Darboux transformations and integrable discretisation

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E andare come spinto dal destino
verso una guerra, verso l'avventura,
e tornare contro ogni vaticino
contro gli Dei e contro la paura

"Odysseus", Francesco Guccini

Declaration of Authorship

This thesis is based on research entirely conducted under the supervision of Professor Jing Ping Wang at the University of Kent. I declare that no part of it has been submitted for any other academic degree or qualification, and I confirm that appropriate credit has been given to other authors where reference is made. A selection of the results in Chapter 4 is presented as the paper [104] published in Physica D.

This dissertation comprises 242 pages, including 188 pages of main text and 28 pages of appendices.

Abstract

Within the Lax-Darboux scheme, Darboux transformations provide discretisations of integrable partial differential equations (PDEs) as integrable differential-difference equations (DΔEs). In the present thesis, we apply this method to the seven non-commutative derivative nonlinear Schrödinger equations (DNLS) identified by Olver and Sokolov [102]. The Lax representations considered here, also reported in [118], arise from solving a classification problem. Focusing on polynomial Darboux matrices of degree $n \in \mathbb{N}$, we construct a model for reduction group-invariant Darboux transformations, namely the rank-1 Darboux matrices $M_{\uparrow}(n)$ and $M_{\downarrow}(n)$, that generates evolutionary systems. We study the constant, linear, and quadratic cases, whose related discretisations, derived through reduction procedures, consist of non-commutative integrable systems with non-commutative constants. We demonstrate that the constant Darboux matrices induce a scaling transformation, the linear Darboux matrices are associated with Volterra-type equations, and reductions of the quadratic Darboux matrices yield two-component systems, including the relativistic Toda, the Merola-Ragnisco-Tu, and the Ablowitz-Ladik equations. Examining the relationship between linear and quadratic Darboux transformations, we provide the necessary conditions for a Darboux matrix to be factorisable with a specific linear Darboux matrix as a factor. Finally, since the DNLS equations are known to be connected by non-local gauge transformations, we extend this correspondence to Darboux transformations and the associated systems of equations. This thesis concludes with four appendices, devoted, respectively, to Lax representations of the non-commutative DNLS equations, the Darboux system associated with the polynomial matrix $M(n)$, the properties of the resulting DΔEs, and the Lax pairs of two novel equations.

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I always find it difficult, at the end of a project, to do justice to all the people who played a role in it: some contributions are evident, others more subtle but no less important. A suggestion, the right comment at the right time, might unlock a situation and bring a week's work to fruition. A simple smile or a gentle word can brighten a day, even if that moment is quickly forgotten.

The past four years at the University of Kent will always have a special place in my heart and I am grateful to all those I met and spent time with. They, directly or indirectly, contributed to this PhD project, but they also managed to shape me into the person I am now. The time spent drinking coffee, watching movies, visiting the countryside or simply drifting away in never-ending conversations was not wasted, but I consider it an integral part of my studies and formation. Unable to enumerate all the merits of the people around me, I will mention just a few, hoping not to wrong anyone.

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Now that I am writing these acknowledgements, a few days before submission, I realise how important it is to meet good friends abroad, but also to maintain the old ones from home. It was reassuring to know that, whatever might happen, I had the strongest group of friends waiting for me in Novara. However far, the presence of Jacopo, Giulio, Raul, Matteo and Simone was constant, their words precious, their jokes unbeatable. We all went through so much in these years, through good and bad times, that I am amazed that we are still so close. I am looking forward to hugging them all. I am sure now that, wherever my life leads me, I will always be able to count on them. As one wrote to me yesterday: “We are bound” and I am proud of it.

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In concluding this book, the product of four years of research and life, I realise the importance of the people I met and I would like once more to offer them my heartfelt thanks.

Edoardo, 30th August 2025

List of Symbols and Notation

In the course of the present work, we use a uniform notation to represent variables, functions, matrices, operators and more. Some of these symbols represent uniquely specific objects, while others are placeholders whose meaning depends on the context. For the sake of clarity, we present a short list of the most frequently used ones: in case of doubt, the reader may refer back here.

The notation has been designed to follow a pattern: the font indicates the type of mathematical object, e.g. scalar, matrix, operator, so that its nature can be recalled at a glance.

Lower case italic and calligraphic letters stand for variables, auxiliary functions and, in general, scalar objects:

- a, b, c, d, … are used to denote generic or auxiliary functions. For instance, we use lower case italic letters for the entries of a Darboux matrix M instead of the heavier notation $M_{ij}^{(k)}$.
- $\mathscr{c}$, $\mathscr{d}$, $\mathscr{e}$, $\mathscr{f}$, … are other generic or auxiliary functions, used when the italic letters are not enough. In most cases, there is an analogy between calligraphic and italic versions of the same characters: for instance, f and $\mathscr{f}$ are different, but they may play a similar role, as in $M(2)$ and $M_{\uparrow}(2)$.
- $\mathscr{a}$, $\mathscr{b}$ are functions that define gauge transformations among DNLS equations (Sections 2.1.4 and 2.2.2).
- $t, x \in \mathbb{C}$ are the continuous independent variables of partial differential equations (PDEs); traditionally, they represent space and time.
- i, j, k, and m, n are integers, usually used to denote indices, points on a lattice or discrete variables in differential-difference equations (DΔEs).
- p, q are smooth functions of x, t used as dependent variables in PDEs.
- r, s, u, v, w are semi-discrete functions of x and n used as dependent variables in DΔEs.

Lower case Greek letters (e.g. ρ, θ, σ, $\dots$) represent non-commutative constants, unless otherwise specified. Exceptions are:

- ι represents the inner shift (Section 3.2.2.5).
- λ is the spectral parameter (Section 1.1).

Upper case italic letters stand for matrices:

- A, B, C are used to represent generic matrices.
- G is the gauge matrix (Section 1.1).
- M, N represent Darboux matrices (Section 3.2.1).
- $M(n)$, defined in Lemma 4.3, represents a polynomial Darboux matrix of degree n, invariant under the reduction group $\mathcal{R}_{\pm}$.
- $M_{\uparrow}(n)$ and $M_{\downarrow}(n)$ in Definition 4.6 represent polynomial Darboux matrices of rank 1 and degree n in which, respectively, the second and the first eigenvalue of the leading term is set to zero.
- I, J, P, Q, and R are matrix coefficients of the Lax representation (2.7) and (2.23) of the DNLS equations.
- U and V form the Lax representation of a PDE (Section 1.1).

Upper case calligraphic letters are operators:

- $\mathcal{D}$ is the total derivative, also written as $\mathcal{D}_x$, $\mathcal{D}_t$.
- $\mathcal{C}$, $\mathcal{P}^n$ and $\mathcal{Q}$ are the operators defined in (2.44), (4.75), and (4.65).
- $\mathcal{R}$, $\mathcal{R}_{\pm}$ represent the actions of the reduction groups (2.42) and (4.4).
- $\mathcal{S}$ is a Darboux transformation and the associated shift operator.
- $\mathcal{T}$ represents the reflection operator.

Other symbols that appear recurrently are:

- $\mathscr{F}$ represents the algebra of differential functions (beginning of Chapter 2).
- $\mathscr{A}$ represents the algebra of difference functions (Section 3.2.2.1).
- $\mathscr{E}$ represents the system of DΔEs associated with a Darboux transformation (Section 3.2.1).
- $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ represent the usual sets of numbers.
- $\Delta_{ij}(A)$ represents the quasideterminant of the matrix A with respect to its entry (i, j).
- Φ represents the fundamental solution of an auxiliary system.
- the superscript $\dagger$ represents the adjoint operator (Section 2.2.3.2).
- the superscript τ represents the transpose of a matrix.
- the superscript $*$ represents the complex conjugation.
- the superscript $\star$ represents the reverse-order involution, defined in (2.21).

A **subscript** has a different meaning according to the fonts and symbols used:

- an integer subscript, e.g. f_n, represents a shift $\mathcal{S}^n(f)$ (Section 3.2.1).
- a continuous subscript, e.g. f_x, denotes a derivative $\mathcal{D}_x(f)$.
- for a matrix M, the subscripts M_{ij} represent its entries.
- for shifted or derived matrix entries, we write $\mathcal{S}(M_{ij})$ or $\mathcal{D}_x(M_{ij}) = M_{ij,x}$.

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Chapter 1

Introduction

Since the first investigations, integrability has appeared to be a quite rare and fragile phenomenon: out of the many equations known, only a few are integrable, and this property is easily destroyed by altering even minimal aspects of them. Nevertheless, integrable equations exhibit remarkably regular and mathematically deep features and appear in many areas of mathematics and physics.

In an empirical sense, integrable equations are often nonlinear equations with precise characteristics, which include, among others, the possibility of constructing exact solutions, the existence of Lax representations, infinite hierarchies of symmetries and conserved densities. On one hand, nonlinearity prevents the use of many techniques available for linear equations, such as the superposition principle and Fourier analysis. On the other hand, analogous concepts have been developed that mimic the latter and allow a systematic treatment of integrable nonlinear equations: examples include the nonlinear superposition principle and the inverse scattering transform.

However, each of these approaches to integrability relies on specific features of the dynamical system, which are not always present. This illustrates why it is so difficult to give a precise definition of integrability: each characterisation captures only one aspect of the theory and every classification depends heavily on the scope and perspective adopted [129].

Originally designed by Darboux [34] for Sturm-Liouville problems, Darboux transformations can also be applied to the auxiliary system of a nonlinear equation. Fundamental references for this theory include the books by Matveev and Salle [83], by Rogers and Schief [105], and by Gu, Hu and Zhou [55]. Chapter 3 is entirely devoted to introduce this topic. In this section, we aim to outline its role in the discretisation of evolutionary partial differential equations (PDEs) into differential-difference equations (DΔEs).

Consider an integrable PDE in two continuous variables $x, t \in \mathbb{C}$, and assume that the pair of matrices (U, V) form its Lax representation. Lax pairs, central in the modern treatment of integrability, are introduced in detail in Section 1.1; if the reader is not familiar with the subject, they are welcome to read that part first. A Darboux transformation $\mathcal{S}$ is a linear invertible transformation

$$\mathcal{S} : \Phi \mapsto \bar{\Phi} = M\Phi,$$

represented by a Darboux matrix M. It maps a fundamental solution Φ of the auxiliary system given by (U, V) into a second fundamental solution $\bar{\Phi}$ of a modified auxiliary system based on a new Lax pair $(\bar{U}, \bar{V})$, i.e.

$$\begin{cases} \Phi_x = U\Phi, \\ \Phi_t = V\Phi, \end{cases} \quad \Rightarrow \quad \begin{cases} \bar{\Phi}_x = \bar{U}\bar{\Phi}, \\ \bar{\Phi}_t = \bar{V}\bar{\Phi}, \end{cases}$$

such that the new Lax pair $(\bar{U}, \bar{V})$ is still a representation of the original PDE [66, 89]. Accordingly, if u is a solution of the PDE, the matrices $U(u)$ and $V(u)$ transform into the new matrices $\bar{U}$ and $\bar{V}$ in the following way

$$\mathcal{S} : U(u) \mapsto \bar{U} = U(\bar{u}), \qquad V(u) \mapsto \bar{V} = V(\bar{u}),$$

where $\bar{u}$ is a different solution of the original PDE.

The Darboux matrix M is an invertible matrix that depends on both solutions u and $\bar{u}$. Therefore, Darboux transformations can be interpreted as auto-Bäcklund transformations of the PDE [73]:

$$\mathcal{S} : u \mapsto \bar{u}.$$

Combining the original auxiliary system with the definition of a Darboux transformation, a new auxiliary system appears:

$$\begin{cases} \mathcal{S}(\Phi) = M\Phi, \\ \Phi_x = U\Phi. \end{cases}$$

Its compatibility condition, often called the Lax-Darboux equation of the Lax representation (M, U), is

$$M_x = \mathcal{S}(U)M - MU.$$

This matrix equation is equivalent to a system $\mathscr{E}$ of nonlinear DΔEs, called the Darboux system: in fact, they contain both differential terms (associated with the partial derivative $\mathcal{D}_x$ and U) and difference terms (associated with the map $\mathcal{S}$ and M) [12, 73, 74]. This procedure is interpreted as a discretisation of the original equation that preserves integrability.

Summarising, a Darboux transformation is characterised by three objects: a map $\mathcal{S}$, the associated Darboux matrix M, and the Darboux system $\mathscr{E}$. A more detailed discussion of the application of Darboux transformations to Lax pairs is presented in Chapter 3.

A standard application of Darboux transformations in integrable systems is the so-called dressing method, in which their iterations generate an infinite sequence of solutions from a given one [130, 133]. An interesting review on Darboux matrices and N-soliton formulas is due to Cieśliński in [31].

Studying Darboux transformations as integrable discretisations of PDEs has the immediate advantage of constructing the related DΔEs, potentially expanding the set of known integrable systems. Moreover, this process naturally produces their Lax representations (M, U), opening the way to further discussions on hierarchies of symmetries, symmetry reductions and other related concepts, see also Section 5.1. The discretisation via Darboux transformations has a standard application to the nonlinear Schrödinger (NLS) equation, as constructed in many research papers [63, 65–67, 90] from various perspectives and with different aims.

The passage from PDEs to DΔEs constitutes the first half of the Lax-Darboux scheme, a general framework that connects different kinds of integrable systems [7, 12]. In general, two Darboux transformations $\mathcal{S}^{(1)}$ and $\mathcal{S}^{(2)}$ of the same PDE do not necessarily commute, meaning that the composition $\mathcal{S}^{(1)}\mathcal{S}^{(2)}$ is not equal to $\mathcal{S}^{(2)}\mathcal{S}^{(1)}$. The second half of the scheme requires that $\mathcal{S}^{(1)}\mathcal{S}^{(2)}$ corresponds to $\mathcal{S}^{(2)}\mathcal{S}^{(1)}$, which consists of a system of partial difference equations (PΔEs), expressed just in terms of the shifts $\mathcal{S}^{(1)}$ and $\mathcal{S}^{(2)}$, with all differential parts disappearing. This property, known as Bianchi permutability, leads to the Lax representations $(M^{(1)}, M^{(2)})$ for PΔEs given by the Darboux matrices of $\mathcal{S}^{(1)}$ and $\mathcal{S}^{(2)}$, as briefly discussed in Sections 3.2.2.6 and 5.2. For a complete introduction to integrable PΔEs and associated Darboux transformations, consult [57].

Thus, the Lax-Darboux scheme achieves a stepwise integrable discretisation of a PDE, in the variables $x, t \in \mathbb{C}$, first into DΔEs in the variables x and $n \in \mathbb{Z}$, and then into PΔEs in the variables $n, m \in \mathbb{Z}$. We present an example in Section 5.2.2, consult [66] as well.

In the present work, we focus on the first half of the Lax-Darboux scheme applied to the non-commutative derivative nonlinear Schrödinger equations (DNLS). Chapter 2 provides a mathematical introduction to these integrable systems. Their commutative versions form an infinite family of evolutionary PDEs depending on two continuous parameters [68, 92, 102], see Section 2.1.2. Each of these equations is related to the others via non-local gauge transformations, see Section 2.1.4. In this sense, the DNLS equations can be regarded either as several independent models, including the famous Kaup-Newell [62], Chen-Lee-Liu [30], and Gerdjikov-Ivanov [50] equations, or as a single, large equivalence class under non-local changes of variables (Miura maps).

The commutative DNLS equations have numerous applications in physics, particularly in the study of plasmas and nonlinear optics. The Kaup-Newell equation, sometimes also called DNLSI, describes small-amplitude Alfvén waves [95, 113], governs large-amplitude magnetohydrodynamic waves (MHD) in plasmas [40, 106], and models pulses in single-mode optical fibres [13, 14, 120].

In [102], Olver and Sokolov classified the non-commutative DNLS equations as seven PDEs, taking values in a non-commutative associative algebra. Their integrability was later fully proved by Tsuchida and Wadati [118], who deduced their Lax representations from the gauge transformations of a given pair, related to a non-commutative form of the Chen-Lee-Liu equation. These results are recalled in Section 2.2. Remarkably, while the commutative DNLS form an infinite family of equations, only three of them have non-commutative versions.

In Chapter 4, we explicitly construct Darboux transformations of all the non-commutative DNLS equations, assuming Darboux matrices that are polynomial in the spectral parameter λ. Considering the invariance under the reduction group of their Lax representations, we identify the evolutionary DΔEs associated with constant, linear, and quadratic Darboux matrices of rank 1, see Theorems 4.13 and 4.21. Part of these results appear in the article [104].

A recurring feature of the present work is the inclusion of non-commutative constants: these are constant elements μ of the (non-commutative) associative algebra such that $\mu_x = 0$. A detailed discussion follows in Section 4.1.4. The common approach [67] is to reduce them to commutative numbers, thereby restricting the generality of the discretisation. To our knowledge, the present work is one of the first studies to preserve these elements naturally appearing in Darboux systems.

The rank-1 linear Darboux transformations of the DNLS equations lead to Volterra-type integrable DΔEs, which are:

- the Volterra equation ($\mathrm{V_a}$),
- the modified Volterra equation ($\mathrm{mV_b}$).

Similarly, rank-1 quadratic Darboux transformations are associated with generalisations of

- the Ablowitz-Ladik equation (AL),
- the Merola-Ragnisco-Tu equation (MRT),
- the relativistic Toda equation (rT),
- the two-component Volterra equations ($2\mathrm{V}_{\mathrm{a}}$) and ($2\mathrm{V}_{\mathrm{b}}$).

We also identify some new models N_i, with $i \in \{1, 2, 3, 4\}$, that, to the best of our knowledge, have not appeared previously in the literature. In particular, as we prove in Appendix C, the models N_1 and N_2 are related to the Ablowitz-Ladik equation, while N_3 is related to the Merola-Ragnisco-Tu equation, via non-invertible Miura transformations. In Appendix B we provide the system of equations associated with a reduction group-invariant polynomial Darboux transformation of degree $n \in \mathbb{N}$.

Along the same line of research, the authors of [65, 66], working on reduction groups, construct the polynomial Darboux transformations of degree $n \leq 2$ for the commutative Kaup-Newell equation. The paper [67] presents results analogous to ours for a non-commutative Kaup-Newell equation in the context of the Yang-Baxter equation. Dressing methods for the DNLS equations are discussed in a wide range of papers, including [126, 128].

We also examine the relations among the obtained Darboux transformations. In particular, we address the problem of whether an arbitrary polynomial Darboux matrix of degree n can be factorised into lower-degree elements. Similar results have been obtained for the NLS equation, leading to the identification of elementary Darboux transformations in [72, 96]. As presented in Section 4.5.2, the quasi-determinants of a Darboux matrix provide the necessary conditions for its factorisation. We apply this theory to the quadratic case, finding a reduction corresponding to the composition of two linear Darboux matrices. These results complete the description of rank-1 Darboux transformations of degree $n \leq 2$.

Given the non-commutative DΔEs obtained above, it is natural to analyse their properties and to proceed with the Lax-Darboux scheme, obtaining the respective PΔEs. We provide some examples in Chapter 5, but this will be the subject of future work.

1.1 Lax representations for integrable PDEs

This thesis introduces a series of Darboux transformations for the DNLS equations, but it also aims to provide a coherent context for these results. The first chapters are therefore supplemented with examples and considerations intended to give the reader a more solid understanding of the subject.

In the present section, we introduce the concept of Lax representations for PDEs, the associated auxiliary systems (already mentioned above) and gauge transformations. As standard references, consult the monographs [15, 53, 135]. Although well-established, this theory is sometimes presented in different terms: we use this introduction to set the conventions adopted here and to keep the exposition self-contained. The reader who is already familiar with these ideas is welcome to skip to the next section.

First introduced by Lax for the KdV equation [71], Lax representations rapidly became not only a fundamental ingredient in the modern formulation of the inverse scattering transform, but also a commonly used criterion for integrability itself (see also Remark 1.1 below). From a Lax pair it is possible to construct Darboux transformations, central in this work, recursion operators (see Section 5.1.1), as well as hierarchies of symmetries, conserved densities and more [116].

Let us consider a system of evolutionary partial differential equations (PDEs), written in terms of two dependent variables $p(x,t)$ and $q(x,t)$, where $x, t \in \mathbb{C}$

represent space and time, respectively:

$$\begin{cases} p_t = f([p,q]), \\ q_t = g([p,q]). \end{cases}$$

The subscripts denote partial derivatives, and the notation $f([p,q])$ indicates that f is a function of a finite number of arguments among p, q and their x-derivatives [125].

In the present case, a *Lax representation* for an evolutionary PDE consists of a pair of square matrices (U, V) such that their *zero-curvature condition*

$$U_t - V_x + [U, V] = 0 \tag{1.1}$$

is equivalent to the original system. Here, the bracket $[\cdot,\cdot]$ of two matrices represents their commutator. The pair (U, V) depends on the variables p, q, their x-derivatives and certain rational (or elliptic) functions of the *spectral parameter* $\lambda \in \mathbb{C}$, which is a characteristic of Lax representations. For different definitions and forms in which Lax pairs appear, consult [39].

An evolutionary nonlinear PDE with a non-trivial Lax representation (U, V) is often called *S-integrable*, according to Calogero's terminology [24, 26]. The pair (U, V) induces a linear *auxiliary system* [131], defined as

$$\begin{cases} \Phi_x = U\Phi, \\ \Phi_t = V\Phi. \end{cases} \tag{1.2}$$

for a *fundamental solution* Φ. The compatibility condition of the auxiliary system, namely the commutativity of the partial derivatives $\mathcal{D}_x(\Phi_t) = \mathcal{D}_t(\Phi_x)$, is equivalent to the zero-curvature condition (1.1) and justifies its name. In this sense, the matrix U is often denoted as the spatial part of the Lax representation, while V is its temporal part.

Let G be a suitable invertible matrix and consider the transformation

$$\Phi = G\Phi'$$

on a fundamental solution Φ of (1.2), yielding a new fundamental solution Φ' of a similar auxiliary system:

$$\begin{cases} \Phi'_x = U'\Phi', \\ \Phi'_t = V'\Phi'. \end{cases} \tag{1.3}$$

The new Lax pair (U', V') is related to the given (U, V) through the following *gauge transformation*:

$$U \mapsto U' = G^{-1}UG - G^{-1}G_x, \qquad V \mapsto V' = G^{-1}VG - G^{-1}G_t. \tag{1.4}$$

Gauge transformations preserve the zero-curvature condition, which is particularly useful for manipulating and simplifying Lax representations.

Remark 1.1. A wide class of PDEs possesses Lax representations without being integrable itself [26]. It is therefore important to distinguish when such a structure is meaningful for integrability from when it is not [23, 109].

1.2 Structure of the thesis

After the current Chapter 1, Chapters 2 and 3 introduce the main topics of this work: the derivative nonlinear Schrödinger equations (DNLS) and the Darboux transformations. In Chapter 4 we construct the constant, linear, and quadratic Darboux transformations of the DNLS equations; we also describe the factorisation of quadratic Darboux matrices into two linear ones. These results partially appear in the paper [104]. Finally, in Chapter 5, we discuss possible future developments and some preliminary results.

In **Chapter 2** we present the DNLS equations, first in the commutative setting, then in the non-commutative one. After a contextualisation with the famous nonlinear Schrödinger equation (NLS) in Section 2.1.1, Section 2.1.2 introduces the commutative DNLS equations, and their three historically relevant cases: the Kaup-Newell, the Chen-Lee-Liu and the Gerdjikov-Ivanov equations. In Section 2.1.3 we present the associated Lax representations, including new Lax pairs for the two-parameter case. We also propose the concept of symmetric DNLS equations, arising from a symmetry constraint on the parametric Lax representation; further details are provided in Appendix A. Section 2.1.4 is devoted to gauge transformations for the commutative DNLS equations, showing a second relevant application of the symmetric DNLS equations.

A similar structure appears in Section 2.2 for the non-commutative DNLS equations: after describing how they were introduced by Olver and Sokolov [102], we present their Lax representations (Section 2.2.1), and we discuss the associated gauge transformations (Section 2.2.2). Additional considerations about parametric Lax pairs in the non-commutative setting are again provided in Appendix A.

Section 2.2.3 is devoted to identifying the reduction groups of both the commutative and non-commutative Lax representations. We observe that all of them are invariant under the action of a transformation $\mathcal{R}$ (Section 2.2.3.1) and the adjoint transformation † (Section 2.2.3.2). We take these characteristics into account when constructing the Darboux matrices in Chapter 4.

Lastly, in Section 2.3, we obtain Olver and Sokolov's classification from the perspective of Lax pairs. We observe that the DNLS Lax representations share some common features, notably homogeneity and reduction group invariance. Then, by imposing these constraints, we identify a finite set of possible Lax pairs, each associated with a non-commutative DNLS equation.

Chapter 3 consists of an introduction to Darboux transformations. Starting in Section 3.1 with the original work of Darboux on Sturm-Liouville equations, we discuss the iteration of a Darboux transformation (see Section 3.1.1) and present Crum's theorem (see Section 3.1.2).

Section 3.2 applies the same concepts to integrable systems, providing a modern definition of Darboux transformations for Lax representations of PDEs. After an introduction (Section 3.2.1), in Section 3.2.2 we describe a discretisation method for obtaining DΔEs from PDEs through Darboux transformations. We define their algebraic framework (Section 3.2.2.1), introduce Lax-Darboux representations and equations (Section 3.2.2.2), and discuss the associated gauge transformations (Section 3.2.2.3). We then examine the iteration (Section 3.2.2.4), composition (Section 3.2.2.5) and inverse (Section 3.2.2.7) of Darboux transformations, briefly mentioning Bianchi permutability (Section 3.2.2.6). In this chapter, we introduce the notions of Darboux inverse matrix M^I and "inner shift" $\iota_{\mathcal{S}}$, which represent more clearly the inverse and composition of Darboux transformations.

The main results of this research are presented in **Chapter 4** and are summarised in the article [104]. Here we explicitly construct the Darboux transformations of the seven non-commutative DNLS equations and analyse the associated Darboux systems.

Since the Lax representations of all DNLS equations consist of 2×2 polynomial matrices in the spectral parameter λ, in Section 4.1 we assume Darboux matrices M to be matrix polynomials in λ of degree $n \in \mathbb{N}$. This condition is sufficient to obtain an initial model, though it remains unsatisfactory. In Section 4.1.1, we refine it by requiring invariance under a reduction group similar to the one for (U, V). The resulting Darboux transformation $M(n)$ is associated with a set $\mathscr{E}(n)$ of non-evolutionary DΔEs. As a final step, in Section 4.1.2, we reduce the rank of $M(n)$, obtaining two systems of evolutionary equations, $\mathscr{E}_{\uparrow}(n)$ and $\mathscr{E}_{\downarrow}(n)$, which form the main focus of this study. We denote the associated Darboux matrices as the up and down rank-1 matrices $M_{\uparrow}(n)$ and $M_{\downarrow}(n)$. In Appendix B we construct the system $\mathscr{E}(n)$ for a generic degree n.

We examine in detail the Darboux transformations for $n \leq 2$. The constant Darboux matrix (Section 4.2) represents the trivial scaling transformation characteristic of homogeneous commutative equations. The linear case (Section 4.3) is the origin of the first integrable DΔEs: the Darboux transformations $M_{\uparrow}(1)$ of the DNLS equations are related to Volterra-type equations, also studied in Appendix C.1. We conclude the section with observations on the down matrix $M_{\downarrow}(1)$ (Section 4.3.2), which is related to the Darboux inverse up matrix $M^I_{\uparrow}(1)$, and on the full-rank matrix $M(1)$ (Section 4.3.3).

The same procedure applies to the quadratic Darboux transformations (Section 4.4). As described in Section 4.4.1, the rank-1 $M_{\uparrow}(2)$ leads to a system of DΔEs including indeterminate auxiliary functions. We identify three possible reductions of $\mathscr{E}_{\uparrow}(2)$, which generalise well-known integrable systems and yield novel ones. The relations among such results and their Lax representations are discussed in Appendix C. The same reductions are significant in Section 4.4.2, where they relate the Darboux inverse up matrix $M^I_{\uparrow}(2)$ to the down matrix $M_{\downarrow}(2)$. The full-rank case $M(2)$, studied in Section 4.4.3, corresponds to a composition of the two rank-1 matrices $M_{\uparrow}(2)$ and $M_{\downarrow}(2)$.

Section 4.5 investigates the relations between the linear $M_{\uparrow}(1)$ and quadratic $M_{\uparrow}(2)$ Darboux transformations. We show (Section 4.5.1) that the composition of two specific linear Darboux matrices corresponds to a reduction of the quadratic Darboux matrix. In Section 4.5.2 we follow the converse: given a Darboux matrix of degree n, we determine a sufficient condition under which it can be expressed as a composition of lower-degree ones. For this purpose, we introduce (Section 4.5.2.1) the notion of quasideterminants, and we present (Section 4.5.2.2) a lemma proving when a Darboux matrix of degree n can be factorised into two matrices with either $M_{\uparrow}(1)$ or $M_{\downarrow}(1)$ as a factor. The section concludes with an application to the quadratic case $M_{\uparrow}(2)$.

Section 4.6 is devoted to gauge transformations of the DΔEs arising from Darboux transformations. In Section 4.6.1 we extend the non-local gauge transformations that link all DNLS equations (commutative and non-commutative) to their respective Darboux matrices and systems. This requires the strong assumption that all variables and auxiliary functions remain local under it, see Remark 4.34. Accepting this ansatz, we present (Section 4.6.2) a useful application as a reduction scheme for Darboux matrices, generating non-local transformations between the associated Darboux systems. This procedure is applied to $M_{\uparrow}(1)$ and $M_{\uparrow}(2)$ in Sections 4.6.2.1 and 4.6.2.2, respectively.

In the concluding **Chapter 5**, we highlight two possible directions for future work. In Section 5.1 we extend some of the algebraic properties of the standard non-commutative Volterra equation [8, 29] to ($\mathrm{V_a}$) by including the non-commutative constants. In Section 5.1.1 we construct its recursion operator and its first symmetry, and in Section 5.1.2 we show that it admits a reduction to a discrete Painlevé equation $\mathrm{dP_1}$, also incorporating non-commutative constants. We complete the study of Volterra equations in Appendix C.1. Similar analyses can be carried out for all the other models.

The natural continuation of the Lax-Darboux scheme is explored in Section 5.2. After a general introduction (Section 5.2.1), Section 5.2.2 outlines the Bianchi permutability of two rank-1 up quadratic Darboux matrices $M_{\uparrow}(2)$ and provides the general system of PΔEs, of which we construct two particular solutions (Sections 5.2.2.1 and 5.2.2.2). An analogous Lax-Darboux scheme applied to the commutative Kaup-Newell equation appears in [66].

As mentioned above, Appendix A considers Lax representations of the commutative DNLS equations depending on the parameters $\alpha, \beta \in \mathbb{C}$ from Section 2.1.3. We discuss also similar Lax pairs for the non-commutative cases with one or two parameters.

In Appendix B we define the system $\mathscr{E}(n)$ associated with the Darboux matrix $M(n)$ of degree $n \geq 1$, generalising the results from Sections 4.3 and 4.4.

In Appendix C we examine the integrable DΔEs found via Darboux transformations. Appendix C.1 considers the Volterra-type equations, namely the potential, modified and two-component Volterra models, and discusses their Lax representations and the corresponding Miura transformations. We analyse the new systems N_i as well: Appendix C.2 proves that N_1 and N_2 are related to the Ablowitz-Ladik equation by a non-invertible Miura transformation, while Appendix C.3 shows that N_3 is linked by similar means to the Merola-Ragnisco-Tu and Kaup equations.

Finally, in Appendix D we construct the Lax representations of two equations described in Chapter 5: these are a non-commutative Painlevé type equation involving a non-commutative constant, and a scalar PΔE obtained as a reduction of the Bianchi permutability of two rank-1 up quadratic Darboux matrices.

Chapter 2

The DNLS equations

Under the evocative name of derivative nonlinear Schrödinger (DNLS) equations lies a class of S-integrable partial differential equations (PDEs) that are variants of the nonlinear Schrödinger equation (NLS) including an x-derivative in the nonlinear term.

Both the commutative and non-commutative DNLS equations are integrable, evolutionary, second-order nonlinear PDEs, defined in terms of two independent variables $x, t \in \mathbb{C}$, originally space and time, and two dependent variables p and q, smooth functions of x and t. By non-commutativity we indicate that the dependent variables p and q are not scalars (as in the commutative case), but they take values in a non-commutative associative algebra.

The most well-known example of non-commutative DNLS equation is a generalisation of the Kaup-Newell equation:

$$\begin{cases} p_t = -p_{xx} + 4(pqp)_x, \\ q_t = q_{xx} + 4(qpq)_x. \end{cases}$$

Here and throughout, subscripts in x and t denote partial derivatives. For several years, the term DNLS was used synonymously with the Kaup-Newell equation [43]. However, other structurally similar models, such as the Chen-Lee-Liu and the Gerdjikov-Ivanov equations, emerged, requiring a broader and more precise characterisation of what constitutes a DNLS equation.

We conclude this introduction by providing the algebraic framework for the current analysis. Let $\mathscr{F}$ be the associative $\mathbb{C}$-algebra of all differential polynomials in p and q, including their derivatives with respect to x, i.e. p_x, q_x, p_{xx} and so on. To emphasise that each $f \in \mathscr{F}$ depends on a finite number of arguments, we write $f([p, q])$. The algebra $\mathscr{F}$ is an example of a differential algebra with respect to the derivation $\mathcal{D}_x$.

The commutativity property extends to the whole algebra $\mathscr{F}$, which is an abelian algebra, meaning that any product of its elements commutes: e.g. $pq = qp$. In the non-commutative case, this is no longer valid, i.e. $pq \neq qp$, and $\mathscr{F}$ is a free associative algebra. In the following parts we introduce and contextualise the commutative and non-commutative DNLS equations in the theory of integrable systems.

2.1 The commutative DNLS equations

The term “derivative nonlinear Schrödinger equation” naturally evokes the idea that it is a modification of the nonlinear Schrödinger equation (NLS). The NLS itself originates from the well-known linear Schrödinger equation, one of the fundamental pillars of modern physics. For further background, the reader may consult any standard text on quantum mechanics such as [32, 54].

2.1.1 The nonlinear Schrödinger equation

The *nonlinear Schrödinger equation* (NLS) is a variant of the Schrödinger equation in which the potential depends cubically on the wave function ψ:

$$2\mathrm{i}\psi_t + \psi_{xx} - \kappa|\psi|^2\psi = 0. \tag{2.1}$$

Here $|\psi|$ denotes the modulus of the complex function $\psi(x,t)$ and $\kappa \in \mathbb{R}$ is a real parameter. The case $\kappa < 0$ corresponds to the *focusing* NLS, while $\kappa > 0$ corresponds to the *defocusing* case.

Some authors argue that the term "nonlinear" is too generic for (2.1) and instead prefer the name *cubic Schrödinger equation*, which emphasizes the specific kind of nonlinearity [76]. In condensed matter physics, equation (2.1) bears the name of *Gross-Pitaevskii equation*, where it models an approximated case of the Bose-Einstein condensation. It further appears in fluid dynamics, nonlinear optics, solid-state physics, and even in mathematical finance [114].

From the perspective of integrability theory, it is convenient to rewrite (2.1) as a system of two complex-valued functions $p = p(x,t)$ and $q = q(x,t)$, representing respectively the wave function ψ and its complex conjugate ψ^*, under the replacement $\mathcal{D}_t \mapsto \mathrm{i}\,\mathcal{D}_t$. In the focusing case, fixing $\kappa = 8$, the corresponding NLS equation assumes the standard form

$$\begin{cases} 2p_t = p_{xx} - 8p^2 q, \\ 2q_t = -q_{xx} + 8q^2 p. \end{cases} \tag{NLS}$$

Its Lax representation, discovered in the seminal work of Zakharov and Shabat [131], is given by the matrices

$$\begin{aligned} U &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lambda + 2 \begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix}, \\ V &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lambda^2 + 2 \begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix} \lambda + \begin{pmatrix} -2pq & p_x \\ -q_x & 2pq \end{pmatrix}. \end{aligned} \tag{2.2}$$

Equation (NLS) is homogeneous if p, q and $\mathcal{D}_x$ are assigned weight 1, and $\mathcal{D}_t$ weight 2. It is also a completely integrable Hamiltonian system that can be solved via the inverse scattering transform (IST) [1, 2, 5, 39].

2.1.2 The derivative nonlinear Schrödinger equations

The *derivative nonlinear Schrödinger equations* (DNLS) are integrable deformations of the NLS (2.1) including a derivative term in x in the potential. A more precise characterisation of the DNLS equations will be provided in Section 2.2 within the non-commutative framework. One of the most natural and widely studied examples of a DNLS equation is

$$\mathrm{i}\psi_t + \psi_{xx} + 4\mathrm{i}(|\psi|^2\psi)_x = 0. \tag{2.3}$$

Let us consider $p = -\psi$ and $q = \psi^*$, as previously done for (NLS), and apply a complex scaling of the independent variables: $\mathcal{D}_t \mapsto -\mathrm{i}\,\mathcal{D}_t$ and $\mathcal{D}_x \mapsto -\mathrm{i}\,\mathcal{D}_x$. We obtain an equivalent formulation of (2.3) as a two-equation system known as the *Kaup-Newell equation* [62]:

$$\begin{cases} p_t = -p_{xx} + 4(p^2q)_x, \\ q_t = q_{xx} + 4(pq^2)_x. \end{cases} \tag{A}$$

A non-commutative analogue of (A) was already presented in the introduction of this chapter. However, other systems structurally similar to (2.3) also qualify as DNLS equations, such as:

$$\mathrm{i}\psi_t + \psi_{xx} + 4\mathrm{i}|\psi|^2\psi_x = 0, \tag{2.4a}$$

$$\mathrm{i}\psi_t + \psi_{xx} - 4\mathrm{i}\psi^2\psi_x^* + 8|\psi|^4\psi = 0. \tag{2.4b}$$

These are also regarded as DNLS equations like (A), and correspond to two well-studied integrable systems. Using the same transformation as above, equation (2.4a) becomes the *Chen-Lee-Liu equation* [30]:

$$\begin{cases} p_t = -p_{xx} + 4pqp_x, \\ q_t = q_{xx} + 4pqq_x, \end{cases} \tag{B}$$

and equation (2.4b) transforms into the *Gerdjikov-Ivanov equation* [50]:

$$\begin{cases} p_t = -p_{xx} - 4p^2 q_x + 8p^3 q^2, \\ q_t = q_{xx} - 4q^2 p_x - 8p^2 q^3. \end{cases} \tag{C}$$

These three systems are historically relevant examples of a broader integrable class, which we call $\mathrm{X}_{\alpha,\beta}(p,q)$, depending on two real parameters α and β [68, 92, 102]:

$$\mathrm{X}_{\alpha,\beta}(p,q): \quad \begin{cases} p_t = -p_{xx} + 2\alpha p^2 q_x + 2\beta pq p_x - \alpha(\beta - 2\alpha) p^3 q^2, \\ q_t = q_{xx} + 2\alpha q^2 p_x + 2\beta pq q_x + \alpha(\beta - 2\alpha) p^2 q^3. \end{cases} \tag{2.5}$$

We refer to all $\mathrm{X}_{\alpha,\beta}(p,q)$ as the *(commutative) derivative nonlinear Schrödinger* equations (DNLS). This class was derived by Kundu [68] from the Kaup-Newell system (A) via a gauge transformation (see later Section 2.1.4). Notice that (A), (B) and (C) correspond to certain choices of parameters:

- the Kaup-Newell system (A) is obtained for $\alpha = 2$ and $\beta = 4$,
- the Chen-Lee-Liu system (B) is obtained for $\alpha = 0$ and $\beta = 2$,
- the Gerdjikov-Ivanov system (C) is obtained for $\alpha = -2$ and $\beta = 0$.

The systems $\mathrm{X}_{\alpha,\beta}(p,q)$ in (2.5) are not the only ones associated with the DNLS equations: an alternative class with similar structural characteristics was reported in [102]:

$$\begin{cases} p_t = -p_{xx} + 2\alpha pq p_x + 2\alpha p^2 q_x + \alpha\beta p^3 q^2, \\ q_t = q_{xx} + 2\beta pq q_x + 2\beta q^2 p_x - \alpha\beta p^2 q^3. \end{cases} \tag{2.6}$$

The integrability of both (2.5) and (2.6) was established in [91], but it is known from [25] that the case $\alpha = \beta = 1$ of (2.6) is linearisable. Later, the authors of [118] showed that (2.6) can be linearised into two heat equations and constructed its general solution. Since we only consider DNLS equations that are nonlinearisable, we exclude (2.6) from our analysis.

Similarly to (NLS), the DNLS system (2.5) is homogeneous if p and q are assigned weight 1, $\mathcal{D}_x$ weight 2, and $\mathcal{D}_t$ weight 4. Consequently, every DNLS equation $\mathrm{X}_{\alpha,\beta}(p,q)$ has total weight 5. This observation will be crucial for Olver and Sokolov's classification [102].

2.1.3 Lax representations

A standard Lax representation of a commutative DNLS equation is given by a pair of 2×2 matrices of the following form:

$$U = I\lambda^2 + 2J\lambda + 2P, \tag{2.7a}$$

$$V = -2I\lambda^4 - 4J\lambda^3 + 4IJJ\lambda^2 + 2Q\lambda - 2R, \tag{2.7b}$$

where I and J are the following matrices:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix}. \tag{2.8}$$

The coefficient I corresponds to the third Pauli matrix. We denote a 2×2 matrix as *anti-diagonal* if all its diagonal entries vanish. The matrices P and R are diagonal, while Q is anti-diagonal. All have entries in $\mathscr{F}$.

Notice that in (2.7) diagonal matrices occur with even powers of λ, whereas anti-diagonal matrices are associated with odd powers. This property reflects an underlying reduction group invariance, as discussed in Section 2.2.3.1, and it plays a central role in the classification in Section 2.3. A more general result on integrable systems admitting Lax pairs of the form (2.7) is presented as Theorem 2.5.

2.1.3.1 List of Lax representations

Following the scheme (2.7) and the zero-curvature condition (1.1), the Kaup-Newell equation (A) is represented by the matrices

$$P = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & 4p^2q - p_x \\ 4pq^2 + q_x & 0 \end{pmatrix}, \qquad R = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.9}$$

The Chen-Lee-Liu equation (B) is represented by the matrices

$$\begin{gathered} P = \begin{pmatrix} 0 & 0 \\ 0 & pq \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & 2p^2q - p_x \\ 2pq^2 + q_x & 0 \end{pmatrix}, \\ R = \begin{pmatrix} 0 & 0 \\ 0 & qp_x - q_xp - 2p^2q^2 \end{pmatrix}. \end{gathered} \tag{2.10}$$

The Gerdjikov-Ivanov equation (C) is represented by the matrices

$$\begin{gathered} P = \begin{pmatrix} -pq & 0 \\ 0 & pq \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & -p_x \\ q_x & 0 \end{pmatrix}, \\ R = \begin{pmatrix} pq_x - p_xq + 2p^2q^2 & 0 \\ 0 & qp_x - q_xp + 2p^2q^2 \end{pmatrix}. \end{gathered} \tag{2.11}$$

These Lax representations are homogeneous with respect to the same set of weights introduced earlier, under the assumption that the spectral parameter λ has weight 1: U has weight 2 and V has weight 4. The homogeneity of (U, V) is another key element in the classification developed in Section 2.3.

Remark 2.1. The operator U in (2.9) is equivalent, up to a rescaling of λ, to its analogue in (2.2) associated with (NLS). This is the most evident example of the relationship between the DNLS equations and the AKNS formulation [2]: other considerations follow in [37, 97]. It appears that both equations (NLS) and (A) are members of a broader family, based on the algebra $\mathfrak{sl}_2(\mathbb{C})$ [42]. The transformations between corresponding solutions are studied in many works, including the papers [60, 121, 123].

We construct a Lax pair for the parametric system (2.5), which, to the best of our knowledge, has not previously appeared in the literature.

Proposition 2.2. *The commutative DNLS equations* (2.5) *admit the following Lax representation:*

$$
\begin{aligned}
U &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lambda^2 + \begin{pmatrix} 0 & (\beta-\alpha)p \\ 2q & 0 \end{pmatrix} \lambda + \frac{\beta-2\alpha}{2(\beta-\alpha)} \begin{pmatrix} \alpha pq & 0 \\ 0 & (2\beta-\alpha)pq \end{pmatrix}, \\
V &= -2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lambda^4 - 2 \begin{pmatrix} 0 & (\beta-\alpha)p \\ 2q & 0 \end{pmatrix} \lambda^3 + 2(\beta-\alpha) \begin{pmatrix} pq & 0 \\ 0 & -pq \end{pmatrix} \lambda^2 + \\
&\qquad + \begin{pmatrix} 0 & (\beta-\alpha)(\beta p^2 q - p_x) \\ 2(\beta p q^2 + q_x) & 0 \end{pmatrix} \lambda + \\
&+ \frac{\beta-2\alpha}{2(\beta-\alpha)} \begin{pmatrix} \begin{matrix} \alpha(pq_x - p_x q)+ \\ +\alpha(\alpha+\beta)p^2q^2 \end{matrix} & 0 \\ 0 & \begin{matrix} (2\beta-\alpha)(q_x p - q p_x)+ \\ +(2\beta-\alpha)(\alpha+\beta)p^2q^2 \end{matrix} \end{pmatrix}.
\end{aligned}
\tag{2.12}
$$

Proof. The result follows directly from the application of the zero-curvature condition (1.1). □

It is straightforward to verify that the Lax pair in (2.12) reduces to the representations (2.9), (2.10) and (2.11) of equations (A), (B), and (C) under the particular choices of parameters shown above.

Note, however, that the pair (U, V) in (2.12) does not follow the pattern presented in (2.7), which is central in the context of reduction groups and classification. In Section 2.2.3.2 we describe this symmetry as the adjoint-invariance of the Lax representations. Furthermore, (U, V) in (2.12) exhibits a singularity at $\alpha = \beta$, which is absent in the system (2.5) itself. In the following section we address both these problems.

2.1.3.2 Symmetric DNLS equations

We restore the structural symmetry in (2.7) for the Lax pair (2.12) by fixing the free parameters α and β in a specific way.

Definition 2.3 (Symmetric DNLS equations). The symmetric DNLS equations are a reduction of the general DNLS equations (2.5) by imposing $\beta = \alpha + 2$:

$$\begin{cases} p_t = -p_{xx} + 2\alpha p^2 q_x + 2(\alpha+2)pqp_x - \alpha(2-\alpha)p^3q^2, \\ q_t = q_{xx} + 2\alpha q^2 p_x + 2(\alpha+2)pqq_x + \alpha(2-\alpha)p^2q^3. \end{cases} \tag{2.13}$$

A Lax representation of the symmetric DNLS equations is given by

$$\begin{aligned} U ={}& \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lambda^2 + 2 \begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix} \lambda + \frac{2-\alpha}{4} \begin{pmatrix} \alpha pq & 0 \\ 0 & (\alpha+4)pq \end{pmatrix}, \\ V ={}& -2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lambda^4 - 4 \begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix} \lambda^3 + 4 \begin{pmatrix} pq & 0 \\ 0 & -pq \end{pmatrix} \lambda^2 + \\ & + 2 \begin{pmatrix} 0 & (\alpha+2)p^2q - p_x \\ (\alpha+2)pq^2 + q_x & 0 \end{pmatrix} \lambda + \\ & + \frac{2-\alpha}{4} \begin{pmatrix} \alpha(pq_x - p_xq + 2(\alpha+1)p^2q^2) & 0 \\ 0 & (\alpha+4)(q_xp - qp_x + 2(\alpha+1)p^2q^2) \end{pmatrix}. \end{aligned} \tag{2.14}$$

The parametric Lax pair (U, V) in (2.14) follows the pattern (2.7). A related result, presented in Appendix A.2, was given by Smirnov in [112]. Symmetric DNLS equations also emerge naturally in the study of gauge transformations: in Section 2.1.4, we show that the condition $\beta = \alpha + 2$ ensures the compatibility of the partial derivatives of the gauge functions required to map one DNLS equation into another.

The classical DNLS equations (A), (B), and (C) are all special cases of symmetric DNLS equations: (A) corresponds to $\alpha = 2$, (B) to $\alpha = 0$, and (C) to $\alpha = -2$. An analogous construction for the non-commutative case is also introduced in Appendix A.2.

2.1.4 Gauge transformations

It is well known that the DNLS equations are equivalent under gauge transformations (see Section 1.1), meaning that it is possible to map any DNLS equation in (2.5) to another [123]. Gauge transformations have been successfully applied by Kundu [68] to the Kaup-Newell equation (A) to generate (2.5) and, as shown in Sections 2.2.2 and 4.6, this property extends to the non-commutative cases and to Darboux transformations.

We are interested in gauge transformations defined by both a gauge matrix G in (1.4) and a change of variables $p, q \mapsto p', q'$. Our goal is to ensure that a Lax representation (U, V) of a DNLS equation $\mathrm{X}_{\alpha,\beta}(p,q)$ in (2.5) is mapped to a second Lax pair (U', V') corresponding to another DNLS equation $\mathrm{X}_{\alpha',\beta'}(p', q')$ within (2.5).

To deduce the conditions on the gauge transformation, we act directly on (2.5), initially ignoring the matrix G. Specifically, we seek a change of variables under which this system remains covariant after a change of the parameters $\alpha, \beta \mapsto \alpha', \beta'$. Subsequently, we construct the associated Lax pair transformation.

Gauge transformation on the equations

Consider a DNLS equation $\mathrm{X}_{\alpha,\beta}(p,q)$ among (2.5) with fixed parameters α and β. We propose the following change of variables involving a certain function $w(p,q)$:

$$p \mapsto e^{w} p', \qquad q \mapsto e^{-w} q'. \tag{2.15}$$

We require that the transformed system of equations $\mathrm{X}_{\alpha,\beta}(e^{w}p', e^{-w}q')$, expressed in terms of p' and q', consists of $\mathrm{X}_{\alpha',\beta'}(p', q')$ for some new parameters α' and β', which means that it is still in the form (2.5).

Substituting (2.15) into (2.5), we obtain the transformed system of equations $\mathrm{X}_{\alpha,\beta}(e^{w}p', e^{-w}q')$:

$$\begin{cases} p'_t = -p'_{xx} - \alpha p'^2\left(2\left(q'w_x - q'_x\right) + \beta p'q'^2\right) + 2\beta p'q'\left(p'w_x + p'_x\right) + \\ \qquad -p'\left(w_t + w_{xx} + w_x^2\right) - 2p'_x w_x + 2\alpha^2 p'^3 q'^2, \\ q'_t = q'_{xx} + \alpha q'^2\left(2\left(p'w_x + p'_x\right) + \beta p'^2 q'\right) + 2\beta p'q'\left(q'_x - q'w_x\right) + \\ \qquad +q'\left(w_t - w_{xx} + w_x^2\right) - 2q'_x w_x - 2\alpha^2 p'^2 q'^3. \end{cases} \tag{2.16}$$

To ensure that (2.16) is a member of (2.5), we constrain w. Since w_x must have weight 2 and w_t weight 4 to preserve the homogeneity of the system, w itself must be of weight 0. Excluding the constant case, this suggests that w is a non-local function, specifically an integral expression. Given the symmetric role played by p and q in (2.5), we adopt the ansatz:

$$w = \gamma \int p'q'\,dx \qquad \Rightarrow \qquad w_x = \gamma\, p'q' \tag{2.17}$$

for a certain constant γ of weight 0. We substitute w into (2.16) and assume that p'_t and q'_t evolve according with (2.5) for some parameters α' and β'. We obtain two differential equations in w_t:

$$\begin{cases} w_t = \left(2\alpha - \gamma - 2\alpha'\right)p'q'_x + \left(2\beta - 3\gamma - 2\beta'\right)q'p'_x + \\ \qquad -\left(2\alpha^2 - 2\alpha'^2 + \alpha'\beta' - \alpha\beta + \gamma\left(2\beta - 2\alpha - \gamma\right)\right)p'^2q'^2, \\ w_t = \left(3\gamma - 2\beta + 2\beta'\right)p'q'_x + \left(\gamma - 2\alpha + 2\alpha'\right)q'p'_x + \\ \qquad +\left(2\alpha^2 - 2\alpha'^2 + \alpha'\beta' - \alpha\beta + \gamma\left(2\beta - 2\alpha - \gamma\right)\right)p'^2q'^2, \end{cases}$$

whose compatibility determines the constant γ:

$$\left(\alpha - \alpha' + \beta - \beta' - 2\gamma\right)\left(p'q'\right)_x = 0 \quad \Rightarrow \quad \gamma = \frac{1}{2}\left(\alpha + \beta - \alpha' - \beta'\right).$$

With the present value of γ, the derivative w_t becomes

$$\begin{aligned} w_t = {} & \frac{1}{2}(3\alpha' - 3\alpha + \beta - \beta')\,(q'p'_x - p'q'_x) + \\ & + \frac{1}{4}\left(3\alpha^2 + 6\alpha\alpha' - 6\alpha\beta + 6\alpha\beta' - 9\alpha'^2 - 2\alpha'\beta + 2\alpha'\beta' + 3\beta^2 - 2\beta\beta' - \beta'^2\right) p'^2 q'^2. \end{aligned}$$

The consistency of the partial derivatives of w, i.e. $\mathcal{D}_t(w_x) = \mathcal{D}_x(w_t)$, leads to the following equation

$$(\alpha - \alpha' + \beta' - \beta)\left(4\,(p'q'_x - q'p'_x) + (7\alpha' + 3\alpha + \beta' - 3\beta)p'^2q'^2\right)_x = 0. \tag{2.18}$$

Even though other solutions are possible, this condition is satisfied by the symmetric DNLS equations (2.13), where $\beta = \alpha + 2$. Hence, the gauge transformation simplifies:

$$\begin{cases} \gamma = \alpha - \alpha', \\ w_x = (\alpha - \alpha')\; p'q', \\ w_t = (\alpha - \alpha')\,(p'q'_x - p'_x q' + 2(\alpha' + 1)p'^2q'^2)\,. \end{cases} \tag{2.19}$$

This shows that the symmetric DNLS equations are related through a particularly simple and elegant class of gauge transformations.

Gauge transformation on the Lax representation

The transformation presented in the previous paragraph is an automorphism of the system (2.5), but it does not include Lax representations. In this section, we justify its interpretation as a gauge transformation by constructing explicitly the associated gauge matrix G.

Although this is not the most general construction, the symmetric DNLS equations satisfy automatically the condition (2.18), hence we focus directly on these models. Consider the Lax representation (U, V) from (2.14) and apply the transformation (2.15), obtaining a modified Lax pair (U', V'). To ensure the transformed pair defines another symmetric DNLS equation, we seek a gauge transformation that maps (U', V') into a pair of the same form (2.14), but with a new parameter α'.

Let G be a diagonal matrix

$$G = \begin{pmatrix} \mathscr{a} & 0 \\ 0 & \mathscr{b} \end{pmatrix} \tag{2.20}$$

for two gauge functions $\mathscr{a}$ and $\mathscr{b}$ defined analogously to w in (2.15).

Applying the gauge transformation (1.4), the new Lax representation becomes

$$\begin{aligned} U' = {} & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lambda^2 + 2 \begin{pmatrix} 0 & \mathscr{a}^{-1}\mathscr{b}\, e^{w} p' \\ \mathscr{a}\,\mathscr{b}^{-1} e^{-w} q' & 0 \end{pmatrix} \lambda + \frac{2-\alpha}{4} \begin{pmatrix} \alpha p'q' & 0 \\ 0 & (\alpha+4)p'q' \end{pmatrix} + \\ & - \begin{pmatrix} \mathscr{a}^{-1}\mathscr{a}_x & 0 \\ 0 & \mathscr{b}^{-1}\mathscr{b}_x \end{pmatrix}, \\ V' = {} & -2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lambda^4 - 4 \begin{pmatrix} 0 & \mathscr{a}^{-1}\mathscr{b}\, e^{w} p' \\ \mathscr{a}\,\mathscr{b}^{-1} e^{-w} q' & 0 \end{pmatrix} \lambda^3 + 4 \begin{pmatrix} p'q' & 0 \\ 0 & -p'q' \end{pmatrix} \lambda^2 + \\ & + 2(\alpha+2) \begin{pmatrix} 0 & \mathscr{a}^{-1}\mathscr{b}\, e^{w}((1-\gamma)p'^2q' - p'_x) \\ \mathscr{a}\,\mathscr{b}^{-1} e^{-w}((1-\gamma)p'q'^2 + q'_x) & 0 \end{pmatrix} \lambda + \\ & + \frac{2-\alpha}{4} \begin{pmatrix} \alpha(p'q'_x - p'_x q' + 2(\alpha-\gamma+1)p'^2q'^2) & 0 \\ 0 & (\alpha+4)(q'_x p' - q'p'_x + 2(\alpha-\gamma+1)p'^2q'^2) \end{pmatrix} + \\ & - \begin{pmatrix} \mathscr{a}^{-1}\mathscr{a}_t & 0 \\ 0 & \mathscr{b}^{-1}\mathscr{b}_t \end{pmatrix}. \end{aligned}$$

We first consider the transformed matrix U': in order to cast it into the form (2.14), it is necessary to require that

$$\mathscr{a} = e^{w}\,\mathscr{b},$$

from which, together with (2.17), we find

$$\mathscr{a}_x = e^{w}\,\mathscr{b}_x + \gamma p'q' e^{w}\,\mathscr{b}\,.$$

Substituting back, U' simplifies to

$$U' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lambda^2 + 2 \begin{pmatrix} 0 & p' \\ q' & 0 \end{pmatrix} \lambda + \frac{2-\alpha}{4} \begin{pmatrix} \alpha p'q' & 0 \\ 0 & (\alpha+4)p'q' \end{pmatrix} - \begin{pmatrix} \mathscr{b}^{-1}\mathscr{b}_x + \gamma p'q' & 0 \\ 0 & \mathscr{b}^{-1}\mathscr{b}_x \end{pmatrix}.$$

To compare it with the Lax pair (2.14) for a new parameter α', we introduce a constant δ and assume

$$\mathscr{b}^{-1}\mathscr{b}_x = \delta p' q'.$$

Matching the coefficients yields the system:

$$\begin{cases} \alpha^2 - 2\alpha + 4(\gamma + \delta) = \alpha'^2 - 2\alpha', \\ \alpha^2 + 2\alpha + 4\delta = \alpha'^2 + 2\alpha', \end{cases}$$

which is solved by the condition $\gamma = \alpha - \alpha'$ in (2.19), producing

$$\delta = \frac{1}{4}\left(\alpha' - \alpha\right)\left(\alpha' + \alpha + 2\right).$$

Thus, the entries of the gauge matrix G depend on w in (2.17) and are

$$\mathscr{a} = e^{-\frac{1}{4}(\alpha' + \alpha - 2)w}, \qquad \mathscr{b} = e^{-\frac{1}{4}(\alpha' + \alpha + 2)w}.$$

The analogous argument for V' confirms the relations in (2.19). Hence the gauge matrix G is determined by the transformation (2.15).

2.2 The non-commutative DNLS equations

In the last decades, alongside developments in the classification of commutative integrable systems, there was a growing interest in identifying their non-commutative versions. These models are often interpreted as multi-component extensions of commutative ones, where the dependent variables are vector-valued or matrix-valued functions. However, integrable equations can also be defined on more general non-commutative associative algebras. In both cases, it is possible to recover all major features of integrability such as Lax representations, symmetries, and conserved densities.

Recall the definition of the algebra of functions $\mathscr{F}$ given at the beginning of this chapter: from this point onward, we consider $\mathscr{F}$ as non-commutative.

In their paper [102], a cornerstone of the present work, Olver and Sokolov addressed the problem of constructing non-commutative analogues of the DNLS equations. The resulting classification focuses on finding non-commutative models that retain key features of the DNLS equations while admitting a higher-order symmetry, as a provisional criterion for integrability. The authors had already applied the same methodology to the NLS equation in [101].

As already noted, the commutative DNLS equations are homogeneous assigning to the dependent variables p, q weight 1, to $\mathcal{D}_x$ weight 2 and to $\mathcal{D}_t$ weight 4. Olver and Sokolov considered similar non-commutative homogeneous equations of the form

$$\begin{cases} p_t = -p_{xx} + f(p, q, p_x, q_x), \\ q_t = q_{xx} + g(p, q, p_x, q_x), \end{cases}$$

where f and g are homogeneous polynomials of weight 5. From this ansatz, the authors worked with generic functions of 56 parameters and investigated the conditions under which such a system admits a homogeneous higher-order symmetry of weight 9, namely:

$$\begin{cases} p_t = -p_{xxxx} + \hat{f}(p, q, p_x, q_x, p_{xx}, q_{xx}, p_{xxx}, q_{xxx}), \\ q_t = q_{xxxx} + \hat{g}(p, q, p_x, q_x, p_{xx}, q_{xx}, p_{xxx}, q_{xxx}), \end{cases}$$

where also $\hat{f}$ and $\hat{g}$ are homogeneous polynomials of weight 9. Equations (A), (B), and (C), as well as systems (2.5), (2.6) and (2.13) all fit this framework for the commutative case.

To reduce the number of admissible systems, Olver and Sokolov classified equations up to rescaling of the variables, interchange $p \leftrightarrow q$, and the application of an involution $\star$, which generalises the transpose of matrices to associative algebras. We define $\star$ as a linear involution on the algebra $\mathscr{F}$ such that, given two generic elements $f, g \in \mathscr{F}$, the $\star$-involution behaves as

$$(f^\star)^\star = f, \qquad (fg)^\star = g^\star f^\star. \tag{2.21}$$

We assume that the dependent variables p, q and any element $k\,\mathbb{1}$ of the commutative subalgebra $\mathbb{C}$ are $\star$-invariant, i.e.

$$p^\star = p, \qquad q^\star = q, \qquad \mathbb{1}^\star = \mathbb{1}, \qquad (k\,\mathbb{1})^\star = k\,\mathbb{1}, \tag{2.22}$$

where $\mathbb{1}$ is the algebra identity and $k \in \mathbb{C}$. Similarly, we require that $\star$ commutes with the derivation $\mathcal{D}_x$. Hence the invariance is extended to the elements p_x, q_{xx}, etc. The main result of [102] follows.

Theorem 2.4. *Up to a scaling of the variables x, t, p, q, the exchange $p \leftrightarrow q$ and the $\star$-involution defined in (2.21) and (2.22), the following list exhausts the non-commutative DNLS equations admitting a higher-order symmetry of weight 9.*

$$\begin{cases} p_t = -p_{xx} + 4(pqp)_x, \\ q_t = q_{xx} + 4(qpq)_x; \end{cases} \tag{A$_1$}$$

$$\begin{cases} p_t = -p_{xx} + 4(q_xp^2 + qpp_x + pqp_x) + 8(q^2p^3 - pq^2p^2), \\ q_t = q_{xx} + 4(q^2p_x + q_xqp + q_xpq) + 8(q^2p^2q - q^3p^2); \end{cases} \tag{A$_2$}$$

$$\begin{cases} p_t = -p_{xx} + 4pqp_x, \\ q_t = q_{xx} + 4q_xpq; \end{cases} \tag{B$_1$}$$

$$\begin{cases} p_t = -p_{xx} + 4\,(pqp)_x - 4(q_xp^2 + qpp_x) + 8(qp^2qp - 2qpqp^2 + q^2p^3), \\ q_t = q_{xx} + 4\,(qpq)_x - 4(q^2p_x + q_xqp) + 8(2q^2pqp - qpq^2p - q^3p^2); \end{cases} \tag{B$_2$}$$

$$\begin{cases} p_t = -p_{xx} + 4(p^2q_x + p_xpq - pq_xp) + 8(p^3q^2 - p^2qpq - p^2q^2p + pqpqp), \\ q_t = q_{xx} + 4(p_xq^2 + pqq_x - qp_xq) + 8(pqpq^2 + qp^2q^2 - p^2q^3 - qpqpq); \end{cases} \tag{B$_3$}$$

$$\begin{cases} p_t = -p_{xx} - 4pq_xp + 8pqpqp, \\ q_t = q_{xx} - 4qp_xq - 8qpqpq; \end{cases} \tag{C$_1$}$$

$$\begin{cases} p_t = -p_{xx} + 4(pqp_x - p^2q_x - p_xpq) + 8(p^3q^2 - p^2qpq + pqp^2q), \\ q_t = q_{xx} + 4(q_xpq - p_xq^2 - pqq_x) + 8(pqpq^2 - p^2q^3 - pq^2pq); \end{cases} \tag{C$_2$}$$

$$\begin{cases} p_t = -p_{xx} + 4(pqp_x + pq_xp), \\ q_t = q_{xx}; \end{cases} \tag{D$_1$}$$

$$\begin{cases} p_t = -p_{xx} + 4(q_xp^2 + qpp_x) + 8(q^2p^3 - qpqp^2), \\ q_t = q_{xx} + 4(q^2p_x + q_xqp - qp_xq - qpq_x) + \\ \qquad + 8(q^2p^2q + q^2pqp - qpq^2p - q^3p^2). \end{cases} \tag{D$_2$}$$

The notation used above is designed to suggest the commutative version of each equation. Specifically, the equations labelled $\mathrm{A}_i, \mathrm{B}_j$ and C_k are non-commutative lifts of the Kaup-Newell equation (A), the Chen-Lee-Liu equation (B), and the Gerdjikov-Ivanov equation (C). It appears that equation (A_1) was introduced earlier by Fordy [43].

The commutative counterpart D of D_1 and D_2 is a special case of the linearisable system (2.6) with $\alpha = 1$ and $\beta = 0$:

$$\begin{cases} p_t = -p_{xx} + 2pqp_x + 2p^2q_x, \\ q_t = q_{xx}. \end{cases} \tag{D}$$

The D_ℓ equations for $\ell = 1, 2$ are C-integrable [25], i.e. linearisable, and will not be treated here. In [118], the authors took advantage of this property to construct a general solution for both D_ℓ.

In the past, it has been a common belief that the existence of a higher-order symmetry implied the presence of an entire hierarchy of symmetries, and hence it was taken as a quite clear indication of integrability. This conjecture was disproved by Bakirov [16], who introduced a fourth-order equation, with a sixth-order symmetry, but no others up to order 56; see Exercise 5.15 in [100], and [19]. Later, in [17], it was theoretically proved that no additional higher symmetry exists for that equation. While aware of this fact, Olver and Sokolov conjectured that the systems they classified were integrable.

In [118], equations $\mathrm{A}_i, \mathrm{B}_j$ and C_k were proved to be S-integrable [25], which means solvable via the inverse scattering transform. Therefore, we refer to these seven models as the *non-commutative derivative nonlinear Schrödinger equations* (DNLS).

It is remarkable that, in the infinite class of commutative DNLS equations, only three models, (A), (B) and (C), admit non-commutative integrable analogues, while other values of the parameters $\alpha, \beta \in \mathbb{R}$ do not lead to such generalisations. In Section 2.2.2 we examine a possible explanation for this phenomenon as proposed in [118].

2.2.1 Lax representations

Although some were already known [43, 119], Tsuchida and Wadati systematically constructed Lax representations for the matrix-valued DNLS equations, thereby proving their complete integrability [118]. The authors considered the Lax representation of the Chen-Lee-Liu system (B_1), already studied in a previous work [119], and used gauge transformations to derive the Lax representations of all the remaining non-commutative DNLS equations, as discussed in Section 2.2.2.

Following the framework of Olver and Sokolov [102], we consider DNLS equations taking values in a non-commutative associative algebra. Nonetheless, many results from the multi-component case treated in [118] still apply. As in the commutative case (2.7), Lax representations of the non-commutative DNLS equations are of the following form:

$$U = I\lambda^2 + 2J\lambda + 2P, \tag{2.23a}$$

$$V = -2I\lambda^4 - 4J\lambda^3 + 4IJJ\lambda^2 + 2Q\lambda - 2R, \tag{2.23b}$$

where the third Pauli matrix I and J are given by

$$I = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix}. \tag{2.24}$$

As in (2.7), the matrices P, Q and R encode the specific model: P and R are diagonal matrices, while Q is anti-diagonal. The bracket $[\cdot,\cdot]$ denotes the usual matrix commutator. The following theorem provides the necessary conditions for such U and V to form a Lax pair.

Theorem 2.5. *The matrices U and V defined in (2.23) form a Lax representation for the system*

$$\begin{aligned} &J_t + IJ_{xx} + 2I[J,P_x] + 4I[J_x,P] - 4(J^3)_x + 8[P,J^3]+ \\ &\quad + 4I[P,[P,J]] - 2[J,R] = 0, \end{aligned} \tag{2.25}$$

if and only if Q and R in (2.23b) satisfy the conditions

$$Q = 2I[P,J] + 4J^3 - IJ_x, \tag{2.26}$$

$$P_t + R_x = 2[P,R]. \tag{2.27}$$

Proof. Recall that P and R are diagonal matrices, while Q is anti-diagonal. The following commutation and anticommutation relations hold

$$\begin{aligned} [I,A] &= IA - AI = 0, && \text{if } A \text{ is a diagonal matrix ;} \\ \{I,B\} &= IB + BI = 0, && \text{if } B \text{ is an anti-diagonal matrix.} \end{aligned}$$

In addition, $I_t = I_x = 0$ and $I^2 = \mathbb{1}$. Substituting (2.23) into the zero-curvature condition (1.1) and collecting the coefficients of the resulting polynomial in λ, we obtain the following system:

$$\lambda^3: \quad J_x - 4IJ^3 + 2JP - 2PJ + IQ = 0; \tag{2.28a}$$

$$\lambda^2: \quad IJJ_x + IJ_xJ + 2IJ^2P - 2IPJ^2 - JQ + QJ = 0; \tag{2.28b}$$

$$\lambda^1: \quad J_t - Q_x - 2JR + 2RJ + 2PQ - 2QP = 0; \tag{2.28c}$$

$$\lambda^0: \quad P_t + R_x - 2PR + 2RP = 0. \tag{2.28d}$$

Multiplying (2.28a) by I on the left yields the expression for Q given in (2.26). Once this condition is imposed, (2.28b) is also satisfied. Moreover, we derive

$$Q_x = -IJ_{xx} + 2I\left([P, J_x] + [P_x, J]\right) + 4(J^3)_x.$$

Substituting this into (2.28c) yields the equation of motion (2.25). Finally, (2.28d) corresponds to the consistency condition (2.27). □

Remark 2.6. In the commutative case, the relations involving I and J remain valid, with the additional property that diagonal matrices commute. Therefore, Theorem 2.5 remains unchanged apart from (2.27), which simplifies to

$$P_t + R_x = 0.$$

2.2.1.1 List of Lax representations

We list below the matrices P, Q and R in the model (2.23) for each non-commutative DNLS equation. Although Q can be deduced from P and R using (2.26), it is included here for completeness.

The Lax representation of (A_1) is

$$P = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & 4pqp - p_x \\ 4qpq + q_x & 0 \end{pmatrix}, \qquad R = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.29}$$

The Lax representation of (A_2) is

$$\begin{gathered} P = \begin{pmatrix} qp & 0 \\ 0 & qp \end{pmatrix}, \qquad Q = \begin{pmatrix} 0 & 2(pqp + qp^2) - p_x \\ 2(qpq + q^2p) + q_x & 0 \end{pmatrix}, \\ R = \begin{pmatrix} qp_x - q_xp - 2(qp)^2 - 4q^2p^2 & 0 \\ 0 & qp_x - q_xp - 2(qp)^2 - 4q^2p^2 \end{pmatrix}. \end{gathered} \tag{2.30}$$

The Lax representation of (B$_1$) is

$$\begin{gathered}
P=\begin{pmatrix} 0 & 0 \\ 0 & qp \end{pmatrix}, \qquad Q=\begin{pmatrix} 0 & 2pqp-p_x \\ 2qpq+q_x & 0 \end{pmatrix}, \\
R=\begin{pmatrix} 0 & 0 \\ 0 & qp_x-q_xp-2qpqp \end{pmatrix}.
\end{gathered} \tag{2.31}$$

The Lax representation of (B$_2$) is

$$\begin{gathered}
P=\begin{pmatrix} -qp & 0 \\ 0 & 0 \end{pmatrix}, \qquad Q=\begin{pmatrix} 0 & 4pqp-2qp^2-p_x \\ 4qpq-2q^2p+q_x & 0 \end{pmatrix}, \\
R=\begin{pmatrix} q_xp-qp_x+6(qp)^2-4q^2p^2 & 0 \\ 0 & 0 \end{pmatrix}.
\end{gathered} \tag{2.32}$$

The Lax representation of (B$_3$) is

$$\begin{gathered}
P=\begin{pmatrix} -pq & 0 \\ 0 & qp-pq \end{pmatrix}, \qquad Q=\begin{pmatrix} 0 & 2p^2q-p_x \\ 2pq^2+q_x & 0 \end{pmatrix}, \\
R=\begin{pmatrix} pq_x-p_xq+4p^2q^2-2(pq)^2 & 0 \\ 0 & \begin{array}{l} (pq_x+qp_x-p_xq-q_xp)+2(qp)^2+ \\ -2(pq)^2-2pq^2p-2qp^2q+4p^2q^2 \end{array} \end{pmatrix}.
\end{gathered} \tag{2.33}$$

The Lax representation of (C$_1$) is

$$\begin{gathered}
P=\begin{pmatrix} -pq & 0 \\ 0 & qp \end{pmatrix}, \qquad Q=\begin{pmatrix} 0 & -p_x \\ q_x & 0 \end{pmatrix}, \\
R=\begin{pmatrix} pq_x-p_xq-2(pq)^2 & 0 \\ 0 & qp_x-q_xp+2(qp)^2 \end{pmatrix}.
\end{gathered} \tag{2.34}$$

The Lax representation of (C$_2$) is

$$\begin{gathered} P=\begin{pmatrix}0 & 0\\ 0 & qp+pq\end{pmatrix}, \qquad Q=\begin{pmatrix}0 & 2(pqp-p^2q)-p_x\\ 2(qpq-pq^2)+q_x & 0\end{pmatrix}, \\ R=\begin{pmatrix}0 & 0\\ 0 & \begin{matrix}qp_x-pq_x+p_xq-q_xp+4p^2q^2+\\ +2pq^2p+2qp^2q-2(pq)^2-2(qp)^2\end{matrix}\end{pmatrix}. \end{gathered} \tag{2.35}$$

Remark 2.7. The non-commutative version of the (NLS) equation and an associated Lax representation are given by

$$\begin{cases} 2p_t=p_{xx}-8pqp,\\ 2q_t=-q_{xx}+8qpq,\end{cases}$$

$$U=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\lambda+2\begin{pmatrix}0 & p\\ q & 0\end{pmatrix},$$

$$V=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\lambda^2+2\begin{pmatrix}0 & p\\ q & 0\end{pmatrix}\lambda+\begin{pmatrix}-2pq & p_x\\ -q_x & 2qp\end{pmatrix}.$$

The U operator above is equivalent, up to a scaling in λ, to the one in (2.29) for (A$_1$). As already discussed in Remark 2.1, this reveals a deep relation between the hierarchies of the NLS and the DNLS equations. The transformations between corresponding multi-component solutions are studied, among others cited above, in the papers [58, 119].

2.2.2 Gauge transformations

As in the commutative case, non-commutative DNLS equations are related to one another via non-local gauge transformations. These were employed in [119] to connect reductions of (B$_1$) with (NLS). Since we do not have a practical collective system of equations analogous to (2.5), it is convenient to introduce gauge transformations directly at the level of Lax representations.

Tsuchida and Wadati [118] applied this approach to the Chen-Lee-Liu representation (2.31), deriving the Lax pairs presented in Section 2.2.1. We extend their construction to the more general setting of a non-commutative associative algebra.

Let us consider the following invertible 2×2 diagonal matrix G, depending on two undetermined non-commutative functions $\mathpzc{a}, \mathpzc{b}$, analogous to the one introduced in (2.20)

$$G = \begin{pmatrix} \mathpzc{a} & 0 \\ 0 & \mathpzc{b} \end{pmatrix}. \tag{2.36}$$

The gauge transformation with G maps the original pair (U, V) from (2.23) into

$$U' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lambda^2 + 2 \begin{pmatrix} 0 & \mathpzc{a}^{-1}\, p\, \mathpzc{b} \\ \mathpzc{b}^{-1}\, q\, \mathpzc{a} & 0 \end{pmatrix} \lambda + 2G^{-1}PG - G^{-1}G_x, \tag{2.37a}$$

$$\begin{aligned} V' = -2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lambda^4 - 4 \begin{pmatrix} 0 & \mathpzc{a}^{-1}\, p\, \mathpzc{b} \\ \mathpzc{b}^{-1}\, q\, \mathpzc{a} & 0 \end{pmatrix} \lambda^3 + \\ + 4 \begin{pmatrix} \mathpzc{a}^{-1}\, pq\, \mathpzc{a} & 0 \\ 0 & -\mathpzc{b}^{-1}\, qp\, \mathpzc{b} \end{pmatrix} \lambda^2 + 2G^{-1}QG\lambda - 2G^{-1}RG - G^{-1}G_t. \end{aligned} \tag{2.37b}$$

The new Lax pair is of the form (2.23) if we define two new dependent variables:

$$p \mapsto p' = \mathpzc{a}^{-1}\, p\, \mathpzc{b}, \qquad q \mapsto q' = \mathpzc{b}^{-1}\, q\, \mathpzc{a}\,. \tag{2.38}$$

In terms of p', q', the Lax pair (2.37) becomes

$$U' = I\lambda^2 + 2J'\lambda + 2P', \tag{2.39a}$$

$$V' = -2I\lambda^4 - 4J'\lambda^3 + 4IJ'J'\lambda^2 + 2Q'\lambda - 2R', \tag{2.39b}$$

where $J' = J(p', q')$ is similar to J in (2.8) with p', q' in place of p, q, and the matrices P', R' are

$$P' = \begin{pmatrix} \mathpzc{a}^{-1}\, \tilde{P}_{11}\, \mathpzc{a} - \frac{1}{2}\, \mathpzc{a}^{-1}\, \mathpzc{a}_x & 0 \\ 0 & \mathpzc{b}^{-1}\, \tilde{P}_{22}\, \mathpzc{b} - \frac{1}{2}\, \mathpzc{b}^{-1}\, \mathpzc{b}_x \end{pmatrix}, \tag{2.40a}$$

$$R' = \begin{pmatrix} \mathscr{a}^{-1}\tilde{R}_{11}\,\mathscr{a} + \frac{1}{2}\mathscr{a}^{-1}\mathscr{a}_t & 0 \\ 0 & \mathscr{b}^{-1}\tilde{R}_{22}\,\mathscr{b} + \frac{1}{2}\mathscr{b}^{-1}\mathscr{b}_t \end{pmatrix}. \tag{2.40b}$$

By convention, a function with a tilde $\tilde{f}(p', q')$ is defined as the same function $f(p, q)$ composed with the transformation (2.38), i.e. $\tilde{f}(p', q') = f(\mathscr{a}\,p'\,\mathscr{b}^{-1}, \mathscr{b}\,q'\,\mathscr{a}^{-1})$.

Remark 2.8. The transformation $Q' = G^{-1}\tilde{Q}G$ is compatible with Theorem 2.5: if Q satisfies (2.26) for given P and R, then the transformed Q' satisfies it with respect to the transformed P' and R'.

For (U', V') to be a Lax representation, it is necessary that the entries of P', R' are closed polynomial functions in p', q' within $\mathscr{F}$, and this applies also to the terms $\mathscr{a}^{-1}\mathscr{a}_x$, $\mathscr{b}^{-1}\mathscr{b}_x$, $\mathscr{a}^{-1}\mathscr{a}_t$ and $\mathscr{b}^{-1}\mathscr{b}_t$, which must be local, closed, and compatible forms [118]. Compatibility here refers to the commutation of partial derivatives, namely $\mathcal{D}_t(G_x) = \mathcal{D}_x(G_t)$ on the set of solutions.

Considering two DNLS equations, X and X′, with respective Lax representations (U, V) and (U', V'), it is possible to construct a gauge transformation that maps X to X′ by solving the following system for the gauge functions $\mathscr{a}, \mathscr{b}$:

$$\mathscr{a}_x = 2(\tilde{P}_{11}\,\mathscr{a} - \mathscr{a}\,P'_{11}), \qquad \mathscr{b}_x = 2(\tilde{P}_{22}\,\mathscr{b} - \mathscr{b}\,P'_{22}), \tag{2.41a}$$

$$\mathscr{a}_t = 2(\mathscr{a}\,R'_{11} - \tilde{R}_{11}\,\mathscr{a}), \qquad \mathscr{b}_t = 2(\mathscr{b}\,R'_{22} - \tilde{R}_{22}\,\mathscr{b}), \tag{2.41b}$$

assuming that all terms such as $\mathscr{a}^{-1}\tilde{P}_{11}\,\mathscr{a}$ are local functions within $\mathscr{F}$. The explicit integration of (2.41) is not required in either the transformed Lax representations or the equations, since only logarithmic derivatives such as $\mathscr{a}^{-1}\mathscr{a}_x$, $\mathscr{b}^{-1}\mathscr{b}_t$ appear.

Example 2.9. *The locality requirement for a gauge transformation is trivially satisfied if* (A_1) *is involved: consider the gauge transformation from* (A_1) *to* (A_2). *We first focus on the spatial part: from* (2.29) *we know that* $P_{11} = P_{22} = 0$, *while from* (2.30) *we have* $P'_{11} = P'_{22} = q'p'$. *Hence* (2.41a) *reduces to*

$$\mathscr{a}^{-1}\mathscr{a}_x = \mathscr{b}^{-1}\mathscr{b}_x = -2q'p'.$$

In the temporal part, $R_{11} = R_{22} = 0$ *and* $R'_{11} = R'_{22} = q'p'_x - q'_x p' - 2(q'p')^2 - 4q'^2 p'^2$*, therefore* (2.41b) *becomes*

$$\mathpzc{a}^{-1}\mathpzc{a}_t = \mathpzc{b}^{-1}\mathpzc{b}_t = 2q'p'_x - 2q'_x p' - 4(q'p')^2 - 8q'^2 p'^2.$$

It is straightforward to verify that these expressions are compatible when p' *and* q' *solve* (A_2)*. Substituting these results into* (2.39) *reproduces the Lax pair* (2.30)*, and applying* (2.38) *to* (A_1) *yields the equation* (A_2)*.*

The compatibility between the logarithmic derivatives $\mathpzc{a}^{-1}\mathpzc{a}_x$, $\mathpzc{a}^{-1}\mathpzc{a}_t$ etc., motivates why only a few commutative DNLS equations admit non-commutative analogues [118]. Let us assume, for instance, that in (2.36) we set $\mathpzc{b} = 1$ and $\mathpzc{a}^{-1}\mathpzc{a}_x = f(p', q')$, for a certain function $f \in \mathscr{F}$. The differential equation for $\mathpzc{a}$ is solved via the Magnus expansion:

$$\mathpzc{a} = \mathcal{E}\ \exp\left(\int_{x_0}^{x} f(p', q')\, dx'\right) = \mathbb{1} + \sum_{i=1}^{\infty} \mathpzc{a}^{(i)},$$

where $\mathcal{E}$ is the path ordering operator and the functions $\mathpzc{a}^{(n)}$, for $n \in \mathbb{N}$, are defined by

$$\mathpzc{a}^{(n)} = \int_{x_0}^{x} dx_1 \int_{x_0}^{x_1} dx_2 \ldots \int_{x_0}^{x_{n-1}} dx_n\ f(p'(x_1), q'(x_1)) \ldots f(p'(x_n), q'(x_n)).$$

To fully construct the transformed Lax representation, we need to build a second form $\mathpzc{a}^{-1}\mathpzc{a}_t$, compatible with $\mathpzc{a}^{-1}\mathpzc{a}_x$. This is in general not possible within $\mathscr{F}$, unless we substantially constrain f, limiting the possible outcomes of a gauge transformation.

Example 2.10. *Let us consider a gauge matrix* G *in* (2.36) *such that* $\mathpzc{a}^{-1}\mathpzc{a}_x = \gamma p'q'$*, for a constant* $\gamma \neq 0$*, and suppose that it is compatible with a certain* $\mathpzc{a}^{-1}\mathpzc{a}_t = g$*, for* $g \in \mathscr{F}$*. This requires that*

$$g_x = \gamma(p'q')_t + \gamma g p'q' - \gamma p'q' g.$$

The existence of g depends on the evolution of p'_t and q'_t. Consider equation (B$_1$), then

$$g_x = \gamma\left(p'q'_{xx} - p'_{xx}q' + 4\left(p'q'p'_xq' + p'q'_xp'q'\right) - p'q'g + gp'q'\right).$$

Assuming an ansatz of the form

$$g = \gamma(p'q'_x - p'_xq') + \delta p'q'p'q',$$

for a constant $\delta \in \mathbb{R}$. We substitute it into the previous equation and find

$$\left(\gamma^2 + \delta\right)p'q'p'q'_x + \left(\gamma^2 + \delta\right)p'_xq'p'q' + \left(\delta - \gamma(\gamma + 4)\right)\left(p'q'p'_xq' + p'q'_xp'q'\right) = 0.$$

This identity is satisfied if and only if $\gamma = -2$ and $\delta = -4$. Therefore, the only compatible forms are

$$\mathscr{a}^{-1}\mathscr{a}_x = -2p'q', \qquad \mathscr{a}^{-1}\mathscr{a}_t = 2(p'_xq' - p'q'_x) - 4p'q'p'q'.$$

In this case, with $\mathscr{b} = 1$, the matrix G generates the gauge transformation from (C$_1$) in the variables p, q to (B$_1$) in the variables p', q'.

2.2.3 Reduction groups

Examining the Lax representations of integrable equations such as (2.2), (2.7) and (2.23), it is often possible to observe certain regularities and patterns. This is because Lax matrices are typically not arbitrary elements of a general matrix algebra, rather they belong to specific subsets of loop algebras, built on what are now known as reduction groups.

The theory of reduction groups originated with Mikhailov in a series of influential works [85–87] and was formalised in [77], introducing the concept of automorphic Lie algebras. The classifications of reduction groups and the deduction of associated new integrable systems are developed in [22, 78, 108]. Further applications include the construction and classification of soliton solutions found via the inverse scattering transform [48, 49], the discretisation of PDEs following the Lax-Darboux scheme [63, 66, 67] and the deduction of recursion operators from a Lax representation [20, 125]. An example of the latter case will be presented in Section 5.1.1, while the application to Darboux transformations is central to this entire work.

Let us consider the *loop algebra* $\mathfrak{g}[\lambda,\lambda^{-1}]$ as the space of Laurent polynomials in λ over the field $\mathbb{C}$ with coefficients in a Lie algebra $\mathfrak{g}$. Hence, a loop algebra is defined as $\mathfrak{g}[\lambda,\lambda^{-1}]=\mathfrak{g}\otimes\mathbb{C}[\lambda,\lambda^{-1}]$, with Lie bracket

$$[X\otimes\lambda^m, Y\otimes\lambda^n]=[X,Y]\otimes\lambda^{m+n},$$

for any $X,Y\in\mathfrak{g}$ and $m,n\in\mathbb{Z}$. Intuitively, a matrix Lax representation with spectral parameter λ belongs to the loop algebra $\mathfrak{g}[\lambda,\lambda^{-1}]$ for a certain matrix algebra $\mathfrak{g}$. For instance, the Lax representations (2.9) and (2.11) take values in the algebra $\mathfrak{sl}_2[\lambda,\lambda^{-1}]$.

A *reduction group* $\mathscr{G}$ corresponding to $\mathfrak{g}[\lambda,\lambda^{-1}]$ is a finite subgroup of the automorphisms of $\mathfrak{g}[\lambda,\lambda^{-1}]$ that act on both the algebra $\mathfrak{g}$ and the Laurent polynomials in $\mathbb{C}[\lambda,\lambda^{-1}]$. An element $T\in\mathscr{G}$ is represented by a pair (Σ_s,s) where s belongs to a subgroup generated by Möbius transformations of $\mathbb{C}$ and complex conjugation [87], and Σ_s is a representation of s in the group of automorphisms of $\mathfrak{g}[\lambda,\lambda^{-1}]$. The transformation T acts on $X(\lambda)\in\mathfrak{g}[\lambda,\lambda^{-1}]$ as

$$T(X(\lambda))=\Sigma_s(X(s^{-1}(\lambda))).$$

We suppose that Σ_s does not depend on λ,;therefore the reduction group $\mathscr{G}$ can be seen as consisting of a subgroup of Möbius transformations (possibly with the complex conjugation) and an associated subgroup of automorphisms of $\mathfrak{g}$. The related *automorphic Lie algebra* is the Lie subalgebra of $\mathfrak{g}[\lambda,\lambda^{-1}]$ that is invariant under the action of $\mathscr{G}$:

$$\mathfrak{g}^{\mathscr{G}}[\lambda,\lambda^{-1}]=\{X(\lambda)\in\mathfrak{g}[\lambda,\lambda^{-1}]\mid T(X(\lambda))=X(\lambda),\forall T\in\mathscr{G}\}.$$

For a complete and more general construction of a reduction group and its automorphic Lie algebra, consult [22, 87]. In the following two sections, we examine the Lax representations (U, V) of the DNLS equations and identify their associated reduction groups.

2.2.3.1 The transformation $\mathcal{R}$

All the Lax representations presented in Sections 2.1.3 and 2.2.1 belong to the loop algebra $\mathfrak{gl}_2[\lambda, \lambda^{-1}]$ and are invariant under the following reduction group, isomorphic to $\mathbb{Z}_2$.

Lemma 2.11. *The Lax representations of the commutative and non-commutative DNLS equations are invariant under the transformation $\mathcal{R}$:*

$$\mathcal{R}: \qquad U(\lambda) \mapsto IU(-\lambda)I, \qquad V(\lambda) \mapsto IV(-\lambda)I, \tag{2.42}$$

where I is the third Pauli matrix defined in (2.24).

Proof. Consider a diagonal matrix B and an anti-diagonal matrix C, the conjugation by I returns

$$IBI = B, \qquad ICI = -C.$$

Since in a Lax matrix of the form (2.23) even powers of λ appear with diagonal terms, while odd powers appear with anti-diagonal terms, the transformation $\lambda \mapsto -\lambda$ combined with conjugation by I leaves U and V invariant. □

This property was already observed for the commutative Kaup-Newell system (A) in [66].

A sufficient condition for a polynomial matrix $A(\lambda) \in \mathfrak{gl}_2[\lambda, \lambda^{-1}]$ to be invariant under the transformation $\mathcal{R}$ is that it has the following form:

$$A(\lambda) = \sum_{i=k}^{\ell} B^{(i)} \lambda^{2i} + \sum_{j=m}^{n} C^{(j)} \lambda^{2j+1}, \tag{2.43}$$

where $B^{(i)}$ are diagonal matrices, $C^{(j)}$ are anti-diagonal matrices, and the ranges in the sums are defined as $k \leq \ell$ and $m \leq n$ in $\mathbb{Z}$.

2.2.3.2 The adjoint

The Lax representations of non-commutative DNLS equations are also invariant under the adjoint transformation, denoted by the superscript †. Traditionally, it corresponds to the Hermitian conjugate of a matrix A, that is its complex-conjugate transpose $A^\dagger = (A^*)^\tau$. With non-commutative entries, the adjoint operation incorporates the $\star$-involution, already defined in (2.21), which reverses the order of multiplication.

Consider the non-commutative algebra $\mathscr{F}$, where p, q are real functions. We define the adjoint of a matrix A as the composition

$$A^\dagger = \mathcal{C}(A^{\star\tau})$$

of the transposition τ, the $\star$-involution and the operator $\mathcal{C}$, which performs the following substitutions:

$$\begin{gathered}\mathcal{C} = \{p \mapsto q, \quad q \mapsto p, \quad p_x \mapsto -q_x, \quad q_x \mapsto -p_x, \quad p_t \mapsto -q_t, \\ q_t \mapsto -p_t, \quad p_{xx} \mapsto q_{xx}, \quad q_{xx} \mapsto p_{xx}\}.\end{gathered} \tag{2.44}$$

This transformation reflects the traditional definition of the adjoint operator since $\mathcal{C}$ mirrors the action of † on $\mathscr{F}$: the functions p and q correspond to ψ and ψ^* of the complex wave function introduced in Section 2.1.2. The minus signs on odd derivatives originate from the definition of the adjoint of the differential operator $\mathcal{D}_x^\dagger = -\mathcal{D}_x$. Note also that the map $\mathcal{C}$, seen as a change of variables, is compatible with the hypotheses of Theorem 2.4.

Example 2.12. *Let us illustrate the definition of adjoint with a simple example:*

$$A = \begin{pmatrix} pq & p_x \\ qqp & 0 \end{pmatrix}, \qquad A^\dagger = \begin{pmatrix} pq & qpp \\ -q_x & 0 \end{pmatrix}.$$

Note that the lower-left entry qqp becomes qpp and the upper-right entry p_x transforms to $-q_x$.

Remark 2.13. Since their Lax representations are adjoint-invariant, any non-commutative DNLS equation X satisfies the symmetry

$$\mathcal{C}\,\mathrm{X}^\star = \mathrm{X}.$$

Considering the commutative case, the invariance under the adjoint transformation motivates the notion of symmetric DNLS equations, defined by enforcing this symmetry on (2.14). A deeper discussion on the non-commutative case is provided in Appendix A.2.

2.3 A classification of the DNLS Lax representations

At the beginning of Section 2.2, we characterised the DNLS equations as homogeneous PDEs, assigning weight 1 to p and q, weight 2 to $\mathcal{D}_x$ and weight 4 to $\mathcal{D}_t$. Similarly, in Section 2.2.1, it was observed that, assigning weight 1 to λ, all DNLS Lax representations (U, V) of the form (2.23) are homogeneous matrices of weights 2 and 4, respectively.

Since there are only finitely many homogeneous polynomials with a fixed weight, we aim to classify all Lax pairs (U, V) satisfying Theorem 2.5 and these weight conditions, and to show that the corresponding integrable equations coincide with the non-commutative DNLS equations in [102].

2.3.1 Background and main computations

Recall the non-commutative polynomial algebra $\mathscr{F}$ introduced in the beginning of Section 2.2. We denote by $\mathscr{F}^{(r)} \subset \mathscr{F}$ the subspace of homogeneous functions of weight $r \in \mathbb{N}$, thus

$$\mathscr{F} = \bigoplus_{r \geq 0} \mathscr{F}^{(r)}$$

is a graded vector space over $\mathbb{C}$. A basis $\mathscr{B}^{(r)}$ of $\mathscr{F}^{(r)}$ consists of all monomials of weight r in $\mathscr{F}$ modulo rescaling.

If we assume that λ has weight 1, the Lax matrix U in (2.23) is homogeneous of weight 2. Accordingly, the matrix P must have entries in $\mathscr{F}^{(2)}$, written as linear combinations of

$$\mathscr{B}^{(2)} = \{p^2, pq, qp, q^2\}.$$

Likewise, since V is a homogeneous matrix of weight 4, the entries of R belong to $\mathscr{F}^{(4)}$ and are linear combinations of

$$\begin{aligned}\mathscr{B}^{(4)} =&\{p^4, p^3q, p^2qp, pqp^2, qp^3, p^2q^2, pqpq, pq^2p, qp^2q, qpqp, q^2p^2, q^3p, q^2pq,\\ &qpq^2, pq^3, q^4, p_xp, pp_x, p_xq, pq_x, q_xp, qp_x, q_xq, qq_x\}.\end{aligned}$$

Using scalar coefficients $\xi^{(i)}$, $\zeta^{(j)} \in \mathbb{C}$, the general form for the entries of P and R is:

$$P_{11} = \xi^{(1)}p^2 + \xi^{(2)}pq + \xi^{(3)}qp + \xi^{(4)}q^2, \tag{2.45a}$$

$$\begin{aligned} R_{11} = \;&\zeta^{(1)}p^4 + \zeta^{(2)}p^3q + \zeta^{(3)}p^2qp + \zeta^{(4)}pqp^2 + \zeta^{(5)}qp^3 + \zeta^{(6)}p^2q^2 +\\ &+ \zeta^{(7)}pqpq + \zeta^{(8)}pq^2p + \zeta^{(9)}qp^2q + \zeta^{(10)}qpqp + \zeta^{(11)}q^2p^2 +\\ &+ \zeta^{(12)}q^3p + \zeta^{(13)}q^2pq + \zeta^{(14)}qpq^2 + \zeta^{(15)}pq^3 + \zeta^{(16)}q^4 +\\ &+ \zeta^{(17)}p_xp + \zeta^{(18)}pp_x + \zeta^{(19)}p_xq + \zeta^{(20)}pq_x + \zeta^{(21)}q_xp +\\ &+ \zeta^{(22)}qp_x + \zeta^{(23)}q_xq + \zeta^{(24)}qq_x,\end{aligned} \tag{2.45b}$$

and the same holds for P_{22} and R_{22} with coefficients $\xi'^{(i)}$ and $\zeta'^{(j)}$. To determine the value of the coefficients, we substitute the ansätze (2.45) into the zero-curvature condition (1.1). According to Theorem 2.5, this is equivalent to checking (2.27) between P and R, using the evolution equations (2.25) to evaluate P_t. Observing the resulting expression, most of the coefficients in (2.45) vanish. The remaining parameters are $\xi^{(2)}, \xi^{(3)}, \xi'^{(2)}$ and $\xi'^{(3)}$, which must satisfy the system

$$\begin{cases} \xi^{(2)}(\xi^{(2)}+1)=0, \\ \xi'^{(3)}(\xi'^{(3)}-1)=0, \\ \xi^{(3)}\xi'^{(2)}=0, \\ \xi'^{(2)}(1+2\xi^{(2)}-\xi'^{(2)})=0, \\ \xi^{(3)}(1+\xi^{(3)}-2\xi'^{(3)})=0. \end{cases} \tag{2.46}$$

Each solution of (2.46) corresponds to a DNLS equation, as analysed in detail in the following section.

2.3.2 Classification theorem

The set of solutions of system (2.46) determines the admissible Lax representations, as stated in the following theorem.

Theorem 2.14. *Up to rescaling of the variables, exchange $p \leftrightarrow q$ and the involution $\star$ in (2.21), the DNLS equations (A_1)-(C_2) are the only systems admitting a Lax representation of the form (2.23), with P and R given as above.*

Proof. We define a solution of (2.46) as a 4-tuple $\Xi \in \mathbb{C}^4$ ordered as $(\xi^{(2)}, \xi'^{(3)}, \xi'^{(2)}, \xi^{(3)})$. From the first equation in (2.46), we notice that $\xi^{(2)}$ must be either 0 or -1, and $\xi'^{(3)}$ must be 0 or 1. The remaining equations lead to 12 distinct solutions:

(i). $\Xi = (0,0,0,0)$,
(ii). $\Xi = (0,0,0,-1)$,
(iii). $\Xi = (0,0,1,0)$,
(iv). $\Xi = (0,1,0,0)$,
(v). $\Xi = (0,1,0,1)$,
(vi). $\Xi = (0,1,1,0)$,
(vii). $\Xi = (-1,0,0,0)$,
(viii). $\Xi = (-1,0,0,-1)$,
(ix). $\Xi = (-1,0,-1,0)$,
(x). $\Xi = (-1,1,0,0)$,
(xi). $\Xi = (-1,1,0,1)$,
(xii). $\Xi = (-1,1,-1,0)$.

Each of these tuples, interpreted as a Lax representation, leads to a DNLS equation:

(i).	A_1,	(ii).	B_2,
(iii).	$\mathrm{B}_2^\star$,	(iv).	B_1,
(v).	A_2,	(vi).	C_2,
(vii).	$\mathrm{B}_1^\star$,	(viii).	$\mathrm{C}_2^\star$,
(ix).	$\mathrm{A}_2^\star$,	(x).	C_1,
(xi).	$\mathrm{B}_3^\star$,	(xii).	B_3.

Several systems X also appear in their $\star$-versions $\mathrm{X}^\star$. Here, as in [102], we identify $\mathrm{X}^\star \sim \mathrm{X}$, although they have different Lax representations. Systems (A_1) and (C_1) do not have a distinct $\star$-reversed counterpart since they are $\star$-invariant. The obtained Lax representations include those appearing in [118]. □

2.3.2.1 List of Lax representations

In the following list, we examine each solution Ξ individually, presenting its Lax representation and the associated DNLS equation. For brevity, we omit the matrix Q in (2.23) since it is determined by P following (2.26).

Representation (i)

$$\begin{cases} p_t = -p_{xx} + 4(pqp)_x, \\ q_t = q_{xx} + 4(qpq)_x, \end{cases}$$
$$P = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad R = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{A_1}$$

Representation (ii)

$$
\begin{cases}
p_t = -p_{xx} + 4(pqp)_x - 4\left(qpp_x + q_xp^2\right) + 8\left(qp^2qp + \right. \\
\qquad \left. -2qpqp^2 + q^2p^3\right), \\
q_t = q_{xx} + 4(qpq)_x - 4\left(q^2p_x + q_xqp\right) - 8\left(qpq^2p + \right. \\
\qquad \left. -2q^2pqp + q^3p^2\right),
\end{cases}
\tag{B$_2$}
$$

$$
P = \begin{pmatrix} -qp & 0 \\ 0 & 0 \end{pmatrix}, \qquad R = \begin{pmatrix} q_xp - qp_x + 6(qp)^2 - 4q^2p^2 & 0 \\ 0 & 0 \end{pmatrix}.
$$

Representation (iii)

$$
\begin{cases}
p_t = -p_{xx} + 4(pqp)_x - 4\left(p^2q_x + p_xpq\right) + 8\left(pqp^2q + \right. \\
\qquad \left. -2p^2qpq + p^3q^2\right), \\
q_t = q_{xx} + 4(qpq)_x - 4\left(pqq_x + p_xq^2\right) - 8\left(p^2q^3 + \right. \\
\qquad \left. -2pqpq^2 + pq^2pq\right),
\end{cases}
\tag{B$_2^\star$}
$$

$$
P = \begin{pmatrix} 0 & 0 \\ 0 & pq \end{pmatrix}, \qquad R = \begin{pmatrix} 0 & 0 \\ 0 & p_xq - pq_x - 6(pq)^2 + 4p^2q^2 \end{pmatrix}.
$$

Representation (iv)

$$
\begin{cases}
p_t = -p_{xx} + 4pqp_x, \\
q_t = q_{xx} + 4q_xpq,
\end{cases}
\tag{B$_1$}
$$

$$
P = \begin{pmatrix} 0 & 0 \\ 0 & qp \end{pmatrix}, \qquad R = \begin{pmatrix} 0 & 0 \\ 0 & qp_x - q_xp - 2(qp)^2 \end{pmatrix}.
$$

Representation (v)

$$\begin{aligned}
&\begin{cases}
p_t = -p_{xx} + 4\left(pqp_x + qpp_x + q_xp^2\right) + 8\left(q^2p^3 - pq^2p^2\right), \\
q_t = q_{xx} + 4\left(q^2p_x + q_xpq + q_xqp\right) + 8\left(q^2p^2q - q^3p^2\right),
\end{cases} \\
&P = \begin{pmatrix} qp & 0 \\ 0 & qp \end{pmatrix}, \\
&R = \begin{pmatrix} qp_x - q_xp - 2(qp)^2 - 4q^2p^2 & 0 \\ 0 & qp_x - q_xp - 2(qp)^2 - 4q^2p^2 \end{pmatrix}.
\end{aligned} \tag{A$_2$}$$

Representation (vi)

$$\begin{aligned}
&\begin{cases}
p_t = -p_{xx} + 4\left(pqp_x - p^2q_x - p_xpq\right) + 8\left(p^3q^2 - p^2qpq + pqp^2q\right), \\
q_t = q_{xx} + 4\left(q_xpq - p_xq^2 - pqq_x\right) + 8\left(pqpq^2 - pq^2pq - p^2q^3\right),
\end{cases} \\
&P = \begin{pmatrix} 0 & 0 \\ 0 & pq + qp \end{pmatrix}, \\
&R = \begin{pmatrix} 0 & 0 \\ 0 & \begin{matrix} qp_x - pq_x + p_xq - q_xp + 4p^2q^2 + 2\left(pq^2p + qp^2q + \right. \\ \left. -(qp)^2 - (pq)^2\right) \end{matrix} \end{pmatrix}.
\end{aligned} \tag{C$_2$}$$

Representation (vii)

$$\begin{aligned}
&\begin{cases}
p_t = -p_{xx} + 4p_xqp, \\
q_t = q_{xx} + 4qpq_x,
\end{cases} \\
&P = \begin{pmatrix} -pq & 0 \\ 0 & 0 \end{pmatrix}, \qquad R = \begin{pmatrix} pq_x - p_xq + 2(pq)^2 & 0 \\ 0 & 0 \end{pmatrix}.
\end{aligned} \tag{B$_1^\star$}$$

Representation (viii)

$$\begin{aligned}
&\begin{cases}
p_t = -p_{xx} + 4\left(p_x qp - qpp_x - q_x p^2\right) + 8\left(qp^2qp - qpqp^2 + q^2p^3\right), \\
q_t = q_{xx} + 4\left(qpq_x - q^2p_x - q_xqp\right) + 8\left(q^2pqp - qpq^2p - q^3p^2\right),
\end{cases} \\
&P = \begin{pmatrix} -pq - qp & 0 \\ 0 & 0 \end{pmatrix}, \\
&R = \begin{pmatrix} \begin{gathered} pq_x - qp_x - p_xq + q_xp - 4q^2p^2 + 2\left((pq)^2 - pq^2p\right. \\ \left. -qp^2q + (qp)^2\right) \end{gathered} & 0 \\ 0 & 0 \end{pmatrix}.
\end{aligned} \tag{$\mathrm{C}_2^\star$}$$

Representation (ix)

$$\begin{aligned}
&\begin{cases}
p_t = -p_{xx} + 4\left(p^2q_x + p_xpq + p_xqp\right) + 8\left(p^3q^2 - p^2q^2p\right), \\
q_t = q_{xx} + 4\left(pqq_x + qpq_x + p_xq^2\right) + 8\left(qp^2q^2 - p^2q^3\right),
\end{cases} \\
&P = \begin{pmatrix} -pq & 0 \\ 0 & -pq \end{pmatrix}, \\
&R = \begin{pmatrix} pq_x - p_xq + 4p^2q^2 + 2(pq)^2 & 0 \\ 0 & pq_x - p_xq + 4p^2q^2 + 2(pq)^2 \end{pmatrix}.
\end{aligned} \tag{$\mathrm{A}_2^\star$}$$

Representation (x)

$$\begin{aligned}
&\begin{cases}
p_t = -p_{xx} - 4pq_xp + 8pqpqp, \\
q_t = q_{xx} - 4qp_xq - 8qpqpq,
\end{cases} \\
&P = \begin{pmatrix} -pq & 0 \\ 0 & qp \end{pmatrix}, \\
&R = \begin{pmatrix} pq_x - p_xq - 2(pq)^2 & 0 \\ 0 & qp_x - q_xp + 2(qp)^2 \end{pmatrix}.
\end{aligned} \tag{C_1}$$

Representation (xi)

$$
\begin{gathered}
\begin{cases}
\begin{aligned}
p_t = {} & -p_{xx} + 4\left(qpp_x - pq_xp + q_xp^2\right) + 8\left(pqpqp - pq^2p^2 + \right.\\
& \left. -qpqp^2 + q^2p^3\right),\\
q_t = {} & q_{xx} + 4\left(q^2p_x - qp_xq + q_xqp\right) + 8\left(q^2p^2q - qpqpq + \right.\\
& \left. +q^2pqp - q^3p^2\right),
\end{aligned}
\end{cases}\\
P = \begin{pmatrix} qp - pq & 0 \\ 0 & qp \end{pmatrix},\\
R = \begin{pmatrix} \begin{matrix} pq_x + qp_x - p_xq - q_xp + \\ +2pq^2p + 2qp^2q + 2(qp)^2 + \\ -2(pq)^2 - 4q^2p^2 \end{matrix} & 0 \\ 0 & qp_x - q_xp + 2(qp)^2 - 4q^2p^2 \end{pmatrix}.
\end{gathered}
\tag{B$_3^\star$}
$$

Representation (xii)

$$
\begin{gathered}
\begin{cases}
\begin{aligned}
p_t = {} & -p_{xx} + 4\left(p^2q_x - pq_xp + p_xpq\right) + 8\left(p^3q^2 + pqpqp + \right.\\
& \left. -p^2q^2p - p^2qpq\right),\\
q_t = {} & q_{xx} + 4\left(pqq_x - qp_xq + p_xq^2\right) + 8\left(pqpq^2 - p^2q^3 + \right.\\
& \left. + qp^2q^2 - qpqpq\right),
\end{aligned}
\end{cases}\\
P = \begin{pmatrix} -pq & 0 \\ 0 & -pq + qp \end{pmatrix},\\
R = \begin{pmatrix} pq_x - p_xq + 4p^2q^2 - 2(pq)^2 & 0 \\ 0 & \begin{matrix} pq_x + qp_x - p_xq - q_xp + \\ +2(2p^2q^2 - pq^2p - qp^2q + \\ +(qp)^2 - (pq)^2) \end{matrix} \end{pmatrix}.
\end{gathered}
\tag{B$_3$}
$$

Remark 2.15. From Remark 2.13, it is always possible to convert a model X* into the standard form X by either applying the $\star$-involution in (2.21) and (2.22), or applying the change of variables $\mathcal{C}$ in (2.44). Similarly, the Lax representation

(U', V') of X* is obtained from the Lax representation (U, V) of X via the transformation

$$U \mapsto U' = KU^\star K, \qquad V \mapsto V' = KV^\star K,$$

where K is the symplectic matrix

$$K = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Note that, since the equations (A$_1$) and (C$_1$) are invariant under the $\star$-involution, their Lax representations are invariant under the transformation above.

Chapter 3

Darboux transformations

In the 1880s Darboux introduced a method for constructing solutions to certain linear differential equations, given particular solutions of associated equations [34]. As Darboux himself explained [35]:

> *“J’ai montré comment, toutes les fois que l’on saura intégrer l’équation*
>
> $$y'' = y[f(x) + m]$$
>
> *pour toutes les valeurs de la constante m, on pourra obtenir une suite illimitée d’équations, contenant de la même manière un paramètre variable, et dont l’intégration sera possible pour toutes les valeurs du paramètre.”*

3.1 Introduction to Darboux transformations

In his original paper [34], Darboux considers a wider class of second-order differential equations. However, the most influential applications of Darboux’s theorem to the theory of integrable systems [55, 57, 83, 105] concern the so-called *Sturm-Liouville eigenvalue problem*, defined as

$$y'' = [u + \lambda]y, \tag{3.1}$$

where we have adapted Darboux's original notation to align with the modern integrability framework. The parameter m corresponds to the spectral parameter λ of a Lax representation, and in place of the potential function f we write the dependent variable u of the associated PDE. A particular solution of (3.1) is represented by the pair $(y^{(1)}, \lambda_1)$, where $y^{(1)}$ satisfies (3.1) with respect to the parameter λ_1.

Definition 3.1 (Darboux transform). Given a Sturm-Liouville problem (3.1) and a particular solution $(y^{(1)}, \lambda_1)$, the Darboux transform of the dependent variable y is defined as

$$\bar{y} = (\mathcal{D}_x - \upsilon_1)y, \qquad \upsilon_1 = \mathcal{D}_x \ln y^{(1)}. \tag{3.2}$$

The Darboux transform $\bar{y}$ solves a new Sturm-Liouville equation of the form (3.1) associated with a new potential $\bar{u}$.

Theorem 3.2. *The Darboux transform $\bar{y}$ defined in (3.2) satisfies the equation*

$$\bar{y}'' = [\bar{u}(x) + \lambda]\bar{y}, \qquad \bar{u} = u - 2\,\upsilon_{1,x}\,. \tag{3.3}$$

Proof. This result can be verified by direct substitution [83], but a more elegant approach involves the factorisation of the differential operator on the RHS of (3.1), as presented in [57]. Substituting the solution $(y^{(1)}, \lambda_1)$ into the Sturm-Liouville problem (3.1), the potential u is given by

$$u = y^{(1)''}(y^{(1)})^{-1} - \lambda_1 = \upsilon_1' + \upsilon_1^2 - \lambda_1.$$

Substituting this in (3.1), the resulting equation can be factorised into a product of two first-order differential operators

$$(\mathcal{D}_x + \upsilon_1)\,(\mathcal{D}_x - \upsilon_1)\,y = (\lambda - \lambda_1)\,y.$$

Applying $(\mathcal{D}_x - \upsilon_1)$ to both sides, we notice that $\bar{y}$ in (3.2) satisfies a second order differential equation with reversed operator order:

$$(\mathcal{D}_x - \upsilon_1)\,(\mathcal{D}_x + \upsilon_1)\,\bar{y} = (\lambda - \lambda_1)\,\bar{y}.$$

Expanding the composition yields a new Sturm-Liouville problem for $\bar{y}$ with the transformed potential $\bar{u}$ as in (3.3). ☐

It is convenient to denote the Darboux transformation with the operator $\mathcal{S}$, viewed as a map between two Sturm-Liouville problems:

$$\mathcal{S}: \qquad y \mapsto \bar{y}, \qquad u \mapsto \bar{u}.$$

In other words, the Darboux theorem asserts that Sturm-Liouville problems are covariant under Darboux transformations: the operator $\mathcal{S}$ links solutions of (3.1) with transformed potentials.

Example 3.3. *A simple case, presented by Darboux himself [34], arises when the potential is zero, i.e.* $u(x) = 0$*. The Sturm-Liouville problem reduces to* $y'' = \lambda y$*, with a particular solution given by* $y^{(1)} = x$ *for* $\lambda_1 = 0$*. From this, we obtain* $\upsilon_1 = x^{-1}$*, the Darboux transform* $\bar{y}$ *and the corresponding equation are*

$$\bar{y} = y_x - \frac{y}{x}, \qquad \bar{y}'' = \left(\frac{2}{x^2} + \lambda\right)\bar{y}.$$

If we set $\lambda = \kappa^2$ *with* $\kappa \in \mathbb{R}$*, and take* y *of the form* $y(x) = e^{\pm\kappa x}$*, then the transformed solution* $\bar{y}$ *becomes*

$$\bar{y} = \left(\pm\kappa - \frac{1}{x}\right)e^{\pm\kappa x}.$$

Example 3.4. *A slight variation of the previous example, discussed in [83], has a remarkable application to quantum mechanics: the RHS of the Sturm-Liouville equation can be interpreted as a 1-dimensional Schrödinger operator* $\mathcal{D}_x^2 - u(x)$*.*

In the zero-potential case, suppose that $\lambda_1 = -\kappa_1^2$*, with* $\kappa_1 > 0$*. The corresponding equation is* $y^{(1)\prime\prime} - \kappa_1^2 y^{(1)} = 0$*. The exponential solution* $y^{(1)} = e^{\pm\kappa_1 x}$ *yields an identical Darboux transformation.*

Consider now the solution $y^{(1)} = \cosh(\kappa_1 x)$*. The consequent transforms of the potential and the dependent variable are*

$$\bar{u} = -\frac{2\kappa_1^2}{\cosh^2(\kappa_1 x)}, \qquad \bar{y} = (\mathcal{D}_x - \kappa_1 \tanh(\kappa_1 x))\, y.$$

Notice that $\bar{u}$ *corresponds to the one-soliton solution of the Korteweg-de Vries (KdV) equation:*

$$u_t - 6uu_x + u_{xxx} = 0. \tag{KdV}$$

Choosing a second solution $y = e^{\mathrm{i}kx}$ *for a certain number* k *such that* $\lambda = k^2$*, then the transformed variable is*

$$\bar{y} = (\mathrm{i}k - \kappa_1 \tanh(\kappa_1 x))\, e^{\mathrm{i}kx}.$$

In the limit $x \to \pm\infty$*, the corresponding Jost solution exhibits zero reflection coefficient. In fact,* u *is one of the simplest examples of a non-singular, real, reflectionless potential.*

3.1.1 Iteration of Darboux transformations: notation

As Darboux himself remarked in the quotation at the beginning of this chapter, iterating a Darboux transformation produces an infinite sequence of Sturm-Liouville systems associated with the original one. In order to clarify this process, we refine our notation. A Darboux transformation $\mathcal{S}$ maps the potential u to the potential

$\bar{u}$, and the variable y to $\bar{y}$. Iterating this transformation yields:

$$u \underset{\mathcal{S}}{\mapsto} \bar{u} \underset{\mathcal{S}}{\mapsto} \bar{\bar{u}} \underset{\mathcal{S}}{\mapsto} \bar{\bar{\bar{u}}} \underset{\mathcal{S}}{\mapsto} \dots$$

$$y \underset{\mathcal{S}}{\mapsto} \bar{y} \underset{\mathcal{S}}{\mapsto} \bar{\bar{y}} \underset{\mathcal{S}}{\mapsto} \bar{\bar{\bar{y}}} \underset{\mathcal{S}}{\mapsto} \dots$$

It is convenient to denote the n-th iteration by the subscript u_n and y_n, where $u_0 = u$, $u_1 = \bar{u}$, $u_2 = \bar{\bar{u}}$ and so on. By convention, the subscript 0 is often omitted.

$$u \underset{\mathcal{S}}{\mapsto} u_1 \underset{\mathcal{S}}{\mapsto} u_2 \underset{\mathcal{S}}{\mapsto} u_3 \underset{\mathcal{S}}{\mapsto} \dots$$

$$y \underset{\mathcal{S}}{\mapsto} y_1 \underset{\mathcal{S}}{\mapsto} y_2 \underset{\mathcal{S}}{\mapsto} y_3 \underset{\mathcal{S}}{\mapsto} \dots$$

Example 3.5. *The iteration of the transformation in Example 3.3 was already discussed by Darboux in [34], where he showed that it produces an infinite sequence of equations.*

In the case $u = 0$*, we start from the solution* $y^{(1)} = x$ *with* $\lambda = 0$ *and we obtain the transformed potential* $u_1 = 2x^{-2}$*. We apply the same transformation again, using the new solution* $y_1^{(1)} = x^2$ *at* $\lambda = 0$*. This gives:*

$$u_2 = 6x^{-2}, \qquad y_2'' = \left(\frac{6}{x^2} + \lambda\right) y_2.$$

The procedure can be iterated indefinitely. If $\mathcal{S}$ *denotes this Darboux transformation, it can be proved by induction that the sequence of Sturm-Liouville problems generated by* $\mathcal{S}^n$ *starting from* $u = 0$ *takes the form*

$$y_n'' = \left(\frac{n(n+1)}{x^2} + \lambda\right) y_n.$$

This process corresponds to the sequence of solutions $y_n^{(1)} = x^{n+1}$*, each defined for* $\lambda = 0$*. The transformation* $\mathcal{S}$ *acts on the potential* u *as the shift*

$$\mathcal{S}: \qquad u_n \mapsto u_{n+1} = u_n + \frac{2(n+1)}{x^2}.$$

3.1.2 Iteration of Darboux transformations: Crum's theorem

In a broader sense, it is possible to construct the n-fold Darboux transformation, being the n-th iteration of $\mathcal{S}$, directly from the initial problem (3.1), without computing the intermediate steps. For this purpose, we introduce the *Wronskian* W of a set of $n \in \mathbb{N}$ differentiable functions $f_1, f_2, \dots, f_n$, defined as the determinant of the matrix formed by f_i and their derivatives up to order $n-1$. More precisely, for $i, j = 1, \dots, n$, we set

$$A_{ij} = \mathcal{D}_x^{i-1}(f_j), \qquad W(f_1, f_2, \dots, f_n) = \det(A), \tag{3.4}$$

where $\mathcal{D}_x^k(f_j)$ denotes the k-th derivative of f_j with respect to x.

We show that the 2-fold Darboux transformation can be expressed in terms of Wronskians. Consider the transformed Sturm-Liouville problem in (3.3), where the dependent variable is y_1 and let $y_1^{(2)}$ be a solution corresponding to λ_2. Using the Darboux transformation (3.2), it is possible to express $y_1^{(2)}$ in terms of two solutions $y^{(1)}$ and $y^{(2)}$ of the original equation (3.1), i.e. $y_1^{(2)} = (\mathcal{D}_x - v_1)y^{(2)}$. From definition (3.2), this gives

$$y_1^{(2)} = \frac{(y_x^{(2)} y^{(1)} - y_x^{(1)} y^{(2)})}{y^{(1)}} = \frac{W(y^{(1)}, y^{(2)})}{W(y^{(1)})},$$

where we expressed the same fraction in terms of Wronskians (3.4). Similarly, after the second iteration, the potential u_2 becomes

$$u_2 = u_1 - 2\,\mathcal{D}_x^2 \ln y_1^{(2)} = u - 2\,\mathcal{D}_x^2 \ln W(y^{(1)}, y^{(2)}).$$

The transformed dependent variable y_2 can also be expressed using Wronskians: defining $v_2 = \mathcal{D}_x \ln y_1^{(2)}$, we compute

$$y_2 = (\mathcal{D}_x - v_2)\,(\mathcal{D}_x - v_1)\,y = \frac{W(y^{(1)}, y^{(2)}, y)}{W(y^{(1)}, y^{(2)})}.$$

Such construction is a special case of a more general result due to Crum [33], who extended Theorem 3.2 to an arbitrary iteration of the same transform in Definition 3.1. In fact, Crum showed that the n-th transform of a Sturm-Liouville problem can be written in terms of Wronskians involving n particular solutions $y^{(1)}, y^{(2)}, \ldots, y^{(n)}$ of the original problem with parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$, without involving intermediate transformed functions.

Theorem 3.6 **(Crum's theorem).** *Let* $y^{(1)}, \ldots, y^{(n)}$ *be solutions of the Sturm-Liouville problem* (3.1)*, corresponding to the parameters* $\lambda_1, \ldots, \lambda_n$*, respectively. The function*

$$y_n = \frac{W(y^{(1)}, \ldots, y^{(n)}, y)}{W(y^{(1)}, \ldots, y^{(n)})} \tag{3.5}$$

is the n*-iterated Darboux transform of* y *and it satisfies the transformed Sturm-Liouville equation*

$$y_n'' = (u_n(x) + \lambda) y_n, \tag{3.6}$$

associated with the potential

$$u_n = u - 2\,\mathcal{D}_x^2 \ln W(y^{(1)}, \ldots, y^{(n)}). \tag{3.7}$$

Both Darboux's original theorem and the example discussed at the beginning of this section are special cases of Crum's theorem.

Proof. Let y be a solution of the Sturm-Liouville problem (3.1). Motivated by the structure of y_1 and y_2, we assume that the function y_n, obtained from an n-fold Darboux transformation, satisfies a transformed Sturm-Liouville problem still of the form (3.1). We suppose that y_n can be written as $y_n = \mathcal{X}^{(n)}(y)$ through a differential operator $\mathcal{X}^{(n)}$ of order n and leading coefficient 1:

$$\mathcal{X}^{(n)} = \sum_{i=0}^{n} \mathscr{d}^{(i)}\, \mathcal{D}_x^i, \qquad \mathscr{d}^{(n)} = 1.$$

The coefficients $\mathscr{d}^{(k)}$ depend on n solutions $y^{(1)}, \ldots, y^{(n)}$ of the original Sturm-Liouville problem associated with the parameters $\lambda_1, \ldots, \lambda_n$, respectively. We require that the expression above vanishes when y is considered among the given $y^{(k)}$. In other words,

$$\mathcal{X}^{(n)}(y)\Big|_{y=y^{(k)}} = 0, \qquad \text{for } k = 1, \dots, n.$$

Substituting, this ansatz yields a linear system in the coefficients $\mathscr{d}^{(i)}$:

$$\sum_{i=1}^{n} \mathscr{d}^{(i-1)}\, \mathcal{D}_x^{i-1}(y^{(j)}) = -\,\mathcal{D}_x^{n}(y^{(j)}),$$

for $j = 1, \dots, n$. Notice that this system can be written compactly in the matrix form $X^\tau A = Y^\tau$, where A is the matrix in (3.4), Y^τ is the row vector $Y^j = \mathcal{D}_x^n(y_j)$, and X^τ is the row vector $X^i = \mathscr{d}^{(i)}$. By Cramer's rule the solution is

$$\mathscr{d}^{(i)} = \frac{\det(A^{(i)})}{\det(A)} = -\frac{W^{(i)}(y^{(1)}, \dots, y^{(n)})}{W(y^{(1)}, \dots, y^{(n)})},$$

where $A^{\tau(i)}$ is the matrix obtained from the transpose of A by replacing the i-th column with the column vector Y, and $W^{(i)}$ is its Wronskian. In this way we determine all the coefficients $\mathscr{d}^{(i)}$.

Notably, due to the properties of the Wronskian, the coefficient $\mathscr{d}^{(n-1)}$ simplifies

$$\mathscr{d}^{(n-1)} = -\frac{W^{(n-1)}(y^{(1)}, \dots, y^{(n)})}{W(y^{(1)}, \dots, y^{(n)})} = -\,\mathcal{D}_x \ln W(y^{(1)}, \dots, y^{(n)}). \tag{3.8}$$

We now substitute $y_n = \mathcal{X}^{(n)}(y)$ into the transformed Sturm-Liouville problem (3.6). Since y satisfies the original equation (3.1), the new potential u_n is completely determined by the coefficient $\mathscr{d}^{(n-1)}$:

$$u_n = u + 2\,\mathcal{D}_x\, \mathscr{d}^{(n-1)} = u - 2\,\mathcal{D}_x^2 \ln W(y^{(1)}, \dots, y^{(n)}),$$

which matches (3.7). Finally, we verify directly that the constructed y_n (3.5) satisfies the transformed equation (3.6), completing the proof [83]. □

After the fundamental results of Darboux and Crum, which revealed the covariance of certain linear problems under Darboux transformations and the fundamental role of Wronskians, many generalisations were proposed. In 1979, Matveev published two papers extending the Darboux framework from Sturm-Liouville problems (3.1) to both more general evolutionary linear partial differential equations (PDEs) [81] and differential-difference equations (DΔEs) [82] of the form

$$\mathcal{D}_t\, y = \sum_{n=0}^{m} u^{(n)}(x,t)\, \mathcal{D}_x^n\, y, \qquad \mathcal{D}_t\, y_m = \sum_{n=k_1}^{k_2} u^{(n)}(m,t) y_{n+m}.$$

Here the coefficients $u^{(n)}$ may be scalars or matrices, and $m \in \mathbb{Z}$. These generalisations have been proved effective in constructing solutions for equations such as the Kadomtsev-Petviashvili equation [61], which can be recast as a Darboux covariant system of PDEs [38, 132]. In a similar way, other authors tried to adapt Crum's theorem beyond Sturm-Liouville-like problems: Wahlquist [124] found a multi-soliton formula for KdV based on Wronskians, and both his and Crum's results are consistent with the general formula in [81]. Moreover, Wronskians are often a fundamental ingredient in obtaining τ-functions and so to generate exact solutions to integrable equations such as the KP, the two-dimensional Toda and the Davey-Stewartson hierarchies [64].

3.2 Darboux transformations and integrable systems

At first glance, it might seem surprising how useful and widely studied Darboux transformations are in integrability theory: integrable systems are often associated with nonlinear dynamics, whereas Darboux transformations act on linear equations.

The key connection lies in the presence of a Lax pair. As introduced in Section 1.1, a Lax representation of an integrable model introduces an auxiliary system, on which the Darboux transformations apply.

Example 3.7. *A fundamental example is provided by the* (KdV) *equation, which admits the Lax pair*

$$L = \mathcal{D}_x^2 - u, \qquad A = -4\,\mathcal{D}_x^3 + 6u\,\mathcal{D}_x + 3u_x, \tag{3.9}$$

with respect to the Lax equation $L_t = [A, L]$*. For an introduction to this traditional form of Lax representations, see [39]. Here* L *is a Schrödinger operator and the associated spectral problem,* $L\Phi = \lambda\Phi$*, is a Sturm-Liouville equation* (3.1)*.*

More generally, as discussed in Chapter 1, consider an integrable PDE with dependent variable u: let the associated Lax representation be (U, V) and the corresponding auxiliary system be (1.2), with fundamental solution Φ. Darboux transformations are typically constructed by acting on the spatial part of (1.2):

$$\Phi_x = U(u, \lambda)\Phi.$$

Comparing this with a Sturm-Liouville equation (3.1), the matrix U plays the role of the potential function u, including the spectral parameter λ, while Φ corresponds to the dependent variable y. A Darboux transformation $\mathcal{S}$, as defined in (3.2), maps $\Phi \mapsto \bar{\Phi}$, $U \mapsto \bar{U}$, and, similarly, $V \mapsto \bar{V}$. We require that $(\bar{U}, \bar{V})$ forms a new Lax representation for the same PDE. More precisely, we assume that $\bar{U}$ and $\bar{V}$ have the same matrix structure as U and V, with $\bar{U} = U(\bar{u}; \lambda)$ and $\bar{V} = V(\bar{u}; \lambda)$, where $\bar{u}$ is another solution of the original PDE.

3.2.1 Introduction to Darboux transformations of PDEs

In the paragraph above, we outlined the relation between Darboux transformations and integrable systems. We now provide a more formal definition [63].

Definition 3.8 (**Darboux transformation**)**.** Given the Lax representation (U, V) of an integrable PDE, a Darboux transformation is an invertible map $\mathcal{S}$, acting linearly on the fundamental solution Φ such that

$$\mathcal{S} : \Phi \mapsto \bar{\Phi} = M\Phi, \qquad \det(M) \neq 0, \tag{3.10}$$

where $\bar{\Phi}$ is the fundamental solution of the transformed auxiliary system

$$\begin{cases} \bar{\Phi}_x = U(\bar{u};\lambda)\bar{\Phi}, \\ \bar{\Phi}_t = V(\bar{u};\lambda)\bar{\Phi}. \end{cases} \tag{3.11}$$

Here $\bar{u}$ is a new solution of the original PDE.

It is customary to write the transformed Lax pair $(U(\bar{u};\lambda), V(\bar{u};\lambda))$ simply as $(\bar{U}, \bar{V})$. The matrix M, called the *Darboux matrix*, is a rational (or sometimes elliptic) function of the spectral parameter λ. It depends on both u and $\bar{u}$, and may also involve auxiliary functions or parameters. By convention, we write $M = M(\bar{u}, u)$, with the new solution $\bar{u}$ listed first.

Substituting the definition of Darboux transformation (3.10) into the transformed auxiliary system (3.11), we obtain two compatibility equations known as the *Lax-Darboux equations*:

$$M_x = \bar{U}M - MU, \tag{3.12a}$$

$$M_t = \bar{V}M - MV. \tag{3.12b}$$

In the literature, other versions of these equations appear depending on the formalism adopted. For instance, assume that the Lax pair is represented by two operators (L, A) [39], where

$$L = \mathcal{D}_x - U, \qquad A = \mathcal{D}_t - V. \tag{3.13}$$

In this case, the zero-curvature condition corresponds to the commutator

$$[L, A] = LA - AL = 0. \tag{3.14}$$

The Darboux transformation of the Lax representation (L, A) with respect to the Darboux matrix M consists of a matrix conjugation:

$$\bar{L} = MLM^{-1}, \qquad \bar{A} = MAM^{-1}. \tag{3.15}$$

Equations (3.12) follow directly from (3.15). This operator formulation is particularly useful for proving the following lemma.

Lemma 3.9. *A Darboux transformation $\mathcal{S}$ maps a Lax representation (L, A) into a new Lax representation $(\bar{L}, \bar{A})$ of the same integrable PDE.*

Proof. If the pair (L, A) in (3.13) satisfies the zero-curvature condition (3.14), then also the transformed pair $(\bar{L}, \bar{A})$ in (3.15) does. ☐

Remark 3.10. Besides the Lax representation (U, V) of an integrable PDE, consider also the Lax representations $(U, V^{(k)})$ of all the members of its hierarchy of symmetries for $k \in \mathbb{N}$. All the previous results remain unchanged via the generalisation $V \mapsto V^{(k)}$, $A \mapsto A^{(k)}$ and $\mathcal{D}_t \mapsto \mathcal{D}_{t^{(k)}}$, for the independent variables $t^{(k)} \in \mathbb{R}$.

Remark 3.11. Darboux transformations (3.10) and (3.12) are in some sense analogous to the gauge transformations (1.3) and (1.4): both cases consist of endomorphisms of Lax representations. However, a Darboux transformation maps a fundamental solution Φ of (U, V) into a second fundamental solution $\bar{\Phi}$ of $(\bar{U}, \bar{V})$, associated with two solutions u and $\bar{u}$ of the same PDE. A gauge transformation either transforms the Lax pair only (Section 1.1), maintaining the equation and the solution fixed, or, when associated with a change of variables, transforms a solution of a given equation into a solution of a different equation (see Sections 2.1.4 and 2.2.2). This double interpretation of gauge transformations is discussed further in Remark 4.36. Moreover, in the present work, Darboux matrices M depend on λ, while gauge matrices G do not.

Example 3.12. *A non-commutative generalisation of the nonlinear Schrödinger equation* (NLS) *is presented in Remark 2.7. A Darboux matrix for this system is*

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \lambda + \begin{pmatrix} h & p \\ \bar{q} & 0 \end{pmatrix}, \tag{3.16}$$

where h is a non-commutative auxiliary function. Substituting it into the Lax-Darboux equation (3.12a)*, we obtain:*

$$p_x = -2hp, \qquad \bar{q}_x = 2\bar{q}h, \qquad h_x = 2(\bar{p}\bar{q} - pq).$$

After simplification, the Darboux transformation yields:

$$\bar{p} = pqp - \frac{1}{4}\mathcal{D}_x\left(p^{-1}p_x\right), \qquad \bar{q} = p^{-1}.$$

We will see in Example 3.16 that this equation corresponds to the Toda lattice [117]. The analogous derivation for the commutative case appears in [63], other Darboux matrices for the NLS equation are discussed in Examples 3.14 and 3.16 [7, 31].

In the commutative setting, the determinant $\det(M)$ of a Darboux matrix M plays a significant role in characterising the properties of the transformation $\mathcal{S}$. The following lemma can be interpreted as a consequence of the classical *Jacobi formula* [31].

Lemma 3.13. *If a Lax representation (U, V) is traceless, then the determinant* $\det(M)$ *of an associated Darboux matrix M is a constant of motion.*

Proof. Consider the spatial part $\bar{\Phi}_x = \bar{U}\bar{\Phi}$ of the transformed auxiliary system (3.11), the Jacobi formula implies that

$$\mathcal{D}_x(\det(\bar{\Phi})) = \operatorname{tr}(\bar{U})\det(\bar{\Phi}).$$

By the definition of Darboux transformation (3.10) and the properties of the determinant, we obtain

$$\det(\Phi)\,\mathcal{D}_x(\det(M)) + \det(M)\,\mathcal{D}_x(\det(\Phi)) = \operatorname{tr}(\bar{U})\det(M)\det(\Phi).$$

Applying the Jacobi formula again, the previous expression is equivalent to

$$\det(M^{-1})\,\mathcal{D}_x(\det(M)) = \operatorname{tr}(\bar{U}) - \operatorname{tr}(U). \tag{3.17}$$

Therefore, if $\operatorname{tr}(U) = 0$, then $\operatorname{tr}(\bar{U}) = 0$ as well, and hence $\mathcal{D}_x(\det(M))$ vanishes. The same argument applies to the temporal part. □

Example 3.14. *Consider the commutative* (NLS) *equation with its Lax representation* (2.2). *Besides the commutative version of the Darboux matrix* (3.16) *from Example 3.12, the* (NLS) *equation also admits the following Darboux matrix:*

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \lambda + \begin{pmatrix} h & p \\ \bar{q} & 1 \end{pmatrix}, \tag{3.18}$$

where h is an auxiliary function. Substituting this into the Lax-Darboux equation (3.12a), *we obtain the system*

$$\begin{cases} h_x = 2(\bar{p}\bar{q} - pq), \\ p_x = 2(\bar{p} - hp), \\ \bar{q}_x = 2(h\bar{q} - q). \end{cases} \tag{3.19}$$

The determinant of the Darboux matrix is

$$\det(M) = \lambda + h - p\bar{q}.$$

Since the Lax representation (2.2) *is traceless, Lemma 3.13 implies that* $\det(M)$ *is constant in x. Thus*

$$\rho = h - p\bar{q}$$

is a constant of motion. By the equations in (3.19)*, it is straightforward to verify that* $\rho_x = 0$*. This allows us to fix* $h = p\bar{q} + \rho$*, which reduces* (3.19) *to*

$$\begin{cases} p_x = 2(\bar{p} - p^2\bar{q} - \rho p), \\ \bar{q}_x = 2(-q + p\bar{q}^2 + \rho\bar{q}). \end{cases}$$

Setting $\rho = 0$ *and shifting the second equation by* $\mathcal{S}^{-1}$ *yields the well-known Merola-Ragnisco-Tu system [84, 134]*

$$\begin{cases} p_x = 2(\bar{p} - p^2\bar{q}), \\ q_x = 2(-\underline{q} + \underline{p}q^2), \end{cases}$$

where $\underline{p} = \mathcal{S}^{-1}(p)$ *denotes the inverse shift. Such a reduction based on constants of motion will be employed repeatedly throughout Chapter 4.*

Remark 3.15. It is important to note that the result above relies on the commutativity of the algebra. In the non-commutative setting, as in Example 3.12, an analogous determinant is not defined. Nevertheless, it is possible to recognise similar constants of motion. This observation motivates the introduction of alternative notions of determinant adapted to non-commutative algebras, such as the quasi-determinants, presented in Section 4.5.2.1, which are central in the factorisation of Darboux matrices in Section 4.5.2.

3.2.2 Darboux transformations as DΔEs

In the previous section, we introduced Darboux transformations in their original sense: as a method for generating solutions of new associated equations. This approach, introduced by Darboux himself, has found many applications in the theory of integrable systems as the dressing method: in principle it enables the construction of sequences of solutions (usually solitonic solutions) of a nonlinear integrable equation starting from a known, even trivial, one, see for instance Example 3.12.

However, in the present work, we focus on a second perspective: we interpret Darboux transformations as providing an integrable discretisation of integrable PDEs that produces corresponding integrable DΔEs. The goal is not to construct explicitly a sequence of solutions, but rather to study the equations (the DΔEs above) that connect its members. We have already encountered an application in Example 3.14.

3.2.2.1 Difference algebra

The discretisation of a PDE into a DΔE is based on the iteration of a Darboux transformation that induces a lattice structure on the generated solutions. We refer to the notation introduced in Section 3.1.1 for a Sturm-Liouville problem.

Let $\mathcal{S}$ be a Darboux transformation and $\mathcal{S}^{-1}$ its inverse. The iteration $\mathcal{S}^m$ on a fundamental solution Φ is denoted by Φ_m, and the iteration $\mathcal{S}^{-n}$ by Φ_{-n}, with $m, n \in \mathbb{N}$. The map $\mathcal{S}$ also acts on the variables u involved in the Darboux transformation:

$$\ldots \underset{\mathcal{S}}{\mapsto} \Phi_{-1} \underset{\mathcal{S}}{\mapsto} \Phi \underset{\mathcal{S}}{\mapsto} \Phi_1 \underset{\mathcal{S}}{\mapsto} \Phi_2 \underset{\mathcal{S}}{\mapsto} \ldots$$

$$\ldots \underset{\mathcal{S}}{\mapsto} u_{-1} \underset{\mathcal{S}}{\mapsto} u \underset{\mathcal{S}}{\mapsto} u_1 \underset{\mathcal{S}}{\mapsto} u_2 \underset{\mathcal{S}}{\mapsto} \ldots$$

It is natural to interpret Φ_n and u_n as functions or dependent variables that depend on a discrete variable n in $\mathbb{Z}$. When the index n is zero we tend to omit it.

Recall from Section 1.1 the general definition of the associative algebra $\mathscr{F}$ of polynomial functions in u and their x-derivatives. We introduce the associative algebra $\mathscr{A}$ consisting of non-commutative polynomial functions in finitely many shifts of the variable u_n with $n \in \mathbb{Z}$.

The Darboux transformation $\mathcal{S}$ is an automorphism of $\mathscr{A}$, increasing the subscript indices of u_n by 1. For a generic function $f(u_m, \ldots, u_n) \in \mathscr{A}$, the shift acts as

$$\mathcal{S} : f(u_m, \ldots, u_n) \mapsto f(u_{m+1}, \ldots, u_{n+1}).$$

Any numerical constant $\alpha \in \mathbb{C}$ is shift-invariant, i.e. $\mathcal{S}(\alpha) = \alpha$. In a more practical notation, we define the composition of $\mathcal{S}$ applied k times as

$$f_k = \mathcal{S}^k(f(u_m, \dots, u_n)) = f(u_{m+k}, \dots, u_{n+k}), \qquad k \in \mathbb{Z}.$$

Similar considerations apply to Lax matrices, where $\mathcal{S}$ acts on their entries.

In this sense, $\mathscr{A}$ has the structure of a *difference algebra* with the Darboux transformation $\mathcal{S}$ acting as a *shift operator*. The iterated action of $\mathcal{S}$ defines the lattice $\mathbb{Z}$, where each vertex corresponds to the independent variable n and each directed edge $n \to n+1$ represents a Darboux transformation $\mathcal{S}$ with Darboux matrix $M_n = M(u_{n+1}, u_n)$.

Besides the shift operator $\mathcal{S}$, the algebra $\mathscr{A}$ admits another automorphism represented by the *reflection operator*:

$$\mathcal{T} : f_k \mapsto f_{-k}, \qquad \mathcal{T} : \alpha \mapsto \alpha, \tag{3.20}$$

for any $f \in \mathscr{A}$ and $\alpha \in \mathbb{C}$. This is an involution on $\mathscr{A}$, i.e. $\mathcal{T}^2 = \mathrm{Id}$. Together, shift and reflection operators generate an infinite dihedral group $\mathbb{D}_\infty$, satisfying the dihedral relation $\mathcal{T}\mathcal{S}\mathcal{T} = \mathcal{S}^{-1}$. The operator $\mathcal{S}$ generates its normal subgroup of translations [27].

In the context of integrable systems, difference algebras are the usual framework to study DΔEs: a rigorous account of their properties is presented in [27], together with derivations and rational operators. The papers [28, 29] are devoted to commutative and non-commutative Hamiltonian structures on DΔEs, while [59, 80] present a modern construction of the variational complex based on $\mathscr{A}$, introduced in [69]. For a more theoretical introduction to difference algebras consult the book [75].

3.2.2.2 Lax-Darboux representations

Consider an integrable PDE with Lax representation (U, V) as in (1.2). We combine the spatial part of the associated auxiliary system with the Darboux transformation in (3.2), obtaining a new auxiliary system:

$$\begin{cases} \mathcal{S}(\Phi) = M\Phi, \\ \Phi_x = U\Phi, \end{cases} \tag{3.21}$$

whose compatibility condition is the Lax-Darboux equation (3.12a)

$$M_x = \mathcal{S}(U)M - MU. \tag{3.22}$$

The equation (3.22) is the zero-curvature condition of the *Lax-Darboux pair* (M, U) and it defines a system $\mathscr{E}$ of DΔEs. We refer to $\mathscr{E}$ as the *Darboux system*, which consists of an auto-Bäcklund transformation of the given equation. It typically involves the dependent variables u and $\bar{u}$, along with possible auxiliary functions. In most cases, $\mathscr{E}$ is redundant or at least it can be reduced using constants of motion, see (3.19) in Example 3.14, or through appropriate changes of variables.

A Darboux transformation maps an integrable PDE with Lax representation (U, V) into an integrable DΔE with Lax-Darboux representation (M, U). Analogously, the dependent variable $u(x, t)$ in $\mathscr{F}$, associated with the PDE, is replaced by the dependent variable $u_n(t)$ in $\mathscr{A}$, associated with the DΔE. In this sense, Darboux transformations act as integrable discretisations.

Example 3.16. *In Example 3.12 we presented the Lax representation of the non-commutative (NLS) equation along with its Darboux matrix M. The system $\mathscr{E}$ in the notation from Section 3.2.2.1 reduces to*

$$p_{xx} = 4pp_{-1}^{-1}p - 4p_1 + p_x p^{-1} p_x. \tag{3.23}$$

We define a non-commutative version of the Manakov-Flaschka coordinates [41, 79]: let $u = -4p_1p^{-1}$ and $v = p_xp^{-1}$, the equation (3.23) is equivalent to the *well-known non-commutative Toda equation [29, 117]:*

$$\begin{cases} u_x = v_1 u - uv, \\ v_x = u - u_{-1}. \end{cases}$$

A similar analysis for the commutative case is carried out in [63]. The corresponding Lax pair is

$$M = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \lambda - \frac{1}{4} \begin{pmatrix} 2v & -4 \\ u & 0 \end{pmatrix}, \qquad U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lambda + \frac{1}{2} \begin{pmatrix} 0 & 4 \\ -u_{-1} & 2v \end{pmatrix}.$$

3.2.2.3 Gauge invariance

As explained in Section 1.1 for the Lax representations (U, V) of PDEs, the Lax-Darboux representations (M, U) for DΔEs are not unique: they are subject to gauge freedom. Let us revisit the same concept in (1.4) applied to the auxiliary system (3.21): suppose the fundamental solution Φ is related to another fundamental solution Φ' by $\Phi = G\Phi'$, where G is an invertible *gauge matrix*. We assume that G is independent of the spectral parameter λ. Then the transformed fundamental solution Φ' satisfies

$$\begin{cases} \mathcal{S}(\Phi') = M'\Phi', \\ \Phi'_x = U'\Phi', \end{cases}$$

and the transformed Lax representation (M', U') is given by

$$M \mapsto M' = G_1^{-1} M G, \qquad U \mapsto U' = G^{-1} U G - G^{-1} G_x. \tag{3.24}$$

It is straightforward to verify that (M', U') provides a Lax-Darboux representation of the same Darboux system $\mathscr{E}$ as (M, U). This gauge freedom will be employed in Chapter 4 to eliminate redundant variables from Lax representations and to identify equivalences among them.

Furthermore, the Lax-Darboux equations (3.22) are linear with respect to the Darboux matrix M. Consequently, it is always possible to rescale M by any non-zero scalar function $\zeta(\lambda)$ of the spectral parameter, i.e.

$$M \mapsto M' = \zeta(\lambda) M. \tag{3.25}$$

3.2.2.4 Iteration of Darboux transformations

Assuming a Darboux transformation $\mathcal{S}$ with Darboux matrix M, the following lemma describes the Darboux matrix associated with $\mathcal{S}^2$.

Lemma 3.17. *Given a Darboux transformation $\mathcal{S}$ with Darboux matrix $M(u_1, u)$, the Darboux matrix associated with $\mathcal{S}^2$ is*

$$\mathcal{S}^2 : \Phi \mapsto \Phi_2 = \mathcal{S}(M) M \Phi,$$

where Φ is a fundamental solution and $\mathcal{S}(M) = M(u_2, u_1)$.

Proof. The Darboux transformation $\mathcal{S}$ maps the fundamental solutions $\Phi \mapsto \Phi_1$, hence the subsequent application maps $\Phi_1 \mapsto \Phi_2$, so that the iteration $\mathcal{S}^2$, seen as a composition, maps $\Phi \mapsto \Phi_2$. Substituting one expression into the other, we obtain the statement. $\square$

It is straightforward to verify that $\mathcal{S}(M)M$ satisfies the Lax-Darboux equation (3.22) with respect to the shift $\mathcal{S}^2$.

We generalise the result of Lemma 3.17 by defining the Darboux matrix corresponding to the n-th iteration $\mathcal{S}^n$ for any $n \in \mathbb{N}$:

$$\mathcal{S}^n : \Phi \mapsto \Phi_n = \prod_{i=0}^{\overset{\curvearrowleft}{n-1}} \mathcal{S}^i(M) \Phi, \tag{3.26}$$

where $\mathcal{S}^0$ denotes the identity transformation. The arrow on the product indicates the multiplication order: since it points to the left, the factors i from 0 to $n-1$ are multiplied on the left.

3.2.2.5 Composition of Darboux transformations

Let us now consider two distinct Darboux transformations $\mathcal{S}_1$ and $\mathcal{S}_2$, associated with the Darboux matrices $M^{(1)}$ and $M^{(2)}$ respectively. Recalling the alternative bar-notation introduced in (3.2), we denote the shifts $\mathcal{S}_1$ and $\mathcal{S}_2$ of a fundamental solution Φ as

$$\mathcal{S}_1 : \Phi \mapsto \bar{\Phi} = M^{(1)}\Phi, \qquad \mathcal{S}_2 : \Phi \mapsto \tilde{\Phi} = M^{(2)}\Phi. \tag{3.27}$$

In general, Darboux transformations do not commute, i.e. $[\mathcal{S}_1, \mathcal{S}_2] \neq 0$. It is therefore imperative to distinguish the order of application in $\bar{\tilde{\Phi}} = \mathcal{S}_1\mathcal{S}_2(\Phi)$ from $\tilde{\bar{\Phi}} = \mathcal{S}_2\mathcal{S}_1(\Phi)$.

For this reason, the difference algebra $\mathscr{A}$ needs to be modified: the generators p and q now admit two independent shifts $\mathcal{S}_1$ and $\mathcal{S}_2$, that are often required to be commutative. Such an extra condition, known as Bianchi permutability [18, 57], is a fundamental ingredient for the Lax-Darboux scheme mentioned in Chapter 1 and will also be briefly discussed in Sections 3.2.2.6 and 5.2. In this part, we do not impose such an additional requirement and we maintain the notation in (3.27).

The Darboux matrix of the composition of two distinct Darboux transformations is given by the next lemma.

Lemma 3.18. *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be the two Darboux transformations defined in (3.27). The Darboux matrix corresponding to the composition $\mathcal{S}_1\mathcal{S}_2$ is*

$$\mathcal{S}_1\mathcal{S}_2 : \Phi \mapsto \bar{\tilde{\Phi}} = M^{(1)}(\bar{\tilde{u}}, \tilde{u})M^{(2)}(\tilde{u}, u)\Phi. \tag{3.28}$$

Proof. As in Lemma 3.17, we apply the two transformations $\mathcal{S}_1$ and $\mathcal{S}_2$ separately:

$$\Phi \underset{\mathcal{S}_2}{\longmapsto} \tilde{\Phi} = M^{(2)}(\tilde{u}, u)\Phi, \qquad \tilde{\Phi} \underset{\mathcal{S}_1}{\longmapsto} \bar{\tilde{\Phi}} = M^{(1)}(\bar{\tilde{u}}, \tilde{u})\, \tilde{\Phi}.$$

Substituting one into the other yields

$$\bar{\tilde{\Phi}} = M^{(1)}(\bar{\tilde{u}}, \tilde{u})\, M^{(2)}(\tilde{u}, u)\, \Phi.$$

$\square$

Similarly, the composition in the reversed order $\mathcal{S}_2 \mathcal{S}_1$ gives

$$\mathcal{S}_2 \mathcal{S}_1 : \Phi \mapsto \tilde{\bar{\Phi}} = M^{(2)}(\tilde{\bar{u}}, \bar{u}) M^{(1)}(\bar{u}, u)\Phi. \tag{3.29}$$

It is straightforward to verify that the composed Darboux matrices above satisfy the Lax-Darboux equations (3.22) with respect to the corresponding shifts. Hence, any composition of Darboux transformations yields a new Darboux transformation for the same Lax representation.

The present bar-tilde notation quickly becomes cumbersome when multiple transformations are involved. We have already seen in (3.26) a more compact notation for iterated Darboux transformations; we now present a general tool to manage also the composition: we introduce the *inner shift* $\iota_{\mathcal{S}}$ of a Darboux transformation $\mathcal{S}$ as

$$\iota_{\mathcal{S}} : \mathscr{S}(u) \mapsto \mathscr{S}(\mathcal{S}(u)), \tag{3.30}$$

where $\mathscr{S}$ denotes any shift operator. The inner shift $\iota_{\mathcal{S}}$ acts on any shift operator $\mathscr{S}$ by composition on the right, while, when applied to an un-shifted quantity, e.g. Φ, it behaves as the standard shift $\mathcal{S}$, i.e. $\iota_{\mathcal{S}}(\Phi) = \mathcal{S}(\Phi)$. Thus it formalises the cumulative action of the preceding transformations.

With this notation, Lemma 3.18 can be rewritten more elegantly:

$$\mathcal{S}_1 \mathcal{S}_2 : \Phi \mapsto \bar{\tilde{\Phi}} = \iota_{\mathcal{S}_2}(M^{(1)}) M^{(2)} \Phi.$$

In the special case where $\mathcal{S}_1 = \mathcal{S}_2$, this formula reduces to Lemma 3.17.

More generally, let $\mathscr{S} = \mathcal{S}_n \mathcal{S}_{n-1} \dots \mathcal{S}_1$ be a composition of n Darboux transformations $\mathcal{S}_i$ with associated Darboux matrices $M^{(i)}$, for $i = 1, \dots, n$. The resulting Darboux transformation is represented by

$$\mathscr{S} : \Phi \mapsto \mathscr{S}(\Phi) = \prod_{i=0}^{\overset{\curvearrowleft}{n-1}} \iota_{\mathcal{S}_i \mathcal{S}_{i-1} \dots \mathcal{S}_1}(M^{(i+1)})\Phi. \tag{3.31}$$

If all shifts $\mathcal{S}_i$ coincide, this expression reduces to (3.26).

Remark 3.19. When constructing the Darboux matrix associated with a composition of shifts, the notation ensures consistency. In all cases, e.g. (3.28) and (3.29), adjacent Darboux matrices share intermediate variables and the outermost arguments correspond to the overall image and pre-image of the composed transformation. For instance, the Darboux matrix R corresponding to $\mathcal{S}_1 \mathcal{S}_2 \mathcal{S}_1$ in (3.27) is given by:

$$R(\bar{\tilde{\bar{u}}}, u) = \iota_{\mathcal{S}_2 \mathcal{S}_1}(M^{(1)}) \iota_{\mathcal{S}_1}(M^{(2)}) M^{(1)} = M^{(1)}(\bar{\tilde{\bar{u}}}, \tilde{\bar{u}}) M^{(2)}(\tilde{\bar{u}}, \bar{u}) M^{(1)}(\bar{u}, u).$$

3.2.2.6 Bianchi permutability

We assume the commutation of the Darboux transformations in (3.27):

$$[\mathcal{S}_1, \mathcal{S}_2] = \mathcal{S}_1 \mathcal{S}_2 - \mathcal{S}_2 \mathcal{S}_1 = 0.$$

This property, known as *Bianchi permutability*, was introduced by Bianchi in [18] and was later widely discussed in books such as [57]. It ensures the consistency of Darboux transformations on a two-dimensional lattice $\mathbb{Z}^2$; more information follows in Section 5.2.

The commutativity condition applied to the associated Darboux matrices reads

$$\iota_{\mathcal{S}_2}(M^{(1)}) M^{(2)} - \iota_{\mathcal{S}_1}(M^{(2)}) M^{(1)} = 0.$$

Since the shifts commute, the inner shift operator $\iota_{\mathcal{S}}$ reduces to the corresponding standard shift $\mathcal{S}$, so the expression simplifies to the classical form of Bianchi permutability:

$$\mathcal{S}_2(M^{(1)})M^{(2)} - \mathcal{S}_1(M^{(2)})M^{(1)} = 0. \tag{3.32}$$

In this setting, the general composition rule (3.31) also simplifies, as all inner shifts are replaced by their corresponding standard shifts. Therefore the composition $\mathscr{S} = \mathcal{S}_n\,\mathcal{S}_{n-1}\ldots\mathcal{S}_1$ in (3.31) becomes

$$\mathscr{S} : \Phi \mapsto \mathscr{S}(\Phi) = \prod_{i=0}^{\overset{\curvearrowleft}{n-1}} \mathcal{S}_i\,\mathcal{S}_{i-1}\ldots\mathcal{S}_1(M^{(i+1)})\Phi,$$

where, as before in (3.26), the arrow pointing to the left denotes that the product is ordered from right to left.

3.2.2.7 Inverse Darboux transformations

Each Darboux transformation $\mathcal{S}$,with matrix M,admits an inverse transformation $\mathcal{S}^{-1}$, which induces negative shifts on the lattice. The Darboux matrix associated with $\mathcal{S}^{-1}$, denoted as the *Darboux inverse matrix* M^I, satisfies

$$\mathcal{S}^{-1} : \Phi \mapsto \Phi_{-1} = M^I\Phi, \qquad \mathcal{S}\,\mathcal{S}^{-1} = \mathcal{S}^{-1}\,\mathcal{S} = \mathrm{Id}, \tag{3.33}$$

where Id represents the identity transformation. The term "Darboux inverse matrix" emphasises that M^I is not simply the matrix inverse of M, but the inverse with respect to the composition of Darboux transformations. In particular, compatibly with Remark 3.19, the matrix M^I depends on u_{-1} and u, i.e. $M^I = M^I(u_{-1}, u)$. We deduce the explicit form of M^I from its action on the auxiliary system:

$$\mathcal{S}^{-1}\,\mathcal{S}\,\Phi = \mathcal{S}^{-1}\,M(u_1, u)\Phi = M^I_1(u, u_1)M(u_1, u)\Phi.$$

Since the composition above must yield the identity transformation, we define

$$M^I = (M_{-1})^{-1}. \tag{3.34}$$

Lemma 3.20. *Both a Darboux transformation $\mathcal{S}$ with matrix M and its inverse $\mathcal{S}^{-1}$ with matrix M^I are associated with the same Darboux system $\mathscr{E}$.*

Proof. Suppose that M satisfies the Lax-Darboux equation (3.22) for a Lax representation (U, V), and $\mathscr{E}$ is the corresponding Darboux system. The inverse matrix M^{-1} satisfies

$$\mathcal{D}_x M^{-1} = UM^{-1} - M^{-1}\mathcal{S}(U).$$

Applying $\mathcal{S}^{-1}$ to both sides yields an equation equivalent to (3.22) for M^I, with $\mathcal{S}^{-1}$ replacing $\mathcal{S}$:

$$M^I_x = \mathcal{S}^{-1}(U)M^I - M^I U.$$

Therefore, the Darboux matrix M^I is associated with the same system $\mathscr{E}$. □

Finally, the Darboux transformation $\mathcal{S}$ and its inverse $\mathcal{S}^{-1}$ commute by construction, i.e. $[\mathcal{S}, \mathcal{S}^{-1}] = 0$: Bianchi permutability (3.32) with (3.34) yields

$$\mathcal{S}(M^I)M - \mathcal{S}^{-1}(M)M^I = \mathcal{S}(M^{-1}_{-1})M - \mathcal{S}^{-1}(M)M^{-1}_{-1} = 0.$$

Chapter 4

Polynomial Darboux transformations of the non-commutative DNLS equations

4.1 Polynomial Darboux transformations

As introduced in Section 3.2, a Darboux transformation is characterised by three objects: a shift $\mathcal{S}$, a Darboux matrix M and a set of equations $\mathscr{E}$. The shift $\mathcal{S}$ is an invertible map acting on dependent variables and auxiliary functions, see Section 3.2.2.1. The Darboux matrix M is an invertible matrix representing the action of $\mathcal{S}$ on a fundamental solution, i.e. $\mathcal{S}(\Phi) = M\Phi$, see Section 3.2.1. Together with a Lax representation (U, V), M satisfies the Lax-Darboux equation, already introduced in (3.22):

$$M_x = \mathcal{S}(U)M - MU. \tag{4.1}$$

The corresponding set of DΔEs is the Darboux system $\mathscr{E}$, which describes the dynamics of the dependent variables and the auxiliary functions involved in M.

In the present work, we consider the Lax representation of the non-commutative DNLS equations, as presented between (2.29) and (2.35), or in their most general form in (2.23). We start by considering generic Darboux matrices, polynomial in the spectral parameter λ, and we refine the ansatz.

Definition 4.1 (Polynomial Darboux transformation). A polynomial Darboux transformation is a Darboux transformation $\mathcal{S}$ associated with a Darboux matrix M of the form

$$M=\sum_{i=0}^{n}M^{(i)}\lambda^i,\qquad n\in\mathbb{N}. \tag{4.2}$$

We denote the entries of M by $M_{ij}^{(k)}$ where k indexes the power of λ, and i,j specify the position of the element in the matrix $M^{(k)}$. The number n is the *order* of the polynomial Darboux transformation and the *rank* is the matrix rank of the leading coefficient $M^{(n)}$, i.e. the coefficient of the highest power of λ.

By equation (3.25), it is always possible to rescale a generic Darboux matrix in a loop algebra $\mathfrak{g}[\lambda,\lambda^{-1}]$, e.g.

$$M=\sum_{i=a}^{b}M^{(i)}\lambda^i,\qquad a\leq b\in\mathbb{Z},$$

into the form (4.2) by multiplying it by λ^{-a}. From this point forward, we adopt (4.2) as the standard definition of a polynomial Darboux matrix. For further discussion on this subject, see [31].

The following lemma defines the necessary conditions for a matrix M as (4.2) to be a Darboux matrix with respect to the Lax pair in (2.23).

Lemma 4.2. *A polynomial Darboux matrix of degree $n>0$ for the DNLS equations is of the form*

$$\begin{aligned}M=&\begin{pmatrix}M_{11}^{(n)} & 0\\ 0 & M_{22}^{(n)}\end{pmatrix}\lambda^n+\begin{pmatrix}M_{11}^{(n-1)} & M_{11}^{(n)}p-p_1M_{22}^{(n)}\\ q_1M_{11}^{(n)}-M_{22}^{(n)}q & M_{22}^{(n-1)}\end{pmatrix}\lambda^{n-1}+\\&+\begin{pmatrix}M_{11}^{(n-2)} & M_{11}^{(n-1)}p-p_1M_{22}^{(n-1)}\\ q_1M_{11}^{(n-1)}-M_{22}^{(n-1)}q & M_{22}^{(n-2)}\end{pmatrix}\lambda^{n-2}+\sum_{i=0}^{n-3}M^{(i)}\lambda^i.\end{aligned}\tag{4.3}$$

The entries $M_{ij}^{(k)}$ are *auxiliary functions* of the Darboux transformation.

Proof. We substitute M and U from (2.23) into the zero-curvature condition (4.1). From the resulting matrix equation the coefficients λ^{n+2}, λ^{n+1} and λ^{n} yield the following algebraic relations (i.e. without derivatives):

$$
\begin{aligned}
&M_{12}^{(n)} = 0, && M_{21}^{(n)} = 0,\\
&M_{12}^{(n-1)} = M_{11}^{(n)}p - p_1 M_{22}^{(n)}, && M_{21}^{(n-1)} = q_1 M_{11}^{(n)} - M_{22}^{(n)}q,\\
&M_{12}^{(n-2)} = M_{11}^{(n-1)}p - p_1 M_{22}^{(n-1)}, && M_{21}^{(n-2)} = q_1 M_{11}^{(n-1)} - M_{22}^{(n-1)}q.
\end{aligned}
$$

These conditions applied directly to (4.2) prove the claim. $\square$

Let $\mathscr{E}$ be the Darboux system related to M in (4.3), the DΔEs associated with the dependent variables p and q consist of two sets of non-evolutionary equations:

$$
\begin{aligned}
&M_{11}^{(n)}p_x - p_{1,x}M_{22}^{(n)} = \mathscr{f}(p,q,M_{ij}^{(k)}), && M_{11}^{(n-1)}p_x - p_{1,x}M_{22}^{(n-1)} = \mathscr{f}'(p,q,M_{ij}^{(\ell)}),\\
&q_{1,x}M_{11}^{(n)} - M_{22}^{(n)}q_x = \mathscr{g}(p,q,M_{ij}^{(k)}), && q_{1,x}M_{11}^{(n-1)} - M_{22}^{(n-1)}q_x = \mathscr{g}'(p,q,M_{ij}^{(\ell)}),
\end{aligned}
$$

where $\mathscr{f}, \mathscr{g}, \mathscr{f}', \mathscr{g}'$ are polynomial functions of p, q, and of the auxiliary functions $M_{ij}^{(k)}$ and $M_{ij}^{(\ell)}$, with $k \in [0,n]$ and $\ell \in [0, n-1]$. These equations are obtained considering the coefficients λ^{n-1} and λ^{n-2} in (4.1).

Reading the systems vertically reveals two distinct subsystems: one based on $M^{(n)}$ (right pair), and the other on $M^{(n-1)}$ (left pair). On the other hand, reading these systems horizontally suggests two potentially distinct dynamics for each of the dependent variables p and q. This raises the issue of compatibility, which we do not address at this stage. These insights will be developed further in Remark 4.5.

4.1.1 Reduction group-invariant polynomials

In Section 2.2.3.1, we observed that the Lax representations of the non-commutative DNLS equations are invariant under the action of the reduction group generated by the transformation $\mathcal{R}$. Thus, we assume that the Darboux matrix M also possesses the same invariance in order to preserve the auxiliary system (3.21).

Instead of requiring the auxiliary system itself to be invariant under $\mathcal{R}$, we impose that the Lax-Darboux equation (4.1) remains invariant when U transforms by $\mathcal{R}$. This defines two automorphisms on the loop algebra $\mathfrak{gl}_2[\lambda, \lambda^{-1}]$:

$$\mathcal{R}_{\pm} : M(\lambda) \mapsto \pm I M(-\lambda) I, \tag{4.4}$$

where $I = \text{diag}(1, -1)$ as defined in (2.24). The Lax representation (U, V) in (2.23) is invariant under $\mathcal{R}_+$, which corresponds to $\mathcal{R}$ in (2.42), while a new transformation $\mathcal{R}_-$ appears.

Lemma 4.3. *A polynomial Darboux matrix for the DNLS equations of degree $n > 0$, invariant under the transformations $\mathcal{R}_{\pm}$, is of the form*

$$\begin{aligned} M(n) = &\begin{pmatrix} M_{11}^{(n)} & 0 \\ 0 & M_{22}^{(n)} \end{pmatrix} \lambda^n + \begin{pmatrix} 0 & M_{11}^{(n)} p - p_1 M_{22}^{(n)} \\ q_1 M_{11}^{(n)} - M_{22}^{(n)} q & 0 \end{pmatrix} \lambda^{n-1} + \\ &+ \begin{pmatrix} M_{11}^{(n-2)} & 0 \\ 0 & M_{22}^{(n-2)} \end{pmatrix} \lambda^{n-2} + \sum_{i=0}^{n-3} M^{(i)} \lambda^i, \end{aligned} \tag{4.5}$$

where $M^{(n-k)}$ is diagonal when k is even, and anti-diagonal when k is odd. We denote by $\mathscr{E}(n)$ the system of equations associated with $M(n)$.

Proof. The result is proven by combining the structural conditions from Lemma 4.2 with the invariance under (4.4). From Lemma 4.2, a necessary condition is that the leading term of M in (4.3) is diagonal. If M has even degree, the invariance under (4.4) is satisfied by $\mathcal{R}_+$. As seen in (2.43), the structure of a $\mathcal{R}_+$-invariant matrix requires diagonal coefficients for even powers of λ and anti-diagonal coefficients for odd powers. Similarly, when its degree is odd, M is invariant under $\mathcal{R}_-$, and thus has diagonal coefficients for odd powers of λ and anti-diagonal ones for even powers. In both cases, Lemma 4.2 fixes the leading term and the second term of $M(n)$. The remaining coefficients alternate between diagonal and anti-diagonal matrices and consist of auxiliary functions. $\square$

We notice that, by accepting only $\mathcal{R}_+$ as reduction group to maintain consistency with the Lax representation (U, V), we would automatically exclude all Darboux matrices of odd degree. The current definition allows a broader class of Darboux transformations.

Remark 4.4. Some references, e.g. [66], consider $\mathcal{R}_+$ as the only reduction group for M. Consequently, only even-degree Darboux matrices $M(2n)$ are admissible, while odd-degree ones are interpreted as degenerate cases with vanishing $M^{(0)}$ coefficient.

Compared with the general case in Lemma 4.2, the evolution of the dependent variables p and q in $\mathscr{E}(n)$ consists of a single pair of non-evolutionary DΔEs of the form

$$\begin{cases} M_{11}^{(n)} p_x - p_{1,x} M_{22}^{(n)} = \mathscr{f}(p, q, M_{ij}^{(k)}), \\ q_{1,x} M_{11}^{(n)} - M_{22}^{(n)} q_x = \mathscr{g}(p, q, M_{ij}^{(k)}), \end{cases} \tag{4.6}$$

where $\mathscr{f}$ and $\mathscr{g}$ are polynomial functions of p, q, and $M_{ij}^{(k)}$, with $k \in [0, n]$. Non-evolutionary DΔEs are considered in [90], where they originate from Darboux transformations applied to a generalisation of the periodic Volterra equation.

Remark 4.5. The system $\mathscr{E}$ associated with the general Darboux matrix M of degree n in (4.3) can be regarded as a union of the systems $\mathscr{E}(n)$ and $\mathscr{E}(n-1)$. This is justified by the fact that M can be expressed as the sum of two $\mathcal{R}_\pm$-invariant Darboux matrices $M(n)$ and $M(n-1)$. However, this property is specific to Darboux matrices: it does not extend to Darboux transformations as a whole, since no analogous decomposition exists at the level of shifts.

4.1.2 Rank-1 polynomials

Let us consider a $\mathcal{R}_\pm$-invariant polynomial Darboux transformation $M(n)$ as in (4.5). Its leading term $M^{(n)}$ is a 2×2 diagonal matrix, whose rank defines the rank of the Darboux transformation.

Definition 4.6 (**Up and down Darboux transformations**)**.** A $\mathcal{R}_\pm$-invariant polynomial Darboux transformation $M(n)$ of degree $n \in \mathbb{N}$ is said to be of type "up", denoted by $M_{\uparrow}(n)$, if it has rank-1 and the null eigenvalue of the leading term is $M_{22}^{(n)}$. Similarly, a Darboux transformation is of type "down", denoted by $M_{\downarrow}(n)$, if it has rank-1 and the null eigenvalue is $M_{11}^{(n)}$. The associated Darboux systems are $\mathscr{E}_{\uparrow}(n)$ and $\mathscr{E}_{\downarrow}(n)$, respectively.

Using (4.5) as a model, we introduce the first examples of rank-1 up Darboux matrices of degree 1 and 2:

$$M_{\uparrow}(1) = \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} \lambda + \begin{pmatrix} 0 & fp \\ q_1 f & 0 \end{pmatrix}, \tag{4.7a}$$

$$M_{\uparrow}(2) = \begin{pmatrix} h & 0 \\ 0 & 0 \end{pmatrix} \lambda^2 + \begin{pmatrix} 0 & hp \\ q_1 h & 0 \end{pmatrix} \lambda + \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \tag{4.7b}$$

where f, h, a and b are auxiliary functions. The corresponding down matrices are

$$M_{\downarrow}(1) = \begin{pmatrix} 0 & 0 \\ 0 & g \end{pmatrix} \lambda - \begin{pmatrix} 0 & p_1 g \\ gq & 0 \end{pmatrix}, \tag{4.8a}$$

$$M_{\downarrow}(2) = \begin{pmatrix} 0 & 0 \\ 0 & \ell \end{pmatrix} \lambda^2 - \begin{pmatrix} 0 & p_1 \ell \\ \ell q & 0 \end{pmatrix} \lambda + \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}, \tag{4.8b}$$

where g, ℓ, c and d are auxiliary functions. As noted in Remark 4.4, some authors regard the matrices $M_{\uparrow}(1)$ and $M_{\downarrow}(1)$ as degenerate forms of $M_{\uparrow}(2)$ and $M_{\downarrow}(2)$ obtained by setting the auxiliary functions a, b and c, d to zero, respectively.

The Darboux systems $\mathscr{E}_{\uparrow}(n)$ and $\mathscr{E}_{\downarrow}(n)$ include evolutionary differential-difference equations in the dependent variables p and q. Specifically:

$$\left\{ \begin{aligned} p_x &= \mathscr{f}(p, q, M_{ij}^{(k)}), \\ q_{1,x} &= \mathscr{g}(p, q, M_{ij}^{(k)}), \end{aligned} \right. \in \mathscr{E}_{\uparrow}(n); \qquad \left\{ \begin{aligned} p_{1,x} &= \mathscr{h}(p, q, M_{ij}^{(k)}), \\ q_x &= \mathscr{k}(p, q, M_{ij}^{(k)}), \end{aligned} \right. \in \mathscr{E}_{\downarrow}(n).$$

Here, $\mathscr{f}, \mathscr{g}, \mathscr{h}$ and $\mathscr{k}$ are polynomial functions. A large part of the present work is devoted to constructing up and down Darboux matrices and analysing the corresponding systems. Similar results on rank-1 Darboux transformations are discussed in [66], especially about the commutative (NLS) equation and the Kaup-Newell equation (A).

Example 4.7. *In certain contexts, rank-1 Darboux transformations are referred to as elementary matrices, since broad classes of Darboux transformations can be constructed via a composition of them [72, 96]. The Darboux transformations introduced in the Examples 3.12 and 3.14 are rank-1 polynomial Darboux transformations of degree 1 and are two of the three elementary Darboux transformations for the* (NLS) *equation [63]. The remaining one represents a trivial, rescaling transformation:*

$$M = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}. \tag{4.9}$$

4.1.3 Reductions

In most cases, the Darboux system $\mathscr{E}$ involves not only the dependent variables, but also the auxiliary functions appearing in the Darboux matrix M. It is rare to eliminate or fully determine all them, more often they act as additional degrees of freedom within the transformations. However, in many cases, it is possible to reduce their number, simplifying $\mathscr{E}$.

We define as a *reduction* of $\mathscr{E}$ any process that decreases the number of variables in it. For our purposes, it is important to distinguish between two types of reductions:

- a reduction is called *fundamental* if the system $\mathscr{E}$ is redundant and some equations can be eliminated without loss of generality.
- a reduction is called *optional* if the system is not redundant and such simplification consists of an admissible choice rather than a necessity.

Fundamental reductions are the most interesting, as they reveal the structure of the system. However, determining whether $\mathscr{E}$ admits one can be difficult. To the best of our knowledge, finding fundamental reductions is largely a matter of intuition.

Optional reductions arise when auxiliary functions (or parameters) are fixed in a way compatible with the Darboux system $\mathscr{E}$, without being required by any necessary condition. This typically means setting such functions to zero. While they simplify the system and reduce the number of equations, they are chosen at the discretion of the author. In this work, optional reductions follow a guiding criterion: they must not change the degree of the Darboux transformation. For instance, when examining $M_{\uparrow}(2)$ in (4.7b), we shall not set $f = 0$ or set both a and b simultaneously to zero, as this would reduce the quadratic matrix to a linear matrix $M_{\uparrow}(1)$ in (4.7a).

We represent optional reductions using a *reduction tree*, a graph where every branch describes a possible choice applied on the system $\mathscr{E}$. A reduction tree is said to be complete if it includes all possible optional reductions compatible with the above criterion. We will show an example applied to the rank-1 quadratic Darboux matrix $M_{\uparrow}(2)$ in (4.7b) in Section 4.4.1.

4.1.4 Non-commutative constants

In Example 3.14, we observed how a constant of motion can be employed to reduce a commutative system of equations. Similar fundamental reductions also apply to non-commutative equations, as we will show in several examples. However, in this framework, the concept of constant is more delicate.

Let μ be a function depending on the variables p, q and on some auxiliary functions in the Darboux matrix M. Given the associated Darboux system $\mathscr{E}$, μ is called a non-commutative constant if it has trivial evolution, i.e. $\mu_x = 0$, and it lies in the kernel of all compatible evolutionary derivations $\mathcal{D}_t$, i.e. $\mu_t = 0$. As the name suggests, μ maintains all the other properties of a function: it does not commute with other elements and transforms under shift $\mathcal{S}(\mu) = \mu_1$ as in Section 3.2.2.1.

Non-commutative constants are a generalisation of the numerical constants $\alpha \in \mathbb{C}$, which have vanishing derivatives, i.e. $\alpha_x = 0$, but they commute with the other variables and are trivial under shift, i.e. $\mathcal{S}(\alpha) = \alpha$.

We argue that non-commutative constants emerge naturally from Darboux systems. Indeed, all the resulting DΔEs admit at least one and are generalisations of other standard non-commutative models, compare for instance with [29]. It is possible to eliminate a non-commutative constant μ by assuming that $\mu = \alpha\,\mathbb{1}$, where $\alpha \in \mathbb{C}$ and $\mathbb{1}$ is the identity element in $\mathscr{A}$. Consequently, μ is substituted by a numerical constant α, which can often be cancelled by rescaling the variables of the equation.

4.1.5 Organisation of the chapter

In the following sections, we analyse in detail the constant, linear and quadratic Darboux transformations of the DNLS equations, together with the associated reductions. This study is conducted by constructing the Darboux matrices individually for all Lax pairs listed in Theorem 2.14.

Some representative results presented in the following parts are published in the paper [104], written together with Professor Jing Ping Wang. In that publication, we consider the Lax representations linked with equations (A_1), ($\mathrm{A}_2^\star$), ($\mathrm{B}_1^\star$), (B_2), (B_3), (C_1) and ($\mathrm{C}_2^\star$) in Section 2.3.2.1. In the following Sections 4.2, 4.3 and 4.4, we present a complete study of the DΔEs associated with all Lax representations obtained in Theorem 2.14 (with and without $\star$). The factorisation of a Darboux matrix (Section 4.5) and Lemma 4.31 also appear briefly in [104], while the considerations in Section 4.6 about gauge transformations and Darboux matrices are introduced for the first time.

When constructing Darboux transformations, we observe recurring structures across the different models. This is due to similarities among the Lax representations from (2.29) to (2.35). In fact, all the DNLS Lax representations presented in Sections 2.2.1.1 and 2.3.2.1 can be characterised using the entries of P in (2.23), as summarised in Table 4.1. Note that the table is symmetric with respect to the diagonal if we identify each model X with its equivalent X*, as discussed, for instance, in [102].

	$P_{11} = 0$	$P_{11} = pq$	$P_{11} =$ else
$P_{22} = 0$	A_1	$\mathrm{B}_1^\star$	B_2, $\mathrm{C}_2^\star$
$P_{22} = qp$	B_1	C_1	A_2, $\mathrm{B}_3^\star$
$P_{22} =$ else	$\mathrm{B}_2^\star$, C_2	$\mathrm{A}_2^\star$, B_3	-

Table 4.1: Characterisation of DNLS Lax representations.

It is often sufficient to consider either P_{11} or P_{22} to determine the associated reduction. For this reason, the same transformation may produce different outcomes, depending on which entry of P is analysed. An example is provided in Section 4.7.1.

A alternative approach relies on gauge transformations. As observed in Sections 2.1.4 and 2.2.2, all the DNLS equations are related by gauge transformations. Under suitable assumptions, this correspondence extends to their Darboux transformations and DΔEs. Consequently, it appears sufficient to construct a Darboux transformation for just one representative DNLS equation to deduce the analogues for all other equations. However, as presented in Section 4.6, the effectiveness of this process is limited by its non-local nature (2.36).

4.2 The constant Darboux transformations $M(0)$

Constant Darboux transformations are represented by polynomial Darboux matrices (4.5) that are independent of the spectral parameter λ. More generally, considering the scaling symmetry (3.25), a constant Darboux matrix $M(0)$ is equivalent to any Darboux matrix M, not necessarily $\mathcal{R}_\pm$-invariant, whose dependence on λ can be completely factorised as a scalar function, i.e. $M(\lambda) = \zeta(\lambda)M'$, where M' does not depend on λ.

Proposition 4.8. *The constant Darboux transformation for a DNLS equation with Lax representation* (U, V) *from* (2.23) *is determined by the following Darboux* *matrix $M(0)$ and system $\mathscr{E}(0)$:*

$$M(0) = \begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix}, \qquad \begin{cases} p_1 = fpg^{-1}, & f_x = 2(\mathcal{S}(P_{11})f - fP_{11}), \\ q_1 = gqf^{-1}, & g_x = 2(\mathcal{S}(P_{22})g - gP_{22}), \end{cases} \tag{4.10}$$

where f, g are auxiliary functions.

The present proof illustrates the guidelines for the equivalent, more elaborate constructions presented in Sections 4.3 and 4.4.

Proof. Let M be a 2×2 matrix with entries M_{ij}. The zero-curvature condition (4.1) for M and U yields the following system of D∆Es:

$$\begin{cases} \mathcal{D}_x(M_{11}) = 2(\mathcal{S}(P_{11})M_{11} - M_{11}P_{11}), \\ \mathcal{D}_x(M_{12}) = 2(\mathcal{S}(P_{11})M_{12} - M_{12}P_{22}), \\ \mathcal{D}_x(M_{21}) = 2(\mathcal{S}(P_{22})M_{21} - M_{21}P_{11}), \\ \mathcal{D}_x(M_{22}) = 2(\mathcal{S}(P_{22})M_{22} - M_{22}P_{22}), \end{cases}$$

together with the algebraic relations:

$$\begin{cases} M_{12}q - p_1 M_{21} = 0, & M_{21}p - q_1 M_{12} = 0, \\ M_{11}p - p_1 M_{22} = 0, & M_{22}q - q_1 M_{11} = 0, \\ M_{12} = 0, & M_{21} = 0. \end{cases}$$

Fixing $M_{12} = 0$ and $M_{21} = 0$, and denoting $M_{11} = f$ and $M_{22} = g$ returns immediately the result. □

Remark 4.9. In a constant Darboux transformation, the new solutions p_1 and q_1 depend explicitly on p, q, and on the two auxiliary functions f and g. If f and g are constants, then the Darboux transformation is equivalent to a rescaling of the DNLS equation. For a discussion of non-commutative constants, refer to Section 4.1.4.

No further reduction is possible: if either f or g vanishes, the whole Darboux matrix $M(0)$ becomes identically zero. Notice also that, even though $M(0)$ is invariant under the action of the reduction group $\mathcal{R}_+$ in (4.4) (compare with Lemma 4.3), it does not correspond to either of the models (4.3) or (4.5).

Example 4.10. *We consider equation* (A_1)*, with Lax representation* (2.29)*. Since both P_{11} and P_{22} vanish, by* (4.10) *the auxiliary functions f and g become the non-commutative constants ϕ and ψ. This leads to the transformation:*

$$\begin{cases} p_1 = \phi p \psi^{-1}, \\ q_1 = \psi q \phi^{-1}, \end{cases}$$

$$M = \begin{pmatrix} \phi & 0 \\ 0 & \psi \end{pmatrix}, \qquad U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lambda^2 + \begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix} \lambda.$$

Notice that f and g being constants corresponds to a fundamental reduction of $\mathscr{E}(0)$ in (4.10) *through constants of motion.*

This Darboux transformation is trivial in the sense that it reduces to a rescaling of the variables p, q. This is particularly evident if ϕ, ψ are commutative scalars.

Such a result is analogous to what is found in [65, 66] for the commutative equation (A)*, and to the Darboux matrix K in* (4.9) *for the NLS equation [63].*

4.2.1 List of constant Darboux transformations

Considering the definitions of p_1 and q_1, it is possible to simplify the evolution of the auxiliary functions f and g in Proposition 4.8. If not both are constants, they satisfy either

$$f_x = \pm 2\left(gqpg^{-1}f - fqp\right), \qquad g_x = 0; \tag{4.11a}$$

$$f_x = 0, \qquad g_x = \pm 2\left(fpqf^{-1}g - gpq\right). \tag{4.11b}$$

We denote these equations as ±(4.11a) or ±(4.11b) to indicate the choice of sign within the equation. Using the same characterisation of Lax representations as in Table 4.1, we present the constant Darboux transformations of all the DNLS equations in Table 4.2.

	$P_{11} = 0$	$P_{11} = pq$	$P_{11} = \text{else}$
$P_{22} = 0$	ϕ, ψ	ϕ, ψ	$-$(4.11a), ψ
$P_{22} = qp$	ϕ, ψ	ϕ, ψ	$+$(4.11a), ψ
$P_{22} = \text{else}$	$\phi, +$(4.11b)	$\phi, -$(4.11b)	-

Table 4.2: The constant Darboux transformations for the DNLS equations.

Remark 4.11. As noted in Table 4.1, the matrix P in (2.23) involves linear combinations of pq and qp, with different signs. The evolutions of f and g in $\mathscr{E}(0)$ are linear, so they depend on the individual components of P. For this reason, Table 4.2 describes the constant Darboux transformations for all Lax representations in the classification of Section 2.3.

4.3 The linear Darboux transformations

We construct three linear Darboux matrices associated with the DNLS equations: the rank-1 up matrix $M_{\uparrow}(1)$ in (4.7a), the rank-1 down matrix $M_{\downarrow}(1)$ in (4.8a), both according to Definition 4.6, and the full-rank matrix $M(1)$ from Lemma 4.3. Considering equation (A), a similar analysis is carried out in [66] yielding analogous results.

All three matrices $M_{\uparrow}(1)$, $M_{\downarrow}(1)$ and $M(1)$ are invariant under the reduction group $\mathcal{R}_{-}$ defined in (4.4). Some publications, including the aforementioned [66], proceed directly from the constant case (presented in Section 4.2) to the quadratic case (see Section 4.4), treating the linear case as a degenerate form of the quadratic one, as discussed in Remark 4.4.

4.3.1 The linear up Darboux transformations $M_{\uparrow}(1)$

The linear up Darboux transformations for the DNLS equations and the corresponding matrices $M_{\uparrow}(1)$ defined in (4.7a) are described in the following proposition.

Proposition 4.12. *The linear up Darboux transformation of a DNLS equation with Lax representation* (U, V) *from* (2.23) *has Darboux matrix*

$$M_{\uparrow}(1) = \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} \lambda + \begin{pmatrix} 0 & fp \\ q_1 f & 0 \end{pmatrix}, \tag{4.12}$$

and it is associated with the following system $\mathscr{E}_{\uparrow}(1)$

$$\begin{cases} f_x = 2\mathcal{S}\left(P_{11} + pq\right) f - 2f\left(P_{11} + pq\right), \\ p_x = 2\left(P_{11}p - pP_{22}\right) + 2\left(pq - f^{-1}p_1q_1f\right)p, \\ q_{1,x} = 2\mathcal{S}\left(P_{22}q - qP_{11}\right) + 2q_1\left(fpqf^{-1} - p_1q_1\right), \end{cases} \tag{4.13}$$

where f *is an auxiliary function.*

Proof. This proposition is proved by direct construction via (4.1), following the approach shown in the proof of Proposition 4.8 for the constant case. □

In Section 4.3.1.1 we discuss the system (4.13) for all possible entries P_{11} and P_{22} associated with the DNLS equations. Analogous considerations apply to the linear down Darboux matrix $M_{\downarrow}(1)$ and to the full-rank Darboux matrix $M(1)$, see Sections 4.3.2 and 4.3.3.

Theorem 4.13. *The linear rank-1 Darboux transformations for the non-commutative DNLS equations are associated with either the Volterra equation* ($\mathrm{V_a}$) *or the modified Volterra equation* ($\mathrm{mV_b}$), *where* μ *is a non-commutative constant.*

$$u_x = 2(\mu_1 u_1 u - u u_{-1}\mu_{-1}), \tag{$\mathrm{V_a}$}$$

$$u_x = 2u(u_1\mu_1 - \mu u_{-1})u. \tag{$\mathrm{mV_b}$}$$

Proof. The theorem is proved by the construction in Section 4.3.1.1 for the up linear Darboux matrix $M_{\uparrow}(1)$, where each DNLS equation is analysed individually. In Section 4.7.1, we provide an alternative proof based on structural similarities within Lax representations. In Section 4.3.2, we demonstrate that the down linear Darboux matrix $M_{\downarrow}(1)$ is equivalent to the Darboux inverse of $M_{\uparrow}(1)$ via a redefinition of the auxiliary functions. □

Remark 4.14. It is known that both the standard and the modified Volterra equations are related to the potential modified Volterra equation via Miura transformations [8]. In the current work, we treat ($\mathrm{V_a}$) and ($\mathrm{mV_b}$) separately to improve readability and avoid numerous complicated changes of variables. We present a full account of the relations among non-commutative Volterra-type equations in Appendix C.1.

4.3.1.1 List of linear up Darboux transformations

We study linear Darboux transformations of type up for each DNLS equation and its $\star$-version, presenting in detail the corresponding integrable DΔEs. We examine both cases separately, since the reduction techniques involved are often different, as are the associated Lax representations. This analysis contributes to proving Theorem 4.13.

Case A_1 The linear up Darboux transformation $M_{\uparrow}(1)$ for the system (A_1) is represented by the following set of equations $\mathscr{E}_{\uparrow}(1)$, derived from (4.13) with P_{11} and P_{22} in the Lax representation (2.29):

$$\begin{cases} f_x = 2(p_1 q_1 f - f p q), \\ p_x = 2(pq - f^{-1} p_1 q_1 f) p, \\ q_{1,x} = 2 q_1 (f p q f^{-1} - p_1 q_1). \end{cases}$$

Its commutative version is discussed in [66] with analogous results. This system admits two non-commutative constants of motion (for a more detailed discussion, see Section 4.1.4):

$$\alpha = fp, \qquad \beta = q_1 f.$$

Reductions obtained from constants of motion are fundamental: they arise naturally from the structure of the Darboux transformation, see Example 3.14. Since this is the first case of linear Darboux transformation, we examine its reductions separately as two intermediate steps with either α or β considered, and the final one where both are used together.

(i) Reduction based on $\boldsymbol{\alpha}$. If the auxiliary function is set to $f = \alpha p^{-1}$, we obtain a two-equation system:

$$\begin{cases} p_x = 2p(qp - \alpha^{-1} p_1 q_1 \alpha), \\ q_{1,x} = 2q_1(\alpha q p \alpha^{-1} - p_1 q_1), \end{cases}$$

$$M = \begin{pmatrix} \alpha p^{-1} & 0 \\ 0 & 0 \end{pmatrix} \lambda + \begin{pmatrix} 0 & \alpha \\ q_1 \alpha p^{-1} & 0 \end{pmatrix},$$

$$U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lambda^2 + 2 \begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix} \lambda.$$

(ii) Reduction based on $\boldsymbol{\beta}$. Setting $f = q_1^{-1}\beta$ leads to a similar result, which corresponds to the $\star$-involution of the previous case with $\beta \mapsto \alpha^{-1}$. The $\star$-involution, introduced in Section 2.2.3.2, consists of reversing the order of multiplication in all products.

(iii) Reduction based on both $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. Using both constants of motion, we define

$$f = \alpha p^{-1}, \qquad q_1 = \beta p \alpha^{-1}, \qquad u = -p\alpha^{-1}.$$

This reduction yields the modified Volterra equation (mV$_\text{b}$) with $\mu = -\alpha\beta_{-1}$. The associated Lax representation is obtained from (M, U) via a gauge transformation with $G = \operatorname{diag}(1, \beta_{-1})$ and the substitution $\lambda \mapsto -\lambda$:

$$
\begin{aligned}
&u_x = 2u(\alpha\beta_{-1}u_{-1} - u_1\alpha_1\beta)u,\\
&M = \begin{pmatrix} u^{-1} & 0 \\ 0 & 0 \end{pmatrix}\lambda + \begin{pmatrix} 0 & \alpha\beta_{-1} \\ 1 & 0 \end{pmatrix},\\
&U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\lambda^2 + 2\begin{pmatrix} 0 & u\alpha\beta_{-1} \\ u_{-1} & 0 \end{pmatrix}\lambda.
\end{aligned}
\tag{4.14}
$$

The same result was obtained in [66] for the commutative Kaup-Newell equation (A), fixing both constants to 1.

Case A$_2$ The linear up Darboux transformation of the system (A$_2$) is given by

$$
\begin{cases}
f_x = 2(p_1q_1 + q_1p_1)f - 2f(pq + qp),\\
p_x = 2\left(qp - f^{-1}p_1q_1f\right)p,\\
q_{1,x} = 2q_1\left(fpqf^{-1} - q_1p_1\right).
\end{cases}
$$

This system admits a single constant of motion:

$$
\rho = q_1fp, \qquad \rho_x = 0. \tag{4.15}
$$

We perform a fundamental reduction of the system, setting $f = q_1^{-1}\rho p^{-1}$. The reduced system is

$$
\begin{aligned}
&\begin{cases}
p_x = 2(qp^2 - p\rho^{-1}q_1p_1\rho),\\
q_{1,x} = 2(\rho qp\rho^{-1}q_1 - q_1^2p_1),
\end{cases}\\
&M = \begin{pmatrix} q_1^{-1}\rho p^{-1} & 0 \\ 0 & 0 \end{pmatrix}\lambda + \begin{pmatrix} 0 & q_1^{-1}\rho \\ \rho p^{-1} & 0 \end{pmatrix},\\
&U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\lambda^2 + 2\begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix}\lambda + 2\begin{pmatrix} qp & 0 \\ 0 & qp \end{pmatrix}.
\end{aligned}
$$

There is no other known constant of motion available, however let us introduce the new dependent variable

$$u = qp\rho^{-1}$$

and let us write $q = u\rho p^{-1}$. In terms of u and p, the Lax representation (M, U) and the Darboux system $\mathscr{E}_{\uparrow}(1)$ are as follows:

$$\begin{cases} u_x = 2(\rho_{-1}u_{-1}u - uu_1\rho_1), \\ p_x = 2\left(u\rho p - p\rho^{-1}u_1\rho_1\rho\right), \end{cases}$$
$$M = \begin{pmatrix} p_1\rho_1^{-1}u_1^{-1}\rho p^{-1} & 0 \\ 0 & 0 \end{pmatrix}\lambda + \begin{pmatrix} 0 & p_1\rho_1^{-1}u_1^{-1}\rho \\ \rho p^{-1} & 0 \end{pmatrix},$$
$$U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\lambda^2 + 2\begin{pmatrix} 0 & p \\ u\rho p^{-1} & 0 \end{pmatrix}\lambda + 2\begin{pmatrix} u\rho & 0 \\ 0 & u\rho \end{pmatrix}.$$

The evolution of the variable u is nonlinear and independent of p: indeed, it forms a subsystem of $\mathscr{E}_{\uparrow}(1)$, whereas the evolution of p is linear and depends also on u. This suggests that the resulting "physical" DΔE is just the u part, while p can be somehow eliminated. Indeed, p can be removed from the Lax representation via a gauge transformation (3.24) using the matrix $G = \operatorname{diag}(p\rho^{-1}, 1)$.

The resulting system corresponds to the $\mathcal{T}$-reflection of ($\mathrm{V_a}$) setting $\mu = \rho$:

$$\begin{aligned} & u_x = 2(\rho_{-1}u_{-1}u - uu_1\rho_1), \\ & M = \begin{pmatrix} u_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}\lambda + \begin{pmatrix} 0 & u_1^{-1}\rho \\ 1 & 0 \end{pmatrix}, \\ & U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\lambda^2 + 2\begin{pmatrix} 0 & \rho \\ u & 0 \end{pmatrix}\lambda + 2\begin{pmatrix} u_1\rho_1 & 0 \\ 0 & u\rho \end{pmatrix}. \end{aligned} \tag{4.16}$$

Remark 4.15. We observed that a specific change of variables causes the system $\mathscr{E}$ to decompose into two subsystems: one nonlinear and defined in its own variables, the other linear and dependent on the first. Moreover, the linear variables are removed from the Lax pair via a gauge transformation.

This phenomenon, noticed in (A_2), reappears throughout the present work. We argue that it consists of a fundamental reduction, similarly to the one induced by a constant of motion. In fact, there are cases when both lead to the same result.

We interpret it by observing that the system $\mathscr{E}$ is constructed from a Lax representation (M, U), which is naturally characterised by the gauge freedom in (3.24). A change of variables of the kind described above isolates within $\mathscr{E}$ a variable corresponding to such gauge freedom and which, for this reason, can be eliminated via a gauge transformation. In the example above, the change of variables defines a "physical" variable u, with nonlinear evolution and associated with the Bäcklund transformation, and a "gauge", "non-physical" variable p, which is natural to remove, since it has no relevance for this study.

Numerous (not only) linear transformations of the DNLS equations admit analogous reductions, changing just the algebraic terms, but maintaining the scheme. To avoid unnecessary repetitions, we will summarise the process without showing the full derivation.

Case $A_2^\star$ The linear up Darboux transformation of the system ($A_2^\star$) admits the constant of motion $\phi = f$ and it corresponds to the following equations:

$$\begin{cases} p_x = 2(p^2 q - \phi^{-1} p_1 q_1 \phi p), \\ q_{1,x} = 2(q_1 \phi p q \phi^{-1} - p_1 q_1^2). \end{cases}$$

Let us consider the new variable:

$$u = -\phi p q.$$

The transformation defined by $q = -p^{-1}\phi^{-1}u$ corresponds to (V_a) with $\mu = \phi^{-1}$. The evolution of u is independent of p, which can be eliminated via a gauge transformation using $G = \mathrm{diag}(1, (\phi p)^{-1})$:

$$\begin{aligned} u_x &= 2(\phi_1^{-1} u_1 u - u u_{-1} \phi_{-1}^{-1}), \\ M &= \begin{pmatrix} \phi & 0 \\ 0 & 0 \end{pmatrix} \lambda + \begin{pmatrix} 0 & 1 \\ -u_1\phi & 0 \end{pmatrix}, \\ U &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lambda^2 + 2 \begin{pmatrix} 0 & \phi^{-1} \\ -u & 0 \end{pmatrix} \lambda + 2 \begin{pmatrix} \phi^{-1} u & 0 \\ 0 & \phi_1^{-1} u_1 \end{pmatrix}. \end{aligned} \tag{4.17}$$

Case B$_1$ The linear up Darboux transformation of the system (B$_1$) is given by

$$\begin{cases} f_x = 2(p_1 q_1 f - f p q), \\ p_x = -2 f^{-1} p_1 q_1 f p, \\ q_{1,x} = 2 q_1 f p q f^{-1}, \end{cases}$$

which admits ρ in (4.15) as a constant of motion. Considering $f = q_1^{-1} \rho p^{-1}$, we reduce the system to:

$$\begin{aligned} &\begin{cases} p_x = -2 p \rho^{-1} q_1 p_1 \rho, \\ q_{1,x} = 2 \rho q p \rho^{-1} q_1, \end{cases} \\ M &= \begin{pmatrix} q_1^{-1} \rho p^{-1} & 0 \\ 0 & 0 \end{pmatrix} \lambda + \begin{pmatrix} 0 & q_1^{-1}\rho \\ \rho p^{-1} & 0 \end{pmatrix}, \\ U &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lambda^2 + 2 \begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix} \lambda + 2 \begin{pmatrix} 0 & 0 \\ 0 & qp \end{pmatrix}. \end{aligned}$$

We now introduce the new variable

$$u = q p \rho^{-1},$$

setting $q = u \rho p^{-1}$, the system $\mathscr{E}_{\uparrow}(1)$ is equivalent to the $\mathcal{T}$-reflection of (V$_\mathrm{a}$) with $\mu = \rho$:

$$u_x = 2(\rho_{-1} u_{-1} u - u u_1 \rho_1).$$

The evolution of the variable u is independent of p, which can be eliminated via a gauge transformation using $G = \operatorname{diag}(p\rho^{-1}, 1)$. The resulting Lax representation is identical to (4.16).

Case $\mathrm{B}_1^\star$ The linear up Darboux transformation of the system ($\mathrm{B}_1^\star$) admits the constant of motion $\phi = f$ and it corresponds to the following equations:

$$\begin{cases} p_x = -2\phi^{-1}p_1q_1\phi p, \\ q_{1,x} = 2q_1\phi pq\phi^{-1}. \end{cases} \tag{4.18}$$

Let us consider the new variable

$$u = -\phi pq,$$

the transformation defined by $q = -p^{-1}\phi^{-1}u$ is equivalent to ($\mathrm{V_a}$) with $\mu = \phi^{-1}$:

$$u_x = 2(\phi_1^{-1}u_1u - uu_{-1}\phi_{-1}^{-1}).$$

The evolution of the variable u is independent of p, which can be eliminated via a gauge transformation using $G = \operatorname{diag}(1, (\phi p)^{-1})$. The resulting Lax representation is identical to (4.17).

Remark 4.16. Equation ($\mathrm{B}_1^\star$) can be treated using a different approach, obtaining as reductions both Volterra equations ($\mathrm{V_a}$) and ($\mathrm{mV_b}$). Indeed, system (4.18) admits a second constant of motion

$$\kappa = q_1\phi\phi_{-1}p_{-1}, \qquad \kappa_x = 0. \tag{4.19}$$

Let us set $q_1 = -\kappa(\phi\phi_{-1}p_{-1})^{-1}$ and substitute it into the system. The associated Lax representation is obtained via a gauge transformation by $G = \operatorname{diag}(\phi_{-1}, 1)$:

$$
\begin{aligned}
&p_x = 2\phi^{-1} p_1 \kappa p_{-1}^{-1} \phi_{-1}^{-1} p, \\
&M = \begin{pmatrix} \phi_{-1} & 0 \\ 0 & 0 \end{pmatrix} \lambda + \begin{pmatrix} 0 & p \\ -\kappa p_{-1}^{-1} & 0 \end{pmatrix}, \\
&U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lambda^2 + 2 \begin{pmatrix} 0 & \phi_{-1}^{-1} p \\ -\kappa_{-1} p_{-2}^{-1} \phi_{-2}^{-1} & 0 \end{pmatrix} \lambda + 2 \begin{pmatrix} \phi_{-1}^{-1} p \kappa_{-1} p_{-2}^{-1} \phi_{-2}^{-1} & 0 \\ 0 & 0 \end{pmatrix}.
\end{aligned}
$$

When $\kappa \mapsto \mathbb{1}$, the identity element of the algebra, the equation above corresponds to the potential modified Volterra equation, see (pmV) in Appendix C.1. From it, we derive two reductions:

- Introducing $u = p\kappa_{-1} p_{-2}^{-1} \phi_{-2}^{-1}$, the given equation corresponds to the Volterra equation ($\mathrm{V_a}$) with $\mu = \phi_{-1}^{-1}$:

  $$u_x = 2\left(\phi^{-1} u_1 u - u u_{-1} \phi_{-2}^{-1}\right).$$

  The remaining p is eliminated by a gauge transformation with $G = \operatorname{diag}(\phi_{-1}, p^{-1})$ and the consequent Lax representation becomes (4.17) shifting $\phi \mapsto \phi_1$.

- Alternatively, by letting $u = \kappa p_{-1}^{-1} \phi_{-1}^{-1} p \kappa_1^{-1}$, the equation corresponds to ($\mathrm{mV_b}$) with $\mu = \kappa_1$:

  $$u_x = 2u\left(u_1 \kappa_2 - \kappa_1 u_{-1}\right) u.$$

  The associated Lax representation is analogous to (4.14) if we perform a gauge transformation by $G = \operatorname{diag}(p\kappa_1^{-1} u^{-1}, 1)$.

Case B_2 The linear up Darboux transformation of the system (B_2) is given by

$$
\begin{cases}
f_x = 2(p_1 q_1 - q_1 p_1) f - 2f(pq - qp), \\
p_x = 2\left(pq - qp - f^{-1} p_1 q_1 f\right) p, \\
q_{1,x} = 2q_1\left(q_1 p_1 - p_1 q_1 + f p q f^{-1}\right).
\end{cases}
$$

The system admits the constant of motion κ introduced in (4.19). We shall use it to determine $q_1 = -\kappa(f f_{-1} p_{-1})^{-1}$, as follows

$$\left\{\begin{aligned} f_x &= 2(fp\kappa_{-1}p_{-2}^{-1}f_{-2}^{-1}f_{-1}^{-1} - p_1\kappa p_{-1}^{-1}f_{-1}^{-1} - f\kappa_{-1}p_{-2}^{-1}f_{-2}^{-1}f_{-1}^{-1}p + \\ &\qquad +\kappa p_{-1}^{-1}f_{-1}^{-1}f^{-1}p_1 f), \\ p_x &= 2(f^{-1}p_1\kappa p_{-1}^{-1}f_{-1}^{-1} + \kappa_{-1}p_{-2}^{-1}f_{-2}^{-1}f_{-1}^{-1}p - p\kappa_{-1}p_{-2}^{-1}f_{-2}^{-1}f_{-1}^{-1})p, \end{aligned}\right.$$

$$M = \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix}\lambda - \begin{pmatrix} 0 & -fp \\ \kappa p_{-1}^{-1}f_{-1}^{-1} & 0 \end{pmatrix},$$

$$U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\lambda^2 - 2\begin{pmatrix} 0 & -p \\ \kappa_{-1}p_{-2}^{-1}f_{-2}^{-1}f_{-1}^{-1} & 0 \end{pmatrix}\lambda + 2\begin{pmatrix} \kappa_{-1}p_{-2}^{-1}f_{-2}^{-1}f_{-1}^{-1}p & 0 \\ 0 & 0 \end{pmatrix}.$$

Let us consider the variable

$$u = \kappa(\kappa_1 p^{-1} f_{-1} p_{-1})^{-1},$$

in terms of which the equation above is equivalent to ($\mathrm{mV_b}$) with $\mu = \kappa_1$:

$$u_x = 2u(u_1\kappa_2 - \kappa_1 u)u.$$

The remaining p is eliminated from the Lax representation via a gauge transformation with $G = \operatorname{diag}(p\kappa_1^{-1}u^{-1}, 1)$, yielding a result identical to (4.14).

Case $\mathrm{B}_2^\star$ The linear up Darboux transformation of the system ($\mathrm{B}_2^\star$) is given by

$$\left\{\begin{aligned} f_x &= 2\left(p_1q_1f - fpq\right), \\ p_x &= 2\left(pqp - p^2q - f^{-1}p_1q_1fp\right), \\ q_{1,x} &= 2\left(p_1q_1^2 - q_1p_1q_1 + q_1fpqf^{-1}\right), \end{aligned}\right.$$

which admits the constant of motion

$$\sigma = fpqf_{-1}, \qquad \sigma_x = 0. \tag{4.20}$$

We use it to express $q = -(f_{-1}\sigma^{-1}fp)^{-1}$, as it follows:

$$\begin{cases} f_x = 2\left(\sigma f_{-1}^{-1} - f_1^{-1}\sigma_1\right), \\ p_x = 2\left(f^{-1}f_1^{-1}\sigma_1 p + pf^{-1}\sigma f_{-1}^{-1} - f^{-1}\sigma f_{-1}^{-1}p\right), \end{cases}$$

$$M = \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix}\lambda - \begin{pmatrix} 0 & -fp \\ p_1^{-1}f_1^{-1}\sigma_1 & 0 \end{pmatrix},$$

$$U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\lambda^2 - 2\begin{pmatrix} 0 & -p \\ p^{-1}f^{-1}\sigma f_{-1}^{-1} & 0 \end{pmatrix}\lambda - 2\begin{pmatrix} 0 & 0 \\ 0 & f^{-1}\sigma f_{-1}^{-1} \end{pmatrix}.$$

We notice that the variable f is already independent of p, which is eliminated by a gauge transformation with $G = \operatorname{diag}(1, p^{-1}f^{-1}\sigma)$. Defining $u = f^{-1}$, the corresponding equation

$$u_x = 2u\left(u_1\sigma_1 - \sigma u_{-1}\right)u$$

is equivalent to (mV$_\mathrm{b}$) with $\mu = \sigma$ and it has (4.14) as Lax representation.

Case B$_3$ The linear up Darboux transformation of the system (B$_3$) admits the constant of motion $\phi = f$ and it corresponds to the following system:

$$\begin{cases} p_x = 2\left(p^2q - pqp - \phi^{-1}p_1q_1\phi p\right), \\ q_{1,x} = 2\left(q_1p_1q_1 - p_1q_1^2 + q_1\phi pq\phi^{-1}\right). \end{cases}$$

Let us consider the new variable

$$u = -\phi pq.$$

The transformation defined by $q = -p^{-1}\phi^{-1}u$ is equivalent to (V$_\mathrm{a}$) with $\mu = \phi^{-1}$:

$$u_x = 2(\phi_1^{-1}u_1u - uu_{-1}\phi_{-1}^{-1}).$$

The evolution of the variable u is independent of p, which can be eliminated via a gauge transformation using $G = \operatorname{diag}(1, (\phi p)^{-1})$. The resulting Lax representation is identical to (4.17).

Case $\mathrm{B}_3^\star$ The linear up Darboux transformation of the system ($\mathrm{B}_3^\star$) is given by

$$\begin{cases} f_x = 2\left(q_1 p_1 f - f q p\right), \\ p_x = 2\left(qp - pq - f^{-1} p_1 q_1 f\right) p, \\ q_{1,x} = 2 q_1 \left(p_1 q_1 - q_1 p_1 + f p q f^{-1}\right). \end{cases}$$

This system admits ρ in (4.15) as a constant of motion. We take advantage of it to set $f = q_1^{-1} \rho p^{-1}$. The system above becomes

$$\begin{cases} p_x = 2\left(q p^2 - p q p - p \rho^{-1} q_1 p_1 \rho\right), \\ q_{1,x} = 2\left(q_1 p_1 q_1 - q_1^2 p_1 - \rho q p \rho^{-1} q_1\right), \end{cases}$$

$$M = \begin{pmatrix} q_1^{-1} \rho p^{-1} & 0 \\ 0 & 0 \end{pmatrix} \lambda + \begin{pmatrix} 0 & q_1^{-1} \rho \\ \rho p^{-1} & 0 \end{pmatrix},$$

$$U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lambda^2 + 2 \begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix} \lambda + 2 \begin{pmatrix} qp - pq & 0 \\ 0 & qp \end{pmatrix}.$$

Let us now introduce the variable

$$u = q p \rho^{-1}.$$

The transformation defined by $q = u \rho p^{-1}$ corresponds to the $\mathcal{T}$-reflection of ($\mathrm{V_a}$) with $\mu = \rho$:

$$u_x = 2\left(\rho_{-1} u_{-1} u - u u_1 \rho_1\right).$$

The evolution of u is independent of p, which can be eliminated via a gauge transformation using $G = \operatorname{diag}(p\rho^{-1}, 1)$. The resulting Lax representation is identical to (4.16).

Case C_1 The linear up Darboux transformation of the system (C_1) is given by

$$\begin{cases} p_x = -2\left(pq + \phi^{-1} p_1 q_1 \phi\right) p, \\ q_{1,x} = 2 q_1 \left(p_1 q_1 + \phi p q \phi^{-1}\right). \end{cases}$$

Let us consider the new variable

$$u = -\phi p q.$$

The transformation defined by $q = -p^{-1}\phi^{-1}u$ corresponds to ($\mathrm{V_a}$) with $\mu = \phi^{-1}$:

$$u_x = 2(\phi_1^{-1}u_1 u - u u_{-1}\phi_{-1}^{-1}).$$

The evolution of the variable u is independent of p, which can be eliminated via a gauge transformation using $G = \mathrm{diag}(1, (\phi p)^{-1})$. The resulting Lax representation is identical to (4.17).

Remark 4.17. The original system admits also an alternative reduction based on the constant of motion ρ in (4.15). Setting $q_1 = -\rho p^{-1}\phi^{-1}$, we determine:

$$\begin{aligned}
&p_x = 2(\phi^{-1}p_1\rho + p\rho_{-1}p_{-1}^{-1}\phi_{-1}^{-1}p),\\
&M = \begin{pmatrix} \phi & 0 \\ 0 & 0 \end{pmatrix}\lambda + \begin{pmatrix} 0 & \phi p \\ -\rho p^{-1} & 0 \end{pmatrix},\\
&U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\lambda^2 - 2\begin{pmatrix} 0 & -p \\ \rho_{-1}p_{-1}^{-1}\phi_{-1}^{-1} & 0 \end{pmatrix}\lambda + 2\begin{pmatrix} p\rho_{-1}p_{-1}^{-1}\phi_{-1}^{-1} & 0 \\ 0 & -\rho_{-1}p_{-1}^{-1}\phi_{-1}^{-1}p \end{pmatrix}.
\end{aligned}$$

Introducing the same variable as above, $u = \phi p \rho_{-1} p_{-1}^{-1}\phi_{-1}^{-1}$, this equation is equivalent to the Volterra equation ($\mathrm{V_a}$) with $\mu = \phi^{-1}$:

$$u_x = 2(\phi_1^{-1}u_1 u - u u_{-1}\phi_{-1}^{-1}).$$

The remaining p can be eliminated via a gauge transformation with $G = \mathrm{diag}(1, (\phi p)^{-1})$ and the resulting Lax representation is identical to (4.17).

Case C_2 The linear up Darboux transformation of the system (C_2) is given by

$$\begin{cases} f_x = 2\left(p_1q_1f - fpq\right), \\ p_x = -2\left(p^2q + f^{-1}p_1q_1fp\right), \\ q_{1,x} = 2\left(p_1q_1^2 + q_1fpqf^{-1}\right). \end{cases}$$

This case admits σ in (4.20) as a constant of motion, we take advantage of it to determine $q = p^{-1}f^{-1}\sigma f_{-1}^{-1}$. The reduced system follows:

$$\begin{cases} f_x = 2(f_1^{-1}\sigma_1 - \sigma f_{-1}^{-1}), \\ p_x = -2(pf^{-1}\sigma f_{-1}^{-1} + f^{-1}f_1^{-1}\sigma_1 p), \end{cases}$$

$$M = \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix}\lambda + \begin{pmatrix} 0 & fp \\ p_1^{-1}f_1^{-1}\sigma_1 & 0 \end{pmatrix},$$

$$U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\lambda^2 + 2\begin{pmatrix} 0 & p \\ p^{-1}f^{-1}\sigma f_{-1}^{-1} & 0 \end{pmatrix}\lambda + 2\begin{pmatrix} 0 & 0 \\ 0 & f^{-1}\sigma f_{-1}^{-1} + p^{-1}f^{-1}\sigma f_{-1}^{-1}p \end{pmatrix}.$$

The evolution of f is already independent of p and is eliminated by a gauge transformation with $G = \mathrm{diag}(1, -p^{-1}f^{-1}\sigma)$. The resulting system corresponds to the modified Volterra equation ($\mathrm{mV_b}$) if we consider $\mu = -\sigma$ and $u = f^{-1}$, and it has (4.14) as Lax representation:

$$u_x = 2u(u_1\sigma_1 - \sigma u_{-1})u.$$

Case $\mathrm{C}_2^\star$ The linear up Darboux transformation of the system ($\mathrm{C}_2^\star$) is given by

$$\begin{cases} f_x = 2(fqp - q_1p_1f), \\ p_x = -2(qp + f^{-1}p_1q_1f)p, \\ q_{1,x} = 2q_1(q_1p_1 + fpqf^{-1}). \end{cases}$$

This system admits the constant of motion κ introduced in (4.19). We take advantage of it to determine $q = -\kappa(f f_{-1} p_{-1})^{-1}$, reducing the system to

$$\begin{cases} f_x = 2\left(\kappa p_{-1}^{-1} f_{-1}^{-1} f^{-1} p_1 f - f \kappa_{-1} p_{-2}^{-1} f_{-2}^{-1} f_{-1}^{-1} p\right), \\ p_x = 2\left(f^{-1} p_1 \kappa p_{-1}^{-1} f_{-1}^{-1} + \kappa_{-1} p_{-2}^{-1} f_{-2}^{-1} f_{-1}^{-1} p\right) p, \end{cases}$$

$$M = \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} \lambda + \begin{pmatrix} 0 & fp \\ -\kappa p_{-1}^{-1} f_{-1}^{-1} & 0 \end{pmatrix},$$

$$U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lambda^2 + 2 \begin{pmatrix} 0 & p \\ -\kappa_{-1} p_{-2}^{-1} f_{-2}^{-1} f_{-1}^{-1} & 0 \end{pmatrix} \lambda + \\ + 2 \begin{pmatrix} p \kappa_{-1} p_{-2}^{-1} f_{-2}^{-1} f_{-1}^{-1} + \kappa_{-1} p_{-2}^{-1} f_{-2}^{-1} f_{-1}^{-1} p & 0 \\ 0 & 0 \end{pmatrix}.$$

Introducing the variable $u = -\kappa p_{-1}^{-1} f_{-1}^{-1} p \kappa_1^{-1}$, the resulting system is equivalent to ($\mathrm{mV_b}$) with $\mu = \kappa_1$:

$$u_x = 2u(u_1 \kappa_2 - \kappa_1 u_{-1})u.$$

The remaining presence of p is eliminated from the Lax representation via a gauge transformation using $G = \operatorname{diag}(p\kappa_1^{-1} u^{-1}, 1)$, which becomes identical to (4.14).

4.3.2 The linear down Darboux transformations $M_\downarrow(1)$

In this section we consider the down type rank-1 linear Darboux transformations, which correspond to the Darboux matrix $M_\downarrow(1)$ defined in (4.8a):

$$M_\downarrow(1) = \begin{pmatrix} 0 & 0 \\ 0 & g \end{pmatrix} \lambda - \begin{pmatrix} 0 & p_1 g \\ gq & 0 \end{pmatrix}. \tag{4.21}$$

The associated Darboux system $\mathscr{E}_\downarrow(1)$ is given by

$$\begin{cases} g_x = 2\,\mathcal{S}(P_{22} - qp)g - 2g(P_{22} - qp), \\ p_{1,x} = 2\,\mathcal{S}(P_{11} p - p P_{22}) + 2p_1(q_1 p_1 - g q p g^{-1}), \\ q_x = 2(P_{22} q - q P_{11}) - 2(qp - g^{-1} q_1 p_1 g) q. \end{cases} \tag{4.22}$$

It is possible to reproduce the same steps used for the up type transformation $M_\uparrow(1)$, eventually reaching similar results. However, there is a deeper relation between the up and down linear cases.

Consider a Darboux matrix $M(u_1, u)$. Its Darboux inverse $M^I(u_{-1}, u)$ (see Section 3.2.2.7) satisfies Lax-Darboux equations analogous to (4.1) with respect to the shift $\mathcal{S}^{-1}$:

$$M^I_\uparrow(1) = -\begin{pmatrix} 0 & 0 \\ 0 & (qf_{-1}p_{-1})^{-1} \end{pmatrix} \lambda + \begin{pmatrix} 0 & (qf_{-1})^{-1} \\ (f_{-1}p_{-1})^{-1} & 0 \end{pmatrix}, \tag{4.23}$$

with the same Darboux system $\mathscr{E}_\uparrow(1)$. Let us recall the reflection operator $\mathcal{T}$ defined in (3.20). Applying $\mathcal{T}$ to $M^I(u_{-1}, u)$ gives $\mathcal{T}(M^I) = \hat{M}(u_1, u)$, which is a solution of the Lax-Darboux equation (4.1) for a new shift $\hat{\mathcal{S}}$. In this way, if $\mathcal{T}(M^I)$ remains polynomial, it can be directly compared with another polynomial Darboux matrix.

Indeed, the matrix $\mathcal{T}(M^I_\uparrow(1))$ has the same structure as a rank-1 down linear Darboux matrix $M_\downarrow(1)$. Therefore, we compare them.

Proposition 4.18. *A rank-1 down linear Darboux transformation for a DNLS equation can be expressed in terms of the Darboux inverse of a rank-1 up linear Darboux transformation, where $g = -(qf_1p_1)^{-1}$:*

$$M_\downarrow(1) = \mathcal{T}(M^I_\uparrow(1)), \qquad \mathscr{E}_\downarrow(1) = \mathcal{T}(\mathscr{E}_\uparrow(1)).$$

Proof. Both $M_\uparrow(1)$ and $M_\downarrow(1)$ are *unimodular polynomial matrices*, i.e. invertible polynomial matrices with polynomial inverses. The matrix $\mathcal{T}(M^I_\uparrow(1))$ has the same structure as a rank-1 down linear Darboux matrix and it matches $M_\downarrow(1)$ provided we set $g = -(qf_1p_1)^{-1}$. It remains to check whether this substitution is compatible with the systems $\mathscr{E}_\downarrow(1)$ and $\mathcal{T}(\mathscr{E}_\uparrow(1))$. Considering the system $\mathscr{E}_\uparrow(1)$ in (4.13), its

$\mathcal{T}$-reflection consists of

$$\begin{cases} f_x = 2\mathcal{S}^{-1}\left(P_{11}+pq\right)f - 2f\left(P_{11}+pq\right), \\ p_x = 2\left(P_{11}p - pP_{22}\right) + 2\left(pq - f^{-1}p_{-1}q_{-1}f\right)p, \\ q_{-1,x} = 2\mathcal{S}^{-1}\left(P_{22}q - qP_{11}\right) + 2q_{-1}\left(fpqf^{-1} - p_{-1}q_{-1}\right). \end{cases}$$

The evolutions of p, q and $g = -(qf_1p_1)^{-1}$, according to the equations above, coincide with those in (4.22). □

4.3.3 The linear full-rank Darboux transformations $M(1)$

We now construct the full-rank linear Darboux transformations for the DNLS equations. According to Lemma 4.3, the Darboux matrix $M(1)$ is given by

$$M(1) = \begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix}\lambda + \begin{pmatrix} 0 & fp - p_1g \\ q_1f - gq & 0 \end{pmatrix}.$$

The Darboux system $\mathscr{E}(1)$, special case of (4.6), contains non-evolutionary DΔEs in the dependent variables:

$$\begin{cases} f_x = 2\mathcal{S}\left(P_{11}+pq\right)f - 2f\left(pq+P_{11}\right), \\ g_x = 2\mathcal{S}\left(P_{22}-qp\right)g - 2g\left(P_{22}-qp\right), \\ p_{1,x}g - fp_x = 2\mathcal{S}\left(P_{11}p - pP_{22}\right)g - 2f\left(P_{11}p - pP_{22}\right) + \\ \qquad +2\left(p_1q_1p_1g + p_1q_1fp - p_1gqp - fpqp\right), \\ q_{1,x}f - gq_x = 2\mathcal{S}\left(P_{22}q - qP_{11}\right)f + 2g\left(P_{22}q - qP_{11}\right) + \\ \qquad +2\left(q_1fpq - q_1p_1gq - q_1p_1q_1f + gqpq\right). \end{cases} \tag{4.24}$$

The case (C_1) is particularly interesting since both auxiliary functions f and g become the constants ϕ and ψ. The Darboux system $\mathscr{E}(1)$ in (4.24) is reduced to the pair of non-evolutionary differential-difference equations

$$\begin{cases} p_{1,x}\psi - \phi p_x = 2\left(\phi pqp + p_1q_1\phi p - p_1\psi qp - p_1q_1p_1\psi\right), \\ \psi q_x - q_{1,x}\phi = 2\left(\psi qpq - q_1\phi pq - q_1p_1q_1\phi + q_1p_1\psi q\right). \end{cases}$$

Another remarkable case is given by equation (A_1), as stated in the following proposition.

Proposition 4.19. *The full-rank linear Darboux transformation for system* (A_1) *is trivial.*

Proof. Considering the Lax representation (2.29), system (4.24) becomes

$$\begin{cases} f_x = 2\left(p_1q_1f - fpq\right), & g_x = 2\left(gqp - q_1p_1g\right), \\ (fp - p_1g)_x = 0, & (q_1f - gq)_x = 0. \end{cases}$$

The last two equations imply

$$p_1 = fpg^{-1} - \chi^{(1)}g^{-1}, \qquad q_1 = gqf^{-1} + \chi^{(2)}f^{-1},$$

for two constants of integration $\chi^{(1)}$ and $\chi^{(2)}$. Substituting into the first equation, we obtain

$$\begin{cases} f_x = 2\chi^{(2)}fpg^{-1} - 2\chi^{(1)}\left(\chi^{(2)}g^{-1} + q\right), \\ g_x = 2\chi^{(1)}gqf^{-1} + 2\chi^{(2)}\left(\chi^{(1)}f^{-1} - p\right). \end{cases}$$

Hence, the evolution of p and q is entirely determined by algebraic relations, and no DΔE is introduced. This result is analogous to $\mathscr{E}(0)$ in (4.10), confirming that the transformation is trivial. $\square$

The triviality of the full-rank linear Darboux transformation has been observed for the NLS equation in [6]. This further highlights the connections between the NLS and the Kaup-Newell systems (A_1) introduced in Remark 2.1.

4.4 The quadratic Darboux transformations

Quadratic Darboux transformations for the DNLS equations are among the most studied Darboux transformations in the literature [65–67]. They represent the first non-trivial class of Darboux matrices that satisfy the same reduction group $\mathcal{R}_+$ (4.4) as the Lax representation (U, V) (2.42). The constant case $M(0)$ in (4.10) is therefore excluded.

We follow the same approach used for linear Darboux matrices in Section 4.3: in Section 4.4.1, we begin by constructing rank-1 up quadratic Darboux matrices $M_{\uparrow}(2)$; the complete list of transformations for the DNLS equations is provided in Section 4.4.1.1. The down case $M_{\downarrow}(2)$ is addressed in Section 4.4.2, and the full-rank quadratic case $M(2)$ is discussed in Section 4.4.3. The relations between linear and quadratic Darboux matrices will be investigated in Section 4.5.

4.4.1 The quadratic up Darboux transformations $M_{\uparrow}(2)$

We construct rank-1 up quadratic Darboux transformations represented by the Darboux matrix $M_{\uparrow}(2)$ in (4.7b). We then discuss the associated system $\mathscr{E}_{\uparrow}(2)$ together with its admissible optional reductions.

Proposition 4.20. *The rank-1 up quadratic Darboux transformation for a DNLS equation with Lax representation* (U, V) *from* (2.23) *has Darboux matrix*

$$M_{\uparrow}(2) = \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} \lambda^2 + \begin{pmatrix} 0 & fp \\ q_1 f & 0 \end{pmatrix} \lambda + \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \tag{4.25}$$

and is associated with the following system $\mathscr{E}_{\uparrow}(2)$*:*

$$\begin{cases} p_x = 2f^{-1}\left(p_1 b - ap\right) + 2\left(P_{11}p - pP_{22}\right) + 2\left(pq - f^{-1}p_1q_1f\right)p, \\ q_{1,x} = 2\left(q_1 a - bq\right)f^{-1} + 2\mathcal{S}\left(P_{22}q - qP_{11}\right) - 2q_1\left(p_1q_1 - fpqf^{-1}\right), \\ f_x = 2\mathcal{S}\left(P_{11} + pq\right)f - 2f\left(P_{11} + pq\right), \\ a_x = 2\mathcal{S}(P_{11})a - 2aP_{11}, \\ b_x = 2\mathcal{S}(P_{22})b - 2bP_{22}, \end{cases} \tag{4.26}$$

where f, a *and* b *are auxiliary functions.*

Proof. This proposition is proved by direct construction via (4.1), following the approach shown in the proof of Proposition 4.8 for the constant case. □

A main difference from the linear case is that, to the best of our knowledge, no complete fundamental reduction of $\mathscr{E}_{\uparrow}(2)$ is available for any DNLS equation. Therefore, it is not possible to determine all the auxiliary functions. Consequently, we also employ optional reductions that are compatible with the guiding principle from Section 4.1.3.

Two natural optional reductions of the transformation $M_{\uparrow}(2)$ are given by setting $a = 0$ or $b = 0$ in $M_{\uparrow}(2)$. Consider also the function

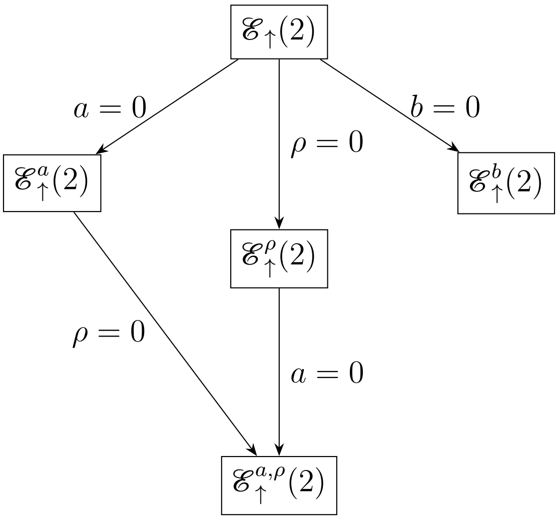
$$\rho = q_1 f p - b, \qquad \rho_x = 2\mathcal{S}(P_{22} - qp)\rho - 2\rho(P_{22} - qp), \tag{4.27}$$

whose evolution is computed from $\mathscr{E}_{\uparrow}(2)$ in (4.26). In a certain sense, ρ in (4.27) extends (4.15) from the linear case and it is a constant of motion when $P_{22} = qp$, i.e. in the Lax pairs of (A_2), (B_1), ($B_3^\star$) and (C_1), mirroring the behaviour observed in Section 4.7.1. Moreover, it is always possible to impose $\rho = 0$ as this condition is compatible with the evolution equation (4.27) for any DNLS equation, obtaining a third reduction.

In summary, the system $\mathscr{E}_{\uparrow}(2)$ in (4.26) admits four optional reductions:

$$\text{(i) } a=0, \qquad \text{(ii) } b=0, \qquad \text{(iii) } \rho=0, \qquad \text{(iv) } a=\rho=0.$$

Each of these respects the guiding principle from Section 4.1.3 and is associated with a reduction of $\mathscr{E}_{\uparrow}(2)$ in (4.26), as illustrated by the following reduction tree:



(4.28)

Other reductions, such as $f=0$ or $a=b=0$, are excluded because they violate the guiding criterion above: In all such cases, the quadratic Darboux matrix becomes equivalent, via (3.25), to the linear one studied in Theorem 4.13.

Before analysing the application of $\mathscr{E}_{\uparrow}(2)$ to the various DNLS equations, we state the main result of this section.

Theorem 4.21. *The reductions in (4.28) of a rank-1 up quadratic Darboux transformation of the non-commutative DNLS equations are either trivial or correspond to one of the following nine integrable DΔEs, which involve non-commutative constants:*

$$\begin{cases} u_x = 2\mu_1 u_1(\nu - vu), \\ v_x = 2(vu-\nu)v_{-1}\mu; \end{cases} \tag{AL}$$

$$\begin{cases} u_x = 2(\mu u_1 \nu - u v u), \\ v_x = 2(-\nu_{-1} v_{-1} \mu_{-1} + v u v); \end{cases} \tag{MRT}$$

$$\begin{cases} u_x = 2v_1(u_1 + \mu_1)u - 2uv(u + \mu), \\ v_x = 2(vu - u_{-1}v_{-1})v; \end{cases} \tag{$\mathrm{N_1}$}$$

$$\begin{cases} u_x = 2(u\mu v - v_1\mu_1 u) + 2\left(v_1 u_1 v_1^{-1} u - u v^{-1} u_{-1} v\right), \\ v_x = 2(vu - u_{-1}v); \end{cases} \tag{$\mathrm{N_2}$}$$

$$\begin{cases} u_x = 2(uv_1 - vu + uu_1\mu_1 - u\mu u), \\ v_x = 2(vu\mu - \mu_{-1}v_{-1}u_{-1}); \end{cases} \tag{$\mathrm{N_3}$}$$

$$\begin{cases} u_x = 2(w_1 u_1 - uw)u, \\ v_x = 2(wuv - u_{-1}v_{-1}w_{-1}), \\ w_x = 2(wv_1 - vw); \end{cases} \tag{$\mathrm{N_4}$}$$

$$\begin{cases} u_x = 2(uv\mu - \mu_1 v_1 u) + 2\left(u\mu^{-1}u_{-1}\mu - \mu_1 u_1 \mu_1^{-1} u\right), \\ v_x = 2(vu_{-1} - uv); \end{cases} \tag{rT}$$

$$\begin{cases} u_x = 2(\mu_1 v_1 u - uv\mu), \\ v_x = 2(uv - vu_{-1}); \end{cases} \tag{$\mathrm{2V_a}$}$$

$$\begin{cases} u_x = 2(v_1\mu_1 u - u\mu v), \\ v_x = 2(vu - u_{-1}v). \end{cases} \tag{$\mathrm{2V_b}$}$$

For a more detailed discussion of the non-commutative constants μ and ν, see Section 4.1.4.

System (AL) generalises the well-known *Ablowitz-Ladik equation* [3, 4] and its non-commutative version given in [29], where the parameter β vanishes.

System (MRT) is a similar generalisation of the *Merola-Ragnisco-Tu equation* [84, 134], already introduced in Example 3.14 for the Darboux transformations of the NLS equation.

System (rT) corresponds to the *relativistic Toda equation* [98, 107, 115] in its non-commutative version, introduced in [45, 70].

Systems ($2\mathrm{V}_\mathrm{a}$) and ($2\mathrm{V}_\mathrm{b}$) are two different non-commutative versions of the *two-component Volterra equation*, analogous to those presented in [8]. For further discussion of the relations among Volterra-type equations, see Appendix C.1.

The remaining systems (N_1), (N_2), (N_3) and (N_4) are to be considered new models, although some of them can be mapped into other known systems via non-invertible Miura transformations (see Appendices C.2 and C.3). Notice that the system (N_2) can be interpreted as a second non-commutative lift of the relativistic Toda equation (rT), since both share equivalent commutative versions (see Remark 4.25). System (N_4) is the only three-equation model in the list and appears independent of any other known equation.

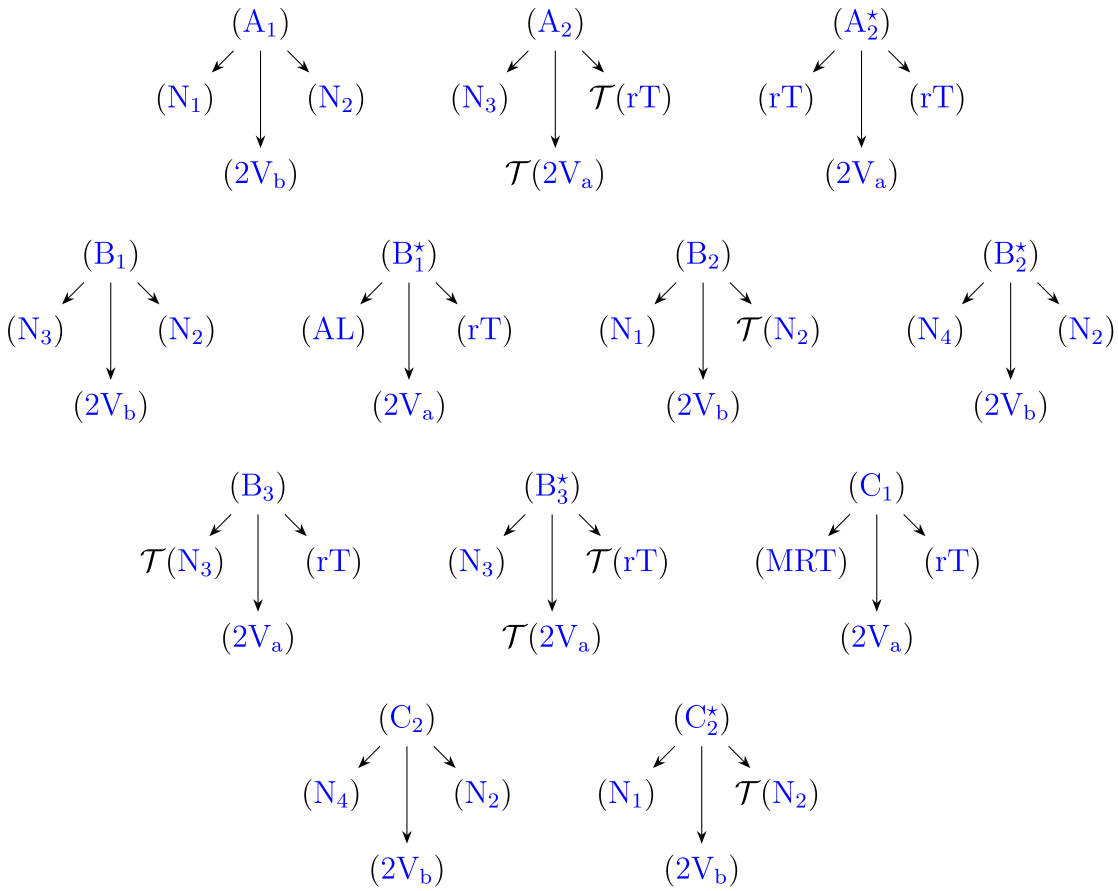
Proof. As with the analogous Theorem 4.13 in the linear case, the proof proceeds by direct construction. In Section 4.4.1.1, we examine the quadratic up Darboux transformation of each DNLS equation individually and present their reductions, which, as shown in Section 4.4.2, are directly related to the down case $M_\downarrow(2)$ in (4.8b). Note that, unlike Theorem 4.13, we do not claim that the full Darboux system $\mathscr{E}_\uparrow(2)$ belongs to the list, only its reductions are: at this stage, we are unable to make a definitive statement regarding the unreduced systems. □

Remark 4.22. Example 4.7 discusses the elementary Darboux transformations of the commutative (NLS) equation, leading to the scaling transformation, the Toda and the Merola-Ragnisco-Tu equations [7, 63]. In the lists above we recovered the scaling transformation (associated with $M(0)$ in Section 4.2), the relativistic Toda equation (rT) and the Merola-Ragnisco-Tu equation (MRT) with non-commutative constants. This, again, suggests the strong relationship between the NLS and DNLS equations already introduced in Remarks 2.1 and 2.7.

4.4.1.1 List of quadratic up Darboux transformations

As for the linear case, we consider all the DNLS equations and their $\star$-involutions presented in Section 2.3, with the aim of proving Theorem 4.21. Each case is studied separately to highlight the reduction procedure and the corresponding Lax representation.

The results of this section are summarised in the following reduction trees, according to the scheme in (4.28). Since in all cases the final reduction $\mathscr{E}_{\uparrow}^{a,\rho}(2)$ is trivial, we omit it.



Case A_1 The quadratic up Darboux transformation of (A_1) admits two constants of motion, α and β, which correspond to the auxiliary functions a and b.

The Lax representation (M, U), constructed from U in (2.29) and $M_{\uparrow}(2)$, is

$$\begin{cases} f_x = 2(p_1 q_1 f - f p q), \\ p_x = 2f^{-1}(p_1\beta - \alpha p) + 2(pq - f^{-1}p_1 q_1 f)p, \\ q_{1,x} = 2(q_1\alpha - \beta q)f^{-1} + 2q_1(fpqf^{-1} - p_1 q_1), \end{cases}$$
$$M = \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix}\lambda^2 + \begin{pmatrix} 0 & fp \\ q_1 f & 0 \end{pmatrix}\lambda + \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \tag{4.30}$$
$$U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\lambda^2 + 2\begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix}\lambda.$$

To the best of our knowledge, no further fundamental reduction is currently known. Therefore the system $\mathscr{E}_{\uparrow}(2)$ includes the auxiliary function f. Nonetheless, it is possible to operate optional reductions following the reduction tree in (4.28).

Remark 4.23. The Lax representation (2.29) is traceless, therefore in the commutative case $\det(M)$ is a constant of motion $\varrho = f^2 p q_1 - bf$ by Lemma 3.13 [66]. As discussed in Remark 3.15, there is no analogous quantity in the non-commutative framework. However, let us consider the non-commutative lift with ρ in (4.27)

$$\varrho = (q_1 f p - b)f, \qquad \varrho_x = 2\left((p_1 q_1 f - f q_1 p_1)\rho - f(pq\rho - \rho q p)\right). \tag{4.31}$$

It is immediate to notice that, in the commutative framework, the RHS of the latter expression vanishes.

(i) **Reduction $\boldsymbol{\alpha = 0}$.** The system $\mathscr{E}^{\alpha}_{\uparrow}(2)$ involves three variables p, q, and f, which can be rewritten in terms of the new variables u, v and p, defined by

$$u = q_1 f p, \qquad v = -q f^{-1} q_1^{-1},$$

Let us define $f = -p_1 u_1^{-1} v_1^{-1} u p^{-1}$ and $q = -vup^{-1}$. The Lax representation (M, U) and the Darboux system $\mathscr{E}_{\uparrow}^{\alpha}(2)$ become the following:

$$\begin{cases} u_x = 2v_1 u_1 (u - \beta) - 2(u - \beta) v u, \\ v_x = 2(v(u - \beta) - (u_{-1} - \beta_{-1}) v_{-1}) v, \\ p_x = 2p \left(u^{-1} v_1 u_1 (u - \beta) - vu \right), \end{cases}$$
$$M = \begin{pmatrix} -p_1 u_1^{-1} v_1^{-1} u p^{-1} & 0 \\ 0 & 0 \end{pmatrix} \lambda^2 + \begin{pmatrix} 0 & -p_1 u_1^{-1} v_1^{-1} u \\ up^{-1} & 0 \end{pmatrix} \lambda + \begin{pmatrix} 0 & 0 \\ 0 & \beta \end{pmatrix},$$
$$U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lambda^2 + 2 \begin{pmatrix} 0 & p \\ -vup^{-1} & 0 \end{pmatrix} \lambda.$$

Notice that the evolutions of the variables u and v form a nonlinear subsystem independent of p, whereas the evolution of p is linear and depends also on u and v. The variable p is eliminated from the Lax representation via a gauge transformation using $G = \operatorname{diag}(pu^{-1}, 1)$.

Remark 4.24. This phenomenon is analogous to what we already noticed when constructing linear Darboux transformations (compare with Remark 4.15). In relation to u and v, we interpret p as a gauge variable: an additional degree of freedom with no physical significance, reflecting the gauge freedom of the Lax representation.

We simplify the evolution of v in the system above by shifting $u \mapsto u + \beta$ and rescaling $M \mapsto -M$. As a consequence, system $\mathscr{E}_{\uparrow}^{\alpha}(2)$ is identified with (N_1), setting $\mu = \beta$.

$$\begin{cases} u_x = 2v_1(u_1 + \beta_1) u - 2uv(u + \beta), \\ v_x = 2(vuv - u_{-1} v_{-1} v), \end{cases}$$
$$M = \begin{pmatrix} v_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} \lambda^2 + \begin{pmatrix} 0 & v_1^{-1}(u + \beta) \\ -1 & 0 \end{pmatrix} \lambda - \begin{pmatrix} 0 & 0 \\ 0 & \beta \end{pmatrix}, \tag{4.32}$$
$$U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lambda^2 + 2 \begin{pmatrix} 0 & u + \beta \\ -v & 0 \end{pmatrix} \lambda + 2 \begin{pmatrix} \beta v & 0 \\ 0 & 0 \end{pmatrix}.$$

In Appendix C.2 we prove that (N_1) is equivalent to (AL) via a non-invertible Miura transformation.

(ii) Reduction $\boldsymbol{\beta = 0}$. As in the previous example, the system $\mathscr{E}^{\beta}_{\uparrow}(2)$ is expressed in terms of three variables p, q and f. It is possible to define a change of variables that separates it into a linear and a nonlinear part: let us introduce

$$u = -p_1 q_1, \qquad v = f^{-1},$$

considering $q = -p^{-1}u_{-1}$ and $f = v^{-1}$. In terms of the variables u, v, and p, the Lax representation (M, U) becomes

$$\begin{cases} u_x = 2\left(u\alpha v - v_1\alpha_1 u\right) + 2\left(v_1 u_1 v_1^{-1} u - u v^{-1} u_{-1} v\right), \\ v_x = 2(vu - u_{-1}v), \\ p_x = 2\left(vuv^{-1} - u_{-1} - v\alpha\right)p, \end{cases}$$
$$M = \begin{pmatrix} v^{-1} & 0 \\ 0 & 0 \end{pmatrix}\lambda^2 + \begin{pmatrix} 0 & v^{-1}p \\ -p_1^{-1}uv^{-1} & 0 \end{pmatrix}\lambda + \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix},$$
$$U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\lambda^2 + 2\begin{pmatrix} 0 & p \\ -p^{-1}u_{-1} & 0 \end{pmatrix}\lambda.$$

System $\mathscr{E}^{\beta}_{\uparrow}(2)$ shares the same features as the previous case: the evolution of the variables u and v is independent of p, which can be eliminated via a gauge transformation with $G = \operatorname{diag}(1, p^{-1}v)$. The reduced system corresponds to (N_2), assuming $\mu = \alpha$:

$$\begin{cases} u_x = 2\left(u\alpha v - v_1\alpha_1 u\right) + 2\left(v_1 u_1 v_1^{-1} u - u v^{-1} u_{-1} v\right), \\ v_x = 2(vu - u_{-1}v), \end{cases}$$
$$M = \begin{pmatrix} v^{-1} & 0 \\ 0 & 0 \end{pmatrix}\lambda^2 + \begin{pmatrix} 0 & 1 \\ -v_1^{-1}uv^{-1} & 0 \end{pmatrix}\lambda + \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix}, \tag{4.33}$$
$$U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\lambda^2 + 2\begin{pmatrix} 0 & v \\ -v^{-1}u_{-1} & 0 \end{pmatrix}\lambda - 2\begin{pmatrix} 0 & 0 \\ 0 & \alpha v \end{pmatrix}.$$

Remark 4.25. This reduction is examined for the commutative case in [66], where it is assumed that $p = r^2$ and $q = s_{-1}^2$. Therefore (4.31) becomes $f^2 r^2 s^2 = \varrho$. Fixing the constant equal to 1 and solving for f, they deduce a two-equation system, which in our convention reads

$$\begin{cases} r_x = (r^2 s_{-1}^2 - r_1^2 s^2 \mp \alpha r s) r, \\ s_x = (r^2 s_{-1}^2 - r_1^2 s^2 \pm \alpha r s) s. \end{cases}$$

Alternatively, we keep the conservation of ϱ and set $q_1 = f^{-2} p^{-1} \varrho$, which leads to

$$u = pq = p f_{-1}^{-2} p_{-1}^{-1} \varrho_{-1}, \qquad v = f_{-1}^{-1}.$$

The commutative system $\mathscr{E}_{\uparrow}^{\beta}(2)$ becomes equivalent to the commutative (rT), with $\mu = \alpha_{-1}$:

$$\begin{cases} u_x = 2u(\alpha_{-1} v - \alpha v_1 + u_{-1} - u_1), \\ v_x = 2v(u_{-1} - u). \end{cases}$$

We thus recognise the commutative relativistic Toda equation in the equation found by [66] via the change of variables $u = r^2 s_{-1}^2$ and $v = \pm r_{-1} s_{-1}$. Notice that the commutative version of (N_2) corresponds indeed to the commutative (rT). However, in Appendix C.2, we show that the non-commutative (N_2) is related to the non-commutative (AL) via a non-invertible Miura transformation.

(iii) Reduction $\boldsymbol{\rho = 0}$. Comparing with (4.27), we set $f = q_1^{-1} \beta p^{-1}$. The resulting system consists of two equations:

$$\begin{cases} p_x = 2p(q - \beta^{-1} q_1 \alpha) p, \\ q_{1,x} = q_1 (\alpha p \beta^{-1} - p_1) q_1, \end{cases}$$

$$M = \begin{pmatrix} q_1^{-1} \beta p^{-1} & 0 \\ 0 & 0 \end{pmatrix} \lambda^2 + \begin{pmatrix} 0 & q_1^{-1} \beta \\ \beta p^{-1} & 0 \end{pmatrix} \lambda + \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix},$$

$$U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lambda^2 + 2 \begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix} \lambda.$$

This evolution corresponds to (2V$_\mathrm{b}$) if we change variables from p and q to u and v:

$$u = -\beta^{-1} q_1 \alpha p, \qquad v = -\beta_{-1}^{-1} q p.$$

We express $\alpha = p_1 v_1^{-1} u p^{-1}$ and $q = -\beta_{-1} v p^{-1}$. The new Lax representation of $\mathscr{E}_{\uparrow}^{\rho}(2)$ is

$$\begin{cases} u_x = 2(v_1 \beta u - u \beta_{-1} v), \\ v_x = 2(vu - u_{-1} v), \\ p_x = 2p\left(u - \beta_{-1} v\right), \end{cases}$$

$$M = \begin{pmatrix} -p_1 v_1^{-1} p^{-1} & 0 \\ 0 & 0 \end{pmatrix} \lambda^2 + \begin{pmatrix} 0 & -p_1 v_1^{-1} \\ \beta p^{-1} & 0 \end{pmatrix} \lambda + \begin{pmatrix} p_1 v_1^{-1} u p^{-1} & 0 \\ 0 & \beta \end{pmatrix},$$

$$U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lambda^2 + 2 \begin{pmatrix} 0 & p \\ -\beta_{-1} v p^{-1} & 0 \end{pmatrix} \lambda.$$

The evolution of the variables u and v is independent of p, which can be eliminated via a gauge transformation using $G = \operatorname{diag}(p, \beta_{-1})$. If we assume $\mu = \beta_{-1}$, we recover the system (2V$_\mathrm{b}$):

$$\begin{cases} u_x = 2(v_1 \beta u - u \beta_{-1} v), \\ v_x = 2(vu - u_{-1} v), \end{cases}$$

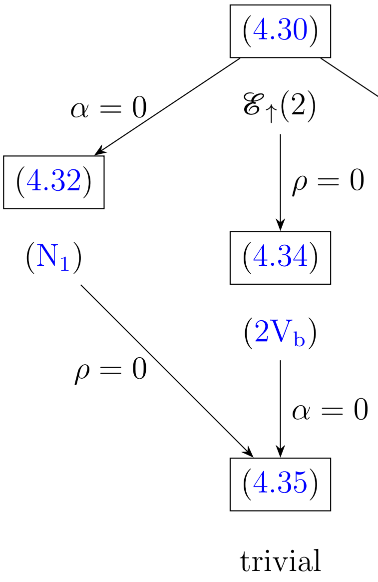
$$M = \begin{pmatrix} v_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} \lambda^2 + \begin{pmatrix} 0 & v_1^{-1} \beta_{-1} \\ -1 & 0 \end{pmatrix} \lambda - \begin{pmatrix} v_1^{-1} u & 0 \\ 0 & \beta_{-1} \end{pmatrix}, \tag{4.34}$$

$$U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lambda^2 + 2 \begin{pmatrix} 0 & \beta_{-1} \\ -v & 0 \end{pmatrix} \lambda + 2 \begin{pmatrix} \beta_{-1} v - u & 0 \\ 0 & 0 \end{pmatrix}.$$

(iv) Reduction $\boldsymbol{\alpha = 0}$ and $\boldsymbol{\rho = 0}$. Finally, we consider the reduction $\mathscr{E}_{\uparrow}^{\alpha,\rho}(2)$, where both α and ρ are set to zero. Starting from $\mathscr{E}_{\uparrow}^{\rho}(2)$ in (4.34) and imposing $\alpha = 0$, it is evident that $u = 0$ and $v_x = 0$. The Lax pair itself is formed by constant matrices. We refer to such a situation as trivial.

$$\begin{aligned} &v_x = 0, \\ &M = \begin{pmatrix} v_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} \lambda^2 + \begin{pmatrix} 0 & v_1^{-1} \\ -\beta & 0 \end{pmatrix} \lambda - \begin{pmatrix} 0 & 0 \\ 0 & \beta \end{pmatrix}, \\ &U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lambda^2 + 2 \begin{pmatrix} 0 & 1 \\ -\beta_{-1} v & 0 \end{pmatrix} \lambda + 2 \begin{pmatrix} \beta_{-1} v & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{4.35}$$

The reduction tree (4.28) associated with the Darboux transformation $M_\uparrow(2)$ of the equation (A_1) is therefore



This derivation, presented in full detail, serves as a paradigm: the other DNLS equations follow a similar pattern. To avoid unnecessary repetitions, we tend to omit the intermediate steps and focus on presenting the final results. For the same reason, we omit the reductions $\mathscr{E}_\uparrow^{\alpha,\rho}(2)$.

Case A_2 The quadratic up Darboux transformation of the system (A_2) is described by the following system of equations:

$$\begin{cases} a_x = 2(q_1p_1a - aqp), \\ b_x = 2(q_1p_1b - bqp), \\ f_x = 2(p_1q_1 + q_1p_1)f - 2f(pq + qp), \\ p_x = 2f^{-1}(p_1b - ap) + 2\left(qp - f^{-1}p_1q_1f\right)p, \\ q_{1,x} = 2(q_1a - bq)f^{-1} - 2q_1\left(q_1p_1 - fpqf^{-1}\right). \end{cases}$$

This system admits a constant of motion ρ given by (4.27). Using it to express $b = q_1fp - \rho$, we derive

$$\begin{cases} a_x = 2(q_1p_1a - aqp), \\ f_x = 2(p_1q_1 + q_1p_1)f - 2f(pq + qp), \\ p_x = -2f^{-1}\left(ap + p_1\rho\right) + 2qp^2, \\ q_{1,x} = 2\left(q_1a + \rho q\right)f^{-1} - 2q_1^2p_1. \end{cases}$$

Since no further fundamental reduction is known, we consider the three optional reductions presented in (4.28).

(i) Reduction $\boldsymbol{a = 0}$. Consider the variables

$$u = -p^{-1}f^{-1}p_1, \qquad v = -qp.$$

The transformation defined by $f = -p_1u^{-1}p^{-1}$ and $q = -vp^{-1}$ corresponds to system (N_3) with $\mu = \rho$. The evolution of the variables u and v is independent of p, which can be eliminated via a gauge transformation using

$G = \mathrm{diag}(p, 1)$.

$$\begin{cases} u_x = 2(uv_1 - vu + uu_1\rho_1 - u\rho u), \\ v_x = 2(vu\rho - \rho_{-1}v_{-1}u_{-1}), \end{cases}$$

$$M = \begin{pmatrix} u^{-1} & 0 \\ 0 & 0 \end{pmatrix} \lambda^2 + \begin{pmatrix} 0 & u^{-1} \\ -v_1u^{-1} & 0 \end{pmatrix} \lambda - \begin{pmatrix} 0 & 0 \\ 0 & v_1u^{-1} - \rho \end{pmatrix}, \tag{4.36}$$

$$U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lambda^2 + 2\begin{pmatrix} 0 & 1 \\ -v & 0 \end{pmatrix} \lambda - 2\begin{pmatrix} u\rho & 0 \\ 0 & v \end{pmatrix}.$$

(ii) Reduction $\boldsymbol{b = 0}$. Consider the variables

$$u = -qp, \qquad v = -\rho^{-1}q_1ap\rho^{-1}.$$

The transformation defined by $q = -up^{-1}$ and $a = p_1u_1^{-1}\rho v\rho p^{-1}$ corresponds to the $\mathcal{T}$-reflection of (rT) with $\mu = \rho$. The evolution of the variables u and v is independent of p, which can be eliminated via a gauge transformation using $G = \mathrm{diag}(p, 1)$.

$$\begin{cases} u_x = 2(uv\rho - \rho_{-1}v_{-1}u + u\rho^{-1}u_1\rho - \rho_{-1}u_{-1}\rho_{-1}^{-1}u), \\ v_x = 2(vu_1 - uv), \end{cases}$$

$$M = \begin{pmatrix} u_1^{-1}\rho & 0 \\ 0 & 0 \end{pmatrix} \lambda^2 + \begin{pmatrix} 0 & u_1^{-1}\rho \\ -\rho & 0 \end{pmatrix} \lambda - \begin{pmatrix} u_1^{-1}\rho v\rho & 0 \\ 0 & 0 \end{pmatrix}, \tag{4.37}$$

$$U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lambda^2 + 2\begin{pmatrix} 0 & 1 \\ -u & 0 \end{pmatrix} \lambda - 2\begin{pmatrix} (v + \rho^{-1}u_1)\rho & 0 \\ 0 & u \end{pmatrix}.$$

(iii) Reduction $\boldsymbol{\rho = 0}$. When $\rho = 0$, a new constant of motion appears:

$$\gamma = q_1ap, \qquad \gamma_x = 0. \tag{4.38}$$

Let us introduce the new dependent variables

$$u = qp, \qquad v = (q_1fp)^{-1}.$$

The transformation defined by $q = up^{-1}$ and $f = p_1(pvu_1)^{-1}$ corresponds to the $\mathcal{T}$-reflection of (2V$_\mathrm{a}$) with $\mu = \gamma$. The evolution of the variables u and v is independent of p, which can be eliminated via a gauge transformation using $G = \operatorname{diag}(p, 1)$.

$$\begin{cases} u_x = 2(\gamma_{-1}v_{-1}u - uv\gamma), \\ v_x = 2(uv - vu_1), \end{cases}$$

$$M = \begin{pmatrix} (vu_1)^{-1} & 0 \\ 0 & 0 \end{pmatrix}\lambda^2 + \begin{pmatrix} 0 & (vu_1)^{-1} \\ v^{-1} & 0 \end{pmatrix}\lambda + \begin{pmatrix} u_1^{-1}\gamma & 0 \\ 0 & v^{-1} \end{pmatrix}, \tag{4.39}$$

$$U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\lambda^2 + 2\begin{pmatrix} 0 & 1 \\ u & 0 \end{pmatrix}\lambda + 2\begin{pmatrix} v\gamma & 0 \\ 0 & u \end{pmatrix}.$$

Case A$_2^\star$ The quadratic up Darboux transformation of (A$_2^\star$) admits the constant of motion $\phi = f$ and is described by the following system of equations:

$$\begin{cases} a_x = 2\left(apq - p_1q_1a\right), \\ b_x = 2\left(bpq - p_1q_1b\right), \\ p_x = 2\phi^{-1}\left(p_1b - ap\right) + 2\left(p^2q - \phi^{-1}p_1q_1\phi p\right), \\ q_{1,x} = 2\left(q_1a - bq\right)\phi^{-1} - 2\left(p_1q_1^2 - q_1\phi pq\phi^{-1}\right). \end{cases}$$

(i) Reduction $\boldsymbol{a = 0}$. Consider the new variables

$$u = \phi^{-1}p_1\left(q_1\phi - bp^{-1}\right), \qquad v = pb_{-1}p_{-1}^{-1}.$$

The transformation defined by $q = p^{-1}(v + \phi_{-1}u_{-1})\phi_{-1}^{-1}$ and $b = p_1^{-1}v_1p$ corresponds to (rT) with $\mu = \phi_{-1}^{-1}$. The evolution of the variables u and v is independent of p, which can be eliminated via a gauge transformation using

$G = \mathrm{diag}(1, p^{-1})$.

$$
\begin{aligned}
&\begin{cases}
u_x = 2(uv\phi_{-1}^{-1} - \phi^{-1}v_1u) + 2\left(u\phi_{-1}u_{-1}\phi_{-1}^{-1} - \phi^{-1}u_1\phi u\right), \\
v_x = 2(vu_{-1} - uv),
\end{cases} \\
&M = \begin{pmatrix} \phi & 0 \\ 0 & 0 \end{pmatrix} \lambda^2 + \begin{pmatrix} 0 & \phi \\ \phi u + v_1 & 0 \end{pmatrix} \lambda + \begin{pmatrix} 0 & 0 \\ 0 & v_1 \end{pmatrix}, \\
&U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lambda^2 + 2 \begin{pmatrix} 0 & 1 \\ \left(v + \phi_{-1}u_{-1}\right)\phi_{-1}^{-1} & 0 \end{pmatrix} \lambda + \\
&\qquad - 2 \begin{pmatrix} \left(v + \phi_{-1}u_{-1}\right)\phi_{-1}^{-1} & 0 \\ 0 & u \end{pmatrix}.
\end{aligned}
\tag{4.40}
$$

(ii) Reduction $b = 0$. Consider the variables

$$u = pq, \qquad v = a_{-1}.$$

The transformation defined by $q = p^{-1}u$ and $a = v_1$ corresponds to (rT) with $\mu = \phi_{-1}^{-1}$. The evolution of the variables u and v is independent of p, which can be eliminated via a gauge transformation using $G = \mathrm{diag}(1, p^{-1})$.

$$
\begin{aligned}
&\begin{cases}
u_x = 2(uv\phi_{-1}^{-1} - \phi^{-1}v_1u) + 2\left(u\phi_{-1}u_{-1}\phi_{-1}^{-1} - \phi^{-1}u_1\phi u\right), \\
v_x = 2(vu_{-1} - uv),
\end{cases} \\
&M = \begin{pmatrix} \phi & 0 \\ 0 & 0 \end{pmatrix} \lambda^2 + \begin{pmatrix} 0 & \phi \\ u_1\phi & 0 \end{pmatrix} \lambda + \begin{pmatrix} v_1 & 0 \\ 0 & 0 \end{pmatrix}, \\
&U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lambda^2 + 2 \begin{pmatrix} 0 & 1 \\ u & 0 \end{pmatrix} \lambda - 2 \begin{pmatrix} u & 0 \\ 0 & \phi^{-1}\left(u_1\phi + v_1\right) \end{pmatrix}.
\end{aligned}
\tag{4.41}
$$

(iii) Reduction $\rho = 0$. Fixing $b = q_1\phi p$, consider the variables

$$u = -pq, \qquad v = -a_{-1}.$$

The transformation defined by $q=-p^{-1}u$ and $a=-v_1$ corresponds to (2V$_\mathrm{a}$) with $\mu=\phi_{-1}^{-1}$. The evolution of the variables u and v is independent of p, which can be eliminated via a gauge transformation using $G=\operatorname{diag}(1,p^{-1})$.

$$\begin{aligned}
&\begin{cases} u_x = 2\left(\phi^{-1}v_1u - uv\phi_{-1}^{-1}\right), \\ v_x = 2\left(uv - vu_{-1}\right), \end{cases}\\
&M = \begin{pmatrix}\phi & 0\\ 0 & 0\end{pmatrix}\lambda^2 + \begin{pmatrix}0 & \phi\\ -u_1\phi & 0\end{pmatrix}\lambda - \begin{pmatrix}v_1 & 0\\ 0 & u_1\phi\end{pmatrix},\\
&U = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\lambda^2 + 2\begin{pmatrix}0 & 1\\ -u & 0\end{pmatrix}\lambda + 2\begin{pmatrix}u & 0\\ 0 & \phi^{-1}v_1\end{pmatrix}.
\end{aligned} \tag{4.42}$$

Case B$_1$ The quadratic up Darboux transformation of (B$_1$) admits the constant of motion $\alpha=a$ and is described by the following system of equations:

$$\begin{cases} b_x = 2(q_1p_1b - bqp), \\ f_x = 2(p_1q_1f - fpq), \\ p_x = 2f^{-1}(p_1b - \alpha p) - 2f^{-1}p_1q_1fp, \\ q_{1,x} = 2(q_1\alpha - bq)f^{-1} + 2q_1fpqf^{-1}, \end{cases}$$

which admits ρ in (4.27) as a constant of motion. Using it to express $b=q_1fp-\rho$, we derive

$$\begin{cases} f_x = 2\left(p_1q_1f - fpq\right), \\ p_x = -2f^{-1}\left(\alpha p + p_1\rho\right), \\ q_{1,x} = 2\left(q_1\alpha + \rho q\right)f^{-1}. \end{cases} \tag{4.43}$$

(i) Reduction $\boldsymbol{\alpha=0}$. Let us consider the variables

$$u=-p^{-1}f^{-1}p_1, \qquad v=-qp.$$

The transformation defined by $q = -vp^{-1}$ and $f = -p_1u^{-1}p^{-1}$ corresponds to (N$_3$) with $\mu = \rho$:

$$\begin{cases} u_x = 2(uv_1 - vu + uu_1\rho_1 - u\rho u), \\ v_x = 2(vu\rho - \rho_{-1}v_{-1}u_{-1}). \end{cases}$$

The evolution of the variables u and v is independent of p, which can be eliminated via a gauge transformation using $G = \operatorname{diag}(p, 1)$. The resulting Lax representation is identical to (4.36).

(ii) **Reduction $\boldsymbol{b = 0}$.** Let us consider the variables

$$u = -p_1q_1, \qquad v = f^{-1}.$$

The transformation defined by $q = -p^{-1}u_{-1}$ and $f = v^{-1}$ corresponds to (N$_2$) with $\mu = \alpha$:

$$\begin{cases} u_x = 2(u\alpha v - v_1\alpha_1u) + 2\left(v_1u_1v_1^{-1}u - uv^{-1}u_{-1}v\right), \\ v_x = 2(vu - u_{-1}v). \end{cases}$$

The evolution of the variables u and v is independent of p, which can be eliminated via a gauge transformation using $G = \operatorname{diag}(1, p^{-1}v)$. The resulting Lax representation is identical to (4.33).

(iii) **Reduction $\boldsymbol{\rho = 0}$.** Let us consider the variables

$$u = -p_1q_1, \qquad v = -f^{-1}.$$

The transformation defined by $q = -p^{-1}u_{-1}$ and $f = -v^{-1}$ corresponds to (2V$_\mathrm{b}$) with $\mu = \alpha$. The evolution of the variables u and v is independent of p, which can be eliminated via a gauge transformation using $G = \mathrm{diag}(1, p^{-1})$.

$$\begin{cases} u_x = 2(v_1\alpha_1 u - u\alpha v), \\ v_x = 2(vu - u_{-1}v), \end{cases}$$
$$M = \begin{pmatrix} v^{-1} & 0 \\ 0 & 0 \end{pmatrix} \lambda^2 + \begin{pmatrix} 0 & v^{-1} \\ -uv^{-1} & 0 \end{pmatrix} \lambda - \begin{pmatrix} \alpha & 0 \\ 0 & uv^{-1} \end{pmatrix}, \tag{4.44}$$
$$U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lambda^2 + 2\begin{pmatrix} 0 & 1 \\ -u_{-1} & 0 \end{pmatrix} \lambda + 2\begin{pmatrix} 0 & 0 \\ 0 & v\alpha - u_{-1} \end{pmatrix}.$$

Case $\mathrm{B}_1^\star$ The quadratic up Darboux transformation of ($\mathrm{B}_1^\star$) admits two constants of motion $\phi = f$ and $\beta = b$, and is described by the following system of equations:

$$\begin{cases} a_x = 2(apq - p_1q_1a), \\ p_x = 2\phi^{-1}(p_1\beta - ap) - 2\phi^{-1}p_1q_1\phi p, \\ q_{1,x} = 2(q_1a - \beta q)\phi^{-1} + 2q_1\phi pq\phi^{-1}. \end{cases} \tag{4.45}$$

(i) **Reduction $\boldsymbol{a = 0}$.** In this case, the system $\mathscr{E}^a_\uparrow(2)$ already consists of two equations. With the change of variables given by $u = \phi p$ and $v = q_1$, it corresponds to (AL) with $\mu = \phi^{-1}$ and $\nu = \beta$:

$$\begin{cases} u_x = 2\phi_1^{-1}u_1(\beta - vu), \\ v_x = 2(vu - \beta)v_{-1}\phi^{-1}, \end{cases}$$
$$M = \begin{pmatrix} \phi & 0 \\ 0 & 0 \end{pmatrix} \lambda^2 + \begin{pmatrix} 0 & u \\ v\phi & 0 \end{pmatrix} \lambda + \begin{pmatrix} 0 & 0 \\ 0 & \beta \end{pmatrix}, \tag{4.46}$$
$$U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lambda^2 + 2\begin{pmatrix} 0 & \phi^{-1}u \\ v_{-1} & 0 \end{pmatrix} \lambda - 2\begin{pmatrix} \phi^{-1}uv_{-1} & 0 \\ 0 & 0 \end{pmatrix}.$$

(ii) **Reduction $\boldsymbol{\beta = 0}$.** Let us consider the variables

$$u = pq, \qquad v = a_{-1}.$$

The transformation defined by $q = p^{-1}u$ and $a = v_1$ corresponds to (rT) with $\mu = \phi_{-1}^{-1}$:

$$\begin{cases} u_x = 2(uv\phi_{-1}^{-1} - \phi^{-1}v_1u) + 2\left(u\phi_{-1}u_{-1}\phi_{-1}^{-1} - \phi^{-1}u_1\phi u\right), \\ v_x = 2(vu_{-1} - uv). \end{cases}$$

The evolution of the variables u and v is independent of p, which can be eliminated via a gauge transformation using $G = \mathrm{diag}(1, p^{-1})$. The resulting Lax representation is identical to (4.41).

(iii) **Reduction $\boldsymbol{\rho = 0}$.** We use this condition to set $q_1 = \beta p^{-1}\phi^{-1}$, eliminating the variable q. We then consider the variables

$$u = -pq = -p\beta_{-1}p_{-1}^{-1}\phi_{-1}^{-1}, \qquad v = -a_{-1}.$$

The resulting transformation, defined by $\beta = -p_1^{-1}u_1\phi p$ and $a = -v_1$, corresponds to ($2\mathrm{V_a}$) with $\mu = \phi_{-1}^{-1}$:

$$\begin{cases} u_x = 2(\phi^{-1}v_1u - uv\phi_{-1}^{-1}), \\ v_x = 2(uv - vu_{-1}). \end{cases}$$

The evolution of the variables u and v is independent of p, which can be eliminated via a gauge transformation using $G = \mathrm{diag}(1, p^{-1})$. The resulting Lax representation is identical to (4.42).

Case B_2 The quadratic up Darboux transformation of (B_2) admits the constant of motion $\beta = b$ and is described by the following system of equations:

$$\begin{cases} a_x = 2(aqp - q_1p_1a), \\ f_x = 2(p_1q_1 - q_1p_1)f - 2f(pq - qp), \\ p_x = 2f^{-1}(p_1\beta - ap) + 2\left(pq - qp - f^{-1}p_1q_1f\right)p, \\ q_{1,x} = 2(q_1a - \beta q)f^{-1} - 2q_1\left(p_1q_1 - q_1p_1 - fpqf^{-1}\right). \end{cases}$$

(i) Reduction $\boldsymbol{a = 0}$. Let us consider the variables

$$u = q_1 f p - \beta, \qquad v = -q f^{-1} q_1^{-1}.$$

The transformation defined by $q = -v(u+\beta)p^{-1}$ and $f = -p_1(u_1+\beta_1)^{-1}v_1^{-1}(u+\beta)p^{-1}$ corresponds to (N_1) with $\mu = \beta$:

$$\begin{cases} u_x = 2v_1(u_1+\beta_1)u - 2uv(u+\beta), \\ v_x = 2(vuv - u_{-1}v_{-1}v). \end{cases}$$

The evolution of the variables u and v is independent of p, which can be eliminated via a gauge transformation using $G = \mathrm{diag}(p(u+\beta)^{-1}, 1)$. The resulting Lax representation is identical to (4.32).

(ii) Reduction $\boldsymbol{\beta = 0}$. The system admits a new constant of motion

$$\theta = (q_1 f a^{-1} f p)^{-1}, \qquad \theta_x = 0. \tag{4.47}$$

We employ θ to determine the auxiliary function a and we introduce the variables

$$u = qp, \qquad v = -q_1 f p.$$

The transformation defined by $q = up^{-1}$ and $f = -p_1 u_1^{-1} v p^{-1}$ corresponds to the $\mathcal{T}$-reflection of the system (N_2) with $\mu = \theta$. The evolution of the variables u and v is independent of p, which can be eliminated via a gauge transformation using $G = \mathrm{diag}(pv^{-1}, 1)$.

$$\begin{cases} u_x = 2(u\theta v - v_{-1}\theta_{-1}u) + 2\left(v_{-1}u_{-1}v_{-1}^{-1}u - uv^{-1}u_1 v\right), \\ v_x = 2(vu - u_1 v), \end{cases}$$

$$M = \begin{pmatrix} v_1 u_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}\lambda^2 + \begin{pmatrix} 0 & v_1 u_1^{-1} v \\ 1 & 0 \end{pmatrix}\lambda - \begin{pmatrix} v_1 u_1^{-1} v\theta & 0 \\ 0 & 0 \end{pmatrix}, \tag{4.48}$$

$$U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\lambda^2 + 2\begin{pmatrix} 0 & v \\ uv^{-1} & 0 \end{pmatrix}\lambda - 2\begin{pmatrix} v\theta & 0 \\ 0 & 0 \end{pmatrix}.$$

(iii) Reduction $\boldsymbol{\rho = 0}$. From the reduction, we set $f = q_1^{-1}\beta p^{-1}$ and introduce the variables

$$u = -\beta^{-1} q_1 a p, \qquad v = -\beta_{-1}^{-1} q p.$$

The transformation defined by $q = -\beta_{-1} v p^{-1}$ and $a = p_1 v_1^{-1} u p^{-1}$ corresponds to (2V$_\mathrm{b}$) with $\mu = \beta_{-1}$:

$$\begin{cases} u_x = 2(v_1 \beta u - u \beta_{-1} v), \\ v_x = 2(vu - u_{-1} v). \end{cases}$$

The evolution of the variables u and v is independent of p, which can be eliminated via a gauge transformation using $G = \mathrm{diag}(p, \beta_{-1})$. The resulting Lax representation is identical to (4.34).

Case B$_2^\star$ The quadratic up Darboux transformation of (B$_2^\star$) admits the constant of motion $\alpha = a$ and is described by the following system of equations:

$$\begin{cases} b_x = 2\left(p_1 q_1 b - b p q\right), \\ f_x = 2\left(p_1 q_1 f - f p q\right), \\ p_x = 2 f^{-1}\left(p_1 b - \alpha p\right) + 2\left(p q p - p^2 q - f^{-1} p_1 q_1 f p\right), \\ q_{1,x} = 2\left(q_1 \alpha - b q\right) f^{-1} - 2\left(q_1 p_1 q_1 - p_1 q_1^2 - q_1 f p q f^{-1}\right). \end{cases}$$

(i) Reduction $\boldsymbol{\alpha = 0}$. Let us consider the variables

$$u = p_1(q_1 f p - b) p^{-1}, \qquad v = -pq, \qquad w = -f^{-1}.$$

The transformation defined by $q = -p^{-1} v$, $b = p_1^{-1}(v_1 w^{-1} - u) p$, and $f = w^{-1}$ corresponds to (N$_4$). The evolution of the variables u and v is independent of p, which can be eliminated via a gauge transformation using

$G = \mathrm{diag}(1, p^{-1})$.

$$\begin{cases} u_x = 2\left(w_1 u_1 - uw\right) u, \\ v_x = 2\left(wuv - u_{-1} v_{-1} w_{-1}\right), \\ w_x = 2(wv_1 - vw), \end{cases}$$

$$M = \begin{pmatrix} w^{-1} & 0 \\ 0 & 0 \end{pmatrix} \lambda^2 + \begin{pmatrix} 0 & w^{-1} \\ -v_1 w^{-1} & 0 \end{pmatrix} \lambda - \begin{pmatrix} 0 & 0 \\ 0 & v_1 w^{-1} - u \end{pmatrix}, \tag{4.49}$$

$$U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lambda^2 + 2 \begin{pmatrix} 0 & 1 \\ -v & 0 \end{pmatrix} \lambda + 2 \begin{pmatrix} 0 & 0 \\ 0 & wu - v \end{pmatrix}.$$

(ii) Reduction $\boldsymbol{b = 0}$. Let us consider the variables

$$u = -p_1 q_1, \qquad v = f^{-1}.$$

The transformation defined by $q = -p^{-1} u_{-1}$ and $f = v^{-1}$ corresponds to (N_2) with $\mu = \alpha$:

$$\begin{cases} u_x = 2\left(u\alpha v - v_1 \alpha_1 u + v_1 u_1 v_1^{-1} u - u v^{-1} u_{-1} v\right), \\ v_x = 2\left(vu - u_{-1} v\right). \end{cases}$$

The evolution of the variables u and v is independent of p, which can be eliminated via a gauge transformation using $G = \mathrm{diag}(1, p_{-1}^{-1} v)$. The resulting Lax representation is identical to (4.33).

(iii) Reduction $\boldsymbol{\rho = 0}$. From the reduction, we set $b = q_1 f p$ and introduce the variables

$$u = -p_1 q_1, \qquad v = -f^{-1}.$$

The transformation defined by $q = -p^{-1} u_{-1}$ and $f = -v^{-1}$ corresponds to ($2\mathrm{V}_\mathrm{b}$) with $\mu = \alpha$:

$$\begin{cases} u_x = 2\left(v_1 \alpha_1 u - u\alpha v\right), \\ v_x = 2\left(vu - u_{-1} v\right). \end{cases}$$

The evolution of the variables u and v is independent of p, which can be eliminated via a gauge transformation using $G = \mathrm{diag}(1, p^{-1})$. The resulting Lax representation is identical to (4.44).

Case B$_3$ The quadratic up Darboux transformation of (B$_3$) admits the constant of motion $\phi = f$ and is described by the following system of equations:

$$\begin{cases} a_x = 2(apq - p_1 q_1 a), \\ b_x = 2(q_1 p_1 - p_1 q_1) b - 2b(qp - pq), \\ p_x = 2\phi^{-1}(p_1 b - ap) + 2\left(p^2 q - pqp - \phi^{-1} p_1 q_1 \phi p\right), \\ q_{1,x} = 2(q_1 a - bq)\phi^{-1} - 2\left(p_1 q_1^2 - q_1 p_1 q_1 - q_1 \phi p q \phi^{-1}\right). \end{cases}$$

(i) Reduction $\boldsymbol{a = 0}$. Let us consider the variables

$$u_1 = (\phi p - q_1^{-1} b) q, \qquad v = pq.$$

The transformation defined by $q = p^{-1} v$ and $b = p_1^{-1} v_1 (\phi - u_1 v^{-1}) p$ corresponds to the $\mathcal{T}$-reflection of the system (N$_3$) with $\mu = \phi_{-1}^{-1}$. The evolution of the variables u and v is independent of p, which can be eliminated via a gauge transformation using $G = \mathrm{diag}(1, p^{-1} v)$.

$$\begin{cases} u_x = 2(uv_{-1} - vu) + 2\left(u u_{-1} \phi_{-2}^{-1} - u \phi_{-1}^{-1} u\right), \\ v_x = 2(vu\phi_{-1}^{-1} - \phi^{-1} v_1 u_1), \end{cases}$$

$$M = \begin{pmatrix} \phi & 0 \\ 0 & 0 \end{pmatrix} \lambda^2 + \begin{pmatrix} 0 & \phi v \\ \phi & 0 \end{pmatrix} \lambda + \begin{pmatrix} 0 & 0 \\ 0 & \phi v - u_1 \end{pmatrix}, \tag{4.50}$$

$$U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lambda^2 + 2 \begin{pmatrix} 0 & v \\ 1 & 0 \end{pmatrix} \lambda - 2 \begin{pmatrix} v & 0 \\ 0 & u\phi_{-1}^{-1} \end{pmatrix}.$$

(ii) Reduction $\boldsymbol{b = 0}$. Let us consider the variables

$$u = pq, \qquad v = a_{-1}.$$

The transformation defined by $q = p^{-1}u$ and $a = v_1$ corresponds to (rT) with $\mu = \phi_{-1}^{-1}$:

$$\begin{cases} u_x = 2(uv\phi_{-1}^{-1} - \phi^{-1}v_1u) + 2\left(u\phi_{-1}u_{-1}\phi_{-1}^{-1} - \phi^{-1}u_1\phi u\right), \\ v_x = 2(vu_{-1} - uv). \end{cases}$$

The evolution of the variables u and v is independent of p, which can be eliminated via a gauge transformation using $G = \operatorname{diag}(1, p^{-1})$. The resulting Lax representation is identical to (4.41).

(iii) **Reduction $\boldsymbol{\rho = 0}$.** From the reduction, we set $b = q_1fp$ and introduce the variables

$$u = -pq, \qquad v = -a_{-1}.$$

The transformation defined by $q = -p^{-1}u$ and $a = -v_1$ corresponds to ($2\mathrm{V_a}$) with $\mu = \phi_{-1}^{-1}$:

$$\begin{cases} u_x = 2(\phi^{-1}v_1u - uv\phi_{-1}^{-1}), \\ v_x = 2(uv - vu_{-1}). \end{cases}$$

The evolution of the variables u and v is independent of p, which can be eliminated via a gauge transformation using $G = \operatorname{diag}(1, p^{-1})$. The resulting Lax representation is identical to (4.42).

Case $\mathrm{B}_3^\star$ The quadratic up Darboux transformation of ($\mathrm{B}_3^\star$) is described by the following system of equations:

$$\begin{cases} a_x = 2\left(q_1p_1 - p_1q_1\right)a - 2a\left(qp - pq\right), \\ b_x = 2\left(q_1p_1b - bqp\right), \\ f_x = 2\left(q_1p_1f - fqp\right), \\ p_x = 2f^{-1}\left(p_1b - ap\right) + 2\left(qp - pq - f^{-1}p_1q_1f\right)p, \\ q_{1,x} = 2\left(q_1a - bq\right)f^{-1} - 2q_1\left(q_1p_1 - p_1q_1 - fpqf^{-1}\right), \end{cases}$$

that admits ρ in (4.27) as a constant of motion. Using ρ to express $b = q_1 f p - \rho$, we derive:

$$\begin{cases} a_x = 2\left(q_1 p_1 - p_1 q_1\right) a - 2a\left(qp - pq\right), \\ f_x = 2\left(q_1 p_1 f - f qp\right), \\ p_x = -2 f^{-1}\left(ap + p_1 \rho\right) + 2\left(qp - pq\right) p, \\ q_{1,x} = 2\left(q_1 a + \rho q\right) f^{-1} - 2 q_1\left(q_1 p_1 - p_1 q_1\right). \end{cases}$$

(i) Reduction $\boldsymbol{a = 0}$. Let us consider the variables

$$u = -p^{-1} f^{-1} p_1, \qquad v = -qp.$$

The transformation defined by $q = -vp^{-1}$ and $f = -p_1 u^{-1} p^{-1}$ yields the system (N$_3$) under $\mu = \rho$:

$$\begin{cases} u_x = 2\left(uv_1 - vu + uu_1\rho_1 - u\rho u\right), \\ v_x = 2\left(vu\rho - \rho_{-1} v_{-1} u_{-1}\right). \end{cases}$$

The evolution of the variables u and v is independent of p, which can be eliminated via a gauge transformation using $G = \operatorname{diag}(p, 1)$. The resulting Lax pair is identical to (4.36).

(ii) Reduction $\boldsymbol{b = 0}$. Let us consider the variables

$$u = -qp, \qquad v = -\rho^{-1} q_1 a p \rho^{-1},$$

with $f = q_1^{-1} \rho p^{-1}$ from (4.27). The transformation defined by $q = -up^{-1}$ and $a = p_1 u_1^{-1} \rho v \rho p^{-1}$ corresponds to the $\mathcal{T}$-reflection of the system (rT) with $\mu = \rho$:

$$\begin{cases} u_x = 2\left(uv\rho - \rho_{-1} v_{-1} u + u\rho^{-1} u_1 \rho - \rho_{-1} u_{-1} \rho_{-1}^{-1} u\right), \\ v_x = 2\left(vu_1 - uv\right). \end{cases}$$

The evolution of the variables u and v is independent of p, which can be eliminated via a gauge transformation using $G = \mathrm{diag}(p, 1)$. The resulting Lax representation is identical to (4.37).

(iii) Reduction $\boldsymbol{\rho = 0}$. This system admits the constant of motion γ defined in (4.38), which allows us to eliminate the auxiliary function a. Let us consider the variables

$$u = (q_1 f p)^{-1}\gamma, \qquad v = \gamma^{-1} q_1 p_1.$$

The transformation defined by $q = \gamma_{-1} v_{-1} p^{-1}$ and $f = p_1 (puv)^{-1}$ corresponds to the $\mathcal{T}$-reflection of the system ($2\mathrm{V_a}$) with $\mu = \gamma$. The evolution of the variables u and v is independent of p, which can be eliminated via a gauge transformation using $G = \mathrm{diag}(pv_{-1}^{-1}, 1)$.

$$\begin{cases} u_x = 2\left(\gamma_{-1} v_{-1} u - uv\gamma\right), \\ v_x = 2\left(uv - vu_1\right), \end{cases}$$

$$M = \begin{pmatrix} (v_{-1}u)^{-1} & 0 \\ 0 & 0 \end{pmatrix} \lambda^2 + \begin{pmatrix} 0 & u^{-1} \\ \gamma (v_{-1}u)^{-1} & 0 \end{pmatrix} \lambda + \begin{pmatrix} v_{-1}^{-1} & 0 \\ 0 & \gamma u^{-1} \end{pmatrix}, \tag{4.51}$$

$$U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lambda^2 + 2 \begin{pmatrix} 0 & v_{-1} \\ \gamma_{-1} & 0 \end{pmatrix} \lambda + 2 \begin{pmatrix} u_{-1} & 0 \\ 0 & \gamma_{-1} v_{-1} \end{pmatrix}.$$

Case C_1 The quadratic up Darboux transformation of (C_1) admits the constant of motion $\phi = f$ and is described by the following system of equations:

$$\begin{cases} a_x = 2(apq - p_1 q_1 a), \\ b_x = 2(q_1 p_1 b - bqp), \\ p_x = 2\phi^{-1}(p_1 b - ap) - 2\left(pq + \phi^{-1} p_1 q_1 \phi\right) p, \\ q_{1,x} = 2(q_1 a - bq)\phi^{-1} + 2 q_1 \left(p_1 q_1 + \phi p q \phi^{-1}\right). \end{cases}$$

Besides ϕ, the system also admits ρ in (4.27) as a constant of motion, which allows us to eliminate b via $b = q_1\phi p - \rho$:

$$\begin{cases} a_x = 2(apq - p_1q_1a), \\ p_x = -2\phi^{-1}\left(ap + p_1\rho + \phi pqp\right), \\ q_{1,x} = 2\left(q_1a + \rho q + q_1p_1q_1\phi\right)\phi^{-1}. \end{cases}$$

(i) Reduction $\boldsymbol{a = 0}$. This system already consists of two equations. The change of variables $u = p$ and $v = q$ clarifies the connection with the system (MRT), given $\mu = \phi^{-1}$ and $\nu = -\rho$:

$$\begin{cases} u_x = -2(\phi^{-1}u_1\rho + uvu), \\ v_x = 2(\rho_{-1}v_{-1}\phi_{-1}^{-1} + vuv), \end{cases}$$

$$M = \begin{pmatrix} \phi & 0 \\ 0 & 0 \end{pmatrix}\lambda^2 + \begin{pmatrix} 0 & \phi u \\ v_1\phi & 0 \end{pmatrix}\lambda + \begin{pmatrix} 0 & 0 \\ 0 & v_1\phi u - \rho \end{pmatrix}, \tag{4.52}$$

$$U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\lambda^2 + 2\begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}\lambda + 2\begin{pmatrix} -uv & 0 \\ 0 & vu \end{pmatrix}.$$

(ii) Reduction $\boldsymbol{b = 0}$. Let us consider the variables

$$u = pq, \qquad v = a_{-1}.$$

The transformation defined by $q = p^{-1}u$ and $a = v_1$ corresponds to (rT) with $\mu = \phi_{-1}^{-1}$:

$$\begin{cases} u_x = 2(uv\phi_{-1}^{-1} - \phi^{-1}v_1u) + 2\left(u\phi_{-1}u_{-1}\phi_{-1}^{-1} - \phi^{-1}u_1\phi u\right), \\ v_x = 2(vu_{-1} - uv). \end{cases}$$

The evolution of the variables u and v is independent of p, which can be eliminated via a gauge transformation using $G = \operatorname{diag}(1, p^{-1})$. The resulting Lax representation coincides with (4.41).

(iii) Reduction $\boldsymbol{\rho = 0}$. Let us consider the variables

$$u = -pq, \qquad v = -a_{-1}.$$

The transformation defined by $q = -p^{-1}u$ and $a = -v_1$ corresponds to (2V$_\mathrm{a}$) with $\mu = \phi_{-1}^{-1}$:

$$\begin{cases} u_x = 2(\phi^{-1}v_1 u - uv\phi_{-1}^{-1}), \\ v_x = 2(uv - vu_{-1}). \end{cases}$$

The evolution of the variables u and v is independent of p, which can be eliminated via a gauge transformation using $G = \mathrm{diag}(1, p^{-1})$. The resulting Lax representation is identical to (4.42).

Case C$_2$ The quadratic up Darboux transformation of (C$_2$) admits the constant of motion $\alpha = a$ and is described by the following system of equations:

$$\begin{cases} b_x = 2\left(p_1 q_1 + q_1 p_1\right) b - 2b\left(pq + qp\right), \\ f_x = 2\left(p_1 q_1 f - fpq\right), \\ p_x = 2f^{-1}\left(p_1 b - \alpha p\right) - 2\left(p^2 q + f^{-1} p_1 q_1 f p\right), \\ q_{1,x} = 2\left(q_1 \alpha - bq\right) f^{-1} + 2\left(p_1 q_1^2 + q_1 f p q f^{-1}\right). \end{cases} \tag{4.53}$$

(i) Reduction $\boldsymbol{\alpha = 0}$. Let us consider the variables

$$u = p_1(q_1 f p - b)p^{-1}, \qquad v = -pq, \qquad w = -f^{-1}.$$

The transformation defined by $b = p_1^{-1}(v_1 w^{-1} - u)p$, $q = -p^{-1}v$, and $f = -w^{-1}$ corresponds to (N$_4$):

$$\begin{cases} u_x = 2(w_1 u_1 - uw)u, \\ v_x = 2(wuv - u_{-1}v_{-1}w_{-1}), \\ w_x = 2(wv_1 - vw). \end{cases}$$

The evolution of the variables u, v, and w is independent of p, which can be eliminated via a gauge transformation using $G = \mathrm{diag}(1, p^{-1})$. The resulting Lax pair is identical to (4.49).

(ii) **Reduction $\boldsymbol{b = 0}$.** Let us consider the variables

$$u = -p_1 q_1, \qquad v = f^{-1}.$$

The transformation defined by $q = -p^{-1}u_{-1}$ and $f = v^{-1}$ corresponds to (N_2) with $\mu = \alpha$:

$$\begin{cases} u_x = 2(u\alpha v - v_1\alpha_1 u) + 2\left(v_1 u_1 v_1^{-1} u - u v^{-1} u_{-1} v\right), \\ v_x = 2(vu - u_{-1}v). \end{cases}$$

The evolution of the variables u and v is independent of p, which can be eliminated via a gauge transformation using $G = \mathrm{diag}(1, p^{-1}v)$. The resulting Lax representation is identical to (4.33).

(iii) **Reduction $\boldsymbol{\rho = 0}$.** From the reduction, we set $b = q_1 f p$ and introduce the variables

$$u = -p_1 q_1, \qquad v = -f^{-1}.$$

The transformation defined by $q = -p^{-1}u_{-1}$ and $f = -v^{-1}$ corresponds to ($2\mathrm{V}_{\mathrm{b}}$) with $\mu = \alpha$:

$$\begin{cases} u_x = 2(v_1\alpha_1 u - u\alpha v), \\ v_x = 2(vu - u_{-1}v). \end{cases}$$

The evolution of the variables u and v is independent of p, which can be eliminated via a gauge transformation using $G = \mathrm{diag}(1, p^{-1})$. The resulting Lax representation is identical to (4.44).

Case $C_2^\star$ The quadratic up Darboux transformation of ($C_2^\star$) admits the constant of motion $\beta = b$ and is described by the following system of equations:

$$\begin{cases} f_x = 2\left(fqp - q_1p_1f\right), \\ a_x = 2a\left(pq + qp\right) - 2\left(p_1q_1 + q_1p_1\right)a, \\ p_x = 2f^{-1}\left(p_1\beta - ap\right) - 2\left(qp + f^{-1}p_1q_1f\right)p, \\ q_{1,x} = 2\left(q_1a - \beta q\right)f^{-1} + 2q_1\left(q_1p_1 + fpqf^{-1}\right). \end{cases}$$

(i) **Reduction $\boldsymbol{a = 0}$.** Let us consider the variables

$$u = q_1fp - \beta, \qquad v = -qp(u+\beta)^{-1}.$$

The transformation defined by $q = -v(u+\beta)p^{-1}$ and $f = -p_1(u_1+\beta_1)^{-1}v_1^{-1}(u+\beta)p^{-1}$ corresponds to (N_1) with $\mu = \beta$:

$$\begin{cases} u_x = 2v_1(u_1+\beta_1)u - 2uv(u+\beta), \\ v_x = 2\left(vu - u_{-1}v_{-1}\right)v. \end{cases}$$

The evolution of the variables u and v is independent of p, which can be eliminated via a gauge transformation using $G = \mathrm{diag}(p(u+\beta)^{-1}, 1)$. The resulting Lax pair is identical to (4.32).

(ii) **Reduction $\boldsymbol{\beta = 0}$.** This system admits θ in (4.47) as constant of motion. We take advantage of it to determine the auxiliary function $a = fp\theta q_1 f$. Let us consider the variables

$$u = qp, \qquad v = -q_1fp.$$

The transformation defined by $q = up^{-1}$ and $f = -p_1u_1^{-1}vp^{-1}$ corresponds to the $\mathcal{T}$-reflection of the system (N_2) with $\mu = \theta$:

$$\begin{cases} u_x = 2\left(u\theta v - v_{-1}\theta_{-1}u - uv^{-1}u_1v + v_{-1}u_{-1}v_{-1}^{-1}u\right), \\ v_x = 2\left(vu - u_1v\right). \end{cases}$$

The evolution of the variables u and v is independent of p, which can be eliminated via a gauge transformation using $G = \mathrm{diag}(pv^{-1}, 1)$. The resulting Lax representation is identical to (4.48).

(iii) **Reduction $\boldsymbol{\rho = 0}$.** From the reduction, we set $f = q_1^{-1}\beta p^{-1}$ and introduce the variables

$$u = -q_1 ap\beta^{-1}, \qquad v = -qp\beta^{-1}.$$

The transformation defined by $q = -v\beta p^{-1}$ and $a = p_1\beta_1^{-1}v_1^{-1}u\beta p^{-1}$ corresponds to ($2\mathrm{V_b}$) with $\mu = \beta$:

$$\begin{cases} u_x = 2\left(v_1\beta_1 u - u\beta v\right), \\ v_x = 2\left(vu - u_{-1}v\right). \end{cases}$$

The evolution of the variables u and v is independent of p, which can be eliminated via a gauge transformation using $G = \mathrm{diag}(p\beta^{-1}, 1)$. The resulting Lax representation is identical to (4.34).

4.4.2 The quadratic down Darboux transformations $M_{\downarrow}(2)$

Recall the rank-1 quadratic down Darboux matrix (4.8b):

$$M_{\downarrow}(2) = \begin{pmatrix} 0 & 0 \\ 0 & g \end{pmatrix}\lambda^2 - \begin{pmatrix} 0 & p_1 g \\ gq & 0 \end{pmatrix}\lambda + \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}, \tag{4.54}$$

with auxiliary functions $g, c,$ and d. The system of equations $\mathscr{E}_{\downarrow}(2)$ associated with a Lax pair of the form (2.23) is

$$\begin{cases} c_x = 2\,\mathcal{S}(P_{11})c - 2cP_{11}, \\ d_x = 2\,\mathcal{S}(P_{22})d - 2dP_{22}, \\ g_x = 2\,\mathcal{S}\left(P_{22} - qp\right)g - 2g\left(P_{22} - qp\right), \\ p_{1,x} = 2\,\mathcal{S}\left(P_{11}p - pP_{22}\right) + 2\left(cp - p_1 d\right)g^{-1} + 2p_1\left(q_1p_1 - gqpg^{-1}\right), \\ q_x = 2\left(P_{22}q - qP_{11}\right) + 2g^{-1}\left(dq - q_1c\right) + 2\left(g^{-1}q_1p_1g - qp\right)q. \end{cases} \tag{4.55}$$

The structure of $\mathscr{E}_{\downarrow}(2)$ mirrors $\mathscr{E}_{\uparrow}(2)$, with g, c and d playing the role of f, a and b. Indeed, the transformation $M_{\downarrow}(2)$ admits optional reductions, fully analogous to (4.28) and based on c, d, and a new function η, analogous to ρ (4.27):

$$\eta = p_1 g q - c, \qquad \eta_x = 2\mathcal{S}(P_{11} + pq)\eta - 2\eta(P_{11} + pq). \tag{4.56}$$

Proposition 4.18 shows that the down linear transformation $M_{\downarrow}(1)$ in (4.21) corresponds to the up case $M_{\uparrow}(1)$ in (4.12) via the inverse matrix (4.23) and a redefinition of the auxiliary function. The Darboux inverse matrix of $M_{\uparrow}(2)$ is

$$M_{\uparrow}^{I}(2) = \begin{pmatrix} \Delta_{-1}^{-1} & -\lambda \Delta_{-1}^{-1} f_{-1} p_{-1} b_{-1}^{-1} \\ -\lambda b_{-1}^{-1} q f_{-1} \Delta_{-1}^{-1} & p_{-1}^{-1} f_{-1}^{-1} (a_{-1} + \lambda^2 f_{-1}) \Delta_{-1}^{-1} f_{-1} p_{-1} b_{-1}^{-1} \end{pmatrix}, \tag{4.57}$$

where Δ is similar to a determinant (see quasi-determinants in Section 4.5.2.1):

$$\Delta = a + \lambda^2 (f - f p b^{-1} q_1 f). \tag{4.58}$$

Although $M_{\uparrow}^{I}(2)$ appears structurally related to $M_{\downarrow}(2)$, since Δ is not polynomial in λ, it cannot be directly compared with $M_{\downarrow}(2)$ in (4.54). This occurs because $M_{\uparrow}(2)$ is not a unimodular polynomial matrix.

However, examining (4.58), there are cases where (4.57) is reduced to a polynomial matrix: it occurs when $a = 0$, $\rho = 0$ and, less evidently, when $b = 0$. These are the optional reductions of $M_{\uparrow}(2)$ highlighted in (4.28). Thus, while $M_{\downarrow}(2)$ is not a unimodular polynomial matrix, all its optional reductions are.

Proposition 4.26. *The optional reductions of a rank-1 down quadratic Darboux transformation for a DNLS equation can be expressed in terms of the respective reductions of the Darboux inverse of a rank-1 up quadratic Darboux transformation with a redefinition of the auxiliary functions.*

$$\begin{aligned} &M_{\downarrow}(2) = \lambda^2 \mathcal{T}(M_{\uparrow}^{I}(2)), && \mathscr{E}_{\downarrow}^{d}(2) = \mathcal{T}(\mathscr{E}_{\uparrow}^{a}(2)), && \textit{when } a = d = 0; \\ &M_{\downarrow}(2) = \lambda^2 \mathcal{T}(M_{\uparrow}^{I}(2)), && \mathscr{E}_{\downarrow}^{c}(2) = \mathcal{T}(\mathscr{E}_{\uparrow}^{b}(2)), && \textit{when } b = c = 0; \\ &M_{\downarrow}(2) = \mathcal{T}(M_{\uparrow}^{I}(2)), && \mathscr{E}_{\downarrow}^{\eta}(2) = \mathcal{T}(\mathscr{E}_{\uparrow}^{\rho}(2)), && \textit{when } \rho = \eta = 0. \end{aligned}$$

Proof. This proof is analogous to the one shown in Proposition 4.18: we consider the action of $\mathcal{T}$ in (3.20) applied to (4.57):

$$\mathcal{T}(M_{\uparrow}^{I}(2)) = \begin{pmatrix} \mathcal{T}(\Delta_{-1}^{-1}) & -\lambda\,\mathcal{T}(\Delta_{-1}^{-1}) f_1 p_1 b_1^{-1} \\ -\lambda b_1^{-1} q f_1\,\mathcal{T}(\Delta_{-1}^{-1}) & p_1^{-1} f_1^{-1}(a_1 + \lambda^2 f_1)\,\mathcal{T}(\Delta_{-1}^{-1}) f_1 p_1 b_1^{-1} \end{pmatrix},$$

together with the three optional reductions mentioned above. Notice that we write $\mathcal{T}(\Delta_{-1}^{-1})$ instead of Δ_1^{-1} because Δ is explicitly a function: indeed $\mathcal{T}(\Delta_{-1}) = a_1 + \lambda^2(f_1 - f_1 p_1 b_1^{-1} q f_1)$, whereas $\Delta_1 = a_1 + \lambda^2(f_1 - f_1 p_1 b_1^{-1} q_2 f_2)$.

(i) Reduction $\boldsymbol{a = 0}$. The function Δ reduces to $\Delta = f - f p b^{-1} q_1 f$, leading to

$$\mathcal{T}(M_{\uparrow}^{I}(2)) = \begin{pmatrix} 0 & 0 \\ 0 & p_1^{-1}\,\mathcal{T}(\Delta_{-1}^{-1}) f_1 p_1 b_1^{-1} \end{pmatrix} - \begin{pmatrix} 0 & \mathcal{T}(\Delta_{-1}^{-1}) f_1 p_1 b_1^{-1} \\ b_1^{-1} q f_1\,\mathcal{T}(\Delta_{-1}^{-1}) & 0 \end{pmatrix} \lambda^{-1} + \\ + \begin{pmatrix} \mathcal{T}(\Delta_{-1}^{-1}) & 0 \\ 0 & 0 \end{pmatrix} \lambda^{-2}.$$

By property (3.25), if the matrix above is rescaled by λ^2, it can be compared with the reduction $d = 0$ of $M_{\downarrow}(2)$. The two matrices are identified by defining $g = (b_1 - q f_1 p_1)^{-1}$ and $c = p_1 g b_1 p_1^{-1} f_1^{-1}$.

(ii) Reduction $\boldsymbol{b = 0}$. All dependence on Δ disappears and we obtain

$$\mathcal{T}(M_{\uparrow}^{I}(2)) = -\begin{pmatrix} 0 & 0 \\ 0 & (q f_1 p_1)^{-1} \end{pmatrix} + \begin{pmatrix} 0 & (q f_1)^{-1} \\ (f_1 p_1)^{-1} & 0 \end{pmatrix} \lambda^{-1} + \\ - \begin{pmatrix} 0 & 0 \\ 0 & (q f_1 a_1^{-1} f_1 p_1)^{-1} \end{pmatrix} \lambda^{-2}.$$

After rescaling it by λ^2, this matrix coincides with the reduction $c = 0$ of $M_{\downarrow}(2)$, setting $g = -(q f_1 p_1)^{-1}$ and $d = -g q a_1 p_1 g$.

(iii) Reduction $\boldsymbol{\rho = 0}$. The function Δ reduces to a, and hence the matrix (4.57) becomes

$$\mathcal{T}(M^I_\uparrow(2)) = \begin{pmatrix} 0 & 0 \\ 0 & (qa_1p_1)^{-1} \end{pmatrix} \lambda^2 - \begin{pmatrix} 0 & (qa_1)^{-1} \\ (a_1p_1)^{-1} & 0 \end{pmatrix} \lambda + \begin{pmatrix} a_1^{-1} & 0 \\ 0 & (qf_1p_1)^{-1} \end{pmatrix},$$

which is equivalent to $M_\downarrow(2)$ when $\eta = 0$, assuming $g = (qa_1p_1)^{-1}$, $c = a_1^{-1}$ and $d = (qf_1p_1)^{-1}$. The relation (4.56) is automatically satisfied under these identifications. Thus both matrices belong to the same reduction.

In case (iii), $M^I_\uparrow(2)$ has no pole at $\lambda = 0$. This behaviour is rather similar to the one for the linear Darboux matrix $M^I_\uparrow(1)$. Indeed, as we prove in Section 4.5, the reduction $\rho = 0$ is a necessary condition for $M_\uparrow(2)$ to be factorisable as a composition of two $M_\uparrow(1)$.

We conclude the proof by verifying that the evolutions of the new functions g, c, and d defined above coincide with $\mathscr{E}_\downarrow(2)$ from (4.55) obtained from the corresponding reductions of $\mathcal{T}(\mathscr{E}_\uparrow(2))$ in (4.26). □

4.4.3 The quadratic full-rank Darboux transformations $M(2)$

The full-rank quadratic Darboux transformation, described by Lemma 4.3, is represented by the Darboux matrix

$$M(2) = \begin{pmatrix} \mathscr{f} & 0 \\ 0 & \mathscr{g} \end{pmatrix} \lambda^2 + \begin{pmatrix} 0 & \mathscr{f} p - p_1 \mathscr{g} \\ q_1 \mathscr{f} - \mathscr{g} q & 0 \end{pmatrix} \lambda + \begin{pmatrix} \mathscr{c} & 0 \\ 0 & \mathscr{d} \end{pmatrix}, \tag{4.59}$$

where $\mathscr{f}, \mathscr{g}, \mathscr{c}$, and $\mathscr{d}$ are auxiliary functions. The Darboux system $\mathscr{E}(2)$, associated with $M(2)$ and (U, V) from (2.23), is

$$\begin{cases} \mathscr{c}_x = 2\,\mathcal{S}(P_{11})\mathscr{c} - 2\mathscr{c}P_{11}, \\ \mathscr{d}_x = 2\,\mathcal{S}(P_{22})\mathscr{d} - 2\mathscr{d}P_{22}, \\ \mathscr{f}_x = 2\,\mathcal{S}\left(pq + P_{11}\right)\mathscr{f} - 2\mathscr{f}\left(pq + P_{11}\right), \\ \mathscr{g}_x = 2\,\mathcal{S}\left(P_{22} - qp\right)\mathscr{g} - 2\mathscr{g}\left(P_{22} - qp\right), \\ p_{1,x}\mathscr{g} - \mathscr{f}p_x = 2\left(\mathscr{c}p - p_1\mathscr{d}\right) + 2\,\mathcal{S}\left(P_{11}p - pP_{22}\right)\mathscr{g} + \\ \qquad -2\mathscr{f}\left(P_{11}p - pP_{22}\right) + 2\left(p_1q_1\mathscr{f}p - \mathscr{f}pqp - p_1\mathscr{g}qp + p_1q_1p_1\mathscr{g}\right), \\ q_{1,x}\mathscr{f} - \mathscr{g}q_x = 2\left(q_1\mathscr{c} - \mathscr{d}q\right) + 2\,\mathcal{S}\left(P_{22}q - qP_{11}\right)\mathscr{f} + \\ \qquad -2\mathscr{g}\left(P_{22}q - qP_{11}\right) + 2\left(q_1\mathscr{f}pq - q_1p_1q_1\mathscr{f} + \mathscr{g}qpq - q_1p_1\mathscr{g}q\right). \end{cases} \tag{4.60}$$

Notice that the dynamics of p and q are non-evolutionary; a similar behaviour was observed in the linear case in Section 4.3.3, and more generally in (4.6). It is possible to discuss the reductions of (4.60) as we did above, identifying its constants of motion and the relevant changes of variables. However, in the following part, we examine just some particular examples.

For the quadratic full-rank Darboux transformation of equation (C_1), the auxiliary functions $\mathscr{f}$ and $\mathscr{g}$ are reduced to the constants ϕ and ψ. The resulting system $\mathscr{E}(2)$ becomes

$$\begin{cases} \mathscr{c}_x = 2(\mathscr{c}pq - p_1q_1\mathscr{c}), \\ \mathscr{d}_x = 2(q_1p_1\mathscr{d} - \mathscr{d}qp), \\ p_{1,x}\psi - \phi p_x = 2\left(\mathscr{c}p - p_1\mathscr{d}\right) + 2\left(\phi pqp + p_1q_1\phi p - p_1\psi qp - p_1q_1p_1\psi\right), \\ q_{1,x}\phi - \psi q_x = 2\left(q_1\mathscr{c} - \mathscr{d}q\right) + 2\left(q_1\phi pq + q_1p_1q_1\phi - \psi qpq - q_1p_1\psi q\right). \end{cases}$$

Similarly, for equation (A_1), the auxiliary functions $\mathscr{c}$ and $\mathscr{d}$ are reduced to the constants γ and δ. The system $\mathscr{E}(2)$ turns into

$$\begin{cases} \mathscr{f}_x = 2(p_1q_1\mathscr{f} - \mathscr{f}pq), \\ \mathscr{g}_x = 2(\mathscr{g}qp - q_1p_1\mathscr{g}), \\ p_{1,x}\mathscr{g} - \mathscr{f}p_x = 2\left(\gamma p - p_1\delta\right) + 2\left(p_1q_1\mathscr{f}p - \mathscr{f}pqp - p_1\mathscr{g}qp + p_1q_1p_1\mathscr{g}\right), \\ q_{1,x}\mathscr{f} - \mathscr{g}q_x = 2\left(q_1\gamma - \delta q\right) + 2\left(q_1\mathscr{f}pq - q_1p_1q_1\mathscr{f} + \mathscr{g}qpq - q_1p_1\mathscr{g}q\right). \end{cases}$$

In both cases, the four auxiliary functions reduce to two, which still need to be determined.

Alternatively, the full-rank case can be described as a composition of two rank-1 up and rank-1 down quadratic Darboux transformations.

Proposition 4.27. *The composition of a rank-1 up and a rank-1 down quadratic Darboux transformations corresponds to the full-rank quadratic Darboux transformation.*

Proof. Let $M_\uparrow(2)$ and $M_\downarrow(2)$ be two rank-1 quadratic Darboux matrices with associated Darboux transformations $\mathcal{S}_\uparrow$ and $\mathcal{S}_\downarrow$. Adopting the same formalism as in Section 3.2.2.5, we define

$$\mathcal{S}_\uparrow : \Phi \mapsto \bar{\Phi} = M_\uparrow(2)\Phi, \qquad \mathcal{S}_\downarrow : \Phi \mapsto \tilde{\Phi} = M_\downarrow(2)\Phi.$$

We do not discuss Bianchi permutability (3.32) of $\mathcal{S}_\uparrow$ and $\mathcal{S}_\downarrow$. Therefore, the order of the composition and the inner shift ι in (3.30) must be considered. The shift $\mathscr{S} = \mathcal{S}_\uparrow\mathcal{S}_\downarrow$ is represented by the following Darboux matrix:

$$\begin{aligned} \iota_{\mathcal{S}_\downarrow}(M_\uparrow(2))M_\downarrow(2) = &\begin{pmatrix} \tilde{f}(c - \tilde{p}gq) & 0 \\ 0 & (\tilde{b} - \bar{\bar{q}}\tilde{f}\tilde{p})g \end{pmatrix}\lambda^2 + \\ &+ \begin{pmatrix} 0 & \tilde{f}\tilde{p}d - \tilde{a}\tilde{p}g \\ \bar{\bar{q}}\tilde{f}c - \tilde{b}gq & 0 \end{pmatrix}\lambda + \begin{pmatrix} \tilde{a}c & 0 \\ 0 & \tilde{b}d \end{pmatrix}. \end{aligned} \tag{4.61}$$

The analogy between $\iota_{\mathcal{S}_\downarrow}(M_\uparrow(2))M_\downarrow(2)$ and $M(2)$ is evident: if we define the new auxiliary functions $\mathscr{f}', \mathscr{g}', \mathscr{c}'$ and $\mathscr{d}'$:

$$\mathscr{c}' = \tilde{a}c, \qquad \mathscr{d}' = \tilde{b}d, \qquad \mathscr{f}' = \tilde{f}(c - \tilde{p}gq), \qquad \mathscr{g}' = (\tilde{b} - \tilde{\tilde{q}}\tilde{f}\tilde{p})g, \tag{4.62}$$

the composed matrix (4.61)

$$\iota_{\mathcal{S}_\downarrow}(M_\uparrow(2))M_\downarrow(2) = \begin{pmatrix} \mathscr{f}' & 0 \\ 0 & \mathscr{g}' \end{pmatrix} \lambda^2 + \begin{pmatrix} 0 & \tilde{f}\tilde{p}d - \tilde{a}\tilde{p}g \\ \tilde{\tilde{q}}\mathscr{f}' - \mathscr{g}'q & 0 \end{pmatrix} \lambda + \begin{pmatrix} \mathscr{c}' & 0 \\ 0 & \mathscr{d}' \end{pmatrix} \tag{4.63}$$

matches the pattern of $M(2)$ for most of its entries. To complete the identification, we impose one further condition on $M^{(1)}_{12}$:

$$\tilde{f}\tilde{p}d - \tilde{a}\tilde{p}g = \mathscr{f}'p - \tilde{\tilde{p}}\mathscr{g}'. \tag{4.64}$$

Thus, the full-rank transformation $\mathscr{S} = \mathcal{S}_\uparrow \mathcal{S}_\downarrow$ is associated with two successive rank-1 steps on the lattice $\mathbb{Z}$. In order to compare the matrix above with $M(2)$ we need to introduce a new operator $\mathcal{Q}$.

Let $\mathscr{f}(u, u_n)$ be a two-point function of the variable u, that is a function defined only on the sites 0 and n of the lattice $\mathbb{Z}$. We define the operator $\mathcal{Q}$ acting on two-point functions by normalising the shift distance to one:

$$\mathcal{Q} : \mathscr{f}(u, u_n) \mapsto \mathscr{f}(u, u_1). \tag{4.65}$$

Intuitively, $\mathcal{Q}$ acts on the lattice $\mathbb{Z}$ by collapsing an arbitrary n-step shift to a single step. If we consider a unit negative shift, as we did for $M^I(u, u_{-1})$ in Sections 4.3.2 and 4.4.2, the action of $\mathcal{Q}$ corresponds to the reflection $\mathcal{T}$ in (3.20).

Following the same procedure, we use $\mathcal{Q}$ to convert the double shift of $\iota_{\mathcal{S}_\downarrow}(M_\uparrow(2))M_\downarrow(2)$ in (4.63) into the single shift of $M(2)$, such that $\mathcal{Q} : \tilde{\tilde{q}} \mapsto q_1$, where we assume that the subscripts are associated with the shift $\mathscr{S}$. Notice that the action of $\mathcal{Q}$ is not defined on the individual $\mathcal{S}_\uparrow$ or $\mathcal{S}_\downarrow$, but only on their composition $\mathcal{S}_\uparrow \mathcal{S}_\downarrow \mapsto \mathcal{S}$.

Considering $\iota_{\mathcal{S}_\downarrow}(\mathscr{E}_\uparrow(2))$, computed from (4.26), and $\mathscr{E}_\downarrow(2)$ in (4.55), with the condition (4.64), we reconstruct the evolution of the functions $\mathscr{f}', \mathscr{g}', \mathscr{c}'$ and $\mathscr{d}'$ in (4.62):

$$
\begin{aligned}
\mathscr{c}'_x &= 2\bar{\tilde{P}}_{11}\mathscr{c}' - 2\mathscr{c}'P_{11},\\
\mathscr{d}'_x &= 2\bar{\tilde{P}}_{22}\mathscr{d}' - 2\mathscr{d}'P_{22},\\
\mathscr{f}'_x &= 2(\bar{\tilde{p}}\bar{\tilde{q}} + \bar{\tilde{P}}_{11})\mathscr{f}' - 2\mathscr{f}'\left(pq + P_{11}\right),\\
\mathscr{g}'_x &= 2(\bar{\tilde{P}}_{22} - \bar{\tilde{q}}\bar{\tilde{p}})\mathscr{g}' - 2\mathscr{g}'\left(P_{22} - qp\right).
\end{aligned}
$$

These equations involve just two-point functions with respect to the shift $\mathscr{S}$. Applying $\mathcal{Q}$ in (4.65), the result is identical to $\mathscr{E}(2)$ in (4.60), therefore we identify $\mathscr{f}', \mathscr{g}', \mathscr{c}', \mathscr{d}'$ with the quantities in $M(2)$, dropping the prime.

Finally, we check whether the dynamics of the dependent variables p and q obtained from (4.61) are compatible with (4.60). From the systems $\iota_{\mathcal{S}_\downarrow}(\mathscr{E}_\uparrow(2))$ and $\mathscr{E}_\downarrow(2)$ we deduce immediately the evolutionary expressions for q_x and $\bar{\tilde{q}}_x$; through the map $\mathcal{Q}$, they satisfy the corresponding non-evolutionary equation in (4.55).

With respect to the dependent variable p, the systems $\iota_{\mathcal{S}_\downarrow}(\mathscr{E}_\uparrow(2))$ and $\mathscr{E}_\downarrow(2)$ are redundant, since both $\iota_{\mathcal{S}_\downarrow}(\mathscr{E}_\uparrow(2))$ and $\mathscr{E}_\downarrow(2)$ yield a DΔE for $\iota_{\mathcal{S}_\downarrow}(p_x) = \tilde{p}_x$. Remarkably, their compatibility condition coincides with (4.64), which we already assumed. The dynamics of p is obtained by differentiating (4.64), where both p and $\tilde{\bar{p}}$ appear, which leads to the non-evolutionary equation (4.55). ☐

Remark 4.28. The composition of Darboux transformations $\mathscr{S} = \mathcal{S}_\uparrow\mathcal{S}_\downarrow$ described above and the consequent action of $\mathcal{Q}$ can be represented on the lattice $\mathbb{Z}$ of Darboux transformations.

Let x_{2n} be a point on the lattice $\mathbb{Z}$, we apply $\mathcal{S}_\downarrow$ to it, reaching the following point, denoted y_{2n+1}, and then applying $\mathcal{S}_\uparrow$ to reach $\mathrm{x}_{2(n+1)}$. Considering the composition of the shifts, the odd nodes y_{2n+1} represent the intermediate steps of the transformation $\mathscr{S}$. The map $\mathcal{Q}$ removes them from the lattice, relabelling

$\mathrm{x}_{2n} \mapsto \mathrm{x}_n$. The following graph helps to visualise this process.

$$\begin{tikzcd}
\cdots \arrow[rr, bend left, "\mathscr{S}"] \arrow[r, bend right, "\mathcal{S}_\downarrow"'] & \mathrm{y}_{2n-1} \arrow[r, bend right, "\mathcal{S}_\uparrow"'] & \mathrm{x}_{2n} \arrow[rr, bend left, "\mathscr{S}"] \arrow[r, bend right, "\mathcal{S}_\downarrow"'] & \mathrm{y}_{2n+1} \arrow[r, bend right, "\mathcal{S}_\uparrow"'] & \mathrm{x}_{2(n+1)} \arrow[r, bend left, "\mathscr{S}"] \arrow[r, bend right, "\mathcal{S}_\downarrow"'] & \cdots \arrow[r, Rightarrow, "\mathcal{Q}"'] & \cdots \arrow[r, bend left, "\mathscr{S}"] & \mathrm{x}_n \arrow[r, bend left, "\mathscr{S}"] & \mathrm{x}_{n+1} \arrow[r, bend left, "\mathscr{S}"] & \cdots
\end{tikzcd}$$

4.5 Relationship between linear and quadratic rank-1 Darboux transformations

From the construction of linear and quadratic Darboux matrices, it is natural to wonder whether the latter correspond to compositions of the former. This would be a first step towards the important problem of the factorisation of Darboux transformations, i.e. determining if certain classes of Darboux matrices can be expressed as compositions of a few "elementary" ones.

Our objective is to investigate whether some polynomial Darboux transformations for the DNLS equations can be written in terms of lower degree ones. A related problem was addressed in [7, 31, 72, 96] for the NLS equation, which introduced the concept of *elementary Darboux transformations*, see Example 4.7.

We approach this problem in two complementary ways. In Section 4.5.1, following the example in Section 4.4.3, we study how the composition of two linear Darboux transformations yields a quadratic one. This leads us to interpret the reduction $\rho = 0$ of a rank-1 quadratic up Darboux transformation $M_\uparrow(2)$ as a composition of two rank-1 up linear Darboux transformations $M_\uparrow(1)$. In Section 4.5.2, we adopt a more general viewpoint: given a Darboux matrix of degree $n \in \mathbb{N}$, we seek its decomposition into a pair of lower-degree Darboux matrices. In order to do that, we briefly introduce the theory of quasi-determinants [46] (in Section 4.5.2.1), then, in Section 4.5.2.2, we present a lemma establishing the necessary conditions for a Darboux matrix of degree $n \in \mathbb{N}$ to be a composition of a Darboux matrix of degree $n-1$ and either $M_\uparrow(1)$ in (4.12) or $M_\downarrow(1)$ in (4.21).

4.5.1 Composition of linear Darboux transformations

In Section 3.2.2.5 we introduced the composition rule for Darboux matrices. We apply it to two rank-1 up linear Darboux matrices $M_{\uparrow}(1)$:

$$M_{\uparrow}(1) = \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} \lambda + \begin{pmatrix} 0 & fp \\ \bar{q} f & 0 \end{pmatrix}, \qquad N_{\uparrow}(1) = \begin{pmatrix} g & 0 \\ 0 & 0 \end{pmatrix} \lambda + \begin{pmatrix} 0 & gp \\ \tilde{q} g & 0 \end{pmatrix},$$

associated with the non-commuting shifts

$$\mathcal{S}_1 : \Phi \mapsto \bar{\Phi} = N_{\uparrow}(1)\Phi, \qquad \mathcal{S}_2 : \Phi \mapsto \tilde{\Phi} = M_{\uparrow}(1)\Phi. \tag{4.66}$$

A similar construction was already discussed in Section 4.4.3 studying the full-rank quadratic Darboux matrix $M(2)$ in (4.59). We do not consider Bianchi permutability (3.32), so the order of $\mathcal{S}_1$ and $\mathcal{S}_2$ must be preserved. We assume a quadratic Darboux transformation $\mathscr{S}$ defined by

$$\mathscr{S} = \mathcal{S}_2 \mathcal{S}_1 \,. \tag{4.67}$$

According to (3.31), the associated Darboux matrix takes the form of

$$\iota_{\mathcal{S}_1}(M_{\uparrow}(1))N_{\uparrow}(1) = \begin{pmatrix} \bar{f} g & 0 \\ 0 & 0 \end{pmatrix} \lambda^2 + \begin{pmatrix} 0 & \bar{f} g p \\ \tilde{\bar{q}} \bar{f} g & 0 \end{pmatrix} \lambda + \begin{pmatrix} \bar{f} \bar{p} \bar{q} g & 0 \\ 0 & \tilde{\bar{q}} \bar{f} g p \end{pmatrix},$$

which has the structure of a rank-1 up quadratic Darboux matrix $M_{\uparrow}(2)$ in (4.25). More specifically, let us introduce the functions

$$h = \bar{f} g, \qquad a = \bar{f} \bar{p} \bar{q} g. \tag{4.68}$$

Recalling the map $\mathcal{Q} : \tilde{\bar{q}} \mapsto q_1$ in (4.65), the transformed $\mathcal{Q}(\iota_{\mathcal{S}_1}(M_{\uparrow}(1))N_{\uparrow}(1))$ corresponds to the reduction $\rho = 0$ of $M_{\uparrow}(2)$:

$$\mathcal{Q}(\iota_{\mathcal{S}_1}(M_{\uparrow}(1))N_{\uparrow}(1)) = \begin{pmatrix} h & 0 \\ 0 & 0 \end{pmatrix} \lambda^2 + \begin{pmatrix} 0 & hp \\ q_1 h & 0 \end{pmatrix} \lambda + \begin{pmatrix} a & 0 \\ 0 & q_1 hp \end{pmatrix}. \tag{4.69}$$

Proposition 4.29. *The composition of two rank-1 up linear Darboux transformations for the DNLS equations corresponds to the reduction* $\rho = 0$ *of a rank-1 quadratic Darboux transformation:*

$$\mathcal{Q}(\iota_{\mathcal{S}_2}(M_\uparrow(1))N_\uparrow(1)) = M_\uparrow(2), \qquad \textit{where } \rho = 0. \tag{4.70}$$

Proof. The matrix (4.69) has the same structure as $M_\uparrow(2)$ with the specific values of h and a given in (4.68), and $b = q_1 hp$ deduced from the reduction $\rho = 0$ (4.27).

Through the action of $\mathcal{Q}$, the dynamics of (4.68), obtained from $\mathscr{E}_\uparrow(1)$ in (4.13), becomes equivalent to that of $\mathscr{E}_\uparrow(2)$ in (4.26): the evolution of f and g from $M_\uparrow(1)$ and $N_\uparrow(1)$ reproduce the corresponding evolution of h and a in $M_\uparrow(2)$. The evolution of the dependent variables p and q is slightly different: from system $\mathscr{E}_\uparrow(1)$ we deduce that

$$N_\uparrow(1) : \begin{cases} p_x = 2(P_{11}p - pP_{22}) + 2(pqp - g^{-1}\bar{p}\bar{q}gp), \\ \tilde{q}_x = 2(\bar{P}_{22}\bar{q} - \bar{q}\bar{P}_{11}) + 2(\bar{q}gpqg^{-1} - \bar{q}\bar{p}\bar{q}); \end{cases}$$

$$\iota_{\mathcal{S}_1}(M_\uparrow(1)) : \begin{cases} \bar{p}_x = 2(\bar{P}_{11}\bar{p} - \bar{p}\bar{P}_{22}) + 2(-\bar{f}^{-1}\tilde{\bar{p}}\tilde{\bar{q}}\bar{f}\bar{p} + \bar{p}\bar{q}\bar{p}), \\ \tilde{\bar{q}}_x = 2(\tilde{\bar{P}}_{22}\tilde{\bar{q}} - \tilde{\bar{q}}\tilde{\bar{P}}_{11}) + 2(\tilde{\bar{q}}\bar{f}\bar{p}\bar{q}\bar{f}^{-1} - \tilde{\bar{q}}\tilde{\bar{p}}\tilde{\bar{q}}). \end{cases}$$

Since the dynamics of $M_\uparrow(2)$ involve p_x and $\tilde{\bar{q}}_x$, we eliminate the intermediate product $\bar{p}\bar{q}$ using the definition of a: setting $\bar{p}\bar{q} = \bar{f}^{-1}ag^{-1}$, we obtain

$$\begin{cases} p_x = 2(P_{11}p - pP_{22}) + 2pqp - 2h^{-1}ap, \\ \tilde{\bar{q}}_x = 2(\tilde{\bar{P}}_{22}\tilde{\bar{q}} - \tilde{\bar{q}}\tilde{\bar{P}}_{11}) + 2\tilde{\bar{q}}ah^{-1} - 2\tilde{\bar{q}}\tilde{\bar{p}}\tilde{\bar{q}}, \end{cases} \tag{4.71}$$

which, through $\mathcal{Q}$, corresponds to the reduction $\mathscr{E}^\rho_\uparrow(2)$ of (4.26). □

Note that in all the rank-1 quadratic Darboux transformations in Section 4.4.1.1 the reduction $\rho = 0$ always corresponds to either ($2\mathrm{V}_\mathrm{a}$) or ($2\mathrm{V}_\mathrm{b}$). This is due to the fact that $\mathscr{E}^\rho_\uparrow(2)$ is a composition of two copies of $\mathscr{E}_\uparrow(1)$, each associated with a Volterra-type equation as (V_a) or (mV_b).

Remark 4.30. Remark 4.28 helps to clarify the role of $a = \bar{f}\bar{p}\bar{q}g$: the product $\bar{p}\bar{q}$ is associated with the intermediate steps y_{2n+1} and is absorbed into the auxiliary function a. In Section 4.5.2.2, we describe this structure from the perspective of the factorisation of a quadratic Darboux matrix.

4.5.2 Factorisation of quadratic Darboux transformations

The relationship between linear and quadratic Darboux matrices can be investigated from a different point of view: we determine under which conditions a generic Darboux matrix M of degree $n \in \mathbb{N}$ can be decomposed into a rank-1 linear Darboux matrix and another of degree $n-1$.

In order to prove such a result, we must introduce the notion of quasi-determinants for matrices with non-commutative entries. We already mentioned the importance of determinants in Lemma 3.13, a corollary to the Jacobi formula. Moreover, in the commutative case, a Darboux matrix M can be factorised into lower-degree Darboux matrices if its determinant $\det(M)$ is a product of the determinants associated with “more elementary” factors.

In the non-commutative case, there is no straightforward analogue of a determinant. Instead, various generalisations exist, each preserving different aspects of it: among the most notable are quasi-determinants, introduced and studied in [46, 47], with applications to Darboux transformations and integrable systems in [51], and the Dieudonné determinant, presented in [36], and applied in integrable systems in [27]. For an introduction to these concepts consult [46]. In this work we focus on quasi-determinants, as they emerge naturally from our framework.

4.5.2.1 Quasi-determinants

Let A be an invertible 2×2 matrix with non-commutative entries. Its inverse can be written as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \qquad A^{-1} = \begin{pmatrix} \Delta_{11}^{-1}(A) & -\Delta_{11}^{-1}(A) A_{12} A_{22}^{-1} \\ -\Delta_{22}^{-1}(A) A_{21} A_{11}^{-1} & \Delta_{22}^{-1}(A) \end{pmatrix}.$$

The matrix A^{-1} involves the inverses of the functions $\Delta_{11}(A)$ and $\Delta_{22}(A)$, which are the quasi-determinants of A with respect to the entries $(1,1)$ and $(2,2)$ [46]. The *quasi-determinants* $\Delta_{ij}(A)$ associated with each entry A_{ij} are

$$\begin{aligned} \Delta_{11}(A) &= A_{11} - A_{12} A_{22}^{-1} A_{21}, \qquad & \Delta_{12}(A) &= A_{12} - A_{11} A_{21}^{-1} A_{22}, \\ \Delta_{21}(A) &= A_{21} - A_{22} A_{12}^{-1} A_{11}, \qquad & \Delta_{22}(A) &= A_{22} - A_{21} A_{11}^{-1} A_{12}. \end{aligned} \tag{4.72}$$

If all the entries of A are non-zero, the quasi-determinants Δ_{ij} are linked by algebraic relations. For example, all quasi-determinants can be expressed in terms of a single one:

$$\begin{aligned} \Delta_{12}(A) &= -\Delta_{11}(A) A_{21}^{-1} A_{22}, \qquad & \Delta_{21}(A) &= -A_{22} A_{12}^{-1} \Delta_{11}(A), \\ \Delta_{22}(A) &= A_{22} A_{12}^{-1} \Delta_{11}(A) A_{11}^{-1} A_{12}, \qquad & \Delta_{22}(A) &= A_{21} A_{11}^{-1} \Delta_{11}(A) A_{21}^{-1} A_{22}, \end{aligned}$$

where $\Delta_{22}(A)$ admits two equivalent representations.

A fundamental property of quasi-determinants is the *hereditary principle* [46]: let A be a square matrix written with block matrices B_{ij}, each of which is both a sub-matrix of A and a formal entry of an equivalent matrix B. The hereditary principle states that

$$\Delta_{ab}(\Delta_{mn}(B)) = \Delta_{ij}(A),$$

meaning that the quasi-determinant $\Delta_{ab}(\cdot)$ of the matrix quasi-determinant $\Delta_{mn}(B)$ is equal to the quasi-determinant $\Delta_{ij}(A)$ provided that A_{ij} corresponds to the same entry (a,b) of the block B_{mn}, i.e. $A_{ij} = (B_{mn})_{ab}$.

It is common to interpret the non-commutative variables p and q as matrix-valued functions with commutative entries [118]. In this sense, quasi-determinants have a natural interpretation: since the non-commutative variables are, in fact, block matrices, by the hereditary property we can “transfer” the quasi-determinants to smaller and smaller blocks until we reach the original commutative entries. At this level, the block structure in A disappears and the standard determinant can be expressed as the product of a quasi-determinant $\Delta_{ij}(A)$ and the minor of the sub-matrix A^{ij} obtained by cancelling the i-th row and j-th column from A [46]:

$$\det(A) = (-1)^{i+j}\Delta_{ij}(A)\det(A^{ij}).$$

The quasi-determinants of the rank-1 up linear Darboux matrices $M_{\uparrow}(1)$ in (4.12) are

$$\nexists\Delta_{11}(M_{\uparrow}(1)), \qquad \Delta_{12}(M_{\uparrow}(1)) = fp, \tag{4.73a}$$

$$\Delta_{21}(M_{\uparrow}(1)) = q_1 f, \qquad \Delta_{22}(M_{\uparrow}(1)) = -q_1 fp\lambda^{-1}. \tag{4.73b}$$

The quasi-determinant $\Delta_{22}(M_{\uparrow}(1))$ coincides with the constant of motion ρ in (4.15) for the systems (A_2), (B_1), ($B_3^\star$) and (C_1).

The quasi-determinants of the rank-1 up quadratic Darboux matrix $M_{\uparrow}(2)$ in (4.25) correspond to

$$\Delta_{11}(M_{\uparrow}(2)) = a + (f - fpb^{-1}q_1 f)\lambda^2, \tag{4.74a}$$

$$\Delta_{12}(M_{\uparrow}(2)) = (fp - q_1^{-1}b)\lambda - af^{-1}q_1^{-1}b\lambda^{-1}, \tag{4.74b}$$

$$\Delta_{21}(M_{\uparrow}(2)) = (q_1 f - bp^{-1})\lambda - bp^{-1}f^{-1}a\lambda^{-1}, \tag{4.74c}$$

$$\Delta_{22}(M_{\uparrow}(2)) = q_1 f(f\lambda^2 + a)^{-1}((q_1^{-1}b - fp)\lambda^2 + af^{-1}q_1^{-1}b). \tag{4.74d}$$

Notice that the quantity Δ in (4.58) is precisely the quasi-determinant $\Delta_{11}(M_{\uparrow}(2))$. Moreover the reductions of $M_{\uparrow}(2)$ described in the reduction tree (4.28) correspond to the cases where $\Delta_{11}(M_{\uparrow}(2))$ becomes monomial in λ. These are also the cases that yield a polynomial Darboux inverse $M_{\uparrow}^{I}(2)$ in Proposition (4.26).

4.5.2.2 Factorisation lemma

In this part, we use quasi-determinants to determine the necessary conditions for the factorisation of a generic Darboux matrix $M_{\uparrow}(n)$ into either $M_{\uparrow}(1)$ in (4.12) or $M_{\downarrow}(1)$ in (4.21), along with another Darboux matrix $M(n-1)$. Before presenting the lemma, we construct the framework necessary for the factorisation.

Consider the lattice $\mathbb{Z}$ from Section 3.2.2.1, where each step is associated with a shift $\mathcal{S}$ linked with the Darboux matrix M. Suppose that M can be factorised into a certain number of lower-degree Darboux matrices. The original lattice $\mathbb{Z}$ can not represent such a factorisation: intuitively, we must "stretch" it first, introducing some intermediate points between every pair of existing sites, each representing a factor of $\mathcal{S}$.

Assume that M admits a factorisation into $m \in \mathbb{N}$ Darboux matrices. We define an operator $\mathcal{P}^m$ that acts on a function h as

$$\mathcal{P}^m : h(u, u_1, u_3, \dots) \mapsto h(u, u_m, u_{3m}, \dots). \tag{4.75}$$

In other words, $\mathcal{P}^m$ maps each point x_n on the lattice $\mathbb{Z}$ to x_{mn}, inserting the intermediate points y_{mn+k}, with $k \in \{1, \dots, m-1\}$, between each pair of the original lattice sites. When restricted to two-point functions, $\mathcal{P}^m$ has the opposite role of $\mathcal{Q}$ in (4.65).

We extend the action of $\mathcal{P}^m$ to the shift operator $\mathcal{S}$ by defining

$$\mathcal{P}^m : \mathcal{S} \mapsto \mathscr{S}^m,$$

where $\mathscr{S}^m$ denotes the composition of m Darboux transformations, written in an order not yet specified. The structure of $\mathscr{S}^m$ is a fundamental ansatz when addressing factorisation: once an explicit composition is fixed, each of the m components is assigned to a corresponding lattice point, either y_{mn+k} or $\mathrm{x}_{m(n+1)}$.

To illustrate this, we consider the case $m = 2$. Given the original lattice indexed by x_n, we assume that the associated shift $\mathcal{S}$ admits a factorisation into two Darboux transformations such as those in (4.66) and (4.67). We apply the map $\mathcal{P}^2$ and, consequently, $\mathcal{S} \mapsto \mathscr{S}^2 = \mathcal{S}_2\mathcal{S}_1$. The intermediate points y_{2n+1} are inserted between each pair of x_{2n} and $\mathrm{x}_{2(n+1)}$ of the original lattice. The following graph can be regarded as the counterpart of that in Remark 4.28.

In the commutative framework, the determinant $\det(M)$ provides useful insights to understand whether a Darboux matrix M admits a factorisation. In the non-commutative case, however, such a process is not straightforward: the quasi-determinants of a product of matrices do not, in general, correspond to the product of the respective quasi-determinants. Indeed, for two $n \times n$ matrices A and B with non-zero entries, the following general relation holds [46]

$$\Delta_{ij}^{-1}(AB) = \sum_{k=1}^{n} \Delta_{kj}^{-1}(B)\Delta_{ik}^{-1}(A). \tag{4.76}$$

Nevertheless, as observed in [51], in certain special cases this expression simplifies to a product of two terms, each corresponding to one of the factors.

We establish the necessary conditions for the factorisation of a $\mathcal{R}_{\pm}$-invariant Darboux matrix of degree $n \in \mathbb{N}$ into a rank-1 linear Darboux transformation and another Darboux matrix of degree $n-1$, considering both left and right compositions.

Lemma 4.31. *Let $\mathcal{S}_{\uparrow}$ and $\mathcal{S}_N$ be two Darboux transformations associated with the Darboux matrices $M_{\uparrow}(1)$ and N, respectively:*

$$\mathcal{S}_{\uparrow} : \Phi \mapsto \tilde{\Phi} = M_{\uparrow}(1)\Phi, \qquad \mathcal{S}_N : \Phi \mapsto \bar{\Phi} = N\Phi.$$

The compositions $\mathcal{S}_\uparrow \mathcal{S}_N$ *and* $\mathcal{S}_N \mathcal{S}_\uparrow$ *are associated with the following relations between quasi-determinants, defined in* (4.72)*, with* $i, j = 1, 2$*:*

$$\Delta_{1i}(\iota_{\mathcal{S}_N}(M_\uparrow(1))N) = \mathcal{S}_N(fp)\Delta_{2i}(N), \tag{4.77a}$$

$$\Delta_{i1}(\iota_{\mathcal{S}_\uparrow}(N)M_\uparrow(1)) = \iota_{\mathcal{S}_\uparrow}(\Delta_{i2}(N))\mathcal{S}_\uparrow(q)f. \tag{4.77b}$$

Proof. Recall the rank-1 up linear Darboux transformation $M_\uparrow(1)$ in (4.12) and let N be a generic Darboux matrix:

$$M_\uparrow(1) = \begin{pmatrix} f\lambda & fp \\ q_1 f & 0 \end{pmatrix}, \qquad N = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

We assume that the composition $\mathscr{S}^2 = \mathcal{S}_\uparrow \mathcal{S}_N$ is associated to a shift $\mathcal{S}$ via the map $\mathcal{Q}$ in (4.65). The Darboux matrix associated with $\mathscr{S}^2$ is

$$M = \iota_{\mathcal{S}_N}(M_\uparrow(1))N = \begin{pmatrix} \lambda \bar{f} A + \bar{f}\bar{p}C & \lambda \bar{f} B + \bar{f}\bar{p}D \\ \bar{\tilde{q}}\bar{f}A & \bar{\tilde{q}}\bar{f}B \end{pmatrix}.$$

According to (4.72), the polynomial quasi-determinants of M are

$$\begin{aligned} \Delta_{11}(M) &= \bar{f}\bar{p}(C - DB^{-1}A) = \mathcal{S}_N(fp)\Delta_{21}(N), \\ \Delta_{12}(M) &= \bar{f}\bar{p}(D - CA^{-1}B) = \mathcal{S}_N(fp)\Delta_{22}(N). \end{aligned}$$

Thus, we obtain (4.77a), which factorises into two terms: one coming from an entry of $M_\uparrow(1)$ and the other being a quasi-determinant of N.

Analogously, we consider the Darboux transformation $\mathscr{S}^2 = \mathcal{S}_N \mathcal{S}_\uparrow$ with the order of composition reversed. The associated Darboux matrix M is given by

$$M = \iota_{\mathcal{S}_\uparrow}(N)M_\uparrow(1) = \begin{pmatrix} \lambda\iota_{\mathcal{S}_\uparrow}(A)f + \iota_{\mathcal{S}_\uparrow}(B)\tilde{q}f & \iota_{\mathcal{S}_\uparrow}(A)fp \\ \lambda\iota_{\mathcal{S}_\uparrow}(C)f + \iota_{\mathcal{S}_\uparrow}(D)\tilde{q}f & \iota_{\mathcal{S}_\uparrow}(C)fp \end{pmatrix}.$$

The polynomial quasi-determinants of M are

$$\Delta_{11}(M) = (\iota_{\mathcal{S}_\uparrow}(B) - \iota_{\mathcal{S}_\uparrow}(AC^{-1}D))\tilde{q}f = \iota_{\mathcal{S}_\uparrow}(\Delta_{12}(N))\tilde{q}f,$$
$$\Delta_{21}(M) = (\iota_{\mathcal{S}_\uparrow}(D) - \iota_{\mathcal{S}_\uparrow}(CA^{-1}B))\tilde{q}f = \iota_{\mathcal{S}_\uparrow}(\Delta_{22}(N))\tilde{q}f.$$

As before, the two quasi-determinants $\Delta_{11}(M)$ and $\Delta_{21}(M)$ have a very simple polynomial form and they correspond to (4.77b) in the thesis. □

Remark 4.32. Considering a rank-1 down linear Darboux transformation $\mathcal{S}_\downarrow$, with Darboux matrix $M_\downarrow(1)$, we prove the analogous relations:

$$\Delta_{2i}(\iota_{\mathcal{S}_N}(M_\downarrow(1))N) = -\mathcal{S}_N(gq)\Delta_{1i}(N), \tag{4.78a}$$
$$\Delta_{i2}(\iota_{\mathcal{S}_\downarrow}(N)M_\downarrow(1)) = -\iota_{\mathcal{S}_\downarrow}(\Delta_{i1}(N))\mathcal{S}_\downarrow(p)g. \tag{4.78b}$$

Example 4.33. *Consider the rank-1 up quadratic Darboux matrix $M_\uparrow(2)$ in (4.25) and suppose it can be factorised into two linear Darboux matrices $M_\uparrow(1)$ in (4.12). Let the shifts be as in (3.27):*

$$\mathcal{S}_1 : \Phi \mapsto \bar{\Phi} = M_\uparrow^{(1)}(1)\Phi, \qquad \mathcal{S}_2 : \Phi \mapsto \tilde{\Phi} = M_\uparrow^{(2)}(1)\Phi,$$

with Darboux matrices given by

$$M_\uparrow^{(1)}(1) = \begin{pmatrix} f\lambda & fp \\ \bar{q}f & 0 \end{pmatrix}, \qquad M_\uparrow^{(2)}(1) = \begin{pmatrix} g\lambda & gp \\ \tilde{q}g & 0 \end{pmatrix}.$$

We apply $\mathcal{P}^2$ to $M_\uparrow(2)$ and to its associated shift $\mathcal{S}$ in order to express them as a composition of $\mathcal{S}_1$ and $\mathcal{S}_2$. The new associated shift is $\mathscr{S}^2$ with Darboux matrix

$$\mathcal{P}^2(M_\uparrow(2)) = \begin{pmatrix} h & 0 \\ 0 & 0 \end{pmatrix}\lambda^2 + \begin{pmatrix} 0 & hp \\ \mathscr{S}^2(q)h & 0 \end{pmatrix}\lambda + \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

for the auxiliary functions h, a and b. Without loss of generality, we assume $\mathscr{S} = \mathcal{S}_2\mathcal{S}_1$. The conditions (4.77a) *correspond to the following system of equations:*

$$\begin{cases} \lambda^2(h - hpb^{-1}\bar{\tilde{q}}h) + a = \bar{f}\bar{p}\bar{q}g, \\ (hp - \bar{\tilde{q}}^{-1}b)\lambda - ah^{-1}\bar{\tilde{q}}^{-1}b\lambda^{-1} = \bar{f}\bar{p}\bar{q}gp\lambda^{-1}, \end{cases}$$

which is solved by the following relations:

$$\begin{cases} b = \bar{\tilde{q}}hp, \\ a = \bar{f}\bar{p}\bar{q}g. \end{cases} \tag{4.79}$$

This situation is analogous to the one described in Section 4.5.1: we recognise in the first condition the reduction $\rho = 0$ of the quadratic Darboux transformations. The second condition depends on the intermediate step of the factorisation and, as discussed in the next section, we interpret it as a new auxiliary function.

4.6 Gauge transformations of Darboux systems

Sections 2.1.4 and 2.2.2 show that the commutative and non-commutative DNLS equations are related by gauge transformations, mapping one equation into another. It is natural to extend this picture to the DΔEs derived from the DNLS equations via Darboux transformations. These are non-local gauge transformations between Lax-Darboux pairs (M, U) and variable transformations between Darboux systems $\mathscr{E}$.

The main advantage of introducing gauge transformations on Darboux matrices is that the Darboux transformations for all the DNLS equations can be obtained from the study of a single model. Therefore, extensive lists such as those in Sections 4.3.1.1 and 4.4.1.1 would no longer be necessary: a single example would be representative for all the others.

A second important consequence is that this method would automatically provide (non-local) changes of variables (Miura maps) between the various integrable

DΔEs associated with Darboux systems, such as the Volterra, relativistic Toda, Ablowitz-Ladik, and other equations, thereby establishing relations among well-studied models.

Finally, gauge transformations offer a systematic reduction scheme for general Darboux systems. Examining the associated DΔEs, it is possible to construct a gauge matrix G that sets specific auxiliary functions to constants. As presented in Section 4.6.2, this observation allows us to identify the best framework to study each Darboux transformation. Then, further gauge transformations map it to all the other DNLS equations. The application of this method to $M_{\uparrow}(1)$ in (4.12) and $M_{\uparrow}(2)$ in (4.25) follows in Sections 4.6.2.1 and 4.6.2.2.

4.6.1 Gauge transformations of Darboux matrices

Let $M(n)$ be a polynomial Darboux matrix (4.5), invariant under the action of the reduction group $\mathcal{R}_{\pm}$ and associated with a DNLS equation with Lax representation (U, V) in (2.23). As shown in Section 2.2.2, a gauge matrix G (2.36) and the associated change of variables (2.38) map (U, V) into a new Lax pair (U', V') corresponding to a different DNLS. By the relation (3.24), this process induces a gauge transformation on $M(n)$:

$$
\begin{aligned}
M'(n) = &\begin{pmatrix} \mathscr{a}_1^{-1}\, M_{11}^{(n)}\, \mathscr{a} & 0 \\ 0 & \mathscr{b}_1^{-1}\, M_{22}^{(n)}\, \mathscr{b} \end{pmatrix} \lambda^{n} + \\
&+ \begin{pmatrix} 0 & \mathscr{a}_1^{-1}\, M_{11}^{(n)}\, \mathscr{a}\, p' - p_1'\, \mathscr{b}_1^{-1}\, M_{22}^{(n)}\, \mathscr{b} \\ q_1'\, \mathscr{a}_1^{-1}\, M_{11}^{(n)}\, \mathscr{a} - \mathscr{b}_1^{-1}\, M_{22}^{(n)}\, \mathscr{b}\, q' & 0 \end{pmatrix} \lambda^{n-1} + \\
&+ \begin{pmatrix} \mathscr{a}_1^{-1}\, M_{11}^{(n-2)}\, \mathscr{a} & 0 \\ 0 & \mathscr{b}_1^{-1}\, M_{22}^{(n-2)}\, \mathscr{b} \end{pmatrix} \lambda^{n-2} + \begin{pmatrix} 0 & \mathscr{a}_1^{-1}\, M_{12}^{(n-3)}\, \mathscr{b} \\ \mathscr{b}_1^{-1}\, M_{21}^{(n-3)}\, \mathscr{a} & 0 \end{pmatrix} \lambda^{n-3} + \dots
\end{aligned}
$$

For notational convenience, we assume that the auxiliary functions $M_{ij}^{(k)}$ do not depend explicitly on p and q, otherwise their arguments would transform accordingly as $M_{ij}^{(k)} \mapsto \tilde{M}_{ij}^{(k)}$. Recall also the tilde notation introduced in Section 2.2.2:

given $f(p,q) \in \mathscr{A}$, the function $\tilde{f}(p',q') = f(\mathscr{a}\, p'\, \mathscr{b}^{-1}, \mathscr{b}\, q'\, \mathscr{a}^{-1})$ is defined by the substitution in (2.38).

The transformed matrix $M'(n)$ retains the same structure as $M(n)$ with new auxiliary functions:

$$\begin{aligned} M_{11}^{(n)\,\prime} &= \mathscr{a}_1^{-1}\, M_{11}^{(n)}\, \mathscr{a}, & \qquad M_{22}^{(n)\,\prime} &= \mathscr{b}_1^{-1}\, M_{22}^{(n)}\, \mathscr{b}, \\ M_{11}^{(n-2)\,\prime} &= \mathscr{a}_1^{-1}\, M_{11}^{(n-2)}\, \mathscr{a}, & \qquad M_{22}^{(n-2)\,\prime} &= \mathscr{b}_1^{-1}\, M_{22}^{(n-2)}\, \mathscr{b}, \end{aligned}$$

and so on. More generally, each entry $M_{ij}^{(k)}$ transforms as

$$M_{ij}^{(k)} \mapsto M_{ij}^{(k)\,\prime} = \mathcal{S}(G_{ii}^{-1}) M_{ij}^{(k)} G_{jj}. \tag{4.80}$$

Thus, a PDE gauge transformation $(U,V) \mapsto (U',V')$ induces a corresponding DΔE gauge transformation $(M,U) \mapsto (M',U')$ through a redefinition of the auxiliary functions.

Remark 4.34. In general, the gauge transformations between DNLS equations are defined by the non-local functions $\mathscr{a}$ and $\mathscr{b}$. In this section, we treat both the dependent variables p and q, and the auxiliary functions $M_{ij}^{(k)}$ as formal variables in the system $\mathscr{E}$. Consequently, the transformed p' and q' from (2.38) and $M_{ij}^{(k)\,\prime}$ from (4.80) are regarded formally as solutions of local DΔEs, even though the transformation might involve non-local terms.

4.6.2 Reductions by gauge transformations

Gauge transformations can be used as a reduction technique for a general Darboux system $\mathscr{E}$ by mapping an auxiliary function f to a constant ϕ. In this way, f is absorbed into a gauge function in G and the number of degrees of freedom is reduced.

More precisely, such a reduction of the auxiliary function f in $\mathscr{E}$ to a constant ϕ in the transformed system $\mathscr{E}'$ does not occur within the original DNLS equation, with respect to which $\mathscr{E}$ was written, but rather requires the identification of a second DNLS equation, associated with the Darboux system $\mathscr{E}'$ for which f is a constant of motion.

The following lemma gives a sufficient condition for an auxiliary function to be reduced to a constant by a gauge transformation.

Lemma 4.35. *Let $\mathcal{S}$ be a Darboux transformation with Darboux matrix M for a DNLS equation with Lax representation (U, V) as in (2.23). Suppose that the evolution of an auxiliary function $M_{ij}^{(k)}$ is given by*

$$\mathcal{D}_x\, M_{ij}^{(k)} = 2\,\mathcal{S}(\mathscr{P}_{ii})M_{ij}^{(k)} - 2M_{ij}^{(k)}\mathscr{P}_{jj}, \tag{4.81}$$

for some functions $\mathscr{P}_{ii}$ and $\mathscr{P}_{jj}$, and let G be a gauge matrix such that

$$G_x G^{-1} = 2\tilde{\mathscr{P}}, \tag{4.82}$$

where $\mathscr{P}$ is the diagonal matrix with entries $\mathscr{P}_{11}$ and $\mathscr{P}_{22}$. If there exists a Lax representation (U', V') with matrix P' defined as

$$P' = G^{-1}(\tilde{P} - \tilde{\mathscr{P}})G, \tag{4.83}$$

for P from (U, V), then there exists a gauge transformation of the DNLS equations, defined by G, in which the transformed auxiliary function $M_{ij}^{(k)}{}'$ is a constant.

Proof. Consider the Darboux system $\mathscr{E}$ associated with a Darboux matrix M for a DNLS equation. Assume that the auxiliary function $M_{ij}^{(k)}$ evolves according to (4.81). Under the gauge transformation defined by (4.80), the transformed auxiliary function becomes

$$\begin{aligned} \mathcal{D}_x\left(M_{ij}^{(k)}{}'\right) = \mathcal{S}\left(2G_{ii}^{-1}\tilde{\mathscr{P}}_{ii}G_{ii} - G_{ii}^{-1}G_{ii,x}\right) M_{ij}^{(k)}{}' + \\ - M_{ij}^{(k)}{}'\left(2G_{jj}^{-1}\tilde{\mathscr{P}}_{jj}G_{jj} - G_{jj}^{-1}G_{jj,x}\right). \end{aligned} \tag{4.84}$$

Recall the relations between gauge functions in (2.41a): in terms of the matrix G, they become

$$\mathcal{D}_x\left(G_{\ell\ell}\right)=2\left(\tilde{P}_{\ell\ell}G_{\ell\ell}-G_{\ell\ell}P'_{\ell\ell}\right), \tag{4.85}$$

for $\ell=1,2$. The matrix $\tilde{P}$ comes from the original Lax pair (U,V) and P' is associated with a transformed Lax pair (U',V') determined by G. If the Lax representation (U',V') defined in (4.83) exists, then by (4.85) the entries of G are

$$\mathcal{D}_x\left(G_{\ell\ell}\right)=2\tilde{\mathscr{P}}_{\ell\ell}G_{\ell\ell}, \tag{4.86}$$

for fixed $\ell=i,j$, and this implies (4.82). Substituting it into (4.84) yields $\mathcal{D}_x(M_{ij}^{(k)}{}')=0$. Therefore, with respect to the new Lax pair (U',V'), the transformed auxiliary function is constant. $\square$

The main application of this lemma is to a general system $\mathscr{E}$, such as (4.13) and (4.26), generated by the undetermined matrix P in (2.23). We fix the auxiliary function f that we wish to reduce; if its evolution in $\mathscr{E}$ is of the form (4.81), equation (4.83) defines a candidate matrix P' with respect to which f is a constant ϕ. Then, using the classification in Section 2.3, we check whether P' is admissible in any Lax pair (U',V') of the DNLS equations. If so, the resulting equations are associated with a Darboux system involving ϕ. In the next sections, we introduce two examples of this process.

4.6.2.1 Gauge transformations of $M_{\uparrow}(1)$

The system $\mathscr{E}_{\uparrow}(1)$ from Proposition 4.12 is defined in terms of three variables: the dependent variables p,q, and the auxiliary function f. The evolution of f is

$$f_x=2\mathcal{S}\left(pq+P_{11}\right)f-2f\left(pq+P_{11}\right),$$

which matches (4.81) in Lemma 4.35. Since $f=M_{11}^{(1)}$, we identify $\mathscr{P}_{11}=pq+P_{11}$. To reduce f to a constant ϕ, equation (4.83) requires a matrix P' such that $P'_{11}=-p'q'$: this condition is met by the Lax representations of $(\mathrm{A}_2^{\star})$, $(\mathrm{B}_1^{\star})$, (B_3) and

(C$_1$). More details about these similarities follow in Section 4.7.1. The gauge fixing equation (4.82) becomes

$$\mathscr{a}_x\,\mathscr{a}^{-1} = 2(pq + P_{11}).$$

The most general DNLS Lax representation where f is reduced to a constant ϕ is given by

$$\begin{aligned}
&\begin{cases}
p'_x = -2p'\,\mathscr{b}^{-1}\,\tilde{P}_{22}\,\mathscr{b} + p'\,\mathscr{b}^{-1}\,\mathscr{b}_x - 2\phi^{-1}p'_1q'_1\phi p', \\
q'_{1,x} = 2\mathcal{S}(\mathscr{b}^{-1}\,\tilde{P}_{22}\,\mathscr{b}\,q') - \mathscr{b}_1^{-1}\,\mathscr{b}_{1,x}\,q'_1 + 2q'_1\phi p' q' \phi^{-1},
\end{cases}\\
&M' = \begin{pmatrix} \phi & 0 \\ 0 & 0 \end{pmatrix}\lambda + \begin{pmatrix} 0 & \phi p' \\ q'_1\phi & 0 \end{pmatrix}, \qquad\qquad (4.87)\\
&U' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\lambda^2 + 2\begin{pmatrix} 0 & p' \\ q' & 0 \end{pmatrix}\lambda - \begin{pmatrix} 2p'q' & 0 \\ 0 & \mathscr{b}^{-1}\,\mathscr{b}_x - 2\,\mathscr{b}^{-1}\,\tilde{P}_{22}\,\mathscr{b} \end{pmatrix}.
\end{aligned}$$

Since P'_{22} is not fixed at this stage, $\mathscr{b}$ remains present both in the reduced system $\mathscr{E}$ and in its Lax representation. However, it is not necessary to specify $\mathscr{b}$, because system (4.87) splits naturally into a nonlinear and a linear part if we introduce the new variable

$$u = -\phi p' q'.$$

Writing $q' = -p'^{-1}\phi^{-1}u$, the elements p', P'_{22} and $\mathscr{b}$ are eliminated via a second gauge transformation with matrix $G = \operatorname{diag}(1, (\phi p')^{-1})$. The resulting system is

$$\begin{aligned}
&u_x = 2\phi_1^{-1}u_1u - 2uu_{-1}\phi_{-1}^{-1}, \\
&M = \begin{pmatrix} \phi & 0 \\ 0 & 0 \end{pmatrix}\lambda + \begin{pmatrix} 0 & 1 \\ -u_1\phi & 0 \end{pmatrix}, \\
&U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\lambda^2 + 2\begin{pmatrix} 0 & \phi^{-1} \\ -u & 0 \end{pmatrix}\lambda + 2\begin{pmatrix} \phi^{-1}u & 0 \\ 0 & \phi_1^{-1}u_1 \end{pmatrix}.
\end{aligned}$$

The equation above coincides with the Volterra equation (V$_\text{a}$) upon identifying $\phi = \mu^{-1}$. It is possible to proceed further by fixing the condition on $\mathscr{b}$ as well: this determines a particular DNLS equation among those listed earlier.

Remark 4.36. To avoid confusion, let us distinguish the two different uses of gauge transformations that appear in this section.

- **(i)** From the general system we consider a non-local gauge transformation, given by (2.36) and (2.38), selecting a specific DNLS equation. This reduces f to the constant ϕ by a non-local change of variables (Miura maps) $p, q \mapsto p', q'$.
- **(ii)** Once the DNLS equation is fixed, we use a local change of variables $p', q' \mapsto u, p'$ to separate the Darboux system into a linear and non-linear part.
- **(iii)** Finally, we employ a local gauge transformation without changing variables, and thus within the same DNLS model, to eliminate the gauge variable p'.

We denote both gauge transformations with a matrix G, but note that one is non-local and implies changing the DNLS equation, the other is local and acts only on the Lax representation of the fixed system.

4.6.2.2 Gauge transformations of $M_{\uparrow}(2)$

We apply the gauge reduction to the Darboux transformation $M_{\uparrow}(2)$, following the same considerations outlined in Remark 4.36. Examining the associated DΔEs $\mathscr{E}_{\uparrow}(2)$ in (4.26), it is straightforward to verify that the evolutions of f, a, and b satisfy the conditions of Lemma 4.35. We may either fix f and b to the constants ϕ and β, or fix a and b to the constants α and β.

- **(i) Fixing *f* and *b*.** The gauge fixing conditions are

$$\mathscr{a}_x \, \mathscr{a}^{-1} = 2(P_{11} + pq), \qquad \mathscr{b}_x \, \mathscr{b}^{-1} = 2P_{22},$$

  leading to a matrix P' associated with equation ($\mathrm{B}_1^{\star}$), with diagonal elements $P'_{11} = -p'q'$ and $P'_{22} = 0$.

- **(ii) Fixing *a* and *b*.** The gauge fixing conditions are

$$\mathscr{a}_x \, \mathscr{a}^{-1} = 2P_{11}, \qquad \mathscr{b}_x \, \mathscr{b}^{-1} = 2P_{22}.$$

The matrix P' vanishes and it is associated with equation (A_1).

Both cases are discussed in detail in Section 4.4.1.1. The corresponding systems of equations $\mathscr{E}$ are transformed analogously, as shown in the following example.

Example 4.37. *We transform* (N_1) *with Lax representation* (4.32) *obtained from* (A_1) *into* (AL) *with Lax representation* (4.46) *obtained from* ($B_1^\star$) *via a gauge transformation* (2.38)*. We first rewrite* (4.32) *in terms of the original variables of* $\mathcal{E}^\alpha_\uparrow(2)$*, including the constants:*

$$\begin{cases} \beta_x = 0, \\ f_x = 2(p_1q_1f - fpq), \\ p_x = 2f^{-1}p_1\beta + 2(pqp - f^{-1}p_1q_1fp), \\ q_{1,x} = -2\beta qf^{-1} + 2(q_1fpqf^{-1} - q_1p_1q_1). \end{cases}$$

We now apply the gauge transformation, defining the new dependent variables p'*,* q'*,* f'*, and* b'*:*

$$\begin{cases} b'_x = b'\,\mathscr{b}^{-1}\,\mathscr{b}_x - \mathscr{b}_1^{-1}\,\mathscr{b}_{1,x}\,b', \\ f'_x = (2p'_1q'_1 - \mathscr{a}_1^{-1}\,\mathscr{a}_{1,x})f' - f'(2p'q' - \mathscr{a}^{-1}\,\mathscr{a}_x), \\ p'_x = 2f'^{-1}p'_1b' + 2(p'q'p' - f'^{-1}p'_1q'_1f'p') + p'\,\mathscr{b}^{-1}\,\mathscr{b}_x - \mathscr{a}^{-1}\,\mathscr{a}_x\,p', \\ q'_{1,x} = -2b'q'f'^{-1} + 2(q'_1f'p'q'f'^{-1} - q'_1p'_1q'_1) + q'_1\,\mathscr{a}_1^{-1}\,\mathscr{a}_{1,x} - \mathscr{b}_1^{-1}\,\mathscr{b}_{1,x}\,q'_1. \end{cases}$$

We fix the gauge functions $\mathscr{a}$ *and* $\mathscr{b}$ *so that the Lax representation* (U, V) *of* (A_1) *is transformed into the Lax representation* (U', V') *of* ($B_1^\star$)*:*

$$\mathscr{a}_x = 2\,\mathscr{a}\,p'q', \qquad \mathscr{b}_x = 0.$$

Under these choices, the new system $\mathcal{E}^{\alpha}_{\uparrow}{}'(2)$ *becomes*

$$\begin{cases} \beta'_x = 0, \\ f'_x = 0, \\ p'_x = 2f'^{-1}p'_1\left(\beta' - q'_1f'p'\right), \\ q'_{1,x} = 2\left(q'_1f'p' - \beta'\right)q'f'^{-1}, \end{cases}$$

which coincides with (AL) *with Lax representation* (4.46)*. The transformation maintains* b' *as a non-commutative constant, which we denote by* β'*, in analogy with* β*. If, instead of transforming the specific reduction* $\mathcal{E}^{\alpha}_{\uparrow}(2)$*, we had considered the full system* $\mathcal{E}_{\uparrow}(2)$ *in* (4.30) *for* (A_1)*, we would have obtained the equivalent system* (4.45) *of* ($\mathrm{B}^{\star}_1$)*.*

4.7 Summary

In this Chapter, we propose an ansatz for polynomial Darboux transformations of the DNLS equations that is invariant under the action of the same reduction group (see Section 2.2.3) of their Lax representations (2.23) and that is associated with a Darboux system $\mathscr{E}$ formed by evolutionary DΔEs. From these properties, we identify the up and down polynomial Darboux matrices $M_{\uparrow}(n)$ and $M_{\downarrow}(n)$ of rank-1 and degree $n \in \mathbb{N}$ presented in Definition 4.6.

The Lax representations of the DNLS equations exhibit remarkable similarities, as shown in the classification work of Section 2.3: the entries of the matrix P in (2.23) can assume only a finite number of admissible forms, that are summarised in Table 4.1.

For all the Lax representations, constant Darboux transformations are always related to trivial transformations, as it appears from Table 4.2. In the following sections, we present similar general considerations about the linear and quadratic cases.

4.7.1 Linear up Darboux transformations

In Section 4.3, we discussed linear up Darboux transformations $M_{\uparrow}(1)$ for each DNLS equation. It is often observed that, when different equations lead to the same reduction, their Lax representations in Table 4.1 share common factors. Taking advantage of that, we examine how system $\mathscr{E}_{\uparrow}(1)$ in (4.13) changes under the different forms of matrix P. Each choice of P_{11} and P_{22} determines a constant of motion among ϕ, ρ, θ and κ, together with a suitably defined dependent variable u. The corresponding results, consistent with the list above, are summarised in Table 4.3.

	$P_{11}=0$	$P_{22}=0$	$P_{11}=-pq$	$P_{22}=qp$
Constant	$\theta=fpqf_{-1}$	$\kappa=q_1ff_{-1}p_{-1}$	$\phi=f$	$\rho=q_1fp$
Variable	$u=f^{-1}$	$u=q_1fp\kappa_1^{-1}$	$u=-\phi pq$	$u=qp\rho^{-1}$
Equation	(mV$_\mathrm{b}$)	(mV$_\mathrm{b}$)	(V$_\mathrm{a}$)	$\mathcal{T}$(V$_\mathrm{a}$)
Lax representation	(4.14)	(4.14)	(4.17)	(4.16)

TABLE 4.3: Reductions of the linear up DNLS Darboux transformations.

We illustrate this construction in detail. Without specifying a given DNLS equation, we fix an admissible entry of P. It is often sufficient to identify the corresponding pair of variable and constant to reduce $\mathscr{E}_{\uparrow}(1)$ to one equation only.

(i) Case $P_{11}=-pq$. We introduce into the Darboux system $\mathscr{E}_{\uparrow}(1)$ (4.13) the new variables

$$u=pq,$$

expressed in terms of the original f, p, and q. A direct substitution yields

$$\begin{cases} f_x = 2\mathcal{S}\left(P_{11}+u\right)f - 2f\left(P_{11}+u\right), \\ u_x = 2\left(P_{11}u - uP_{11}\right) + 2\left(uf_{-1}u_{-1}f_{-1}^{-1} - f^{-1}u_1fu\right). \end{cases}$$

Notice that the entry P_{22} does not appear. In the Lax representations where $P_{11}=-pq$, as in (A$_2^\star$), (B$_1^\star$), (B$_3$) and (C$_1$), the variable f becomes the constant of motion ϕ. Transforming $u\mapsto -\phi^{-1}u$ reduces the system to the

Volterra equation ($\mathrm{V_a}$), with Lax representation (4.17), obtained via a gauge transformation with $G = \operatorname{diag}(1, (\phi p)^{-1})$.

(ii) Case $\boldsymbol{P_{22} = qp}$. As in the previous case, we introduce from (4.13)

$$u = qp, \qquad \rho = q_1 f p,$$

where ρ, interpreted as a variable, follows (4.15). The Darboux system $\mathscr{E}_{\uparrow}(1)$ becomes

$$\begin{cases} \rho_x = 2\,\mathcal{S}(u - P_{22})\rho - 2\rho(u - P_{22}), \\ u_x = 2(P_{22}u - uP_{22}) + 2\rho_{-1}u_{-1}\rho_{-1}^{-1}u - 2u\rho^{-1}u_1\rho. \end{cases}$$

All Lax representations with $P_{22} = qp$, namely the models ($\mathrm{A_2}$), ($\mathrm{B_1}$), ($\mathrm{B_3^\star}$), and ($\mathrm{C_1}$), admit ρ as a constant of motion. Rescaling $u \mapsto u\rho$, the evolution of u corresponds to the $\mathcal{T}$-reflection of the Volterra equation ($\mathrm{V_a}$), with $\mu = \rho$ and Lax representation (4.16), obtained via a gauge transformation with $G = \operatorname{diag}(p\rho^{-1}, 1)$.

(iii) Case $\boldsymbol{P_{11} = 0}$. From the system (4.13), we introduce the variables

$$\sigma = -fpqf_{-1}, \qquad u = f^{-1},$$

where σ is defined analogously to (4.20). The corresponding evolutions are

$$\begin{cases} \sigma_x = 2\,\mathcal{S}\left(P_{11}\right)\sigma - 2\sigma\,\mathcal{S}^{-1}\left(P_{11}\right), \\ u_x = 2u\left(u_1\sigma_1 u - \mathcal{S}(P_{11})\right) - 2\left(u\sigma u_{-1} - P_{11}\right)u. \end{cases}$$

Indeed, in the cases ($\mathrm{A_1}$), ($\mathrm{B_1}$), ($\mathrm{B_2^\star}$) and ($\mathrm{C_2}$), where $P_{11} = 0$, the variable σ is a constant of motion. In this situation, the evolution of u corresponds to ($\mathrm{mV_b}$) with $\mu = \sigma$. The associated Lax representation, obtained via a gauge transformation with $G = \operatorname{diag}(1, p^{-1}u\sigma)$, is identical to (4.14). Compare also with Example 4.38.

(iv) Case $P_{22} = 0$. Similarly, the function κ in (4.19) induces two new variables

$$\kappa = -q_1 f f_{-1} p_{-1}, \qquad u = q_1 f_1^{-1} q_2^{-1}.$$

The system (4.13) reduces to

$$\begin{cases} \kappa_x = 2\,\mathcal{S}\,(P_{22})\,\kappa - 2\kappa\,\mathcal{S}^{-1}\,(P_{22})\,, \\ u_x = 2u(u_1\kappa_2 u - \kappa_1 P_{22}\kappa_1^{-1}) - 2(u\kappa_1 u_{-1} - \kappa\,\mathcal{S}^{-1}(P_{22})\kappa^{-1})u. \end{cases}$$

The variable κ is a constant of motion for all cases where $P_{22} = 0$, consisting of (A_1), ($\mathrm{B}_1^\star$), (B_2) and ($\mathrm{C}_2^\star$). In this situation, the evolution of u corresponds to ($\mathrm{mV_b}$) with $\mu = \kappa_1$. The associated Lax representation, identical to (4.14), is obtained via a gauge transformation with $G = \mathrm{diag}(p\kappa_1^{-1}u_1^{-1}, 1)$.

By combining Table 4.1 and Table 4.3, we notice that many DNLS equations allow different constants of motion and reductions depending on the choice of u.

Example 4.38. *We present two distinct reductions of equation* (B_1) *based on two distinct constants of motion. First, the conserved quantity ρ in* (4.15) *leads to the choice of variable $u = -qp\rho^{-1}$ and to the $\mathcal{T}$-reflection of the Volterra equation* ($\mathrm{V_a}$) *with Lax representation* (4.16)*. Alternatively, if we exploit the conserved constant σ in* (4.56)*, the Darboux system becomes*

$$\begin{cases} p_x = 2\left(f^{-1}f_1^{-1}\sigma_1 p - f^{-1}\sigma f_1^{-1}p - pP_{22}\right), \\ f_x = 2\left(\sigma f_{-1}^{-1} - f_1^{-1}\sigma_1\right), \end{cases} \tag{4.88}$$

where the evolution of f is independent of p. Setting $u = f^{-1}$, this Darboux transformation corresponds to ($\mathrm{mV_b}$)*. The associated Lax representation, after a gauge transformation with $G = \mathrm{diag}(1, p^{-1}u\sigma)$ is identical to* (4.14) *for $\sigma = \alpha\beta_{-1}$.*

4.7.2 Quadratic up Darboux transformations

The numerous reductions of $M_{\uparrow}(2)$ presented in Section 4.4 often exhibit analogous structures: they involve the same changes of variables and constants of motion, and lead to the same DΔE. As discussed for the linear case in Section 4.7.1, this depends on the entries of the matrix P, as classified in Table 4.1. Notable analogies appear also between the reductions $b = 0$ and $\rho = 0$. In this section, we justify these observations by computing some general reductions of $M_{\uparrow}(2)$, independently of any specific choice of P and DNLS equation.

We consider the following assumption: let $b = \varepsilon q_1 f p$, where $\varepsilon \in \{1, 0\}$. Clearly, the case $\varepsilon = 0$ leads to the reduction $b = 0$, while $\varepsilon = 1$ corresponds to the reduction $\rho = 0$. To treat both cases simultaneously, we introduce two new dependent variables

$$u = p_1 q_1, \qquad w = f^{-1}.$$

Under these substitutions, the Darboux system $\mathscr{E}_{\uparrow}(2)$ in (4.26) becomes

$$\begin{cases} a_x = 2\,\mathcal{S}(P_{11})a - 2aP_{11}, \\ w_x = 2(P_{11} + u_{-1})w - 2w\,\mathcal{S}(P_{11} + u_{-1}), \\ u_x = 2\left(\mathcal{S}(P_{11})u - u\,\mathcal{S}(P_{11}) + uaw - w_1 a_1 u\right) + \\ \qquad +2(\varepsilon - 1)\left(w_1 u_1 w_1^{-1} u - u w^{-1} u_{-1} w\right). \end{cases} \tag{4.89}$$

Since the entry P_{22} does not appear in the system, we analyse the role of P_{11} in the reductions for b and ρ.

(i) Case $P_{11} = 0$. The auxiliary function a reduces to the constant α. The equations (A_1), (B_1), ($B_2^\star$) and (C_2) belong to this case. The system (4.89) reduces to two equations:

$$\begin{cases} w_x = 2\left(u_{-1} w - w u\right), \\ u_x = 2\left(u \alpha w - w_1 \alpha_1 u\right) + 2(\varepsilon - 1)\left(w_1 u_1 w_1^{-1} u - u w^{-1} u_{-1} w\right). \end{cases}$$

(i.a) Case $\boldsymbol{b = 0}$. When $\varepsilon = 0$, the resulting system is (N_2) for $u \mapsto -u$.

(i.b) Case $\boldsymbol{\rho = 0}$. When $\varepsilon = 1$, the resulting system is ($2V_b$) for $u \mapsto -u$ and $w \mapsto -v$.

(ii) Case $\boldsymbol{P_{11} = -pq}$. The auxiliary function f reduces to the constant ϕ, therefore $w = \phi^{-1}$. The equations ($A_2^\star$), ($B_1^\star$), (B_3) and (C_1) belong to this case. The system (4.89) is reduced to two equations:

$$\begin{cases} a_x = 2(au_{-1} - ua), \\ u_x = 2\left(ua\phi^{-1} - \phi_1^{-1}a_1u + (\varepsilon - 1)\left(\phi_1^{-1}u_1\phi_1u - u\phi u_{-1}\phi^{-1}\right)\right). \end{cases} \tag{4.90}$$

(ii.a) Case $\boldsymbol{b = 0}$. When $\varepsilon = 0$, the resulting system is (rT) for $a \mapsto v$.

(ii.b) Case $\boldsymbol{\rho = 0}$. When $\varepsilon = 1$, the resulting system is ($2V_a$) for $u \mapsto -u$ and $v \mapsto -v$.

The reductions are summarised in the following table, where the symbol $\propto$ denotes that the given variable on the left is proportional to the element on the right up to left or right multiplication by numbers or non-commutative constants.

	$P_{11} = 0$	$P_{11} = -pq$
Constant	$\alpha = a$	$\phi = f$
Variables	$u \propto p_1q_1,\ v \propto f^{-1}$	$u \propto p_1q_1,\ v \propto a$
Reduction $b = 0$	(N_2)	(rT)
Reduction $\rho = 0$	($2V_b$)	($2V_a$)

TABLE 4.4: Some reductions of the quadratic DNLS Darboux transformations.

Remark 4.39. As shown in Example 4.38 for the linear case, a given DNLS equation may admit different quadratic reductions. For instance, equation (A_1) admits a Lax representation with $P_{11} = 0$, which leads to the reduction of the system $\mathscr{E}^{\rho}_{\uparrow}(2)$ involving the variables $u \propto pq$ and $v \propto f^{-1}$. However, as discussed above, an alternative reduction is given by the variables $u \propto q_1\alpha p$ and $v \propto qp$. This is a consequence of $P_{22} = 0$ in the Lax representation. Regardless of the choice, both approaches lead to the same outcome: the two-component Volterra equation ($2V_b$).

4.7.3 Factorisation of quadratic Darboux transformations

Section 4.5 shows that the necessary conditions for $M_{\uparrow}(2)$ to be a composition of two linear up matrices $M_{\uparrow}(1)$ are met under the reduction $\rho = 0$. Indeed, the lists of linear and quadratic Darboux transformations in Sections 4.3.1.1 and 4.4.1.1 confirm this deduction: they show that both $\mathscr{E}_{\uparrow}(1)$ and $\mathscr{E}^{\rho}_{\uparrow}(2)$ are related to Volterra-type equations. This correspondence can be explained by the fact that $\mathscr{E}^{\rho}_{\uparrow}(2)$ is a composition of two (not necessarily identical) copies of $\mathscr{E}_{\uparrow}(1)$.

Let us consider the associated changes of variables. Following scheme (i) in Section 4.7.1, the transformation $M_{\uparrow}(1)$ for the equations ($\mathrm{A}_2^{\star}$), ($\mathrm{B}_1^{\star}$), (B_3) and (C_1) is associated with (V_{a}) via the dependent variable $u = pq$. According to scheme (ii-b) in Section 4.7.2, the same equations admit the reduction in ($2\mathrm{V}_{\mathrm{a}}$) of the system $\mathscr{E}^{\rho}_{\uparrow}(2)$ through the variables $u = -p_1 q_1$ and $v = -a$. In the latter case, we interpret the variable u as inherited from the corresponding variable u in $\mathscr{E}_{\uparrow}(1)$, while the variable v arises as a consequence of (4.79), since $v \propto \bar{p}\bar{q}$. In the quadratic transformation, v encapsulates the disappeared intermediate step between the two linear transformations.

4.7.4 Gauge transformations of Darboux systems

As introduced in Sections 2.1.4 and 2.2.2, all non-commutative DNLS equations are related by certain gauge transformations consisting of a change of variables (2.38) and a matrix transformation (2.36) of their Lax representation (U, V). In Section 4.6, we found that similar considerations also hold for Darboux matrices M and the associated Darboux systems $\mathscr{E}$ through a redefinition of the dependent variables (2.38) and the auxiliary functions (4.80). This allows the transformation between two Lax representations (M, U) and (M', U'), and the associated DΔEs among the ones examined above. An application of this relation is given in Example 4.37.

Moreover, the same relations are employed as a reduction of the generic system $\mathscr{E}$, finding a gauge fixing that reduces a given auxiliary function to a non-commutative constant. The necessary conditions for such a gauge reduction to exist are presented in Lemma 4.35. It is important to note that such gauge transformation changes the considered DNLS equation, therefore the reduction consists of selecting which system among those listed in Theorem 2.4 treats the chosen auxiliary function as a constant. In conclusion, we provide some examples of gauge reductions applied to the linear $\mathscr{E}_{\uparrow}(1)$ and quadratic $\mathscr{E}_{\uparrow}(2)$ up Darboux systems.

Chapter 5

Outlooks

The present work is devoted to the construction of Darboux transformations of the non-commutative DNLS equations and to the derivation of the associated DΔEs. These are integrable generalisations of well-known equations that include non-commutative constants, see Theorems 4.13 and 4.21, each equipped with a Lax representation. Such results open several relevant directions for further research. In the present chapter, we discuss two main possibilities: we extend some algebraic and geometric properties of the non-commutative Volterra equation to the generalised equation ($\mathrm{V_a}$), and we present several partial-difference equations (PΔEs) associated with Bianchi permutability of Darboux transformations and the second part of the Lax-Darboux scheme.

Since each DΔE obtained from a Darboux transformation is an integrable system on its own, it is in principle possible to construct the associated Hamiltonian structures and recursion operators, as well as their symmetry reductions to Painlevé-type discrete equations. As an example, in Section 5.1 we focus on the Volterra equation ($\mathrm{V_a}$). We present an associated recursion operator, constructed following the scheme in [20, 56, 125], and we use it to deduce its first symmetry. Subsequently, we follow the construction in [8] to compute a reduction leading to a generalisation of a previously known discrete Painlevé equation $\mathrm{dP_1}$. Analogous generalisations of the standard commutative discrete Painlevé equations include the study of matrix Pailevé hierarchies in [52].

As introduced in Chapter 1 and in Section 3.2.2.6, Darboux transformations constitute the first part of the Lax-Darboux scheme, that is the initial stage of a complete integrable discretisation of a PDE. In Chapter 4, we performed a discretisation of the DNLS equations into DΔEs, constructing various Darboux transformations. Now, in Section 5.2, we outline the natural continuation of this project: the commutativity of two Darboux transformations leads to a full discretisation in terms of PΔEs. In the present case, the Bianchi permutability (3.32) of the constant, and rank-1 linear and quadratic Darboux matrices introduces a wide class of PΔEs, of which we present some examples. A full construction will be the subject of future work.

5.1 Recursion operator and reduction for equation $\mathrm{V_a}$

The equations in Theorems 4.13 and 4.21 are integrable generalisations of well-studied DΔEs, which have known Hamiltonian structures, recursion operators, and first symmetries, both in the commutative [63] and non-commutative [29] settings. These results can also be extended to the equations obtained in the present work, including the non-commutative constants. Here, we focus on the non-commutative Volterra equation ($\mathrm{V_a}$): we construct its recursion operator $\mathfrak{R}$, deduce its first symmetry $g^{(1)}$ and, following [8], use these to compute a reduction to a Painlevé-type discrete equation similar to $\mathrm{dP_1}$.

5.1.1 Recursion operator

Before stating the main result of this section, we briefly recall the notions of generalised symmetries and recursion operators for evolutionary equations. For a complete discussion, we refer to the monograph [100] or the papers [27, 125], which focus more specifically on the geometric structures of DΔEs.

Consider an evolutionary equation, e.g. $u_x = f([u])$, where f is a function of the variable u and its shifts. Let $u_{x'} = g([u])$ be another evolutionary DΔE, where $x' \in \mathbb{R}$ is an additional continuous parameter. The function g is a (generator of an infinitesimal) symmetry of f if the following Lie bracket vanishes

$$[f, g] = g_*[f] - f_*[g], \tag{5.1}$$

where f_* and g_* denote the Fréchet derivatives of the functions f and g, e.g.

$$f_* = \sum_{j\in\mathbb{Z}} \frac{\partial f}{\partial u_j} \mathcal{S}^j.$$

Generalised symmetries are a fundamental ingredient in the modern theory of integrability: integrable evolutionary DΔEs often present infinite-dimensional Abelian subalgebras of such symmetries with respect to (5.1). A similar construction applies to evolutionary PDEs and it is widely accepted as a criterion for testing integrability and solving classification problems [88, 127].

A recursion operator $\mathfrak{R}$ [44, 99, 100] is a linear operator that maps symmetries of an evolutionary equation to new symmetries. Fixing $f = g^{(0)}$, a recursion operator $\mathfrak{R}$ of f generates the infinite hierarchy of symmetries

$$\mathfrak{R} : g^{(i)} \mapsto \mathfrak{R}g^{(i)} = g^{(i+1)},$$

for all $i \in \mathbb{N}$. The structure of recursion operators and their relations with Nijenhuis, Hamiltonian and pre-Hamiltonian operators are described in [27, 28, 111].

For operators acting on non-commutative variables, it is essential to distinguish between left and right multiplication [27, 29]. Let us define the left multiplication of a function f as l_f and its right multiplication as r_f such that $l_f g = fg$ and $r_f g = gf$ for any other function g.

Theorem 5.1. *A recursion operator of the Volterra equation* (V$_\mathrm{a}$) *is*

$$\begin{aligned}\mathfrak{R} = &\left(r_u\,\mathcal{S} - l_u\,\mathcal{S}^{-1}\,l_{\mu^{-1}} r_\mu\right)\left(r_{\mu^{-1}}\,\mathcal{S} - l_{\mu^{-1}}\right)^{-1}\left(r_{\mu^{-1}}\,\mathcal{S}\,r_u - l_u\,\mathcal{S}^{-1}\,l_{\mu^{-1}}\right)\times\\ &\times\left(l_{\mu^{-1}} r_u - l_u\,\mathcal{S}^{-1}\,l_{\mu^{-1}}\right)^{-1}.\end{aligned}$$

Proof. The construction of the recursion operator $\mathfrak{R}$ follows the procedure described for evolutionary equations in [56]. The same approach has been successfully applied in [20, 63, 103, 125] to reduction group-invariant Lax pairs.

We consider the Lax representation (M, U) in (4.17) of (V$_\mathrm{a}$) with $\mu = \alpha\beta_{-1}$. We assume that the hierarchy of symmetries $g^{(i)}$ is associated with Lax pairs $(M, U^{(i)})$ of the form

$$U^{(i+1)} = \lambda^2 U^{(i)} + B^{(i)}. \tag{5.2}$$

Note that λ^2 is the primitive automorphic function of the reduction group $\mathcal{R}$ in (2.42) [22]. We assume that the undetermined matrix $B^{(i)}$ follows the same reduction group symmetry as U. Therefore, we write

$$B^{(i)} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\lambda^2 + \begin{pmatrix} 0 & c \\ d & 0 \end{pmatrix}\lambda + \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix},$$

where the entries are functions of u and its shifts. Combining (5.2) with the Lax-Darboux equation (4.1) yields

$$M_{x^{(i+1)}} = \lambda^2 M_{x^{(i)}} + \mathcal{S}(B^{(i)})M - MB^{(i)}, \tag{5.3}$$

which connects the symmetry $g^{(i+1)}$ with the symmetry $g^{(i)}$, providing the implicit structure of a recursion operator. Expanding the matrix equation (5.3) gives the following system:

$$
\begin{cases}
f - e_1 = 0,\\
u_1\mu^{-1}c + d_1 = 0,\\
\mu^{-1}a - a_1\mu^{-1} = 0,\\
a_1 - b - \mu^{-1}c = 0,\\
c_1u_1\mu^{-1} + d - e_1\mu^{-1} + \mu^{-1}e = 0,\\
u_{1,x^{(i)}}\mu^{-1} - u_1\mu^{-1}a + b_1u_1\mu^{-1} - d_1\mu^{-1} = 0,\\
u_{1,x^{(i+1)}}\mu^{-1} + u_1\mu^{-1}e - f_1u_1\mu^{-1} = 0.
\end{cases}
$$

Therefore $f = e_1$, $d = -u\mu_{-1}^{-1}c_{-1}$ and $b = a_1 - \mu^{-1}c$. After straightforward algebraic manipulations, the system above reduces to

$$
\begin{cases}
\mu^{-1}a - a_1\mu^{-1} = 0,\\
(r_{\mu^{-1}}\mathcal{S} - l_{\mu^{-1}})e = (r_{u_1\mu^{-1}}\mathcal{S} - l_{u\mu_{-1}^{-1}}\mathcal{S}^{-1})c,\\
u_{1,x^{(i)}} = (\mathcal{S}\, l_{\mu^{-1}}r_u - l_{u_1\mu^{-1}})c,\\
u_{1,x^{(i+1)}} = (\mathcal{S}\, r_u\,\mathcal{S} - l_{u_1\mu^{-1}}r_\mu)e.
\end{cases}
$$

Assuming $a = 0$, we combine the remaining equations in the form of $u_{x^{(i+1)}} = \mathfrak{R}\, u_{x^{(i)}}$, where $\mathfrak{R}$ is the recursion operator presented in the statement. □

If we first consider μ as a complex number, and then assume that all variables are commutative, the operator $\mathfrak{R}$ is equivalent to those in [29] and [63]:

$$
\mathfrak{R} = \left(r_u\,\mathcal{S} - l_u\,\mathcal{S}^{-1}\right)(\mathcal{S} - 1)^{-1}\left(\mathcal{S}\, r_u - l_u\,\mathcal{S}^{-1}\right)\left(r_u - l_u\,\mathcal{S}^{-1}\right)^{-1}, \tag{5.4a}
$$

$$
\mathfrak{R} = u\left(\mathcal{S} - \mathcal{S}^{-1}\right)(\mathcal{S} - 1)^{-1}\left(\mathcal{S}\, u - u\,\mathcal{S}^{-1}\right)\left(1 - \mathcal{S}^{-1}\right)^{-1}u^{-1}. \tag{5.4b}
$$

Applying $\mathfrak{R}$ from Theorem 5.1 to $(\mathrm{V_a})$, we obtain the first higher symmetry of the equation:

$$
\begin{aligned}
g^{(1)} = {} & 2\mu_1\left(\mu_2u_2u_1 + u_1\mu_1u_1 + u_1u\mu\right)u+\\
& - 2u\left(\mu uu_{-1} + u_{-1}\mu_{-1}u_{-1} + u_{-1}u_{-2}\mu_{-2}\right)\mu_{-1}.
\end{aligned} \tag{5.5}
$$

We present a Lax representation for (5.5) in Proposition D.1. Denoting the RHS of ($\mathrm{V_a}$) as $g^{(0)}$:

$$g^{(0)} = 2(\mu_1 u_1 u - u u_{-1} \mu_{-1}),$$

it is straightforward to check that $[g^{(0)}, g^{(1)}] = 0$ according to the Lie bracket defined in (5.1).

5.1.2 Painlevé-type reduction

The Volterra equation ($\mathrm{V_a}$) is homogeneous, thus it is associated with the scaling symmetry u. This is a special kind of symmetry that does not commute with the original equation, but instead satisfies

$$[u, g^{(0)}] = g^{(0)}. \tag{5.6}$$

The Lax representation of the scaling symmetry is provided in D.2. Since also $\mathfrak{R}$ is homogeneous, all the related symmetries $g^{(i)}$ of ($\mathrm{V_a}$) are homogeneous. Therefore u is a scaling symmetry for all members of the hierarchy, i.e.

$$[u, g^{(i)}] = (i+1) g^{(i)}.$$

Notice that ($\mathrm{V_a}$) and (5.5) satisfy the commutation relations above.

Recall that reductions associated with (standard) symmetries $g^{(i)}$ lead to algebro-geometric solutions (solitons). Following [8, 10], we focus on a combination of symmetries $g^{(i)}$ with the scaling symmetry u:

$$\not{\ell} = g^{(1)} + \xi(u + x g^{(0)}), \tag{5.7}$$

where $\xi \in \mathbb{C}$ is a constant. In explicit terms, we write

$$\begin{aligned} \not{\ell} = {} & 2\mu_1 \left(\xi u_1 x + u_1 u \mu + u_1 \mu_1 u_1 + \mu_2 u_2 u_1 \right) u \\ & - 2u \left(\xi u_{-1} x + u_{-1} u_{-2} \mu_{-2} + u_{-1} \mu_{-1} u_{-1} + \mu u u_{-1} \right) \mu_{-1} + \xi u. \end{aligned}$$

We apply the constraint $\mathscr{f} = 0$ to the Volterra equation (V_{a}). By the properties of symmetries and the scaling symmetry, it is simple to see that $\mathscr{f}$ commutes with $g^{(0)}$; therefore $\mathscr{f} = 0$ is stable under time evolution. At a general lattice site $n \in \mathbb{Z}$, $\mathscr{f}$ can be written as

$$\mathscr{f}_n = 2\left(\mu_{n+1}\, \mathscr{h}_{n+1}\, u_n - u_n\, \mathscr{h}_{n-1}\, \mu_{n-1}\right),$$

where the function $\mathscr{h}_n$ takes the general form

$$\mathscr{h}_n = u_n u_{n-1}\mu_{n-1} + u_n \mu_n u_n + \mu_{n+1} u_{n+1} u_n + \xi x u_n + \mu_n^{-1}\left(\frac{\xi n}{4} + \nu + \sigma(-1)^n\right),$$

where ν and σ are commutative constants. The new constraint $\mathscr{h}_n = 0$ implies the previous one $\mathscr{f}_n = 0$, and is also stable under the evolution in x. Indeed

$$\mathscr{h}_{n,x} = 2\left(u_n \mu_n\, \mathscr{h}_n + \mu_{n+1}\, \mathscr{h}_{n+1}\, u_n - \mathscr{h}_n\, \mu_n u_n - u_n\, \mathscr{h}_{n-1}\, \mu_{n-1}\right),$$

therefore, if $\mathscr{h}_n = 0$ holds, also $\mathscr{h}_{n,x} = 0$ holds. The constraint $\mathscr{h}_n = 0$ corresponds to the equation

$$\mu_{n+1} u_{n+1} u_n + u_n \mu_n u_n + u_n u_{n-1} \mu_{n-1} + \xi x u_n + \mu_n^{-1}\left(\frac{\xi n}{4} + \nu + (-1)^n \sigma\right) = 0.$$

Note that the constant ξ can be eliminated by rescaling both x and u. Setting

$$x \mapsto x/\sqrt{\xi}, \qquad \mu_n \mapsto \sqrt{\xi}\mu_n/2, \qquad \nu \mapsto \xi\, \nu/4, \qquad \sigma \mapsto \xi\, \sigma/4 \tag{5.8}$$

yields a version of the non-commutative discrete Painlevé equation dP_1 including the non-commutative constant μ:

$$\mu_{n+1} u_{n+1} u_n + u_n \mu_n u_n + u_n u_{n-1} \mu_{n-1} + 2 x u_n + \left(n + \nu + (-1)^n \sigma\right) \mu_n^{-1} = 0. \tag{5.9}$$

Indeed, when μ is made commutative and we replace $u_n \mapsto \mu_n^{-1} u_n$, equation (5.9) reduces to the discrete Painlevé equation dP_1 discussed in [8, 10]:

$$u_{n+1} u_n + u_n^2 + u_n u_{n-1} + 2 x u_n + n + \nu + (-1)^n \sigma = 0.$$

As shown in Proposition D.3, a Lax representation of (5.9) is obtained if we combine, according to (5.7), the Lax representations of $g^{(1)}$ and $g^{(0)}$ with the non autonomous Lax representation of the scaling symmetry u and apply the same rescaling as above.

5.2 Insights into the Lax-Darboux scheme

Given an integrable PDE, a first discretisation is obtained by defining the integrable DΔEs associated with Darboux transformations. As briefly shown in Chapter 1 and Section 3.2.2.6, this is the first half of the so-called Lax-Darboux scheme [7, 12, 66, 89]. The pairwise commutativity of Darboux transformations yields a full discretisation of the PDE, resulting in integrable partial-difference equations (PΔEs).

These equations play a central role in the theory of integrability [57]: many integrable PΔEs depending on four lattice points (i.e. quadrilateral PΔEs) are classified in the well-known ABS lists [9]. Full constructions of the Lax-Darboux scheme for the commutative (NLS) equation and the Kaup-Newell equation (A) are presented in [65, 66].

From the perspective of Lax representations, from the Lax pair (U, V) in (1.1) of a PDE, a Darboux matrix induces a Lax pair (M, U) in (3.21) of a DΔE, and the compatibility of two Darboux matrices, namely Bianchi permutability (3.32), leads to the Lax pair $(M^{(1)}, M^{(2)})$ of a PΔE.

We consider the commutativity of the Darboux transformations of the DNLS equations found in Chapter 4. By systematically applying (3.32) to pairs of constant, linear, and quadratic Darboux matrices, it is possible to deduce the respective systems of PΔEs. This also simplifies the composition rule discussed in Section 3.2.2.5: whenever two Darboux transformations commute, the associated inner shifts coincide with the respective standard shifts.

However, the Bianchi permutability of matrices with non-commutative entries leads to a large number of equations, which are often difficult to solve or simplify, even after considering optional reductions. In the following sections we introduce some relevant examples.

5.2.1 Bianchi permutability and integrable PΔEs

Following the same convention as in (3.27), let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two Darboux transformations associated to the following Darboux matrices:

$$\mathcal{S}_1 : \Phi \mapsto \bar{\Phi} = M^{(1)}\Phi, \qquad \mathcal{S}_2 : \Phi \mapsto \tilde{\Phi} = M^{(2)}\Phi. \tag{5.10}$$

As already introduced in Section 3.2.2.6, their commutativity, i.e. $[\mathcal{S}_1, \mathcal{S}_2] = 0$, corresponds to the Bianchi permutability (3.32) of the respective Darboux matrices, see [18, 57]:

$$\mathcal{S}_2(M^{(1)})M^{(2)} - \mathcal{S}_1(M^{(2)})M^{(1)} = 0. \tag{5.11}$$

The Darboux transformations $\mathcal{S}_1$ and $\mathcal{S}_2$ define a new over-determined auxiliary system that is analogous to (1.2) for PDEs and (3.21) for DΔEs:

$$\begin{cases} \mathcal{S}_1(\Phi) = M^{(1)}\Phi, \\ \mathcal{S}_2(\Phi) = M^{(2)}\Phi, \end{cases} \tag{5.12}$$

where Φ denotes the fundamental solution. Bianchi permutability is interpreted as the compatibility equation of (5.12), i.e. a fully discrete zero-curvature condition for the Lax pair $(M^{(1)}, M^{(2)})$. In this sense, the PΔEs obtained from (5.11) are considered integrable.

This also ensures the consistency of the transformations (5.10) along a "square path", such as $\mathcal{S}_1\mathcal{S}_2(\Phi) = \mathcal{S}_2\mathcal{S}_1(\Phi)$, making the order of the shifts irrelevant.

This idea is illustrated in the following graph:

$$\begin{tikzcd}
 & \bar{\Phi} \arrow[dr, "\mathcal{S}_2"] & \\
\Phi \arrow[ur, "\mathcal{S}_1"] \arrow[dr, "\mathcal{S}_2"'] & & \tilde{\bar{\Phi}} = \bar{\tilde{\Phi}} \\
 & \tilde{\Phi} \arrow[ur, "\mathcal{S}_1"'] &
\end{tikzcd}$$

Therefore, by (5.10) we extend the lattice $\mathbb{Z}$ of a single Darboux transformation to admit two independent shifts. If these shifts commute, i.e. if (5.11) holds, $\mathcal{S}_1$ and $\mathcal{S}_2$ define a lattice $\mathbb{Z}^2$, where each direction represents one of the two shifts. The fundamental solution of (5.12) is therefore $\Phi_{m,n} = \Phi(m, n)$, depending on the discrete variables $m, n \in \mathbb{Z}$ and

$$\mathcal{S}_1 : \Phi_{m,n} \mapsto \Phi_{m+1,n}, \qquad \mathcal{S}_2 : \Phi_{m,n} \mapsto \Phi_{m,n+1}. \tag{5.13}$$

As in the differential-difference case, we usually omit the subscripts when they are both zero. Therefore, we write $\Phi_{m,n} = \mathcal{S}_1^m \mathcal{S}_2^n(\Phi)$. Bianchi permutability ensures that the definition of $\Phi_{m+1,n+1}$ is unique. This notation is illustrated by modifying the previous graph:

$$\begin{tikzcd}
 & \Phi_{1,0} \arrow[dr, "\mathcal{S}_2"] & \\
\Phi \arrow[ur, "\mathcal{S}_1"] \arrow[dr, "\mathcal{S}_2"'] & & \Phi_{1,1} \\
 & \Phi_{0,1} \arrow[ur, "\mathcal{S}_1"'] &
\end{tikzcd}$$

Applying the same reasoning, we define $p_{m,n} = p(m, n)$ and $q_{m,n} = q(m, n)$ as the dependent variables of the PΔE with $m, n \in \mathbb{Z}$, as in (5.13). This is well posed at each point of $\mathbb{Z}^2$, as it does not depend on the chosen path [93, 94].

The Lax-Darboux scheme consists in transforming a PDE in terms of $p(x, t)$ and

$q(x,t)$, into a DΔE in terms of $p_n(t)$ and $q_n(t)$, and finally into a PΔE in terms of $p_{m,n}$ and $q_{m,n}$. In the next section we examine an application of this construction to the DNLS equations.

5.2.2 Bianchi permutability of $M_{\uparrow}(2)$

Several interesting full discretisations of the DNLS equations arise from the compatibility of two quadratic Darboux matrices $M_{\uparrow}(2)$ in (4.7b). Let us consider two Darboux transformations $\mathcal{S}_1$ and $\mathcal{S}_2$ as in (5.10), associated with

$$M^{(1)} = \begin{pmatrix} f & 0 \\ 0 & 0 \end{pmatrix} \lambda^2 + \begin{pmatrix} 0 & fp \\ q_{1,0}f & 0 \end{pmatrix} \lambda + \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \tag{5.14a}$$

$$M^{(2)} = \begin{pmatrix} g & 0 \\ 0 & 0 \end{pmatrix} \lambda^2 + \begin{pmatrix} 0 & gp \\ q_{0,1}g & 0 \end{pmatrix} \lambda + \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}, \tag{5.14b}$$

where f, g, a, b, c and d are auxiliary functions defined on $\mathbb{Z}^2$. Note that the shift q_1 in the original $M_{\uparrow}(2)$ is replaced by the respective generalisation on $\mathbb{Z}^2$ according to (5.13).

Imposing Bianchi permutability (5.11) on these transformations leads to the following system of PΔEs:

$$\begin{cases} a_{0,1}c - c_{1,0}a = 0, \\ f_{0,1}g - g_{1,0}f = 0, \\ b_{0,1}d - d_{1,0}b = 0, \\ (a_{0,1}g - c_{1,0}f)p + f_{0,1}p_{0,1}d - g_{1,0}p_{1,0}b = 0, \\ q_{1,1}(f_{0,1}c - g_{1,0}a) - b_{0,1}q_{0,1}g - d_{1,0}q_{1,0}f = 0, \\ f_{0,1}p_{0,1}q_{0,1}g - g_{1,0}p_{1,0}q_{1,0}f + a_{0,1}g - g_{1,0}a + f_{0,1}c - c_{1,0}f = 0. \end{cases} \tag{5.15}$$

The same situation is examined for the commutative Kaup-Newell equation (A) in [66]. The first five equations in (5.15) define the shifts of the variables in a specific direction

$$\begin{cases} c_{1,0} = a_{0,1} c a^{-1}, \\ g_{1,0} = f_{0,1} g f^{-1}, \\ d_{1,0} = b_{0,1} d b^{-1}, \\ p_{1,0} = f g^{-1} (p_{0,1} d - f_{0,1}^{-1} a_{0,1} \ell a^{-1} f p) b^{-1}, \\ q_{1,0} = b d^{-1} (q_{0,1} g + b_{0,1}^{-1} q_{1,1} f_{0,1} \ell) f^{-1}, \end{cases} \tag{5.16}$$

where, for convenience, we set

$$\ell = c - g f^{-1} a. \tag{5.17}$$

The last term in (5.15) is interpreted as a further condition on the equations above:

$$\begin{gathered} f_{0,1} \left(1 - p_{0,1} b_{0,1}^{-1} q_{1,1} f_{0,1}\right) \ell - a_{0,1} \ell a^{-1} f \left(1 - p d^{-1} q_{0,1} g\right) + \\ + a_{0,1} \ell a^{-1} f p d^{-1} b_{0,1}^{-1} q_{1,1} f_{0,1} \ell = 0. \end{gathered} \tag{5.18}$$

We present two particular solutions of the system (5.15).

5.2.2.1 The trivial solution

The system (5.15) admits a trivial solution: the compatibility condition (5.18) vanishes if we set $\ell = 0$, which means that $c = g f^{-1} a$. The entire system simplifies and becomes equivalent to

$$\begin{cases} c_{1,0} = f_{1,0} f g^{-1} f_{0,1}^{-1} a_{0,1} g f^{-1}, \\ g_{1,0} = f_{0,1} g f^{-1}, \\ d_{1,0} = b_{0,1} d b^{-1}, \\ p_{1,0} = f g^{-1} p_{0,1} d b^{-1}, \\ q_{1,0} = b d^{-1} q_{0,1} g f^{-1}, \end{cases} \tag{5.19}$$

where the auxiliary functions f, g, b and d remain undefined. We refer to this solution as "trivial" because it directly relates the shifts $\mathcal{S}_1$ and $\mathcal{S}_2$ of the dependent variables p and q without involving any additional lattice point.

Observing the first three equations in (5.16), it is common to solve them introducing the so-called potential variables $\mathscr{f}$, $\mathscr{c}$ and $\mathscr{d}$:

$$f = \mathscr{f}_{1,0}\,\mathscr{f}^{-1}, \qquad g = \mathscr{f}_{0,1}\,\mathscr{f}^{-1}, \tag{5.20a}$$

$$a = \mathscr{c}_{1,0}\,\mathscr{c}^{-1}, \qquad c = \mathscr{c}_{0,1}\,\mathscr{c}^{-1}, \tag{5.20b}$$

$$b = \mathscr{d}_{1,0}\,\mathscr{d}^{-1}, \qquad d = \mathscr{d}_{0,1}\,\mathscr{d}^{-1}\,. \tag{5.20c}$$

In this way the number of equations is reduced and system (5.19) becomes

$$\begin{cases} \mathscr{c}_{1,0} = \mathscr{f}_{1,0}\,\mathscr{f}_{0,1}^{-1}\,\mathscr{c}_{0,1}, \\ p_{1,0} = \mathscr{f}_{1,0}\,\mathscr{f}_{0,1}^{-1}\,p_{0,1}\,\mathscr{d}_{0,1}\,\mathscr{d}_{1,0}^{-1}, \\ q_{1,0} = \mathscr{d}_{1,0}\,\mathscr{d}_{0,1}^{-1}\,q_{0,1}\,\mathscr{f}_{0,1}\,\mathscr{f}_{1,0}^{-1}, \end{cases} \tag{5.21}$$

where the first equation corresponds to the condition $\mathscr{l} = 0$.

5.2.2.2 A reduction of the general solution

Excluding the trivial solution, i.e. assuming $\mathscr{l} \neq 0$, we solve the compatibility equation (5.18) in full generality using the rational expression:

$$\begin{aligned} \mathscr{h} &= 1 - p_{0,1}b_{0,1}^{-1}q_{1,1}f_{0,1}, \\ \mathscr{k} &= 1 - pd^{-1}\left(q_{0,1}g + b_{0,1}^{-1}q_{1,1}f_{0,1}\mathscr{l}\right), \\ a_{0,1} &= f_{0,1}\mathscr{h}\mathscr{l}\mathscr{k}^{-1}f^{-1}a\mathscr{l}^{-1}. \end{aligned} \tag{5.22}$$

Substituting it into (5.16) and using the potential variables (5.20), the PΔEs reduce to the following equations:

$$\begin{cases} a_{0,1} = f_{0,1}\mathscr{h}\ell\mathscr{k}^{-1}f^{-1}a\ell^{-1}, \\ p_{1,0} = \mathscr{f}_{1,0}\,\mathscr{f}_{0,1}^{-1}\,(p_{0,1}\,\mathscr{d}_{0,1} - \mathscr{h}\ell\mathscr{k}^{-1}p\,\mathscr{d})\,\mathscr{d}_{1,0}^{-1}, \\ q_{1,0} = \mathscr{d}_{1,0}\left(\mathscr{d}_{0,1}^{-1}\,q_{0,1}\,\mathscr{f}_{0,1} + \mathscr{d}_{1,1}^{-1}\,q_{1,1}\,\mathscr{f}_{1,1}\,\mathscr{f}_{0,1}^{-1}\,\ell\,\mathscr{f}\right)\mathscr{f}_{1,0}^{-1}\,. \end{cases} \tag{5.23}$$

This system is analogous to a general Darboux system $\mathscr{E}$. To the best of our knowledge, there are no fundamental reductions that eliminate undetermined degrees of freedom, so we instead consider optional reductions. However, the situation is more complicated than in Darboux transformations: the number of free parameters is considerably larger and the fully discrete setting makes solving the equations more difficult.

Nevertheless, we present a surprisingly simple reduction of (5.23). Assume that

$$d = 0 \quad \text{and} \quad b = q_{1,0}fp. \tag{5.24}$$

Note that the second assumption corresponds to the reduction $\rho = 0$ for a matrix $M_{\uparrow}(2)$. The general system (5.15) becomes equivalent to

$$\begin{cases} a_{0,1}c - c_{1,0}a = 0, \\ f_{0,1}g - g_{1,0}f = 0, \\ g_{1,0}p_{1,0}q_{1,0}f + c_{1,0}f - a_{0,1}g = 0, \\ f_{0,1}p_{0,1}q_{0,1}g + f_{0,1}c - g_{1,0}a = 0, \\ a_{0,1}g - g_{1,0}a - c_{1,0}f + f_{0,1}c + f_{0,1}p_{0,1}q_{0,1}g - g_{1,0}p_{1,0}q_{1,0}f = 0. \end{cases} \tag{5.25}$$

The fifth equation above, which corresponds to the compatibility condition (5.18), vanishes identically assuming the other equations. The first two relations can be solved by the potential variables $\mathscr{f}$ and $\mathscr{c}$ in (5.20), and it is convenient to introduce the new variable

$$u = pq$$

in place of p and q. This transformation is particularly significant as it appears frequently in the present work: see for instance Sections 4.7.1 and 4.7.2. The system of PΔEs above becomes equivalent to

$$\begin{cases} u_{1,0} = \cancel{\ell}_{1,0}\,\cancel{\ell}_{1,1}^{-1}\,\cancel{c}_{1,1}(\cancel{c}_{0,1}^{-1}\,\cancel{\ell}_{0,1}\,\cancel{\ell}_{1,0}^{-1} - \cancel{c}_{1,0}^{-1}), \\ u_{0,1} = (\cancel{\ell}_{0,1}\,\cancel{\ell}_{1,0}^{-1}\,\cancel{c}_{1,0} - \cancel{c}_{0,1})\,\cancel{c}^{-1}\,\cancel{\ell}\,\cancel{\ell}_{0,1}^{-1}. \end{cases} \tag{5.26}$$

This system is over-determined: by imposing the consistency condition $\mathcal{S}_1(u_{0,1}) = \mathcal{S}_2(u_{1,0})$, it becomes equivalent to the following scalar equation

$$\cancel{\ell}_{1,2}^{-1}\,\cancel{c}_{1,2}(\cancel{c}_{0,2}^{-1}\,\cancel{\ell}_{0,2} - \cancel{c}_{1,1}^{-1}\,\cancel{\ell}_{1,1}) = (\cancel{\ell}_{2,0}^{-1}\,\cancel{c}_{2,0} - \cancel{\ell}_{1,1}^{-1}\,\cancel{c}_{1,1})\,\cancel{c}_{1,0}^{-1}\,\cancel{\ell}_{1,0},$$

which can be further simplified by letting $w = \cancel{\ell}^{-1}\,\cancel{c}$:

$$w_{1,2}(w_{0,2}^{-1} - w_{1,1}^{-1}) - (w_{2,0} - w_{1,1})w_{1,0}^{-1} = 0. \tag{5.27}$$

This is a particularly simple example of PΔE associated with the Bianchi permutability of two rank-1 up quadratic Darboux transformations.

Equation (5.27) is integrable in the sense that it possesses a Lax representation as constructed in Proposition D.4 from the Darboux matrices (5.14). This model is remarkable since it is a five-point equation and so it does not belong to the ABS classification of quadrilateral integrable PΔEs [9].

Appendix A

Parametric DNLS Lax representations

In this appendix, we revisit some Lax representations introduced in Chapter 2 and we present equivalent formulations in the non-commutative setting, highlighting their respective advantages and disadvantages. To the best of our knowledge, with the exception of (A.7) in [112], the Lax pairs presented here have not previously appeared in the literature.

In particular, Section A.1 provides a non-commutative Lax representation, analogous to the commutative (2.12), but related only with some non-commutative DNLS equations. In Section A.2 we introduce two one-parameter families of Lax representations in the non-commutative setting: the first is related to the symmetric DNLS equations (see Section 2.1.3.2), and the second consists of a generalisation of a result from Smirnov [112], which can be expressed entirely in terms of the matrices I and J in (2.24). Finally, in Section A.3, we construct a two-parameter Lax representation that is invariant under both reduction groups introduced in Section 2.2.3.

A.1 General non-commutative Lax representations

The construction of a general Lax representation, such as the one in Proposition (2.2), is in principle possible also for all non-commutative DNLS equations. However, it would require a large number of parameters or, at the very least, result in a high-degree polynomial. Therefore, the practical utility of such a representation is debatable.

Nevertheless, a simpler and more straightforward non-commutative lift of the Lax representation (2.12) is

$$
\begin{aligned}
U =& \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lambda^2 + \begin{pmatrix} 0 & (\beta-\alpha)p \\ 2q & 0 \end{pmatrix} \lambda + \frac{\beta-2\alpha}{2(\beta-\alpha)} \begin{pmatrix} \alpha pq & 0 \\ 0 & (2\beta-\alpha)qp \end{pmatrix}, \\
V =& -2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lambda^4 - 2 \begin{pmatrix} 0 & (\beta-\alpha)p \\ 2q & 0 \end{pmatrix} \lambda^3 + 2(\beta-\alpha) \begin{pmatrix} pq & 0 \\ 0 & -qp \end{pmatrix} \lambda^2 + \\
&+ \begin{pmatrix} 0 & (\beta-\alpha)(\beta pqp - p_x) \\ 2(\beta qpq + q_x) & 0 \end{pmatrix} \lambda + \\
&+ \frac{\beta-2\alpha}{2(\beta-\alpha)} \begin{pmatrix} \alpha(pq_x - p_xq + (\alpha+\beta)pqpq) & 0 \\ 0 & (2\beta-\alpha)(q_xp - qp_x + (\alpha+\beta)qpqp) \end{pmatrix},
\end{aligned}
\tag{A.1a}
$$

where we assume that $\alpha \neq \beta$. Imposing the zero-curvature condition (1.1), the Lax pair above yields the following system of equations:

$$
\begin{cases}
p_t = -p_{xx} + 2\alpha pq_xp + \frac{\beta(2\beta-3\alpha)}{\beta-\alpha} pqp_x + \frac{\alpha\beta}{\beta-\alpha} p_xqp - \alpha(\beta-2\alpha)pqpqp, \\
q_t = q_{xx} + 2\alpha qp_xq + \frac{\alpha\beta}{\beta-\alpha} qpq_x + \frac{\beta(2\beta-3\alpha)}{\beta-\alpha} q_xpq + \alpha(\beta-2\alpha)qpqpq,
\end{cases}
\tag{A.2}
$$

together with the following compatibility conditions:

$$
\begin{cases}
\frac{\alpha\beta(\beta-2\alpha)(2\beta-3\alpha)}{4(\beta-\alpha)^2} (pqpq_x - pqp_xq - pq_xpq + p_xqpq) = 0, \\
\frac{\alpha\beta(\beta-2\alpha)(2\beta-\alpha)}{4(\beta-\alpha)^2} (qpqp_x - qpq_xp - qp_xqp + q_xpqp) = 0.
\end{cases}
\tag{A.3}
$$

The compatibility conditions are satisfied only in three cases: when $\alpha = 0$, when $\beta = 0$, and when $\beta = 2\alpha$, which correspond, up to rescaling, to the non-commutative lifts (B_1), (C_1) and (A_1) of the three historically important DNLS.

Observe that, in contrast to the commutative case, both the Lax representations and their associated equations are formally singular when $\alpha = \beta$. However, if commutativity is imposed, the compatibility conditions (A.3) vanish identically and system (A.2) becomes regular.

A.2 One-parameter Lax representations

The symmetric DNLS equations (2.13) are defined to render adjoint-invariant the general Lax representation (2.12) for the commutative DNLS equations (2.5). They also reappear in Section 2.1.4 as DNLS equations satisfying the consistency condition (2.18) of gauge transformations.

We obtain a non-commutative version of the symmetric DNLS equation imposing the constraint $\beta = \alpha + 2$ to (A.1), as in Section 2.1.3.2. The resulting Lax representation, analogous to (2.14), is a one-parameter Lax pair, invariant under the $\mathcal{R}$ transformation in Section 2.2.3.1 and the adjoint transformation in Section 2.2.3.2:

$$U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lambda^2 + 2 \begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix} \lambda + \frac{2-\alpha}{4} \begin{pmatrix} \alpha pq & 0 \\ 0 & (\alpha+4)qp \end{pmatrix}, \tag{A.4a}$$

$$\begin{aligned} V = &-2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lambda^4 - 4 \begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix} \lambda^3 + 4 \begin{pmatrix} pq & 0 \\ 0 & -qp \end{pmatrix} \lambda^2 + \\ &+ 2 \begin{pmatrix} 0 & (\alpha+2)pqp - p_x \\ (\alpha+2)qpq + q_x & 0 \end{pmatrix} \lambda + \\ &+ \frac{2-\alpha}{4} \begin{pmatrix} \alpha(pq_x - p_x q + 2(\alpha+1)pqpq) & 0 \\ 0 & (\alpha+4)(q_x p - qp_x + 2(\alpha+1)qpqp) \end{pmatrix}. \end{aligned} \tag{A.4b}$$

The associated system of equations, which is a reduction of (A.2), becomes

$$\begin{cases} p_t = -p_{xx} - \frac{1}{2}(\alpha+2)(\alpha-4)pqp_x + 2\alpha pq_xp + \frac{1}{2}\alpha(\alpha+2)p_xqp + \\ \qquad +(\alpha-2)\alpha pqpqp, \\ q_t = q_{xx} + \frac{1}{2}\alpha(\alpha+2)qpq_x + 2\alpha qp_xq - \frac{1}{2}(\alpha-4)(\alpha+2)q_xpq + \\ \qquad -(\alpha-2)\alpha qpqpq, \end{cases} \tag{A.5}$$

along with the compatibility conditions, derived from (A.3):

$$\begin{cases} \alpha(\alpha-2)(\alpha+2)(\alpha-4)\left(pqpq_x - pqp_xq - pq_xpq + p_xqpq\right) = 0, \\ \alpha(\alpha-2)(\alpha+2)(\alpha+4)\left(qpqp_x - qpq_xp - qp_xqp + q_xpqp\right) = 0. \end{cases} \tag{A.6}$$

As in Appendix A.1, this Lax representation holds in three specific cases: when $\alpha = -2, 0, 2$, corresponding respectively to (B_1), (A_1), and (C_1).

In [112], Smirnov presents a Lax representation for the commutative DNLS equations (2.5) depending on a single parameter $s \in \mathbb{R}$. The spatial part U is deduced from the Lax representations of (A), (B) and (C), while the temporal part V is constructed from U via the zero-curvature condition. Smirnov's Lax pair follows the pattern (2.7) and it can be expressed in terms of the matrices I and J (2.8):

$$P = -2sIJ^2, \qquad R = 2s\left([J, J_x] + 2(3-8s)IJ^4\right). \tag{A.7}$$

The matrix Q appearing in the Lax pair is deduced from (2.26). The associated equations of motion, expressed in terms of I and J, are

$$J_t = -IJ_{xx} + 8(1-2s)J^2J_x + 4(1-4s)JJ_xJ - 16s(1-4s)IJ^5. \tag{A.8}$$

Expressed in terms of the dependent variables p and q, these matrix equations become the following system:

$$\begin{cases} p_t = -p_{xx} + 4(1-4s)p^2q_x + 8(1-2s)pqp_x - 16s(1-4s)p^3q^2, \\ q_t = q_{xx} + 4(1-4s)q^2p_x + 8(1-2s)pqq_x + 16s(1-4s)p^2q^3, \end{cases}$$

which is equivalent to the symmetric DNLS (2.13) upon setting $\alpha = 2(1-4s)$. For special values of s, the standard DNLS equations are recovered: setting $s = 0$, it corresponds to (A), with $s = 1/4$ to (B) and with $s = 1/2$ to (C).

Remark A.1. It is possible to reduce the two parameters α and β in system (2.5) to a single parameter $\zeta = \beta\alpha^{-1}$ by rescaling $p \mapsto \alpha^{-1}p$, provided that $\alpha \neq 0$. This yields

$$\begin{cases} p_t = -p_{xx} + 2p^2q_x + 2\zeta pqp_x - (\zeta-2)p^3q^2, \\ q_t = q_{xx} + 2q^2p_x + 2\zeta pqq_x + (\zeta-2)p^2q^3. \end{cases}$$

The three cases (A), (B) and (C) correspond to $\zeta = 1, \infty, 0$, respectively. The parameter s in (A.8) and ζ are related by the Möbius transformation

$$\zeta = \frac{1-2s}{1-4s}.$$

A generalisation of Smirnov's representation (A.7) to the non-commutative case is possible. However, as in Appendix A.1, this construction is valid only for (A_1), (B_1) and (C_1). Consider the following matrices in (2.23):

$$P = 2s\,(2(1-2s)\mathbb{1} - I)\,J^2,$$
$$R = 2s\,(\mathbb{1} - 2(1-2s)I)\left([J,J_x] + 2(3-8s)IJ^4\right).$$

In this case, alongside I and J from (2.24), we include the identity matrix $\mathbb{1}$. The zero-curvature condition (1.1) yields the equation of motion

$$J_t = -IJ_{xx} + 4(1-2s)(\mathbb{1}+4sI)J^2J_x + 4(1-2s)(\mathbb{1}-4sI)J_xJ^2 + \\ + 4(1-4s)JJ_xJ - 16s(1-4s)IJ^4$$

and the compatibility condition

$$s(1-2s)(1-4s)(2\,\mathbb{1} - (1-4s)I)\left(J^3J_x - J^2J_xJ - JJ_xJ^2 + J_xJ^3\right) = 0.$$

While the latter is trivially satisfied in the commutative case, in the non-commutative case it restricts s to the values $0, 1/4, 1/2$, which correspond to the three main cases

(A_1), (B_1), and (C_1).

Remark A.2. Notably, the Lax representation above is expressed entirely in terms of the matrices $\mathbb{1}$, I and J, which can be interpreted as generators of a non-commutative associative algebra, satisfying the anti-commutation relation

$$\{I, J\} = IJ + JI = 0.$$

The evolution of the dependent variables J expressed in its components p and q corresponds to the system of equations:

$$\begin{cases} p_t = -p_{xx} + 4(1-2s)(1+4s)pqp_x + 4(1-4s)pq_xp + \\ \qquad +4(1-2s)(1-4s)p_xqp - 16s(1-4s)pqpqp, \\ q_t = q_{xx} + 4(1-2s)(1-4s)qpq_x + 4(1-4s)qp_xq + \\ \qquad +4(1-2s)(1+4s)q_xpq + 16s(1-4s)qpqpq, \end{cases}$$

and the compatibility conditions become

$$\begin{cases} s(1-2s)(1-4s)(1+4s)\left(pqpq_x - pqp_xq - pq_xpq + p_xqpq\right) = 0, \\ s(1-2s)(1-4s)(3-4s)\left(qpqp_x - qpq_xp - qp_xqp + q_xpqp\right) = 0. \end{cases}$$

A.3 Two-parameter Lax representation

Assuming that $\alpha \neq \beta$, we construct a Lax representation of the same form as (2.23) and (A.4), depending on two parameters and invariant under both reduction groups in Section 2.2.3. Redefining the dependent variables $p \mapsto 2p$ and $q \mapsto (\beta-\alpha)q$, we obtain

$$U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lambda^2 + 2(\beta-\alpha) \begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix} \lambda + (\beta - 2\alpha) \begin{pmatrix} \alpha pq & 0 \\ 0 & (2\beta-\alpha)qp \end{pmatrix},$$

$$V = -2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lambda^4 - 4(\beta-\alpha) \begin{pmatrix} 0 & p \\ q & 0 \end{pmatrix} \lambda^3 + 4(\beta-\alpha)^2 \begin{pmatrix} pq & 0 \\ 0 & -qp \end{pmatrix} \lambda^2 +$$

$$+\,2(\beta-\alpha)\begin{pmatrix} 0 & 2\beta(\beta-\alpha)pqp-p_x \\ 2\beta(\beta-\alpha)qpq+q_x & 0 \end{pmatrix}\lambda+$$

$$+\,(\beta-2\alpha)\begin{pmatrix} \alpha(pq_x-p_xq)+2\alpha(\beta^2-\alpha^2)pqpq & 0 \\ 0 & (2\beta-\alpha)(q_xp-qp_x+2(\beta^2-\alpha^2)qpqp) \end{pmatrix}.$$

The associated system of equations is

$$\begin{cases} p_t=-p_{xx}+2\beta(2\beta-3\alpha)pqp_x+4\alpha(\beta-\alpha)pq_xp+2\alpha\beta p_xqp+ \\ \qquad -4\alpha(\beta-2\alpha)(\beta-\alpha)^2pqpqp, \\ q_t=q_{xx}+2\alpha\beta qpq_x+4\alpha(\beta-\alpha)qp_xq+2\beta(2\beta-3\alpha)q_xpq+ \\ \qquad +4\alpha(\beta-2\alpha)(\beta-\alpha)^2qpqpq, \end{cases}$$

with the following compatibility conditions:

$$\begin{cases} \alpha\beta(\beta-2\alpha)(2\beta-3\alpha)\left(pqpq_x-pqp_xq-pq_xpq+p_xqpq\right)=0, \\ \alpha\beta(\beta-2\alpha)(2\beta-\alpha)\left(qpqp_x-qpq_xp-qp_xqp+q_xpqp\right)=0. \end{cases}$$

The compatibility conditions above allow three cases only: $\alpha=0$, $\beta=0$ and $\beta=2\alpha$, which correspond, up to rescaling by the remaining parameter, to (B_1), (C_1), and (A_1), respectively.

Appendix B

The general polynomial Darboux transformation $M(n)$

Chapter 4 is devoted to the construction and analysis of the constant, linear and quadratic Darboux transformations of the DNLS equations. This is done using the Lax representations (U, V) of the form (2.23) and the polynomial Darboux matrices defined in (4.5). The Darboux systems $\mathscr{E}(1)$ in (4.24) and $\mathscr{E}(2)$ in (4.60) are sufficient to reveal an emerging pattern in the Darboux system $\mathscr{E}(n)$ associated with a full-rank Darboux transformation $M(n)$ of degree $n \geq 0$. The rank-1 versions, $\mathscr{E}_{\uparrow}(n)$ and $\mathscr{E}_{\downarrow}(n)$, together with all the respective optional reductions, follow directly from this structure.

We consider the DNLS Lax representation (U, V) given in (2.23) and a polynomial Darboux matrix $M(n)$ from (4.5) of degree $n \in \mathbb{N}$ and invariant under the reduction group $\mathcal{R}_{\pm}$ in (4.4). Substituting these elements into the Lax-Darboux equation (4.1), we prove the following theorem.

Theorem B.1. *Let $M(n)$ be a polynomial Darboux matrix of degree $n > 0$ as in (4.5) for the DNLS Lax representation (2.23). The Darboux system $\mathscr{E}(n)$ consists of the differential-difference equations*

$$\mathscr{E}(n) = \{\mathscr{e}_i \mid i = 0, 1, 2 \ldots n\} \tag{B.1}$$

defined below. For convenience, we assume that $M^{(k)}$ vanishes for all $k < 0$. The special case $n = 0$ is irregular since the corresponding matrix $M(0)$ does not follow (4.5). The Darboux system $\mathscr{E}(0)$ takes the form

$$\mathscr{E}(0) = \left\{ M_{11}^{(0)} p - p_1 M_{22}^{(0)} = 0, \quad M_{22}^{(0)} q - q_1 M_{11}^{(0)} = 0, \quad \mathscr{e}_{2k}|_{k=0} \right\}, \tag{B.2}$$

where $\mathscr{e}_{2k}|_{k=0}$ denotes the term $\mathscr{e}_{2k}$ evaluated at $k = 0$.

The explicit form of the equations $\mathscr{e}_i$ depending on $k = 1, \dots \left\lfloor \frac{n}{2} \right\rfloor$ is as follows, where $\lfloor \cdot \rfloor$ stands for the floor function.

$$\mathscr{e}_0 : \begin{cases} \mathcal{D}_x\, M_{11}^{(n)} = 2\mathcal{S}\,(P_{11} + pq)\, M_{11}^{(n)} - 2M_{11}^{(n)}\,(P_{11} + pq)\,, \\ \mathcal{D}_x\, M_{22}^{(n)} = 2\mathcal{S}\,(P_{22} - qp)\, M_{22}^{(n)} - 2M_{22}^{(n)}\,(P_{22} - qp)\,; \end{cases}$$

$$\mathscr{e}_1 : \begin{cases} p_{1,x} M_{22}^{(n)} - M_{11}^{(n)} p_x = 2(M_{11}^{(n-2)} p - p_1 M_{22}^{(n-2)}) + 2\mathcal{S}(P_{11} p - p P_{22}) M_{22}^{(n)} + \\ \qquad -2M_{11}^{(n)} (P_{11} p - p P_{22}) + 2(p_1 q_1 p_1 M_{22}^{(n)} + p_1 q_1 M_{11}^{(n)} p - p_1 M_{22}^{(n)} q p + \\ \qquad - M_{11}^{(n)} p q p) - 2M_{12}^{(n-3)}, \\ q_{1,x} M_{11}^{(n)} - M_{22}^{(n)} q_x = 2(q_1 M_{11}^{(n-2)} - M_{22}^{(n-2)} q) + 2\mathcal{S}(P_{22} q - q P_{11}) M_{11}^{(n)} + \\ \qquad -2M_{22}^{(n)} (P_{22} q - q P_{11}) + 2(q_1 M_{11}^{(n)} p q - q_1 p_1 q_1 M_{11}^{(n)} - q_1 p_1 M_{22}^{(n)} q + \\ \qquad + M_{22}^{(n)} q p q) - 2M_{21}^{(n-3)}; \end{cases}$$

$$\mathscr{e}_{2k} : \begin{cases} \mathcal{D}_x\, M_{11}^{(n-2k)} = 2\mathcal{S}(P_{11}) M_{11}^{(n-2k)} - 2M_{11}^{(n-2k)} P_{11} + \\ \qquad +2(p_1 M_{21}^{(n-2k-1)} - M_{12}^{(n-2k-1)} q), \\ \mathcal{D}_x\, M_{22}^{(n-2k)} = 2\mathcal{S}(P_{22}) M_{22}^{(n-2k)} - 2M_{22}^{(n-2k)} P_{22} + \\ \qquad +2(q_1 M_{12}^{(n-2k-1)} - M_{21}^{(n-2k-1)} p); \end{cases}$$

$$\mathscr{e}_{2k+1} : \begin{cases} \mathcal{D}_x\, M_{12}^{(n-2k-1)} = 2\mathcal{S}(P_{11}) M_{12}^{(n-2k-1)} - 2M_{12}^{(n-2k-1)} P_{22} + \\ \qquad +2(p_1 M_{22}^{(n-2k-2)} - M_{11}^{(n-2k-2)} p) + 2M_{12}^{(n-2k-3)}, \\ \mathcal{D}_x\, M_{21}^{(n-2k-1)} = 2\mathcal{S}(P_{22}) M_{21}^{(n-2k-1)} - 2M_{21}^{(n-2k-1)} P_{11} + \\ \qquad +2(q_1 M_{11}^{(n-2k-2)} - M_{22}^{(n-2k-2)} q) - 2M_{21}^{(n-2k-3)}. \end{cases}$$

By convention, when $i > n$, the corresponding set of equations $\mathscr{e}_i$ is empty.

Proof. We consider the Lax representation (U, V) in (2.23) and the Darboux matrix $M(n)$ from Lemma 4.3 expressed as

$$U = \sum_{i=0}^{2} U^{(i)}\lambda^i, \qquad M(n) = \sum_{i=0}^{n} M^{(i)}\lambda^i. \tag{B.3}$$

Substituting these into the Lax-Darboux equation (4.1), we collect the coefficients of each power of λ. By Lemma 4.2, the leading term of the resulting polynomial is proportional to λ^n. The coefficient of λ^k for each $k = 0, \ldots, n$ corresponds to the following matrix equation

$$\mathcal{D}_x(M^{(k)}) = \sum_{j=0}^{2} \mathcal{S}(U^{(j)})M^{(k-j)} - M^{(k-j)}U^{(j)}. \tag{B.4}$$

Notice that each equation for fixed k couples only the coefficients $M^{(k)}$, $M^{(k-1)}$, and $M^{(k-2)}$. Using the explicit terms of U from (2.23) and matrix M in Lemma 4.3, it follows that (B.4) reproduces the terms $\mathscr{e}_{n-k}$ described in the theorem.

In particular, substituting the leading term $M^{(n)}$ and $M^{(n-1)}$, which is algebraically determined by the former, into (B.4) yields the terms $\mathscr{e}_0$ and $\mathscr{e}_1$. The remaining $\mathscr{e}_k$ follow similarly.

The irregular case for $n = 0$ must be computed directly, as done in Section 4.2. ☐

Remark B.2. Since, by convention, $M^{(k)} = 0$ for all $k < 0$, the expressions for $\mathscr{e}_n$ and $\mathscr{e}_{n-1}$ are simplified from the general case $\mathscr{e}_i$.

The theorem above provides a direct construction of the system $\mathscr{E}(n)$ associated with the polynomial Darboux matrix $M(n)$. The rank-1 cases $\mathscr{E}_\uparrow(n)$ and $\mathscr{E}_\downarrow(n)$ are obtained by setting $M^{(n)}_{22} = 0$ or $M^{(n)}_{11} = 0$, respectively.

Appendix C

Transformation-related integrable DΔEs

In Chapter 4, especially in Sections 4.3 and 4.4, we obtained a variety of integrable differential-difference equations (DΔEs) by applying specific changes of variables to the Darboux systems $\mathscr{E}$. In this way we found links with the Volterra, Toda, Ablowitz-Ladik, Merola-Ragnisco-Tu equations. However, different choices of variables yield different equations from the same system.

As a change of variables, we consider a transformation that involves a finite number of shifts of the dependent variables, i.e. a Miura transformation. In this appendix, we consider specific Miura transformations of the Volterra equations and the systems (N_1), (N_2) and (N_3).

In Section C.1, we focus on the various non-commutative generalisations of the Volterra equation. Generalising the results of [8], we introduce the potential modified Volterra equation, the modified and two-component Volterra equations, each discussed with the associated Miura transformations and Lax representations.

The next two sections examine the systems (N_1), (N_2) and (N_3), linking them with other well-known models. Section C.2 proves that the equations (N_1) and (N_2) are both associated with (AL) via a non-invertible Miura transformation. Similarly, Section C.3 connects the equation (N_3) both to (MRT) and to the following generalisation of Kaup equation [12, 29] with non-commutative constants:

$$\begin{cases} u_x = 2(\mu_1 u_1 - u\mu)(u+v), \\ v_x = 2(u+v)(\mu v - v_{-1}\mu_{-1}). \end{cases} \tag{K}$$

The lists presented here are not intended to be comprehensive: the classification of Miura transformations is still an open problem, and thus a full classification lies beyond the scope of this work.

At the end of each section, we include a graph summarising the discussed relations: an arrow indicates the direction of the corresponding Miura transformation, while a double-headed one denotes when it is invertible.

C.1 The Volterra equations

The Volterra-type equations appear repeatedly throughout this dissertation: linear Darboux transformations of the DNLS equations yield the Volterra equation ($\mathrm{V_a}$) and the modified Volterra equation ($\mathrm{mV_b}$), while quadratic transformations are the origin of give rise to the two-component versions ($\mathrm{2V_a}$) and ($\mathrm{2V_b}$). In this section we show how all Volterra equations are linked by Miura transformations. The relation among respective Darboux transformations is addressed in Section 4.5.

As observed by Adler [8], in the non-commutative setting there are two forms of the (commutative) Volterra equation, which are denoted as VL^1 and VL^2:

$$u_x = u_1 u - u u_{-1}, \tag{VL^1}$$

$$u_x = u_1^\star u - u u_{-1}^\star, \tag{VL^2}$$

where $\star$ is the involution introduced in (2.21). Adler proposed a scheme that connects (VL^1) and (VL^2) with the potential modified Volterra equation, the two respective versions of modified Volterra equation and their associated two-component Volterra systems. Here we extend these results to the generalised equations involving a non-commutative constant μ.

The potential modified Volterra equation corresponds to the following equation

$$v_x = \mu_{-1} v_1 v_{-1}^{-1} \mu_{-2} v, \tag{pmV}$$

whose Lax representation is given by

$$\begin{aligned} M &= 2 \begin{pmatrix} v_1^{-1}\mu_{-1}^{-1} v & 0 \\ 0 & 0 \end{pmatrix} \lambda + \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}, \\ U &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lambda^2 + \begin{pmatrix} 0 & v^{-1}\mu_{-1} v_1 \\ -2 v_{-1}^{-1}\mu_{-2} v & 0 \end{pmatrix} \lambda. \end{aligned}$$

The pair (M, U) originates from a gauge transformation of (4.17) with $G = \operatorname{diag}(v_1, v_1)$. Equation (pmV) is pivotal in the present scheme: it generates two distinct branches of Volterra equations, labelled a and b. Branch a consists of the following models:

$$w_x = \mu w_1 \mu^{-1} w^2 - w^2 \mu_{-1}^{-1} w_{-1} \mu_{-1}; \tag{mV$_\mathrm{a}$}$$

$$u_x = \mu_1 u_1 u - u u_{-1} \mu_{-1}; \tag{V$_\mathrm{a}$}$$

$$r_x = \theta_1 s_1 r - r s \theta, \qquad s_x = r s - s r_{-1}. \tag{2V$_\mathrm{a}$}$$

When μ commutes with all variables, equation (V$_\mathrm{a}$) represents the traditional non-commutative Volterra equation (VL^1) [110, 122], equation (mV$_\mathrm{a}$) represents the modified Volterra equation, and equation (2V$_\mathrm{a}$) represents the two-component Volterra equation.

Branch b consists of the following analogous models:

$$w_x = w(w_1 \eta_1 - \eta w_{-1}) w; \tag{mV$_\mathrm{b}$}$$

$$u_x = u_1^\star u - u u_{-1}^\star; \tag{VL^2}$$

$$r_x = s_1\theta_1 r - r\theta s, \qquad s_x = sr - r_{-1}s. \tag{2V$_\mathrm{b}$}$$

Here, we use directly Adler's (VL^2) in place of a counterpart of ($\mathrm{V_a}$): a generalisation with non-commutative constants requires non-trivial conditions, as shown in Remark C.2. When the constant μ is commutative, equation ($\mathrm{mV_b}$) corresponds to the modified Volterra equation [11] and, ($2\mathrm{V_b}$) corresponds to the two-component Volterra equations.

In what follows, we present the Lax representations and the Miura transformations among the given models. From equation (pmV), we obtain ($\mathrm{mV_a}$) by the substitution

$$(\mathrm{pmV}) \to (\mathrm{mV_a}): \qquad w = v_1 v^{-1}\mu_{-1}.$$

The equation ($\mathrm{mV_a}$) has the following Lax representation:

$$M = \begin{pmatrix} \mu^{-1} & 0 \\ 0 & 0 \end{pmatrix}\lambda + \frac{1}{2}\begin{pmatrix} 0 & w_1\mu^{-1} \\ -2w_1\mu^{-1} & 0 \end{pmatrix},$$

$$U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\lambda^2 + \begin{pmatrix} 0 & \mu w_1\mu^{-1} \\ -2w & 0 \end{pmatrix}\lambda + \begin{pmatrix} \mu w_1\mu^{-1}w & 0 \\ 0 & \mu w_1\mu^{-1}w \end{pmatrix}.$$

Similarly, the modified Volterra equation ($\mathrm{mV_a}$) can be converted into ($\mathrm{V_a}$) considering

$$(\mathrm{mV_a}) \to (\mathrm{V_a}): \qquad u = w_1\mu^{-1}w.$$

The map between ($\mathrm{V_a}$) and ($\mathrm{mV_a}$), when μ is commutative, is a rescaling of the original Miura transformation [21, 29, 122]. The Lax representation of ($\mathrm{V_a}$) can be deduced from (4.17) by setting $\phi = 2\mu^{-1}$, while the Lax representation of its $\mathcal{T}$-reflection is (4.16) assuming $\mu = 2\rho$.

From ($\mathrm{V_a}$) it is possible to deduce ($2\mathrm{V_a}$):

$$(\mathrm{V_a}) \to (2\mathrm{V_a}): \qquad r_n = \mu_{2n+1}u_{2n+1}, \qquad s_n = u_{2n}\mu_{2n-1}^{-1}, \qquad \theta_n = \mu_{2n-1}\mu_{2n}$$

The Lax representation of ($2\mathrm{V_a}$) is (4.42) setting $\phi = 2\mu^{-1}$, and its $\mathcal{T}$-reflection is either (4.39) or (4.51) with $\mu = 2\gamma$.

Remark C.1. The Volterra equation (2V$_\mathrm{a}$) is invariant under the change of variables given by

$$r' = \theta_1 s_1, \qquad s' = r\theta^{-1}.$$

This property is motivated by the bijection between (2V$_\mathrm{a}$) and (V$_\mathrm{a}$): for the latter, the same transformation corresponds to the invariance $u_n \mapsto \mu_{n+1} u_{n+1} \mu_n^{-1}$.

The Miura transformation from the potential modified Volterra (pmV) to the modified Volterra equation (mV$_\mathrm{b}$) in branch b is

$$(\mathrm{pmV}) \to (\mathrm{mV_b}): \qquad w = \mu^{-1} v^{-1} \mu_{-1} v_1 \mu_1, \qquad \eta = \mu_1^{-1} \mu_{-1}.$$

The Lax representation of (mV$_\mathrm{b}$) is given by (4.14) with $\mu = 2\alpha\beta_{-1}$. From (mV$_\mathrm{b}$) we obtain the two-component system (2V$_\mathrm{b}$) defining

$$(\mathrm{mV_b}) \to (\mathrm{2V_b}): \qquad r_n = w_{2n+1}\eta_{2n+1}w_{2n}, \qquad s_n = w_{2n-1}w_{2n}, \qquad \theta_n = \eta_{2n}.$$

The Lax representation of (mV$_\mathrm{b}$) is given by either (4.34) with $\mu = 2\beta$, or by (4.44) with $\mu = 2\alpha$.

There is no known generalisation V$_\mathrm{b}$ of Adler's (VL^2) including non-commutative constants. In fact, our attempts to deduce a model analogue to (V$_\mathrm{a}$) from (mV$_\mathrm{b}$) or (2V$_\mathrm{b}$) either force the non-commutative constant to reduce to the identity $\mathbb{1}$ or require additional constraints. Let us show that in detail: given (mV$_\mathrm{b}$), assume the following change of variables

$$u_{2n} = w_{2n}\mu_{2n+1}^{-1}\mu_{2n-1}w_{2n-1}, \qquad u^{\star}_{2n+1} = w_{2n}\mu_{2n}\mu_{2n+2}^{-1}w_{2n+1}.$$

The evolution of the new variable u is given by

$$\begin{aligned} u_{2n,x} =& w_{2n}w_{2n+1}\eta_{2n+1}w_{2n}\eta_{2n}w_{2n-1} - w_{2n}\eta_{2n}w_{2n-1}\eta_{2n-1}w_{2n-2}w_{2n-1} = \\ =& w_{2n}w_{2n+1}\eta_{2n+1}u_{2n} - u_{2n}\eta_{2n-1}w_{2n-2}w_{2n-1}. \end{aligned}$$

The remaining terms in w can be identified with the $\star$-involution $u^\star_{2n} = w_{2n-1}\mu_{2n-1}\mu^{-1}_{2n+1}w_{2n}$ of u_{2n}, imposing that $\mu_{2n} = \mu_{2n+2}$. In such case

$$u_{2n,x} = u^\star_{2n+1}\eta_{2n+1}u_{2n} - u_{2n}\eta_{2n-1}u^\star_{2n-1}.$$

We now turn to the evolution of $u^\star_{2n+1} = w_{2n}w_{2n+1}$: from the given assumptions we obtain

$$u^\star_{2n+1,x} = u^\star_{2n+1}u_{2n+2} - u_{2n}u^\star_{2n+1}.$$

Here any dependence on η vanishes: shifting it back and applying the $\star$-involution reduces it to

$$u_{2n,x} = u^\star_{2n+1}u_{2n} - u_{2n}u^\star_{2n-1},$$

which coincides with the previous result only if η_n equals the identity $\mathbb{1}$ for all n, yielding Adler's (VL^2). A similar argument holds if we start from ($2\mathrm{V}_\mathrm{b}$).

Remark C.2. A non-commutative constant can be introduced in (VL^2), by requiring additional algebraic conditions. Let us consider the following transformation of ($2\mathrm{V}_\mathrm{b}$):

$$u_{2n} = \theta^{-1}_n r_n, \qquad u^\star_{2n-1} = \theta^{-1}_{n-1}s_n.$$

The associated evolutions are the following:

$$\begin{aligned} u_{2n,x} &= u^\star_{2n+1}\theta_{n+1}\theta_n u_{2n} - u_{2n}\theta_n\theta_{n-1}u^\star_{2n-1}, \\ u^\star_{2n-1,x} &= u^\star_{2n-1}\theta_n u_{2n} - u_{2n-2}\theta_{n-1}u^\star_{2n-1}. \end{aligned}$$

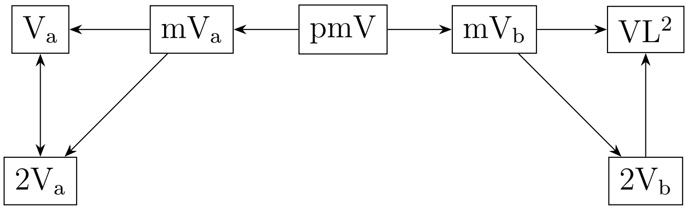
Shifting the latter one unit forward and taking its $\star$-involution yields

$$u^\star_{2n,x} = u^\star_{2n+1}\theta^\star_{n+1}u_{2n} - u_{2n}\theta^\star_n u^\star_{2n-1}, \tag{C.3}$$

which is consistent with the given dynamics if we impose that $\theta^\star_n = \theta_n\theta_{n-1}$. This leads to a version of (VL^2) with a non-commutative constant $\kappa = \theta\theta_{-1}$:

$$u_x = u^\star_1\kappa_1 u - u\kappa u^\star_{-1}.$$

The relationship between these models is represented by the following graph, analogous to one in [8]:



C.2 The Ablowitz-Ladik equation

The Ablowitz-Ladik equation (AL) is related to the systems (N_1) and (N_2) by non-invertible Miura transformations. System (N_1), expressed in the variables u and v, is related to (N_2) in the variables w and v through

$$(\mathrm{N}_1) \to (\mathrm{N}_2): \qquad w = uv, \qquad v \mapsto -v.$$

The Lax representation of (N_1) is given in (4.32). The Lax pair of (N_2) is presented in (4.33), while its $\mathcal{T}$-reflection is in (4.48). We introduce a new system, denoted (N_1'), which serves to connect (AL) with this framework:

$$\begin{cases} r_x = 2r(\mu - s_1)^{-1} r_1 s_1 - 2s r_{-1}(\mu_{-1} - s)^{-1} r, \\ s_x = 2(s r_{-1} - r s), \end{cases} \tag{N$_1'$}$$

with Lax representation given by

$$M = \begin{pmatrix} r_1^{-1}(\mu - s_1) & 0 \\ 0 & 0 \end{pmatrix} \lambda^2 - \begin{pmatrix} 0 & r_1^{-1}(\mu - s_1) \\ \mu - s_1 & 0 \end{pmatrix} \lambda + \begin{pmatrix} 0 & 0 \\ 0 & \mu \end{pmatrix},$$

$$U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lambda^2 - 2 \begin{pmatrix} 0 & 1 \\ r & 0 \end{pmatrix} \lambda - 2 \begin{pmatrix} r + (\mu - s_1)^{-1} r_1 s_1 & 0 \\ 0 & 0 \end{pmatrix}.$$

The evolution of the variable s in (N_1') resembles that of the relativistic Toda equation. The system (N_1') is obtained from (N_1) via the substitution

$$(\mathrm{N}_1) \to (\mathrm{N}_1') : \qquad r = -v(u+\mu), \qquad s = -u_{-1}.$$

The Ablowitz-Ladik equation (AL), expressed in the variables u and v, is related to (N_1') by the non-invertible Miura transformation

$$(\mathrm{AL}) \to (\mathrm{N}_1') : \qquad r = v_{-1}\mu u, \qquad s_1 = \nu - vu, \qquad \nu \mapsto \mu.$$

The Lax representation of (AL) is given in (4.46). The relationship between the systems above is summarised by the following graph. Although not included in the diagram, system (N_1') is related to (N_1) by an invertible Miura transformation.

$$\begin{tikzcd}
\boxed{(\mathrm{N}_2)} \arrow[r, leftrightarrow] & \boxed{(\mathrm{N}_1)} & \boxed{(\mathrm{AL})} \arrow[l]
\end{tikzcd}$$

C.3 The Merola-Ragnisco-Tu equation

The Merola-Ragnisco-Tu equation (MRT), expressed in the variables r and s, is related to the $\mathcal{T}$-reflection of (N_3), in the variables u and v, via the non-invertible Miura transformation

$$(\mathrm{MRT}) \to (\mathrm{N}_3) : \qquad u_1 = -s_1^{-1}\nu s, \qquad v = rs, \qquad \mu \mapsto \mu_{-1}.$$

The Lax representation of (MRT) is given in (4.52), while the Lax representation of (N_3) is in (4.36) and its $\mathcal{T}$-reflection is in (4.50).

We present two equations related to (N_3), which we denote ($\mathrm{N}_3^{\mathrm{a}}$) and ($\mathrm{N}_3^{\mathrm{b}}$):

$$\begin{cases} w_x = 2(wv_1 - vw + wv_1^{-1}w_1\mu_1 - \mu_{-1}w_{-1}v^{-1}w), \\ v_x = 2(w\mu - \mu_{-1}w_{-1}); \end{cases} \tag{$\mathrm{N}_3^{\mathrm{a}}$}$$

$$\begin{cases} v_x = 2v(\mu - s_1)^{-1}v_1\mu - 2\mu_{-1}v_{-1}(\mu_{-1} - s)^{-1}v, \\ s_x = 2(sv_{-1} - vs). \end{cases} \tag{$\mathrm{N}_3^{\mathrm{b}}$}$$

Notice that the evolution of the variable v in ($\mathrm{N}_3^{\mathrm{a}}$) is linear and depends only on w and its shifts, whereas in ($\mathrm{N}_3^{\mathrm{b}}$) the evolution of s has the same structure as the relativistic Toda equation. The Lax representation of ($\mathrm{N}_3^{\mathrm{a}}$) is

$$\begin{aligned} M &= \begin{pmatrix} w^{-1}v & 0 \\ 0 & 0 \end{pmatrix}\lambda^2 + \begin{pmatrix} 0 & w^{-1}v \\ -v_1w^{-1}v & 0 \end{pmatrix}\lambda + \begin{pmatrix} 0 & 0 \\ 0 & \mu - v_1w^{-1}v \end{pmatrix}, \\ U &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\lambda^2 + 2\begin{pmatrix} 0 & 1 \\ -v & 0 \end{pmatrix}\lambda - 2\begin{pmatrix} v^{-1}w\mu & 0 \\ 0 & v \end{pmatrix}. \end{aligned}$$

The Lax representation of ($\mathrm{N}_3^{\mathrm{b}}$) is

$$\begin{aligned} M &= \begin{pmatrix} v_1^{-1}(\mu - s_1) & 0 \\ 0 & 0 \end{pmatrix}\lambda^2 + \begin{pmatrix} 0 & v_1^{-1}(\mu - s_1) \\ s_1 - \mu & 0 \end{pmatrix}\lambda + \begin{pmatrix} 0 & 0 \\ 0 & s_1 \end{pmatrix}, \\ U &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\lambda^2 + 2\begin{pmatrix} 0 & 1 \\ -v & 0 \end{pmatrix}\lambda - 2\begin{pmatrix} (\mu - s_1)^{-1}v_1\mu & 0 \\ 0 & v \end{pmatrix}. \end{aligned}$$

Both models originate from (N_3), expressed in the variables u and v, by the following Miura transformations

$$\begin{aligned} &(\mathrm{N}_3) \to (\mathrm{N}_3^{\mathrm{a}}) : \qquad && w = vu, \qquad && v \mapsto v; \\ &(\mathrm{N}_3) \to (\mathrm{N}_3^{\mathrm{b}}) : \qquad && s_1 = \mu - v_1u^{-1}, \qquad && v \mapsto v. \end{aligned}$$

From the system (N_3), written in the variables u and v, it is also possible to obtain the $\mathcal{T}$-reflection of the Kaup equation (K), written in terms of r, s:

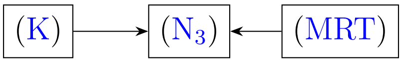
$$(\mathrm{K}) \to (\mathrm{N}_3) : \qquad u = -(r + s), \qquad v = r\mu - \mu_{-1}r_{-1}.$$

The Lax representation of the $\mathcal{T}$-reflection of (K) is

$$M = \begin{pmatrix} (r+s)^{-1} & 0 \\ 0 & 0 \end{pmatrix} \lambda^2 + \begin{pmatrix} 0 & (r+s)^{-1} \\ (\mu r - r_1 \mu_1)(r+s)^{-1} & 0 \end{pmatrix} \lambda + \\ + \begin{pmatrix} 0 & 0 \\ 0 & (\mu r - r_1 \mu_1)(r+s)^{-1} - \mu \end{pmatrix},$$

$$U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lambda^2 + 2 \begin{pmatrix} 0 & 1 \\ (\mu_{-1} r_{-1} - r\mu) & 0 \end{pmatrix} \lambda + 2 \begin{pmatrix} (r+s)\mu & 0 \\ 0 & (\mu_{-1} r_{-1} - r\mu) \end{pmatrix}.$$

The relations among the present equations are summarised by the following graph. Although not included in the diagram, the systems ($\mathrm{N}_3^{\mathrm{a}}$) and ($\mathrm{N}_3^{\mathrm{b}}$) are related to ($\mathrm{N}_3$) by invertible Miura transformations.



Appendix D

Paralipomena about Lax representations

In this Appendix, we collect some additional results omitted from Chapter 5 of this dissertation. In particular, in Section 5.1.2, we present equation (5.9) as a new version of the discrete non-commutative Painlevé equation dP_1 that includes a non-commutative constant. In Section 5.2.2.2, we show a reduction of the Bianchi permutability of two quadratic Darboux transformations $M_{\uparrow}(2)$ that leads to an integrable P∆E (5.27) not included in the ABS list. In what follows, we construct a Lax representation for both equations, proving their integrability.

Both Lax pairs are obtained by standard methods, building on the material introduced above. Nevertheless, to the best of our knowledge, they are novel objects and do not appear in the paper [104], hence we decided to list them here.

D.1 Lax representation of the Painlevé type equation (5.9)

The non-commutative equation (5.9) is analogous to the Painlevé equation dP_1 and includes a non-commutative constant μ. As discussed in Section 5.1.2, it originates from a symmetry reduction of the Volterra equation (V_a) based on its first symmetry (5.5) and its scaling symmetry (5.6). In order to deduce the Lax representation of (5.9), we first provide the Lax pairs of the latter two equations. Let us also recall that (V_a) is obtained from (4.17), i.e. from the matrices

$$\begin{aligned} M &= \begin{pmatrix} \mu^{-1} & 0 \\ 0 & 0 \end{pmatrix} \lambda + \begin{pmatrix} 0 & 1 \\ -u_1\mu^{-1} & 0 \end{pmatrix}, \\ U &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lambda^2 + 2 \begin{pmatrix} 0 & \mu \\ -u & 0 \end{pmatrix} \lambda + 2 \begin{pmatrix} \mu u & 0 \\ 0 & \mu_1 u_1 \end{pmatrix}. \end{aligned} \tag{D.1}$$

As showed in Theorem 5.1, the construction of the recursion operator $\mathfrak{R}$ for equation (V_a) is based on the ansatz (5.2) that the Lax representation $(M, U^{(i+1)})$ of the symmetry $g^{(i+1)}$ is related with the Lax representation $(M, U^{(i)})$ of the previous symmetry $g^{(i)}$. In Section 5.1.1, by applying $\mathfrak{R}$ directly to $g^{(0)}$, i.e. (V_a), we obtained its first symmetry $g^{(1)}$ (5.5). The same process also provides instructions to deduce the corresponding Lax representation.

Proposition D.1. *A Lax representation* $(M, U^{(1)})$ *of the first symmetry* $g^{(1)}$ (5.5) *of the Volterra equation* (V_a) *is given by the same* M *in the Lax pair* (D.1) *of* (V_a) *and*

$$\begin{aligned} U^{(1)} = & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lambda^4 + 2 \begin{pmatrix} 0 & \mu \\ -u & 0 \end{pmatrix} \lambda^3 + 2 \begin{pmatrix} \mu u & 0 \\ 0 & -u\mu \end{pmatrix} \lambda^2 + \\ & + 2 \begin{pmatrix} 0 & \mu\left(\mu_1 u_1 + u\mu\right) \\ -u\left(\mu u + u_{-1}\mu_{-1}\right) & 0 \end{pmatrix} \lambda + \\ & + 2 \begin{pmatrix} \mu\left(\mu_1 u_1 u + u\mu u + u u_{-1}\mu_{-1}\right) & 0 \\ 0 & \mu_1\left(\mu_2 u_2 u_1 + u_1\mu_1 u_1 + u_1 u\mu\right) \end{pmatrix}. \end{aligned} \tag{D.2}$$

Proof. We consider the proof of Theorem 5.1. By the ansatz (5.2), the Lax matrix $U^{(1)}$ is constructed as

$$U^{(1)} = \lambda^2 U + B,$$

where some of the entries of the matrix B are determined by (5.3):

$$B = \begin{pmatrix} 0 & 0 \\ 0 & -\mu^{-1}c \end{pmatrix} \lambda^2 + \begin{pmatrix} 0 & c \\ -u\mu_{-1}^{-1}c_{-1} & 0 \end{pmatrix} \lambda + \begin{pmatrix} e & 0 \\ 0 & e_1 \end{pmatrix}.$$

We find an explicit form for the remaining terms c and e by applying this construction to the case when $u_x^{(i)} = g^{(0)}$. Hence,

$$\begin{aligned} c &= 2(\mu\mu_1 u_1 + \mu u \mu), \\ e &= 2\left(\mu^{-1}uu_{-1}\mu_{-1}^{-1} + \mu^{-1}u\mu^{-1}u + \mu^{-1}\mu_1^{-1}u_1 u\right). \end{aligned}$$

Therefore, B is fully determined and this completes the proof. ☐

We determine a Lax representation (M, W) also for the scaling symmetry (5.6) of ($\mathrm{V_a}$). Since this is not a commuting symmetry, it follows a variation of the standard Lax-Darboux equation (4.1), which involves a scalar function $\kappa(\lambda)$ and the partial derivative M_λ of M with respect to the spectral parameter λ:

$$M_\tau + \kappa(\lambda) M_\lambda = W_1 M - MW, \tag{D.3}$$

We observe that the pair of matrices (M_n, W_n) is non-autonomous, i.e. it depends explicitly on the independent variable $n \in \mathbb{Z}$. We define the action of the shift $\mathcal{S}$ on n and $x \in \mathbb{R}$ as

$$\mathcal{S}(n) = n + 1, \qquad \mathcal{S}(x) = x.$$

Proposition D.2. *A Lax representation* (M_n, W_n) *of the scaling symmetry* (5.6) *of* ($\mathrm{V_a}$) *at a site* $n \in \mathbb{Z}$ *is given by* M_n *in* (D.1) *and*

$$W_n = \frac{1}{2}\begin{pmatrix} n & 0 \\ 0 & n+1 \end{pmatrix}, \qquad \kappa = \frac{1}{2}\lambda. \tag{D.4}$$

Proof. It is straightforward to check that the pair (M_n, W_n) represents the scaling symmetry (5.6) through the modified Lax-Darboux equation (D.3). ☐

The reduction of (V_a) to equation (5.9) is originated from the constraint $\mathscr{f} = 0$ in (5.7). This can be interpreted as the stationary condition on the equation $u_{t'} = \mathscr{f}$, which is a symmetry of (V_a) for $t' \in \mathbb{R}$. We obtain its Lax representation (M, F) as the same linear combination in (5.7) of the respective Lax representations (D.1), (D.2) and (D.4):

$$F_n = U_n^{(1)} + \xi \left(W_n + x\, U_n \right). \tag{D.5}$$

Since the pair above involves (M_n, W_n), we consider it with respect to the modified version of the Lax-Darboux equation in (D.3).

Proposition D.3. *A Lax representation* (M'_n, F'_n) *of equation* (5.9) *is given by the function* $\kappa = \lambda/2$ *and the matrices*

$$M'_n = \begin{pmatrix} \mu_n^{-1} & 0 \\ 0 & 0 \end{pmatrix} \lambda + \begin{pmatrix} 0 & \frac{\sqrt{\xi}}{2} \\ -u_{n+1}\mu_n^{-1} & 0 \end{pmatrix},$$

$$\begin{aligned} F'_n = & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lambda^4 + 2 \begin{pmatrix} 0 & \frac{\sqrt{\xi}}{2}\mu_n \\ -u_n & 0 \end{pmatrix} \lambda^3 + \sqrt{\xi} \begin{pmatrix} \mu_n u_n + x & 0 \\ 0 & -u_n\mu_n - x \end{pmatrix} \lambda^2 + \\ & + \sqrt{\xi} \begin{pmatrix} 0 & \frac{\sqrt{\xi}}{2}\mu_n \left(\mu_{n+1}u_{n+1} + u_n\mu_n + 2x \right) \\ \mu_{n+1}u_{n+1}u_n + (n + \nu + (-1)^n\sigma)\mu_n^{-1} & 0 \end{pmatrix} \lambda + \\ & - \frac{\xi}{2} \begin{pmatrix} \nu + (-1)^n\sigma & 0 \\ 0 & \nu - (-1)^n\sigma \end{pmatrix}. \end{aligned}$$

Proof. The linear combination (D.5) together with $\kappa(\lambda) = \lambda/2$ from (D.4) yields a Lax representation (M_n, F_n) of the stationary equation $\mathscr{f} = 0$:

$$\begin{aligned} F_n = & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \lambda^4 + 2 \begin{pmatrix} 0 & \mu_n \\ -u_n & 0 \end{pmatrix} \lambda^3 + \begin{pmatrix} \xi x + 2\mu_n u_n & 0 \\ 0 & -\xi x - 2u_n\mu_n \end{pmatrix} \lambda^2 + \\ & + 2 \begin{pmatrix} 0 & \mu_n \left(\xi x + u_n\mu_n + \mu_{n+1}u_{n+1} \right) \\ -u_n \left(\xi x + u_{n-1}\mu_{n-1} + \mu_n u_n \right) & 0 \end{pmatrix} \lambda + \end{aligned}$$

$$
+\begin{pmatrix} \begin{array}{l} 2\mu_n\left(\xi x u_n + u_n u_{n-1}\mu_{n-1}+ \right. \\ \left. +u_n\mu_n u_n + \mu_{n+1}u_{n+1}u_n\right) + \frac{\xi n}{2} \end{array} & 0 \\ 0 & \begin{array}{l} 2\mu_{n+1}\left(\xi x u_{n+1} + u_{n+1}u_n\mu_n + \right. \\ \left. +u_{n+1}\mu_{n+1}u_{n+1} + \mu_{n+2}u_{n+2}u_{n+1}\right) + \\ +\frac{1}{2}\xi(n+1) \end{array} \end{pmatrix},
$$

via the stationary version of the Lax-Darboux equation (D.3)

$$
\kappa(\lambda) M_\lambda = W_1 M - M W, \tag{D.6}
$$

The condition is already satisfied by the Painlevé equation (5.9), since related with $\mathscr{h} = 0$, that implies $\mathscr{f} = 0$. However, (M_n, F_n) above depends on the variables u_{n-1}, u_n, u_{n+1}, and u_{n+2}, while (5.9) depends on u_{n-1}, u_n, and u_{n+1}. Therefore, it is possible to reduce the given Lax pair by cancelling two dependent variables. Shifting (5.9), we write

$$
\begin{aligned}
u_{n+2} = -\mu_{n+2}^{-1}\Big(\Big(\frac{1}{4}\xi(n+1) + (-1)^{n+1}\sigma + \nu\Big)\mu_{n+1}^{-1}u_{n+1}^{-1} + \xi x + \\
+ u_{n+1}u_n\mu_n u_{n+1}^{-1} + u_{n+1}\mu_{n+1}\Big), \\
u_{n-1} = -\Big(\Big(\frac{\xi n}{4} + (-1)^n\sigma + \nu\Big)u_n^{-1}\mu_n^{-1} + \xi x + \mu_n u_n + u_n^{-1}\mu_{n+1}^{-1}u_{n+1}u_n\Big)\mu_{n-1}^{-1}.
\end{aligned}
$$

Substituting the latter in the pair (M_n, F_n), we eliminate the variables u_{n+2} and u_{n-1}, obtaining a new Lax representation $(M_n, \hat{F}_m)$ of $\mathscr{h} = 0$:

$$
M_n = \begin{pmatrix} \mu_n^{-1} & 0 \\ 0 & 0 \end{pmatrix}\lambda + \begin{pmatrix} 0 & 1 \\ -u_{n+1}\mu_n^{-1} & 0 \end{pmatrix},
$$

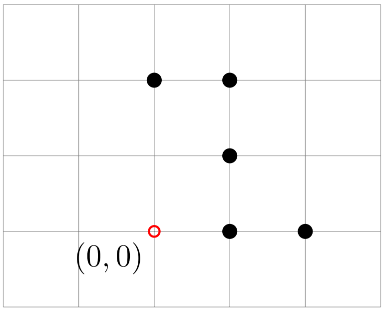
$$
\begin{aligned}
\hat{F}_n = & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\lambda^4 + 2\begin{pmatrix} 0 & \mu_n \\ -u_n & 0 \end{pmatrix}\lambda^3 + \begin{pmatrix} 2\mu_n u_n + \xi x & 0 \\ 0 & -2u_n\mu_n - \xi x \end{pmatrix}\lambda^2 + \\
& + 2\begin{pmatrix} 0 & \mu_n\left(u_n\mu_n + \mu_{n+1}u_{n+1} + \xi x\right) \\ (\nu + \frac{n\xi}{4} + (-1)^n\sigma)\mu_n^{-1} + \mu_{n+1}u_{n+1}u_n & 0 \end{pmatrix}\lambda + \\
& - 2\begin{pmatrix} \nu + (-1)^n\sigma & 0 \\ 0 & \nu - (-1)^n\sigma \end{pmatrix},
\end{aligned}
$$

which is related to (5.9) by the change of variables in (5.8). The same transformation applied to $(M_n, \hat{F}_n)$ yields the Lax representation (M'_n, F'_n) in the proposition. ☐

A direct computation shows that the latter pair of matrices satisfies (5.9) with respect to (D.6) and $\kappa(\lambda) = \lambda/2$.

D.2 Lax representation of equation (5.27)

As discussed in Section 5.2.2.2, equation (5.27) is a reduction of the Bianchi permutability (5.11) of two Darboux matrices $M_{\uparrow}(2)$ (4.7b) of the DNLS equations. This is a five-point scalar PΔE in the variable $w_{m,n}$ with $m, n \in \mathbb{Z}$, which does not belong to the ABS classification [9]. In the stencil below, the empty red dot marks the origin of the axes.



Starting from the Darboux matrices $M_{\uparrow}(2)$, we construct a Lax representation for (5.27) using Bianchi permutability (5.11) as a zero-curvature condition. General considerations on this approach can be found in Section 5.2.1 and in the monograph [57].

Proposition D.4. *A Lax representation* (A, B) *for equation* (5.27) *is given by*

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \lambda^2 + \begin{pmatrix} 0 & w_{0,1}(w_{-1,1}^{-1} - w^{-1}) \\ 1 & 0 \end{pmatrix} \lambda + \begin{pmatrix} w_{1,0} w^{-1} & 0 \\ 0 & w_{0,1}(w_{-1,1}^{-1} - w^{-1}) \end{pmatrix},$$

$$B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \lambda^2 + \begin{pmatrix} 0 & w_{0,1}(w_{-1,1}^{-1} - w^{-1}) \\ 1 & 0 \end{pmatrix} \lambda + \begin{pmatrix} w_{0,1} w^{-1} & 0 \\ 0 & 0 \end{pmatrix}.$$

Proof. We adapt the same passages in Section 5.2.2.2 to the Darboux matrices $M^{(1)}$ and $M^{(2)}$ in (5.14). We consider the reduction $\rho = 0$ of $M^{(1)}$ and we set $d = 0$ in $M^{(2)}$. The variables f, g, a and c are expressed through the potential variables $\not{f}$ and $\not{c}$ in (5.20). The resulting matrices are

$$M^{(1)} = \begin{pmatrix} \not{f}_{1,0}\not{f}^{-1} & 0 \\ 0 & 0 \end{pmatrix}\lambda^2 + \begin{pmatrix} 0 & \not{f}_{1,0}\not{f}^{-1}p \\ q_{1,0}\not{f}_{1,0}\not{f}^{-1} & 0 \end{pmatrix}\lambda + \begin{pmatrix} \not{c}_{1,0}\not{c}^{-1} & 0 \\ 0 & q_{1,0}\not{f}_{1,0}\not{f}^{-1}p \end{pmatrix},$$
$$M^{(2)} = \begin{pmatrix} \not{f}_{0,1}\not{f}^{-1} & 0 \\ 0 & 0 \end{pmatrix}\lambda^2 + \begin{pmatrix} 0 & \not{f}_{0,1}\not{f}^{-1}p \\ q_{0,1}\not{f}_{0,1}\not{f}^{-1} & 0 \end{pmatrix}\lambda + \begin{pmatrix} \not{c}_{0,1}\not{c}^{-1} & 0 \\ 0 & 0 \end{pmatrix}.$$

We change the dependent variables to $u_{m,n} = p_{m,n}q_{m,n}$, eliminating the redundant degree of freedom with a gauge transformation with $G = \operatorname{diag}(\not{c}, q)$. This yields

$$M^{(1)} = \begin{pmatrix} \not{c}_{1,0}^{-1}\not{f}_{1,0}\not{f}^{-1}\not{c} & 0 \\ 0 & 0 \end{pmatrix}\lambda^2 + \begin{pmatrix} 0 & \not{c}_{1,0}^{-1}\not{f}_{1,0}\not{f}^{-1}u \\ \not{f}_{1,0}\not{f}^{-1}\not{c} & 0 \end{pmatrix}\lambda + \begin{pmatrix} 1 & 0 \\ 0 & \not{f}_{1,0}\not{f}^{-1}u \end{pmatrix},$$
$$M^{(2)} = \begin{pmatrix} \not{c}_{0,1}^{-1}\not{f}_{0,1}\not{f}^{-1}\not{c} & 0 \\ 0 & 0 \end{pmatrix}\lambda^2 + \begin{pmatrix} 0 & \not{c}_{0,1}^{-1}\not{f}_{0,1}\not{f}^{-1}u \\ \not{f}_{0,1}\not{f}^{-1}\not{c} & 0 \end{pmatrix}\lambda + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Finally, we introduce the variable w, while the compatibility condition (5.26) allows to eliminate the remaining u:

$$w = \not{f}^{-1}\not{c}, \qquad u = \not{f}w_{0,1}(w_{-1,1}^{-1} - w^{-1})\not{f}^{-1},$$

deducing the pair of matrices

$$\begin{aligned} A = {} & \begin{pmatrix} w_{1,0}^{-1}w & 0 \\ 0 & 0 \end{pmatrix}\lambda^2 + \begin{pmatrix} 0 & w^{-1}w_{0,1}(w_{-1,1}^{-1} - w^{-1})\not{f}^{-1} \\ \not{f}_{1,0}w & 0 \end{pmatrix}\lambda + \\ & + \begin{pmatrix} 1 & 0 \\ 0 & \not{f}_{1,0}w_{0,1}(w_{-1,1}^{-1} - w^{-1})\not{f}^{-1} \end{pmatrix}, \\ B = {} & \begin{pmatrix} w_{0,1}^{-1}w & 0 \\ 0 & 0 \end{pmatrix}\lambda^2 + \begin{pmatrix} 0 & (w_{-1,1}^{-1} - w^{-1})\not{f}^{-1} \\ \not{f}_{0,1}w & 0 \end{pmatrix}\lambda + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

This corresponds to the result of the proposition after applying a gauge transformation with $G = \mathrm{diag}(w^{-1}, \not\ell)$. $\square$

It is straightforward to verify that the pair (A, B) reproduces equation (5.27) via Bianchi permutability (5.11). The existence of a nontrivial Lax representation is a standard indicator of integrability.

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