1}^{(\kappa)}; -m - \kappa - 1, n + \kappa + 1, m + n + 2). \end{aligned}$$

we obtain the sequence of solutions

$$\widehat{w}_{m,n}^{(\kappa)} \xrightarrow{\mathcal{R}_2} \widehat{u}_{m,n}^{(2\kappa+1)} \xrightarrow{\mathcal{R}_2} \widehat{w}_{m+1,n+1}^{(\kappa)} \xrightarrow{\mathcal{R}_2} \widehat{u}_{m+1,n+1}^{(2\kappa+1)} \xrightarrow{\mathcal{R}_2} \widehat{w}_{m+2,n+2}^{(\kappa)}$$

from which the result follows. We note that

$$\mathcal{R}_2^2(\widehat{w}_{m,n}^{(\kappa)}) = \widehat{w}_{m+1,n+1}^{(\kappa)}, \quad \mathcal{R}_2^2(\widehat{u}_{m,n}^{(2\kappa+1)}) = \widehat{u}_{m+1,n+1}^{(2\kappa+1)}.$$

□

When $\kappa = 0$ then there is no alternating term and so

$$\mathcal{Q}_{2N}(z) = \frac{\widehat{w}_{N+m,N+n}^{(0)}(z) + 1}{\widehat{w}_{N+m,N+n}^{(0)}(z) - 1}, \quad \mathcal{Q}_{2N+1}(z) = \frac{\widehat{u}_{N+m,N+n}^{(1)}(z) + 1}{\widehat{u}_{N+m,N+n}^{(1)}(z) - 1},$$

satisfy

$$\frac{2N + 2m + 2n + 1}{\mathcal{Q}_{N+1} + \mathcal{Q}_N} + \frac{2N + 2m + 2n - 1}{\mathcal{Q}_N + \mathcal{Q}_{N-1}} = z - \frac{2[(N + m + n)\mathcal{Q}_N + m - n]}{1 - \mathcal{Q}_N^2},$$

which is equation (3.18) with $\lambda = m + n$, $\rho = m - n$ and $\varphi = 0$. As previously, although there is no alternating term, the structure of the even and odd terms is different.

Symmetric case. In the symmetric case when $\varphi = 0$ in (3.19), from the transformations

$$\begin{aligned} & \mathcal{R}_2(\widehat{w}_{m,0}^{(\kappa)}; -m - \kappa, -\kappa, m) = (\widehat{w}_{m+1,0}^{(\kappa-1/2)}; -m - \kappa - \frac{1}{2}, -\kappa + \frac{1}{2}, m + 1), \\ & \mathcal{R}_2(\widehat{w}_{0,n}^{(\kappa)}; n, n + \frac{1}{2}, n + \kappa) = (\widehat{w}_{0,n+1}^{(\kappa-1/2)}; \kappa - \frac{1}{2}, n + \kappa + \frac{1}{2}, n + 1), \end{aligned}$$

recall (4.21c) and (4.21d), we obtain the respective sequences

$$\begin{aligned} \widehat{w}_{m,0}^{(\kappa)} &\xrightarrow{\mathcal{R}_2} \widehat{w}_{m+1,0}^{(\kappa-1/2)} \xrightarrow{\mathcal{R}_2} \widehat{w}_{m+2,0}^{(\kappa-1)} \\ \widehat{w}_{0,n}^{(\kappa)} &\xrightarrow{\mathcal{R}_2} \widehat{w}_{0,n+1}^{(\kappa-1/2)} \xrightarrow{\mathcal{R}_2} \widehat{w}_{0,n+2}^{(\kappa-1)}. \end{aligned}$$

If we define

$$q_n(z) = \frac{\widehat{w}_{m+n,0}^{(\kappa-n/2)}(z) + 1}{\widehat{w}_{m+n,0}^{(\kappa-n/2)}(z) - 1}, \quad p_n(z) = \frac{\widehat{w}_{0,n+m}^{(\kappa-n/2)}(z) + 1}{\widehat{w}_{0,n+m}^{(\kappa-n/2)}(z) - 1},$$

then $q_n(z)$ and $p_n(z)$ satisfy

$$\begin{aligned} \frac{2n+2m+1}{q_{n+1}+q_n} + \frac{2n+2m-1}{q_n+q_{n-1}} &= z - \frac{2(n+m)q_n + 2m + 4\kappa}{1 - q_n^2}, \\ \frac{2n+2m+1}{p_{n+1}+p_n} + \frac{2n+2m-1}{p_n+p_{n-1}} &= z - \frac{2(n+m)p_n - 2m - 4\kappa}{1 - p_n^2}, \end{aligned}$$

which are the discrete equation (3.19) with $\lambda = m$, $\rho = 2\kappa + m$, $\varphi = 0$ and $\lambda = m$, $\rho = -2\kappa - m$, $\varphi = 0$, respectively.

5.2.3 Third Discrete Equation: Ternary Discrete P_I

In the following example, we illustrate how using the Bäcklund transformation $\mathcal{R}_3$ (3.24) gives rise to a hierarchy of solutions of ternary dP_I (3.28) in terms of generalised Umemura polynomials.

Example 5.27 Applying the Bäcklund transformation $\mathcal{R}_3$ (3.24), with $(a, b, c) = (\kappa + 1, -\kappa - m, m + 1)$ to the solution $w_{1,m}^{(\kappa)}$ gives the sequence of rational solutions

$$\begin{aligned} \widehat{w}_{1,m}^{(\kappa)} &\xrightarrow{\mathcal{R}_3} \widehat{v}_{1,m}^{(\kappa+m+1/2)} \xrightarrow{\mathcal{R}_3} \widehat{u}_{1,m}^{(2\kappa+1)} \xrightarrow{\mathcal{R}_3} \widehat{w}_{2,m}^{(\kappa)} \xrightarrow{\mathcal{R}_3} \widehat{v}_{2,m}^{(\kappa+m+1/2)} \\ &\xrightarrow{\mathcal{R}_3} \widehat{u}_{2,m}^{(2\kappa+1)} \end{aligned}$$

so that

$$\mathcal{R}_3^3(\widehat{w}_{1,m}^{(\kappa)}) = \widehat{w}_{2,m}^{(\kappa)}, \quad \mathcal{R}_3^3(\widehat{v}_{1,m}^{(\kappa+m+1/2)}) = \widehat{v}_{2,m}^{(\kappa+m+1/2)}, \quad \mathcal{R}_3^3(\widehat{u}_{1,m}^{(2\kappa+1)}) = \widehat{u}_{2,m}^{(2\kappa+1)}.$$

Hence if we define the rational functions

$$x_{3n}(z) = X_n(z) = \frac{1}{\widehat{w}_{n+1,m}^{(\kappa)}(z) - 1} = -\frac{1}{2} + \frac{d}{dz} \ln \frac{U_{n+1,m}^{(2\kappa)}(z)}{U_{n,m-1}^{(2\kappa+2)}(z)},$$

$$x_{3n+1}(z) = Y_n(z) = \frac{1}{\widehat{v}_{n+1,m}^{(\kappa+m+1/2)}(z) - 1} = -\frac{1}{2} + \frac{d}{dz} \ln \frac{U_{n+1,m-1}^{(2\kappa+2)}(z)}{U_{n,m}^{(2\kappa+2)}(z)},$$

$$x_{3n+2}(z) = Z_n(z) = \frac{1}{\widehat{u}_{n+1,m}^{(2\kappa+1)}(z) - 1} = -\frac{1}{2} + \frac{n+m+2\kappa+2}{z} + \frac{d}{dz} \ln \frac{U_{n+1,m}^{(2\kappa+2)}(z)}{U_{n+1,m}^{(2\kappa)}(z)},$$

then $x_n(z)$ satisfies ternary dP₁

$$x_n(x_{n+1} + x_{n-1} + 1) + \frac{a_n}{z} = 0,$$

with

$$a_n = \frac{1}{3}(n+m+2\kappa) + 1 + \frac{1}{3}(\kappa-m)\cos\left(\frac{2}{3}\pi n\right) + \frac{1}{9}(3m+3\kappa+1)\sqrt{3}\sin\left(\frac{2}{3}\pi n\right),$$

so

$$a_{3n} = n + \kappa + 1, \quad a_{3n+1} = n + m + \kappa + \frac{3}{2}, \quad a_{3n+2} = n + \frac{3}{2},$$

whilst $X_n(z)$, $Y_n(z)$ and $Z_n(z)$ satisfy

$$X_n(Y_n + Z_{n-1} + 1) + \frac{n + \kappa + 1}{z} = 0,$$

$$Y_n(Z_n + X_n + 1) + \frac{n + m + \kappa + \frac{3}{2}}{z} = 0,$$

$$Z_n(X_{n+1} + Y_n + 1) + \frac{n + \frac{3}{2}}{z} = 0.$$

More generally, using the solution $\widehat{w}_{m,n}^{(\kappa)}$, for fixed m and n , then we have the following Lemma.

Lemma 5.28 *Consider the rational functions*

$$X_N(z) = \frac{1}{\widehat{w}_{N+m,n}^{(\kappa)}(z) - 1}, \quad Y_N(z) = \frac{1}{\widehat{v}_{N+m,n}^{(\kappa+n+1/2)}(z) - 1}, \quad Z_N(z) = \frac{1}{\widehat{u}_{N+m,n}^{(2\kappa+1)}(z) - 1} \quad (5.30)$$

with $\widehat{w}_{m,n}^{(\kappa)}(z)$, $\widehat{v}_{m,n}^{(\kappa)}(z)$ and $\widehat{u}_{m,n}^{(\kappa)}(z)$ as given in Theorem 4.9, then $X_N(z)$, $Y_N(z)$ and $Z_N(z)$ satisfy

$$X_N(Y_N + Z_{N-1} + 1) + \frac{N + m + \kappa}{z} = 0, \quad (5.31a)$$

$$Y_N(Z_N + X_N + 1) + \frac{N + m + n + \kappa + \frac{1}{2}}{z} = 0, \quad (5.31b)$$

$$Z_N(X_{N+1} + Y_N + 1) + \frac{N + m + \frac{1}{2}}{z} = 0, \quad (5.31c)$$

which is the system (3.29) with $\lambda = \frac{1}{3}(\kappa - n)$, $\varphi = n + \kappa + \frac{1}{3}$ and $\rho = m + \frac{1}{3}n + \frac{2}{3}\kappa$.

Proof From the transformations

$$\begin{aligned}\mathcal{R}_3(\widehat{w}_{m,n}^{(\kappa)}; m + \kappa, -n - \kappa, m + n) &= (\widehat{v}_{m,n}^{(\kappa+n+1/2)}; m + n + \kappa + \frac{1}{2}, \kappa + \frac{1}{2}, m - n), \\ \mathcal{R}_3(\widehat{v}_{m,n}^{(\kappa)}; m + \kappa, -n + \kappa, m - n) &= (\widehat{u}_{m,n}^{(2\kappa-2n)}; m + n + \frac{1}{2}, \frac{1}{2}, m - n + 2\kappa), \\ \mathcal{R}_3(\widehat{u}_{m,n}^{(2\kappa)}; m + \frac{1}{2}, n + \frac{1}{2}, m + n + 2\kappa) &= (\widehat{w}_{m+1,n}^{(\kappa-1/2)}; m + \kappa + \frac{1}{2}, -n - \kappa + \frac{1}{2}, m + n + 1),\end{aligned}$$

we obtain the sequence of solutions

$$\begin{array}{ccccccc}\widehat{w}_{m,n}^{(\kappa)} & \xrightarrow{\mathcal{R}_3} & \widehat{v}_{m,n}^{(\kappa+n+1/2)} & \xrightarrow{\mathcal{R}_3} & \widehat{u}_{m,n}^{(2\kappa+1)} & \xrightarrow{\mathcal{R}_3} & \widehat{w}_{m+1,n}^{(\kappa)} \\ & & \xrightarrow{\mathcal{R}_3} & \widehat{v}_{m+1,n}^{(\kappa+n+1/2)} & \xrightarrow{\mathcal{R}_3} & \widehat{u}_{m+1,n}^{(2\kappa+1)} & \end{array}$$

from which the result follows. We note that

$$\mathcal{R}_3^3(\widehat{w}_{m,n}^{(\kappa)}) = \widehat{w}_{m+1,n}^{(\kappa)}, \quad \mathcal{R}_3^3(\widehat{v}_{m,n}^{(\kappa)}) = \widehat{v}_{m+1,n}^{(\kappa)}, \quad \mathcal{R}_3^3(\widehat{u}_{m,n}^{(2\kappa+1)}) = \widehat{u}_{m+1,n}^{(2\kappa+1)}.$$

□

Remarks 5.29 It is interesting to note that $X_N(z)$, $Y_N(z)$ and $Z_N(z)$ arise from the three different sets of rational solutions of P_V that are expressed in terms of the generalised Umemura polynomials.

As was the case for $\mathcal{R}_2$, it is also possible to obtain another sequence of solutions as illustrated in the following Lemma.

Lemma 5.30 Consider the rational functions

$$X_N(z) = \frac{1}{\widehat{w}_{N+m,n}^{(\kappa)}(z) - 1}, \quad Y_N(z) = \frac{1}{\widehat{u}_{N+m,n-1}^{(2\kappa+1)}(z) - 1}, \quad Z_N(z) = \frac{1}{\widehat{v}_{N+m+1,n}^{(\kappa+n-1/2)}(z) - 1} \quad (5.32)$$

with $\widehat{w}_{m,n}^{(\kappa)}(z)$, $\widehat{v}_{m,n}^{(\kappa)}(z)$ and $\widehat{u}_{m,n}^{(\kappa)}(z)$ as given in Theorem 4.9, then $X_N(z)$, $Y_N(z)$ and $Z_N(z)$ satisfy

$$X_N(Y_N + Z_{N-1} + 1) + \frac{N + m + \kappa}{z} = 0, \quad (5.33a)$$

$$Y_N(Z_N + X_N + 1) + \frac{N + m + \frac{1}{2}}{z} = 0, \quad (5.33b)$$

$$Z_N(X_{N+1} + Y_N + 1) + \frac{N + m + n + \kappa + \frac{1}{2}}{z} = 0, \quad (5.33c)$$

which is the system (3.29) with $\lambda = \frac{1}{3}(\kappa - n)$, $\varphi = -n - \kappa + \frac{1}{3}$ and $\rho = m + \frac{1}{3}n + \frac{2}{3}\kappa$.

Proof From the transformations

$$\begin{aligned}\mathcal{R}_3(\widehat{w}_{m,n}^{(\kappa)}; m + \kappa, n + \kappa, m + n) &= (\widehat{u}_{m,n-1}^{(2\kappa+1)}; m + \tfrac{1}{2}, -n + \tfrac{1}{2}, m + n + 2\kappa), \\ \mathcal{R}_3(\widehat{u}_{m,n}^{(2\kappa)}; m + \tfrac{1}{2}, -n - \tfrac{1}{2}, m + n + 2\kappa) &= (\widehat{v}_{m+1,n+1}^{(\kappa+n)}; m + n + \kappa + 1, -\kappa + 1, m - n), \\ \mathcal{R}_3(\widehat{v}_{m,n}^{(\kappa)}; m + \kappa, n - \kappa, m + n) &= (\widehat{w}_{m,n}^{(\kappa-n+1/2)}; m + \kappa + \tfrac{1}{2}, -n + \kappa + \tfrac{1}{2}, m + n),\end{aligned}$$

we obtain the sequence of solutions

$$\widehat{w}_{m,n}^{(\kappa)} \xrightarrow{\mathcal{R}_3} \widehat{u}_{m,n-1}^{(2\kappa+1)} \xrightarrow{\mathcal{R}_3} \widehat{v}_{m+1,n}^{(\kappa+n-1/2)} \xrightarrow{\mathcal{R}_3} \widehat{w}_{m+1,n}^{(\kappa)} \xrightarrow{\mathcal{R}_3} \widehat{u}_{m+1,n-1}^{(2\kappa+1)}$$

from which the result follows. We note that

$$\mathcal{R}_3^3(\widehat{w}_{m,n}^{(\kappa)}) = \widehat{w}_{m+1,n}^{(\kappa)}, \quad \mathcal{R}_3^3(\widehat{v}_{m,n}^{(\kappa)}) = \widehat{v}_{m+1,n}^{(\kappa)}, \quad \mathcal{R}_3^3(\widehat{u}_{m,n}^{(2\kappa+1)}) = \widehat{u}_{m+1,n}^{(2\kappa+1)}.$$

□

Remarks 5.31

- (i) There are no solutions in terms of generalised Umemura polynomials in the symmetric case when $\lambda = \varphi = 0$.
- (ii) Lemmas 5.28 and 5.30 imply that there are hierarchies rational solutions of the fourth discrete equation (3.34) in terms of generalised Umemura polynomials.

5.2.4 Partial difference systems

In this subsection we derive some partial difference systems which have solutions in terms of generalised Umemura polynomials.

Lemma 5.32 *If we define*

$$\widehat{\mathcal{W}}_{m,n}^{(\kappa)}(z) = \frac{\widehat{w}_{m,n}^{(\kappa)}(z) + 1}{\widehat{w}_{m,n}^{(\kappa)}(z) - 1}, \quad \widehat{\mathcal{V}}_{m,n}^{(\kappa)}(z) = \frac{\widehat{v}_{m,n}^{(\kappa)}(z) + 1}{\widehat{v}_{m,n}^{(\kappa)}(z) - 1}, \quad \widehat{\mathcal{U}}_{m,n}^{(\kappa)}(z) = \frac{\widehat{u}_{m,n}^{(\kappa)}(z) + 1}{\widehat{u}_{m,n}^{(\kappa)}(z) - 1}, \quad (5.34)$$

with $\widehat{w}_{m,n}^{(\kappa)}(z)$, $\widehat{v}_{m,n}^{(\kappa)}(z)$ and $\widehat{u}_{m,n}^{(\kappa)}(z)$ as given in Theorem 4.9, then they satisfy the partial difference systems

$$\widehat{\mathcal{W}}_{m,n}^{(\kappa+1)} + \widehat{\mathcal{W}}_{m,n}^{(\kappa)} = \frac{4(m+n+2\kappa+1)\widehat{\mathcal{V}}_{m,n}^{(\kappa+n+1/2)} + m+n}{z \left(1 - \left(\widehat{\mathcal{V}}_{m,n}^{(\kappa+n+1/2)}\right)^2\right)}, \quad (5.35a)$$

$$\widehat{\mathcal{V}}_{m,n}^{(\kappa+n+1/2)} + \widehat{\mathcal{V}}_{m,n}^{(\kappa+n-1/2)} = \frac{4(m+n+2\kappa)\widehat{\mathcal{W}}_{m,n}^{(\kappa)} + m-n}{z \left(1 - \left(\widehat{\mathcal{W}}_{m,n}^{(\kappa)}\right)^2\right)}, \quad (5.35b)$$

and

$$\widehat{\mathcal{V}}_{m,n}^{(\kappa+n)} + \widehat{\mathcal{V}}_{m+1,n+1}^{(\kappa+n)} = \frac{4(m+n+1)\widehat{\mathcal{U}}_{m,n}^{(2\kappa)} + m-n}{z \left(1 - \left(\widehat{\mathcal{U}}_{m,n}^{(2\kappa)}\right)^2\right)}, \quad (5.36a)$$

$$\widehat{\mathcal{U}}_{m,n}^{(2\kappa)} + \widehat{\mathcal{U}}_{m+1,n+1}^{(2\kappa-2)} = \frac{4(m+n+2)\widehat{\mathcal{V}}_{m+1,n+1}^{(\kappa+n)} + m+n+2\kappa}{z \left(1 - \left(\widehat{\mathcal{V}}_{m+1,n+1}^{(\kappa+n)}\right)^2\right)}. \quad (5.36b)$$

Proof These follow from setting $N = 0$ in (5.24) and (5.25), respectively. $\square$

Lemma 5.33 If $\widehat{\mathcal{W}}_{m,n}^{(\kappa)}(z)$ and $\widehat{\mathcal{U}}_{m,n}^{(\kappa)}(z)$ are given by (5.34),

$$\widehat{\mathcal{W}}_{m+1,n}^{(\kappa)} + \widehat{\mathcal{W}}_{m,n+1}^{(\kappa)} = \frac{4(m-n)\widehat{\mathcal{U}}_{m,n}^{(2\kappa+1)} + m+n+1}{z \left(1 - \left(\widehat{\mathcal{U}}_{m,n}^{(2\kappa+1)}\right)^2\right)}, \quad (5.37a)$$

$$\widehat{\mathcal{U}}_{m-1,n}^{(2\kappa+1)} + \widehat{\mathcal{U}}_{m,n-1}^{(2\kappa+1)} = \frac{4(m-n)\widehat{\mathcal{W}}_{m,n}^{(\kappa)} + m+n+2\kappa}{z \left(1 - \left(\widehat{\mathcal{W}}_{m,n}^{(\kappa)}\right)^2\right)}. \quad (5.37b)$$

Proof From the transformations (5.26) we obtain the sequences

$$\begin{array}{ccccc} \widehat{w}_{m,n+1}^{(\kappa)} & \xrightarrow{\mathcal{R}_1} & \widehat{u}_{m,n}^{(2\kappa+1)} & \xrightarrow{\mathcal{R}_1} & \widehat{w}_{m+1,n}^{(\kappa)} \\ \widehat{u}_{m-1,n}^{(2\kappa+1)} & \xrightarrow{\mathcal{R}_1} & \widehat{w}_{m,n}^{(\kappa)} & \xrightarrow{\mathcal{R}_1} & \widehat{u}_{m,n-1}^{(2\kappa+1)} \end{array}$$

from which the result follows. We note that

$$\mathcal{R}_1^3\left(\widehat{w}_{m,n}^{(\kappa)}\right) = \widehat{w}_{m+1,n}^{(\kappa)}, \quad \mathcal{R}_1^3\left(\widehat{u}_{m,n}^{(\kappa)}\right) = \widehat{u}_{m+1,n}^{(\kappa)}, \quad \mathcal{R}_1^3\left(\widehat{u}_{m,n}^{(2\kappa+1)}\right) = \widehat{u}_{m+1,n}^{(2\kappa+1)}.$$

$\square$

Lemma 5.34 If $\widehat{\mathcal{W}}_{m,n}^{(\kappa)}(z)$, $\widehat{\mathcal{V}}_{m,n}^{(\kappa)}(z)$ and $\widehat{\mathcal{U}}_{m,n}^{(\kappa)}(z)$ are given by (5.34), then they satisfy the partial difference systems

$$\begin{aligned} \frac{2m+1}{\widehat{\mathcal{W}}_{m+1,n}^{(\kappa-1)} + \widehat{\mathcal{V}}_{m,n}^{(\kappa+n-1/2)}} + \frac{2m-1}{\widehat{\mathcal{W}}_{m-1,n}^{(\kappa)} + \widehat{\mathcal{V}}_{m,n}^{(\kappa+n-1/2)}} &= z - \frac{2\left[(m+n)\widehat{\mathcal{V}}_{m,n}^{(\kappa+n-1/2)} + m+n+2\kappa-1\right]}{1 - \left(\widehat{\mathcal{V}}_{m,n}^{(\kappa+n-1/2)}\right)^2}, \\ \frac{2m+1}{\widehat{\mathcal{W}}_{m,n}^{(\kappa)} + \widehat{\mathcal{V}}_{m+1,n}^{(\kappa+n-1/2)}} + \frac{2m-1}{\widehat{\mathcal{W}}_{m,n}^{(\kappa)} + \widehat{\mathcal{V}}_{m-1,n}^{(\kappa+n+1/2)}} &= z - \frac{2\left[(m-n)\widehat{\mathcal{W}}_{m,n}^{(\kappa)} + m+n+2\kappa\right]}{1 - \left(\widehat{\mathcal{W}}_{m,n}^{(\kappa)}\right)^2}, \end{aligned}$$

and

$$\frac{2m+2n+2\kappa+3}{\widehat{\mathcal{W}}_{m+1,n+1}^{(\kappa)} + \widehat{\mathcal{U}}_{m,n}^{(2\kappa+1)}} + \frac{2m+2n+2\kappa+1}{\widehat{\mathcal{W}}_{m,n}^{(\kappa)} + \widehat{\mathcal{U}}_{m,n}^{(2\kappa+1)}} = z - \frac{2\left[(m+n+1)\widehat{\mathcal{U}}_{m,n}^{(2\kappa+1)} + m-n\right]}{1 - \left(\widehat{\mathcal{U}}_{m,n}^{(2\kappa+1)}\right)^2},$$

$$\frac{2m+2n+2\kappa+1}{\widehat{\mathcal{W}}_{m,n}^{(\kappa)} + \widehat{\mathcal{U}}_{m,n}^{(2\kappa+1)}} + \frac{2m+2n+2\kappa-1}{\widehat{\mathcal{W}}_{m,n}^{(\kappa)} + \widehat{\mathcal{U}}_{m-1,n-1}^{(2\kappa+1)}} = z - \frac{2[(m+n+2\kappa)\widehat{\mathcal{W}}_{m,n}^{(\kappa)} + m-n]}{1 - (\widehat{\mathcal{W}}_{m,n}^{(\kappa)})^2}.$$

Proof These follow from setting $N = 0$ in (5.28) and (5.29), respectively. $\square$

Lemma 5.35 *If we define*

$$\widehat{\mathcal{X}}_{m,n}^{(\kappa)}(z) = \frac{1}{\widehat{w}_{m,n}^{(\kappa)}(z) - 1}, \quad \widehat{\mathcal{Y}}_{m,n}^{(\kappa)}(z) = \frac{1}{\widehat{v}_{m,n}^{(\kappa)}(z) - 1}, \quad \widehat{\mathcal{Z}}_{m,n}^{(\kappa)}(z) = \frac{1}{\widehat{u}_{m,n}^{(\kappa)}(z) - 1}, \quad (5.38)$$

with $\widehat{w}_{m,n}^{(\kappa)}(z)$, $\widehat{v}_{m,n}^{(\kappa)}(z)$ and $\widehat{u}_{m,n}^{(\kappa)}(z)$ as given in Theorem 4.9, then they satisfy the partial difference system

$$\widehat{\mathcal{X}}_{m,n}^{(\kappa)} \left(\widehat{\mathcal{Y}}_{m,n}^{(\kappa+n+1/2)} + \widehat{\mathcal{Z}}_{m-1,n}^{(2\kappa+1)} + 1 \right) + \frac{m+\kappa}{z} = 0, \quad (5.39a)$$

$$\widehat{\mathcal{Y}}_{m,n}^{(\kappa+n+1/2)} \left(\widehat{\mathcal{Z}}_{m,n}^{(2\kappa+1)} + \widehat{\mathcal{X}}_{m,n}^{(\kappa)} + 1 \right) + \frac{m+n+\kappa+\frac{1}{2}}{z} = 0, \quad (5.39b)$$

$$\widehat{\mathcal{Z}}_{m,n}^{(2\kappa+1)} \left(\widehat{\mathcal{X}}_{m+1,n}^{(\kappa)} + \widehat{\mathcal{Y}}_{m,n}^{(\kappa+n+1/2)} + 1 \right) + \frac{m+\frac{1}{2}}{z} = 0. \quad (5.39c)$$

Proof This follows from setting $N = 0$ in (5.30) and (5.8). $\square$

5.3 Non-Unique Solutions of the Discrete Equations

Kitaev, Law and McLeod [48, Theorem 1.2] prove that if $\gamma \notin \mathbb{Z}$ in case (i), then the rational solution of P_V is unique and if $\gamma \in \mathbb{Z}$ then there are at most two rational solutions. Clarkson and Dunning [17] gave examples of non-unique rational solutions which arise for solutions expressed in terms of generalised Laguerre polynomials discussed in §5.1. This is illustrated in the following example.

Example 5.36 The rational solutions

$$w_{1,1}^{(1)}(z) = -\frac{(z-3)(z^2-8z+20)}{(z-2)(z-6)}, \quad u_{1,2}^{(1)}(z) = \frac{(z^2+4z+6)(z^3+9z^2+36z+60)}{z^4+12z^3+54z^2+96z+72}, \quad (5.40)$$

both satisfy P_V (1.2) for the parameters $(\alpha, \beta, \gamma) = (\frac{1}{2}, -\frac{25}{2}, 1)$. Hence the solutions

$$q_1(z) = \frac{w_{1,1}^{(1)}(z) + 1}{w_{1,1}^{(1)}(z) - 1} = \frac{z^3 - 12z^2 + 52z - 72}{(z-4)(z^2-6z+12)}, \quad (5.41a)$$

$$p_1(z) = \frac{u_{1,2}^{(1)}(z) + 1}{u_{1,2}^{(1)}(z) - 1} = \frac{z^5 + 14z^4 + 90z^3 + 312z^2 + 552z + 432}{(z^2+6z+12)(z^3+6z^2+18z+24)}, \quad (5.41b)$$

both satisfy equation (4.11) for the parameters $(\alpha, \beta, \gamma) = (\frac{1}{2}, -\frac{25}{2}, 1)$.

Remarks 5.37

- (i) The rational solutions (5.41) are special cases of the solution of equation (4.11) for the parameters $(\alpha, \beta, \gamma) = (\frac{1}{2}, -\frac{25}{2}, 1)$ given by

$$q(z) = \frac{f(z; C_1, C_2)}{g(z; C_1, C_2)}, \quad (5.42a)$$

where

$$\begin{aligned} f(z; C_1, C_2) = & 2C_1^2(z^3 - 12z^2 + 52z - 72)e^{2z} - C_1C_2(z^5 - 2z^4 + 36z^3 \\ & + 48z^2 - 240z - 864)e^z - 3C_2^2(z^5 + 14z^4 + 90z^3 \\ & + 312z^2 + 552z + 432), \end{aligned} \quad (5.42b)$$

$$\begin{aligned} g(z; C_1, C_2) = & 2C_1^2(z - 4)(z^2 - 6z + 12)e^{2z} + C_1C_2(z^5 - 12z^3 - 24z^2 \\ & + 144z + 576)e^z - 3C_2^2(z^2 + 6z + 12)(z^3 + 6z^2 + 18z + 24), \end{aligned} \quad (5.42c)$$

with C_1 and C_2 arbitrary constants.

- (ii) Special function solutions of P_V (1.2) are known to be expressible in terms of *Kummer functions* $M(a, b, z)$ and $U(a, b, z)$, or equivalently *Whittaker functions* $M_{\kappa, \mu}(z)$ and $W_{\kappa, \mu}(z)$, see, e.g. [53, 66, 85], also [39, §40] and [69, §32.10(v)]. Special cases of the Kummer functions $M(a, b, z)$ and $U(a, b, z)$, are expressed in terms of *modified Bessel functions* $I_\nu(z)$ and $K_\nu(z)$, see [69, §13.6(iii)]. Further, when $\nu = n + \frac{1}{2}$, with $n \in \mathbb{Z}$, then modified Bessel functions become *spherical Bessel functions* $I_{\pm(n+1/2)}(z)$, with $n \in \mathbb{N}$, which have the form

$$I_{\pm(n+1/2)}(z) = \frac{1}{\sqrt{2\pi z}} \sum_{k=0}^n \frac{(n+k)!}{(2z)^k k!(n-k)!} \left\{ (-1)^k e^z \pm (-1)^{n+1} e^{-z} \right\}, \quad (5.43)$$

see [69, §10.49(ii)]. The solution

$$q(z) = \frac{(2\nu + 3)\mathcal{Z}_{\nu+1}^2(\frac{1}{2}z) + (4\nu + 4 + z)\mathcal{Z}_{\nu+1}(\frac{1}{2}z)\mathcal{Z}_\nu(\frac{1}{2}z) + z\mathcal{Z}_\nu^2(\frac{1}{2}z)}{\mathcal{Z}_{\nu+1}(\frac{1}{2}z)\{(2\nu + 1 + z)\mathcal{Z}_{\nu+1}(\frac{1}{2}z) - z\mathcal{Z}_\nu(\frac{1}{2}z)\}}, \quad (5.44)$$

where

$$\mathcal{Z}_\nu(z) = \begin{cases} c_1 I_\nu(z) + c_2 I_{-\nu}(z), & \text{if } \nu \notin \mathbb{Z}, \\ c_1 I_n(z) + c_2 (-1)^n K_n(z), & \text{if } \nu = n \in \mathbb{Z}, \end{cases} \quad (5.45)$$

with c_1 and c_2 arbitrary constants, satisfies (4.11) for the parameters $(\alpha, \beta, \gamma) = (\frac{1}{2}, -2(\nu + 1)^2, 1)$. Setting $\nu = \frac{3}{2}$ in (5.44) gives (5.42), with C_1 and C_2 expressed in terms of c_1 and c_2 .

In this subsection, we use pairs of solutions such as (5.41) to derive examples of distinct hierarchies of rational solutions which satisfy the same discrete equation.

5.3.1 First Discrete Equation: Asymmetric Discrete P_{II}

Applying the Bäcklund transformation $\mathcal{R}_1$ (3.3), with $(a, b, c) = (1, -5, 1)$, to the solutions (5.40) gives the sequences of rational solutions

$$\begin{aligned} w_{1,1}^{(1)} &\xrightarrow{\mathcal{R}_1} \frac{1}{v_{1,1}^{(4)}} \xrightarrow{\mathcal{R}_1} w_{2,1}^{(1)} \xrightarrow{\mathcal{R}_1} \frac{1}{v_{2,1}^{(4)}} \xrightarrow{\mathcal{R}_1} w_{3,1}^{(1)} \\ u_{1,2}^{(1)} &\xrightarrow{\mathcal{R}_1} v_{2,1}^{(4)} \xrightarrow{\mathcal{R}_1} u_{2,2}^{(1)} \xrightarrow{\mathcal{R}_1} v_{2,2}^{(4)} \xrightarrow{\mathcal{R}_1} u_{3,2}^{(1)} \end{aligned}$$

Hence we obtain the hierarchies

$$q_{2n}(z) = \frac{1 + v_{n,1}^{(4)}(z)}{1 - v_{n,1}^{(4)}(z)} = 1 - \frac{8}{z} + 2 \frac{d}{dz} \ln \frac{T_{n,1}^{(1)}(z)}{T_{n-1,2}^{(1)}(z)}, \quad (5.46a)$$

$$q_{2n+1}(z) = \frac{w_{n+1,1}^{(1)}(z) + 1}{w_{n+1,1}^{(1)}(z) - 1} = 1 + 2 \frac{d}{dz} \ln \frac{T_{n,1}^{(2)}(z)}{T_{n,2}^{(0)}(z)}, \quad (5.46b)$$

and

$$p_{2n}(z) = \frac{v_{2,n}^{(-4)}(z) + 1}{v_{2,n}^{(-4)}(z) - 1} = -1 - \frac{8}{z} + 2 \frac{d}{dz} \ln \frac{T_{1,n+1}^{(-2n-5)}(z)}{T_{2,n}^{(-2n-5)}(z)}, \quad (5.47a)$$

$$p_{2n+1}(z) = \frac{u_{n+1,2}^{(1)}(z) + 1}{u_{n+1,2}^{(1)}(z) - 1} = 1 + 2 \frac{d}{dz} \ln \frac{T_{2,n+1}^{(-2n-6)}(z)}{T_{1,n+1}^{(-2n-6)}(z)}, \quad (5.47b)$$

where q_{2n} and p_{2n} both satisfy equation (4.11) for the parameters $(\alpha, \beta, \gamma) = (\frac{1}{2}(n+3)^2, -\frac{1}{2}(n+2)^2, -4)$ whilst q_{2n+1} and p_{2n+1} both satisfy equation (4.11) for the parameters $(\alpha, \beta, \gamma) = (\frac{1}{2}(n+1)^2, -\frac{1}{2}(n+5)^2, 1)$. We note that there is a difference in the structure of the solutions in the hierarchies (5.46) and (5.47). The solutions q_n (5.46) are expressed in terms of the generalised Laguerre polynomials $T_{n,1}^{(\mu)}(z)$ and $T_{n,2}^{(\mu)}(z)$ which are determinants of fixed size, whereas the solutions p_n (5.47) are expressed in terms of the generalised Laguerre polynomials $T_{1,n}^{(\mu)}(z)$ and $T_{2,n}^{(\mu)}(z)$ which are determinants that increase in size as n increases.

Both q_n and p_n satisfy the discrete equation

$$q_{n+1} + q_{n-1} = \frac{4(n+5)q_n - \frac{3}{2} + \frac{5}{2}(-1)^n}{1 - q_n^2}, \quad (5.48)$$

which is asymmetric dP_{II} (3.7) with $\lambda = 5$, $\rho = -\frac{3}{2}$ and $\varphi = \frac{5}{2}$. Further, if either $(x_n, y_n) = (q_{2n}, q_{2n+1})$ or $(x_n, y_n) = (p_{2n}, p_{2n+1})$ then we obtain solutions to the discrete system

$$x_{n+1} + x_n = \frac{4}{z} \frac{(2n+6)y_n - 4}{1 - y_n^2}, \quad y_n + y_{n-1} = \frac{4}{z} \frac{(2n+5)x_n + 1}{1 - x_n^2}.$$

The first few solutions in the hierarchies (5.46) and (5.47) are given by

$$\begin{aligned} q_0 &= 1 - \frac{8}{z} + 2 \frac{d}{dz} \ln(z-3), \\ q_2 &= 1 - \frac{8}{z} + 2 \frac{d}{dz} \ln \frac{z^2 - 8z + 12}{z^2 - 8z + 20}, \\ q_4 &= 1 - \frac{8}{z} + 2 \frac{d}{dz} \ln \frac{z^3 - 15z^2 + 60z - 60}{z^4 - 20z^3 + 150z^2 - 480z + 600}, \\ q_6 &= 1 - \frac{8}{z} + 2 \frac{d}{dz} \ln \frac{z^4 - 24z^3 + 180z^2 - 480z + 360}{z^6 - 36z^5 + 522z^4 - 3840z^3 + 15120z^2 - 30240z + 25200}, \\ q_1 &= 1 + 2 \frac{d}{dz} \ln \frac{z-4}{z^2 - 6z + 12}, \\ q_3 &= 1 + 2 \frac{d}{dz} \ln \frac{z^2 - 10z + 20}{z^4 - 16z^3 + 96z^2 - 240z + 240}, \\ q_5 &= 1 + 2 \frac{d}{dz} \ln \frac{z^3 - 18z^2 + 90z - 120}{z^6 - 30z^5 + 360z^4 - 2160z^3 + 6840z^2 - 10800z + 7200}, \\ q_7 &= 1 \\ &\quad + 2 \frac{d}{dz} \ln \frac{z^4 - 28z^3 + 252z^2 - 840z + 840}{z^8 - 48z^7 + 960z^6 - 10320z^5 + 64800z^4 - 241920z^3 + 524160z^2 - 604800z + 302400}, \end{aligned}$$

and

$$\begin{aligned} p_0 &= -1 - \frac{8}{z} + 2 \frac{d}{dz} \ln(z^2 + 4z + 6), \\ p_2 &= -1 - \frac{8}{z} + 2 \frac{d}{dz} \ln \frac{z^4 + 12z^3 + 54z^2 + 96z + 72}{z^3 + 9z^2 + 36z + 60}, \\ p_4 &= -1 - \frac{8}{z} + 2 \frac{d}{dz} \ln \frac{z^6 + 24z^5 + 228z^4 + 1056z^3 + 2520z^2 + 2880z + 1440}{z^6 + 24z^5 + 252z^4 + 1440z^3 + 4680z^2 + 8640z + 7200}, \\ p_1 &= 1 + 2 \frac{d}{dz} \ln \frac{z^3 + 6z^2 + 18z + 24}{z^2 + 6z + 12}, \\ p_3 &= 1 + 2 \frac{d}{dz} \ln \frac{z^6 + 18z^5 + 144z^4 + 624z^3 + 1512z^2 + 2160z + 1440}{z^4 + 16z^3 + 96z^2 + 240z + 240}, \\ p_5 &= 1 \\ &\quad + 2 \frac{d}{dz} \ln \frac{z^9 + 36z^8 + 576z^7 + 5280z^6 + 30240z^5 + 112320z^4 + 270720z^3 + 414720z^2 + 388800z + 172800}{z^6 + 30z^5 + 360z^4 + 2160z^3 + 6840z^2 + 10800z + 7200}, \end{aligned}$$

with $q_{-1}(z) = p_{-1}(z) = 1$.

Remarks 5.38 The solutions q_0 and p_0 are special cases of the solution

$$q(z) = \frac{c_1(z^2 - 9z + 24)e^z - c_2(z^3 + 8z^2 + 30z + 48)}{c_1z(z-3)e^z + c_2z(z^2 + 4z + 6)},$$

which satisfies equation (4.11) for the parameters $(\alpha, \beta, \gamma) = (\frac{9}{2}, -2, -4)$ and is obtained by setting $\nu = \frac{3}{2}$ is the solution

$$q(z) = \frac{(4\nu + 2 + z)\mathcal{Z}_{\nu+1}(\frac{1}{2}z) - z\mathcal{Z}_{\nu}(\frac{1}{2}z)}{z\{\mathcal{Z}_{\nu+1}(\frac{1}{2}z) - \mathcal{Z}_{\nu}(\frac{1}{2}z)\}},$$

which satisfies equation (4.11) for the parameters $(\alpha, \beta, \gamma) = (\frac{1}{2}(\nu + \frac{3}{2})^2, -\frac{1}{2}(\nu + \frac{1}{2})^2, -2\nu - 1)$.

5.3.2 Second discrete equation

The rational solutions

$$v_{0,n}^{(m+n+1)}(z) = (-1)^n \frac{m+1}{n+1} \frac{T_{0,n}^{(m-n+1)}(z)}{T_{0,n+1}^{(m-n-1)}(z)}, \quad \frac{1}{v_{m,0}^{(-m-n-1)}(z)} = -\frac{m+1}{n+1} \frac{T_{m,1}^{(-m-n-3)}(z)}{T_{m-1,1}^{(-m-n-3)}(z)},$$

both satisfy P_V (1.2) for the parameters $(\alpha, \beta, \gamma) = (\frac{1}{2}(n+1)^2, -\frac{1}{2}(m+1)^2, m+n+1)$. In the case when $m = n - 1$, applying the Bäcklund transformation $\mathcal{R}_2$ (3.13) gives the hierarchy of solutions

$$q_{2n}(z) = \frac{1 + v_{n,0}^{(-2n)}(z)}{1 - v_{n,0}^{(-2n)}(z)} = 1 + \frac{4n}{z} - 2 \frac{d}{dz} \ln T_{n-1,1}^{(-2n-1)}(z), \quad (5.49a)$$

$$q_{2n+1}(z) = \frac{1 + \tilde{v}_{n,0}^{(-2n)}(z)}{1 - \tilde{v}_{n,0}^{(-2n)}(z)} = -1 + 2 \frac{d}{dz} \ln T_{n,1}^{(-2n-3)}(z), \quad (5.49b)$$

and

$$p_{2n}(z) = \frac{v_{0,n-1}^{(2n)}(z) + 1}{v_{0,n-1}^{(2n)}(z) - 1} = -1 + \frac{4n}{z} - 2 \frac{d}{dz} \ln T_{0,n-1}^{(1)}(z) \quad (5.50a)$$

$$p_{2n+1}(z) = \frac{\tilde{v}_{0,n-1}^{(2n)}(z) - 1}{\tilde{v}_{0,n-1}^{(2n)}(z) + 1} = 1 + 2 \frac{d}{dz} \ln T_{0,n}^{(1)}(z), \quad (5.50b)$$

where

$$v_{n,0}^{(\mu)}(z) = \frac{n + \mu}{n + 1} \frac{T_{n-1,1}^{(\mu-2)}(z)}{T_{n,1}^{(\mu-2)}(z)} = \frac{n + \mu}{n + 1} \frac{L_n^{(\mu-1)}(z)}{L_{n+1}^{(\mu-1)}(z)},$$

$$\tilde{v}_{n,0}^{(\mu)}(z) = \frac{T_{n,1}^{(\mu-2)}(z)}{T_{n-1,1}^{(\mu-2)}(z)} = \frac{L_{n+1}^{(\mu-1)}(z)}{L_n^{(\mu-1)}(z)},$$

$$v_{0,n}^{(\mu)}(z) = (-1)^n \frac{\mu - n}{n+1} \frac{T_{0,n}^{(\mu-2n)}(z)}{T_{0,n+1}^{(\mu-2n-2)}(z)} = \frac{n - \mu}{n+1} \frac{L_n^{(-\mu-1)}(-z)}{L_{n+1}^{(-\mu-1)}(-z)},$$

$$\tilde{v}_{0,n}^{(\mu)}(z) = (-1)^{n+1} \frac{T_{0,n+1}^{(\mu-2n-2)}(z)}{T_{0,n}^{(\mu-2n)}(z)} = \frac{L_{n+1}^{(-1-\mu)}(-z)}{L_n^{(-1-\mu)}(-z)}.$$

The solutions q_{2n} and p_{2n} both satisfy equation (4.11) for the parameters $(\alpha, \beta, \gamma) = (\frac{1}{2}n^2, -\frac{1}{2}(n+1)^2, 2n)$ whilst q_{2n+1} and p_{2n+1} both satisfy equation (4.11) for the parameters $(\alpha, \beta, \gamma) = (\frac{1}{2}n^2, -\frac{1}{2}(n+1)^2, 2n+2)$. Again there is a difference in the structure of the solutions in the hierarchies (5.49) and (5.50).

Both q_n and p_n satisfy the discrete equation

$$\frac{2n+2}{q_{n+1}+q_n} + \frac{2n}{q_n+q_{n-1}} = z - \frac{[2n+1+(-1)^n]q_n-2}{1-q_n^2},$$

which is equation (3.18) with $\lambda = \frac{1}{2}$, $\rho = -2$ and $\varphi = \frac{1}{2}$. Further, if either $(x_n, y_n) = (q_{2n}, q_{2n+1})$ or $(x_n, y_n) = (p_{2n}, p_{2n+1})$ then we obtain solutions of the discrete system

$$\frac{2n+2}{x_{n+1}+y_n} + \frac{2n+1}{x_n+y_n} = \frac{1}{2}z - \frac{(2n+1)y_n-1}{1-y_n^2},$$

$$\frac{2n+1}{x_n+y_n} + \frac{2n}{x_n+y_{n-1}} = \frac{1}{2}z - \frac{(2n+1)x_n-1}{1-x_n^2}.$$

The first few solutions in the hierarchies (5.49) and (5.50) are given by

$$\begin{aligned} q_2 &= 1 + \frac{4}{z} - \frac{2}{z+1}, & q_1 &= -1 + \frac{2}{z+1}, \\ q_4 &= 1 + \frac{8}{z} - \frac{4(2+z)}{z^2+4z+6}, & q_3 &= -1 + \frac{4(z+2)}{z^2+4z+6}, \\ q_6 &= 1 + \frac{12}{z} - \frac{6(z^2+6z+12)}{z^3+9z^2+36z+60}, & q_5 &= -1 + \frac{6(z^2+6z+12)}{z^3+9z^2+36z+60}, \\ q_8 &= 1 + \frac{16}{z} - \frac{8(z^3+12z^2+60z+120)}{z^4+16z^3+120z^2+480z+840}, & q_7 &= -1 + \frac{8(z^3+12z^2+60z+120)}{z^4+16z^3+120z^2+480z+840}, \end{aligned}$$

and

$$\begin{aligned} p_2 &= -1 + \frac{4}{z}, & p_1 &= 1, \\ p_4 &= -1 + \frac{8}{z} - \frac{2}{z-3}, & p_3 &= 1 + \frac{2}{z-3}, \\ p_6 &= -1 + \frac{12}{z} - \frac{4(z-4)}{z^2-8z+20}, & p_5 &= 1 + \frac{4(z-4)}{z^2-8z+20}, \\ p_8 &= -1 + \frac{16}{z} - \frac{6(z^2-10z+30)}{z^3-15z^2+90z-210}, & p_7 &= 1 + \frac{6(z^2-10z+30)}{z^3-15z^2+90z-210}. \end{aligned}$$

5.3.3 Third discrete equation: ternary discrete P_I

The rational solutions

$$v_{3,1}^{(-1)}(z) = -\frac{(z-2)(z-6)}{(z-3)(z^2-8z+20)}, \quad v_{2,2}^{(-1)}(z) = \frac{z^4+12z^3+54z^2+96z+72}{(z^2+4z+6)(z^3+9z^2+36z+60)} \quad (5.51)$$

both satisfy P_V (1.2) for the parameters $(\alpha, \beta, \gamma) = (\frac{25}{2}, -\frac{1}{2}, -1)$ and so

$$x(z) = \frac{1}{v_{3,1}^{(-1)}(z) - 1} = -\frac{(z-3)(z^2-8z+20)}{(z-4)(z^2-6z+12)}, \quad (5.52a)$$

$$y(z) = \frac{1}{v_{2,2}^{(-1)}(z) - 1} = -\frac{(z^2+4z+6)(z^3+9z^2+36z+60)}{(z^2+6z+12)(z^3+6z^2+18z+24)}, \quad (5.52b)$$

both satisfy equation (4.12) for the parameters $(\alpha, \beta, \gamma) = (\frac{25}{2}, -\frac{1}{2}, -1)$. We note that

$$v_{3,1}^{(-1)}(z) = 1/w_{1,1}^{(1)}(z), \quad v_{2,2}^{(-1)}(z) = 1/u_{1,2}^{(1)}(z).$$

Applying the Bäcklund transformation $\mathcal{R}_3$ (3.24), with $(a, b, c) = (5, 1, -1)$, to the solutions (5.51) gives the sequences of rational solutions

$$\begin{aligned} v_{3,1}^{(-1)} &\xrightarrow{\mathcal{R}_3} u_{2,3}^{(6)} \xrightarrow{\mathcal{R}_3} \frac{1}{w_{3,2}^{(-5)}} \xrightarrow{\mathcal{R}_3} v_{3,2}^{(0)} \xrightarrow{\mathcal{R}_3} u_{3,3}^{(7)} \xrightarrow{\mathcal{R}_3} \frac{1}{w_{3,3}^{(-6)}} \\ &\xrightarrow{\mathcal{R}_3} v_{3,3}^{(1)} \\ v_{2,2}^{(-1)} &\xrightarrow{\mathcal{R}_3} \frac{1}{w_{3,2}^{(-6)}} \xrightarrow{\mathcal{R}_3} u_{3,2}^{(5)} \xrightarrow{\mathcal{R}_3} v_{2,3}^{(0)} \xrightarrow{\mathcal{R}_3} \frac{1}{w_{3,3}^{(-7)}} \\ &\xrightarrow{\mathcal{R}_3} u_{4,2}^{(6)} \xrightarrow{\mathcal{R}_3} v_{2,4}^{(1)} \end{aligned}$$

Hence we obtain the solution hierarchies

$$x_{3n}(z) = -1 + \frac{n+3}{z} - \frac{d}{dz} \ln T_{0,n}^{(2-n)}(z) = -1 + \frac{n+3}{z} - \frac{d}{dz} \ln L_n^{(-n-3)}(-z), \quad (5.53a)$$

$$x_{3n+1}(z) = -1 + \frac{d}{dz} \ln \frac{T_{0,n+1}^{(1-n)}(z)}{T_{0,n}^{(3-n)}(z)} = -1 + \frac{d}{dz} \ln \frac{L_{n+1}^{(-n-4)}(-z)}{L_n^{(-n-4)}(-z)}, \quad (5.53b)$$

$$x_{3n+2}(z) = \frac{d}{dz} \ln T_{0,n+1}^{(2-n)}(z) = \frac{d}{dz} \ln L_{n+1}^{(-n-5)}, \quad (5.53c)$$

and

$$y_{3n}(z) = \frac{n+3}{z} - \frac{d}{dz} \ln T_{1,1}^{(-n-4)}(z) = \frac{n+3}{z} - \frac{d}{dz} \ln L_2^{(-n-3)}(z), \quad (5.54a)$$

$$y_{3n+1}(z) = -1 + \frac{d}{dz} \ln \frac{T_{1,1}^{(-n-5)}(z)}{T_{2,1}^{(-n-5)}(z)} = -1 + \frac{d}{dz} \ln \frac{L_2^{(-n-4)}(z)}{L_3^{(-n-4)}(z)}, \quad (5.54b)$$

$$y_{3n+2}(z) = -1 + \frac{d}{dz} \ln T_{2,1}^{(-n-6)}(z) = -1 + \frac{d}{dz} \ln L_3^{(-n-5)}(z), \quad (5.54c)$$

where x_n and y_n both satisfy equation (4.12) for the parameters $(\alpha, \beta, \gamma) = (\frac{1}{2}(n+1)^2, -\frac{9}{2}, n+3)$, x_{3n+1} and y_{3n+1} for the parameters $(\alpha, \beta, \gamma) = (\frac{1}{2}(n+4)^2, -\frac{1}{2}, n-2)$ and x_{3n+2} and y_{3n+2} for the parameters $(\alpha, \beta, \gamma) = (\frac{1}{2}(n+1)^2, -\frac{9}{2}, n+5)$. We remark that whilst x_n is only defined for $n \geq -1$, y_n is defined for all $n \in \mathbb{Z}$. Also the solutions x_n only involve the generalised Laguerre polynomials $T_{0,n}^{(\mu)}(z) = (-1)^{[n/2]} L_n^{(-\mu-2n-1)}(-z)$, whereas the solutions y_n only involve the generalised Laguerre polynomials $T_{1,1}^{(\mu)}(z) = L_2^{(\mu+1)}(z)$ and $T_{2,1}^{(\mu)}(z) = L_3^{(\mu+1)}(z)$. Both x_n and y_n satisfy the ternary dP₁ equation

$$x_n(x_{n+1} + x_{n-1} + 1) + \frac{a_n}{z} = 0,$$

and the equation

$$\frac{a_{n+1}}{x_{n+2} + x_n + 1} + \frac{a_{n-1}}{x_n + x_{n-2} + 1} = z + \frac{a_n}{x_n},$$

with

$$a_n = \frac{1}{3}n + \frac{5}{3} - \frac{2}{3} \cos(\frac{2}{3}\pi n) + \frac{10}{9}\sqrt{3} \sin(\frac{2}{3}\pi n).$$

Further if $(X_n, Y_n, Z_n) = (x_{3n}, x_{3n+1}, x_{3n+2})$ or $(X_n, Y_n, Z_n) = (y_{3n}, y_{3n+1}, y_{3n+2})$, then we obtain solutions of the discrete system

$$\begin{aligned} X_n(Y_n + Z_{n-1} + 1) + \frac{n+1}{z} &= 0, & Y_n(Z_n + X_n + 1) + \frac{n+4}{z} &= 0, \\ Z_n(X_{n+1} + Y_n + 1) + \frac{n+1}{z} &= 0. \end{aligned}$$

The first few solutions in the hierarchies (5.53) and (5.54) are given by

$$\begin{aligned} x_0 &= -1 + \frac{3}{z}, & x_1 &= -1 + \frac{1}{z-3}, & x_2 &= \frac{1}{z-4}, \\ x_3 &= -1 + \frac{4}{z} - \frac{1}{z-3}, & x_4 &= -1 + \frac{d}{dz} \ln \frac{z^2 - 6z + 12}{z-4}, & x_5 &= \frac{2(z-4)}{z^2 - 8z + 20}, \\ x_6 &= -1 + \frac{5}{z} - \frac{2(z-3)}{z^2 - 6z + 12}, & x_7 &= -1 + \frac{d}{dz} \ln \frac{z^3 - 9z^2 + 36z - 60}{z^2 - 8z + 20}, & x_8 &= \frac{3(z^2 - 8z + 20)}{z^3 - 12z^2 + 60z - 120}, \end{aligned}$$

and

$$\begin{aligned}
y_0 &= \frac{3}{z} - \frac{2(z+1)}{z^2+2z+2}, & y_1 &= -1 + \frac{d}{dz} \ln \frac{z^2+4z+6}{z^3+3z^2+6z+6}, & y_2 &= -1 + \frac{3(z^2+4z+6)}{z^3+6z^2+18z+24}, \\
y_3 &= \frac{4}{z} - \frac{2(z+2)}{z^2+4z+6}, & y_4 &= -1 + \frac{d}{dz} \ln \frac{z^2+6z+12}{z^3+6z^2+18z+24}, & y_5 &= -1 + \frac{3(z^2+6z+12)}{z^3+9z^2+36z+60}, \\
y_6 &= \frac{5}{z} - \frac{2(z+3)}{z^2+6z+12}, & y_7 &= -1 + \frac{d}{dz} \ln \frac{z^2+8z+20}{z^3+9z^2+36z+60}, & y_8 &= -1 + \frac{3(z^2+8z+20)}{z^3+12z^2+60z+120}.
\end{aligned}$$

6 Discussion

In this paper we have derived discrete equations from the Bäcklund transformations of P_V (1.2) and obtained hierarchies of rational solutions of the discrete equations. Whilst the asymmetric dP_{II} (3.7), the discrete equation (3.18) and the ternary dP_I (3.28) were derived by Tokihiro, Grammaticos and Ramani [80] from Bäcklund transformations of P_V (1.2), our approach gives the explicit relationship between solutions of the discrete equations and solutions of P_V . This enabled us to derive the hierarchies of rational solutions of the discrete equations.

Tokihiro, Grammaticos and Ramani [80] also derive the discrete system

$$x_n + x_{n-1} = \frac{1}{y_n} + \frac{n + \lambda + \mu}{1 - y_n}, \quad y_{n+1}y_n = \frac{x_n - n - \lambda}{x_n^2 - \varphi^2}, \quad (6.1)$$

with λ , μ and φ constants, which was derived earlier by Fokas, Grammaticos and Ramani [25] from the Schlesinger transformations of P_V (1.2). Tokihiro, Grammaticos and Ramani [80] state that whilst y_n satisfies an equation which is equivalent to P_V , x_n satisfies a second-order, fourth-degree equation. Schlesinger transformations of P_V (1.2) are transformations which are quadratic in dw/dz . For example, consider the Schlesinger transformations R_1 and R_2 of P_V in [28, 56], which Fokas, Grammaticos and Ramani [25] use to derive the discrete system (6.1). Suppose that $w_0 = w(z; a_0, b_0, c_0)$ satisfies P_V for the parameters $(\alpha, \beta, \gamma) = (\frac{1}{2}a_0^2, -\frac{1}{2}b_0^2, c_0)$, then the Schlesinger transformations R_1 and R_2 are given by

$$\begin{aligned}
w_1 &= R_1(w_0) = \frac{\{zw'_0 + a_0w_0^2 - (a_0 + b_0 + z)w_0 + b_0\}\{zw'_0 + a_0w_0^2 - (a_0 - b_0 + z)w_0 - b_0\}}{z^2(w'_0)^2 - 2a_0zw_0(w_0 - 1)w'_0 + (a_0^2w_0^2 - b_0^2)(w_0 - 1)^2 - 2z(a_0 - c_0 + 1)w_0(w_0 - 1) - z^2w_0^2}, \\
w_2 &= R_2(w_0) = \frac{\{zw'_0 - a_0w_0^2 + (a_0 + b_0 + z)w_0 - b_0\}\{zw'_0 - a_0w_0^2 + (a_0 - b_0 + z)w_0 + b_0\}}{z^2(w'_0)^2 + 2a_0zw_0(w_0 - 1)w'_0 + (a_0^2w_0^2 - b_0^2)(w_0 - 1)^2 - 2z(a_0 - c_0 - 1)w_0(w_0 - 1) - z^2w_0^2},
\end{aligned}$$

with $' = d/dz$, and the effect on the parameters is

$$R_1(a_0, b_0, c_0) = (a_0 + 1, b_0, c_0 - 1), \quad R_2(a_0, b_0, c_0) = (a_0 - 1, b_0, c_0 + 1).$$

Hence w_1 and w_2 respectively satisfy P_V (1.2) for the parameters $(\alpha, \beta, \gamma) = (\frac{1}{2}(a_0 + 1)^2, -\frac{1}{2}b_0^2, c_0 - 1)$ and $(\alpha, \beta, \gamma) = (\frac{1}{2}(a_0 - 1)^2, -\frac{1}{2}b_0^2, c_0 + 1)$, so that $R_1 R_2 = R_2 R_1 = \mathcal{I}$, the identity transformation. In terms of the Bäcklund transformations $\mathcal{T}_{\varepsilon_1, \varepsilon_2, \varepsilon_3}$ defined in §2, it is straightforward to show that

$$R_1 \equiv \mathcal{T}_{1,1,1}^2 \circ \mathcal{T}_{-1,-1,-1}, \quad R_2 \equiv \mathcal{T}_{-1,-1,-1}^2 \circ \mathcal{T}_{1,1,1}.$$

All the Schlesinger transformations of P_V (1.2) given in [28, 56] can be written as a sequence of the Bäcklund transformations $\mathcal{T}_{\varepsilon_1, \varepsilon_2, \varepsilon_3}$. The Schlesinger transformations in [28, 56] are written in terms of the monodromy parameters θ_0, θ_1 and θ_∞ which are related to the parameters a_0, b_0 and c_0 through (2.5), i.e.

$$\theta_0 = \frac{1}{2}(a_0 + b_0 - c_0 - 1), \quad \theta_1 = -\frac{1}{2}(a_0 + b_0 + c_0 - 1), \quad \theta_\infty = a_0 - b_0,$$

so that

$$a_0 = \frac{1}{2}(\theta_0 - \theta_1 + \theta_\infty), \quad b_0 = \frac{1}{2}(\theta_0 - \theta_1 - \theta_\infty), \quad c_0 = 1 - \theta_0 - \theta_1.$$

Exact solutions of the discrete equations (3.7), (3.18) and (3.28) in terms of Kummer functions and modified Bessel functions, which are a special case of Kummer functions, cf. [69, §13.6(iii)], using the explicit relationship between these discrete equations and P_V (1.2), is currently under investigation, and we do not pursue this further here.

A Properties of the Generalised Laguerre Polynomials and the Generalised Umemura Polynomials

For convenience, in this appendix we list various differential-difference equations and discrete equations satisfied by the generalised Laguerre polynomials $T_{m,n}^{(\mu)}(z)$ and the generalised Umemura polynomials $U_{m,n}^{(\kappa)}(z)$.

A.1 Properties of the Generalised Laguerre Polynomials

Clarkson and Dunning [17, §3] prove the following differential-difference equations for the generalised Laguerre polynomials

$$D_z(T_{m,n-1}^{(\mu+1)} \cdot T_{m,n}^{(\mu)}) = T_{m+1,n-1}^{(\mu)} T_{m-1,n}^{(\mu+1)}, \quad (\text{A.1a})$$

$$D_z(T_{m,n-1}^{(\mu+1)} \cdot T_{m+1,n}^{(\mu-1)}) = T_{m+1,n-1}^{(\mu)} T_{m,n}^{(\mu)}, \quad (\text{A.1b})$$

$$D_z(T_{m,n-1}^{(\mu+1)} \cdot T_{m,n}^{(\mu-1)}) = T_{m+1,n-1}^{(\mu)} T_{m-1,n}^{(\mu)}, \quad (\text{A.1c})$$

$$D_z(T_{m+1,n}^{(\mu)} \cdot T_{m,n}^{(\mu+1)}) = T_{m+1,n-1}^{(\mu+1)} T_{m,n+1}^{(\mu)}, \quad (\text{A.1d})$$

$$D_z(T_{m,n}^{(\mu)} \cdot T_{m,n}^{(\mu+1)}) = T_{m+1,n-1}^{(\mu+1)} T_{m-1,n+1}^{(\mu)}, \quad (\text{A.1e})$$

$$D_z(T_{m+1,n}^{(\mu)} \cdot T_{m,n}^{(\mu)}) = T_{m+1,n-1}^{(\mu+1)} T_{m,n+1}^{(\mu-1)}, \quad (\text{A.1f})$$

where $D_z(f \cdot g)$ is the Hirota operator (4.16), and the discrete equations

$$T_{m,n+1}^{(\mu-1)} T_{m,n-1}^{(\mu+1)} = T_{m+1,n}^{(\mu-1)} T_{m-1,n}^{(\mu+1)} - (T_{m,n}^{(\mu)})^2, \quad (\text{A.2a})$$

$$T_{m+1,n}^{(\mu)} T_{m,n}^{(\mu)} = T_{m+1,n}^{(\mu-1)} T_{m,n}^{(\mu+1)} - T_{m,n+1}^{(\mu-1)} T_{m+1,n-1}^{(\mu+1)}, \quad (\text{A.2b})$$

$$T_{m,n}^{(\mu-1)} T_{m,n-1}^{(\mu+1)} = T_{m,n}^{(\mu)} T_{m,n-1}^{(\mu)} - T_{m-1,n}^{(\mu)} T_{m+1,n-1}^{(\mu)}. \quad (\text{A.2c})$$

The following differential-difference equations arise from the definitions of $w_{m,n}^{(\mu)}$, $v_{m,n}^{(\mu)}$ and $u_{m,n}^{(\mu)}$ in Theorem 4.4 and Remarks 4.5

$$zD_z(T_{m-1,n}^{(\mu)} \cdot T_{m+1,n-1}^{(\mu)}) = (m+1)T_{m,n-1}^{(\mu)} T_{m,n}^{(\mu)} + (z-m-2n-\mu)T_{m-1,n}^{(\mu)} T_{m+1,n-1}^{(\mu)}, \quad (\text{A.3a})$$

$$zD_z(T_{m-1,n-1}^{(\mu+1)} \cdot T_{m,n}^{(\mu-1)}) = (m+n+\mu)T_{m,n-1}^{(\mu+1)} T_{m-1,n}^{(\mu-1)} - (m+n)T_{m,n}^{(\mu-1)} T_{m-1,n-1}^{(\mu+1)}, \quad (\text{A.3b})$$

$$zD_z(T_{m,n-1}^{(\mu+1)} \cdot T_{m-1,n+1}^{(\mu-1)}) = nT_{m-1,n}^{(\mu+1)} T_{m,n}^{(\mu-1)} - (z-n-\mu)T_{m,n-1}^{(\mu+1)} T_{m-1,n+1}^{(\mu-1)}, \quad (\text{A.3c})$$

and so from (A.1b) and (A.3b)

$$zT_{m,n-1}^{(\mu)} T_{m-1,n}^{(\mu)} = (m+n+\mu)T_{m,n-1}^{(\mu+1)} T_{m-1,n}^{(\mu-1)} - (m+n)T_{m,n}^{(\mu-1)} T_{m-1,n-1}^{(\mu+1)}. \quad (\text{A.4})$$

The generalised Laguerre polynomials $T_{m,n}^{(\mu)}(z)$ satisfy the differential-difference equations

$$zD_z^2(T_{m,n}^{(\mu)} \cdot T_{m,n}^{(\mu)}) + 2T_{m,n}^{(\mu)} \frac{dT_{m,n}^{(\mu)}}{dz} = 2(\mu+m+n+1)T_{m+1,n-1}^{(\mu+1)} T_{m-1,n+1}^{(\mu-1)}, \quad (\text{A.5a})$$

$$zD_z^2(T_{m,n}^{(\mu)} \cdot T_{m,n}^{(\mu)}) + 2T_{m,n}^{(\mu)} \frac{dT_{m,n}^{(\mu)}}{dz} = 2(\mu+m+n+1) \left\{ T_{m,n}^{(\mu+1)} T_{m,n}^{(\mu-1)} - (T_{m,n}^{(\mu)})^2 \right\}, \quad (\text{A.5b})$$

where

$$D_z^2(f \cdot g) = g \frac{d^2 f}{dz^2} - 2 \frac{df}{dz} \frac{dg}{dz} + f \frac{dg}{dz},$$

from which we obtain the identity

$$T_{m+1,n-1}^{(\mu+1)} T_{m-1,n+1}^{(\mu-1)} = T_{m,n}^{(\mu+1)} T_{m,n}^{(\mu-1)} - (T_{m,n}^{(\mu)})^2. \quad (\text{A.6})$$

A.2 Properties of Generalised Umemura Polynomials

Masuda, Ohta and Kajiwara [55], see also [54], prove the following discrete equations for the generalised Umemura polynomials

$$U_{m+1,n}^{(\kappa-1)} U_{m-1,n+1}^{(\kappa+1)} = U_{m+1,n+1}^{(\kappa-1)} U_{m-1,n}^{(\kappa+1)} + 2(2m+1) U_{m,n+1}^{(\kappa-1)} U_{m,n}^{(\kappa+1)}, \quad (\text{A.7a})$$

$$U_{m,n+1}^{(\kappa-1)} U_{m+1,n-1}^{(\kappa+1)} = U_{m+1,n+1}^{(\kappa-1)} U_{m,n-1}^{(\kappa+1)} - 2(2n+1) U_{m+1,n}^{(\kappa-1)} U_{m,n}^{(\kappa+1)}, \quad (\text{A.7b})$$

$$U_{m-1,n}^{(\kappa)} U_{m+1,n-1}^{(\kappa)} = z U_{m,n}^{(\kappa-2)} U_{m,n-1}^{(\kappa+2)} - 2(2m+\kappa) U_{m,n}^{(\kappa)} U_{m,n-1}^{(\kappa)}, \quad (\text{A.7c})$$

$$U_{m,n-1}^{(\kappa)} U_{m-1,n+1}^{(\kappa)} = z U_{m,n}^{(\kappa-2)} U_{m-1,n}^{(\kappa+2)} + 2(2n+\kappa) U_{m,n}^{(\kappa)} U_{m-1,n}^{(\kappa)}, \quad (\text{A.7d})$$

$$U_{m+1,n}^{(\kappa-2)} U_{m-1,n-1}^{(\kappa+2)} = z U_{m,n}^{(\kappa-2)} U_{m,n-1}^{(\kappa+2)} - 2(\kappa-1) U_{m,n}^{(\kappa)} U_{m,n-1}^{(\kappa)}, \quad (\text{A.7e})$$

$$U_{m+1,n-1}^{(\kappa-2)} U_{m-1,n-1}^{(\kappa+2)} = z U_{m,n}^{(\kappa-2)} U_{m-1,n}^{(\kappa+2)} + 2(\kappa-1) U_{m,n}^{(\kappa)} U_{m-1,n}^{(\kappa)}, \quad (\text{A.7f})$$

$$U_{m-1,n+1}^{(\kappa-1)} U_{m,n-1}^{(\kappa+1)} = z U_{m-1,n}^{(\kappa-1)} U_{m,n}^{(\kappa+1)} + 2(2m+\kappa-1) U_{m,n}^{(\kappa-1)} U_{m-1,n}^{(\kappa+1)}, \quad (\text{A.7g})$$

$$U_{m+1,n-1}^{(\kappa-1)} U_{m,n-1}^{(\kappa+1)} = z U_{m,n-1}^{(\kappa-1)} U_{m,n}^{(\kappa+1)} - 2(2n+\kappa-1) U_{m,n}^{(\kappa-1)} U_{m,n-1}^{(\kappa+1)}, \quad (\text{A.7h})$$

$$U_{m+1,n}^{(\kappa-1)} U_{m-1,n-1}^{(\kappa+1)} = z U_{m,n-1}^{(\kappa-1)} U_{m,n}^{(\kappa+1)} - 2(2m+2n+\kappa) U_{m,n}^{(\kappa-1)} U_{m,n-1}^{(\kappa+1)}, \quad (\text{A.7i})$$

$$U_{m,n+1}^{(\kappa-1)} U_{m-1,n-1}^{(\kappa+1)} = z U_{m-1,n}^{(\kappa-1)} U_{m,n}^{(\kappa+1)} + 2(2m+2n+\kappa) U_{m,n}^{(\kappa-1)} U_{m-1,n}^{(\kappa+1)}, \quad (\text{A.7j})$$

where $D_z(f \cdot g)$ is the Hirota operator (4.16), see equation (4.16) and Proposition 4.1 in [55]. From equations (A.12) and (A.13) in [54, Appendix A] one can derive the following differential-difference equations for the generalised Umemura polynomials

$$2D_z(U_{m,n-1}^{(\kappa)} \cdot U_{m-1,n}^{(\kappa)}) = U_{m,n-1}^{(\kappa)} U_{m-1,n}^{(\kappa)} - U_{m,n}^{(\kappa)} U_{m-1,n-1}^{(\kappa)}, \quad (\text{A.8a})$$

$$2D_z(U_{m,n-1}^{(\kappa)} \cdot U_{m-1,n}^{(\kappa)}) = U_{m,n}^{(\kappa-2)} U_{m-1,n-1}^{(\kappa+2)} - U_{m,n-1}^{(\kappa)} U_{m-1,n}^{(\kappa)}, \quad (\text{A.8b})$$

$$2D_z(U_{m,n}^{(\kappa-1)} \cdot U_{m-1,n-1}^{(\kappa+1)}) = U_{m,n}^{(\kappa-1)} U_{m-1,n-1}^{(\kappa+1)} - U_{m,n-1}^{(\kappa+1)} U_{m-1,n}^{(\kappa-1)}, \quad (\text{A.8c})$$

$$2D_z(U_{m,n}^{(\kappa-1)} \cdot U_{m-1,n-1}^{(\kappa+1)}) = U_{m,n-1}^{(\kappa-1)} U_{m-1,n}^{(\kappa+1)} - U_{m,n}^{(\kappa-1)} U_{m-1,n-1}^{(\kappa+1)}. \quad (\text{A.8d})$$

By adding and subtracting these equations we obtain the differential-difference equations

$$4D_z(U_{m,n-1}^{(\kappa)} \cdot U_{m-1,n}^{(\kappa)}) = U_{m,n}^{(\kappa-2)} U_{m-1,n-1}^{(\kappa+2)} - U_{m,n}^{(\kappa)} U_{m-1,n-1}^{(\kappa)}, \quad (\text{A.9a})$$

$$4D_z(U_{m,n}^{(\kappa-1)} \cdot U_{m-1,n-1}^{(\kappa+1)}) = U_{m,n-1}^{(\kappa-1)} U_{m-1,n}^{(\kappa+1)} - U_{m,n-1}^{(\kappa+1)} U_{m-1,n}^{(\kappa-1)}, \quad (\text{A.9b})$$

and the discrete equations

$$2U_{m,n-1}^{(\kappa)} U_{m-1,n}^{(\kappa)} = U_{m,n}^{(\kappa)} U_{m-1,n-1}^{(\kappa)} + U_{m,n}^{(\kappa-2)} U_{m-1,n-1}^{(\kappa+2)}, \quad (\text{A.10a})$$

$$2U_{m,n}^{(\kappa-1)} U_{m-1,n-1}^{(\kappa+1)} = U_{m,n-1}^{(\kappa+1)} U_{m-1,n}^{(\kappa-1)} + U_{m,n-1}^{(\kappa-1)} U_{m-1,n}^{(\kappa+1)}. \quad (\text{A.10b})$$

Further differential-difference equations are given by

$$D_z(U_{m+1,n}^{(\kappa-1)} \cdot U_{m-1,n}^{(\kappa+1)}) = (2m+1) U_{m,n}^{(\kappa-1)} U_{m,n}^{(\kappa+1)}, \quad (\text{A.11a})$$

$$D_z(U_{m,n+1}^{(\kappa-1)} \cdot U_{m,n-1}^{(\kappa+1)}) s = (2n+1) U_{m,n}^{(\kappa-1)} U_{m,n}^{(\kappa+1)}. \quad (\text{A.11b})$$

The following differential-difference equations arise from the definitions of $\widehat{w}_{m,n}^{(\kappa)}$, $\widehat{v}_{m,n}^{(\kappa)}$ and $\widehat{u}_{m,n}^{(\kappa)}$ in Theorem 4.9 and Remark 4.10

$$zD_z(U_{m-1,n}^{(2\kappa)} \cdot U_{m,n-1}^{(2\kappa+2)}) = (n + \kappa)U_{m-1,n}^{(2\kappa)}U_{m,n-1}^{(2\kappa+2)} - (m + \kappa)U_{m,n-1}^{(2\kappa)}U_{m-1,n}^{(2\kappa+2)}, \quad (\text{A.12a})$$

$$zD_z(U_{m,n-1}^{(2\kappa)} \cdot U_{m-1,n}^{(2\kappa+2)}) = (m + \kappa)U_{m,n-1}^{(2\kappa)}U_{m-1,n}^{(2\kappa+2)} - (n + \kappa)U_{m-1,n}^{(2\kappa)}U_{m,n-1}^{(2\kappa+2)}, \quad (\text{A.12b})$$

$$zD_z(U_{m-1,n-1}^{(2\kappa+1)} \cdot U_{m,n}^{(2\kappa+1)}) = \kappa U_{m-1,n-1}^{(2\kappa+1)}U_{m,n}^{(2\kappa+1)} - (m + n + \kappa)U_{m-1,n-1}^{(2\kappa+3)}U_{m,n}^{(2\kappa-1)}, \quad (\text{A.12c})$$

$$2zD_z(U_{m+1,n}^{(\kappa-1)} \cdot U_{m-1,n}^{(\kappa+1)}) = (2m + 1) \left\{ U_{m-1,n}^{(\kappa+1)}U_{m+1,n}^{(\kappa-1)} + U_{m,n-1}^{(\kappa+1)}U_{m,n+1}^{(\kappa-1)} \right\}, \quad (\text{A.12d})$$

$$2zD_z(U_{m,n+1}^{(\kappa-1)} \cdot U_{m,n-1}^{(\kappa+1)}) = (2n + 1) \left\{ U_{m-1,n}^{(\kappa+1)}U_{m+1,n}^{(\kappa-1)} + U_{m,n-1}^{(\kappa+1)}U_{m,n+1}^{(\kappa-1)} \right\}. \quad (\text{A.12e})$$

Consequently from (A.11a) and (A.12d), or from (A.11b) and (A.12e), we obtain

$$2zU_{m,n}^{(\kappa-1)}U_{m,n}^{(\kappa+1)} = U_{m-1,n}^{(\kappa+1)}U_{m+1,n}^{(\kappa-1)} + U_{m,n-1}^{(\kappa+1)}U_{m,n+1}^{(\kappa-1)}. \quad (\text{A.13})$$

The generalised Umemura polynomials $U_{m,n}^{(\kappa)}(z)$ satisfy the differential-difference equations

$$U_{m+1,n}^{(\kappa)}U_{m-1,n}^{(\kappa)} = 4zD_z^2(U_{m,n}^{(\kappa)} \cdot U_{m,n}^{(\kappa)}) + 8U_{m,n}^{(\kappa)} \frac{dU_{m,n}^{(\kappa)}}{dz} + (z - 6m - 2n - 2\kappa - 2)(U_{m,n}^{(\kappa)})^2, \quad (\text{A.14a})$$

$$U_{m,n+1}^{(\kappa)}U_{m,n-1}^{(\kappa)} = 4zD_z^2(U_{m,n}^{(\kappa)} \cdot U_{m,n}^{(\kappa)}) + 8U_{m,n}^{(\kappa)} \frac{dU_{m,n}^{(\kappa)}}{dz} + (z + 2m + 6n + 2\kappa + 2)(U_{m,n}^{(\kappa)})^2, \quad (\text{A.14b})$$

and

$$U_{m+1,n}^{(\kappa-2)}U_{m-1,n}^{(\kappa+2)} = 4zD_z^2(U_{m,n}^{(\kappa)} \cdot U_{m,n}^{(\kappa)}) + 8U_{m,n}^{(\kappa)} \frac{dU_{m,n}^{(\kappa)}}{dz} + (z + 2m - 2n - 2\kappa + 2)(U_{m,n}^{(\kappa)})^2, \quad (\text{A.15a})$$

$$U_{m,n+1}^{(\kappa-2)}U_{m,n-1}^{(\kappa+2)} = 4zD_z^2(U_{m,n}^{(\kappa)} \cdot U_{m,n}^{(\kappa)}) + 8U_{m,n}^{(\kappa)} \frac{dU_{m,n}^{(\kappa)}}{dz} + (z + 2m - 2n + 2\kappa - 2)(U_{m,n}^{(\kappa)})^2, \quad (\text{A.15b})$$

with $U_{-1,-1}^{((\kappa))}(z) = U_{-1,0}^{((\kappa))}(z) = U_{0,-1}^{((\kappa))}(z) = U_{0,0}^{((\kappa))}(z) = 1$. Eliminating the derivative terms in (A.14) and (A.15) gives

$$U_{m,n+1}^{(\kappa)}U_{m,n-1}^{(\kappa)} - U_{m+1,n}^{(\kappa)}U_{m-1,n}^{(\kappa)} = 4(2m + 2n + \kappa + 1)(U_{m,n}^{(\kappa)})^2, \quad (\text{A.16a})$$

$$U_{m,n+1}^{(\kappa-2)}U_{m,n-1}^{(\kappa+2)} - U_{m+1,n}^{(\kappa-2)}U_{m-1,n}^{(\kappa+2)} = 4(\kappa - 1)(U_{m,n}^{(\kappa)})^2. \quad (\text{A.16b})$$

Also from (A.14) and (A.15) we obtain the difference equations

$$U_{m,n+1}^{(\kappa-2)}U_{m,n-1}^{(\kappa+2)} - U_{m+1,n}^{(\kappa)}U_{m-1,n}^{(\kappa)} = 4(2m + \kappa)(U_{m,n}^{(\kappa)})^2, \quad (\text{A.17a})$$

$$U_{m,n+1}^{(\kappa)} U_{m,n-1}^{(\kappa)} - U_{m+1,n}^{(\kappa-2)} U_{m-1,n}^{(\kappa+2)} = 4(2n + \kappa) \left(U_{m,n}^{(\kappa)} \right)^2, \quad (\text{A.17b})$$

$$U_{m+1,n}^{(\kappa-2)} U_{m-1,n}^{(\kappa+2)} - U_{m+1,n}^{(\kappa)} U_{m-1,n}^{(\kappa)} = 4(2m + 1) \left(U_{m,n}^{(\kappa)} \right)^2, \quad (\text{A.17c})$$

$$U_{m,n+1}^{(\kappa)} U_{m,n-1}^{(\kappa)} - U_{m,n+1}^{(\kappa-2)} U_{m,n-1}^{(\kappa+2)} = 4(2n + 1) \left(U_{m,n}^{(\kappa)} \right)^2. \quad (\text{A.17d})$$

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Data Availability No data were used for the research described in the article.

Declarations

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