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On the Dimension Spectrum of Infinite Subsystems of Continued Fractions and Perron-Frobenius Operators

Author:
Painos Chitanga

Supervisor:
Bas Lemmens

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Declaration of Authorship

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Painos Chitanga

UNIVERSITY OF KENT

Abstract

School of Engineering, Mathematics and Physics

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On the Dimension Spectrum of Infinite Subsystems of Continued Fractions and Perron-Frobenius Operators

by Painos CHITANGA

The main topic of this thesis centres around the dimension spectrum of continued fractions expansions and the use of the spectral theory of Perron-Frobenius operators in this context. The study of the dimension spectrum of iterated function systems has a long history going back to the famous *Texan conjecture*, which asserts that the dimension spectrum of the iterated function system $\Theta = \{\theta_n : n \in \mathbb{N}\}$, where $\theta_n(x) = \frac{1}{n+x}$, is full, i.e., for each $0 \leq s \leq 1$, there is an $F \subseteq \mathbb{N}$ such that $\dim_{\mathcal{H}}(J_F) = s$. Here J_F is the collection of all irrationals $x \in (0,1)$ whose continued expansion digits belong to F, so

$$J_F = \{x = [a_1, a_2, a_3, \ldots] : a_i \in F, i \in \mathbb{N}\} \subseteq [0,1].$$

This conjecture was settled by Kesseböhmer and Zhu in [30]. Later, other interesting iterated function systems were shown to have a full dimension spectrum, including, reverse continued fractions expansions [18] and complex continued fractions expansions [8]. Since then, the structure of the dimension spectrum $\mathcal{DS}(\Theta)$ has been studied for various classes of iterated function systems, see [7, 8, 10, 19, 29]. Among other results it was shown [8] that, the dimension spectrum of a conformal iterated function system is a compact and perfect set.

Recently the dimension spectrum of infinite subsystems $\Theta_A = \{\theta_n : n \in A\}$ of Θ has been investigated by Chousionis, Leykekhman and Urbański in [7, 8] for different subsets $A \subseteq \mathbb{N}$. In particular, they considered the set of powers $P_q = \{q^n : n \in \mathbb{N}\}$, $q \geq 2$ and asked among other questions if it has a full dimension. We give an affirmative answer to their question in Chapter 4. A significant part of this thesis is devoted to answering open questions and extending results from [7].

It is well known that the Hausdorff dimension of the invariant set of a subsystem of Θ can be analysed using Perron-Frobenius operators, see [7, 8, 16, 17, 20, 21, 23–25, 28, 39]. Interestingly, some of the properties of the dimension spectrum such as compactness and perfectness can be studied and analysed at a purely operator-theoretical level, by considering the spectrum of a family of Perron-Frobenius type operators, without having to make a direct reference to the dimension spectrum of the iterated function. One objective of this thesis is to establish this observation.

The structure of the thesis will be as follows: In Chapter 1, we recall the relevant theory of general iterated function systems and summarise known results on the dimension spectrum.

In Chapter 2, we analyse Perron-Frobenius operators and introduce the notion of dimension spectrum of a class of Perron-Frobenius operators. We prove a new result Theorem 2.30 showing that, in general, the dimension spectrum of these classes of Perron-Frobenius operators is a compact and perfect set, and we relate this result to existing results in the literature concerning the dimension spectrum of iterated function systems.

In Chapter 3, we recall the connection between the Hausdorff dimension of invariant sets and Perron-Frobeinus operators, and the computational methods of Falk and Nussbaum [16, 17] to find rigorous estimates for the Hausdorff dimension of continued fraction expansions. We will also present some new ways of estimating the Hausdorff dimensions for certain families of continued fraction expansions, see Theorem 3.6.

Chapter 4 is devoted to the study of concrete infinite subsystems of continued fraction expansions and extending the results from [7] by using the theory developed in Chapter 2 to analyse the dimension spectrum. We prove that the set of powers $P_q = \{q^n : n \in \mathbb{N}\}$ has a full dimension spectrum for $q \geq 2$, see Theorem 4.1, answering a question of Chousionis, Leykekhman and Urbański [7]. In contrast, in Theorem 4.3, we show that the dimension spectrum of $P_q^* = \{q^n : n \in \mathbb{N}\} \cup \{1\}$ has infinitely many gaps and regions where it is nowhere dense. We also investigate the case where the infinite subsystem is generated by monomials, $M_q = \{n^q : n \in \mathbb{N}\}$, see Theorem 4.4. For these sets, we show that the dimension spectrum of M_q is, in general, a finite union of disjoint closed intervals. In particular, we show that it is full, that is, consists of a single closed interval, for $q \in \{1, 2, 3, 4, 5\}$. It consists of two intervals for $q \in \{6, 7, 8\}$, three intervals for $q \in \{9, 10, 11, 12\}$ and four intervals for $q = 19$, see Theorem 4.4. We also study the case where $A = \{2^{2^n} : n \in \mathbb{N}\}$, and prove that its dimension spectrum is a nowhere dense set, Theorem 4.6. Most of these results are written up in [6].

Finally, in Chapter 5, we revisit the set P_q^*, and provide some partial results towards obtaining a complete understanding of its dimension spectrum. We also present several open problems that lie beyond the scope of our methods.

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Contents

To the memory of dad, Painos Hlulani Chitanga

Chapter 1

Iterated Function Systems

The study of iterated function systems gained popularity in the analysis of sets or solid objects whose shapes are extremely irregular yet exhibit self-similar patterns at arbitrarily small scales. Such sets are called *fractals*, a name coined by Mandelbrot [31]. These sets usually have a non-integral Hausdorff dimension; see for instance [2, 6–10, 12, 14, 20, 23–25, 28, 29, 32, 33, 39, 41–43].

The focus of this thesis is on the Hausdorff dimension of sets of continued fractions expansions. These sets can be described as invariant sets of iterated function systems. In this chapter we will recall some of the basic theory of iterated function systems that is relevant for this thesis. Many of the results can be found in Falconer's books [12, 13] and in the works by Mauldin and Urbaǹski [32–34].

1.1 Hausdorff Dimension and Hausdorff Measure

Let (X, ρ) be a compact metric space. A δ-cover of $A \subseteq X$ is a countable collection $\{A_k \subseteq X : k \in I\}$ such that

$$A \subseteq \bigcup_{k \in I} A_k \text{ and } \operatorname{diam}(A_k) < \delta \text{ for each } k \in I,$$

where $\operatorname{diam}(B)$ denotes the diameter of B, i.e.,

$$\operatorname{diam}(B) = \sup\{\rho(x, y) \colon x, y \in B\}.$$

For $s \geq 0$ and $\delta > 0$, let

$$\mathcal{H}^s_\delta(A) = \inf \left\{ \sum_{k \in I} (\operatorname{diam}(A_k))^s \colon \{A_k \colon k \in I\} \text{ is a } \delta\text{-cover of } A \right\}.$$

If $\delta_1 \leq \delta_2$, then every δ_1-cover is also a δ_2-cover, so $\mathcal{H}^s_\delta$ is decreasing in δ. We define

$$\mathcal{H}^s(A) = \sup \{\mathcal{H}^s_\delta(A) \colon \delta > 0\} = \lim_{\delta \to 0} \mathcal{H}^s_\delta(A).$$

This defines an outer measure called the *s-dimensional Hausdorff measure* of A.

If $\{A_k \colon k \in I\}$ is a δ-cover of A, then for $t > s$ we have

$$\sum_{k \in I} (\operatorname{diam}(A_k))^t \leq \delta^{t-s} \sum_{k \in I} (\operatorname{diam}(A_k))^s.$$

Taking the infimum gives $\mathcal{H}^t_\delta(A) \leq \delta^{t-s} \mathcal{H}^s_\delta(A)$, and letting $\delta \to 0$ shows that if $\mathcal{H}^s(A) < \infty$, then $\mathcal{H}^t(A) = 0$.

Thus, there is a critical value of s at which the Hausdorff measure jumps from ∞ to 0 which is called the *Hausdorff dimension* of A, denoted by $\dim_{\mathcal{H}}(A)$, see [12] for more details on the Hausdorff measure. Formally

$$\dim_{\mathcal{H}}(A) = \inf \{s \geq 0 \colon \mathcal{H}^s(A) = 0\} = \sup \{s \geq 0 \colon \mathcal{H}^s(A) = \infty\}.$$

Therefore,

$$\mathcal{H}^s(A) = \begin{cases} \infty & \text{if } s < \dim_{\mathcal{H}}(A), \\ 0 & \text{if } s > \dim_{\mathcal{H}}(A). \end{cases}$$

It is straightforward to verify that $\dim_{\mathcal{H}}(\emptyset) = 0$. For $s = \dim_{\mathcal{H}}(A)$, the value of $\mathcal{H}^s(A)$ may be infinite, finite and positive, or zero.

Computing the Hausdorff dimension is generally difficult, both theoretically and computationally. Unlike classical measures, the Hausdorff measure involves covering a set with arbitrarily small subsets and taking a limit, which rarely simplifies. Considerable work has been done on computing the Hausdorff dimension of various sets; see for instance [5, 13, 17, 20, 24, 25, 28]. For this thesis [17, 25] are particularly relevant.

1.2 Hausdorff Metric

Let (X, ρ) be a compact metric space. If $A \subseteq X$ is non-empty and compact, and $y \in X$, define the *distance from y to the set A* as

$$\rho(y, A) = \inf\{\rho(y, x) \colon x \in A\}.$$

If $A, B \subseteq X$ are non-empty and compact, the *distance from A to B* is given by

$$D(A, B) = \sup\{\rho(a, B) \colon a \in A\}.$$

Note that $D(A, B)$ and $D(B, A)$ are not necessarily equal. The *Hausdorff distance* between A and B is then given by

$$\rho_{\mathcal{H}}(A, B) = \max\{D(A, B), D(B, A)\}.$$

This defines a metric on the collection $\mathcal{H}(X)$ of all non-empty compact subsets of X. In fact, $(\mathcal{H}(X), \rho_{\mathcal{H}})$ is a complete metric space; see for instance [36, Chapter 7, Exercise 7].

1.3 Iterated Function Systems and Invariant Sets

Let (X, ρ) be a compact metric space. A map $\theta\colon X \to X$ is called a *contraction* if there exists $c \in (0,1)$ such that

$$\rho(\theta(x), \theta(y)) \leq c\rho(x, y) \quad \text{for all } x, y \in X.$$

If equality holds for all $x, y \in X$, i.e.,

$$\rho(\theta(x), \theta(y)) = c\rho(x, y),$$

then θ is called a *similarity map* or a *similitude*.

Let $E \subseteq \mathbb{N}$, with $|E| \geq 2$ and (X, ρ) be a compact metric space. Consider a collection of injective maps

$$\Theta_E = \{\theta_\alpha\colon X \to X \mid \alpha \in E\},$$

and suppose that there exists a $c \in (0,1)$ such that

$$\rho(\theta_\alpha(x), \theta_\alpha(y)) \leq c\,\rho(x, y), \quad \text{for all } \; x, y \in X, \quad \text{and all } \alpha \in E. \tag{1.1}$$

Such a collection is called an *iterated function system* (IFS).

We will simply write Θ_E to denote an IFS. If E is finite i.e., $|E| < \infty$, then Θ_E is a finite IFS, otherwise it is an infinite IFS.

Let Θ_E be a finite IFS. Define the *Hutchinson map* $\theta : \mathcal{H}(X) \to \mathcal{H}(X)$ by

$$\theta(A) = \bigcup_{\alpha \in E} \theta_\alpha(A), \quad A \in \mathcal{H}(X). \tag{1.2}$$

Hutchinson [25] showed that θ is a contraction on $(\mathcal{H}(X), \rho_{\mathcal{H}})$ with the same contraction factor as the one for Θ_E. To see this let $A, B \in \mathcal{H}(X)$, so $\theta_\alpha(A)$ and $\theta_\alpha(B)$ are compact. Since E is finite, the unions $\theta(A)$ and $\theta(B)$ are also compact. Let $\rho_{\mathcal{H}}(A,B) < \delta$. For $\alpha \in E$ and $x \in A$, since $\rho_{\mathcal{H}}(A,B) < \delta$ it follows that there exists a $y \in B$ such that $\rho(x,y) < \delta$. Thus

$$\rho(\theta_\alpha(x), \theta_\alpha(y)) \leq c\rho(x,y) < c\delta.$$

This shows that $\cup_{a\in E}\theta_a(A) = \theta(A)$ is contained in a $c\delta$-neighbourhood of $\theta(B)$. Using a similar argument one deduces that $\theta(B)$ is contained in a $c\delta$-neighbourhood of $\theta(A)$. This implies that $\rho_{\mathcal{H}}(\theta(A),\theta(B)) \leq c\delta$. Since this holds for all δ, we have that $\rho_{\mathcal{H}}(\theta(A),\theta(B)) \leq c\rho_{\mathcal{H}}(A,B)$. Thus, θ is a contraction map on a complete metric space $\mathcal{H}(X)$.

By Banach's contraction mapping theorem, θ has a unique fixed point $C \in \mathcal{H}(X)$, which satisfies

$$C = \theta(C) = \bigcup_{\alpha\in E} \theta_\alpha(C). \tag{1.3}$$

Moreover, for any $A \in \mathcal{H}(X)$

$$\lim_{n\longrightarrow\infty} \theta^n(A) = C. \tag{1.4}$$

This set C is called the *invariant set* or *attractor* of the IFS Θ_E.

If Θ_E is an infinite IFS, define $\theta\colon \mathcal{H}(X) \to \mathcal{H}(X)$ by

$$\theta(A) = \overline{\bigcup_{\alpha\in E} \theta_\alpha(A)}, \quad A \in \mathcal{H}(X).$$

Then θ is still a contraction and there exist a unique set C such that

$$C = \overline{\bigcup_{\alpha\in E} \theta_\alpha(C)}$$

There exists an alternative way to define the invariant set. Let

$$E_k = E^k, \quad E_* = \bigcup_{k\in\mathbb{N}} E_k \quad \text{and} \quad E_\infty = E^{\mathbb{N}}.$$

For $\omega = (\omega_1, \omega_2, \ldots, \omega_m) \in E_*$, the *length* of ω is denoted by $|\omega| = m$. If $\omega \in E_\infty$, then $|\omega| = \infty$. Let $k \in \mathbb{N}$ and $\omega \in E_* \cup E_\infty$. If $|\omega| \geq k$, $\omega|_k$ denotes the prefix consisting of the first k terms of ω. Define

$$\theta_{\omega|_k} := \theta_{\omega_1} \circ \theta_{\omega_2} \circ \ldots \circ \theta_{\omega_k}.$$

Using (1.1) we have that,

$$\operatorname{diam}\left(\theta_{\omega|_k}(X)\right) \leq c^k \operatorname{diam}(X). \tag{1.5}$$

Observe that if $\omega \in E_\infty$, then

$$\theta_{\omega|_{k+1}}(X) \subseteq \theta_{\omega|_k}(X) \quad \text{for each } k \in \mathbb{N},$$

so $(\theta_{\omega|_k}(X))_{k\in\mathbb{N}}$ is a decreasing sequence of subsets of X. Since X is non-empty and compact, $(\theta_{\omega|_k}(X))_{k\in\mathbb{N}}$ is a sequence of non-empty compact sets with diameters tending to zero by (1.5). Hence the intersection of the collection $\{\theta_{\omega|_k}(X) \colon k \in \mathbb{N}\}$ consists of a single point. Thus we can define the *coding map* as follows. For $\omega \in E_\infty$, define

$$\pi(\omega) = \bigcap_{k\in\mathbb{N}} \theta_{\omega|_k}(X) \tag{1.6}$$

so $\pi \colon E_\infty \to X$ is a well-defined map. The main set of interest is defined as

$$J_E := \pi(E_\infty) = \bigcup_{\omega\in E_\infty} \bigcap_{k\in\mathbb{N}} \theta_{\omega|_k}(X). \tag{1.7}$$

For $\omega, \tau \in E_\infty$, let $|\omega \wedge \tau|$ denote the length of the initial segment common to both ω and τ. If $\omega_1 \neq \tau_1$, then $|\omega \wedge \tau| = 0$, otherwise $\omega_i = \tau_i$ for all $0 < i \leq |w \wedge \tau|$. Define $d_E(\omega, \tau) = e^{-|w\wedge\tau|}$. Then (E_∞, d_E) is a metric space.

Lemma 1.1. *If E is finite, then (E_∞, d_E) is a compact metric space.*

Proof. Let $\left(\omega^{(k)}\right)_{k\in\mathbb{N}}$ be a sequence in E_∞ where $\omega^{(k)} = (\omega_1^{(k)}, \omega_2^{(k)}, \ldots)$. We wish to show that $\left(\omega^{(k)}\right)_{k\in\mathbb{N}}$ has a convergent subsequence.

Since $|E|$ is finite, we know there exists $a_1 \in E$ such that $\omega_1^{(k_i)} = a_1$ for infinitely many i. Extract a subsequence $(\omega_1^{(k_i^1)})_{i\in\mathbb{N}}$ of $\left(\omega^{(k)}\right)_{i\in\mathbb{N}}$ such that $\omega_1^{(k_i^1)} = a_1$ for all $i \in \mathbb{N}$. Next we know that there exists $a_2 \in E$ such that $\omega_2^{(k_i^1)} = a_2$ for infinitely many i. Extract a subsequence $\left(\omega^{(k_i^2)}\right)_{i\in\mathbb{N}}$ of $\left(\omega^{(k_i^1)}\right)_{i\in\mathbb{N}}$ such that $\omega_2^{(k_i^2)} = a_2$ for all i. Repeating this process gives for each $m \in \mathbb{N}$ an $a_m \in E$ and a subsequence $\left(\omega^{k_i^m}\right)_{i\in\mathbb{N}}$ of $(\omega^{(k_i^{m-1})})_{i\in\mathbb{N}}$ such that $(\omega_1^{(k_i^m)}, \ldots, \omega_m^{(k_i^m)}) = (a_1 \ldots a_m)$. Let $\bar{\omega}$ be a word in E_∞ such that $\bar{\omega}|_m = (a_1, a_2, \ldots, a_m)$ for all m. The sequence $(\omega^{(k_i^i)})_{i\in\mathbb{N}}$ is a subsequence of $(\omega^{(k)})_{k\in\mathbb{N}}$ by construction and for all i we have that $\omega^{(k_i^i)}|_i = (a_1, a_2, \ldots, a_i)$. It follows that $\omega^{(k_i^i)} \to \bar{\omega}$ as $i \to \infty$. Indeed, given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that $e^{-N} < \varepsilon$. If $i \geq N$, then $|\omega^{(k_i^i)} \wedge \bar{\omega}| \geq i \geq N$ so $d_E(\omega^{(k_i^i)}, \bar{\omega}) \leq e^{-N} < \varepsilon$. □

By using (1.5), π is a continuous map from (E_∞, d_E) to (X, ρ). Now let $\sigma \colon E_\infty \to E_\infty$ be a shift map defined by

$$\sigma(\omega) = \omega_2\omega_3 \ldots \text{ for each } \omega \in E_\infty.$$

We write $\alpha\omega = \alpha\omega_1\omega_2\ldots$, i.e. it is the word formed by concatenating $\alpha \in E$ to ω. Note that $\theta_\alpha(\pi(\omega)) = \pi(\alpha\omega)$, and $\pi(\omega) = \theta_{\omega_1}(\pi(\sigma(\omega)))$.

Lemma 1.2. *Let E be finite and J_E be as defined in (1.7). Then J_E satisfies,*

$$J_E = \bigcup_{\alpha\in E} \theta_\alpha(J_E).$$

Proof. Let $x \in J_E$. Then by definition of J_E in (1.7), there exists $\omega \in E_\infty$ such that $\pi(\omega) = x$. So,

$$x = \pi(\omega) = \theta_{\omega_1}(\pi(\sigma(\omega))).$$

Thus, $x \in \theta_{\omega_1}(J_E)$ and hence $x \in \bigcup_{\alpha \in E} \theta_\alpha(J_E)$. For the reverse inclusion, if $x \in \bigcup_{\alpha \in E} \theta_\alpha(J_E)$, then $x \in \theta_\alpha(J_E)$ for some $\alpha \in E$, so there exists $y \in J_E$ such that $x = \theta_\alpha(y)$. Since $y \in J_E$, there exists an $\omega \in E_\infty$ such that $\pi(\omega) = y$. Now

$$x = \theta_\alpha(y) = \theta_\alpha(\pi(\omega)) = \pi(\alpha\omega) \in J_E,$$

therefore $x \in J_E$, proving the result. □

Remark 1.3. Observe that J_E coincides with the unique invariant set C from (1.3) of the IFS Θ_E.

It suffices to show that $J_E \in \mathcal{H}(X)$ as C is the unique fixed point in (1.3). Indeed, (E_∞, d_∞) is a compact metric space using Lemma 1.1, and the map $\pi : E_\infty \to X$ is continuous so it follows that $\pi(E_\infty) = J_E$ is compact.

An important class of IFS which was studied by Hutchinson [25] and Falconer [15], and many others are affine IFS.

Example 1.4 (Affinities)**.** Consider maps of the form

$$\theta_i(x) = R_i x + b_i, \; x \in X, \quad i \in E. \tag{1.8}$$

Here $X \subseteq \mathbb{R}^n$, and R_i is an $n \times n$ non-singular transformation matrix and $r(R_i^T R_i) < 1$, where $r(A)$ denotes the spectral radius of A. Then Θ_E is an affine IFS.

If in addition, $R_i = a_i T_i$ for an $n \times n$ orthogonal transformation matrix T_i so $T_i^* T_i = I_n$, the $n \times n$ identity matrix, then Θ_E is an affine IFS consisting of similitudes. If $X \subset \mathbb{R}$, then $R_i = \pm 1$ and a_i is the similitude ratio. Note that the assumption that $r(R_i^T R_i) < 1$ ensures that Θ_E consist of contraction maps.

An interesting example in $\mathbb{R}$ is given by

$$\theta_i(x) = \frac{x+i}{m}, \quad x \in [0,1] \quad \text{for } i = 0, \ldots m-1.$$

Note that $a_i = m^{-1}$ for all $i \in E$. If we take $E \subseteq \{1, \ldots m-1\}$, then the invariant set of this IFS is the collection of all number $x \in [0,1]$ whose base m expansion only contains digits in E.

The analysis of the Hausdorff dimension of an invariant set of an IFS Θ_E is often complicated if the components $\{\theta_\alpha(C) \colon \alpha \in E\}$ overlap. As a result, one often makes assumptions on separation conditions.

An IFS Θ_E with invariant set C is said to satisfy the *open set condition* (OSC) if there exists an open set $U \subset X$ such that

$$\bigcup_{\alpha \in E} \theta_\alpha(U) \subseteq U,$$

and $\theta_\alpha(U) \cap \theta_\beta(U) = \emptyset$ for $\alpha, \beta \in E$ with $\alpha \neq \beta$. In addition, if $C \cap U \neq \emptyset$, then Θ_E satisfies the *strong open set condition* (SOSC).

In Example 1.4, the translation b_i can be chosen such that the IFS satisfies the OSC. For example

$$\theta_i(x) = \frac{1}{i(i+1)} x + \frac{1}{1+i}, \quad x \in [0,1] \quad i \in \mathbb{N}.$$

If $U = (0,1)$, then

$$\theta_i(U) = \left(\frac{1}{i+1}, \frac{1}{i} \right) \subset U$$

and these sets are disjoint. Therefore $\{\theta_i \colon i \in \mathbb{N}\}$ is an IFS consisting of similitudes that satisfies the OSC. In fact it satisfies the SOSC as $C = [0,1]$ so $C \cap U = U$.

Methods for computing the Hausdorff dimension of invariant sets of IFS go back to Moran [35], who showed that if $X \subseteq \mathbb{R}^n$ and $\Theta = \{\theta_i \colon X \to X \mid i = 1, \ldots, k\}$ is an IFS consisting of similitudes satisfying the OSC, then the

Hausdorff dimension of the invariant set is the unique value of s such that

$$\sum_{i=1}^{k} a_i^s = 1. \tag{1.9}$$

See also [25] for the same result. Using this result, we can compute the Hausdorff dimension of the two famous sets, Cantor's middle third set and the Sierpiński triangle.

In the case of the Cantor set, the similitudes are

$$\theta_1(x) = \frac{x}{3} \quad \text{and } \theta_2(x) = \frac{2+x}{3}, \quad x \in [0,1].$$

Both have contraction ratio 3^{-1}. Clearly θ_i is an injective map for $i = 1, 2$ and if we take $U = (0, 1)$, then $\theta_1(U) \cup \theta_2(U) \subset U$ and $\theta_1(U) \cap \theta_2(U) = \emptyset$, so the IFS satisfies the OSC. It follows that the Hausdorff dimension of the Cantor set is the unique value of s satisfying

$$2\left(\frac{1}{3}\right)^s = 1,$$

so $s = \frac{\ln 2}{\ln 3}$.

For the Sierpiński triangle, let $X \subset \mathbb{R}^2$ be the triangle with vertices $(0, 0)$, $(0, 1)$ and $(1, 0)$. Then the Sierpiński triangle is the invariant set of the following IFS $\Theta = \{\theta_i \colon i = 1, 2, 3\}$ where;

$$\theta_1(x,y) = \left(\frac{x}{2}, \frac{y}{2}\right), \quad \theta_2(x,y) = \left(\frac{x}{2} + \frac{1}{2}, \frac{y}{2}\right) \quad \text{and} \quad \theta_3(x,y) = \left(\frac{x}{2}, \frac{y}{2} + \frac{1}{2}\right).$$

This IFS consists of three similitudes with same contraction constant 2^{-1}. The map θ_i is an injective map for $i = 1, 2, 3$. For $U = X^\circ$, the interior of the triangle, we have that $\theta_i(U) \subseteq U$ and $\theta_i(U) \cap \theta_j(U) = \emptyset$ for $i, j \in \{1, 2, 3\}$ and $i \neq j$. Hence the IFS satisfies the OSC is satisfied so the Hausdorff dimension is determined by

$$3\left(\frac{1}{2}\right)^s = 1,$$

which yields $s = \frac{\ln 3}{\ln 2}$.

Moran's result gives a formula for computing the Hausdorff dimension of an IFS consisting of similitudes on $\mathbb{R}^n$ satisfying the OSC; Schief [41] also showed that if X is a complete metric space, then the OSC is not sufficient and has to be replaced with the SOSC. In that case, the Hausdorff dimension is still given by the equation $\sum_{i=1}^{k} a_i^s = 1$.

The results of Moran, Hutchinson and Schief have been extended in various directions. In particular, they have been generalised to settings where there are infinitely many θ_i making up the IFS, see Mauldin and Urbaǹiski [32–34]. Further extensions, incorporating so called infinitesimal similitudes, have been obtained in [16, 17, 39, 43], which will be the main setting of this thesis.

1.4 Infinitesimal Similitudes

Let (X, ρ) be a perfect, compact metric space. If $\theta\colon X \to X$, we shall say that θ is an infinitesimal similitude at $x \in X$ if for any sequences (x_n) and (y_n) with $x_n \neq y_n$ for all $n \in \mathbb{N}$ and $x_n, y_n \to x$, the limit

$$\lim_{n\to\infty} \frac{\rho(\theta(x_n), \theta(y_n))}{\rho(x_n, y_n)} =: (D\theta)(x)$$

exists and is independent of the particular sequence (x_n) and (y_n). If θ is an infinitesimal similitude at every point, we say that θ is an *infinitesimal similitude on* X.

This abstract definition becomes concrete through differentiability. In fact, if $X \subseteq \mathbb{R}$ and $\theta\colon X \to X$ is a differentiable function, then the infinitesimal similitude can be identified with the absolute value of its derivative. Note that in [39, Lemma 4.1], it was shown that if $D\theta$ exists at x, then it is continuous at x. Moreover, [43] established the following result.

Theorem 1.5. *Let $\theta\colon \mathbb{R} \to \mathbb{R}$ be a differentiable function. Then $D\theta$ exists at x if and only if the function $|\theta'|\colon \mathbb{R} \to \mathbb{R}$ is continuous at x. Moreover, $(D\theta)(x) = |\theta'(x)|$.*

Proof. Assume $D\theta$ exists at x. Let $x_n \to x$ and $x_n \neq x$ for all $n \in \mathbb{N}$. Let $y_n = x$ for all $n \in \mathbb{N}$. Since $(D\theta)(x)$ exists, it follows that $(D\theta)$ is continuous at x and

$$\begin{aligned}(D\theta)(x) &= \lim_{n\to\infty} \frac{|\theta(x_n) - \theta(y_n)|}{|x_n - y_n|} \\ &= \lim_{n\to\infty} \frac{|\theta(x_n) - \theta(x)|}{|x_n - x|} \\ &= \left| \lim_{n\to\infty} \frac{\theta(x_n) - \theta(x)}{x_n - x} \right| \\ &= |\theta'(x)|.\end{aligned}$$

Thus, $(D\theta)(x) = |\theta'(x)|$ and $|\theta'|$ is continuous at x.

On the other hand, assume that $|\theta'|\colon \mathbb{R} \to \mathbb{R}$ is continuous at x. Let $x_n \neq y_n$ be such that $x_n \to x$ and $y_n \to x$. Suppose that $x_n < y_n$. Since θ is continuous on $[x_n, y_n]$, the mean value theorem implies that there exists a $z_n \in (x_n, y_n)$ such that

$$\left| \frac{\theta(y_n) - \theta(x_n)}{y_n - x_n} \right| = |\theta'(z_n)|. \tag{1.10}$$

Like wise if $y_n < x_n$, there exists $z_n \in [y_n, x_n]$ for which the equality (1.10) holds. Since $x_n < z_n < y_n$ or $y_n < z_n < x_n$ for all $n \in \mathbb{N}$ and $x_n, y_n \to x$ we have that $z_n \to x$. The continuity of θ' implies that the limit exists and is $|\theta'(x)|$. Therefore, $D\theta$ exists at x. □

This characterisation provides a practical criterion to check whether $D\theta$ exists: one simply needs to verify the continuity of $|\theta'|$. In particular, the maps involved in the IFS for continued fraction expansions satisfy this property, allowing us to apply the general theory of infinitesimal similitudes developed in [39].

Remark 1.6. In general, affine maps of Example 1.4 are not infinitesimal similitudes.

Indeed, let $X = [0,1]^2$, and $\theta : X \to X$ be such that $\theta(x) = Rx$ with

$$R = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$$

for some $0 < \alpha < \beta < 1$. Then θ is not an infinitesimal similitude. If we let $x_n = 0$, $y_n = (n^{-1}, 0)$ and $z_n = (0, n^{-1})$, then

$$\frac{\rho(\theta(x_n), \theta(y_n))}{\rho(x_n, y_n)} = \alpha \quad \text{and} \quad \frac{\rho(\theta(x_n), \theta(z_n))}{\rho(x_n, z_n)} = \beta \text{ for all } n.$$

So $(D\theta)(0)$ does not exist.

1.5 Continued Fractions

As mentioned earlier, the sets of continued fraction expansions can be studied using the theory of IFS. In fact, the maps involved are infinitesimal similitudes.

Every irrational number x in the unit interval has a unique representation of the form

$$x = [a_1, a_2, a_3, \ldots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \frac{1}{a_3 + \ldots}}},$$

with $a_i \in \mathbb{N}$ for $i \in \mathbb{N}$. For example

$$\sqrt{2} - 1 = [2, 2, 2, \ldots] = \cfrac{1}{2 + \cfrac{1}{2 + \frac{1}{2 + \ldots}}}.$$

Note that $\sqrt{2}-1$ is also a fixed point in $[0,1]$ of the map $x \mapsto \frac{1}{2+x}$.

For each $a \in \mathbb{N}$, let

$$\theta_a(x) = \frac{1}{a+x}, \quad x \in [0,1].$$

Clearly θ_a is injective. We have that $|\theta_a'(x)| = (a+x)^{-2}$ which is continuous, hence θ_a is an infinitesimal similitude by Theorem 1.5 and $||\theta_a'||_\infty = a^{-2}$. Therefore

$$\sup\left\{\frac{|\theta_a(x)-\theta_a(y)|}{|x-y|} : x, y \in [0,1], x \neq y\right\} \leq a^{-2}.$$

Thus θ_a is a strict contraction for all $a \geq 2$. For $a = 1$, the derivative satisfies $|\theta_1'(0)| = 1$, hence θ_1 is not strictly contracting. The issue can be resolved by considering compositions $\theta_{a,b} = \theta_a \circ \theta_b$ for $a, b \in \mathbb{N}$, which are strict contractions, as

$$\theta_{a,b}'(x) = (\theta_a \circ \theta_b)'(x) = \frac{1}{(a(b+x)+1)^2} \leq \frac{1}{(ab+1)^2} \leq \frac{1}{4}.$$

Thus, $\Theta' = \{\theta_{a,b} \colon [0,1] \to [0,1] \mid a, b \in \mathbb{N}\}$ is an IFS for which the general theory applies and the invariant set of Θ' coincides with the invariant set of $\Theta = \{\theta \colon [0,1] \to [0,1] \mid a \in \mathbb{N}\}$. So, to analyse the Hausdorff dimension of this invariant set we can, and will, simply work with the IFS $\Theta = \{\theta_a \colon [0,1] \to [0,1] \mid a \in \mathbb{N}\}$ even though θ_1 is not a strict contraction.

For $a \in \mathbb{N}$ and $U = (0,1)$, we have that

$$\theta_a(U) = \left(\frac{1}{a+1}, \frac{1}{a}\right) \subset U,$$

and these sets are disjoint. Thus, $\Theta = \{\theta_a \colon [0,1] \to [0,1] \colon a \in \mathbb{N}\}$ is an IFS consisting of infinitesimal similitudes which satisfies the OSC.

We are particularly interested in invariant sets of subsystems of the IFS $\Theta = \{\theta_n \colon n \in \mathbb{N}\}$. Given $E \subseteq \mathbb{N}$, we work with $\Theta_E = \{\theta_a \colon a \in E\}$. There exists a non-empty compact set, which is the invariant set of IFS Θ_E, which we denote by J_E and J_E is the collection of all irrationals $x \in (0,1)$ whose continued expansion digits belong to E, so

$$J_E = \{x = [a_1, a_2, a_3, \ldots] \colon a_i \in E, i \in \mathbb{N}\} \subseteq [0,1].$$

These sets typically have a non-integral Hausdorff dimension, which we denote by $\dim_{\mathcal{H}}(J_E)$. They have been studied extensively; see for instance [4, 5, 13, 16, 17, 20, 23, 24, 28, 32, 33, 37, 39]. In particular, the Hausdorff dimension of $J_{\{1,2\}}$ was studied extensively. Good [20] showed that $0.5306 \leq \dim_{\mathcal{H}}(J_{\{1,2\}}) \leq 0.5320$. Falk and Nussbaum [16], also provided an approach to numerical computation of Hausdorff dimension improving the bounds to 8 decimal places. Finally, Jenkinson and Pollicott [28] computed the Hausdorff dimension of J_{E_2} accurately to over 100 decimal.

1.6 The Dimension Spectrum of an Infinite IFS

A central theme of this thesis is the notion of the dimension spectrum of an infinite IFS. Let Θ be an infinite IFS. For each $F \subseteq \mathbb{N}$, Θ_F is a subsystem of Θ. It follows that Θ_F has an invariant set J_F given by (1.7) with Hausdorff dimension $\dim_{\mathcal{H}}(J_F)$. One can then collect the Hausdorff dimensions of the invariant sets of all subsystems of Θ into a set and define the *dimension spectrum* of Θ,

$$\mathcal{DS}(\Theta) = \{\dim_{\mathcal{H}}(J_F) \colon F \subseteq \mathbb{N}\}. \tag{1.11}$$

The structure of $\mathcal{DS}(\Theta)$ has been studied recently for a variety of IFS; see for instance [7–10, 19, 29, 30].

Given $A \subseteq \mathbb{N}$ infinite, and $\Theta_A = \{\theta_a \colon a \in A\}$ with $\theta_a \colon x \to (a+x)^{-1}$, $x \in [0,1]$. We denote the dimension spectrum of Θ_A by

$$\mathcal{DS}(A) \coloneqq \mathcal{DS}(\Theta_A) = \{\dim_{\mathcal{H}}(J_F) \colon F \subseteq A\}.$$

If F is a singleton, then $\dim_{\mathcal{H}}(J_F) = 0$, hence $0 \in \mathcal{DS}(A)$. Also $\dim_{\mathcal{H}}(J_A) \in \mathcal{DS}(A)$, so

$$\mathcal{DS}(A) \subseteq [0, \dim_{\mathcal{H}}(J_A)].$$

A priori, there are no other elements of $\mathcal{DS}(A)$ which are known explicitly. For the case when $A = \mathbb{N}$, Hensley [23] conjectured that

$$\mathcal{FDS}(\mathbb{N}) \coloneqq \{\dim_{\mathcal{H}}(J_F) \colon F \subseteq \mathbb{N} \text{ with } |F| < \infty\}$$

is dense in $[0, 1]$. Independently, Mauldin and Urbański [32] made the same conjecture; this is known as the *Texan Conjecture*. Jenkinson proved that $\mathcal{FDS}(\mathbb{N})$ intersect $[0, \frac{1}{2}]$ densely, see [27, Theorem, 1], and he provided additional evidence for the Texan conjecture. Kesseböhmer and Zhu [30] showed that $\mathcal{DS}(\mathbb{N})$ has full dimension spectrum, i.e.,

$$[0, 1] = \mathcal{DS}(\mathbb{N}),$$

and confirmed the Texan conjecture.

Recently, dimension spectrum of A has been investigated by Chousionis, Leykekhman and Urbański in [7, 8] for different infinite subsets A of $\mathbb{N}$, see also [10, 29]. In [7] the dimension spectrum of the set of powers of integers $q \geq 2$ and the set of squares were analysed among other sets, which motivate the results in Chapter 3, 4 and 5 in this thesis.

Although our primary focus is on IFS associated with continued fraction expansions, it is useful to briefly mention results of the structure of the dimension spectrum for other IFS.

In [30] Kesseböhmer and Zhu also investigated the dimension spectrum of 1-dim affine IFS $\Theta = \{\theta_i \colon X \to X \mid i \in \mathbb{N}\}$ with $X \subseteq \mathbb{R}$ where $\theta_i(x) = a_i x + b_i$.

In particular, for $X = [0, 1]$, $a_i = i^{-1}(i+1)^{-1}$ and $b_i = (i+1)^{-1}$ for $i \geq 1$. They showed [30, Example 3.1] that the dimension spectrum $\mathcal{DS}(\Theta) = [0, 1]$, so that structure is a closed unit interval.

They further demonstrated in [30, Example 3.3] that if we take $a_i = 2 \cdot 3^{-i}$, $b_i = \sum_{j=1}^{i-1} a_j$ for $i \geq 1$ and $X = [0,1]$, then the dimension spectrum $\mathcal{DS}(\Theta)$ contains a closed interval $[0, \frac{\ln 2}{\ln 3}]$ and it is a nowhere dense set in $(\frac{\ln 2}{\ln 3}, 1)$. So it contains a closed interval followed by a nowhere dense part.

Another interesting example they considered [30, Example 3.4], where $a_i = 2^{-2^i}$, $b_i = \sum_{j=1}^{i-1} a_j$, $i \geq 1$ and $X = [0,1]$. In this case the dimension spectrum is a nowhere dense set, which is a Cantor set.

The parabolic backward continued fraction system $\Theta = \{\theta_b \colon b \in \mathbb{N} \text{ and } b \geq 2\}$ where $\theta_b(x) = \frac{1}{b-x}$ for $x \in [0,1]$ was also studied [18]. Every irrational number in $[0,1]$ has a unique representation of the form

$$x = [b_1, b_2, b_3, \ldots] = \cfrac{1}{b_1 - \cfrac{1}{b_2 - \frac{1}{b_3 - \ldots}}},$$

with $b_i \in \mathbb{N}$ and $b_i \geq 2$. It was shown [18] that $\mathcal{DS}(\Theta) = [0,1]$.

A further example of interest is the complex continued fractions, which can be represented as an infinite IFS $\Theta_E = \{\theta_a \colon a \in E\}$, where

$$E = \{m + ni \colon (m,n) \in \mathbb{Z} \times \mathbb{N}\}$$

and $X \subseteq \mathbb{C}$ is the closed disc centred at $\frac{1}{2}$ with radius $\frac{1}{2}$ and

$$\theta_b(z) = \frac{1}{b+z}, \ z \in X, \text{ and } b \in E.$$

In this case, Θ_E has a full dimension spectrum [8, Theorem 1.4], i.e.,

$$\mathcal{DS}(\Theta_E) = [0, \dim_{\mathcal{H}}(J_E)].$$

There also exist some general results describing the topological structures of the dimension spectrum covering a variety of these settings:

Let X be a compact and connected subset of $\mathbb{R}^n$. Following [9, p.8], a C^1 diffeomorphism $\theta\colon X \to X$ is called *conformal* if its derivative at every point of X is a similitude.

An IFS $\Theta = \{\theta_\alpha\colon X \to X \mid \alpha \in \mathbb{N}\}$ with X a compact connected subset of $\mathbb{R}^n$, is *conformal*, if it satisfies the following conditions:

(i) $X = \overline{\mathrm{Int}(X)}$, i.e., X is regular.

(ii) Θ satisfies the OSC with U, the interior of X.

(iii) θ_α is a conformal map for each α, and there exists $V \subseteq \mathbb{R}^n$ open connected such that $X \subset V \subset \mathbb{R}^n$ such that θ_α extends to a $C^{1+\varepsilon}$ diffeomorphism and is conformal on V

(iv) *Bounded Distortion Property.* There exists $M \geq 1$ such that for $\omega \in \cup_{k=1}^{\infty} E^k$,

$$||\theta'_\omega(y)|| \leq M||\theta'_\omega(x)||$$

for all $x, y \in X$, where $||\theta'_\omega(x)||$ denotes the Euclidean norm of the derivative.

Chousionis, Leykekhman and Urbaǹski proved that if Θ is a conformal IFS, then $\mathcal{DS}(\Theta)$ is compact and perfect set, see [8, Theorem 1.2].

In the same paper, they conjectured that every compact subset of $\mathbb{R}$ containing 0 can be realised as a dimension spectrum of some conformal IFS [8, Conjecture 1.3]. However Das and Simmons [10, Theorem 2.7] showed that this is not the case. In the same paper [10, Theorem 2.7], they showed that $\mathcal{DS}(\Theta)$ is not necessarily a uniformly perfect set.

Jurga [29] investigated a different class of infinite non-conformal IFS. We say Θ_E is an affine IFS if for each $\alpha \in E$, the map θ_α is a affine map defined by (1.8). Jurga [29, Theorem 1.4] showed that the dimension spectrum is not necessarily compact. In particular, there exists Θ an affine IFS satisfying the SOSC such that $\mathcal{DS}(\Theta)$ is not compact. In the same paper, it was also shown that there exists an affine IFS Θ_E such that $\mathcal{DS}(\Theta_E)$ contains isolated points hence $\mathcal{DS}(\Theta_E)$ is not necessarily perfect.

The Hausdorff dimension of invariant sets of IFS with infinitesimal similitudes can be analysed by using so called Perron-Frobenius or transfer operators. These operators are discussed in Chapter 2.

Chapter 2

Perron-Frobenius Operators

It is well known that the Hausdorff dimension and the dimension spectrum of IFSs with infinitesimal similitudes can be studied by analysing the spectrum of Perron-Frobenius operators; see for instance [4, 7–10, 17, 21, 23, 24, 28, 29, 33, 39]. Many of the proofs of results concerning such IFSs play out at the level of the spectral theory of these Perron-Frobenius operators and do not rely on their connection with the Hausdorff dimension. In particular the compactness and perfectness of the dimension spectrum for certain IFSs can be viewed as purely operator-theoretical results, without making any link to the Hausdorff dimension of the underlying IFS. The main goal of this chapter is to discuss this fact. We analyse a subclass of Perron-Frobenius operators and introduce the dimension spectrum of this class of operators. We show that it is, in general, a compact and perfect set.

At the end of the chapter we compare our results to known results from the literature [8, 29] on the compactness and perfectness of infinite IFSs.

2.1 Introduction

Throughout this chapter we shall assume that (X, ρ) is a compact metric space with diameter

$$d \coloneqq \operatorname{diam}(X) = \sup\{\rho(x, y)\colon x, y \in X\},$$

and that $C(X)$ denotes the space of all real-valued continuous functions on X. We will start by recalling some properties of the space of continuous functions which we will use in our arguments.

Equip $C(X)$ with the $||\cdot||_\infty$ norm so,

$$||f||_\infty = \sup\{|f(x)|\colon x \in X\},$$

together with the pointwise ordering, i.e., for $f, g \in C(X)$,

$$f \leq g \quad \Longleftrightarrow \quad f(x) \leq g(x) \text{ for all } x \in X.$$

In this setup, the Banach space $(C(X), ||\cdot||_\infty)$ is a complete order-unit space with cone

$$C(X)_+ = \{f \in C(X)\colon f(x) \geq 0 \text{ for all } x \in X\}$$

of all positive continuous functions on X, and order-unit $u\colon x \mapsto 1$ for all x. We shall consistently use the notation u to denote this constant function.

We say a function $f \in C(X)$ is *strictly positive* if $f(x) > 0$ for all $x \in X$.

Lemma 2.1. *Let $f, g \in C(X)$ be strictly positive. For each $0 < \lambda < 1$, there exists a $\mu \in [\lambda, 1)$ such that $f + \lambda g \leq \mu(f + g)$. Likewise for each $\lambda > 1$, there exists a $\mu \in (1, \lambda]$ such that $\mu(f + g) \leq f + \lambda g$.*

Proof. Since f and g are strictly positive on X, the function $h(x) = \frac{f(x)+\lambda g(x)}{f(x)+g(x)}$ is well defined, positive and continuous. So, by the Extreme Value Theorem h attains a maximum, say at $x_0 \in X$. Set $\mu = h(x_0)$. Then, for $0 < \lambda < 1$,

$$\mu = h(x_0) = \frac{f(x_0) + \lambda g(x_0)}{f(x_0) + g(x_0)} < \frac{f(x_0) + g(x_0)}{f(x_0) + g(x_0)} = 1.$$

Thus $\mu < 1$ and $f + \lambda g \leq \mu(f + g)$. As $\lambda(f + g) < f + \lambda g \leq \mu(f + g)$, we also have that $\lambda \leq \mu$. The second assertion can be derived in the same way by considering the minimum of h. $\square$

Let X be a compact metric space, a map $T\colon C(X) \to C(X)$ is a *positive operator* if for every $f \geq 0$ it follows that $Tf \geq 0$. The *operator norm of* T is defined by

$$||T|| = \sup\left\{\frac{||Tf||_\infty}{||y||_\infty}\colon f \in C(X), ||y||_\infty \neq 0\right\} = \sup\{||Tf||_\infty\colon f \in C(X), ||y||_\infty = 1\},$$

and finally we define the *spectral radius of* T as

$$r(T) = \lim_{n\to\infty}(||T^n||)^{\frac{1}{n}}.$$

We will consistently use $r(T)$ to denote the spectral radius of T.

If $K \subset C(X)$ is a cone, we define the *cone operator norm* of T as

$$\begin{aligned} ||T||_K &= \sup\left\{\frac{||Tf||_\infty}{||f||_\infty}\colon f \in K, ||f||_\infty \neq 0\right\} \\ &= \sup\{||Ty||_\infty\colon f \in C(X) \cap K, ||f||_\infty = 1\}, \end{aligned}$$

and we define the *cone spectral radius of* T as

$$r_K(T) = \lim_{n\to\infty}(||T^n||_K)^{\frac{1}{n}}.$$

It follows that $r_K(T) \leq r(T)$.

The following result gives bounds on the spectral radius of an operator.

Lemma 2.2. *Let* (X, ρ) *be a compact metric space and* $T\colon C(X) \to C(X)$ *a positive bounded linear operator. If* $w \in C(X)$ *is strictly positive and there exists* $\mu, \lambda > 0$ *such that* $\mu w \leq Tw \leq \lambda w$*, then*

$$\mu \leq r(T) \leq \lambda.$$

Proof. Assume $Tw \geq \mu w$. Then

$$T^2 w = T(Tw) \geq \mu Tw \geq \mu^2 w.$$

For each $k \in \mathbb{N}$, it can be shown that $T^k w \geq \mu^k w$. The norm is monotone, so $||T^k w||_\infty \geq \mu^k ||w||_\infty$. Let $u\colon x \mapsto 1$ be an order unit on $C(X)$. Then $w \leq ||w||_\infty u$ and

$$||T^k|| = ||T^k u||_\infty \geq \frac{||T^k w||_\infty}{||w||_\infty} \geq \mu^k.$$

It follows that

$$r(T) = \lim_{k\to\infty} ||T^k||^{1/k} \geq \mu.$$

For the other inequality, using arguments as in the first case, we can establish $T^k w \leq \lambda^k w$. So

$$||T^k w||_\infty \leq \lambda^k ||w||_\infty \quad \text{and} \quad r(T) = \lim_{k\to\infty} ||T^k w||_\infty^{1/k} \leq \lambda.$$

□

Note that this result holds for any positive linear operator on a complete normed space.

If $(Y, ||.||_Y)$ is a complete normed space, an operator $T\colon Y \to Y$ is *compact* if, for every bounded set $A \subset Y$, the set $\overline{T(A)}$ is compact in Y. Furthermore, we say that an operator $T\colon Y_+ \to Y_+$ is *compact in the sense of Bonsall* [3] if the sequence (Ty_n) has a convergent subsequence whenever (y_n) is a bounded sequence in Y_+.

There exist examples of linear operators $T\colon Y_+ \to Y_+$ that are compact in the sense of Bonsall, but do not admit a compact extension on the whole of Y see [3, Section 4].

In [3], the following result was proved.

Theorem 2.3. *Let Y be a complete normed space. If $T\colon Y_+ \to Y_+$ is compact in the sense of Bonsall and has a non-zero cone spectral radius μ, then there exists a non-zero vector v in Y_+ such that $Tv = \mu v$.*

2.2 The Cone $K(M,\lambda)$

Recall that a function $f\colon X \to \mathbb{R}$ is said to be *Hölder continuous* if there exist $C, \lambda > 0$ such that

$$|f(x) - f(y)| \leq C\rho(x,y)^{\lambda}.$$

The subset $C(X)_+ = \{f \in C(X)\colon f(x) \geq 0, \text{ for all x} \in X\}$ of $C(X)$ is the standard cone of $C(X)$. In the analysis, the following sub-cone will play an important role.

Fix $M > 0$ and $\lambda > 0$. Define

$$K(M,\lambda) = \left\{ f \in C(X)\colon 0 \leq f(x) \leq f(y)e^{M\rho(x,y)^{\lambda}} \quad \text{for } x, y \in X \right\}.$$

Remark 2.4. If $f \in K(M,\lambda)$, then $f = 0$ or $f(x) > 0$ for all $x \in X$. Indeed, if $f \in K(M,\lambda)$ and $f(y) = 0$ for some $y \in X$, then for $x \in X$ we have

$$0 \leq f(x) \leq f(y)e^{M\rho(x,y)^{\lambda}} = 0.$$

Thus $f(x) = 0$ for all $x \in X$, so $f = 0$. If $f(y) > 0$ for some $y \in X$ then $f(x) \geq f(y)e^{-Md^{\lambda}} > 0$, where $d = \operatorname{diam}(X)$. Thus, $f \in K(M,\lambda)$, implies $f = 0$ or $f(x) > 0$ for all $x \in X$.

For a fixed $\lambda > 0$, let $M_2 > M_1 > 0$, if $f \in K(M_1,\lambda)$, then for $x, y \in X$ we have that

$$0 \leq f(x) \leq f(y)e^{M_1\rho(x,y)^{\lambda}} \leq f(y)e^{M_2\rho(x,y)^{\lambda}}.$$

Thus $K(M_1,\lambda) \subseteq K(M_2,\lambda)$. We shall always think of λ as fixed in our arguments.

Remark 2.5. $f \in K(M,\lambda) \setminus \{0\}$ if and only if $\ln(f)$ is Hölder continuous with parameters M and λ.

Indeed if $f \in K(M,\lambda)$ and $f > 0$ then by Remark 2.4. It follows that

$$\begin{aligned} 0 < f(x) \le f(y)e^{M\rho(x,y)^\lambda} \quad &\Longleftrightarrow \quad 0 < \frac{f(x)}{f(y)} \le e^{M\rho(x,y)^\lambda} \\ &\Longleftrightarrow \quad |\ln(f(x)) - \ln(f(y))| \le M\rho(x,y)^\lambda. \end{aligned}$$

This implies that

$$f(x) \le f(y)e^{M\rho(x,y)^\lambda} \quad \Longleftrightarrow \quad |\ln(f(y)) - \ln(f(x))| \le M\rho(x,y)^\lambda. \tag{2.1}$$

Thus $\ln(f)$ is Hölder continuous. The following lemma can be found in [39, Lemma 3.2]. It shows that every bounded subset of $K(M,\lambda)$ is an equicontinuous family.

Lemma 2.6. *Let $M, \lambda > 0$. Then $K(M,\lambda)$ is a closed cone in $(C(X), ||.||_\infty)$ and for any $R > 0$, the set*

$$\{f \in K(M,\lambda) \colon ||f||_\infty \le R\}$$

is an equicontinuous family.

Proof. If $f, g \in K(M,\lambda)$ and $\mu > 0$, then for $x, y \in X$ we have that

$$\begin{aligned} 0 \le (f+\mu g)(x) = f(x) + \mu g(x) \quad &\le \quad f(y)e^{M\rho(x,y)^\lambda} + \mu g(y)e^{M\rho(x,y)^\lambda} \\ &= \quad (f+\mu g)(y)e^{M\rho(x,y)^\lambda}. \end{aligned}$$

Thus, $f + \mu g \in K(M,\lambda)$, hence it is a cone.

For closedness, let (f_n) be a sequence in $K(M, \lambda)$ such that $f_n \to f$ uniformly. Then f is continuous and for $x, y \in X$ we have that

$$0 \leq \lim_{n \to \infty} \left(f_n(y) e^{M\rho(x,y)^\lambda} - f_n(x) \right) = f(y) e^{M\rho(x,y)^\lambda} - f(x).$$

Hence, $K(M, \lambda)$ is closed in $C(X)$. It remains to show equicontinuity. To this end, let $f \in K(M, \lambda)$, $f \neq 0$, so $f(x) > 0$ for all $x \in X$ by Remark 2.4. Now using (2.1), we have that

$$f(x) \leq f(y) e^{M\rho(x,y)^\lambda} \iff |\ln(f(y)) - \ln(f(x))| \leq M\rho(x,y)^\lambda.$$

Set $u_x = \ln(f(x))$ and $u_y = \ln(f(y))$. We may assume $u_x < u_y$. Using (2.1) and the Mean Value Theorem there exist $z_0 \in [u_x, u_y]$ such that

$$\begin{aligned} |f(y) - f(x)| &= |e^{u_y} - e^{u_x}| \\ &= e^{z_0} |u_y - u_x| \\ &\leq e^{u_y} |u_y - u_x| \\ &\leq f(y) |\ln(f(y)) - \ln(f(x))| \\ &\leq f(y) M\rho(x,y)^\lambda \\ &\leq M||f||_\infty \rho(x,y)^\lambda, \end{aligned}$$

As $||f||_\infty \leq R$, it follows that

$$|f(y) - f(x)| \leq RM\rho(x,y)^\lambda.$$

Thus, the set $\{f \in K(M, \lambda) \colon ||f||_\infty \leq R\}$ is an equicontinuous family. □

2.3 Perron-Frobenius Operators

Let (X, ρ) be a compact metric space, and let $\Theta = \{\theta_n \colon n \in \mathbb{N}\}$ be a collection of injective, uniformly contracting maps on X with contraction constant $c < 1$. Let $L \colon C(X) \to C(X)$ be a linear operator of the form

$$(Lf)(x) = \sum_{n=1}^{\infty} a_n(x) f(\theta_n(x)). \tag{2.2}$$

These operators are called *Perron-Frobenius operators or transfer operators*, and have been widely studied in a variety of contexts see for instance [4, 5, 7, 8, 16, 17, 23, 39, 43].

We shall make the following assumption:

(**A1**) For each $n \in \mathbb{N}$, $a_n \colon X \to \mathbb{R}$ is a positive continuous function, and there exist constants $0 < \beta_n < \eta_n < 1$ such that

$$\beta_n \leq a_n(x) \leq \eta_n,$$

and $\eta = \sup\{\eta_n \colon n \in \mathbb{N}\} < 1$. Moreover, for each $x \in X$,

$$a(x) \coloneqq \sum_{n=1}^{\infty} a_n(x) < \infty, \quad \text{and} \quad a \colon x \mapsto a(x) \text{ is continuous.}$$

Remark 2.7. By Dini's Theorem, the assumption in (**A1**) that $a \colon X \to \mathbb{R}$ is continuous is equivalent to assuming that the sequence

$$\left(\sum_{k=1}^{n} a_k \right)_{n \in \mathbb{N}}$$

converges uniformly to a.

The following result shows that if **(A1)** is satisfied, then L defines a bounded linear operator from $C(X)$ into $C(X)$, see [38, Section 5].

Lemma 2.8. *If (**A1**) is satisfied, then $L\colon C(X) \to C(X)$ defined in (2.2) is bounded linear operator.*

Proof. Note for $f \in C(X)$ and $x \in X$ fixed, the series $\sum_{n=1}^{\infty} a_n(x) f(\theta_n(x))$ is absolutely convergent as

$$\sum_{n=1}^{\infty} |a_n(x) f(\theta_n(x))| = \sum_{n=1}^{\infty} a_n(x) |f(\theta_n(x))| \leq ||f||_\infty ||a||_\infty < \infty.$$

So we can define

$$(Lf)(x) = \sum_{n=1}^{\infty} a_n(x) f(\theta_n(x)).$$

Let $g_n(x) = \sum_{k=1}^{n} a_k(x) f(\theta_k(x))$. Then (g_n) is a sequence of continuous functions and $g_n \to Lf$ pointwise. Since $(C(X), ||.||)_\infty$ is complete, it suffices to show that (g_n) is Cauchy.

Fix $\varepsilon > 0$. Using Remark 2.7, assumption (**A1**) implies that the sequence $(\sum_{k=1}^{n} a_k)_n$ converges uniformly to $a = \sum_{k=1}^{\infty} a_k$. So there exists an $N \in \mathbb{N}$ such that if $n \geq N$, then

$$\sup_{x \in X} \left| a(x) - \sum_{k=1}^{n} a_k(x) \right| = \sup_{x \in X} \sum_{k>n} a_k(x) < \frac{\varepsilon}{||f||_\infty + 1}. \tag{2.3}$$

Now if $n, m \geq N$, we may assume $m > n$, then using (2.3)

$$|g_m(x) - g_n(x)| = \left| \sum_{n+1}^{m} a_k(x) f(\theta_k(x)) \right| \leq \sum_{n+1}^{m} a_k(x) |f(\theta_k(x))| \leq ||f||_\infty \sup_{x \in X} \sum_{k>n} a_k(x) < \varepsilon$$

Thus (g_n) is Cauchy so that $Lf \in C(X)$. □

We shall also make the following assumption:

(**A2**) For $n \in \mathbb{N}$, the maps $a_n\colon X \to \mathbb{R}$ are uniform Lipschitz contractions, i.e, there exists a constant $c < 1$, independent of n such that $|a_n(x) - a_n(y)| \leq c\rho(x, y)$ for all $x, y \in X$, and there exist constants $M_0 > 0$ and $\lambda > 0$ such that $a_n \in K(M_0, \lambda)$ for all n.

The following lemma was proved in [39, Lemma 5.3].

Lemma 2.9. *Let (X,ρ) be a compact metric space and let $L\colon C(X)\to C(X)$ be defined by (2.2). Assume that (**A1**) and (**A2**) are satisfied. Then there exists $M>0$ such that*

$$L(K(M,\lambda))\subset K(M,\lambda),$$

and L has a strictly positive eigenvector $v\in K(M,\lambda)$ with eigenvalue $r(L)>0$.

Proof. Let $M=\frac{M_0}{1-c^\lambda}$, which is well defined since $c<1$ and $\lambda>0$. Then $M_0+c^\lambda M=M$. We will show that with this choice of M, one has

$$L(K(M,\lambda))\subseteq K(M,\lambda).$$

Let $f\in K(M,\lambda)$ and $x,y\in X$. Since θ_n is a contraction map, we have

$$f(\theta_n(x))\leq f(\theta_n(y))\,e^{M\rho(\theta_n(x),\theta_n(y))^\lambda}\leq f(\theta_n(y))\,e^{Mc^\lambda\rho(x,y)^\lambda},$$

and

$$a_n(x)\leq a_n(y)\,e^{M_0\rho(x,y)^\lambda}.$$

It follows that

$$\begin{aligned}0\leq (Lf)(x)&=\sum_{n=1}^{\infty}a_n(x)f(\theta_n(x))\\&\leq\sum_{n=1}^{\infty}a_n(y)f(\theta_n(y))\,e^{(M_0+Mc^\lambda)\rho(x,y)^\lambda}\\&=(Lf)(y)\,e^{M\rho(x,y)^\lambda}.\end{aligned}$$

This proves that $L(K(M,\lambda))\subseteq K(M,\lambda)$.

Let $L|_K\colon K(M,\lambda)\to K(M,\lambda)$ be the restriction of L to $K(M,\lambda)$. Using Lemma 2.6, the set $\{f\in K(M,\lambda)\colon ||f||_\infty\leq R\}$, $R>0$ is an equicontinuous, uniformly bounded family. So by Ascoli-Arzelà theorem, it is compact. It then follows that if (f_n) is a bounded sequence in $K(\lambda, M)$, then $\{Lf_n : n \in \mathbb{N}\}$ is an equicontinuous uniformly bounded family so it has a convergent subsequence. Thus $L|_K$ is compact in the sense of Bonsall on $K(\lambda, M)$. Note that $u : x \mapsto 1$ is in $K(M, \lambda)$, so

$$r_K(L) \geq \lim_{k\to\infty} ||L^k u||_\infty^{1/k} = r(L).$$

As $L_{|_K}$ is a restriction of L we have that $r(L) \geq r_K(L)$, thus $r_K(L) = r(L)$. From assumption (**A1**), $0 < \beta := \sum_{n=1}^{\infty} \beta_n$ so

$$(Lu)(x) = \sum_{n=1}^{\infty} a_n(x) \geq \beta > 0.$$

Lemma 2.2 implies that $r_K(L) = r(L) \geq \beta > 0$. Using Theorem 2.3, there exists $v \in K(M, \lambda)$ such that $||v||_\infty = 1$ and $Lv = r(L)v$. As $v \neq 0$, $v(x) > 0$ for all $x \in X$. □

2.3.1 Main Assumptions

Motivated by the study of the dimension spectrum for infinite IFS see [6–10, 27, 29, 30, 32, 33] we consider the following abstract setup at operator level.

Let (X, ρ) be a compact metric space and $\Theta = \{\theta_n : X \to X : n \in \mathbb{N}\}$ be a collection of injective uniform contraction maps on X with Lipschitz constant c and $B = \{b_n : X \to \mathbb{R} : n \in \mathbb{N}\}$ be a collection of real-valued uniformly Lipschitz functions on X with Lipschitz constant c such that:

(**B1**) For each n, there exist constants β_n and η_n such that $0 < \beta_n \leq b_n(x) \leq \eta_n < 1$ for all $x \in X$ and $\eta = \sup\{\eta_n : n \in \mathbb{N}\} < 1$.

(**B2**) $\lim_n ||b_n||_\infty = 0$.

(**B3**) There exists a $y \in X$ and $\sigma > 0$ such that $\sum_{n\in\mathbb{N}} b_n^\sigma(y) < \infty$.

(**B4**) There exist $M_0 > 0$ and $\lambda > 0$ such that $b_n \in K(M_0, \lambda)$ for all $n \in \mathbb{N}$.

We will simply write (Θ, B) for a collection satisfying assumptions (**B1**) - (**B4**). If $E \subseteq \mathbb{N}$, we write (Θ_E, B_E) to denote the subsystem of (Θ, B) indexed by E.

Remark 2.10. The assumptions (**B1**)-(**B4**) are satisfied for many classical IFS's.

1. Let $\Theta = \{\theta_n \colon X \to X \colon n \in \mathbb{N}\}$ be an infinite conformal IFS, and $b_n(x) = ||\theta_n'(x)||$ the norm-derivative of θ_n. Then the OSC together with the bounded distortion property implies that $\lim_n ||b_n||_\infty = 0$, see [33, Lemma 2.5]. The bounded distortion property also ensures (**B4**) see for instance [34, Chapter 4]. Also, [33, Lemma 2.5] ensures that (**B3**) holds. Thus infinite conformal IFSs satisfy (**B1**)-(**B4**) with $B = \{b_n \colon n \in \mathbb{N}\}$ where $b_n(x) = ||\theta_n'(x)||$.

2. Let $\Theta = \{\theta_n \colon X \to X \colon n \in \mathbb{N}\}$ be an infinite IFS consisting of infinitesimal similitudes where each $(D\theta_n)$ is a strictly positive Hölder continuous function with a same Lipschitz constant as that of θ. In that case we can take $b_n(x) = (D\theta_n)(x)$, $x \in X$, see [39]. Then from Hypothesis **H5.1** and **H5.2** in [39, Section 5], we immediately have (**B1**) and (**B2**). Also from **H5.3** in [39] we deduce (**B3**) and (**B4**). Thus an IFS consisting of similitudes satisfying hypothesis **H5.1** - **H5.4** of [39, Section 5], also satisfied (**B1**)-(**B4**).

In the remainder of the chapter we assume the collection (Θ, B) satisfies (**B1**) - (**B4**).

2.3.2 Perron-Frobenius Operators over Finite Sets

Let $F \subset \mathbb{N}$ finite. For $s \geq 0$, define

$$(L_{s,F} f)(x) = \sum_{n \in F} b_n^s(x) f(\theta_n(x)). \tag{2.4}$$

The following result is found in [39, Lemma 3.3] and [5, Corollary 4.1].

Lemma 2.11. *Let (Θ, B) be given satisfying $(\boldsymbol{B1}) - (\boldsymbol{B4})$ and $F \subset \mathbb{N}$ finite. If $L_{s,F} \colon C(X) \to C(X)$ is defined as in (2.4), then there exists an $M = M_s > 0$ such that*

$$L_{s,F}\left(K(M,\lambda)\right) \subseteq K(M,\lambda).$$

Let $F_\infty = F^{\mathbb{N}}$. If $\omega = (n_1, n_2, ...) \in F_\infty$, define

$$\theta_{\omega|_k}(x) = \theta_{n_k} \circ \theta_{n_{k-1}} \circ \ldots \circ \theta_{n_1}(x)$$

and define $b_{\omega|_k}(x)$ inductively as

$$\begin{aligned} b_{\omega|_k}(x) &= b_{n_1}(x) && \text{if } k = 1 \\ b_{\omega|_k}(x) &= b_{n_k}(\theta_{\omega|_{k-1}}(x))b_{\omega|_{k-1}}(x) && \text{if } k \geq 2. \end{aligned}$$

Computing $L^2_{s,F}$ we have that

$$\begin{aligned} (L^2_{s,F}f)(x) &= (L_{s,F}(L_{s,F}f))(x) \\ &= \sum_{n_1 \in F} b^s_{n_1}(x)(L_{s,F}f)(\theta_{n_1}(x)) \\ &= \sum_{n_1 \in F} b^s_{n_1}(x) \sum_{n_2 \in F} b^s_{n_2}(\theta_{n_1}(x)) f(\theta_{n_2}(\theta_{n_1}(x))) \\ &= \sum_{n_1 \in F} \sum_{n_2 \in F} b^s_{n_1}(x) b^s_{n_2}(\theta_{n_1}(x)) f(\theta_{n_2} \circ \theta_{n_1}(x)) \\ &= \sum_{(n_1,n_2) \in F \times F} b^s_{n_2}(\theta_{n_1}(x)) b^s_{n_1}(x) f(\theta_{n_2} \circ \theta_{n_1}(x)) \\ &= \sum_{\omega|_2 \in F^2} b^s_{\omega|_2}(x) f(\theta_{\omega|2}(x)). \end{aligned}$$

In [39, Lemma 3.4] it was shown that the k^{th} iterates of $L_{s,F}$ satisfies

$$(L^k_{s,F}f)(x) = \sum_{\omega|_k \in F^k} b^s_{\omega|_k}(x) f(\theta_{\omega|_k}(x)). \tag{2.5}$$

If we take the order-unit $u \colon x \mapsto 1$ for all $x \in X$, we have that

$$(L^k_{s,F}u)(x) = \sum_{\omega|_k \in F^k} b^s_{\omega|_k}(x) \quad \text{and} \quad \left\| L^k_{s,F} \right\| = \sup_{x \in X} \sum_{\omega|_k \in F^k} b^s_{\omega|_k}(x).$$

The following result is [39, Theorem 3.6], and was also proved earlier in [5, Proposition 4.1].

Theorem 2.12. *Let* (Θ, B) *be given satisfying* (***B1***) – (***B4***). *For* $F \subseteq \mathbb{N}$ *finite and* $s > 0$, *if* $L_{s,F} \colon C(X) \to C(X)$ *is defined as in* (2.4), *then* $L_{s,F}$ *has a strictly positive eigenvector* v_s *with eigenvalue* $r(L_{s,F})$, *the spectral radius of* $L_{s,F}$.

Furthermore, we have the following result [39, Lemma 4.6].

Lemma 2.13. *Let* (Θ, B) *be given satisfying* (***B1***) – (***B4***). *For* $F \subset \mathbb{N}$ *finite, the map* $s \mapsto r(L_{s,F})$ *is continuous and strictly decreasing on* $(0, \infty)$. *Moreover, there exists a unique number* $s_F \geq 0$ *such that* $r(L_{s_F,F}) = 1$.

2.3.3 Perron-Frobenius Operators over Infinite Sets

In the previous section, we considered $F \subset \mathbb{N}$ finite. In this section, we consider $E \subseteq \mathbb{N}$ infinite, and we show that the Perron-Frobenius operator is well defined for all sufficiently large s and the same results hold see [39, Section 5].

Note that assumption (**B3**) gives the existence of a $\sigma > 0$ and $y \in X$ such that $\sum_n b_n^\sigma(y) < \infty$.

Using this, we have the following result [39, Lemma 5.4].

Lemma 2.14. *Let* (Θ, B) *be given satisfying* (***B1***) – (***B4***). *If* $\sigma > 0$ *is such that* $\sum_{n=1}^\infty b_n^\sigma(y) < \infty$ *for some* $y \in X$, *then for any* $t > \sigma$ *and* $x \in X$,

$$\sum_{n \in \mathbb{N}} b_n^t(x) < \infty.$$

Proof. Since $b_n(x) < 1$, we have that $b_n^t(x) \leq b_n^\sigma(x)$ for all $x \in X$. So it suffices to show that $\sum_{n\in\mathbb{N}} b_n^\sigma(x) < \infty$. We know that $\sum_{n=1}^\infty b_n^\sigma(y) < \infty$. As $b_n \in K(M_0, \lambda)$, we have that $b_n^\sigma \in K(\sigma M_0, \lambda)$. So

$$b_n^\sigma(x) \leq b_n(y)^\sigma e^{\sigma M_0 \rho(x,y)^\lambda} \leq b_n(y)^\sigma e^{\sigma M_0 d^\lambda},$$

and the result follows. $\square$

Observe that $\sum_{n=1}^{\infty} b_n^s(y) < \infty$ for some $y \in X$ if and only if $\sum_{n=1}^{\infty} ||b_n||_\infty^s < \infty$. Thus the set

$$\left\{ s \geq 0 \colon \sum_{n\in\mathbb{N}} ||b_n||_\infty^s < \infty \right\}$$

is non-empty. Define

$$\sigma_0 \coloneqq \inf\{s \geq 0 \colon \sum_{n\in\mathbb{N}} ||b_n||_\infty^s < \infty\}. \tag{2.6}$$

Note that at σ_0 we have two cases to consider.

Case 1: $\sum_{n\in\mathbb{N}} ||b_n||_\infty^s < \infty$ for $s > \sigma_0$ and $\sum_{n\in\mathbb{N}} ||b_n||_\infty^{\sigma_0} = \infty$.

Case 2: $\sum_{n\in\mathbb{N}} ||b_n||_\infty^s < \infty$ for $s \geq \sigma_0$.

Both cases can occur:

Example 2.15. Consider the system (Θ, B), where $B = \{b_n \colon n \in \mathbb{N}\}$ is defined by

$$b_n(x) = (2^n + x)^{-2}, x \in [0,1], \text{ for all } n \in \mathbb{N}.$$

We claim that we are in the first case.

To see this we first have to show that B is a collection of Lipschitz maps with factor $c < 1$. So

$$|b_n'(x)| = 2 \cdot \left(\frac{1}{2^n + x}\right)^3 \leq \frac{2}{2^{3n}}.$$

Thus by the Mean Value Theorem

$$|b_n(x) - b_n(y)| \leq \frac{2}{2^{3n}}|x-y| \leq \frac{1}{4}|x-y|, \quad x, y \in [0,1],$$

for all n. Next, we will show that the assumptions (**B1**)-(**B4**) are satisfied and then we determine the infimum and interval of convergence. Indeed,

$$0 < (2^n + 1)^{-2} \leq b_n(x) \leq (2^n)^{-2} < 1,$$

and $\lim_n ||b_n||_\infty = \lim_n (2^n)^{-2} = 0$ so (**B1**) and (**B2**) hold. For $s > 0$ we have that

$$\sum_{n=1}^{\infty} ||b_n||^s = \sum_{n=1}^{\infty} \left(\frac{1}{2^n}\right)^{2s} = \frac{1}{4^s - 1} < \infty.$$

Thus (**B3**) holds. Using the Mean Value Theorem on $\ln(b_n)$ again we have

$$\ln\left(\frac{b_n(x)}{b_n(y)}\right) = 2|\ln(2^n + x) - \ln(2^n + y)| \leq 2(2^n)^{-1}|x - y|,$$

so $0 \leq b_n(x) \leq e^{|x-y|} b_n(y)$. Thus $b_n \in K(1, 1)$. So B satisfies the assumptions (**B1**)- (**B4**). Clearly at $s = 0$ the sum is infinity, so $\sigma_0 = 0$.

Example 2.16. Consider the system (Θ, B), where $B = \{b_n : n \in \mathbb{N}\}$ is defined by

$$b_n(x) = \frac{1}{((n+3) + x) \ln^2((n+3) + x)}, \quad x \in [0, 1] \text{ for all } n \in \mathbb{N}.$$

We claim we are in the second case.

As in the first case we show that B is a collection of Lipschitz maps with constant $c < 1$ and then show that the assumptions (**B1**)- (**B4**) are satisfied. Taking the derivatives we have

$$\begin{aligned} |b_n'(x)| &= \frac{1}{(n+3+x)^2 \ln^2(n+3+x)} \left(1 + \frac{1}{\ln(n+3+x)}\right) \\ &\leq \frac{1}{16 \ln^2 4} \left(1 + \frac{1}{\ln 4}\right). \end{aligned}$$

Hence B is collection of Lipschitz maps with constant $c < 1$. Next, we show it satisfies the assumptions. Observe that

$$0 < \frac{1}{(n+4)\ln^2(n+4)} \le b_n(x) \le \frac{1}{(n+3)\ln^2(n+3)} < 1$$

and $\lim_n b_n = 0$, so (**B1**) and (**B2**) are satisfied.

To show that $b_n \in K(M_0, \lambda)$ for some $M_0, \lambda > 0$, set

$$g_n(x) = \ln\left(b_n(x)\right) = -\ln((3+n)+x) - 2\ln\left(\ln((3+n)+x)\right).$$

Then

$$\begin{aligned} g_n'(x) &= -\frac{1}{(3+n)+x} - \frac{2}{((3+n)+x)\ln((3+n)+x)} \\ &= -\frac{1}{(3+n)+x}\left(1 + \frac{2}{\ln((3+n)+x)}\right). \end{aligned}$$

Hence

$$|g_n'(x)| \le \frac{1}{4}\left(1 + \frac{1}{\ln 2}\right) \le \frac{3}{4} =: M_0.$$

The last inequality comes from the relation $1 - \frac{1}{z} \le \ln z$ for all $z > 0$.

Given $x, y \in [0,1]$, we can use the Mean Value Theorem to show that

$$\begin{aligned} \left|\ln\left(\frac{b_n(x)}{b_n(y)}\right)\right| &= |\ln(b_n(x)) - \ln(b_n(y))| \\ &= |g_n(x) - g_n(y)| \\ &\le M_0|x-y|. \end{aligned}$$

Therefore,

$$\frac{b_n(x)}{b_n(y)} \le e^{M_0|x-y|}, \quad \text{so} \quad b_n \in K(M_0, 1).$$

To determine the interval of convergence, consider $s \geq 1$, and note that

$$((3+n)+x)^s \ln^{2s}((3+n)+x) \geq (3+n)\ln^2(3+n),$$

so

$$\begin{aligned}\sum_{n\in\mathbb{N}} b_n^s(x) &\leq \sum_{n\in\mathbb{N}} ||b_n||_\infty^s \\ &\leq \sum_{n\in\mathbb{N}} \frac{1}{(3+n)\ln^2(n+3)} \\ &\leq \int_3^\infty \frac{1}{x\ln^2 x}dx = \frac{1}{\ln 3}.\end{aligned}$$

Hence $\sum_{n\in\mathbb{N}} b_n^s(x) < \infty$ for $s \geq 1$.

On the other hand, if $s < 1$, let $\delta > 0$ be such that $s \leq 1-\delta$. Since $\frac{\ln^r k}{k^\lambda} \to 0$ as $k \to \infty$ for all $r, \lambda > 0$, we have that there exists an N such that

$$||b_n||_\infty^s = \frac{1}{(3+n)^s \ln^{2s}(3+n)} \geq \frac{1}{\frac{(3+n)(\ln(3+n))^{2(1-\delta)}}{(3+n)^\delta}} \geq \frac{1}{3+n}$$

for all $n \geq N$. Thus,

$$\sum_{n>N} ||b_n||_\infty^s \geq \sum_{n>N} \frac{1}{3+n}$$

which diverges. In this case $\sigma_0 = 1$, and we are in the second case.

For any $s > \sigma_0$, if we are in Case 1, or, $s \geq \sigma_0$, if we are in Case 2, define

$$(L_s f)(x) = \sum_{n=1}^\infty b_n^s(x) f(\theta_n(x)) \quad \text{for all } x \in X. \tag{2.7}$$

Note that for $s \geq \sigma_0$, we have that $\beta_s := \sum_{n=1}^\infty ||b^s||_\infty < \infty$. Furthermore, as $b_n \in K(M,\lambda)$, it follows that $b_n^s \in K(sM,\lambda)$ and for each $k \in \mathbb{N}$,

$$\sum_{n=1}^k b_n^s(x) \leq \sum_{n=1}^k b_n^s(y) e^{sM\rho(x,y)^\lambda} \leq \beta_s e^{sMd^\lambda},$$

where $d = \text{diam}(X)$. Therefore we can define

$$g_k(x) = \sum_{n=1}^{k} b_n^s(x),$$

so $g_k \in K(sM, \lambda)$ and $||g_k||_\infty \leq \beta_s e^{sMd^\lambda}$ for each k. It follows from Lemma 2.6, that $\{g_k : k \in \mathbb{N}\}$ is a collection of bounded equicontinuous functions, so by the Ascoli-Arzelà Theorem it has a convergent subsequence with limit say

$$g(x) := \sum_{n=1}^{\infty} b_n^s(x). \tag{2.8}$$

Since the convergence is uniform, it follows that $g : X \to \mathbb{R}$ is continuous. Using Lemma 2.8, L_s defines a bounded linear operator from $C(X)$ into itself. Furthermore, Lemma 2.9 implies that there exists an $M > 0$ satisfying $sM_0 + Mc^\lambda \leq M$ and a positive eigenvector $v_s \in K(M, \lambda) \setminus \{0\}$ of L_s with corresponding eigenvalue $r(L_s) > 0$.

The following result is [39, Lemma 5.5].

Theorem 2.17. *Let (Θ, B) be given satisfying $(\boldsymbol{B1}) - (\boldsymbol{B4})$. Let σ_0 be defined as in (2.6). If $s > \sigma_0$ and L_s is defined by (2.7), then the map $s \mapsto \lambda_s = r(L_s)$ is continuous and strictly decreasing. Also $\lambda_s \to 0$ as $s \to \infty$.*

Proof. We first show that $s \mapsto \lambda_s$ is decreasing.

Let $\sigma_0 < s_1 < s_2$. Since $b_n(x) \leq \eta_n \leq \eta < 1$ by (**B1**) for all $x \in X$, we have the following

$$b_n^{s_2}(x) = b_n^{s_1}(x) b_n^{s_2 - s_1}(x) \leq \eta^{s_2 - s_1} b_n^{s_1}(x).$$

As $\eta < 1$ and $s_2 - s_1 > 0$, $\mu := \frac{1}{\eta^{s_2 - s_1}} > 1$ and $b_n^{s_1}(x) \geq \mu b_n^{s_2}(x)$. Let v_{s_2} be an eigenvector of L_{s_2} with eigenvalue $\lambda_{s_2} = r(L_{s_2}) > 0$. Then

$$\begin{aligned}(L_{s_1} v_{s_2})(x) &= \sum_{n=1}^{\infty} b_n^{s_1}(x) v_{s_2}(\theta_n(x)) \\ &\geq \mu \sum_{n=1}^{\infty} b_n^{s_2}(x) v_{s_2}(\theta_n(x)) \\ &= \mu (L_{s_2} v_s)(x) = \mu \lambda_{s_2} v_{s_2}(x).\end{aligned}$$

Thus, $L_{s_1} v_{s_2} \geq \mu \lambda_{s_2} v_{s_2}$. As $v_{s_2} \in K(M, \lambda) \setminus \{0\}$, it follows from Lemma 2.2 that $\lambda_{s_1} = r(L_{s_1}) \geq \mu \lambda_{s_2}$. As $\mu > 1$, we have that $\lambda_{s_1} > \lambda_{s_2}$. Therefore, $s \mapsto \lambda_s$ is strictly decreasing.

To show continuity, note that as λ_s is strictly positive and decreasing, it suffices to show that $\ln(\lambda_s)$ is a convex function, since a finite valued convex function is continuous on an open interval. Let $\sigma_0 < s_1 < s_2$ and take $\beta \in (0, 1)$. Set $s = \beta s_2 + (1 - \beta) s_1$. For $f \in C(X)$ and $f \geq 0$, using the Hölder's inequality we have

$$\begin{aligned}(L_s f)(x) &= \sum_{n=1}^{\infty} b_n^{s}(x) f(\theta_n(x)) \\ &= \sum_{n=1}^{\infty} (b_n^{s_1} f(\theta(x)))^{1-\beta} (b_n^{s_2}(x) f(\theta_n(x)))^{\beta} \\ &\leq \left(\sum_{n=1}^{\infty} b_n^{s_1}(x) f(\theta_n(x)) \right)^{1-\beta} \left(\sum_{n=1}^{\infty} b_n^{s_2}(x) f(\theta_n(x)) \right)^{\beta}.\end{aligned}$$

Thus $L_s f \leq (L_{s_1 f})^{1-\beta}(L_{s_2} f)^{\beta}$. for any $f \geq 0$. Now using this estimate to compute bounds for L_s^2 we obtain

$$
\begin{aligned}
(L_s^2 f)(x) &= (L_s(L_s f))(x) \\
&\leq \left(L_s(L_{s_1} f)^{1-\beta}(L_{s_2} f)^{\beta}\right)(x) \\
&= \sum_{n=1}^{\infty} b_s(x)((L_{s_1} f)(\theta_n(x)))^{1-\beta}((L_{s_2} f)(\theta_n(x)))^{\beta} \\
&= \sum_{n=1}^{\infty} (b_n^{s_1}(x)(L_{s_1} f)(\theta_n(x)))^{1-\beta}(b_n^{s_2}(x)(L_{s_2} f)(\theta_n(x)))^{\beta} \\
&\leq \left(\sum_{n=1}^{\infty} b_n^{s_1}(x)(L_{s_1} f)(\theta_n(x))\right)^{1-\beta} \left(\sum_{n=1}^{\infty} b_n^{s_2}(x)(L_{s_2} f)(\theta_n(x))\right)^{\beta} \\
&= ((L_{s_1}^2 f)(x))^{1-\beta}((L_{s_2}^2 f)(x))^{\beta}
\end{aligned}
$$

So $L_s^2 f \leq (L_{s_1}^2 f)^{1-\beta}(L_{s_2} f)^{\beta}$. In the same way using (2.5), one can show that for each $k \in \mathbb{N}$,

$$L_s^k f \leq (L_{s_1}^k)^{1-\beta}(L_{s_2}^k f)^{\beta}.$$

Since this holds for any $f \geq 0$, it follows that

$$||L_s^k|| \leq ||L_{s_1}^k||^{1-\beta}||L_{s_2}^k||^{\beta}.$$

It follows that

$$r(L_s) \leq (r(L_{s_1}))^{1-\beta}(r(L_{s_2}))^{\beta}.$$

Thus, $\ln(r(L_s)) = \ln(\lambda s) \leq (1-\beta)\ln(\lambda_{s_1}) + \beta \ln(\lambda_{s_2})$. So the map $s \mapsto \ln(\lambda_s)$ is convex, hence $s \mapsto \ln(\lambda_s)$ is continuous.

To show that $\lambda_s \to 0$ as $s \to \infty$, fix $t > \sigma_0$ with $\sum_{n=1}^{\infty} b_n^t(x) < M$ for some M large. As $b_n(x) < \eta_n \leq \eta < 1$ by (**B1**), we have

$$\sum_{n=1}^{\infty} b_n^s(x) \leq \eta^{s-t} \sum_{n=1}^{\infty} b_n^t(x).$$

Therefore

$$||L_s|| \leq \sup_{x \in X} \sum_{n=1}^{\infty} b_n^s(x) \leq \eta^{s-t} M.$$

This implies that $||L_s|| \to 0$ as $s \to \infty$. Since $\lambda_s \leq ||L_s||$ we have the result.

□

Note that Theorem 2.17 implies that the set $\{s \geq \sigma_0 \colon r(L_s) \leq 1\}$ is non-empty. Finally, for L_s in (2.7), define

$$s_{\mathbb{N}} \coloneqq \inf\{s \geq \sigma_0 \colon r(L_s) \leq 1\}. \tag{2.9}$$

Remark 2.18. Note that there does not necessarily exists an $s > 0$ such that $r(L_s) = 1$.

To see this let $\theta_n(x) = \frac{1}{\ln((n+3)+x)}$, $x \in [0, 1]$ and $n \in \mathbb{N}$. Then

$$|\theta_n'(x)| = \frac{1}{((3+n)+x)\ln^2((3+n)+x)} \leq \frac{1}{(3+n)\ln^2(3+n)} \leq \frac{1}{4\ln^2 4} =: c.$$

Let $\Theta = \{\theta_n \colon n \in \mathbb{N}\}$, then Θ is a collection of injective uniform contractions with a factor $c < 1$.

For $n \in \mathbb{N}$, define

$$b_n(x) \coloneqq |\theta_n'(x)| = \frac{1}{((3+n)+x)\ln^2((3+n)+x)}.$$

From Example 2.16, the set $B = \{b_n \colon n \in \mathbb{N}\}$ is a collection of uniform Lipschitz functions with Lipschitz constant at most

$$\frac{1}{16\ln^2 4}\left(1 + \frac{1}{\ln 4}\right) < c < 1.$$

Furthermore, (Θ, B) satisfies assumption (**B1**)-(**B4**), and in particular $\sigma_0 = 1$. Example 2.16 implies that for $s < 1$, the infinite sum $\sum_{n=1}^{\infty} ||b_n||^s$ diverges, and for $s \geq 1$,

$$\sum_{n=1}^{\infty} b^s(x) \leq \frac{1}{\ln 3} < 1.$$

Note that for $u\colon x \mapsto 1$, $x \in [0,1]$ we have that

$$L_s u \leq \frac{1}{\ln 3} u.$$

It follows from Lemma 2.2 that $r(L_s) \leq \frac{1}{\ln 3} < 1$. So, L_s is defined for all $s \geq 1$ and $r(L_s) < 1$ but L_s is not defined for $s < 1$. This shows that there is no s such that $r(L_s) = 1$. In this case $s_{\mathbb{N}} = \sigma_0 = 1$.

We will consider Perron-Frobenius operators,

$$(L_s f)(x) = \sum_{n \in E} b_n^s(x) f(\theta_n(x)),$$

with infinitely many terms in the sum. For those s for which L_s is defined, the operator still has positive eigenvector corresponding to $r(L_s)$. This is a result from [39, Section 5]. For completeness we include the proof.

Let $F_1 \subset F_2 \subset \ldots$ be a collection of subsets of $\mathbb{N}$ such that $|F_k| < \infty$ for each $k \in \mathbb{N}$. Set

$$E = \bigcup_{k=1}^{\infty} F_k. \tag{2.10}$$

For each $k \in \mathbb{N}$, F_k is finite, so the operator L_{s,F_k} is defined for all $s > 0$ and using Lemma 2.13, there exist an $s_{F_k} =: s_k$ such that $r(L_{s_k,F_k}) = 1$. Observe that $L_{s,F_k} \leq L_{s,F_{k+1}}$ so $r(L_{s,F_k}) \leq r(L_{s,F_{k+1}})$. In particular $r(L_{s_k,F_k}) = 1 \leq r(L_{s_k,F_{k+1}})$. The decreasing property of $s \mapsto r(L_{s,F_{k+1}})$ from Lemma 2.13 implies that $s_k \leq s_{k+1}$. Thus, the sequence $(s_k)_{k \in \mathbb{N}}$ is monotonic and bounded from above by $s_{\mathbb{N}}$, where $s_{\mathbb{N}}$ is defined by (2.9). So by monotone convergence, say

$$\lim_{k \to \infty} s_k =: s_E. \tag{2.11}$$

Now for $s \geq s_E$, define

$$(L_{s,E}f)(x) = \sum_{n \in E} b_n^s(x) f(\theta_n(x)). \tag{2.12}$$

Theorem 2.19 ([39]). *For $s \geq s_E$, if $L_{s,E}$ is defined as in (2.12), then $L_{s,E}$ is a bounded linear operator from $C(X)$ into $C(X)$ and there exists an $M > 0$, such that, $L_{s,E}(K(M,\lambda)) \subseteq K(M,\lambda)$. In addition, $L_{s,E}$ has an eigenvector $w_s \in K(M,\lambda)$ with corresponding eigenvalue $\mu_s := r(L_{s,E})$ and $\mu_s = \lim_k r(L_{s,F_k})$. Furthermore, the map $s \mapsto \mu_s$ is strictly decreasing and if s_E is defined by (2.11), then*

$$s_E = \inf\{s > 0 \colon r(L_{s,E}) \leq 1\}.$$

We will prove this result by breaking it up into several lemmas. We are only interested in $\sigma_E \leq s \leq s_{\mathbb{N}}$. Since for each $s > 0$, $b_n^s \in K(s_{\mathbb{N}} M_0, \lambda)$ by assumption (**B4**), we will use the largest cone $K(s_{\mathbb{N}} M_0, \lambda)$. We will also choose an $M > 0$ satisfying $s_{\mathbb{N}} M_0 + Mc^\lambda < M$. It immediately follows that $b_n^s \in K(M,\lambda)$ for all $s \leq s_{\mathbb{N}}$ and n.

Lemma 2.20. *Let $F_1 \subset F_2 \subset \ldots \subset \mathbb{N}$ be such that $|F_k| < \infty$ and $E = \cup_k F_k$. Let s_E be defined as in (2.11) and $s \geq s_E$. For $k \in \mathbb{N}$ and $x \in X$ define*

$$g_k(x) = \sum_{n \in F_k} b_n^s(x).$$

Then $g_k(x) \leq e^{Md^\lambda}$ for all $x \in X$ and all $k \in \mathbb{N}$; that is $\{g_k \colon k \in \mathbb{N}\}$ is uniformly bounded, where $d = \operatorname{diam}(X)$.

Proof. Fix $k \in \mathbb{N}$, so $|F_k| < \infty$. Using Theorem 2.12, the operator L_{s_k,F_k} has an positive eigenvector $v_{s_k,F_k} \in K(M,\lambda) \setminus \{0\}$ with eigenvalue $r(L_{s_k,F_k}) = 1$. So $L_{s_k,F_k} v_{s_k F_k} = v_{s_k F_k}$. We have that for each $n \in F_k$,

$$\frac{v_{s_k,F_k}(\theta_n(x))}{v_{s_k,F_k}(x)} \geq e^{-M\rho(x,\theta_n(x))^\lambda} \geq e^{-Md^\lambda}.$$

Since $0 < b_n < 1$ and $s_k \leq s_E \leq s$, it follows that $b_n^s \leq b_n^{s_k}$ for $n \in F_k$. So for $x \in X$, we have the following

$$1 = \frac{(L_{s_k,F_k} v_{s_k,F_k})(x)}{v_{s_k,F_k}(x)} = \sum_{n \in F_k} b_n^{s_k}(x) \frac{v_{s_k,F_k}(\theta_n(x))}{v_{s_k,F_k}(x)}$$
$$\geq e^{-Md^\lambda} \sum_{n \in F_k} b_n^{s_k}(x)$$
$$\geq e^{-Md^\lambda} \sum_{n \in F_k} b_n^{s}(x).$$

Thus, for each $k \in \mathbb{N}$ we have that

$$g_k(x) = \sum_{n \in F_k} b_n^s(x) \leq e^{Md^\lambda}.$$

which shows that $\{g_k \colon k \in \mathbb{N}\}$ is uniformly bounded. □

As each b_n is positive, $(g_k(x))_{k \in \mathbb{N}}$ is an increasing bounded sequence so it converges pointwise to $g(x) := \sum_{n \in E} b_n^s(x) < \infty$.

Lemma 2.21. *Let $F_1 \subset F_2 \subset \ldots \subset \mathbb{N}$ be such that $|F_k| < \infty$ and $E = \cup_k F_k$. Let s_E be defined as in (2.11) and $s \geq s_E$. For each $k \in \mathbb{N}$ define*

$$g_k(x) = \sum_{n \in F_k} b_n^s(x) \quad \textit{and} \quad g(x) = \sum_{n \in E} b_n^s(x).$$

Then $g_k \to g$ uniformly.

Proof. It suffices to show that $\{g_k \colon k \in \mathbb{N}\}$ is an equicontinuous collection of functions for it will follow that the convergence is uniform. Let $d = \operatorname{diam}(X)$ and fix $x, y \in X$. Since $b_n^s \in K(M, \lambda)$ for $n \in E$, we have that

$$0 < b_n^s(x) \leq e^{M\rho(x,y)^\lambda} b_n^s(y).$$

Thus, for $k \in \mathbb{N}$ fixed, we have

$$0 < \sum_{n \in F_k} b_n^s(x) \leq e^{M\rho(x,y)^\lambda} \sum_{n \in F_k} b_n^s(y).$$

Therefore $g_k \in K(M, \lambda)$. As $\{g_k : k \in \mathbb{N}\}$ is uniformly bounded, it follows from Lemma 2.6 together with the Ascoli-Arzelà Theorem that $g_k \to g$ uniformly. □

Lemma 2.22. *Let $F_1 \subset F_2 \subset \ldots \subset \mathbb{N}$ be such that $|F_k| < \infty$ and $E = \cup_k F_k$. Let s_E be defined by (2.11). For $s \geq s_E$, we have that $L_{s,E} \colon C(X) \to C(X)$ defined by (2.12) is a bounded linear operator, with*

$$\lim_{k \to \infty} ||L_{s,E} - L_{s,F_k}|| = 0. \tag{2.13}$$

Proof. Let $s \geq s_E$ and $f \in C(X)$. Lemma 2.20 implies that

$$(L_{s,E} f)(x) = \sum_{n \in E} b_n^s(x) f(\theta_n(x)) \leq e^{Md^\lambda} ||f||_\infty < \infty.$$

So $L_{s,E}$ is well defined.

Now let $f \in C(X)$ with $||f||_\infty \leq 1$ and consider $x, y \in X$. Then

$$|(L_{s,E} f)(x) - (L_{s,E} f)(y)| = \left| \sum_{n \in E} b_n^s(x) f(\theta_n(x)) - b_n^s(y) f(\theta_n(y)) \right|$$

$$\leq \sum_{n \in E} b_n^s(x) \left| f(\theta_n(x)) - f(\theta_n(y)) \right| + \sum_{n \in E} \left| f(\theta_n(y)) \right| \left| b_n^s(x) - b_n^s(y) \right|.$$

Let $\varepsilon > 0$. Since f is continuous on X, it is uniformly continuous on X, so there exists $\delta_1 > 0$ such that if $\rho(x, y) < \delta_1$, then $|f(x) - f(y)| < \frac{\varepsilon}{2e^{Md^\lambda}}$. As θ_n is a contraction map on X, it follows that $|f(\theta_n(x)) - f(\theta_n(y))| < \frac{\varepsilon}{2e^{Md^\lambda}}$ for each $n \in E$. Thus

$$\sum_{n \in E} b_n^s(x) \left| f(\theta_n(x)) - f(\theta_n(y)) \right| \leq \frac{\varepsilon}{2},$$

by Lemma 2.21, $\left(\sum_{n\in F_k} b_n^s\right)_k$ converges uniformly to a continuous function $g(x) = \sum_{n\in E} b_n^s(x)$, so, there exits a $\delta_2 > 0$ such that if $|x-y| < \delta_2$, then

$$\sum_{n\in E} |f(\theta_n(y))|\,|b_n^s(x) - b_n^s(y)| \leq \sum_{n\in E} |b_n^s(x) - b_n^s(y)| < \frac{\varepsilon}{2}.$$

Let $\delta = \min\{\delta_1, \delta_2\}$. So, if $\rho(x,y) < \delta$, then $|(L_{s,E}f)(x) - (L_{s,E}f)(y)| < \varepsilon$, which shows that $L_{s,E}f \in C(X)$. Thus $L_{s,E}$ is a bounded linear map that maps $C(X)$ into itself.

Also, the uniform convergence of $\left(\sum_{n\in F_k} b_n^s\right)_k$ implies that there exists a sequence $\varepsilon_k \downarrow 0$ such that

$$0 < \sum_{n\in E} b_n^s(x) - \sum_{j\in F_k} b_n^s(x) = \sum_{j\in E\setminus F_k} b_n^s(x) < \varepsilon_k, \quad \text{for all} \;\; x \in X.$$

So for $x \in X$ and $k \in \mathbb{N}$,

$$|(L_{s,E}f)(x) - (L_{t,F_k}f)(x)| \leq ||f||_\infty \sum_{j\in E\setminus F_k} b_j^s(x) < ||f||_\infty \varepsilon_k$$

Thus, $\lim_{k\to\infty} ||L_{s,E} - L_{t,F_k}||_\infty = 0$, which shows that (2.13) holds. □

Note that in general, if $(L_k)_{k\in\mathbb{N}}$ is a sequence of bounded operators on a Banach lattice Y, and L is also a bounded linear operator on Y, such that $||L - L_k|| \to 0$ as $k \to \infty$, then it is not necessarily true that $r(L_k) \to r(L)$. This problem was considered by Kakutani [40, pp. 282-283] who gave an example of a bounded linear map on ℓ^2 whose spectral radius is discontinuous in the C^*-algebra of bounded linear maps on ℓ^2. See also an example in [1, pp 49] for the same result.

In our setup, this is not the case. In fact we have the following result. In the case of infinite conformal IFS, this result is related to [33, Theorem 3.15]. We will discuss this result in Chapter 3.

Theorem 2.23. *Let $F_1 \subset F_2 \subset \ldots \mathbb{N}$ be such that $|F_k| < \infty$ and $E = \cup_k F_k$. Let s_E be defined by (2.11). For $s \geq s_E$, $L_{s,E}$ defined by (2.12) has a strictly positive eigenvector $v_{s,E} =: w_s \in K(M,\lambda) \setminus 0$ with corresponding eigenvalue $r(L_{s,E}) =: \mu_s$ that is,*

$$L_{s,E} w_s = \mu_s w_s. \tag{2.14}$$

The map $s \mapsto \mu_s$ is strictly decreasing and continuous. Also

$$\lim_{k\to\infty} r(L_{s,F_k}) = \mu_s \quad \textit{and} \quad s_E = \inf\{s \geq 0 \colon r(L_{s,E}) \leq 1\}.$$

Proof. It follows directly from Lemma 2.9, as b_n^s satisfies (**A1**) and (**A2**) that $L_{s,E}$ has an eigenvector $w_s \in K(M,\lambda) \setminus \{0\}$ with corresponding eigenvalue $r(L_{s,E}) = \mu_s$. Thus (2.14) holds.

The subsystem (Θ_E, B_E) of (Θ, B) satisfies (**B1**) – (**B4**) so it follows from Theorem 2.17, that $s \to \mu_s$ is a strictly decreasing function.

Now we show that $\lim_{k\to\infty} r(L_{s,F_k}) = \mu_s$.

Note that $w_s \in K(M,\lambda) \setminus \{0\}$ implies that $w_s(x) > 0$ for all $x \in X$ by Remark 2.4. Because $||L_{s,E} - L_{s,F_k}|| \to 0$ as $k \to \infty$ by (2.13), it holds that $||L_{s,E} w_s - L_{s,F_k} w_s||_\infty = ||\mu_s w_s - L_{s,F_k} w_s||_\infty \to 0$ as $k \to \infty$. Since $w_s(x) > 0$ for all $x \in X$, it follows that given $\varepsilon > 0$, there exists a $k_0 \in \mathbb{N}$ such that

$$L_{s,F_k} w_s \geq (1-\varepsilon)\mu_s w_s$$

for all $k \geq k_0$. It then follows from Lemma 2.2 that $r(L_{s,F_k}) \geq (1-\varepsilon)\mu_s$ for $k \geq k_0$. Since this holds for all $\varepsilon > 0$ we have that, $\liminf_k r(L_{s,F_k}) \geq \mu_s$. Since $L_{s,F_k} f \leq L_{s,E} f$ for all $f \geq 0$, we have $||L_{s,F_k}|| \leq ||L_{s,F}||$ for all k, thus, $\limsup_k r(L_{s,F_k}) \leq r(L_{s,E}) = \mu_s$. Hence $\lim_k r(L_{s,F_k}) = r(L_{s,E}) = \mu_s$.

Finally we show that $s_E = \inf\{s \geq 0 \colon r(L_{s,E}) \leq 1\}$. Fix $k \in \mathbb{N}$. As $F_k \subset F_{k+1}$, it follows that for any $f \in C(X)$ with $f \geq 0$ that $L_{s,F_k} f \leq L_{s,F_{k+1}} f$, so that $r(L_{s,F_k}) =: \lambda_{s,F_k} \leq \lambda_{s,F_{k+1}} := r(L_{s,F_{k+1}})$. The map $s \mapsto \lambda_{s,F_k}$ is strictly decreasing by Lemma 2.13, and, since $s_k < s_E$, we have that

$$\lambda_{s_E,F_k} < \lambda_{s_k,F_k} = 1.$$

So, $\lim_k \lambda_{s_E,F_k} = \lim_k r(L_{s_E,F_k}) = \mu_{S_E} \leq 1$.

This implies that $s_E \in \{s \geq 0\colon r(L_{s,E}) \leq 1\}$. It only remains to show that s_E is the infimum of $\{s \geq 0\colon r(L_{s,E}) \leq 1\}$. Let $t = \inf\{s \geq 0\colon r(L_{s,E}) \leq 1\}$ so $t \leq s_E$. Arguing by contradiction suppose $t < s_E$. Then $L_{s,E}$ is defined for $s \in (t, s_E)$. Pick $s^* \in (t, s_E)$ and as $s_k \uparrow s_E$, we have that $s^* < s_k < s_E$ for all k sufficiently large. The map $s \mapsto r(L_{s,F_k})$ is strictly decreasing by Lemma 2.13, this implies that

$$r(L_{s^*,F_k}) > r(L_{s_k,F_k}) = 1$$

for all k sufficiently large. So $r(L_{s^*,E}) \geq r(L_{s^* F_k}) > 1$ for all k sufficiently large. This implies that $s^* \notin \{s > 0 : r(L_{s,E}) \leq 1\}$. This is a contradiction as the map $s \mapsto r(L_{s,E})$ is decreasing. Thus $t = s_E = \inf\{s \geq 0\colon r(L_{s,E}) \leq 1\}$. □

2.4 Dimension Spectrum of Families of Perron-Frobenius Operators

Recall that a subset $C \subseteq \mathbb{R}$ is called *perfect* if every point of C is a limit point, that is, if C' is the set of all limit points of C, then $C = C'$.

Let (Θ, B) be given satisfying (**B1**) - (**B4**). We define the class of operators as

$$\mathcal{L}(\Theta, B) = \{L_{s,F}\colon \quad F \subseteq \mathbb{N} \text{ and } L_{s,F} \text{ is defined}\}.$$

Given $F \subseteq \mathbb{N}$, Let

$$s_F = \inf\{s \geq 0\colon r(L_{s,F}) \leq 1\}$$

Remark 2.24. Note that for $F \subset \mathbb{N}$ finite, Lemma 2.13 implies that $r(L_{t,F}) = 1$ for a unique value t, and the decreasing and continuity of $s \to r(L_{s,F})$ implies $t = s_F$.

Finally define the *dimension spectrum* of (Θ, B) as

$$\mathcal{DS}(\mathcal{L}(\Theta, B)) \coloneqq \{s_F \colon F \subseteq \mathbb{N}\}\,. \tag{2.15}$$

Note that $0 \in \mathcal{DS}(\mathcal{L}(\Theta, B))$, corresponding to a singleton. Indeed, if $F = \{n\}$, then $(L_{s,F}f)(x) = |b_n(x)|^s f(\theta_n(x))$ and $L_{0,F}u = u$, with $u \colon x \mapsto 1$ the order-unit of $C(X)$.

Also $s_{\mathbb{N}} \in \mathcal{DS}(\mathcal{L}(\Theta, B))$, where $s_{\mathbb{N}}$ is defined by (2.9). So,

$$\mathcal{DS}(\mathcal{L}(\Theta, B)) \subseteq [0, s_{\mathbb{N}}].$$

We will establish that $\mathcal{DS}(\mathcal{L}(\Theta, B))$ is a perfect set, hence it is compact as perfect sets are closed.

Definition 2.25. Let (Θ, B) be given, $F \subset \mathbb{N}$ finite and $0 < s < s_{\mathbb{N}}$ where $s_{\mathbb{N}}$ is given by (2.9), we say n is a *break point* for (F, s) if $n > \max F$ and

$$r(L_{s,F}) < 1 \leq r(L_{s,F\cup\{n\}}).$$

Observe that if (Θ, B) and $F \subset \mathbb{N}$ finite is given then for any $n > \max F$, if v_s is the eigenvector of $L_{s,F}$ with eigenvalue $\lambda_s = r(L_{s,F})$ then

$$(L_{s,F\cup\{n\}}v_s)(x) = \lambda_s v_s + b_n^s(x) v_s(\theta_n(x)) \leq (\lambda_s + ||b_n||^s e^{Md^\lambda}) v_s(x).$$

As $v_s > 0$ it follows from Lemma 2.2 that $r(L_{s,F\cup\{n\}}) \leq \lambda_s + ||b_n||^s e^{Md^\lambda}$, where $d = \operatorname{diam}(X)$. Clearly if $\lambda_s < 1$, then $\lambda_s + ||b_n||^s e^{Md^\lambda} < 1$ for sufficiently large n as $\lim_n ||b_n|| = 0$ by (**B2**). So if (F, s) has a break point, then there exists a largest break point $n_0 \in \mathbb{N}$, which is called the *strict break point* for (F, s). So

$$r(L_{s,F\cup\{n_0\}}) \geq 1 \text{ and } r(L_{s,F\cup\{n_0+1\}}) < 1.$$

Remark 2.26. Suppose (Θ, B) is given and $0 < s < s_{\mathbb{N}}$ where $s_{\mathbb{N}}$ is defined by (2.9). Then there exists $n_1 \geq 1$ such that $F_1 = \{1, 2, \ldots, n_1\}$ satisfies

$$r(L_{s,F_1}) < 1 \quad\text{and}\quad r(L_{s,F_1\cup\{n_1+1\}}) \geq 1.$$

Indeed, if this is not the case, then for each $n \in \mathbb{N}$, we have that $r(L_{s,\{1,\ldots,n\}}) \leq 1$ for all $n \in \mathbb{N}$ which would imply that $r(L_s) \leq 1$ by Theorem 2.23, contradicting the fact that $s_{\mathbb{N}}$ is the infimum.

This remark tells us that if (Θ, B) is given and $E = \{n_1, n_2, \ldots\} \subseteq \mathbb{N}$ infinite, if we restrict (Θ, B) to (Θ_E, B_E), and if $s_E = \inf\{s \geq 0 \colon r(L_{s,E}) \leq 1\}$ then for any $s < s_E$, by taking the first initial part of E, we know that there exists a n_k such that

$$r(L_{s,\{n_1,\ldots,n_k\}}) \geq 1.$$

We give a characterisation for $s \in (0, s_{\mathbb{N}})$ to be in $\mathcal{DS}(\mathcal{L}(\Theta, B))$. This result is similar to the one given by Kesseböhmer and Zhu in [30, Theorem 2.2] for the dimension spectrum of conformal maps which was used to settle the Texan conjecture. In [7] in the study of the dimension spectrum of some infinite IFS, the same result was used. This result will play a crucial role in Chapter 4.

Lemma 2.27. *Let (Θ, B) be given satisfying (**B1**) − (**B4**) and $0 < s < s_{\mathbb{N}}$, where $s_{\mathbb{N}}$ is given by (2.9). If for each $F \subset \mathbb{N}$ finite with strict break point $n_0 \in \mathbb{N}$ for (F, s) there exists an $M > n_0$ such that $r(L_{s,F\cup T_M}) > 1$, where $T_M = \{n \in \mathbb{N} \colon n_0 < n \leq M\}$, then $s \in \mathcal{DS}(\mathcal{L}(\Theta, B))$.*

Proof. As $0 < s < s_{\mathbb{N}}$, Remark 2.26 implies that there exists a $n_1 \geq 1$ such that $F_1 = \{1, 2, \ldots, n_1\}$ satisfies

$$r(L_{s,F_1}) < 1 \quad \text{and} \quad r(L_{s,F_1\cup\{n_1+1\}}) \geq 1.$$

Now, let $m_1 \geq n_1 + 1$ be a strict breakpoint for (F_1, s). If $r(L_{s,F_1\cup\{m_1\}}) = 1$, then we are done, because $s = s_{F_1\cup\{m_1\}}$ by Remark 2.24. Otherwise, it follows from assumption that there exists an $M_1 > m_1$ such that $r(L_{s,F_1\cup T_{M_1}}) > 1$, where $T_{M_1} = \{n \in \mathbb{N}\colon m_1 < n \leq M_1\}$. In this case we can use Remark 2.26 again to find an $n_2 > m_1$ such that $F_2 = F_1 \cup \{m_1 + 1, \ldots, n_2\}$ satisfies

$$r(L_{s,F_2}) < 1 \text{ and } r(L_{s,F_2\cup\{n_2+1\}}) \geq 1.$$

Now, let $m_2 \geq n_2 + 1$ be a strict breakpoint for (F_2, s). If $r(L_{s,F_2\cup\{m_2\}}) = 1$ then, we are done. Otherwise, we know there exists $M_2 > m_2$ such that $r(L_{s,F_2\cup T_{M_2}}) > 1$ where $T_{M_2} = \{n \in \mathbb{N}\colon m_2 < n \leq M_2\}$ by assumption.

Repeating this process, we either find, after finitely many steps, a subset F_n of $\mathbb{N}$ and a strict break point m_n for (F_n, s) with $r(L_{s,F_n\cup\{m_n\}}) = 1$, or, we get a sequence $F_1 \subset F_2 \subset F_3 \subset \ldots \subset \mathbb{N}$ and indices $m_1 < m_2 < \ldots < m_n < \ldots$ such that m_n is a strict break point for (F_n, s) and $r(L_{s,F_n\cup\{m_n\}}) > 1$ for each $n \in \mathbb{N}$.

Let σ_n be the unique value satisfying $r(L_{\sigma_n,F_n}) = 1$. Then $\sigma_n < s$ for all $n \geq 1$. As $1 = r(L_{\sigma_n,F_n}) \leq r(L_{\sigma_n,F_{n+1}})$, the sequence (σ_n) is increasing so by monotonic convergence (σ_n) converges say σ, and necessarily $\sigma \leq s$. If $\sigma = s$, then we are done by Theorem 2.23, because if we take $E = \cup_n F_n$, then $\sigma = \inf\{s \geq 0\colon r(L_{s,E}) \leq 1\}$. Thus, to complete the proof it suffices to show that $\sigma = s$. Arguing by contradiction, assume that $\sigma < s$.

For $n \geq 1$, let $G_n = F_n \cup \{m_n\}$, so $r(L_{s,G_n}) > 1$. By assumption (**B1**), the maps b_k satisfies

$$0 < b_k(x) \leq \eta_k \leq \eta < 1.$$

Let $v_{\sigma_n} \in C(X)$ be the strictly positive eigenvector of L_{σ_n, F_n}. So, $L_{\sigma_n, F_n} v_{\sigma_n} = v_{\sigma_n}$. Then

$$\begin{aligned}(L_{s,F_n} v_{\sigma_n})(x) &= \sum_{k \in F_n} b_k(x)^s v_{\sigma_n}(\theta_k(x)) \\ &= \sum_{k \in F_n} b_k(x)^{s-\sigma_n} b_k(x)^{\sigma_n} v_{\sigma_n}(\theta_k(x)) \\ &\leq \sum_{k \in F_n} \eta^{s-\sigma_n} b_k(x)^{\sigma_n} v_{\sigma_n}(\theta_k(x)) \\ &\leq \sum_{k \in F_n} \eta^{s-\sigma} b_k(x)^{\sigma_n} v_{\sigma_n}(\theta_k(x)) \\ &= \eta^{s-\sigma} v_{\sigma_n}(x),\end{aligned}$$

so that

$$r(L_{s,F_n}) \leq \eta^{s-\sigma}.$$

Let $w_s \in C(X)$ be a strictly positive eigenvector of L_{s,F_n} such that $L_{s,F_n} w_s = r(L_{s,F_n}) w_s$. It then follows that

$$\begin{aligned}(L_{s,G_n} w_s)(x) &= (L_{s,F_n} w_s)(x) + b_{m_n}^s(x) w_s(\theta_{m_n}(x)) \\ &\leq \eta^{s-\sigma} w_s(x) + ||b_{m_n}||_\infty^s e^{Md^\lambda} w_s(x).\end{aligned}$$

Thus $r(L_{s,G_n}) \leq \eta^{s-\sigma} + ||b_{m_n}||_\infty^s e^{Md^\lambda}$, where $d = \operatorname{diam}(X)$. We know that $r(L_{s,G_n}) > 1$, which gives

$$1 < r(L_{s,G_n}) \leq \eta^{s-\sigma} + ||b_{m_n}||_\infty^s e^{Md^\lambda}$$

for all $n \geq 1$. This is impossible, since $||b_{m_n}||_\infty \to 0$ as $n \to \infty$, $s - \sigma > 0$ and $\eta < 1$. Thus $s = \sigma$, which completes the proof. □

As a consequence of this result we have the following result, which is related to [8, Theorem 4.9] concerning infinite conformal graph directed systems. Note that the parameter σ_0 defined in (2.6) is called the finiteness parameter in the general conformal IFS.

Lemma 2.28. *Let* (Θ, B) *be given satisfying* $(\boldsymbol{B1}) - (\boldsymbol{B4})$*. If for* $0 < s_* < s_{\mathbb{N}}$*, where* $s_{\mathbb{N}}$ *is defined by* (2.9)*, we have that*

$$\sum_{n=1}^{\infty} ||b_n||_{\infty}^{s_*} = \infty, \text{ then } s_* \in \mathcal{DS}(\mathcal{L}(\Theta, B)).$$

In particular, if $\sigma_0 > 0$ *where* σ_0 *is defined by* (2.6)*, then*

$$[0, \sigma_0) \subseteq \mathcal{DS}(\mathcal{L}(\Theta, B)).$$

Proof. Assume $\sum_{n\in\mathbb{N}} ||b_n||_{\infty}^{s_*} = \infty$. Let $F \subset \mathbb{N}$ finite and $k_0 > \max F$ be a strict breakpoint for (s_*, F). Let $H_m = F \cup \{k_0 + 1, k_0 + 2, \ldots, , k_0 + m\}$. Then the operator $L_{s_*, F\cup\{k_0\}}$ has eigenvector v_{s_*} with spectral radius $\lambda_{s_*} := r(L_{s_*, F\cup\{k_0\}}) \geq 1$. Note that $v_{s_*} \in K(M, \lambda)$, so for any $j \in \mathbb{N}$ we have

$$\frac{v_{s_*}(\theta_{k_0+j}(x))}{v_{s_*}(\theta_{k_0}(x))} \geq e^{-Md^{\lambda}}.$$

Also $b_{k_0+j}^{s_*} \in K(M, \lambda)$, so

$$\frac{b_{k_0+j}^{s_*}(x)}{b_{k_0}^{s_*}(x)} \geq e^{-Md^{\lambda}} \frac{||b_{k_0+j}||_{\infty}^{s_*}}{b_{k_0}^{s_*}(x)} \geq e^{-Md^{\lambda}} \frac{||b_{k_0+j}||_{\infty}^{s_*}}{||b_{k_0}||_{\infty}^{s_*}},$$

hence

$$\frac{b_{k_0+j}^{s_*}(x)}{b_{k_0}^{s_*}(x)} \frac{v_{s_*}(\theta_{k_0+j}(x))}{v_{s_*}(\theta_{k_0}(x))} \geq e^{-2Md^{\lambda}} \frac{||b_{k_0+j}||_{\infty}^{s_*}}{||b_{k_0}||_{\infty}^{s_*}}.$$

Using this inequality, we have the following,

$$\begin{aligned}(L_{s_*,H_m}v_{s_*})(x) &= (L_{s_*,F}v_{s_*})(x) + \sum_{j=1}^{m} b_{k_0+j}^{s_*}(x)_{s_*}^{v}(\theta_{k_0+j}(x)) \\ &= (L_{s_*,F}v_{s_*})(x) + b_{k_0}^{s_*}(x)v_{s_*}(\theta_{k_0}(x)) \sum_{j=1}^{m} \frac{b_{k_0+j}^{s_*}(x)}{b_{k_0}^{s_*}(x)} \frac{v_{s_*}(\theta_{k_0+j}(x))}{v_{s_*}(\theta_{k_0}(x))} \\ &\geq (L_{s_*,F}v_{s_*})(x) + b_{k_0}^{s_*}(x)v_{s_*}(\theta_{k_0}(x)) \frac{e^{-2Md^\lambda}}{||b_{k_0}||_\infty^{s_*}} \sum_{j=1}^{m} ||b_{k_0+j}||_\infty^{s_*}.\end{aligned}$$

As $\sum_{n\in\mathbb{N}} ||b_n||_\infty^{s_*} = \infty$, for all m sufficiently large, there exists a constant $\rho > 1$ such that

$$(L_{s_*,H_m}v_{s_*})(x) \geq (L_{s_*,F}v_{s_*})(x) + \rho b_{k_0}^{s_*}(x)v_{s_*}(\theta_{k_0}(x))$$

for all m large. This implies that there exists a constant $\mu > 1$ by Lemma 2.1 such that

$$(L_{s_*,H_m}v_{s_*})(x) \geq \mu\left((L_{s_*,F}v_{s_*})(x) + b_{k_0}^{s_*}(x)v_{s_*}((\theta_{k_0}(x))\right) = \mu\lambda_{s_*}v_{s_*}(x) \geq \mu v_{s_*}(x)$$

for all m large, hence $r(L_{s,H_m}) > 1$ for all m large. Take $M > 0$ large enough and set $T_M = \{n \in \mathbb{N} \colon k_0 < n \leq M\}$. Then $H_M = F \cup T_M$, so we have that $r(L_{s,F\cup T_M}) > 1$. The result now follows from Lemma 2.27. □

As we have seen in Example 2.15, σ_0 can be zero. This means that we cannot Lemma 2.28 to determine whether the dimension spectrum contains an interval. On the other hand, if $\sigma > 0$ then we can use Lemma 2.15 to conclude that $\mathcal{DS}(\mathcal{L}(\Theta, B))$ contains an interval. In Example 2.16, we have that $\sigma_0 = s_\mathbb{N} = 1$, so $[0,1) \subseteq \mathcal{DS}(\mathcal{L}(\Theta, B))$ in that case. In fact $\mathcal{DS}(\mathcal{L}(\Theta, B)) = [0,1]$ as $s_\mathbb{N} \in \mathcal{DS}(\mathcal{L}(\Theta, B))$.

The following result establishes the closedness of $\mathcal{DS}(\mathcal{L}(\Theta, B))$. In the context of infinite conformal IFS, this result is related to [8, Theorem 1.2].

Lemma 2.29. *Let* (Θ, B) *be given satisfying* (***B1***) – (***B4***)*. Then* $\mathcal{DS}(\mathcal{L}(\Theta, B))$ *is a closed set.*

Proof. Let $s_{\mathbb{N}}$ be defined as in (2.9) and let $0 < s_* < s_{\mathbb{N}}$ be an accumulation point of $\mathcal{DS}(\mathcal{L}(\Theta, B))$. Then there exists a sequence (F_k) of sets with $F_k \subseteq \mathbb{N}$ and $|F_k| < \infty$ for all k such that $r(L_{s_k,F_k}) = 1$, satisfies $s_k \to s_*$ and $s_k \neq s_*$ for all k. There are two cases to consider.

(i) $\sum_{n\in\mathbb{N}} ||b_n||_\infty^{s_*} < \infty$ and

(ii) $\sum_{n\in\mathbb{N}} ||b_n||_\infty^{s_*} = \infty$.

Note that if we are in the second case where $\sum_{n\in\mathbb{N}} ||b_n||_\infty^{s} = \infty$. Then it follows from Lemma 2.28 that $s_* \in \mathcal{DS}(\mathcal{L}(\Theta, B))$.

In the first case, we may assume, without loss of generality, that we can split into two subcases:

(a) $s_k \downarrow s_*$ and

(b) $s_k \uparrow s_*$.

Suppose we are in case (a). Then $s_k > s_* \geq \sigma_0$ for all $k \in \mathbb{N}$, which implies that $r(L_{s_*,F_k}) > 1$ for all $k \in \mathbb{N}$.

Claim 1: There exists $n_1 \in \mathbb{N}$ such that $n_1 \in F_k$ for infinitely many k.

Indeed, if this is not the case, then for each $n \in \mathbb{N}$ there exists a $K_n \geq 1$ such that $n \notin F_k$ for all $k \geq K_n$. Let $M_n = \max\{K_1, K_2, \ldots, K_n\}$. Then for all $k \geq M_n$, $F_k \subseteq T_{n+1} \coloneqq \{n+1, n+2, \ldots\}$. Hence for any positive function $f \in C(X)$ we have that

$$L_{s_*,F_k} f \leq L_{s_*,T_{n+1}} f,$$

so $r(L_{s_*,F_k}) \leq r(L_{s_*,T_{n+1}})$ for all $k \geq M_n$. For $u\colon x \mapsto 1$, we have that

$$(L_{s_*,T_{n+1}} u)(x) \leq \sum_{r\geq n+1} ||b_r||_\infty^{s_*} \to 0, \text{ as } n \to \infty.$$

Thus, $r(L_{s_*,F_k}) \leq r(L_{s_*,T_{n+1}}) < 1$ for all n large and $k \geq M_n$, a contradiction, since $r(L_{s_*,F_k}) \geq r(L_{s_k,F_k}) = 1$.

Take n_1 to be the smallest element which appears infinitely often and set $G_1 = \{n_1\}$. By taking a subsequence of (F_k) we may assume that $\min F_k = n_1$ for all $k \in \mathbb{N}$.

Claim 2: For this subsequence, there exists $n_2 > n_1$ such that $n_2 \in F_k$ for infinitely many k.

Indeed, if not, then for all $n_2 \in \mathbb{N}$ there exists K_{n_2} such that $n_2 \notin F_k$ for all $k \geq K_{n_2}$. Again take $M_n = \max\{K_{n_1+1}, \dots K_{n_1+n}\}$. Then for all $k \geq M_n$,

$$F_k \subseteq G_1 \cup T_{n_1+n+1} = G_1 \cup \{n_1 + n + 1, n_1 + n + 2, \dots\}.$$

So, $r(L_{s_*,F_k}) \leq r(L_{s_*,G_1 \cup T_{n_1+n+1}})$. Note that $r(L_{s_k,G_1}) \leq r(L_{s_k,F_k}) = 1$ for all k. Also, $r(L_{s_k,G_1}) \to r(L_{s_*,G_1})$, as the map $s \mapsto r(L_{s,G_1})$ is continuous by Lemma 2.13. So, $r(L_{s_*,G_1}) \leq 1$. If $r(L_{s_*,G_1}) = 1$, then we are done. So, suppose $r(L_{s_*,G_1}) < 1$. Now consider $v^1_{s_*}$ the strictly positive eigenvector of L_{s_*,G_1} with eigenvalue $\lambda_{s_*} < 1$. Then there exists a $\lambda < 1$ such that

$$(L_{s_*,G_1 \cup T_{n_1+n+1}} v^1_{s_*})(x) \leq \lambda_{s_*} v^1_{s_*}(x) + \sum_{r \geq n_1+n+1} ||b_r||^{s_*}_\infty < \lambda v^1_{s_*}(x),$$

for all n large. This implies that $r(L_{s_*,F_k}) \leq r(L_{s_*,G_1 \cup T_{n_1+n+1}}) \leq \lambda < 1$ for all $k \geq M_n$ and all n large, a contradiction.

Take $n_2 > n_1$ to be the smallest such that $n_2 \in F_k$ for infinitely many k. Set $G_2 = \{n_1, n_2\}$. Taking a further subsequence (F_k) we may assume that each F_k satisfies $\min F_k \backslash \{n_1\} = n_2$. Now using the eigenvector $v^2_{s_*}$ of L_{s_*,G_2} with eigenvalue $\mu_{s_*} < 1$ we can argue in the same way that there exists a smallest $n_3 > n_2$ such that $n_3 \in F_k$ for infinitely many k.

Thus, either after finitely many repetitions, we find a $G_N = \{n_1, n_2, \dots, n_N\}$ with $r(L_{s_*,G_N}) = 1$ or we obtain a sequence (G_p) with

$$G_p \subseteq G_{p+1} \text{ and } r(L_{s_*,G_p}) < 1 \text{ for all } p \in \mathbb{N}.$$

Set

$$G = \cup_{p\in\mathbb{N}} G_p.$$

Claim 3: $s_* = \inf\{s \geq 0 \colon r(L_{s,G}) \leq 1\}$.

Since G_p is finite for each p, there exists a unique t_p such that $r(L_{t_p,G_p}) = 1$. Set $t = \lim_{p\to\infty} t_p$ and note that $t_p \leq s_*$ for all p, as the map $s \mapsto r(L_{s,G_p})$ is decreasing and continuous by Lemma 2.13, so, $t \leq s_*$. By Theorem 2.23, $L_{t,G}$ is defined and $t = \inf\{s \geq 0 \colon r(L_{s,G}) \leq 1\}$. Hence, it suffices to show that $t = s_*$.

Suppose $t < s_*$. For each $p \geq 1$ there exists F_{m_p} with $m_p < m_{p+1}$ such that

$$G_p \subseteq F_{m_p} \subseteq G_p \cup \{n \colon n > n_p\}.$$

Let $t < \tilde{s} < s_*$. As $t < \tilde{s}$, it follows from Theorem 2.23 that $r(L_{\tilde{s},G}) = \rho < 1$. Let $w_{\tilde{s}}$ be the eigenvector of $L_{\tilde{s},G}$ with corresponding eigenvalue ρ. Then there exists a $\mu < 1$ such that

$$\begin{aligned}
(L_{\tilde{s},G_p\cup T_{n_p+1}} w_{\tilde{s}})(x) &\leq (L_{\tilde{s},G} w_{\tilde{s}})(x) + \sum_{r\geq n_p+1} ||b_r||_\infty^{\tilde{s}} \\
&= \rho w_{\tilde{s}}(x) + \sum_{r\geq n_p+1} ||b_r||_\infty^{\tilde{s}} \\
&\leq \mu w_{\tilde{s}}(x)
\end{aligned}$$

for all p large. Thus $r(L_{\tilde{s},G_p\cup T_{n_p+1}}) < 1$. So for p large, $r(L_{\tilde{s},F_{m_p}}) < 1$. However, for $s_{m_p} > \tilde{s}$, we have that $r(L_{\tilde{s},F_{m_p}}) \geq r(L_{s_{m_p},F_{m_p}}) = 1$, a contraction.

Case (b): $s_k \uparrow s_*$.

In this case, $s_k < s_*$ and as the map $s \mapsto r(L_{s,F_k})$ is strictly decreasing by Lemma 2.13, it follows that $r(L_{s_*,F_k}) < 1$ for all $k \in \mathbb{N}$.

Claim 4: There exists an n_1 such that $n_1 \notin F_k$ for infinitely many k.

Indeed, if for each $m \geq 1$, there exists $K_m \geq 1$ such that $m \in F_k$ for all $k \geq K_m$ then we take $M_n^1 = \max\{K_1, \ldots, K_n\}$ and get that $F_k \supseteq \{1, \ldots, n\} =: A_n^1$ for all $k \geq M_n^1$. If σ_n is such that $r(L_{\sigma_n,A_n^1}) = 1$ which exists by Lemma 2.13

because A_n^1 is finite, then $\sigma_n \to s_{\mathbb{N}}$ where $s_{\mathbb{N}}$ is given by (2.9). As $s_* < s_{\mathbb{N}}$, this implies that for large n, $s_* < \sigma_n$ so that $r(L_{s_*,F_k}) \geq r(L_{\sigma_n,F_k}) \geq r(L_{\sigma_n,A_n^1}) = 1$ for $k \geq M_n^1$ and n sufficiently large, which is impossible.

Let $n_1 \in \mathbb{N}$ be the smallest such element of $\mathbb{N}$. Set $H_1 = \mathbb{N} \setminus \{n_1\}$. Taking a further subsequence, we may assume that $n_1 \notin F_k$ and $A^1 \colon = \{1, 2, ..., n_1 - 1\} \subseteq F_k$ for all $k \in \mathbb{N}$. Note that $r(L_{s_*,H_1}) \geq 1$ as each $F_k \subseteq H_1$ and $1 = r(L_{s_k,F_k}) \leq r(L_{s_k,H_1})$ and $r(L_{s_k,H_1}) \to r(L_{s_*,H_1})$ by Theorem 2.23. If $s_* = \inf\{s \geq 0 \colon r(L_{s,H_1}) \leq 1\}$, then we are done. Otherwise $r(L_{s_*,H_1}) > 1$.

Claim 5 : There exists $n_2 > n_1$ such that $n_2 \notin F_k$ for infinitely many k.

Indeed, if for all $m > n_1$ there exists $K_m \geq 1$ such that $n \in F_k$ for all $k \geq K_m$ then we can let $M_n^2 = \max\{K_{n_1+1}, \ldots, K_{n_1+n}\}$ and get that $F_k \supseteq \{1, \ldots, n_1 - 1\} \cup \{n_1 + 1, \ldots, n_1 + n\} =: A_n^2$ for all $k \geq M_n^2$. If σ_n is such that $r(L_{\sigma_n,A_n^2}) = 1$, which exists by Lemma 2.13 because A_n^2 is finite, then $\sigma_n \to s_{H_1}$ and $s_{H_1} = \inf\{s \geq 0 \colon r(L_{s,H_1}) \leq 1\}$ by Theorem 2.23, and $s_{H_1} > s_*$, since $r(L_{s_*,H_1}) > 1$. This implies that $\sigma_n > s_*$ for sufficiently large n. But $\sigma_n < s_k < s_*$ for all $k \geq M_n^2$ and n sufficiently large which is impossible.

Let $n_2 > n_1$ be the smallest such element and let $H_2 = \mathbb{N} \setminus \{n_1, n_2\}$. Taking subsequences, we may assume for all $k \in \mathbb{N}$ that $n_2 \notin F_k$ and $A^2 := \{1, \ldots, n_1 - 1\} \cup \{n_1 + 1, \ldots, n_2 - 1\} \subseteq F_k$. Note that $r(L_{s,H_2}) \geq 1$ as $F_k \subseteq H_2$ and $1 = r(L_{s_k,F_k}) \leq r(L_{s_k,H_2})$ and $r(L_{s_k,H_2}) \to r(L_{s_*,H_2})$ by Theorem 2.23. If $s_* = \inf\{s > 0 \colon r(L_{s,H_2}) \leq 1\}$, then we done. Otherwise $r(L_{s_*,H_2}) > 1$. So assume $r(L_{s_*,H_2}) > 1$.

Repeating this process we either find after finitely many iterations, a set $H_p = \mathbb{N} \setminus \{n_1, n_2, \ldots, n_p\}$ such that $s_* = \inf\{s \geq 0 \colon r(L_{s,H_p}) \leq 1\}$ or we have a sequence of sets $(H_p)_{p \in \mathbb{N}}$ with $H_p \supseteq H_{p+1}$ and $r(L_{s_*,H_p}) > 1$ for all $p \in \mathbb{N}$. Moreover for $p \geq 2$, there exists a $k_p > k_{p-1}$ such that

$$F_{k_p} \supseteq \{1, 2, \ldots, n_p\} \setminus \{n_1, n_2, \ldots, n_p\} =: A^p.$$

In the latter case, let $H = \cap_{p\in\mathbb{N}} H_p$. Note that $H \neq \emptyset$. Indeed if $H = \emptyset$, then for each $n \in \mathbb{N}$ there exists a P_n such that $n \notin H_p$ for all $p \geq P_n$. Let $M_n = \max\{P_1, P_2, \ldots, P_n\}$. Then for all $p \geq P_n$, $H_p \subseteq T_{n+1} := \{n+1, n+2, \ldots\}$. So $r(L_{s_*,H_p}) \leq r(L_{s_*,T_{n+1}})$, but

$$(L_{s_*,T_{n+1}}u)(x) \leq \sum_{r\geq n+1} ||b_r||_\infty^{s_*} \to 0 \quad \text{as} \quad n \to \infty.$$

Thus $r(L_{s_*,T_{n+1}}) < 1$. So, $r(L_{s_*,H_p}) < 1$ for $p \geq P_n$ and n sufficiently large. A contradiction since $r(L_{s_*,H_p}) > 1$. Thus $H \neq \emptyset$. Let t be the unique value such that $t = \inf\{s \geq 0\colon r(L_{s,H}) \leq 1\}$.

Claim 6 : $t = s_*$.

We first show that $t \geq s_*$. Suppose $t < s_*$. Let w_{s_*} be the normalised eigenvector of $L_{s_*,H}$ with $||w_{s_*}||_\infty = 1$ which exists by Theorem 2.23, with eigenvalue $\tau_{s_*} < 1$ because the map $s \mapsto r(L_{s,H})$ is decreasing. Then there exists a sequence $\varepsilon_p \downarrow 0$ such that

$$(L_{s_*,H_p} w_{s_*})(x) \leq (L_{s_*,H} w_{s_*})(x) + \sum_{j\geq p} ||b_j||_\infty^{s_*} \leq \tau_{s_*} w_{s_*}(x) + \varepsilon_p.$$

As $\varepsilon_p \downarrow 0$, there exists $\tau < 1$ such that $L_{s_*,H_p} w_s \leq \tau w_{s_*}$ for all p sufficiently large. But this implies that $r(L_{s_*,H_p}) \leq \tau < 1$ for all p sufficiently large, a contradiction since $r(L_{s_*,H_p}) > 1$ for all p, thus $t \geq s_*$.

Now suppose that $t > s_*$. Then let $\hat{w}_{s_*}$ be the normalised eigenvector of $L_{s_*,H}$ with eigenvalue $\hat{\tau}_{s_*} > 1$ such that $L_{s_*,H}\hat{w}_{s_*} = \hat{\tau}_{s_*}\hat{w}_{s_*}$. Note that for each

$p \in \mathbb{N}$, there exists $k_p \in \mathbb{N}$ such that $A^p \subseteq F_{k_p}$ so that

$$\begin{aligned}(L_{s_*,F_{k_p}}\hat{w}_{s_*})(x) &\geq (L_{s_*,A^p}\hat{w}_{s_*})(x)\\ &= \sum_{k\in A^p} b_k^{s_*}(x)\hat{w}_{s_*}(\theta_k(x))\\ &= \sum_{k\in H_p} b_k^{s_*}(x)\hat{w}_{s_*}(\theta_k(x)) - \sum_{j>n_p} b_j^{s_*}(x)\hat{w}_{s_*}(\theta_j(x))\\ &\geq \hat{\tau}_{s_*}\hat{w}_{s_*}(x) - \sum_{j>n_p} ||b_j||_\infty^{s_*}.\end{aligned}$$

It follows that there exists $\hat{\tau} > 1$ such that

$$L_{s_*,F_{k_p}}\hat{w}_{s_*} \geq \hat{\tau}\hat{w}_{s_*},$$

for all p large. This implies that $r(L_{s_*,F_{k_p}}) > 1$ for all p large, contradicting $s_{k_p} \leq s_*$. Thus $t = s_*$ so that $s_* = \inf\{s \geq 0\colon r(L_{s,H}) \leq 1\}$.
Hence $\mathcal{DS}(\mathcal{L}(\Theta, B))$ is a closed set. □

Finally we establish the perfectness of $\mathcal{DS}(\mathcal{L}(\Theta, B))$.

Theorem 2.30. *Let (Θ, B) be given satisfying $(\boldsymbol{B1})$–$(\boldsymbol{B4})$. Then $\mathcal{DS}(\mathcal{L}(\Theta, B))$ is a perfect set.*

Proof. Note that $\mathcal{DS}(\mathcal{L}(\Theta, B))$ is close by Lemma 2.29, so it suffices to show that $\mathcal{DS}(\mathcal{L}(\Theta, B))$ has no isolated points. Let $0 < s_* < s_{\mathbb{N}}$ where $s_{\mathbb{N}}$ is defined by (2.9) and assume that $s_* \in \mathcal{DS}(\mathcal{L}(\Theta, B))$. It follows that there exists an $F \subseteq \mathbb{N}$ such that $s_* = \inf\{s > 0\colon r(L_{s,F}) \leq 1\}$. There are two cases to consider namely $|F| = \infty$ or $|F| < \infty$. If $|F| = \infty$ then by taking F_k to be the first k elements of F we have that $F_k \subset F_{k+1}$ and $F = \cup_{k\in\mathbb{N}}F_k$. Let s_k be the unique number such that $r(L_{s_k,F_k}) = 1$ which exists by Lemma 2.13. Then $s_k < s_{k+1}$ and $\lim_k s_k = s_*$ by Theorem 2.23.

On the other hand, if $|F| < \infty$, then we know that $r(L_{s_*,F}) = 1$ by Lemma 2.13. For $k \geq \max F$, let $F_k = F \cup \{k\}$ and let s_k be a unique number such that $r(L_{s_k,F_k}) = 1$. As $F \subset F_k$, we know $s_* \leq s_k$ and also $s_{k+1} \leq s_k$.

Claim: $\lim_k s_k = s_*$.

Note that by the Monotonic Convergence Theorem, (s_k) converges, say $s_k \to \sigma$, and we have $s_* \leq \sigma$. To establish equality assume that $s_* < \sigma$ and let $v_{\sigma,F}$ be the eigenvector of $L_{\sigma,F}$ with eigenvalue $\lambda_\sigma < 1$, then

$$(L_{\sigma,F_k} v_{\sigma,F})(x) = (L_{\sigma,F} v_{\sigma,F})(x) + b_k^\sigma(x) v_{\sigma,F}(\theta_k(x)) \leq (\lambda_\sigma + ||b_k||_\infty^\sigma e^{2Md^\lambda}) v_{\sigma,F}(x),$$

where $d = \mathrm{diam}(X)$. Hence

$$r(L_{\sigma,F_k}) \leq \lambda_\sigma + ||b_k||_\infty^\sigma e^{2Md^\lambda}.$$

Since $\lambda_\sigma < 1$ and $\lim_k ||b_k||_\infty = 0$ by assumption (**B3**), we have that $r(L_{\sigma,F_k}) < 1$ for all sufficiently large k. However $\sigma \leq s_k$ for all k, and since the map $s \mapsto r(L_{s,F_k})$ is strictly decreasing we have that $r(L_{\sigma,F_k}) \geq r(L_{s_k,F_k}) = 1$, a contradiction.

This implies that for any $\varepsilon > 0$, $(s_* - \varepsilon, s_* + \varepsilon) \cap \mathcal{DS}(\mathcal{L}(\Theta, B)) \neq \emptyset$. □

In [29], Jurga gave an example of an infinite non-conformal IFS whose dimension spectrum is not compact. It is interesting to compare this example with the result presented here. Jurga's example in [29, Section 5.1] is an affine IFS, defined as follows.

Let $3 < \beta < \gamma$ and $\frac{1}{2} < b < d < 1$. Set

$$A_1^{\beta,\gamma} = A_2^{\beta,\gamma} = A_3^{\beta,\gamma} = \begin{pmatrix} 1 & -1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\beta} & 0 \\ 0 & \frac{1}{\gamma} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 1 \\ -\frac{1}{2} & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \frac{1}{2}\left(\frac{1}{\beta} + \frac{1}{\gamma}\right) & \frac{1}{\beta} - \frac{1}{\gamma} \\ \frac{1}{4}\left(\frac{1}{\beta} - \frac{1}{\gamma}\right) & \frac{1}{2}\left(\frac{1}{\beta} + \frac{1}{\gamma}\right) \end{pmatrix}$$

and

$$A_n^{\beta,\gamma} = \begin{pmatrix} \frac{1}{\beta^n} & \frac{b}{\gamma^n} \\ \frac{1}{\beta^n} & \frac{d}{\gamma^n} \end{pmatrix} \quad n \geq 5.$$

Consider $X = [0,1]^2$ and $\Theta = \{\theta_n \colon X \to X \colon n \in \mathbb{N} \setminus \{4\}\}$ where $\theta_n(x) = A_n(x)$. Suppose we take $b_n(x) = ||\theta'(x)|| = ||A_n|| = \eta_n$.

Then Θ is an IFS and $B = \{b_n \colon n \in \mathbb{N} \setminus \{4\}\}$ satisfies the assumptions (**B1**)-(**B4**). It follows from Theorem 2.30 that $\mathcal{DS}(\mathcal{L}(\Theta, B))$ is a compact and perfect set.

On the other hand, we know from [29, Section 5] that the (Hausdorff) dimension spectrum $\mathcal{DS}(\Theta)$ is not compact. As the IFS is non-conformal, the maps θ_n are not infinitesimal similitudes and the results from [39] do not apply. This means that if $s \in \mathcal{DS}(\Theta, B)$, i.e., there exists $E \subseteq \mathbb{N}$ such that $s = \inf\{t \geq 0 \colon r(L_{t,E}) \leq 1\}$, then s does not necessarily correspond to the Hausdorff dimension of the subsystem Θ_E. So there is no connection between the spectrum of the Perron-Frobenius operator and the Hausdorff dimension of the invariant set, i.e., there is no link between $\mathcal{DS}(\mathcal{L}(\Theta, B))$ and $\mathcal{DS}(\Theta)$ for this system.

It is import to note that Theorem 2.30 is satisfied in many classical IFS settings. For example, this holds when the maps are infinitesimal similitudes, see for instance [39]. More precisely, let $\Theta = \{\theta_n \colon X \to X \colon n \in \mathbb{N}\}$ be an IFS with contraction factor c satisfying the OSC, such that θ_n is an infinitesimal similitude for each $n \in \mathbb{N}$ i.e. $|\theta'|$ exists and is continuous. In addition, suppose that the map $x \mapsto (D\theta_n)(x) = |\theta'(x)|$ is strictly positive Hölder continuous, and Lipschitz with constant at most c. Then for any finite subset $F \subset \mathbb{N}$ define a map

$$(L_s)f(x) = \sum_{n \in F} (D\theta_n)^s(x) f(\theta(x)) = \sum_{n \in F} |\theta_n'(x)|^s f(\theta_n(x)).$$

It is known that the Hausdorff dimension of the invariant set of the IFS $\{\theta_n \colon X \to X \mid n \in F\}$ is connected to the spectral radius of $L_{s,F}$ via the following result which is due to Mauldin and Urbaǹski [33] and can also can be found in [39].

Theorem 2.31. *Let $\Theta_F = \{\theta_n \colon X \to X \colon n \in F\}$ be a finite IFS consisting of infinitesimal similitudes with uniform contraction constant c. Assume θ_F satisfies the OSC with invariant set J_E and assume the map $x \mapsto (D\theta_n)(x)$ is a strictly positive Hölder continuous function on X. Then the Hausdorff dimension of J_E is given by the unique value s_E such that $r(L_{s_E,E}) = 1$.*

Thus, in this setup of Theorem 2.31, $\mathcal{DS}(\mathcal{L}(\Theta, B)) = \mathcal{DS}(\Theta)$.

Another classic example is conformal IFS. It was shown in [8, Theorem 1.2] that if Θ is a conformal IFS on $\mathbb{R}^n$ or $\mathbb{C}^n$ satisfying the OSC, then $\mathcal{DS}(\Theta)$ is a perfect set. In that case Theorem 2.31 also applies. We simply take $b_n(x) = ||\theta_n'(x)||$, and get the relationship with the Hausdorff dimension, $\mathcal{DS}(\mathcal{L}(\Theta, B)) = \mathcal{DS}(\Theta)$.

Chapter 3

Hausdorff Dimension of Continued Fraction Expansions

It is well known that the Hausdorff dimension of sets defined by continued fraction expansions can be analysed using the Perron-Frobenius operators, see for instance [6–9, 16, 17, 20–24, 28, 30, 37, 39]. There exist several rigorous and efficient algorithms to approximate the Hausdorff dimension of invariant sets of finite conformal IFSs, see [8, 16, 17, 20, 28]. We will rely on the method developed by Falk and Nussbaum in [16, 17], which provides a rigorous framework applicable to a wide variety of sets.

We start this chapter by recalling results that link the Hausdorff dimension of invariant set of finite IFS with spectral properties of the Perron-Frobenius operator. We then describe the numerical methods of Falk and Nussbaum for computing the Hausdorff dimension, and provide explicit bounds that will be needed in Chapter 4 and 5.

We provide upper and lower bounds for the Hausdorff dimension of certain families of continued fractions expansion. We also discuss a number of preliminary results that will be used to analyse the Hausdorff dimension of infinite subsystems.

3.1 Continued Fraction Expansions

In the remainder of this thesis, we will consider the map $\theta_n \colon x \mapsto (n+x)^{-1}$ and the system $\Theta = \{\theta_n \colon n \in \mathbb{N}\}$. By Theorem 1.5, Θ is an IFS of infinitesimal similitudes and in addition it satisfies the OSC.

For each $n \in \mathbb{N}$, define

$$b_n(x) = |\theta_n'(x)| = \left(\frac{1}{n+x}\right)^2, \quad x \in [0,1].$$

We first verify that the collection $B = \{b_n \colon n \in \mathbb{N}\}$ satisfies the assumption (**B2**) - (**B4**) from Section 2.2 of the Chapter 2 and, then discuss assumption (**B1**). First

$$|b_n'(x)| = \frac{2}{(n+x)^3} \leq \frac{2}{n^3}$$

for all n so B is a collection of Lipschitz functions. Moreover,

$$\frac{1}{(n+1)^2} \leq b_n(x) \leq \frac{1}{n^2}.$$

Since $((n+1)+x)^{-2} < (n+x)^{-2}$ for all x, we have $||b_{n+1}||_\infty < ||b_n||_\infty = n^{-2}$, and hence

$$\lim_{n\to\infty} ||b_n||_\infty = 0,$$

so (**B2**) is satisfied. For $s = 1$, we have

$$\sum_{n=1}^{\infty} ||b_n||_\infty = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

Thus, (**B3**) is satisfied. Next, consider

$$\begin{aligned}
\left|\ln\left(\frac{b_n(x)}{b_n(y)}\right)\right| &= |\ln(b_n(x)) - \ln(b_n(y))| \\
&= \left|\ln\left(\frac{1}{n+x}\right)^2 - \ln\left(\frac{1}{n+y}\right)^2\right| \\
&= 2\left|\ln(n+x) - \ln(n+y)\right|.
\end{aligned}$$

By the mean value theorem, for some $c \in (x, y)$,

$$|\ln(n+x) - \ln(n+y)| = \frac{1}{n+c}|x-y| \le \frac{1}{n}|x-y|.$$

Thus,

$$\left|\ln\left(\frac{b_n(x)}{b_n(y)}\right)\right| \le \frac{2}{n}|x-y|.$$

Hence

$$0 < b_n(x) = \left(\frac{1}{n+x}\right)^2 \le \left(\frac{1}{n+y}\right)^2 e^{\frac{2}{n}|x-y|} = b_n(y)e^{\frac{2}{n}|x-y|},$$

which implies that $b_n \in K(\frac{2}{n}, 1)$. Thus (Θ, B) satisfies the conditions (**B2**) - (**B4**).

Note that $||b_1||_\infty = 1$, so $\eta_1 = 1$, thus (**B1**) is not necessarily satisfied. We use the compositions $\Theta' = \{\theta_{n,m} \colon n, m \in \mathbb{N}\}$ where $\theta_{n,m} = \theta_n \circ \theta_m$. The derivatives satisfy

$$\theta'_{n,m}(x) = \frac{1}{(n(m+x)+1)^2}.$$

Consider the set $\mathbb{N}^2 = \{1, 2, 3, \dots\}^2$. We will use the Cantor's diagonal enumeration of $\mathbb{N}^2$. To enumerate all ordered pairs (n, m), we first group them by the value of the sum $s = n + m$.

For each integer $s \ge 2$, the pairs satisfying $n + m = s$ are

$$(1, s-1),\ (2, s-2),\ \dots,\ (s-1, 1).$$

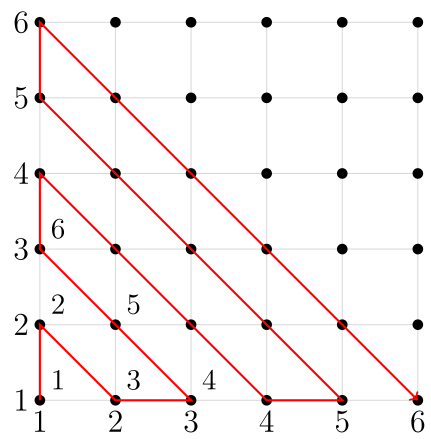
Thus the first few diagonals are:

$$
\begin{aligned}
s &= 2 : (1,1),\\
s &= 3 : (1,2),\ (2,1),\\
s &= 4 : (3,1),\ (2,2),\ (1,3),\\
s &= 5 : (1,4),\ (2,3),\ (3,2),\ (4,1),\\
&\vdots
\end{aligned}
$$

As a second step, we impose an order on each s-level as follows. If $s \geq 3$ is odd, we declare $(n,m) < (n',m')$ if and only if $n < n'$. If $s \geq 4$ is even,we declare $(n,m) < (n',m')$ if and only if $n > n'$. This produces an enumeration $\pi\colon \mathbb{N}^2 \to \mathbb{N}$ defined by

$$\pi((1,1)) = 1, \quad \pi((1,2)) = 2, \quad \pi((2,1)) = 3, \quad ((3,1)) = 4, \quad \pi((2,2)) = 5, \quad \ldots.$$

The picture below gives an idea of this enumeration



Every ordered pair (n,m) appears exactly once, on the s-level where $s = n+m$ and has the unique position on the list of all pairs with sum s. Thus, this gives a bijection between $\mathbb{N}$ and $\mathbb{N}^2$. In an abuse of notation we will write k to actually mean $\pi^{-1}(k) = (n,m)$. Let

$$b_k(x) = b_{n,m}(x) = \theta'_{n,m}(x) = \frac{1}{(n(m+x)+1)^2}, \quad x \in [0,1],$$

and $B' = \{b_k : k \in \mathbb{N}\}$. Then

$$0 < \beta_k = \frac{1}{(n(m+1)+1)^2} \le b_k(x) \le \frac{1}{(nm+1)^2} = \eta_k < 1.$$

Thus (**B1**) is satisfied.

We claim that it satisfies (**B1**) - (**B4**) and for completeness we prove it.

Observe that if $\pi(n,m) = k \ge 3$ and $s = m+n$, then $\sum_{j=1}^{s-2} j \le k \le \sum_{j=1}^{s-1} j$. This implies that

$$k \to \infty \text{ if and only if } s \to \infty. \tag{3.1}$$

As $nm \ge 1 \cdot (s-1) = s-1$, we have $mn + 1 \ge s$, so $||b_k||_\infty = ||b_{n,m}||_\infty \le s^{-1}$. It follows that if $k \to \infty$, then $s \to \infty$ by (3.1), so that $\lim_k ||b_k|| = 0$. Thus (**B2**) is satisfied.

Using the Mean Value Theorem it is straightforward to show that

$$\left| \ln \left(\frac{b_{n,m}(x)}{b_{n,m}(y)} \right) \right| = 2|\ln(n(m+y)+1) - \ln(n(m+x)+1)| \le \frac{2}{m}|x-y|,$$

for $x, y \in [0,1]$ so (**B4**) holds. Finally the sum

$$\sum_{n,m \in \mathbb{N}} \frac{1}{(mn+1)^2} \le \sum_n \sum_m \frac{1}{n^2 m^2} = \left(\sum_n \frac{1}{n^2} \right)^2 < \infty.$$

Thus, (Θ', B') satisfies the assumptions (**B1**)-(**B4**). It follows that the theory developed in the Chapter 2 is directly applicable.

Let $F \subset \mathbb{N}$ be finite and write $\gamma = \min\{n : n \in F\}$. For $s \ge 0$ we then define the associated Perron-Frobenius operator $L'_{s,F} : C([0,\gamma^{-1}]) \to C([0,\gamma^{-1}])$ as

$$(L'_{s,F} f)(x) = \sum_{n,m \in F} (\theta'_{n,m}(x))^s f(\theta_{m,n}(x)). \tag{3.2}$$

From Theorem 2.12, $L'_{s,F}$ has a strictly positive eigenvector v_s with an eigenvalue $r(L'_{s,F}) = \lambda_s$, furthermore the map $s \mapsto r(L'_{s,F})$ is strictly decreasing by Lemma

2.13.

On the other hand, for (Θ, B) with $\theta_n(x) = (n+x)^{-1}$ and $b_n(x) = (n+x)^{-2}$, define the operator $L_{s,F}\colon C([0,\gamma^{-1}]) \to C([0,\gamma^{-1}])$ as

$$(L_{s,F}f)(x) = \sum_{n\in F} b_n^s(x) f(\theta_n(x)) = \sum_{n\in F} \left(\frac{1}{n+x}\right)^{2s} f(\theta_n(x)). \tag{3.3}$$

It is easy to verify that

$$\begin{aligned}(L_{s,F}^2 f)(x) &= \sum_{(n,m)\in F^2} |\theta_n'(\theta_m(x))|^s |\theta_m'(x)|^s f(\theta_n \circ \theta_m(x)) \\ &= \sum_{n,m\in F} (\theta_{nm}'(x))^s f(\theta_{nm}(x)) \\ &= (L_{s,F}' f)(x).\end{aligned}$$

Thus $L_{s,F}^2 = L_{s,F}'$. Hence $L_{s,F}^2$ has eigenvector v_s with eigenvalue λ_s so $L_{s,F}^2 v_s = \lambda_s v_s$. Let

$$w_s = v_s + \frac{1}{\sqrt{\lambda_s}} L_s v_s,$$

then

$$L_{s,F} w_s = L_{s,F} v_s + \frac{1}{\sqrt{\lambda_s}} L_{s,F}^2 v_s = L_{s,F} v_s + \sqrt{\lambda_s} v_s = \sqrt{\lambda_s}\left(v_s + \frac{1}{\sqrt{\lambda_s}} L_{s,F} v_s\right) = \sqrt{\lambda_s} w_s.$$

So, w_s is an eigenvector of $L_{s,F}$ with eigenvalue $\sqrt{\lambda_s}$ and

$$r(L_{s,F}) = \sqrt{r(L_{s,F}')} = \sqrt{\lambda_s}.$$

As the root function is monotonic, the map $s \mapsto r(L_{s,F})$ is strictly decreasing and continuous. Note that

$$r(L_{s,F}') \le 1 \text{ if and only if } r(L_{s,F}) \le 1,$$

and

$$r(L'_{s,F}) \geq 1 \text{ if and only if } r(L_{s,F}) \geq 1.$$

In the sequel we will need the fact that $\dim_{\mathcal{H}}(J_F) \leq s$ if and only if $r(L'_{s,F}) \leq 1$. So we can use $L_{s,F}$ instead of $L'_{s,F}$ to get these inequalities.

Remark 3.1. If $E \subseteq \mathbb{N}$ is infinite and $L_{s,E}$ is well defined then $\sum_{n\in E} n^{-2s} < \infty$. It follows that the double sum

$$\sum_{n,m\in E} \left(\frac{1}{n(m+1)}\right)^{2s} \leq \sum_{n,m\in E} \left(\frac{1}{nm}\right)^{2s} \leq \left(\sum_{n\in E} \left(\frac{1}{n}\right)^{2s}\right)^2,$$

is finite. Thus $L_{s,E}$ is defined if and only if $L'_{s,E}$ is defined, and therefore, Theorem 2.23 can also be used.

It follows that the dimension spectrum of the class of operators associated with (Θ, B) and (Θ', B') agrees i.e.,

$$\mathcal{DS}(\mathcal{L}(\Theta, B)) = \mathcal{DS}(\mathcal{L}(\Theta', B')),$$

where $\mathcal{DS}(\mathcal{L}(\Theta, B))$ is defined by (2.15). Due to this observation, we will simply work with

$$(L_{s,F}f)(x) = \sum_{n\in F} b_n^s(x) f(\theta_n(x)) = \sum_{n\in F} \left(\frac{1}{n+x}\right)^{2s} f(\theta_n(x)),$$

and assume that the operator satisfies the results established in the previous chapter. It may also be considered on other Banach spaces, see for instance [4, 7, 8, 16, 17, 24], such as the real Banach space $C^\alpha([0,1])$ consisting of functions $f\colon [0,1] \to \mathbb{R}$ (respectively the complex Banach space $C^\alpha_{\mathbb{C}}$ $f\colon [0,1] \to \mathbb{C}$) which are Hölder continuous with Hölder exponent $0 < \alpha \leq 1$.

It can also be considered on Banach space $C^k([0,1])$ (respectively $C^k_{\mathbb{C}}([0,1])$) consisting of k-times continuously differentiable real (respectively complex) functions on $[0,1]$ for $k \in \mathbb{N}$. Indeed, $L_{s,F}$ is a bounded real linear operator from

$C^\alpha([0,1])$ to itself and also from $C^k([0,1])$ to itself. The operator can be extended in the usual way to a complex linear operator on $C^\alpha_{\mathbb{C}}([0,1])$ or $C^k_{\mathbb{C}}([0,1])$.

Of course, if we considers these alternative domains, then further assumptions on the functions b_n are needed to ensure that $L_{s,F}$ maps the domain into itself. In our case we shall study it in $C([0,1])$.

Note that we have not chosen any other space above because we want a method that can be used to rigorously estimates the Hausdorff dimension of J_F for various sets $F \subset \mathbb{N}$. The other methods in the literature, which study the operator on different domains, are only applicable to specific sets and it is not clear how these methods can be generalised to arbitrary subsets of $\mathbb{N}$. For example, in [28], in order to approximate the Hausdorff dimension of $J_{\{1,2\}}$, they studied the authors study the operator on the space of analytical functions on and open disc $D \subset \mathbb{C}$ of radius r that have a continuous extension to the closed disc $\overline{D}$. This forms a Banach space with respect to the uniform norm. In this setup, the operator $L_{s,F}$ is of trace class and they used the Fredholm determinant

$$\det(1 - L_{s,F}).$$

This approach provides very good estimates for the Hausdorff dimension of $J_{\{1,2\}}$, see [28] .

Recall that J_F denotes the set of irrational numbers in the unit interval whose continued fraction expansion digits belong to F. It is the invariant set of the iterated function system $\{\theta_n : n \in F\}$. We now state the crucial theorem which links the Hausdorff dimension of J_F to the spectral radius of $L_{s,F}$. This result plays a key role in the sequel, the result we present here is a special case of a more general theorem that can be found in [16, Theorem 3.1] and has been established in varying degrees of generality; see for instance [32, Section 2.2], [37, Theorem 5.4] and [39, Theorem 6.5], also [4, 7, 8, 16, 17, 24].

Theorem 3.2. *Let $F \subseteq \mathbb{N}$, finite with $\gamma = \min\{n : n \in F\}$ and $\Gamma = \max\{n : n \in F\}$. Let $s > 0$ and $L_{s,F}$ be defined by (3.3), then the following holds:*

1. *The operator $L_{s,F}$ has a unique strictly positive normalised eigenvector $v_s \in C([0, \gamma^{-1}])$ with $L_{s,F} = \lambda_s v_s$ where $\lambda_s = r(L_{s,F})$, the spectral radius of $L_{s,F}$.*

2. *The function $s \mapsto \lambda_s$ is strictly decreasing and continuous.*

3. *The function v_s is decreasing on $[0, \gamma^{-1}]$ and*

$$v_s(x) \le v_s(y) e^{\frac{2s}{\gamma}|x-y|} \quad \text{for all } x, y \in [0, \gamma^{-1}] \quad \text{and} \tag{3.4}$$

$$\begin{aligned} 2s(2s+1)\dots(2s+p-1)(2\gamma^{-1}+\Gamma)^{-p} &\le (-1)^p \frac{D^p[v_s(x)]}{v_s(x)} \\ &\le 2s(2s+1)\dots(2s+p-1)\gamma^{-p}. \end{aligned} \tag{3.5}$$

In particular, we have that

$$2s(2s+1)(2\gamma^{-1}+\Gamma)^{-2} v_s(x) \le v_s''(x) \le 2s(2s+1)\gamma^{-2} v_s(x) \tag{3.6}$$

4. *The unique value s_F such that $\lambda_{s_F} = 1$ is equal to $\dim_{\mathcal{H}}(J_F)$.*

Note that (3.4) follows from (3.5) by setting $p = 1$. Indeed if $p = 1$, then

$$2s(2\gamma^{-1}+\Gamma)^{-1} \le -\frac{v_s'(x)}{v_s(x)} \le 2s\gamma^{-1}.$$

For $x, y \in [0, \gamma^{-1}]$, we obtain

$$\begin{aligned} \ln\left(\frac{v_s(x)}{v_s(y)}\right) = \ln v_s(x) - \ln v_s(y) &= \int_y^x \frac{d}{dz} \ln(v_s(z)) \\ &= \int_y^x \frac{v_s'(z)}{v_s(z)} dz \\ &\ge \int_y^x -\frac{2s}{\gamma} dz = -\frac{2s}{\gamma}|x-y|. \end{aligned}$$

Thus

$$-\frac{2s}{\gamma}|x-y| \leq \ln\left(\frac{v_s(x)}{v_s(y)}\right) \quad \text{for all} \quad x, y \in [0, \gamma^{-1}],$$

which implies that $v_s(x) \leq v_s(y)e^{\frac{2s}{\gamma}|x-y|}$ so that $v_s \in K\left(\frac{2s}{\gamma}, 1\right)$.

Remark 3.3. Note that v_s is not merely continuous but at least twice differentiable and (3.6) gives us the bounds of v_s''

$$2s(2s+1)(2\gamma^{-1}+\Gamma)^{-2}v_s(x) \leq v_s''(x) \leq 2s(2s+1)\gamma^{-2}v_s(x).$$

We will use this fact in our arguments.

3.2 The Falk and Nussbaum Numerical Method

In order to fully characterise the dimension spectrum of infinite subsystems of Θ, it is often useful to have explicit bounds for certain finite subsets of $\mathbb{N}$. These bounds will be obtained using the computational method developed by Falk and Nussbaum in [16], which combines operator theory with rigorous numerical approximation. In this section, we briefly discuss the Falk and Nussbaum method and explain how it is used to develop a computational approach for approximating the Hausdorff dimension of invariant sets with a finite alphabet. We use their numerical methods to get rigorous approximations accurate to six decimal places. Even higher precision can be obtained by employing computational methods from [28].

Given $F = \{n_1, n_2, \ldots n_p\} \subseteq \mathbb{N}$ finite, Theorem 3.2 guarantees the existence and uniqueness of $s_F \geq 0$ such that

$$r(L_{s_F,F}) = 1 \quad \text{and} \quad \dim_{\mathcal{H}}(J_F) = s_F.$$

The goal is to approximate s_F, the Hausdorff dimension of the set J_F. The idea is to approximate $L_{s,F}$ by $q \times q$ non-negative matrices A_s and B_s, such that

$$r(A_s) \leq r(L_{s,F}) \leq r(B_s).$$

We introduce the interpolation framework. Write

$$\gamma = \min\{n \colon n \in F\} \quad \text{and} \quad \Gamma = \max\{n \colon n \in F\}.$$

Consider a function $f \in C^2([0,\gamma^{-1}])$, meaning f is twice continuously differentiable. We approximate f using a piece-wise linear interpolation f^I.

Fix an integer $N \geq 2$ which denotes the number of intervals in a partition of $[0,\gamma^{-1}]$. Set

$$h = \frac{1}{\gamma N}, \quad x_0 = 0, \quad x_k = x_0 + kh, \quad k = 1,\ldots,N,$$

so that $x_{k+1} - x_k = h$ and $x_N = \gamma^{-1}$. Then $P = \{x_0, x_1, \ldots, x_N\}$ forms a partition of $[0,\gamma^{-1}]$. For $x \in [x_k, x_{k+1}]$, define

$$f^I(x) = \frac{x_{k+1} - x}{h} f(x_k) + \frac{x - x_k}{h} f(x_{k+1}). \tag{3.7}$$

A standard interpolation result ensures that for each $x \in [x_k, x_{k+1}]$, there exists $y \in [x_k, x_{k+1}]$ such that

$$f^I(x) - f(x) = \frac{1}{2}(x_{k+1} - x)(x - x_k) f''(y). \tag{3.8}$$

As the operator $L_{s,F}$ has eigenvector $v_s := v_{s,F} \in K\left(\frac{2s}{\gamma}, 1\right)$, it follows that for $y \in [x_k, x_{k+1}]$,

$$v_s(x_k)e^{-\frac{2s}{\gamma}h} \leq v_s(x_k)e^{-\frac{2s}{\gamma}|y-x_k|} \leq v_s(y)$$

and

$$v_s(y) \le v_s(x_k)e^{\frac{2s}{\gamma}|y-x_k|} \le v_s(x_k)e^{\frac{2s}{\gamma}h}.$$

Thus

$$v_s(x_k)e^{-\frac{2s}{\gamma}h} \le v_s(y) \le v_s(x_k)e^{\frac{2s}{\gamma}h}.$$

Using the same argument, we have that

$$v_s(x_{k+1})e^{-\frac{2s}{\gamma}h} \le v_s(y) \le v_s(x_{k+1})e^{\frac{2s}{\gamma}h}.$$

Taking $v_s = f$ in (3.7) we have that

$$v_s^I(x) = \frac{x_{k+1}-x}{h}v_s(x_k) + \frac{x-x_k}{h}v_s(x_{k+1}).$$

For $x \in [x_k, x_{k+1}]$, let $\alpha = \frac{x-x_k}{h}$, so that $1-\alpha = \frac{x_{k+1}-x}{h}$. Then

$$\begin{aligned}
v_s^I(x)e^{-\frac{2s}{\gamma}h} &= (1-\alpha)v_s(x_k)e^{-\frac{2s}{\gamma}h} + \alpha v_s(x_{k+1})e^{-\frac{2s}{\gamma}h} \\
&\le (1-\alpha)v_s(x) + \alpha v_s(x) \\
&= v_s(x) \\
&\le (1-\alpha)v_s(x_k)e^{\frac{2s}{\gamma}h} + \alpha v_s(x_{k+1})e^{\frac{2s}{\gamma}h} \\
&= v_s^I(x)e^{\frac{2s}{\gamma}h}.
\end{aligned}$$

Taking $f = v_s$ in (3.8), and combining this with the above inequalities and (3.6), we deduce that

$$\begin{aligned}
&[x_{k+1}-x][x-x_k]s(2s+1)(2\gamma^{-1}+\Gamma)^{-2}e^{-2s\frac{h}{\gamma}}v_s^I(x) \\
&\le v_s^I(x) - v_s(x) \\
&\le [x_{k+1}-x][x-x_k]s(2s+1)\gamma^{-2}e^{2s\frac{h}{\gamma}}v_s^I(x).
\end{aligned}$$

Thus we obtain upper and lower bound on the interpolation error on each subinterval $[x_k, x_{k+1}]$ for each k. For $x \in [x_k, x_{k+1}]$, define the error functions

$$\mathrm{err}^1(x) = [x_{k+1} - x][x - x_k]s(2s+1)\gamma^{-2}e^{2s\frac{h}{\gamma}},$$

$$\mathrm{err}^2(x) = [x_{k+1} - x][x - x_k]s(2s+1)(2\gamma^{-1} + \Gamma)^{-2}e^{-2s\frac{h}{\gamma}}.$$

Note that err^1 and err^2 depend on the subinterval containing x. Consequently,

$$[1 - \mathrm{err}^1(x)]v_s^I(x) \leq v_s(x) \leq [1 - \mathrm{err}^2(x)]v_s^I(x).$$

Next we construct two matrices. For fixed $k \in \{0, 1, 2, ..., N\}$, replacing x with $\theta_{n_j}(x_k)$ and sum over the $j's$ we obtain

$$\begin{aligned}
&\sum_{j=1}^{p} b_{n_j}^s(x_k)[1 - \mathrm{err}^1(\theta_{n_j}(x_k))]v_s^I(\theta_{n_j}(x_k)) \\
&\leq \sum_{j=1}^{p} b_{n_j}^s(x_k)v_s(\theta_{n_j}(x_k)) \\
&= L_{s,F}v_s(x_k) = \lambda_s v_s(x_k) \\
&\leq \sum_{j=1}^{p} b_{n_j}^s(x_k)[1 - \mathrm{err}^2(\theta_{n_j}(x_k))]v_s^I(\theta_{n_j}(x_k)).
\end{aligned}$$

Let $w = (w_0, w_1, \ldots, w_N) \in \mathbb{R}^{N+1}$ be defined by

$$w_k = v_s(x_k) = v_s^I(x_k) \quad \text{for} \quad k = 0, 1, \ldots, N.$$

We now define two $(N+1) \times (N+1)$ matrices A_s and B_s such that

$$(A_s w)_k = \sum_{j=1}^{p} b_{n_j}^s(x_k)[1 - \mathrm{err}^1(\theta_{n_j}(x_k))]v_s^I(\theta_{n_j}(x_k))$$

and

$$(B_s w) = \sum_{j=1}^{p} b_{n_j}^s(x_k)[1 - \mathrm{err}^2(\theta_{n_j}(x_k))]v_s^I(\theta_{n_j}(x_k)).$$

Note that $w > 0$ and we have that $(A_s w)_k \le L_s v_s(x_k) \le (B_s w)_k$. It then follows that

$$r(A_s) \le r(L_s) = \lambda_s \le r(B_s).$$

In practice, to construct A_s and B_s, fix $k \in \{0, \ldots, N\}$ and consider each $n_j \in F$. Compute $\theta_{n_j}(x_k)$, which lies in some interval $[x_i, x_{i+1}]$. To get the entries of $B_s = (b_{ki})$

$$\begin{aligned} b_{ki} &= \sum_{j:\, \theta_{n_j}(x_k) \in [x_i, x_{i+1}]} b^s_{n_j}(x_k)[1 - \mathrm{err}^2(\theta_{n_j}(x_k))] \frac{x_{x_{i+1}} - \theta_{n_j}(x_k)}{h} \\ &\quad + \sum_{j:\, \theta_{n_j}(x) \in [x_{i-1}, x_i]} b^s_{n_j}(x_k)[1 - \mathrm{err}^2(\theta_{n_j}(x_k))] \frac{\theta_{n_j}(x_k) - x_{i-1}}{h}. \end{aligned}$$

In an analogous way we get the entries of A_s.

Given finite $F \subset \mathbb{N}$ and an integer $N \ge 2$, the matrices A_s and B_s can be computed for each $s > 0$. The goal is to estimate a value s_l such that $r(A_{s_l}) \ge 1$, so $r(L_{s_l,F}) \ge 1$. Since the map $s \mapsto r(L_{s,F})$ is decreasing by Theorem 3.2, it follows that $s_l \le s_F$, where s_F denote the Hausdorff dimension of J_F. Thus s_l is the lower bound for s_F.

Similarly, we estimate a value s_u such that $r(B_{s_u}) < 1$, which yields $r(L_{s_u,F}) < 1$ and hence $s_F < s_u$. Thus, s_u is an upper bound for s_F. Consequently, we obtain rigorous interval bounds

$$s_l \le s_F \le s_u.$$

In our arguments we occasionally require upper and lower bounds for $\dim_{\mathcal{H}}(J_F)$ for specific finite sets $F \subset \mathbb{N}$. Whenever we refer to the computational methods of Falk and Nussbaum, we mean precisely this approach to bounding $\dim_{\mathcal{H}}(J_F)$ for $F \subset \mathbb{N}$ finite. The MATLAB codes for implementing this method for finite subsets of $\mathbb{N}$ are available at:

https://sites.math.rutgers.edu/~falk/hausdorff/codes.html

Table 3.1 lists the bounds sufficient for our purposes. We used $N = 100$ as the number of intervals on the partition of $[0, \gamma^{-1}]$. It should however, be noted that much sharper bounds can be obtained by using the numerical methods from [16]. In some cases, for instance $F = \{1, 2\}$, very sharp estimates exist. For example in [28].

TABLE 3.1: Upper and lower bounds for Hausdorff dimension

$\{1,2\}$	[0.531277, 0.531281]	$\{1,2^{10}\}$	[0.150819, 0.150820]
$\{1,3\}$	[0.454487, 0.454490]	$\{1,2^{11}\}$	[0.140914, 0.140915]
$\{1,2^2\}$	[0.411181, 0.411183]	$\{1,2,4\}$	[0.669217, 0.669223]
$\{1,2^3\}$	[0.333644, 0.333646]	$\{1,2^5,3^5\}$	[0.272593, 0.272595]
$\{1,2^4\}$	[0.280974, 0.280976]	$\{1,2^6,3^6\}$	[0.238624, 0.238626]
$\{1,2^5\}$	[0.243375, 0.243377]	$\{1,2^7,3^7\}$	[0.212932, 0.212933]
$\{1,2^6\}$	[0.215370, 0.215371]	$\{1,2^8,3^8\}$	[0.192784, 0.192786]
$\{1,2^7\}$	[0.193748, 0.193749]	$\{1,2^9,3^9\}$	[0.176528, 0.176529]
$\{1,2^8\}$	[0.176544, 0.176545]	$\{1,2^{10},3^{10}\}$	[0.163106, 0.163107]
$\{1,2^{12}\}$	[0.132398, 0.132400]	$\{1,2^{12},3^{12}\}$	[0.142168, 0.142170]
$\{1,2^{11}\}$	[0.140913, 0.140915]	$\{1,2^{11},3^{11}\}$	[0.151814, 0.151816]
$\{1,2^9\}$	[0.162508, 0.162510]	$\{1,3^5,\dots,100^5\}$	[0.243455, 0.243456]
$\{1,3^2,3^3\}$	[0.380856, 0.380863]	$\{1,3^2,3^4,3^5,3^6\dots,3^{10}\}$	[0.371828, 0.371833]
$\{1,5^3\}$	[0.194405, 0.194408]	$\{1,5^3,5^4\}$	[0.223875, 0.223878]
$\{1,5^3,5^5\}$	[0.211228, 0.211234]	$\{1,5^3,5^5,5^6\}$	[0.217632, 0.217634]

Remark 3.4. It is worth noting, the rigorous approach of Falk and Nussbaum does not require one to know the explicit form of v_s but rather its behaviour.

Besides the bounds listed in Table 3.1, we will also need bounds for families of sets. In particular, for $F = \{1, n\}$, we will need a generic lower bound for the Hausdorff dimension of $J_{\{1,n\}}$.

3.3 Bounds for $\dim_{\mathcal{H}}(J_{\{1,n\}})$

The main idea is to use the strictly positive eigenvector of the operator

$$(L_{s,\{1\}}f)(x) = \left(\frac{1}{1+x}\right)^{2s} f\left(\frac{1}{1+x}\right),$$

as an approximation of the eigenvector of $L_{s,\{1,n\}}$ to derive the bounds for the eigenvalue of $L_{s,\{1,n\}}$.

Lemma 3.5. *Let $\mu > 0$ and $s \geq 0$. The operator $L_{s,\{\mu\}} \colon [0, \frac{1}{\mu}] \to [0, \frac{1}{\mu}]$ given by*

$$(L_{s,\{\mu\}} f)(x) = \left(\frac{1}{\mu + x}\right)^{2s} f\left(\frac{1}{\mu + x}\right)$$

has $v_s(x) = \left(\frac{1}{\lambda + x}\right)^{2s}$ as a strictly positive eigenvector with eigenvalue λ^{-2s}, where

$$\lambda = \frac{\mu + \sqrt{\mu^2 + 4}}{2}.$$

Moreover, $r(L_{s,\{\mu\}}) = \lambda^{-2s}$.

Proof. Note that λ satisfies $\lambda^2 - \mu\lambda - 1 = 0$, hence

$$v_s\left(\frac{1}{\mu + x}\right) = \left(\frac{1}{\lambda + \frac{1}{\mu + x}}\right)^{2s} = \left(\frac{\mu + x}{\mu\lambda + 1 + \lambda x}\right)^{2s} = \left(\frac{\mu + x}{\lambda^2 + \lambda x}\right)^{2s}.$$

Thus,

$$v_s\left(\frac{1}{\mu + x}\right) = \lambda^{-2s}(\mu + x)^{2s} v_s(x).$$

This implies that $(L_{s,\{\mu\}} v_s)(x) = \lambda^{-2s} v_s(x)$. As v_s is strictly positive, $r(L_{s,\{\mu\}}) = \lambda^{-2s}$ by Lemma 2.2. □

Using this result, we now prove the following estimates for the Hausdorff dimension of $J_{\{1,n\}}$.

Theorem 3.6. *For $n \geq 1$ let*

$$s_-(n) = \max\left\{ s \geq 0 \colon \lambda^{-2s}\left(1 + \left(\frac{\lambda}{n + \lambda - 1}\right)^{2s}\right) \geq 1 \right\}$$

and

$$s_+(n) = \min\left\{ s \geq 0 \colon \lambda^{-2s}\left(1 + \left(\frac{\lambda + 1}{n + \lambda}\right)^{2s}\right) \leq 1 \right\},$$

where $\lambda = \frac{1+\sqrt{5}}{2}$*. Then*

$$s_-(n) \leq \dim_{\mathcal{H}}(J_{\{1,n\}}) \leq s_+(n).$$

Proof. Note that if $v_s(x) = \left(\frac{1}{\lambda+x}\right)^{2s}$, so $L_{s,\{1\}}v_s = \lambda^{-2s}v_s$, then

$$\begin{aligned} v_s\left(\frac{1}{x+n}\right) &= \left(\frac{1}{\lambda + \frac{1}{n+x}}\right)^{2s} \\ &= \left(\frac{n+x}{\lambda(n+x)+1}\right)^{2s} \\ &= \frac{(n+x)^{2s}}{\lambda^{2s}(n+x+\lambda^{-1})^{2s}} = \frac{(n+x)^{2s}}{\lambda^{2s}(n+x+\lambda-1)^{2s}}, \end{aligned}$$

as $\lambda^{-1} = \lambda - 1$. This implies that

$$(L_{s,\{1,n\}}v_s)(x) = \lambda^{-2s}\left(1 + \left(\frac{\lambda+x}{n+x+\lambda-1}\right)^{2s}\right)v_s(x).$$

For $n > 1$ and $x \in [0,1]$ the continuous function,

$$s \mapsto \lambda^{-2s}\left(1 + \left(\frac{\lambda+x}{n+x+\lambda-1}\right)^{2s}\right),$$

is strictly decreasing, positive, and at $s = 0$ takes the value 2. Moreover, for $n > 1$ and $s > 0$, the function

$$x \mapsto \lambda^{-2s}\left(1 + \left(\frac{\lambda+x}{n+x+\lambda-1}\right)^{2s}\right)$$

is strictly increasing on $[0,1]$. Thus, its maximum is attained at $x = 1$, and its minimum is attained at $x = 0$.

It follows that for $s \geq s_+(n)$, we have $(L_{s,\{1,n\}}v_s)(x) \leq v_s(x)$ and $r(L_{s,\{1,n\}}) \leq 1$ for all $s \geq s_+(n)$. Hence, by Theorem 3.2 we get that $\dim_{\mathcal{H}}(J_{\{1,n\}}) \leq s_+(n)$. Similarly, for $s \leq s_-(n)$ we have that $(L_{s,\{1,n\}}v_s)(x) \geq v_s(x)$ and $r(L_{s,\{1,n\}}) \geq 1$ for all $s \leq s_-(n)$. So, by Theorem 3.2 we get that $\dim_{\mathcal{H}}(J_{\{1,n\}}) \geq s_-(n)$. □

We can use the previous theorem to derive a general lower bound for $n \geq 4$.

Corollary 3.7. *For each $n \geq 4$ we have*

$$\dim_{\mathcal{H}}(J_{\{1,n\}}) > \frac{0.52679}{\ln(n)}.$$

Proof. We need to show for each integer $n \geq 4$ that $\frac{0.52679}{\ln(n)} < s_-(n)$. For $x \geq 4$ let

$$s(x) = \frac{c}{\ln x} \quad \text{and} \quad h(x) = \left(\frac{1}{\lambda}\right)^{2s(x)} + \left(\frac{1}{x+\lambda-1}\right)^{2s(x)}.$$

Here $c > 0$ is a constant which will be chosen later to get the lower bound for $x \geq 4$. But for the moment it is useful to work with any c and any $x \geq 4$, because the method of proof gives a way to get a better constant if one has that $x \geq N$ for some fixed N.

By Theorem 3.6 we need to show that $h(x) > 1$ for all $x \geq 4$. We will show that $h'(x) > 0$ for all $x \geq 4$ and subsequently find a suitable constant $c > 0$ such that $h(4) > 1$. Note that

$$s'(x) = -\frac{c}{x\ln^2(x)} < 0$$

for all $x > 0$, and

$$h'(x) = 2s'(x)\left(\frac{1}{x+\lambda-1}\right)^{2s(x)} \cdot \left(\left(\frac{x+\lambda-1}{\lambda}\right)^{2s(x)} \ln\left(\frac{1}{\lambda}\right) + \ln\left(\frac{1}{x+\lambda-1}\right) - \frac{s(x)}{(x+\lambda-1)s'(x)}\right).$$

So, $2s'(x)\left(\frac{1}{x+\lambda-1}\right)^{s(x)} < 0$ and $\left(\frac{x+\lambda-1}{\lambda}\right)^{2s(x)}\ln\left(\frac{1}{\lambda}\right) < 0$. Moreover, $-\frac{s(x)}{s'(x)} = x\ln(x)$, so that

$$-\frac{s(x)}{(x+\lambda-1)s'(x)} = \frac{x}{(x+\lambda-1)}\ln x < \frac{x}{(x+\lambda-1)}\ln(x+\lambda-1).$$

This implies that

$$\ln\left(\frac{1}{x+\lambda-1}\right) - \frac{s(x)}{(x+\lambda-1)s'(x)} < -\ln(x+\lambda-1)\left(1-\frac{x}{x+\lambda-1}\right) < 0,$$

so $h'(x) > 0$ for all $x > 0$. For $x = 4$ and $s(4) = 0.52679/\ln(4)$ a direct calculation shows that

$$h(4) = \left(\frac{1}{\lambda}\right)^{\frac{0.52679}{\ln 2}} + \left(\frac{1}{3+\lambda}\right)^{\frac{0.52679}{\ln 2}} > 1.$$

Thus $\dim_{\mathcal{H}}(J_{\{1,n\}}) > \frac{0.52679}{\ln(n)}$. □

We will also need a generic lower bound of the Hausdorff dimension of the sets of type $\{1, 2^q\}$ for $q \geq 12$ particularly Theorem 4.5 in the next chapter. Using the same method as in the proof of Corollary 3.7 we need to first find a constant $c > 0$ such that for $x = 2^{12}$ and $s(2^{12}) = \frac{c}{12\ln 2}$ we have that

$$h(2^{12}) = \left(\frac{1}{\lambda}\right)^{\frac{2c}{12\ln 2}} + \left(\frac{1}{2^{12}+\lambda-1}\right)^{\frac{2c}{12\ln 2}} > 1$$

In this case one can check that $c = 1.0571$ gives $h(2^{12}) > 1.005$, hence we have for $q \geq 12$ that

$$\dim_{\mathcal{H}}(J_{\{1,2^q\}}) \geq \frac{1.0571}{q\ln 2} \geq \frac{1.525}{q}. \tag{3.9}$$

The following statement can be found in [10, Claim 3.1] but contains an inaccuracy in its proof. More precisely, the assertion on [10, page 80] that g is an eigenvector of L' seems unjustified. We give a proof of this result. It tells us how the Hausdorff dimension of J_F changes by adding a single element to the alphabet. It is also related to [8, Proposition 4.9]. Given $F \subseteq \mathbb{N}$ finite, it gives bounds on how $\dim_{\mathcal{H}}(J_{F\cup\{n\}})$ changes for $n \notin F$.

Lemma 3.8. *If $F \subseteq \mathbb{N}$ is finite with $|F| \geq 2$ and $\sigma = \dim_{\mathcal{H}}(J_F)$, then there exists a $C_F > 1$ such that for all $n \in \mathbb{N} \setminus F$ we have that*

$$\sigma + C_F^{-1} n^{-2\sigma} \leq \dim_{\mathcal{H}}(J_{F\cup\{n\}}) \leq \sigma + C_F n^{-2\sigma}. \tag{3.10}$$

Moreover if $|F| = 1$, then $\lim_{n\to\infty} \dim_{\mathcal{H}}(J_{F\cup\{n\}}) = 0$.

Proof. Note that to establish (3.10) it suffices to show that there exists a constant $C_F > 1$ such that (3.10) holds for all n sufficiently large. Let v_s be the strictly positive eigenvector of $L_{s,F}$ with eigenvalue $\lambda_s = r(L_{s,F})$, and let w_s be the strictly positive eigenvector of $L_{s,F\cup\{n\}}$ with eigenvalue $\mu_s = r(L_{s,F\cup\{n\}})$ for $\sigma < s < 1$.

If we can show that there exists $C_1 > 1$ such that for all sufficiently large n, $\mu_s < 1$ for $s = \sigma + C_1 n^{2\sigma}$, then we know by Theorem 3.2 that $\dim_{\mathcal{H}}(J_{F\cup\{n\}}) < \sigma + C_1 n^{-2\sigma}$ for all n large. By (3.4),

$$v_s\left(\frac{1}{n+x}\right) \leq v_s(x) e^{\frac{2s}{\gamma}} \leq v_s(x) e^{2s}$$

for all $x \in [0, \gamma^{-1}]$. Thus

$$(L_{s,F\cup\{n\}} v_s)(x) \leq \lambda_s v_s(x) + n^{-2s} v_s(x) e^{2s} = (\lambda_s + n^{-2s} e^{2s}) v_s(x),$$

so that $r(L_{s,F\cup\{n\}}) \leq \lambda_s + n^{-2s} e^{2s}$. For $n \in \mathbb{N}$ let $\theta_n \colon x \mapsto \frac{1}{n+x}$. We know for $s > \sigma$, that $((\theta_n \circ \theta_m)'(x))^{s-\sigma} \leq 4^{-(s-\sigma)}$ for all $x \in [0, \gamma^{-1}]$. Thus,

$$\begin{aligned}
(L_{s,F}^2 v_\sigma)(x) &= \sum_{m,n\in F} ((\theta_n \circ \theta_m)'(x))^s v_\sigma((\theta_n \circ \theta_m)(x)) \\
&\leq 4^{-(s-\sigma)} \sum_{m,n\in F} ((\theta_n \circ \theta_m)'(x))^\sigma v_\sigma((\theta_n \circ \theta_m)(x)) \\
&= 4^{-(s-\sigma)} v_\sigma(x),
\end{aligned}$$

which gives $r(L_{s,F}^2) \leq 4^{-(s-\sigma)}$. Hence $\lambda_s = r(L_{s,F}) \leq 2^{-(s-\sigma)}$. Thus

$$r(L_{s,F\cup\{n\}}) \leq 2^{-(s-\sigma)} + n^{-2s}e^{2s}$$

and we see $\mu_s < 1$ if $2^{-(s-\sigma)} + n^{-2s}e^{2s} < 1$. As $2^{s-\sigma}e^{2s} < e^3$, this inequality holds if

$$n^{-2\sigma}e^3 < 2^{s-\sigma} - 1. \tag{3.11}$$

We now wish to show that there exists a $C_1 > 1$ such that $s = \sigma + C_1 n^{-2\sigma}$ satisfies (3.11) and $\sigma < s < 1$. Note that (3.11) holds if

$$n^{-2\sigma}e^3 < 2^{C_1 n^{-2\sigma}} - 1 = e^{C_1 n^{-2\sigma}\ln 2} - 1. \tag{3.12}$$

As $e^x - 1 > x$ for all $x \geq 0$, we see that (3.12) hold if $n^{-2\sigma}e^3 < C_1 n^{-2\sigma}\ln 2$ which gives $C_1 \geq \frac{e^3}{\ln 2}$. To ensure that $s < 1$ for $s = \sigma + C_1 n^{-2\sigma}$ we also require that $n > \left(\frac{C_1}{1-\sigma}\right)^{1/2\sigma}$. Thus for all n sufficiently large, $\mu_s < 1$ for all $s = \sigma + C_1 n^{-2\sigma}$ where $C_1 > \frac{e^3}{\ln 2}$ which establishes the upperbound for $\dim_{\mathcal{H}}(J_{F\cup\{n\}})$.

To show that $\lim_n \dim_{\mathcal{H}}(J_{F\cup\{n\}}) = 0$ for $|F| = 1$, we note that $|F| = 1$ then $\sigma = 0$ by Lemma 3.5. So $\mu_s < 1$ if $2^{-s} + n^{-2s}e^{2s} < 1$ in that case, which is equivalent to $e/n < (1 - 2^{-s})^{1/2s}$. Clearly for each $\varepsilon > 0$, there exists an $N > 1$ such that $e/n < (1 - 2^{-\varepsilon})^{1/2\varepsilon}$ for all $n > N$, hence $\mu_\varepsilon < 1$ for all $n > N$. This implies that $\dim_{\mathcal{H}}(J_{F\cup\{n\}}) \to 0$ as $n \to \infty$.

To obtain the lower bound for $\dim_{\mathcal{H}}(J_{F\cup\{n\}})$, we need the fact that $s \mapsto \ln \mu_s$ is strictly decreasing and convex, see for instance [16, Theorem 8.1]. If we can show that there exists $C_2 < 1$ such that for all n sufficiently large, $\mu_s > 1$ for $s = \sigma + C_2 n^{-2\sigma}$, then it follows from Theorem 3.2 that $\dim_{\mathcal{H}}(J_{F\cup\{n\}}) > \sigma + C_2 n^{-2\sigma}$ for all n sufficiently large.

Using the Mean Value Theorem we know for $0 \leq y \leq z \leq 1$ that

$$\ln\left(\frac{n+z}{n+y}\right)^2 = 2(\ln(n+z) - \ln(n+y)) \leq \frac{2}{n}(z-y),$$

so

$$\left(\frac{1}{n+y}\right)^2 \leq \left(\frac{1}{n+z}\right)^2 e^{\frac{2}{n}(z-y)}.$$

It follows that $n^{-2}e^{-2} \leq (n+x)^{-2}$ for all $x \in [0, \gamma^{-1}]$. We also know from (3.6) that

$$e^{-2}v_\sigma(x) \leq v_\sigma\left(\frac{1}{n+x}\right) \quad \text{for all } x \in [0, \gamma^{-1}].$$

Thus,

$$n^{-2\sigma}e^{-4}v_\sigma(x) \leq \left(\frac{1}{n+x}\right)^{2\sigma} v_\sigma\left(\frac{1}{n+x}\right),$$

so that

$$(L_{\sigma,F\cup\{n\}}v_\sigma)(x) = v_\sigma(x) + \left(\frac{1}{n+x}\right)^{2\sigma} v_\sigma\left(\frac{1}{n+x}\right) \geq (1+n^{-2\sigma}e^{-4})v_\sigma(x),$$

Hence $\mu_\sigma \geq 1 + n^{-2\sigma}e^{-4}$.

Let $u\colon x \mapsto 1$ be the order unit on $C([0,1])$. Then $L_{0,F\cup\{n\}}u = (|F|+1)u$ hence $r(L_{0,F\cup\{n\}}) = |F|+1$. Set $\rho(s) = \ln \mu_s$, which is a strictly decreasing convex function with

$$\rho(0) = \ln(|F|+1) > \rho(\sigma) \geq \ln(1+n^{-2\sigma}e^{-4}) > 0.$$

Let $s_1 > \sigma$ be the unique value such that $\rho(s_1) = 0$. The straight-line through $(0, \ln(|F|+1))$ and $(\sigma, \ln(1+n^{-2\sigma}e^{-4}))$ intersects the s-axis at s_2 with $\sigma < s_2 \leq s_1$ by convexity. A simple computation gives

$$s_2 = \sigma\left(\frac{\ln(|F|+1)}{\ln(|F|+1) - \ln(1+n^{-2\sigma}e^{-4})}\right) > \sigma\left(1 + \frac{\ln(1+n^{-2\sigma}e^{-4})}{\ln(|F|+1)}\right).$$

The power series for the function $x \mapsto \ln(1+x)$ for $0 \leq x < 1$, give us

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - ... \geq x - \frac{x^2}{2} > \frac{x}{2}.$$

Using this, we find that

$$s_2 > \sigma\left(1 + \frac{\frac{1}{2}n^{-2\sigma}e^{-4}}{\ln(|F|+1)}\right) = \sigma + \frac{\sigma}{2e^4\ln(|F|+1)}n^{-2\sigma}.$$

Thus, if we take $C_2 = \frac{\sigma}{2e^4\ln(|F|+1)} < 1$ and set $s = \sigma + C_2 n^{-2\sigma}$ we have that $\ln\mu_s > 0$, hence $\mu_s > 1$. Take $C_F = \max\{C_1, C_2^{-1}\}$ we conclude that

$$\sigma + C_F^{-1}n^{-2\sigma} \leq \dim_{\mathcal{H}}(J_{F\cup\{n\}}) < \sigma + C_F n^{-2\sigma}$$

for all n large, which completes the proof. □

The following basic observation will be useful in the sequel. For completeness we include a proof.

Lemma 3.9. *Let F be a non-empty and finite subset of $\mathbb{N}$ and $m, n \in \mathbb{N}$ with $m > n$ such $m, n \notin F$, then*

$$\dim_{\mathcal{H}}(J_{F\cup\{n\}}) > \dim_{\mathcal{H}}(J_{F\cup\{m\}}).$$

Proof. There are two cases to consider namely $\min F < n$ or $\min F > n$. In the first case,

$$\min\{k\colon k \in F\cup\{n\}\} = \min\{k\colon k \in F\cup\{m\}\} := \gamma.$$

This means that the interval $[0,\gamma^{-1}]$ is the same and we are working on $C([0,\gamma^{-1}])$. So both $L_{s,F\cup\{m\}}$ and $L_{s,F\cup\{n\}}$ are well defined.

The operator $L_{s,F\cup\{m\}}$ has eigenvector v_s and eigenvalue λ_s. If $\dim_{\mathcal{H}}(J_{F\cup\{m\}}) = s_1$, then $\lambda_{s_1} = 1$. Fix $x \in [0,\gamma^{-1}]$ and using (3.4), we have that

$$\begin{aligned}(L_{s_1,F\cup\{n\}}v_{s_1})(x) &= (L_{s_1,F}v_{s_1})(x) + \left(\frac{1}{n+x}\right)^{2s_1} v_{s_1}\left(\frac{1}{n+x}\right)\\ &\geq (L_{s_1,F}v_{s_1})(x) + \left(\frac{m+1}{n+1}\right)^{2s_1} e^{-2s_1\left(\frac{m-n}{mn}\right)}\left(\frac{1}{m+x}\right)^{2s_1} v_{s_1}\left(\frac{1}{m+x}\right).\end{aligned}$$

If $\eta_{s_1} := \left(\frac{m+1}{n+1}\right)^{2s_1} e^{-2s_1\left(\frac{m-n}{mn}\right)} > 1$, then there exists $\mu > 1$ such that

$$L_{s_1,F\cup\{n\}}v_{s_1} \geq \mu L_{s,F\cup\{m\}}v_{s_1} = \mu v_{s_1}.$$

Thus $r(L_{s_1,F\cup\{n\}}) > 1$, and therefore $\dim_{\mathcal{H}}(J_{F\cup\{n\}}) > s_1$. So it suffices to show that $\eta_{s_1} > 1$. This holds provided $\ln(\eta_{s_1}) > 0$, which is satisfied if

$$2s_1\left(\ln\left(\frac{m+1}{n+1}\right) - \left(\frac{m-n}{mn}\right)\right) > 0,$$

that is

$$\ln\left(\frac{m+1}{n+1}\right) > \frac{m-n}{mn},$$

as $s_1 > 0$. Using the estimate $\ln x \geq 1 - \frac{1}{x}$, we obtain

$$\ln\left(\frac{m+1}{n+1}\right) \geq \frac{m-n}{m+1}.$$

Hence the required inequality holds if $\frac{m-n}{m+1} > \frac{m-n}{mn}$. Since $n > \gamma$, it follows that $n \geq 2$, and therefore

$$mn \geq 2m > m+1 \quad \text{as} \quad m > n.$$

Thus

$$\ln\left(\frac{m+1}{n+1}\right) > \frac{m-n}{mn}, \text{ so } \eta_{s_1} > 1.$$

In the second case if $n < \min F$ then the operator $L_{s,F\cup\{n\}}$ is defined on $C([0,n^{-1}])$ whereas $L_{s,F\cup\{m\}}$ is defined on $C([0,\gamma^{-1}])$, where $\gamma = \min\{\min F, m\}$ and necessarily $\gamma \geq 2$. The idea is to extend $v_s \in C([0,\gamma^{-1}])$ to the interval $[0,n^{-1}]$ and use the same argument as in the first case.

If $s_1 = \dim_{\mathcal{H}}(J_{F\cup\{m\}})$ then the operator $L_{s_1,F\cup\{m\}}$ has eigenvector v_{s_1} defined on $[0,\gamma^{-1}]$. In this case, $\frac{1}{n+x} > \frac{1}{\gamma}$ for $x > 0$ so v_{s_1} is not defined on $(\gamma^{-1}, n^{-1}]$.

We extend it by defining

$$w_{s_1}(x) = \begin{cases} v_{s_1}(x) & 0 \le x < \frac{1}{\gamma} \\ v_{s_1}\left(\frac{1}{\gamma}\right) & \frac{1}{\gamma} \le x \le \frac{1}{n} \end{cases}$$

Clearly w_{s_1} is continuous on $\left[0, \frac{1}{n}\right]$ and for all $x \in \left[0, \frac{1}{n}\right]$ we have that $w_{s_1}\left(\frac{1}{n+x}\right) = v_s\left(\frac{1}{\gamma}\right)$. As

$$\frac{m-n}{mn} \ge \frac{1}{n+x} - \frac{1}{m+x} \ge \frac{1}{\gamma} - \frac{1}{m+x}.$$

For $x \in [0, n^{-1}]$, we have

$$\begin{aligned} (L_{s_1,F\cup\{n\}}w_{s_1})(x) &= (L_{s_1,F}w_{s_1})(x) + \left(\frac{1}{n+x}\right)^{2s_1} w_{s_1}\left(\frac{1}{n+x}\right) \\ &= (L_{s_1,F}w_{s_1})(x) + \left(\frac{1}{n+x}\right)^{2s_1} v_{s_1}\left(\frac{1}{\gamma}\right) \\ &\ge (L_{s_1,F}w_{s_1})(x) + \left(\frac{m+1}{n+1}\right)^{2s_1} e^{-s_1\left(\frac{m-n}{mn}\right)} \left(\frac{1}{m+x}\right)^{2s_1} v_{s_1}\left(\frac{1}{m+x}\right). \end{aligned}$$

Arguing as in the first case, if

$$\eta_{s_1} = \left(\frac{m+1}{n+1}\right)^{2s_1} e^{-s_1\left(\frac{m-n}{mn}\right)} > 1,$$

then we are done. This holds if

$$2\ln\left(\frac{m+1}{n+1}\right) > \frac{m-n}{mn},$$

i.e., $\ln\left(\frac{m+1}{n+1}\right) > \frac{m-n}{2mn}$. This is satisfied since $\frac{m-n}{m+1} > \frac{m-n}{2mn}$ as $m \ge 2$, $2m > m+1$. Thus we have the result. □

This result implies the following result, which can be found in [7, Proposition 2.7].

Proposition 3.10. *If $A, B \subseteq \mathbb{N}$ and there exists a non-decreasing bijection $\tau\colon A \to B$, i.e. $\tau(n) \geq n$ for all $n \in A$, then*

$$\dim_{\mathcal{H}}(J_B) \leq \dim_{\mathcal{H}}(J_A).$$

3.4 Hausdorff Dimension Estimates of Infinite Subsystems for Continued Fractions

According to [23], it was already known to Gauss that if

$$(L_s f)(x) = \sum_{n=1}^{\infty} \left(\frac{1}{n+x}\right)^{2s} f\left(\frac{1}{n+x}\right),$$

then at $s = 1$, $v_1(x) = (1+x)^{-1}$ is an eigenvector of L_1 with eigenvalue 1 which is the spectral radius of L_1. Indeed

$$\begin{aligned}(L_1 v_1)(x) &= \sum_{n=1}^{\infty} \left(\frac{1}{n+x}\right)^{2} \left(\frac{1}{1+\frac{1}{n+x}}\right) \\ &= \sum_{n=1}^{\infty} \frac{1}{(n+x)(n+1+x)} \\ &= \sum_{n=1}^{\infty} \left(\frac{1}{n+x} - \frac{1}{n+1+x}\right) = \frac{1}{x+1}.\end{aligned}$$

So $s_{\mathbb{N}} = 1$, where $s_{\mathbb{N}}$ is defined by (2.9).

On the other hand, by a classical result of Jarnìk, see for instance [11, 30] and references therein, we know that

$$\lim_n \dim_{\mathcal{H}}(J_{\{1,2,\ldots,n\}}) = 1.$$

Thus $s_{\mathbb{N}} = \dim_{\mathcal{H}}(J_{\mathbb{N}}) = 1$. This is not a coincidence but in fact a consequence of a more general result by Mauldin and Urbaǹski [32, Theorem 1.1]. Similar results exist for conformal IFS and for infinitesimal similitudes [39, Theorem 5.12]. Here we prove a particular case of this result applicable to infinite subsystems for continued fractions. It provides a method of estimating the Hausdorff dimension of infinite sets.

Theorem 3.11. *Suppose* $E = \{n_1, n_2, \ldots\} \subset \mathbb{N}$ *with* $n_k < n_{k+1}$ *for all* $k \in \mathbb{N}$ *and* $F_k = \{n_j : j \leq k\}$*. Then*

$$\dim_{\mathcal{H}}(J_E) = \lim_{k\to\infty} \dim_{\mathcal{H}}(J_{F_k}) = s_E,$$

where s_E *is defined by* (2.11)*. If* $r(L_{t,E}) = 1$*, then* $t = s_E$*.*

Proof. Let $E = \{n_1, n_2, \ldots\} \subseteq \mathbb{N}$ and $F_k = \{n_j : j \leq k\}$. Then $E = \cup_{k=1}^{\infty} F_k$. Define $s_k := s_{F_k} = \dim_{\mathcal{H}}(J_{F_k})$, so, $r(L_{s_k,F_k}) = 1$ by Theorem 3.2 and also $r(L'_{s_k,F_k}) = 1$ where L'_{s,F_k} is defined by (3.2). Set $s_E := \lim_{k\to\infty} s_k$. It follows from Lemma 2.22 that $L'_{s,E}$ is well defined for all $s \geq s_E$, and s_E satisfies $s_E = \inf\{s > 0 : r(L'_{s,E}) \leq 1\}$ by Theorem 2.23. It then follows from [39, Theorem 5.12] that the Hausdorff dimension of the invariant set of the IFS $\{\theta_n \circ \theta_m : n, m \in E\}$ is s_E. Since $\{\theta_n \circ \theta_m : n, m \in E\}$ and $\{\theta_n : n \in E\}$ has the same invariant set J_E, we have $\dim_{\mathcal{H}}(J_E) = s_E$. If $r(L_{t,E}) = 1$, then $r(L'_{t,E}) = 1$, the decreasing property of $s \mapsto r(L'_{s,E})$ by Theorem 2.23 implies that $t = \inf\{s \geq 0 : r(L'_{s,E}) \leq 1\} = s_E$. □

Remark 3.12. We note that, for the class of infinite subsets $E \subseteq \mathbb{N}$ considered in this thesis, there exists a unique value s_E such that

$$r(L_{s_E,E}) = 1 \quad \text{and} \quad \dim_{\mathcal{H}}(J_E) = s_E.$$

Indeed, in the case of the powers $P_q^* = \{1\} \cup \{q^n : n \in \mathbb{N}\}$, we have $\sigma_0 = 0$, where σ_0 is defined in (2.6). Let $E \subseteq P_q^*$ infinite and let $n_1 < n_2$ denote the two smallest elements of E. For $s > 0$ and $u\colon x \mapsto 1$, we obtain

$$(L_{s,E}u)(x) = \sum_{n\in E} \left(\frac{1}{n+x}\right)^{2s} \geq \left(\frac{1}{n_1+1}\right)^{2s} + \left(\frac{1}{n_1+1}\right)^{2s}.$$

Consequently,

$$r(L_{s,E}) \geq \left(\frac{1}{n_1+1}\right)^{2s} + \left(\frac{1}{n_1+1}\right)^{2s}.$$

As $s \to 0^+$, the right hand side converges to 2 and hence $r(L_{s,E}) > 1$ for sufficiently small $s > 0$. Since the map $s \mapsto r(L_{s,E})$ is strictly decreasing, this implies the existence and uniqueness of $s_E > 0$ such that $r(L_{s_E,E}) = 1$.

When we assume the existence of such value s_E for $E \subseteq \mathbb{N}$ infinite in what follows, this remark serves as motivation.

We finish the chapter by using the eigenvector of the operator $L_{s,\{1\}}$ to give bounds of infinite subsystems which we will use in Chapter 4 and 5. Although these estimates are crude, they are easy to obtain and sufficient for our purposes. Again we will exploit the eigenvector of the operator $L_{s,\{1\}}$.

Lemma 3.13. *Let $M_q = \{n^q\colon n \in \mathbb{N}\}$. We have that*

$$\dim_{\mathcal{H}}(J_{M_2}) < 0.67, \qquad \dim_{\mathcal{H}}(J_{M_3}) < 0.485, \qquad \dim_{\mathcal{H}}(J_{M_4}) < 0.38,$$
$$\dim_{\mathcal{H}}(J_{M_5}) < 0.31, \qquad \dim_{\mathcal{H}}(J_{M_6}) < 0.265, \qquad \dim_{\mathcal{H}}(J_{M_7}) < 0.234,$$
$$\dim_{\mathcal{H}}(J_{M_8}) < 0.21, \qquad \dim_{\mathcal{H}}(J_{M_9}) < 0.191, \qquad \dim_{\mathcal{H}}(J_{M_{10}}) < 0.175,$$
$$\dim_{\mathcal{H}}(J_{M_{11}}) < 0.163, \qquad \dim_{\mathcal{H}}(J_{M_{12}}) < 0.1505.$$

Proof. For $m \geq 1$ let $M_q^m = \{1^q, 2^q, \ldots, m^q\}$ so $M_q = \cup_m M_q^m$. Let $v_s(x) = (\lambda + x)^{-2s}$ be the eigenvector of $L_{s,\{1\}}$ given in Lemma 3.5 with eigenvalue λ^{-2s}, where $\lambda = (1+\sqrt{5})/2$. Then

$$\begin{aligned}
L_{s,M_q^m} v_s(x) &= \lambda^{-2s} v_s(x) + \sum_{n\geq 2}^{m} \left(\frac{1}{n^q+x}\right)^{2s} v_s\left(\frac{1}{n^q+x}\right) \\
&= \lambda^{-2s}\left(\frac{1}{\lambda+x}\right)^{2s} + \sum_{n\geq 2}^{m}\left(\frac{1}{n^q+x}\right)^{2s}\left(\frac{1}{\lambda+(n^q+x)^{-1}}\right)^{2s} \\
&\leq \lambda^{-2s}\left(1+\sum_{n\geq 2}^{m}\left(\frac{\lambda+x}{n^q+x}\right)^{2s}\right) v_s(x) \\
&\leq \lambda^{-2s}\left(1+\sum_{n\geq 2}^{m}\left(\frac{\lambda+1}{n^q+1}\right)^{2s}\right) v_s(x) \\
&\leq \lambda^{-2s}\left(1+\left(\frac{\lambda+1}{2^q+1}\right)^{2s}+\left(\frac{\lambda+1}{3^q+1}\right)^{2s}+\left(\frac{\lambda+1}{4^q+1}\right)^{2s}+(\lambda+1)^{2s}\int_4^\infty x^{-2qs}\mathrm{d}x\right) v_s(x) \\
&= \lambda^{-2s}\left(1+\left(\frac{\lambda+1}{2^q+1}\right)^{2s}+\left(\frac{\lambda+1}{3^q+1}\right)^{2s}+\left(\frac{\lambda+1}{4^q+1}\right)^{2s}+\frac{(\lambda+1)^{2s}}{2qs-1}4^{-2qs+1}\right) v_s(x).
\end{aligned}$$

Now set

$$\alpha(q,s) = \lambda^{-2s}\left(1+\left(\frac{\lambda+1}{2^q+1}\right)^{2s}+\left(\frac{\lambda+1}{3^q+1}\right)^{2s}+\left(\frac{\lambda+1}{4^q+1}\right)^{2s}+\frac{(\lambda+1)^{2s}}{2qs-1}4^{-2qs+1}\right).$$

Note that if $\alpha(q,s) < 1$, then $r(L_{s,M_q^m}) < 1$ for all m, hence $\dim_{\mathcal{H}}(J_{M_q^m}) < s$ for all m. This implies that $\dim_{\mathcal{H}}(J_{M_q}) \leq s$ by Theorem 3.11. Using a calculator we find that

$$\begin{aligned}
&\alpha(2,0.67) < 0.986, \quad \alpha(3,0.485) < 0.967, \quad \alpha(4,0.38) < 0.975, \\
&\alpha(5,0.31) < 0.995, \quad \alpha(6,0.265) < 0.99985, \quad \alpha(7,0.234) < 0.9983, \\
&\alpha(8,0.21) < 0.998, \quad \alpha(9,0.191) < 0.998, \quad \alpha(10,0.175) < 0.995, \\
&\alpha(11,0.163) < 0.996, \quad \alpha(12,0.152) < 0.9994.
\end{aligned}$$

□

We like to point out that the following much sharper bound, $\dim_{\mathcal{H}}(J_{M_2}) < 0.59825579$, can be found in [7, Table 1].

Chapter 4

Dimension Spectrum of Continued Fractions

The central theme of this Chapter is to analyse the dimension spectrum of continued fraction expansions with coefficients restricted to infinite subsets of $\mathbb{N}$. Part of this chapter is based on our paper [6] with additional results.

We prove that the set of powers $P_q = \{q^n : n \in \mathbb{N}\}$ has full dimension spectrum answering a question by Chousionis, Leykekhman and Urbański. On the other hand, we show that the dimension spectrum for $P_q^* = \{1\} \cup P_q$ has many gaps and regions where it is nowhere dense.

We also investigate the case where A is generated by monomials $M_q = \{n^q : n \in \mathbb{N}\}$. For M_q, we prove that the dimension spectrum is full for $q \in \{1, 2, 3, 4, 5\}$ and it is a finite union of disjoint closed intervals for all $q \in \mathbb{N}$. In particular it is a union of two disjoint intervals for $q \in \{6, 7, 8\}$ and three disjoint intervals for $q \in \{9, 10, 11, 12\}$. we also show that for $q = 19$ it consists of four disjoint intervals.

Furthermore, we provide an example of A where the dimension spectrum is a nowhere dense set.

4.1 Dimension Spectrum of Continued Fractions

Recall that for $A \subseteq \mathbb{N}$, the *dimension spectrum of* A is the collection of all realisable Hausdorff dimension of subsets F of A, that is,

$$\mathcal{DS}(A) = \{\dim_{\mathcal{H}}(J_F) \colon F \subseteq A\}.$$

We note that if A is finite, then $\mathcal{DS}(A)$ is a finite set, but if A is infinite the structure of $\mathcal{DS}(A)$ is not well understood. By taking F to be a singleton, we have that $\dim_{\mathcal{H}}(J_F) = 0$, so $0 \in \mathcal{DS}(A)$, and $\dim_{\mathcal{H}}(J_A) \in \mathcal{DS}(A)$ so

$$\mathcal{DS}(A) \subseteq [0, \dim_{\mathcal{H}}(J_A)].$$

Recently, dimension spectrum of A has been investigated by Chousionis, Leykekhman and Urbański in [7, 8] for different infinite subsets A of $\mathbb{N}$. The case where $A = \mathbb{N}$ was studied earlier by Kesseböhmer and Zhu [30], who showed that it has full dimension spectrum, i.e., $[0, 1] = \mathcal{DS}(\mathbb{N})$, which confirmed a conjecture by Hensley [21] and Mauldin and Urbański [32] known as the Texan conjecture, see also [27]. In [7] the dimension spectrum of the set of powers of integers $q \geq 2$ and the set of squares was analysed among other sets, which motivate the results presented here.

We analyse the dimension spectrum for a variety of natural choices of A including the set of powers of integers

$$q \geq 2 : \quad P_q = \{q^n \colon n \in \mathbb{N}\} \text{ and } P_q^* = P_q \cup \{1\}.$$

In [7, Theorem 1.3] the set $M_2 = \{n^2 \colon n \in \mathbb{N}\}$ was considered and they showed that the dimension spectrum is full i.e.,

$$\mathcal{DS}(M_2) = [0, \dim_{\mathcal{H}}(J_{M_2})]$$

Also, in [7, Theorem 1.4] the dimension spectrum of P_q was considered, and for each $q \geq 2$ it was shown that there exists an $s(q) > 0$ such that

$$[0, \min\{s(q), \dim_{\mathcal{H}}(J_{P_q})\}] \subseteq \mathcal{DS}(P_q).$$

Their result shows that the dimension spectrum of P_q always contains an interval. Contrary to what was suggested in [7], we show that P_q has full dimension spectrum for all $q \geq 2$. In fact, we will prove the following more general result.

Theorem 4.1. *If $A = \{a_1, a_2, \ldots\} \subset \mathbb{N}$ with $2 \leq a_1 < a_2 < \ldots$ and $a_n a_m \geq a_{n+m}$ for all $m, n \in \mathbb{N}$, then*

$$[0, \dim_{\mathcal{H}}(J_A)] = \mathcal{DS}(A). \tag{4.1}$$

Note that this implies that P_q has full dimension spectrum for all $q \geq 2$. This result also implies several results from [7], in particular [7, Theorem 4.11] and [7, Theorem 1.2]. We summarise some of its consequences in the following corollary.

Corollary 4.2. *The following subsets of $\mathbb{N}$ have a full dimension spectrum:*

(a) $A = \{a + bn \colon n \in \mathbb{N}\}$ *for integers* $0 \leq a < b$ *with* $b \geq 2$.

(b) $A = \{a + bn \colon n \in \mathbb{N} \cup \{0\}$ *for integers* $2 \leq a \leq b$.

(c) $A_{primes} = \{p \colon p \text{ prime}\}$.

(d) $P_q = \{q^n \colon n \in \mathbb{N}\}$ *for* $q \geq 2$.

The condition in Theorem for the set of prime is due to Isikawa [26].

As we shall see, the fullness of the dimension spectrum of P_q is in stark contrast with the dimension spectrum of P_q^*, which has many gaps. More specifically, given $q \geq 2$ and $k \geq 0$ let

$$I_k = \{1, \ldots, q^k\} \quad \text{and} \quad T_k = \{q^{k+1}, q^{k+2}, \ldots\},$$

and set

$$\mu_k = \dim_{\mathcal{H}}(J_{I_{k-1}\cup T_k}) = \dim_{\mathcal{H}}(J_{P_q^*\setminus\{q^k\}}) \quad \text{and} \quad \nu_k = \dim_{\mathcal{H}}(J_{I_k}) \qquad \text{for } k \geq 0.$$

Note that $\mu_0 = \dim_{\mathcal{H}}(J_{P_q})$ and $\nu_0 = \dim_{\mathcal{H}}(J_{\{1\}}) = 0$. We have the following result.

Theorem 4.3. *For all $q \geq 3$ and $k \geq 1$,*

(i) $\mu_k < \nu_k$ and $(\mu_k, \nu_k) \cap \mathcal{DS}(P_q^*) = \emptyset$.

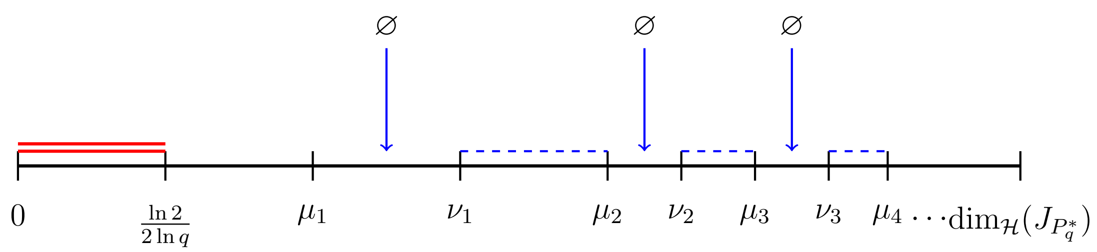
(ii) $\mathcal{DS}(P_q^*)$ is nowhere dense in (ν_k, μ_{k+1}).

For $q = 2$, assertions (i) and (ii) hold for all $k \geq 2$.

Furthermore, the dimension spectrum of P_q^* contains an initial interval.

Theorem 4.4. *The interval $[0, \frac{\ln 2}{2\ln q}]$ is contained in $\mathcal{DS}(P_q^*)$ for each $q \geq 2$.*

Thus, for $q \geq 2$ the dimension spectrum contains the interval $[0, \frac{\ln 2}{2\ln q}]$ and is nowhere dense in $[\mu^1, \dim_{\mathcal{H}}(J_{P_q^*})]$. However, at present, the exact structure of the dimension spectrum in the interval $(\frac{\ln 2}{2\ln q}, \mu^1)$ is unclear for $q \geq 3$. The picture below illustrates the structure of the dimension spectrum of the set P_q^*. In the next chapter, we will revisit the interval $(\frac{\ln 2}{2\ln q}, \mu_1)$ and provide evidence for the structure.



We will also analyse the dimension spectrum for sets generated by a monomial, $M_q = \{n^q : n \in \mathbb{N}\}$, and prove the following result.

Theorem 4.5. *The dimension spectrum of M_q satisfies:*

(i) For $q \in \{1,2,3,4,5\}$ *we have that* $\mathcal{DS}(M_q) = [0, \dim_{\mathcal{H}}(J_{M_q})]$.

(ii) For $q \geq 6$ *we have*

$$\dim_{\mathcal{H}}(J_{M_q \setminus \{2^q\}}) < \dim_{\mathcal{H}}(J_{\{1,2^q\}})$$

and $\mathcal{DS}(M_q) \cap (\dim_{\mathcal{H}}(J_{M_q \setminus \{2^q\}}), \dim_{\mathcal{H}}(J_{\{1,2^q\}}))$ *is empty.*

(iii) For $q \in \{6,7,8\}$ *we have that*

$$\mathcal{DS}(M_q) = [0, \dim_{\mathcal{H}}(J_{M_q \setminus \{2^q\}})] \cup [\dim_{\mathcal{H}}(J_{\{1,2^q\}}), \dim_{\mathcal{H}}(J_{M_q})].$$

(iv) For $q \in \{9,10,11,12\}$ *we have that* $\dim_{\mathcal{H}}(J_{M_q \setminus \{3^q\}}) < \dim_{\mathcal{H}}(J_{\{1,2^q,3^q\}})$ *and*

$$\mathcal{DS}(M_q) = [0, \dim_{\mathcal{H}}(J_{M_q \setminus \{2^q\}})] \cup [\dim_{\mathcal{H}}(J_{\{1,2^q\}}), \dim_{\mathcal{H}}(J_{M_q \setminus \{3^q\}})] \\ \cup [\dim_{\mathcal{H}}(J_{\{1,2^q,3^q\}}), \dim_{\mathcal{H}}(J_{M_q})].$$

(v) For $q = 19$ *we have that* $\dim_{\mathcal{H}}(J_{M_q \setminus \{4^q\}}) < \dim_{\mathcal{H}}(J_{\{1,2^q,4^q\}})$ *and*

$$\mathcal{DS}(M_q) = [0, \dim_{\mathcal{H}}(J_{M_q \setminus \{2^q\}})] \cup [\dim_{\mathcal{H}}(J_{\{1,2^q\}}), \dim_{\mathcal{H}}(J_{M_q \setminus \{3^q\}})] \cup \\ [\dim_{\mathcal{H}}(J_{\{1,2^q,3^q\}}), \dim_{\mathcal{H}}(J_{M_q \setminus \{4^q\}})] \cup [\dim_{\mathcal{H}}(J_{\{1,2^q,3^q,4^q\}}), \dim_{\mathcal{H}}(J_{M_q})].$$

(vi) $\mathcal{DS}(M_q)$ *is a disjoint union of finitely many nontrivial closed intervals for each* $q \in \mathbb{N}$.

The case $q = 1$ is the Texan conjecture established in [30], and the case $q = 2$, i.e., the set of squares, was treated in [7, Theorem 1.3]. It seems that the number of intervals increases with q, but it is not clear if there exists an a priori upper bound for the number of distinct intervals that holds for all q. It would be interesting to understand at which values of q the number of intervals in the dimension spectrum of M_q jumps. For instance the first jump from 1 to

2 intervals occurs at $q = 6$, and at $q = 9$ it jumps from 2 to 3 intervals and at $q = 19$ it jumps to 4.

To prove the final statement in Theorem 4.5 we will establish a general criterion on $A \subseteq \mathbb{N}$ that implies that its dimension spectrum consists of finitely many non-trivial disjoint closed interval.

We also provide an example of a dimension spectrum which is nowhere dense. In fact we will establish a general criterion on A that implies A has a nowhere dense dimension spectrum, and derive the following result.

Theorem 4.6. *Let*

$$A = \{2^{2^n} : n \in \mathbb{N}\}.$$

Then $\mathcal{DS}(A)$ *is nowhere dense in* $[0, \dim_{\mathcal{H}}(J_A)]$.

Instead of taking powers of 2 in Theorem 4.6, we could have used any $q \geq 2$ and get the same result.

4.2 Strict Break Points

We introduced the notion of break points in Chapter 2 where we discussed properties of Perron-Frobenius operators. In this section we put it in context of Hausdorff dimension and show how we use it to analyse the dimension spectrum of continued fractions.

As mentioned before, the idea goes back to the work by Kesseböhmer and Zhu [30, Theorem 2.2], and the same idea was also used in [7].

Let $A = \{a_1, a_2, \ldots\} \subseteq \mathbb{N}$ with $a_1 < a_2 < \ldots$. Given $F \subset A$ finite and $0 < s < \dim_{\mathcal{H}}(J_A)$, we say that $a_k \in A$ is a *break point for* (F, s) if $a_k > \max F$ and

$$\dim_{\mathcal{H}}(J_F) < s \leq \dim_{\mathcal{H}}(J_{F \cup \{a_k\}}).$$

If (F, s) has a break point, then by Lemma 3.8 there exists a largest break point $a_{k_0} \in A$, which is called the *strict break point* for (F, s). So, $\dim_{\mathcal{H}}(J_{F\cup\{a_{k_0}\}}) \geq s$ and $\dim_{\mathcal{H}}(J_{F\cup\{a_{k_0+1}\}}) < s$.

Strict break points can be used to show that $s \in (0, 1)$ is in the dimension spectrum of A as the following lemma shows, which is similar to [30, Theorem 2.2].

Lemma 4.7. *Let $A \subseteq \mathbb{N}$ be infinite and $F_1 \subset F_2 \ldots \subset A$ be a nested sequence of finite subsets with $\max F_n < \max F_{n+1}$ for all $n \geq 1$. If $0 < s < \dim_{\mathcal{H}}(J_A)$ and for each n, there exists a strict break point a_{m_n} for (F_n, s), then $s \in \mathcal{DS}(A)$.*

Proof. Let $\sigma_n = \dim_{\mathcal{H}}(J_{F_n}) < s$ for $n \geq 1$. and let $\sigma = \dim_{\mathcal{H}}(J_{F_\infty})$, where $F_\infty = \cup_{n\in\mathbb{N}} F_n$. From Theorem 3.11 we know that $\sigma_n \to \sigma$. Also $\sigma \leq s$, as $\sigma_n < s$ for all $n \in \mathbb{N}$. To complete the proof we show that $\sigma = s$. Suppose, by way of contradiction, that $\sigma < s$.

For $n \geq 1$ let $G_n = F_n \cup \{a_{m_n}\}$, so $\dim_{\mathcal{H}}(J_{G_n}) \geq s$ for each n. For $a, b \in \mathbb{N}$ the maps $\theta_a \colon x \mapsto \frac{1}{a+x}$ and $\theta_b \colon x \mapsto \frac{1}{b+x}$ satisfy

$$(\theta_a \circ \theta_b)'(x) = (a(b + x) + 1)^{-2} \quad \text{for } x \in [0, \gamma^{-1}].$$

So,

$$((\theta_a \circ \theta_b)'(x))^{s-\sigma_n} = (a(b + x) + 1)^{-2(s-\sigma_n)} \leq 2^{-2(s-\sigma)} = 4^{-(s-\sigma)}.$$

We know, see for instance [39, Lemma 3.4], that

$$(L^2_{s,F_n} f)(x) = \sum_{a,b\in F_n} ((\theta_a \circ \theta_b)'(x))^s f((\theta_a \circ \theta_b)(x)) \quad \text{for } f \in C([0, \gamma^{-1}]).$$

Now let $v_n \in C([0,\gamma^{-1}])$ be the strictly positive eigenvector of L_{σ_n,F_n} with $L_{\sigma_n,F_n} v_n = v_n$. Then

$$\begin{aligned}(L^2_{s,F_n} v_n)(x) &= \sum_{a,b\in F_n} ((\theta_a\circ\theta_b)'(x))^s v_n((\theta_a\circ\theta_b)(x)) \\ &\le 4^{-(s-\sigma)} \sum_{a,b\in F_n} ((\theta_a\circ\theta_b)'(x))^{\sigma_n} v_n((\theta_a\circ\theta_b)(x)) \\ &= 4^{-(s-\sigma)} L^2_{\sigma_n,F_n} v_n(x) \\ &= 4^{-(s-\sigma)} v_n(x),\end{aligned}$$

hence $r(L^2_{s,F_n}) \le 4^{-(s-\sigma)}$. By Gelfand's formula, $r(L_{s,F_n}) = \lim_k \|L^k_{s,F_n}\|^{1/k}$, so that

$$r(L_{s,F_n}) = \lim_k \left(\|L^{2k}_{s,F_n}\|^{1/k}\right)^{1/2} = r(L^2_{s,F_n})^{1/2} \le 2^{-(s-\sigma)}. \tag{4.2}$$

We know from Theorem 3.2 that there exists a strictly positive function $w_s \in C([0,\gamma^{-1}])$ where $\gamma = \min F_n$ such that $L_{s,F_n} w_s = r(L_{s,F_n}) w_s$. Now using (4.2) and together with (3.4) in Theorem 3.2 we get that

$$\begin{aligned}(L_{s,G_n} w_s)(x) &= (L_{s,F_n} w_s)(x) + \left(\frac{1}{a_{m_n}+x}\right)^{2s} w_s\left(\frac{1}{a_{m_n}+x}\right) \\ &\le 2^{-(s-\sigma)} w_s(x) + \left(\frac{1}{a_{m_n}}\right)^{2s} e^{2s} w_s(x),\end{aligned}$$

hence $r(L_{s,G_n}) \le 2^{-(s-\sigma)} + a_{m_n}^{-2s} e^{2s}$. As $\dim_{\mathcal{H}}(J_{G_n}) \ge s$, we know that $r(L_{s,G_n}) \ge 1$, which gives

$$1 \le r(L_{s,G_n}) \le 2^{-(s-\sigma)} + a_{m_n}^{-2s} e^{2s}$$

for $n \ge 1$. This is impossible, since $a_{m_n} \to \infty$ and $s - \sigma > 0$. □

The following Lemma is a special case of Theorem 2.27. In this context we make a link to the Hausdorff dimension it is similar to [30, Theorem 2.2].

Lemma 4.8. *Suppose $A \subseteq \mathbb{N}$ is infinite and $0 < s < \dim_{\mathcal{H}}(J_A)$. If for each $F \subseteq A$ finite with strict break point $a_{k_0} \in A$ for (F, s) we have that $s < \dim_{\mathcal{H}}(J_{F\cup T})$, where $T = \{a_n \in A \colon n > k_0\}$, then $s \in \mathcal{DS}(A)$.*

Proof. Let $A = \{a_1, a_2, \ldots\} \subseteq \mathbb{N}$ with $a_1 < a_2 < \ldots$. As $0 < s < \dim_{\mathcal{H}}(J_A)$, it follows from Theorem 3.11 that there exists a $k_1 \geq 1$ such that $F_1 = \{a_1, \ldots, a_{k_1}\}$ satisfies

$$\dim_{\mathcal{H}}(J_{F_1}) < s \quad \text{and} \quad \dim_{\mathcal{H}}(J_{F_1 \cup \{a_{k_1+1}\}}) \geq s.$$

Now let $m_1 \geq k_1 + 1$ be such that a_{m_1} is a strict break point for (F_1, s). It follows from the assumption that $\dim_{\mathcal{H}}(J_{F_1 \cup T_1}) \geq s$, where $T_1 = \{a_k \in E \colon k > m_1\}$. In that case we can use Theorem 3.11 again and find a $k_2 > m_1$ such that $F_2 = F_1 \cup \{a_{m_1+1}, \ldots, a_{k_2}\}$ satisfies

$$\dim_{\mathcal{H}}(J_{F_2}) < s \quad \text{and} \quad \dim_{\mathcal{H}}(J_{F_2 \cup \{a_{k_2+1}\}}) \geq s.$$

Now let $m_2 \geq k_2 + 1$ be such that a_{m_2} is a strict break point for (F_2, s). Thus $\dim_{\mathcal{H}}(J_{F_2 \cup T_2}) \geq s$, where $T_2 = \{a_k \in E \colon k > m_2\}$ by the assumption. Repeating this process, we find a nested sequence $F_1 \subset F_2 \subset \ldots \subset A$, with $\max F_n < \max F_{n+1}$ for all $n \in \mathbb{N}$, and indices $m_1 < m_2 < \ldots$ such that $a_{m_n} \in A$ is strict break point for (F_n, s) for all n. It now follows from Lemma 4.7 that $s \in \mathcal{DS}(A)$. □

We will also need a general criterion to identify gaps in the dimension spectrum. This criterion is similar to the one given by Kesseböhmer and Zhu in [30, Theorem 2.4]. For completeness we include a proof of the statement we will need for our purposes. To formulate it, we introduce some notation.

Let $A = \{a_1, a_2, \ldots\} \subseteq \mathbb{N}$, with $a_1 < a_2 < \ldots$, $I_k = \{a_1, \ldots, a_k\}$, and $T_k = \{a_{k+1}, a_{k+2}, \ldots\}$ for $k \geq 1$. Denote $\alpha_k = \dim_{\mathcal{H}}(J_{I_{k-1} \cup T_k}) = \dim_{\mathcal{H}}(J_{A \setminus \{a_k\}})$ and $\beta_k = \dim_{\mathcal{H}}(J_{I_k})$ for $k \geq 1$. Here $I_0 = \emptyset$. Given $F \subset A$ finite, we write

$$F^{\sharp} = (F \setminus \max F) \cup \{a_n \in A \colon a_n > \max F\}. \tag{4.3}$$

Lemma 4.9. *If $\alpha_k < \beta_k$ for some $k \geq 2$, and for each finite $F \subset A$ with $\beta_k < \dim_{\mathcal{H}}(J_F) < \alpha_{k+1}$ we have that*

$$\dim_{\mathcal{H}}(J_{F^\sharp}) < \dim_{\mathcal{H}}(J_F),$$

then $\mathcal{DS}(A)$ is nowhere dense in (β_k, α_{k+1}).

Proof. Let $F \subset A$ finite with $\dim_{\mathcal{H}}(J_F) = s$ and $\beta_k < s < \alpha_{k+1}$. We claim that there exists no $G \subset A$ finite with $\dim_{\mathcal{H}}(J_G) \in (\beta_k, \alpha_{k+1})$ such that

$$\dim_{\mathcal{H}}(J_{F^\sharp}) < \dim_{\mathcal{H}}(J_G) < \dim_{\mathcal{H}}(J_F).$$

Suppose that $G \subset A$ finite with $\dim_{\mathcal{H}}(J_G) \in (\beta_k, \alpha_{k+1})$. Let $a_q = \min(F \cup G) \setminus (F \cap G)$. We note that $I_k \subseteq F, G$, since $\alpha_k < \beta_k \leq \dim_{\mathcal{H}}(J_F), \dim_{\mathcal{H}}(J_G)$ and the fact that $\dim_{\mathcal{H}}(J_{A \setminus \{a_k\}}) \geq \dim_{\mathcal{H}}(J_{A \setminus \{a_m\}})$ for $m \leq k$ by Proposition 3.10 and Theorem 3.11. So, $q > k \geq 2$.

There are four cases to consider. Firstly, $a_q = \max F$. In that case, $a_q \notin G$ so $G \supseteq F \setminus \max F$, hence $G \subseteq F^\sharp$. As $\dim_{\mathcal{H}}(J_{F^\sharp}) < \dim_{\mathcal{H}}(J_F)$, we conclude that $\dim_{\mathcal{H}}(J_G) \leq \dim_{\mathcal{H}}(J_{F^\sharp})$.

The second case to consider is $a_q > \max F$. In that case $F \subset G$, hence $\dim_{\mathcal{H}}(J_F) \leq \dim_{\mathcal{H}}(J_G)$.

As a third case we suppose that $a_q < \max F$ and $a_q \in F$. Let $F_* = F \cap \{a_1, \ldots, a_q\} \supset I_k$. Then $F_* \setminus \{a_q\} = F_* \setminus \max F_*$, so that $G \subset F_*^\sharp$ and $F_* \subseteq F \setminus \max F \subset F^\sharp$. As

$$\alpha_{k+1} > \dim_{\mathcal{H}}(J_F) \geq \dim_{\mathcal{H}}(J_{F_*}) > \dim_{\mathcal{H}}(J_{I_k}) = \beta_k,$$

it follows from the assumption that

$$\dim_{\mathcal{H}}(J_G) \leq \dim_{\mathcal{H}}(J_{F_*^\sharp}) < \dim_{\mathcal{H}}(J_{F_*}) \leq \dim_{\mathcal{H}}(J_{F^\sharp}),$$

which settles this case.

For the remaining case we need to consider $a_q < \max F$ and $a_q \in G$. In that case we consider $G_* = G \cap \{a_1, \ldots, a_q\} \supset I_k$. Then $F \subset G_*^\sharp$, and

$$\beta_k < \dim_{\mathcal{H}}(J_{G_*}) \leq \dim_{\mathcal{H}}(J_G) < \alpha_{k+1}.$$

So, using the assumption we find that

$$\dim_{\mathcal{H}}(J_F) < \dim_{\mathcal{H}}(J_{G_*^\sharp}) < \dim_{\mathcal{H}}(J_{G_*}) \leq \dim_{\mathcal{H}}(J_G),$$

which completes the proof of the claim.

It follows from the claim that any open interval $I \subseteq (\beta_k, \alpha_{k+1})$ contains an open interval I_0 such that $\mathcal{DS}(A) \cap I_0$ is empty. Indeed, if $\mathcal{DS}(A) \cap I$ is non-empty, then there exists $B \subset A$ with $\dim_{\mathcal{H}}(J_B) \in I$. By Theorem 3.11 we know that there exists $F \subset B$ finite with $\dim_{\mathcal{H}}(J_F) \in I$. From the claim we know that there exists no $G \subset A$ finite with

$$\dim_{\mathcal{H}}(J_{F^\sharp}) < \dim_{\mathcal{H}}(J_G) < \dim_{\mathcal{H}}(J_F).$$

So, if we put $I_0 = (\dim_{\mathcal{H}}(J_{F^\sharp}), \dim_{\mathcal{H}}(J_F))$, then $\mathrm{DS}(A) \cap I_0$ is empty by Theorem 3.11. This shows that $\mathcal{DS}(A)$ is nowhere dense in (β_k, α_{k+1}). □

The following result allows us to characterise the dimension spectrum of A near zero. It was motivated by [30, Example 3.4]. This observation enabled us to give an example in the continued fraction case of a set whose dimension spectrum is nowhere dense in $[0, \dim_{\mathcal{H}}(J_A)]$.

Lemma 4.10. *Let $A = \{a_1, a_2, \ldots\} \subseteq \mathbb{N}$ with $a_1 < a_2 < \ldots$. If for any $F \subset A$ finite with $|F| \geq 2$, we have that*

$$\dim_{\mathcal{H}}(J_{F^\sharp}) < \dim_{\mathcal{H}}(J_F),$$

then $\mathcal{DS}(A)$ *is nowhere dense in* $[0, \dim_{\mathcal{H}}(J_{\{a_1,a_2\}})]$.

Proof. Let $s_n = \dim_{\mathcal{H}}(J_{\{a_n,a_{n+1}\}})$ for $n \geq 1$. From Lemma 3.8, we know that if $|F| = 1$, then $\lim_n \dim_{\mathcal{H}}(J_{F\cup\{n\}}) = 0$. Using this argument together with Proposition 3.10 we have that $0 \leq s_n \leq \dim_{\mathcal{H}}(J_{\{a_1,a_{n+1}\}}) \to 0$. Thus $s_n \to 0$. We will show that $\mathcal{DS}(A)$ is nowhere dense in (s_{n+1}, s_n) for all $n \geq 1$. Suppose $G \subseteq A$ is such that $\dim_{\mathcal{H}}(J_G) \in (s_{n+1}, s_n)$.

Claim 1: Then

$$\min G \leq a_{n+1} \quad \text{and} \quad \min(G \setminus \{\min G\}) > a_{n+1}.$$

Indeed, if $\min(G \setminus \{\min G\}) < a_{n+1}$, then by replacing smallest two elements of G by $\{a_n, a_{n+1}\}$, we have that

$$\dim_{\mathcal{H}}(J_G) \geq \dim_{\mathcal{H}}(J_{\{a_n,a_{n+1}\}}) = s_n,$$

by Proposition 3.10, which is impossible. If $\min G > a_{n+1}$, then $G \subseteq \{a_k \colon k \geq n+2\}$. It now follows from Proposition 3.10 and the assumption that

$$\dim_{\mathcal{H}}(J_G) \leq \dim_{\mathcal{H}}(J_{\{a_k \colon k\geq n+2\}}) \leq \dim_{\mathcal{H}}(J_{\{a_{n+1},a_{n+2}\}^\sharp}) < \dim_{\mathcal{H}}(J_{\{a_{n+1},a_{n+2}\}}) = s_{n+1}$$

Thus any $G \subset A$ such that $\dim_{\mathcal{H}}(J_G) \in (s_{n+1}, s_n)$ satisfies Claim 1.

Next we show that for any interval $(a, b) \subseteq (s_{n+1}, s_n)$ there exists an open interval $I_{n+1} \subseteq (a, b)$ such that

$$I_{n+1} \cap \{\dim_{\mathcal{H}}(J_F) \colon F \subseteq A \text{ finite and} \min F \leq a_{n+1}\} = \emptyset.$$

To do this, we will construct I_{n+1} iteratively. Define for $1 \leq k \leq n+1$

$$B_k = \{\dim_{\mathcal{H}}(J_F) \in (s_{n+1}, s_n) \,:\, \min F = a_k \text{ and } F \text{ finite }\} \quad \text{and} \quad B = \bigcup_{k=1}^{n+1} B_k.$$

If $(a,b) \cap B_1 = \emptyset$ we set $I_1 = (a,b)$ otherwise there exists a finite set $F \subset A$ with $\min F = a_1$ such that $\dim_{\mathcal{H}}(J_F) \in (a,b)$.

Claim 2: There is no $G \subset A$ finite such that $a_1 = \min G$ and

$$\dim_{\mathcal{H}}(J_{F^\sharp}) < \dim_{\mathcal{H}}(J_G) < \dim_{\mathcal{H}}(J_F).$$

Let $G \subseteq A$ with $\min G = a_1$ and set $a_q = \min(F \cup G) \setminus (F \cap G)$. So $a_q > a_1$. There are four cases to consider. Firstly $a_q = \max F$. In that case $G \subseteq F^\sharp$ and hence $\dim_{\mathcal{H}}(J_G) \leq \dim_{\mathcal{H}}(J_{F^\sharp})$.

The second case we consider $a_q > \max F$. In that case $F \subset G$, hence $\dim_{\mathcal{H}}(J_F) \leq \dim_{\mathcal{H}}(J_G)$.

As a third case we assume that $a_1 < a_q < \max F$ and $a_q \in F$. Let $F_* = \{a_1, \ldots, a_q\} \cap F$. Then $F_* \setminus \{a_q\} = F_* \setminus \max F_*$, so that $G \subseteq F_*^\sharp$ and $F_* \subset F \setminus \max F \subset F^\sharp$. As F_* is finite, by the assumption we have that $\dim_{\mathcal{H}}(J_{F_*^\sharp}) < \dim_{\mathcal{H}}(J_{F_*})$, so

$$\dim_{\mathcal{H}}(J_G) \leq \dim_{\mathcal{H}}(J_{F_*^\sharp}) < \dim_{\mathcal{H}}(J_{F_*}) \leq \dim_{\mathcal{H}}(J_{F^\sharp}),$$

which settles the case.

For the remaining case we need to consider $a_1 < a_q < \max F$ and $a_q \in G$. In that case we consider $G_* = \{a_1, a_2, \ldots, a_q\} \cap G$. Then $F \subseteq G_*^\sharp$ and $G_* \subseteq G$, so

$$\dim_{\mathcal{H}}(J_F) \leq \dim_{\mathcal{H}}(J_{G_*^\sharp}) < \dim_{\mathcal{H}}(J_{G_*}) \leq \dim_{\mathcal{H}}(J_G)$$

which complete the proof of the claim.

Thus if we set $I_1 = (\dim_{\mathcal{H}}(J_{F^\sharp}), \dim_{\mathcal{H}}(J_F))$, then

$$I_1 \cap B_1 = \emptyset.$$

Now assume that for some $1 \leq k \leq n$ we have constructed an open interval $I_k \subseteq (a, b)$ such that

$$I_k \cap \left(\bigcup_{j=1}^{k} B_j \right) = \emptyset.$$

If $I_k \cap B_{k+1} = \emptyset$, then define $I_{k+1} := I_k$. Otherwise we assume that there exists some finite $F \subset A$ such that $\dim_{\mathcal{H}}(J_F) \in I_k$ and $\min F = a_{k+1}$. By Claim 1

$$\min(F \setminus \min F) > a_{n+1}.$$

Again using the hypothesis and considering the four cases of Claim 2 it can be shown that for $I_{k+1} = (\dim_{\mathcal{H}}(J_{F^\sharp}), \dim_{\mathcal{H}}(J_F))$, we have that

$$I_{k+1} \cap \left(\bigcup_{j=1}^{k+1} B_j \right) = \emptyset.$$

This shows that there exists an interval $I_{n+1} \subseteq (a, b)$ with $I_{n+1} \cap B = \emptyset$.
To complete the proof we need to show that $\mathcal{DS}(A) \cap I_{n+1} = \emptyset$.
If $\mathcal{DS}(A) \cap I_{n+1} \neq \emptyset$ then there exists $G \subseteq A$ infinite such that $\dim_{\mathcal{H}}(J_G) \in I_{n+1}$ and Claim 1 holds on G. Using Theorem 3.11, there exists an $F \subseteq G$ finite such that $\min F \leq a_{n+1}$ and $\min(F \setminus \{\min F\}) > a_{n+1}$ and $\dim_{\mathcal{H}}(J_F) \in I_{n+1}$, but there is no such finite set F, a contradiction. □

Combining Lemma 4.9 and Lemma 4.10 we immediately have the following result.

Corollary 4.11. *Let* $A = \{a_1, a_2, \ldots\} \subseteq \mathbb{N}$ *with* $a_1 < a_2 < \ldots$. *If for any* $F \subset A$ *finite, with* $|F| \geq 2$ *we have that*

$$\dim_{\mathcal{H}}(J_{F^\sharp}) < \dim_{\mathcal{H}}(J_F),$$

then $\mathcal{DS}(A)$ *is nowhere dense in* $[0, \dim_{\mathcal{H}}(J_A)]$.

Proof. By setting

$$\beta_k = \dim_{\mathcal{H}}(J_{\{a_1,\ldots,a_k\}}),\ \alpha_k = \dim_{\mathcal{H}}(J_{A\setminus\{a_k\}}) \text{ and } s_k = \dim_{\mathcal{H}}(J_{\{a_k,a_{k+1}\}})$$

for all k, we have a an infinite partition of $[0, \dim_{\mathcal{H}}(J_A)]$ defined by

$$0 < \ldots < s_3 < s_2 < s_1 = \beta_2 < \alpha_3 < \beta_3 < \alpha_4 < \ldots < \dim_{\mathcal{H}}(J_A),$$

see diagram for the illustration.

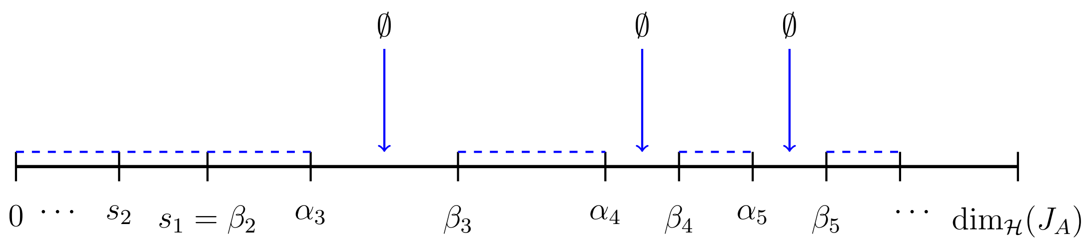
Using Lemma 4.10 we know that $\mathcal{DS}(A)$ is nowhere dense in $[0, s_1]$. For each $k \geq 2$, using Lemma 4.9, we have that $\mathcal{DS}(A)$ is nowhere dense (β_k, α_{k+1}) for all $k \geq 2$. Clearly $\mathcal{DS}(A) \cap (\alpha_k, \beta_k) = \emptyset$ for all $k \geq 2$. To see that $(\alpha_k, \beta_k) \cap \mathcal{DS}(A) = \emptyset$ we argue by contradiction. Suppose that $F \subseteq A$ is such that $\alpha_k < \dim_{\mathcal{H}}(F) < \beta_k$. We claim that $\{a_1 \ldots, a_{k-1}\} \subset F$, as otherwise $F \subseteq A \setminus \{a_m\}$ for some $m \leq k-1$. In that case we get by assumption

$$\dim_{\mathcal{H}}(J_F) < \alpha_m < \beta_m < \beta_{k-1} < \alpha_k,$$

which is impossible. As $\{a_1, \ldots, a_{k-1}\} \subset F$ and $\dim_{\mathcal{H}}(J_F) < \beta_k$, we know that $a_k \notin F$. Thus, $F \subseteq A \setminus \{a_k\}$, which contradicts the fact that $\alpha_k < \dim_{\mathcal{H}}(J_F)$. This implies that $\mathcal{DS}(A)$ is nowhere dense in

$$[\beta_2, \dim_{\mathcal{H}}(J_A))].$$

Thus $\mathcal{DS}(A)$ is nowhere dense in $[0, \dim_{\mathcal{H}}(J_A)]$. □



4.3 Proof of Theorem 4.1

Proof of Theorem 4.1. Clearly 0 and $\sigma = \dim_{\mathcal{H}}(J_A)$ are in the dimension spectrum of A. Take $0 < s < \sigma$. We will use Lemma 4.8. For $m \geq 1$ let $I_m = \{a_1, \ldots, a_m\}$ and let $u\colon x \mapsto 1$ be the order unit on $C([0,1])$. By Theorem 3.11 we know that $\sigma_m = \dim_{\mathcal{H}}(J_{I_m}) \to \sigma$.

Note that for each $m \geq 1$ and $x \in [0,1]$ we have that

$$(L_{s,I_m}u)(x) \leq \sum_{j=1}^{m} \left(\frac{1}{a_j}\right)^{2s} =: \alpha_m(s).$$

We claim that $\alpha_m(s) > 1$ for all m sufficiently large. Indeed, if $\alpha_m(s) \leq 1$ for all m, then $r(L_{s,I_m}) \leq 1$ for all $m \geq 1$. As $0 < s < \sigma$, we know from Theorem 3.2 that

$$1 = r(L_{\sigma_m,I_m}) < r(L_{s,I_m}) \leq \alpha_m(s) \leq 1$$

for all m sufficiently large, since $\sigma_m > s$ for all m large. This is impossible, hence $\alpha_m(s) > 1$ for all m sufficiently large.

Now let $F \subset A$ finite and $a_{k_0} \in A$ be a strict break point for (F, s). So, $r(L_{s,F\cup\{a_{k_0}\}}) \geq 1$. Let v_s be the strictly positive eigenvector for $L_{s,F\cup\{a_{k_0}\}}$, and set $H_m = F \cup \{a_{k_0+j}\colon j = 1, \ldots, m\}$. For $x \in [0,1]$, we have that

$$\frac{a_{k_0} + x}{a_{k_0+j} + x} \geq \frac{a_{k_0}}{a_{k_0+j}},$$

so that

$$\left(\frac{1}{a_{k_0+j} + x}\right)^{2s} \geq \left(\frac{a_{k_0}}{a_{k_0+j}}\right)^{2s} \left(\frac{1}{a_{k_0} + x}\right)^{2s} \quad \text{for } j = 1, \ldots, m.$$

By Theorem 3.2, v_s is a decreasing function on $[0,1]$. This implies that

$$\begin{aligned}(L_{s,H_m}v_s)(x) &= (L_{s,F}v_s)(x)+\sum_{j=1}^{m}\left(\frac{1}{a_{k_0+j}+x}\right)^{2s} v_s\left(\frac{1}{a_{k_0+j}+x}\right)\\ &\geq (L_{s,F}v_s)(x)+\left(\frac{1}{a_{k_0}+x}\right)^{2s} v_s\left(\frac{1}{a_{k_0}+x}\right)\sum_{j=1}^{m}\left(\frac{a_{k_0}}{a_{k_0+j}}\right)^{2s}.\end{aligned}$$

Using the assumption, $a_m a_n \geq a_{m+n}$ for all $m,n\geq 1$, we find that

$$\sum_{j=1}^{m}\left(\frac{a_{k_0}}{a_{k_0+j}}\right)^{2s} \geq \sum_{j=1}^{m}\left(\frac{1}{a_j}\right)^{2s} = \alpha_m(s).$$

As $\alpha_m(s) > 1$ for all $m \geq 1$ sufficiently large, there exists an $M \geq 1$ and a constant $\lambda > 1$ such that

$$(L_{s,H_M}v_s)(x) \geq (L_{s,F}v_s)(x)+\lambda\left(\frac{1}{a_{k_0}+x}\right)^{2s} v_s\left(\frac{1}{a_{k_0}+x}\right).$$

Now using Lemma 2.1 we conclude that there exists $\mu > 1$ such that

$$(L_{s,H_M}v_s)(x) \geq \mu\left((L_{s,F}v_s)(x)+\left(\frac{1}{a_{k_0}+x}\right)^{2s} v_s\left(\frac{1}{a_{k_0}+x}\right)\right) \geq \mu v_s(x).$$

Note that if $m \geq M$, then $H_m \supseteq H_M$. This implies that $r(L_{s,H_m}) > 1$ for all $m \geq M$, hence $\dim_{\mathcal{H}}(J_{H_m}) > s$ for all $m \geq M$. As $F \cup T \supset H_m$, where $T = \{a_n\colon n > k_0\}$, we have that $\dim_{\mathcal{H}}(J_{F\cup T}) > s$. The result now follows from Lemma 4.8. □

4.4 Gaps in $\mathcal{DS}(P_q^*)$: Proof of Theorem 4.3

To establish the structure of the dimension spectrum for P_q^*, the following result is useful.

Theorem 4.12. *Suppose that $F \subset P_q^*$ is finite. If $q \geq 3$ and $\{1, q\} \subseteq F$, or, $q = 2$ and $\{1, 2, 4\} \subseteq F$, then*

$$\dim_{\mathcal{H}}(J_{F^\sharp}) < \dim_{\mathcal{H}}(J_F).$$

where $F^\sharp$ is given by (4.3).

Proof. Suppose that $F \subset P_q^*$ is finite with $\max F = q^k$. Set $G = F \setminus \max F$ and, for $0 < s \leq 1$, let v_s be the positive eigenvector of $L_{s,F}$ with eigenvalue $\lambda_s = r(L_{s,F})$.

Then for each $m \geq q^k$ and $x \in [0,1]$ we have that $\frac{q^k+x}{m+x} \leq \frac{q^k+1}{m+1}$. Furthermore, v_s satisfies

$$v_s\left(\frac{1}{m+x}\right) \leq e^{2s\left(\frac{1}{q^k+x}-\frac{1}{m+x}\right)} v_s\left(\frac{1}{q^k+x}\right) \leq e^{\frac{2s}{q^k}} v_s\left(\frac{1}{q^k+x}\right).$$

Thus,

$$\begin{aligned}(L_{s,F^\sharp}v_s)(x) &= (L_{s,G}v_s)(x) + \sum_{j=1}^{\infty}\left(\frac{1}{q^{k+j}+x}\right)^{2s} v_s\left(\frac{1}{q^{k+j}+x}\right) \\ &\leq (L_{s,G}v_s)(x) + \left(\frac{1}{q^k+x}\right)^{2s} v_s\left(\frac{1}{q^k+x}\right) e^{\frac{2s}{q^k}} \sum_{j=1}^{\infty}\left(\frac{q^k+x}{q^{k+j}+x}\right)^{2s}.\end{aligned}$$

We have that

$$\sum_{j=1}^{\infty}\left(\frac{q^k+x}{q^{k+j}+x}\right)^{2s} \leq \sum_{j=1}^{\infty}\left(\frac{q^k+1}{q^{k+j}+1}\right)^{2s} \leq \left(\frac{q^k+1}{q^k}\right)^{2s} \sum_{j=1}^{\infty}\left(\frac{1}{q^j}\right)^{2s} = \frac{\left(1+\frac{1}{q^k}\right)^{2s}}{q^{2s}-1}.$$

Now let

$$\gamma(k,q,s) = \frac{\left(e^{\frac{1}{q^k}}\left(1+\frac{1}{q^k}\right)\right)^{2s}}{q^{2s}-1} \leq \frac{e^{\frac{4s}{q^k}}}{q^{2s}-1},$$

as $e^x > 1 + x$. Note that if $\gamma(k,q,s) < 1$, then there exists by Lemma 2.1 a $\mu < 1$ such that $L_{s,F^\sharp}v_s \leq \mu L_{s,F}v_s = \mu\lambda_s v_s$. In particular, if this holds for

$s = \dim_{\mathcal{H}}(J_F)$, we get that $L_{s,F^\sharp} v_s \le \mu v_s$. This would imply that $r(L_{s,F^\sharp}) \le \mu < 1$, hence $\dim_{\mathcal{H}}(J_{F^\sharp}) < s$. So we need to show that $\gamma(k,q,s_0) < 1$ where $s_0 = \dim_{\mathcal{H}}(J_F)$.

Firstly suppose that $q \ge 4$ and $k = 1$, so $F = \{1, q\}$. By Corollary 3.7, $\frac{0.52679}{\ln(q)} < s_0 \le 1/2$, so that

$$\gamma(1,q,s_0) \le \frac{e^{4s_0/q}}{q^{2s_0}-1} \le \frac{e^{2/q}}{q^{2s_0}-1} < 1,$$

as $q^{\frac{1.05356}{\ln q}} - 1 = e^{1.05385} - 1 > e^{0.5} \ge e^{2/q}$ for $q \ge 4$.

Likewise, if $q \ge 4$ and $k \ge 2$, then $\frac{0.52679}{\ln q} \le \dim_{\mathcal{H}}(J_{\{1,q\}}) \le s_0 = \dim_{\mathcal{H}}(J_F) \le 1$ and $q^k \ge 2q$, so that

$$\gamma(k,q,s_0) \le \frac{e^{4s_0/q^k}}{q^{2s_0}-1} \le \frac{e^{2/q}}{q^{2s_0}-1} < 1.$$

Let us now consider the case $q = 3$ and $k \ge 2$. In that case

$$\gamma(k,3,s_0) \le \frac{e^{4s_0/3^k}}{3^{2s_0}-1} \le \frac{e^{4/9}}{3^{2s_0}-1} < 0.92 < 1,$$

since $\dim_{\mathcal{H}}(J_{\{1,3\}}) = s_0 \ge 0.454$, see Table 3.1.

The case $q = 3$ and $k = 1$ requires a more refined estimate than $\gamma(1,3,s_0)$. In that case we have that

$$\begin{aligned}(L_{s,F^\sharp} v_s)(x) &= (L_{s,G} v_s)(x) + \sum_{j=1}^{\infty} \left(\frac{1}{3^{1+j}+x}\right)^{2s} v_s\left(\frac{1}{3^{1+j}+x}\right) \\ &\le (L_{s,G} v_s)(x) + \left(\frac{1}{3+x}\right)^{2s} v_s\left(\frac{1}{3+x}\right) \sum_{j=1}^{\infty} \left(\frac{4}{3^{j+1}+1}\right)^{2s} e^{2s\left(\frac{1}{3}-\frac{1}{3^{j+1}}\right)}.\end{aligned}$$

Note that

$$\begin{aligned}
&\sum_{j=1}^{\infty}\left(\frac{4}{3^{j+1}+1}\right)^{2s} e^{2s\left(\frac{1}{3}-\frac{1}{3^{j+1}}\right)} \\
&\quad\le 4^{2s}\left(\left(\frac{e^{2/9}}{10}\right)^{2s}+\left(\frac{e^{8/27}}{28}\right)^{2s}+e^{2s/3}\sum_{j=3}^{\infty}\left(\frac{1}{3^{j+1}}\right)^{2s}\right) \\
&\quad= 4^{2s}\left(\left(\frac{e^{2/9}}{10}\right)^{2s}+\left(\frac{e^{8/27}}{28}\right)^{2s}+\left(\frac{e^{1/3}}{27}\right)^{2s}\left(\frac{1}{3^{2s}-1}\right)\right).
\end{aligned}$$

Now using the fact that $0.454 \le s_0 = \dim_{\mathcal{H}}(J_{\{1,3\}}) \le 0.455$, we get that

$$4^{2s_0}\left(\left(\frac{e^{2/9}}{10}\right)^{2s_0}+\left(\frac{e^{8/27}}{28}\right)^{2s_0}+\left(\frac{e^{1/3}}{27}\right)^{2s_0}\left(\frac{1}{3^{2s_0}-1}\right)\right) < 0.899 < 1,$$

which gives the desired inequality.

Finally let us consider the case $q = 2$ and $\{1, 2, 4\} \subseteq F$. If $k \ge 3$, then

$$\gamma(k, 2, s_0) \le \frac{e^{4s_0/2^k}}{2^{2s_0}-1} \le \frac{e^{s_0/2}}{2^{2s_0}-1} < 0.9 < 1,$$

since $0.669 \le s_0 = \dim_{\mathcal{H}}(J_{\{1,2,4\}}) \le 0.67$, see Table 3.1.

If $k = 2$, then $F = \{1, 2, 4\}$ and $G = \{1, 2\}$, so that

$$\begin{aligned}
&(L_{s,F^\sharp}v_s)(x) = (L_{s,G}v_s)(x) + \sum_{j=1}^{\infty}\left(\frac{1}{2^{2+j}+x}\right)^{2s} v_s\left(\frac{1}{2^{2+j}+x}\right) \\
&\le (L_{s,G}v_s)(x) + \left(\frac{1}{4+x}\right)^{2s} v_s\left(\frac{1}{4+x}\right)\sum_{j=1}^{\infty}\left(\frac{5}{2^{2+j}+1}\right)^{2s} e^{2s\left(\frac{1}{4}-\frac{1}{2^{2+j}}\right)}.
\end{aligned}$$

Note that

$$\begin{aligned}
&\sum_{j=1}^{\infty}\left(\frac{5}{2^{j+2}+1}\right)^{2s} e^{2s\left(\frac{1}{4}-\frac{1}{2^{j+2}}\right)} \\
&\leq 5^{2s}\left(\left(\frac{e^{1/8}}{9}\right)^{2s}+\left(\frac{e^{3/16}}{17}\right)^{2s}+e^{s/2}\sum_{j=3}^{\infty}\left(\frac{1}{2^{j+2}}\right)^{2s}\right) \\
&= 5^{2s}\left(\left(\frac{e^{1/8}}{9}\right)^{2s}+\left(\frac{e^{3/16}}{17}\right)^{2s}+\left(\frac{e^{1/4}}{16}\right)^{2s}\left(\frac{1}{2^{2s}-1}\right)\right).
\end{aligned}$$

Now using the fact that $0.669 \leq s_0 = \dim_{\mathcal{H}}(J_{\{1,2,4\}}) \leq 0.67$, we get that

$$5^{2s_0}\left(\left(\frac{e^{1/8}}{9}\right)^{2s_0}+\left(\frac{e^{3/16}}{17}\right)^{2s_0}+\left(\frac{e^{1/4}}{16}\right)^{2s_0}\left(\frac{1}{2^{2s_0}-1}\right)\right) < 0.984 < 1,$$

which gives the desired inequality. □

Using the previous Theorem it is now easy to prove Theorem 4.3.

Proof of Theorem 4.3. Suppose that $q \geq 3$ and $k \geq 1$. To prove assertion (i) we first note that we can take $F = I_k = \{1, \dots, q^k\}$ in Theorem 4.12 and conclude that $\mu_k < \nu_k$. To see that $(\mu_k, \nu_k) \cap \mathcal{DS}(P_q^*) = \emptyset$ we argue by contradiction. So, suppose that $F \subseteq P_q^*$ is such that $\mu_k < \dim_{\mathcal{H}}(F) < \nu_k$. We claim that $\{1, \dots, q^{k-1}\} \subset F$, as otherwise $F \subseteq P_q^* \setminus \{q^m\}$ for some $m \leq k-1$. In that case we get that $\dim_{\mathcal{H}}(J_F) < \mu_m < \nu_m < \nu_{k-1} < \mu_k$, which is impossible. As $\{1, \dots, q^{k-1}\} \subset F$ and $\dim_{\mathcal{H}}(J_F) < \nu_k$, we know that $q^k \notin F$. Thus, $F \subseteq P_q^* \setminus \{q^k\}$, which contradicts the fact that $\mu_k < \dim_{\mathcal{H}}(J_F)$.

To prove assertion (ii) let $F \subset P_q^*$ be finite with $\nu_k < \dim_{\mathcal{H}}(J_F) < \mu_{k+1}$. Then $\{1, \dots, q^k\} \subset F$, as otherwise $F \subset P_q^* \setminus \{q^m\}$ for some $m \leq k$, which would imply that $\dim_{\mathcal{H}}(J_F) \leq \mu_m < \nu_m \leq \nu_k$. As $\mu_k < \nu_k$ for all $k \geq 1$, we can combine Lemma 4.9 and Theorem 4.12 and conclude that $\mathcal{DS}(P_q^*)$ is nowhere dense in (ν_k, μ_{k+1}) for $k \geq 1$ Note that $\alpha_k = \mu_{k_1}$ and $\beta - k = \nu_{k_1}$ for all $k \geq 1$.

The proof for $q = 2$ can be derived in the same way from Theorem 4.12 and Lemma 4.9. □

4.5 Proof of Theorem 4.4

Proof of Theorem 4.4. Let $0 < s < \frac{\ln 2}{2\ln q}$. To show that s is in the dimension spectrum we verify the condition Lemma 4.8. So, suppose that $F \subset P_q^*$ is finite with strict break point say q^{k_0} for (F, s). Let v_s be the strictly positive eigenvector of $L_{s,F\cup\{q^{k_0}\}}$ with eigenvalue $\lambda_s = r(L_{s,F\cup\{q^{k_0}\}}) \geq 1$ and let $T = \{q^k : k > k_0\}$. Set $T_m = \{q^{k_0+j} : 1 \leq j \leq m\}$.

We know from Theorem 3.2 that v_s is decreasing on $[0, 1]$. Using this fact, we have that for $x \in [0, 1]$,

$$\begin{aligned}(L_{s,F\cup T_m} v_s)(x) &= (L_{s,F} v_s)(x) + \sum_{j=1}^{m} \left(\frac{1}{q^{k_0+j} + x}\right)^{2s} v_s \left(\frac{1}{q^{k_0+j} + x}\right) \\ &\geq (L_{s,F} v_s)(x) + \left(\frac{1}{q^{k_0} + x}\right)^{2s} v_s \left(\frac{1}{q^{k_0} + x}\right) \sum_{j=1}^{m} \left(\frac{q^{k_0} + x}{q^{k_0+j} + x}\right)^{2s} \\ &\geq (L_{s,F} v_s)(x) + \left(\frac{1}{q^{k_0} + x}\right)^{2s} v_s \left(\frac{1}{q^{k_0} + x}\right) \sum_{j=1}^{m} \left(\frac{1}{q^j}\right)^{2s}.\end{aligned}$$

As $s < \frac{\ln 2}{2\ln q}$, we know that $\frac{1}{q^{2s}-1} > 1$, hence there exists an M such that $\sum_{j=1}^{M} \left(\frac{1}{q^j}\right)^{2s} > 1$. So, there exists a $\lambda > 1$ such that for $x \in [0, 1]$,

$$(L_{s,F\cup T_M} v_s)(x) > (L_{s,F} v_s)(x) + \lambda \left(\frac{1}{q^{k_0} + x}\right)^{2s} v_s \left(\frac{1}{q^{k_0} + x}\right).$$

Now using Lemma 2.1 we conclude that there exists $\mu > 1$ such that

$$L_{s,F\cup T_M} v_s(x) \geq \mu \lambda_s v_s(x) \geq \mu v_s,$$

hence $r(L_{s,F\cup T_M}) \geq \mu > 1$, which implies that $\dim_{\mathcal{H}}(J_{F\cup T}) \geq \dim_{\mathcal{H}}(J_{F\cup T_M}) > s$ by Theorem 3.2. Which implies $s \in \mathcal{DS}(P_q^*)$ by Lemma 4.8.

To complete the proof note that clearly 0 is in the dimension spectrum, but also $\frac{\ln 2}{2\ln q}$, as the dimension spectrum is closed by Theorem 2.29 see also [8, Theorem 1.2]. □

4.6 The Dimension Spectrum of M_q: Proof of Theorem 4.5

We will first prove the final statement in Theorem 4.5. In fact, we will show that the following general condition on $A \subseteq \mathbb{N}$ implies that its dimension spectrum is a finite union of disjoint closed intervals.

Definition 4.13. Given an infinite set $A = \{a_1, a_2, \ldots\} \subseteq \mathbb{N}$ with $a_1 < a_2 < \ldots$, we say that A has a *critical break point value* k^* if for each $t \in \mathcal{DS}(A)$ with $0 < t < \dim_{\mathcal{H}}(J_A)$ and each finite set $F \subset A$ with a strict break point a_m for (F, t) and $m > k^*$ we have that

$$\dim_{\mathcal{H}}(J_{F \cup \{a_n : n > m\}}) > t.$$

Proposition 4.14. *If $A = \{a_1, a_2, \ldots\} \subseteq \mathbb{N}$ with $a_1 < a_2 < \ldots$, has a critical break point value, then for each $s \in \mathcal{DS}(A)$ there exists a $\delta > 0$ such that $[s - \delta, s] \subseteq \mathcal{DS}(A)$ or $[s, s + \delta] \subseteq \mathcal{DS}(A)$.*

Proof. Let $s \in \mathcal{DS}(A)$ and $F \subseteq A$, with $\dim_{\mathcal{H}}(J_F) = s$. Suppose first that F is finite. Take $m > k^*$ such that $a_m > \max F$, where k^* is the critical break point value for A. Set $t_1 = \dim_{\mathcal{H}}(J_{F \cup \{a_n : n \geq m\}}) > s$. We will show that each $s < t < t_1$ is in $\mathcal{DS}(A)$. As $t_1 > s$, we know from Theorem 3.11 that either $\dim_{\mathcal{H}}(J_{F \cup \{a_m\}}) \geq t$, in this case we set $F_1 = F$, or, there exists a $k_1 \geq m$ such that $F_1 = F \cup \{a_m, \ldots, a_{k_1}\}$ satisfies

$$\dim_{\mathcal{H}}(J_{F_1}) < t \quad \text{and} \quad \dim_{\mathcal{H}}(J_{F \cup \{a_{k_1+1}\}}) \geq t.$$

In both cases we find that (F_1, t) has a strict break point say a_{m_1} with $m_1 \geq m$. Now using that $m_1 > k^*$ we see that $\dim_{\mathcal{H}}(J_{F_1 \cup \{a_k : k > m_1\}}) > t$. It again follows from Theorem 3.11 that there exists a $k_2 > m_1$ such that $F_2 = F_1 \cup \{a_{m_1+1}, \dots, a_{k_2}\}$ satisfies

$$\dim_{\mathcal{H}}(J_{F_2}) < t \quad \text{and} \quad \dim_{\mathcal{H}}(J_{F \cup \{a_{k_2+1}\}}) \geq t.$$

Let a_{m_2} be a strict break point for (F_2, s). Again as $m_2 > k^*$, we have that $\dim_{\mathcal{H}}(J_{F_2 \cup \{a_k : k > m_2\}}) > t$. It again follows from Theorem 3.11 that there exists a $k_3 > m_2$ such that $F_3 = F_2 \cup \{a_{m_2+1}, \dots, a_{k_3}\}$ satisfies

$$\dim_{\mathcal{H}}(J_{F_3}) < t \quad \text{and} \quad \dim_{\mathcal{H}}(J_{F_3 \cup \{a_{k_3+1}\}}) \geq t.$$

Let a_{m_3} be a strict break point for (F_3, t). By repeating this process we find a nested sequence of sets $F_1 \subset F_2 \subset \dots \subset A$ with $\max F_n < \max F_{n+1}$ and strict break points a_{m_n} for (F_n, t) for each n. It now follows from Lemma 4.7 that $t \in \mathcal{DS}(A)$.

In the case when F is infinite we take $m > k^*$, such that $F' = \{a_k \in F : k < m\}$ is non-empty, so $s_0 := \dim_{\mathcal{H}}(J_{F'}) < s$. Set $s_1 = \dim_{\mathcal{H}}(J_{F' \cup \{a_k : k \geq m\}}) \geq s$. Then using exactly the same reasoning as in the first case with F' instead of F it can be shown that each $s_0 < t < s_1$ is in $\mathcal{DS}(A)$. □

Theorem 4.15. *If $A = \{a_1, a_2, \dots, \} \subseteq \mathbb{N}$ with $a_1 < a_2 < \dots$ has a critical break point value, then $\mathcal{DS}(A)$ is a disjoint union of finitely many closed intervals.*

Proof. We know from Proposition 4.14 that each connected component of $\mathcal{DS}(A)$ is a closed interval, as $\mathcal{DS}(A)$ is closed from Theorem 2.29, it remains to show that it only has finitely many connected components. Suppose by a way of contradiction that it consists of infinitely many connected components say $[\alpha_i, \beta_i]$ for $i \in I$. Let $F_i \subset A$ be such that $\dim_{\mathcal{H}}(J_{F_i}) = \alpha_i$. Note that $\alpha_0 = 0$ and $|F_0| = 1$. For each $i \in I$ we have that $|F_i| \geq 2$.

As there are infinitely many $F_i's$ we know there exists an F_j containing $a_{j_1} < a_{j_2}$ with $j_2 > k^*$ where k^* is the critical break point value of A. Now let $F = F_j \cap \{a_k \colon k < j_2\}$ and set $s_0 = \dim_{\mathcal{H}}(J_F) < \alpha_j$ and $s_1 = \dim_{\mathcal{H}}(J_{F\cup\{a_n \colon n\geq j_2\}})$. To get the contradiction we now use the same argument as in the proof of Proposition 4.14 to show that each $s_0 < t < \alpha_j$ is in $\mathcal{DS}(A)$.

As $s_0 < t < \alpha_j \leq s_1$, we know from Theorem 3.11 that either $\dim_{\mathcal{H}}(J_{F\cup\{a_{j_2}\}}) \geq t$, in this case we set $F_1 = F$, or, there exists a $k_1 \geq j_2$ such that $F_1 = F \cup \{a_{j_2}, \ldots, a_{k_1}\}$ satisfies

$$\dim_{\mathcal{H}}(J_{F_1}) < t \quad \text{and} \quad \dim_{\mathcal{H}}(J_{F\cup\{a_{k_1+1}\}}) \geq t.$$

In both cases we find that (F_1, t) has a strict break point, say a_{m_1}, with $m_1 \geq j_2$. As $m_1 > k^*$ we see that $\dim_{\mathcal{H}}(J_{F_1\cup\{a_k \colon k>m_1\}}) > t$. It now follows from Theorem 3.11 that there exists a $k_2 > m_1$ such that $F_2 = F_1 \cup \{a_{m_1+1}, \ldots, a_{k_2}\}$ satisfies

$$\dim_{\mathcal{H}}(J_{F_2}) < t \quad \text{and} \quad \dim_{\mathcal{H}}(J_{F\cup\{a_{k_2+1}\}}) \geq t.$$

Let a_{m_2} be a strict break point for (F_2, t). Iteratively repeating this process yields a nested sequence of sets $F_1 \subset F_2 \subset \ldots \subset A$ with $\max F_n < \max F_{n+1}$ and strict break point a_{m_n} for (F_n, t) for each n. It now follows from Lemma 4.7 that $t \in \mathcal{DS}(A)$ which contradicts the fact that $[\alpha_j, \beta_j]$ is a connected component of $\mathcal{DS}(A)$. □

We will see that M_q has a critical break point value for $q \geq 11$, namely $k^* = 2q$. To show this we need an upper bound for $\dim_{\mathcal{H}}(J_{M_q})$ for $q \geq 11$. The following bound, which is not sharp will be sufficient for our purpose.

Lemma 4.16. *For $q \geq 11$ we have that $\dim_{\mathcal{H}}(J_{M_q}) \leq \frac{2}{\sqrt{q}}$.*

Proof. Let $q \geq 11$ and $\frac{1}{2q} < s \leq \frac{2}{\sqrt{q}}$. For $k > 2$, let $M_q^k = \{1^q, 2^q, \ldots, k^q\}$. Using the positive eigenvector $v_s(x) = (\lambda + x)^{-2s}$ of $L_{s,1}$ with eigenvalue λ^{-2s}, where

$\lambda = (1+\sqrt{5})/2$ from Lemma 3.5 together with the idea used in the proof of Theorem 3.6 we know

$$v_s\left(\frac{1}{x+n^q}\right) = \frac{(n^q+x)^{2s}}{\lambda^{2s}(n^q+x+\lambda^{-1})^{2s}} = \frac{(n^q+x)^{2s}}{\lambda^{2s}(n^q+x+\lambda-1)^{2s}},$$

as $\lambda^{-1} = \lambda - 1$. This implies

$$\begin{aligned}(L_{s,M_q^m} v_s)(x) &= \lambda^{-2s} v_s(x) + \sum_{n\geq 2}^{m} \left(\frac{1}{n^q+x}\right)^{2s} v_s\left(\frac{1}{n^q+x}\right) \\ &= \lambda^{-2s}\left(1+\sum_{n=2}^{m}\left(\frac{\lambda+x}{n^q+x+\lambda-1}\right)^{2s}\right) v_s(x) \\ &\leq \lambda^{-2s}\left(1+\sum_{n=2}^{m}\left(\frac{\lambda+1}{n^q+\lambda}\right)^{2s}\right) v_s(x).\end{aligned}$$

The last inequality come from the fact that $x \mapsto \frac{\lambda+x}{n^q+x+\lambda-1}$ is increasing in x so it attains is maximum at $x=1$. Thus,

$$\begin{aligned}\lambda^{-2s}\left(1+\sum_{n=2}^{k}\left(\frac{\lambda+1}{n^q+\lambda}\right)^{2s}\right) &= \lambda^{-2s} + \sum_{n=2}^{k}\left(\frac{\lambda}{n^q+\lambda}\right)^{2s} \\ &\leq \lambda^{-2s} + \left(\frac{\lambda}{2^q}\right)^{2s} + \lambda^{2s}\int_2^\infty \frac{1}{x^{2qs}}dx \\ &= \lambda^{-2s} + \left(\frac{\lambda}{2^q}\right)^{2s}\left(1+\frac{2}{2sq-1}\right) \\ &=: \mu(s). \end{aligned} \tag{4.4}$$

Our goal is to show that $\mu(s) < 1$ for $s = \frac{2}{\sqrt{q}}$ and $q \geq 11$. To establish this inequality set

$$h(x) = \lambda^{-4/\sqrt{x}} + \left(\frac{\lambda}{2^x}\right)^{4/\sqrt{x}}\left(1+\frac{2}{4\sqrt{x}-1}\right)$$

for all $x \geq 1$. We need to show that $h(x) < 1$ for all $x \geq 11$. Since $h(x) \to 1$ as $x \to \infty$ it suffices to show that h is strictly increasing for $x \geq 11$.

A direct computation gives

$$h'(x) = \frac{2\ln\lambda}{x\sqrt{x}}\lambda^{-4/\sqrt{x}} - \left(\frac{\lambda}{2^x}\right)^{4/\sqrt{x}}\left[\left(1+\frac{2}{4\sqrt{x}-1}\right)\left(\frac{2\ln 2}{\sqrt{x}}+\frac{2\ln\lambda}{x\sqrt{x}}\right)+\frac{4}{\sqrt{x}(4\sqrt{x}-1)^2}\right].$$

To prove that $h'(x) > 0$ for all $x \geq 11$ we will show that

$$\frac{x\sqrt{x}}{2\ln\lambda}\left(\frac{2^x}{\lambda}\right)^{4/\sqrt{x}} h'(x) \geq 0 \quad \text{for } x \geq 11.$$

Note that

$$\begin{aligned}
&\frac{x\sqrt{x}}{2\ln\lambda}\left(\frac{2^x}{\lambda}\right)^{4/\sqrt{x}} h'(x) \\
= \;& \lambda^{-8/\sqrt{x}}2^{4\sqrt{x}} - \left[\left(1+\frac{2}{4\sqrt{x}-1}\right)\left(\frac{\ln 2}{\ln\lambda}x+1\right)+\frac{2x}{\ln\lambda(4\sqrt{x}-1)^2}\right] \\
\geq \;& \lambda^{-8/\sqrt{11}}2^{4\sqrt{x}} - \left[\left(1+\frac{2}{4\sqrt{11}-1}\right)\left(\frac{\ln 2}{\ln\lambda}x+1\right)+\frac{2}{\ln\lambda(16-8/\sqrt{11})}\right] \\
=: \;& g(x).
\end{aligned}$$

As $g(11) > 0$. Using the derivative of g it is easy to see that g is increasing function for $x \geq 11$, so $h'(x) > 0$ for all $x \geq 11$.

Thus, if we take $s = 2/\sqrt{q}$ for $q \geq 11$ in (4.4), then $\mu(s) < 1$. This implies that $r(L_{s,M_q^k}) \leq \mu(s) < 1$, hence $\dim_{\mathcal{H}}(J_{M_q^k}) < s$ for all k and $q \geq 11$ by Theorem 3.2. It now follows from Theorem 3.11 that $\dim_{\mathcal{H}}(J_{M_q}) \leq s$ for $s = 2/\sqrt{q}$. □

Theorem 4.17. *The set M_q has a critical break point value $k^* = 2q$ for $q \geq 11$.*

Proof. Suppose that $s \in \mathcal{DS}(M_q)$ with $0 < s < \dim_{\mathcal{H}}(J_{M^q})$ and $q \geq 11$. Let k_0^q be strict break point for (F, s) where $F \subset M_q$ is a finite set and $k_0 > 2q$. Let $H_m = F \cup \{k^q \colon k_0 < k \leq m\}$ for $m > k_0$. Consider the operator $L_{s,F\cup\{k_0^q\}}$ with positive eigenvector v_s and eigenvalue $r(L_{s,F\cup\{k_0^q\}}) \geq 1$ as k_0^q is a strict break point for (F,s). Then

$$\begin{aligned}(L_{s,H_m}v_s)(x) &= (L_{s,F}v_s)(x) + \sum_{k=k_0+1}^{m}\left(\frac{1}{k^q+x}\right)^{2s} v_s\left(\frac{1}{k^q+x}\right) \\ &\geq (L_{s,F}v_s)(x) + \left(\frac{1}{k_0^q+x}\right)^{2s} v_s\left(\frac{1}{k_0^q+x}\right)\sum_{k=k_0+1}^{m}\left(\frac{k_0^q}{k^q}\right)^{2s},\end{aligned}$$

as v_s is decreasing by Theorem 3.2 and $\frac{k_0^q+x}{k^q+x} \geq \frac{k_0^q}{k^q}$ for all $x \in [0,1]$ and $k \geq k_0$.

We will now show that $\sum_{k=k_0+1}^{m}\left(\frac{k_0^q}{k^q}\right)^{2s} > 1$ for all m sufficiently large. Note that

$$\sum_{k=k_0+1}^{\infty}\left(\frac{k_0^q}{k^q}\right)^{2s} \geq k_0^{2qs}\int_{k_0+1}^{\infty} x^{-2qs}dx = \left(\frac{k_0}{k_0+1}\right)^{2sq}\frac{k_0+1}{2sq-1}.$$

The map

$$k_0 \mapsto \left(\frac{k_0}{k_0+1}\right)^{2sq}\frac{k_0+1}{2sq-1}$$

is increasing in k_0. Since $k_0 > 2q$ and $s \leq \dim_{\mathcal{H}}(J_{M_q}) \leq 2/\sqrt{q}$ by Lemma 4.16 we find that

$$\sum_{k=k_0+1}^{\infty}\left(\frac{k_0^q}{k^q}\right)^{2s} \geq \left(\frac{2q+1}{2q+2}\right)^{2sq}\frac{2q+2}{2sq-1} \geq \left(\frac{2q+1}{2q+2}\right)^{4\sqrt{q}}\frac{2q+2}{4\sqrt{q}-1} =: \tau(q).$$

We will show that $\tau(q) > 1$ for all $q \geq 11$. Note that the function $g(x) = \left(\frac{2x+1}{2x+2}\right)^{4\sqrt{x}}$ has the property that

$$\ln(g(x)) = 4\sqrt{x}\ln(1 - 1/(2x+2))$$

is increasing, so g is increasing as well. Also the function $x \mapsto \frac{2x+2}{4\sqrt{x}-1}$ is increasing for $x \geq 11$. Thus $\tau(q) \geq \tau(11) \geq 1.112$ for all $q \geq 11$. It follows that for all m sufficiently large that

$$\sum_{k=k_0+1}^{m}\left(\frac{k_0^q}{k^q}\right)^{2s} > 1.$$

Thus there exists a $\mu > 1$ such that

$$(L_{s,H_m} v_s)(x) \geq (L_{s,F} v_s)(x) + \mu \left(\frac{1}{k_0^q + x} \right)^{2s} v_s \left(\frac{1}{k_0^q + x} \right)$$

for all m large. Using Lemma 2.1 we conclude that there exists a $\lambda > 1$ such that $L_{s,H_m} v_s(x) \geq \lambda v_s(x)$, hence $r(L_{s,H_m}) \geq \lambda > 1$ for all m sufficiently large so $\dim_{\mathcal{H}}(J_{F \cup \{k^q \,:\, k > k_0\}}) > s$ which completes the proof. □

As a consequence we find that the final assertion in Theorem 4.5 holds for $q \geq 11$.

Corollary 4.18. *For $q \geq 11$ we have that $\mathcal{DS}(M_q)$ is the disjoint union of finitely many closed intervals*

Proof. Simply combine Theorem 4.15 and 4.17 . □

To complete the proof of Theorem 4.5 we need to establish the first four assertions concerning $\mathcal{DS}(M_q)$ where $1 \leq q \leq 10$. To begin we show that the dimension spectrum of M_q is full for $q \in \{1, 2, 3, 4, 5\}$, which is statement (i) in Theorem 4.5.

Theorem 4.19. *For $q \in \{1, 2, 3, 4, 5\}$ we have that* $\mathrm{DS}(M_q) = [0, \dim_{\mathcal{H}}(J_{M_q})]$.

Proof. Given $0 < s < \dim_{\mathcal{H}}(J_{M_q})$, we will use Lemma 4.8 to show that $s \in \mathrm{DS}(M_q)$. Let n_0^q be a strict break point for (F, s), so $n_0 > 1$. We know that the operator $L_{s,F \cup \{n_0^q\}}$ has a positive eigenvector v_s with eigenvalue

$$\lambda_s = r(L_{s,F \cup \{n_0^q\}}) \geq 1.$$

For $m > n_0$, let $T_m = \{(n_0+1)^q, \dots, m^q\}$ and set $H_m = F \cup T_m$. Then

$$\begin{aligned} L_{s,H_m} v_s(x) &= L_{s,F} v_s(x) + \sum_{k=n_0+1}^{m} \left(\frac{1}{k^q+x}\right)^{2s} v_s\left(\frac{1}{k^q+x}\right) \\ &\geq L_{s,F} v_s(x) + \left(\frac{1}{n_0^q+x}\right)^{2s} v_s\left(\frac{1}{n_0^q+x}\right) \sum_{k=n_0+1}^{m} \left(\frac{n_0^q}{k^q}\right)^{2s}, \end{aligned}$$

as v_s is decreasing by Theorem 3.2 part (3). Set

$$\gamma_m = \sum_{k=n_0+1}^{m} \left(\frac{n_0^q}{k^q}\right)^{2s}. \tag{4.5}$$

If $0 < s \leq (2q)^{-1}$, the sum diverges as $m \to \infty$, hence there exists an $M > n_0$ such that $\gamma_M > 1$. This implies that there exists a $\mu > 1$ such that

$$L_{s,H_M} v_s(x) \geq L_{s,F} v_s(x) + \gamma_M \left(\frac{1}{n_0^q+x}\right)^{2s} v_s\left(\frac{1}{n_0^q+x}\right) \geq \mu L_{s,F\cup\{n_0^q\}} v_s(x) \geq \mu v_s(x)$$

by Lemma 2.1. Thus, $r(L_{s,H_M}) > 1$, which implies that

$$\dim_{\mathcal{H}}(J_{F\cup\{k^q:\, k>n_0\}}) \geq \dim_{\mathcal{H}}(J_{F\cup H_M}) > s,$$

so $s \in \mathcal{DS}(M_q)$ by Lemma 4.8.

Now if $(2q)^{-1} < s < \dim_{\mathcal{H}}(J_{M_q})$, then we consider the following estimate:

$$\begin{aligned} \sum_{k=n_0+1}^{\infty} \left(\frac{n_0^q}{k^q}\right)^{2s} &\geq \left(\frac{n_0}{n_0+1}\right)^{2qs} + \left(\frac{n_0}{n_0+2}\right)^{2qs} + \left(\frac{n_0}{n_0+3}\right)^{2qs} + n_0^{2qs} \int_{n_0+4}^{\infty} x^{-2qs} \mathrm{d}x \\ &= \left(\frac{n_0}{n_0+1}\right)^{2qs} + \left(\frac{n_0}{n_0+2}\right)^{2qs} \\ &\quad + \left(\frac{n_0}{n_0+3}\right)^{2qs} + \left(\frac{n_0}{n_0+4}\right)^{2qs} \frac{n_0+4}{2qs-1} \\ &=: \gamma(q, n_0, s). \end{aligned} \tag{4.6}$$

Note that $n_0 \mapsto \gamma(q, n_0, s)$ is increasing and $s \mapsto \gamma(q, n_0, s)$ is decreasing so we consider respective bounds for the mimimum values of the function $\gamma(q, n_0, s)$. Reasoning as above, it suffices to prove that $\gamma(q, n_0, s) > 1$.

We first consider the case $q = 1$. For $n_0 \geq 2$ and $0 < s \leq 1$ have that

$$\gamma(1, n_0, s) \geq \gamma(1, 2, 1) = 1369/900 > 1.$$

Now consider the case $q = 2$. By Lemma 3.13 we know that $s < \dim_{\mathcal{H}}(J_{M_2}) < 0.67$, and for each $n_0 \geq 3$ we have that

$$\gamma(2, n_0, s) \geq \gamma(2, 3, 0.67) \geq 1.3.$$

If $n_0 = 2$, the estimate $s \leq \dim_{\mathcal{H}}(J_{\{1,2^2\}}) \leq 0.4112$ in Table 3.1 gives

$$\gamma(2, 2, 0.4112) \geq 2.5.$$

The next case is $q = 3$. By Lemma 3.13 we know that $s \leq \dim_{\mathcal{H}}(J_{M_3}) < 0.485$, and for each $n_0 \geq 3$ we have that

$$\gamma(3, n_0, s) \geq \gamma(3, 3, 0.485) \geq 1.1.$$

In case $n_0 = 2$, the estimate $s \leq \dim_{\mathcal{H}}(J_{\{1,2^3\}}) \leq 0.334$ in Table 3.1 gives $\gamma(3, 2, 0.334) \geq 1.5$.

Now consider the case $q = 4$. By Lemma 3.13 we know that $s < \dim_{\mathcal{H}}(J_{M_4}) \leq 0.38$, and for each $n_0 \geq 3$ we have that

$$\gamma(4, n_0, s) \geq \gamma(4, 3, 0.38) \geq 1.01.$$

For $n_0 = 2$, the estimate $s \leq \dim_{\mathcal{H}}(J_{\{1,2^4\}}) \leq 0.281$ in Table 3.1 gives

$$\gamma(4, 2, 0.281) \geq 1.14.$$

Finally we need to check the case $q = 5$. By Lemma 3.13 we know that $s < \dim_{\mathcal{H}}(J_{M_5}) \le 0.31$, and for each $n_0 \ge 4$ we have that

$$\gamma(5, n_0, s) \ge \gamma(5, 4, 0.31) \ge 1.4.$$

For $n_0 = 3$, we have that $s \le \dim_{\mathcal{H}}(J_{\{1,2^5,3^5\}}) \le 0.273$ from Table 3.1, which gives $\gamma(5, 3, 0.273) \ge 1.25$. If $n_0 = 2$, then we cannot use $\gamma(5, n_0, s)$ so we need a different argument. If $n_0 = 2$, then $F = \{1\}$, hence it is sufficient to know that $\dim_{\mathcal{H}}(J_{\{1,2^5\}}) < \dim_{\mathcal{H}}(J_{M_5\setminus\{2^5\}})$. From the estimates in Table 3.1 we see that

$$\dim_{\mathcal{H}}(J_{\{1,2^5\}}) < \dim_{\mathcal{H}}(J_{\{1,3^5,4^5,\dots,100^5\}}) \le \dim_{\mathcal{H}}(J_{M_5\setminus\{2^5\}}),$$

which completes the proof.

□

Next we prove the second statement in Theorem 4.5.

Theorem 4.20. *For $q \ge 6$ we have that*

$$\dim_{\mathcal{H}}(J_{M_q\setminus\{2^q\}}) < \dim_{\mathcal{H}}(J_{\{1,2^q\}}) \tag{4.7}$$

and $\mathcal{DS}(M_q) \cap (\dim_{\mathcal{H}}(J_{M_q\setminus\{2^q\}}), \dim_{\mathcal{H}}(J_{\{1,2^q\}}))$ is empty

Proof. For $q \ge 6$ set $s_q = \dim_{\mathcal{H}}(J_{\{1,2^q\}})$. We will first consider the case where $q \ge 12$. Recall that $s_q \ge 1.525/q > (2q)^{-1}$ by (3.9) for $q \ge 12$. Let v_q be a positive eigenvector of $L_{s_q,\{1,2^q\}}$ with eigenvalue 1. Set $H = M_q \setminus \{2^q\}$ and note that $L_{s,H}$ is a bounded linear operator for all $s > (2q)^{-1}$. Let $H_m = \{1\} \cup \{n^q : 3 \le n \le m\}$ for $m \ge 3$. Using (3.4),

$$\begin{aligned}(L_{s_q,H_m} v_q)(x) &= (L_{s_q,\{1\}} v_q)(x) + \sum_{n=3}^{m} \left(\frac{1}{n^q + x}\right)^{2s_q} v_q\left(\frac{1}{n^q+x}\right) \\ &\le (L_{s_q,\{1\}} v_q)(x) + \left(\frac{1}{2^q + x}\right)^{2s_q} v_q\left(\frac{1}{2^q+x}\right) \sum_{n=3}^{m} \left(\frac{2^q + x}{n^q + x}\right)^{2s_q} e^{2s_q\left(\frac{1}{2^q+x} - \frac{1}{n^q+x}\right)}.\end{aligned}$$

Note that

$$\begin{aligned}\sum_{n=3}^{\infty}\left(\frac{2^q+x}{n^q+x}\right)^{2s_q} e^{2s_q\left(\frac{1}{2^q+x}-\frac{1}{n^q+x}\right)} &\le e^{\frac{2s_q}{2^q}}(2^q+1)^{2s_q}\sum_{n=3}^{\infty} n^{-2s_q q}\\ &\le e^{\frac{2s_q}{2^q}}(2^q+1)^{2s_q}\int_2^{\infty} x^{-2s_q q}\mathrm{d}x\\ &= e^{\frac{2s_q}{2^q}}\left(1+\frac{1}{2^q}\right)^{2s_q}\left(\frac{2}{2s_q q-1}\right)\\ &\le \frac{2e^{\frac{4s_q}{2^q}}}{2s_q q-1},\end{aligned}$$

as $(1+1/n)^n \le e$ for all n.

The map $s \in ((2q)^{-1}, 1] \mapsto \frac{2e^{\frac{4s}{2^q}}}{2sq-1}$ is decreasing for all $q \ge 6$, as

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{2e^{\frac{4s}{2^q}}}{2sq-1}\right) = \frac{4e^{\frac{4s}{2^q}}}{(2sq-1)^2}\left((2sq-1)/2^{q-1}-q\right) \le \frac{4e^{\frac{4s}{2^q}}}{(2sq-1)^2}(q/2^{q-2}-q) < 0.$$

Moreover, the map $q \mapsto \frac{2e^{\frac{4s}{2^q}}}{2sq-1}$ is decreasing as well.

Now using using (3.9), we find that for $q \ge 12$ that

$$\frac{2e^{\frac{4s_q}{2^q}}}{2s_q q-1} \le \frac{2e^{\frac{4(1.525)}{12\cdot 2^{12}}}}{2(1.525)-1} \le 0.98 < 1.$$

This implies for each m that

$$(L_{s_q,H_m}v_q)(x) \le \left(\frac{1}{1+x}\right)^{2s_q} v_q\left(\frac{1}{1+x}\right) + 0.98\left(\frac{1}{2^q+x}\right)^{2s_q} v_q\left(\frac{1}{2^q+x}\right).$$

By Lemma 2.1 there exists a $\mu < 1$ such that $L_{s_q,H_m}v_q \le \mu L_{s_q,\{1,2^q\}}v_q = \mu v_q$, hence $r(L_{s_q,H_m}) \le \mu < 1$ for all m. It now follows from Theorem 3.11 that

$$\dim_{\mathcal{H}}(J_H) < s_q = \dim_{\mathcal{H}}(J_{\{1,2^q\}}),$$

which completes the case where $q \ge 12$.

To deal with the other cases we use the bounds for $s_q = \dim_{\mathcal{H}}(J_{\{1,2^q\}})$ given in Table 3.1 and the following refined estimate,

$$\begin{aligned}
\sum_{n=3}^{\infty} \left(\frac{2^q + x}{n^q + x}\right)^{2s_q} & e^{2s_q\left(\frac{1}{2^q+x} - \frac{1}{n^q+x}\right)} \\
&\leq e^{\frac{2s_q}{2^q}} \left(\left(\frac{2^q+1}{3^q+1}\right)^{2s_q} + \left(\frac{2^q+1}{4^q+1}\right)^{2s_q} + (2^q+1)^{2s_q} \sum_{n=5}^{\infty} n^{-2s_q q} \right) \\
&\leq e^{\frac{2s_q}{2^q}} \left(\left(\frac{2^q+1}{3^q+1}\right)^{2s_q} + \left(\frac{2^q+1}{4^q+1}\right)^{2s_q} + (2^q+1)^{2s_q} \int_4^{\infty} x^{-2s_q q} \mathrm{d}x \right) \\
&= e^{\frac{2s_q}{2^q}} \left(\left(\frac{2^q+1}{3^q+1}\right)^{2s_q} + \left(\frac{2^q+1}{4^q+1}\right)^{2s_q} + \left(\frac{2^q+1}{4^q}\right)^{2s_q} \left(\frac{4}{2s_q q - 1}\right) \right).
\end{aligned}$$

Set

$$\gamma(s,q) = e^{\frac{2s}{2^q}} \left(\left(\frac{2^q+1}{3^q+1}\right)^{2s} + \left(\frac{2^q+1}{4^q+1}\right)^{2s} + \left(\frac{2^q+1}{4^q}\right)^{2s} \left(\frac{4}{2sq - 1}\right) \right).$$

To complete the proof of inequality (4.7), we check for $q \in \{6, \dots, 11\}$ that $\gamma(s_q, q) < 1$. Using the upper and lower bounds in Table 3.1 for $s_q = \dim_{\mathcal{H}}(J_{\{1,2^q\}})$ we see that $\gamma(s_{11}, 11) < 0.63$, $\gamma(s_{10}, 10) < 0.67$, $\gamma(s_9, 9) < 0.72$, $\gamma(s_8, 8) < 0.78$, $\gamma(s_7, 7) < 0.85$, and $\gamma(s_6, 6) < 0.96$.

To show for $q \geq 6$ that $\mathcal{DS}(M_q) \cap (\dim_{\mathcal{H}}(J_{M_q \setminus \{2^q\}}), \dim_{\mathcal{H}}(J_{\{1,2^q\}}))$ is empty, let $\dim_{\mathcal{H}}(J_{M_q \setminus \{2^q\}}) < s < \dim_{\mathcal{H}}(J_{\{1,2^q\}})$. Suppose by way of contradiction that $\dim_{\mathcal{H}}(J_F) = s$ for some $F \subset M_q$. Note that if $2^q \notin F$, then $F \subset M_q \setminus \{2^q\}$, hence $s \leq \dim_{\mathcal{H}}(J_{M_q \setminus \{2^q\}})$, which is impossible. Thus, $2^q \in F$. Now if $1 \notin F$, then $G = (F \setminus \{2^q\}) \cup \{1\} \subset M_q \setminus \{2^q\}$. So, Proposition 3.10 gives

$$s \leq \dim_{\mathcal{H}}(J_G) \leq \dim_{\mathcal{H}}(J_{M_q \setminus \{2^q\}}),$$

which is impossible. So, $\{1, 2^q\} \subseteq F$, hence $\dim_{\mathcal{H}}(J_{\{1,2^q\}}) \leq s$, which is a contradiction. □

Let us now prove the third statement in Theorem 4.5.

Theorem 4.21. *For $q \in \{6,7,8\}$ we have that*

$$\mathrm{DS}(M_q) = [0, \dim_{\mathcal{H}}(J_{M_q \setminus \{2^q\}})] \cup [\dim_{\mathcal{H}}(J_{\{1,2^q\}}), \dim_{\mathcal{H}}(J_{M_q})].$$

Proof. We will use Lemma 4.8.

Suppose first that $s \in [\dim_{\mathcal{H}}(J_{\{1,2^q\}}), \dim_{\mathcal{H}}(J_{M_q})]$ and n_0^q is a strict break point for (F, s), so $n_0 \geq 3$. Reasoning as in the proof of Theorem 4.19 we see that it suffices to show for $(2q)^{-1} < s < \dim_{\mathcal{H}}(J_{M_q})$ that $\gamma(q, n_0, s) > 1$ in (4.6). If $n_0 \geq 4$, then using the estimates in Lemma 3.13 we find that

$$\gamma(6,4,0.265) > 1.3, \quad \gamma(7,4,0.234) > 1.2, \text{ and } \gamma(8,4,0.208) > 1.2.$$

On the other hand if $n_0 = 3$, then we know that $s \leq \dim_{\mathcal{H}}(J_{\{1,2^q,3^q\}})$ and we can use the upper bounds in Table 3.1 to get that

$$\gamma(6,3,0.238636) > 1.3, \text{ and } \gamma(7,3,0.212933) > 1.2.$$

For $q = 8$, we need to expand the sum on the left-hand-side in (4.6) and consider

$$\gamma'(q, n_0, s) := \left(\frac{n_0}{n_0+1}\right)^{2qs} + \left(\frac{n_0}{n_0+2}\right)^{2qs} + \left(\frac{n_0}{n_0+3}\right)^{2qs} + \left(\frac{n_0}{n_0+4}\right)^{2qs} + \left(\frac{n_0}{n_0+5}\right)^{2qs} + \left(\frac{n_0}{n_0+6}\right)^{2qs} \frac{n_0+6}{2qs-1},$$

which satisfies $\gamma'(8,3,0.197286) > 1.004$.

If $s \in [0, \dim_{\mathcal{H}}(J_{M_q \setminus \{2^q\}})]$, then we can use Lemma 4.8 with respect to $A = M_q \setminus \{2^q\}$. So, if n_0^q is a strict break point for (F, s), then $n_0 \geq 3$. Now the same inequalities for $\gamma(q, n_0, s)$ and $\gamma'(q, n_0, s)$ as above imply that

$$s \in \mathcal{DS}(M_q \setminus \{2^q\}) \subset \mathcal{DS}(M_q).$$

□

To complete the proof of Theorem 4.5 it remains to show the fourth assertion.

Theorem 4.22. *For* $q \in \{9, 10, 11, 12\}$ *we have that* $\dim_{\mathcal{H}}(J_{M_q \setminus \{3^q\}}) < \dim_{\mathcal{H}}(J_{\{1,2^q,3^q\}})$ *and*

$$\begin{aligned}\mathcal{DS}(M_q) &= [0, \dim_{\mathcal{H}}(J_{M_q \setminus \{2^q\}})] \\ &\quad \cup [\dim_{\mathcal{H}}(J_{\{1,2^q\}}), \dim_{\mathcal{H}}(J_{M_q \setminus \{3^q\}})] \cup [\dim_{\mathcal{H}}(J_{\{1,2^q,3^q\}}), \dim_{\mathcal{H}}(J_{M_q})].\end{aligned}$$

Proof. To establish the inequality we reason as in the proof of Theorem 4.20. Let $s_q = \dim_{\mathcal{H}}(J_{\{1,2^q,3^q\}})$ and v_q be the strictly positive eigenvector of $L_{s_q,\{1,2^q,3^q\}}$ with eigenvalue 1. So $s_q > 1.525/q > (2q)^{-1}$ by (3.9) and $L_{s,H}$ with $H = M_q \setminus \{3^q\}$, is a bounded linear operator for $s > (2q)^{-1}$. Let $H_m = \{1, 2^q\} \cup \{n^q : 4 \leq n \leq m\}$ for $m \geq 4$. Using (3.6),

$$\begin{aligned}(L_{s_q,H_m} v_q)(x) &= \left(\frac{1}{1+x}\right)^{2s_q} v_q\left(\frac{1}{1+x}\right) + \left(\frac{1}{2^q+x}\right)^{2s_q} v_q\left(\frac{1}{2^q+x}\right) \\ &\quad + \sum_{n=4}^{m} \left(\frac{1}{n^q+x}\right)^{2s_q} v_q\left(\frac{1}{n^q+x}\right) \\ &\leq \left(\frac{1}{1+x}\right)^{2s_q} v_q\left(\frac{1}{1+x}\right) + \left(\frac{1}{2^q+x}\right)^{2s_q} v_q\left(\frac{1}{2^q+x}\right) \\ &\quad + \left(\frac{1}{3^q+x}\right)^{2s_q} v_q\left(\frac{1}{3^q+x}\right) \sum_{n=4}^{m} \left(\frac{3^q+x}{n^q+x}\right)^{2s_q} e^{2s_q\left(\frac{1}{3^q+x} - \frac{1}{n^q+x}\right)}.\end{aligned}$$

Note that for $k \geq 4$ we have that

$$\begin{aligned}&\sum_{n=4}^{\infty} \left(\frac{3^q+x}{n^q+x}\right)^{2s_q} e^{2s_q\left(\frac{1}{3^q+x} - \frac{1}{n^q+x}\right)} \\ &\quad \leq e^{\frac{2s_q}{3^q}} \left(\left(\frac{3^q+1}{4^q+1}\right)^{2s_q} + \cdots + \left(\frac{3^q+1}{k^q+1}\right)^{2s_q} + (3^q+1)^{2s_q} \sum_{n=k+1}^{\infty} n^{-2s_q q} \right) \\ &\quad \leq e^{\frac{2s_q}{3^q}} \left(\left(\frac{3^q+1}{4^q+1}\right)^{2s_q} + \cdots + \left(\frac{3^q+1}{k^q+1}\right)^{2s_q} + (3^q+1)^{2s_q} \int_k^{\infty} x^{-2s_q q} \mathrm{d}x \right) \\ &\quad = e^{\frac{2s_q}{3^q}} \left(\left(\frac{3^q+1}{4^q+1}\right)^{2s_q} + \cdots + \left(\frac{3^q+1}{k^q+1}\right)^{2s_q} + \left(\frac{3^q+1}{k^q}\right)^{2s_q} \left(\frac{k}{2s_q q - 1}\right) \right) \\ &\quad =: \beta(s_q, q, k).\end{aligned}$$

Using the upper and lower bounds for s_q in Table 3.1 and taking $k = 8$, we find that $\beta(s_9, 9, 8) < 0.99$, $\beta(s_{10}, 10, 8) < 0.94$, $\beta(s_{11}, 11, 8) < 0.9$, $\beta(s_{11}, 11, 8) < 0.9$ and $\beta(s_{12}, 12, 8) \leq 0.89$. It now follows from Lemma 2.1 that there exists a $\mu < 1$ such that $L_{s_q,H_m} v_q \leq \mu L_{s_q,\{1,2^q,3^q\}} v_q = \mu v_q$. So, $r(L_{s_q,H_m}) \leq \mu < 1$ for all m, which implies that $\dim_{\mathcal{H}}(F_H) < s_q = \dim_{\mathcal{H}}(J_{\{1,2^q,3^q\}})$ by Theorem 3.11 as $s_q > (2q)^{-1}$.

Reasoning in the same way as in the proof of Theorem 4.20 it can easily be shown for $q \in \{9, 10, 11, 12\}$ that there is no $s \in \mathrm{DS}(M_q)$ between the closed intervals.

To show that each element in the intervals belongs to the dimension spectrum we will use Lemma 4.8. Suppose first that $s \in [\dim_{\mathcal{H}}(J_{\{1,2^q,3^q\}}), \dim_{\mathcal{H}}(J_{M_q})]$ and n_0^q is a strict break point for (F, s), so $n_0 \geq 4$. Using the same arguments as in the proof of Theorem 4.19 we see that it suffices to show for $(2q)^{-1} < s < \dim_{\mathcal{H}}(J_{M_q})$ that $\gamma(q, n_0, s) > 1$ in (4.6) to conclude that $s \in \mathrm{DS}(M_q)$. If $n_0 \geq 4$, we can use the upper bounds in Lemma 3.13 to get that $\gamma(9, 4, 0.191) > 1.1$, $\gamma(10, 4, 0.175) > 1.1$, $\gamma(11, 4, 0.163) > 1.15$ and $\gamma(12, 4, 0.152) > 1.1$.

On the other hand, if $s \in [\dim_{\mathcal{H}}(J_{\{1,2^q\}}), \dim_{\mathcal{H}}(J_{M_q \setminus \{3^q\}})]$, we can apply Lemma 4.8 with $A = M_q \setminus \{3^q\}$. In that case, if n_0^q is a strict break point for (F, s), then $n_0 \geq 4$, and the same estimates as above hold. So, $s \in \mathcal{DS}(M_q \setminus \{3^q\}) \subset \mathcal{DS}(M_q)$. Finally, for $s \in [0, \dim_{\mathcal{H}}(J_{M_q \setminus \{2^q\}})]$ we apply Lemma 4.8 with $A = M_q \setminus \{2^q\}$. So, if n_0^q is a strict break point for (F, s), then $n_0 \geq 3$. Using the upper bound for $\dim_{\mathcal{H}}(J_{\{1,2^q\}})$ in Table 3.1 for $q = 9, 10$ and $q = 11$, we get that

$$\gamma(9, 3, 0.162510) > 1.09, \quad \gamma(10, 3, 0.150820) > 1.02.$$

The case when $q = 11$, we expand the terms to the right hand in ((4.6)) and consider

$$\gamma'(q, n_0, s) := \left(\frac{n_0}{n_0+1}\right)^{2qs} + \left(\frac{n_0}{n_0+2}\right)^{2qs} + \ldots + \left(\frac{n_0}{n_0+7}\right)^{2qs} + \left(\frac{n_0}{n_0+8}\right)^{2qs} \frac{n_0+8}{2qs-1},$$

and we get $\gamma'(11, 3, 0.140915) > 1.002$, If $q = 12$, we need a different argument, with $A = M_q \setminus \{2^q\}$. If $n_0 \geq 4$, then the same upper bound $\gamma(12, 4, 0.152) > 1.1$ can be used but if $n_0 = 3$, then we should have that $s \leq \dim_{\mathcal{H}}(J_{\{1,3^q\}})$ because 2^q is not part of the alphabet and we have that $\gamma(12, 3, 0.094745) > 1.9$. It follows that $s \in \mathcal{DS}(M_q \setminus \{2^q\}) \subset \mathcal{DS}(M_q)$ and we are done. □

The following result show that there exists four intervals in the dimension spectrum of M_q.

Theorem 4.23. *For $q = 19$ we have that*

$$\dim_{\mathcal{H}}(J_{M_q \setminus \{3^q\}}) < \dim_{\mathcal{H}}(J_{\{1,2^q,3^q\}}) \quad \textit{and} \quad \dim_{\mathcal{H}}(J_{M_q \setminus \{4^q\}}) < \dim_{\mathcal{H}}(J_{\{1,2^q,3^q,4^q\}}).$$

In addition,

$$\begin{aligned}\mathcal{DS}(M_q) =& [0, \dim_{\mathcal{H}}(J_{M_q \setminus \{2^q\}})] \cup [\dim_{\mathcal{H}}(J_{\{1,2^q\}}), \dim_{\mathcal{H}}(J_{M_q \setminus \{3^q\}})] \\ & \cup [\dim_{\mathcal{H}}(J_{\{1,2^q,3^q\}}), \dim_{\mathcal{H}}(J_{M_q \setminus \{4^q\}})] \cup [\dim_{\mathcal{H}}(J_{\{1,2^q,3^q,4^q\}}), \dim_{\mathcal{H}}(J_{M_q})].\end{aligned}$$

Proof. Again to establish the inequalities we reason as in the proof of Theorem 4.22. Let $s_q = \dim_{\mathcal{H}}(J_{\{1,2^q,3^q\}})$ and v_q be the strictly positive eigenvector of $L_{s_q,\{1,2^q,3^q\}}$ with eigenvalue 1. So $s_q > 1.525/q > (2q)^{-1}$ by (3.9) and $L_{s,H}$ with $H = M_q \setminus \{3^q\}$, is a bounded linear operator for $s > (2q)^{-1}$. Let $H_m = \{1, 2^q\} \cup \{n^q \colon 4 \leq n \leq m\}$ for $m \geq 4$. Using (3.6), together with $\beta(s_q, q, k)$

defined in the proof of Theorem 4.22, we have that

$$\begin{aligned}(L_{s_q,H_m}v_q)(x) &= \left(\frac{1}{1+x}\right)^{2s_q} v_q\left(\frac{1}{1+x}\right) + \left(\frac{1}{2^q+x}\right)^{2s_q} v_q\left(\frac{1}{2^q+x}\right) \\ &\quad + \sum_{n=4}^{m}\left(\frac{1}{n^q+x}\right)^{2s_q} v_q\left(\frac{1}{n^q+x}\right) \\ &\leq \left(\frac{1}{1+x}\right)^{2s_q} v_q\left(\frac{1}{1+x}\right) + \left(\frac{1}{2^q+x}\right)^{2s_q} v_q\left(\frac{1}{2^q+x}\right) \\ &\quad + \left(\frac{1}{3^q+x}\right)^{2s_q} v_q\left(\frac{1}{3^q+x}\right)\sum_{n=4}^{m}\left(\frac{3^q+x}{n^q+x}\right)^{2s_q} e^{2s_q\left(\frac{1}{3^q+x}-\frac{1}{n^q+x}\right)}. \\ &\leq \left(\frac{1}{1+x}\right)^{2s_q} v_q\left(\frac{1}{1+x}\right) + \left(\frac{1}{2^q+x}\right)^{2s_q} v_q\left(\frac{1}{2^q+x}\right) \\ &\quad + \beta(s_q,q,k)\left(\frac{1}{3^q+x}\right)^{2s_q} v_q\left(\frac{1}{3^q+x}\right)\end{aligned}$$

Using the upper and lower bounds for s_q in Table 3.1 and taking $k = 8$, we find that $\beta(s_{19}, 19, 8) < 0.7$, It now follows from Lemma 2.1 that there exists a $\mu < 1$ such that $L_{s_q,H_m}v_q \leq \mu L_{s_q,\{1,2^q,3^q\}}v_q = \mu v_q$. So, $r(L_{s_q,H_m}) \leq \mu < 1$ for all m, which implies that $\dim_{\mathcal{H}}(J_{M_q\setminus\{3^q\}}) < s_q = \dim_{\mathcal{H}}(J_{\{1,2^q,3^q\}})$ by Theorem 3.11 as $s_q > (2q)^{-1}$.

Next using $s_q = \dim_{\mathcal{H}}(J_{\{1,2^q,3^q,4^q\}})$, and taking

$$\beta(s_q,q,k) = e^{\frac{2s_q}{4^q}}\left(\left(\frac{4^q+1}{5^q+1}\right)^{2s_q} + \cdots + \left(\frac{4^q+1}{k^q+1}\right)^{2s_q} + \left(\frac{4^q+1}{k^q}\right)^{2s_q}\left(\frac{k}{2s_q q-1}\right)\right)$$

we have that if v_{s_q} is the eigenvector of $L_{s_q,\{1,2^q,3^q,4^q\}}$ with the corresponding eigenvalue 1 then

$$\begin{aligned}(L_{s_q,M_q\setminus\{4^q\}}v_{s_q})(x) &\leq \sum_{n\leq 3}\left(\frac{1}{n^q+x}\right)^{2s_q} v_{s_q}\left(\frac{1}{n^q+x}\right) \\ &\quad + \beta(s_q,q,k)\left(\frac{1}{4^q+x}\right)^{2s_q} v_{s_q}\left(\frac{1}{4^q+x}\right)\end{aligned}$$

and with $k = 9$ we have $\beta(s_q,q,k) \leq 0.995$. It now follows from Lemma 2.1 that there exists a $\mu < 1$ such that $L_{s_q,M_q\setminus\{4^q\}}v_q \leq \mu L_{s_q,\{1,2^q,3^q,4^q\}}v_q =$

μv_q. So, $r(L_{s_q, M_q \setminus \{4^q\}}) \leq \mu < 1$, which implies that $\dim_{\mathcal{H}}(J_{M_q \setminus \{4^q\}}) < s_q = \dim_{\mathcal{H}}(J_{\{1,2^q,3^q,4^q\}})$. Reasoning in the same way as in the proof of Theorem 4.20 it can easily be shown for $q = 19$ that there is no $s \in \mathrm{DS}(M_q)$ between the closed intervals.

To show that each element in the intervals belongs to the dimension spectrum we will use Lemma 4.8. Suppose first that $s \in [\dim_{\mathcal{H}}(J_{\{1,2^q,3^q,4^q\}}), \dim_{\mathcal{H}}(J_{M_q})]$ and n_0^q is a strict break point for (F, s), so $n_0 \geq 5$. Using the same arguments as in the proof of Theorem 4.19 we see that it suffices to show for

$$(2q)^{-1} < s < \dim_{\mathcal{H}}(J_{M_q})$$

that $\gamma(q, n_0, s) > 1$ in (4.6) to conclude that $s \in \mathrm{DS}(M_q)$. If $n_0 \geq 5$, we can first compute the upper bound of $\dim_{\mathcal{H}}(J_{M_{19}})$ as in Lemma 3.13 and obtain that $\alpha(19, 0.105) \leq 0.998$ so that $\dim_{\mathcal{H}}(J_{M_{19}}) \leq 0.105$. Using this in (4.6) we get that $\gamma(19, 5, 0.105) > 1.2$.

On the other hand, if $s \in [\dim_{\mathcal{H}}(J_{\{1,2^q,3^q\}}), \dim_{\mathcal{H}}(J_{M_q \setminus \{4^q\}})]$, we can apply Lemma 4.8 with $A = M_q \setminus \{4^q\}$. In that case, if n_0^q is a strict break point for (F, s), then $n_0 \geq 5$, and the same estimates as above hold. So, $s \in \mathcal{DS}(M_q \setminus \{4^q\}) \subset \mathcal{DS}(M_q)$.

Also if $s \in [\dim_{\mathcal{H}}(J_{\{1,2^q\}}), \dim_{\mathcal{H}}(J_{M_q \setminus \{3^q\}})]$, then the break point is at least 4 if we restrict the alphabet to $M_q \setminus \{3^q\}$. Using the upper bound on the $\dim_{\mathcal{H}}(J_{\{1,2^q,3^q\}}) \leq$ we have that $\gamma(19, 4, 0.10025757) \geq 1.008$. Finally, for $s \in [0, \dim_{\mathcal{H}}(J_{M_q \setminus \{2^q\}})]$ we apply Lemma 4.8 with $A = M_q \setminus \{2^q\}$. So, if n_0^q is a strict break point for (F, s), then $n_0 \geq 3$. If $n_0 \geq 4$ we can use the previous bounds to get $\gamma(19, 4, 0.10025757) \geq 1.008$. However, if the break point is $n_0 = 3$, then we need the upper bound of $\dim_{\mathcal{H}}(J_{\{1,3^q\}}) \leq 0.06732451$ which implies that $\gamma(19, 3, 0.06732451) \geq 1.15$. It follows that $s \in \mathcal{DS}(M_q \setminus \{2^q\}) \subset \mathcal{DS}(M_q)$ and we are done. □

4.7 Nowhere Dense Dimension Spectrum: Proof of Theorem 4.6

Before we prove Theorem 4.6, we need a lower bound on the Hausdorff dimension of the pair $\{a_n, a_{n+1}\}$ for each n.

Lemma 4.24. *If* $a_n = 2^{2^n}$ *for* $n \in \mathbb{N}$, *then*

$$\dim_{\mathcal{H}}(J_{\{a_n,a_{n+1}\}}) \geq \frac{1}{2^{n+2}}.$$

Proof. Using Lemma 3.5 with $\mu = a_n$ and setting $\lambda = \frac{a_n+\sqrt{a_n^2+4}}{2}$. Then $v_s(x) = (\lambda + x)^{-2s}$ is an eigenvector of $L_{s,\{a_n\}}$ with eigenvalue λ^{-2s}. Also

$$(L_{s,\{a_n,a_{n+1}\}}v_s)(x) = \lambda^{-2s}\left(1 + \left(\frac{\lambda + x}{a_{n+1} + x + \lambda - 1}\right)^{2s}\right) v_s(x).$$

As the map $x \mapsto \frac{\lambda+x}{a_{n+1}+x+\lambda-1}$ is increasing, we have that

$$(L_{s,\{a_n,a_{n+1}\}}v_s)(x) \geq \left(\left(\frac{1}{\lambda}\right)^{2s} + \left(\frac{1}{a_{n+1} + \lambda - 1}\right)^{2s}\right) v_s(x)$$

If we can show for $t_n = \frac{1}{2^{n+2}}$, that

$$\left(\frac{1}{\lambda}\right)^{2t_n} + \left(\frac{1}{a_{n+1} + \lambda - 1}\right)^{2t_n} \geq 1, \tag{4.8}$$

then it follows from Lemma 2.2 that $r(L_{t_n,\{a_n,a_{n+1}\}}) \geq 1$ so that $\dim_{\mathcal{H}}(J_{\{a_n,a_{n+1}\}}) \geq t_n$ by Theorem 3.2 and we are done. Assume $n \geq 1$ and substituting y for a_n in (4.8) to get a formulation

$$h(y) = \left(\frac{2}{y + \sqrt{y^2 + 4}}\right)^{\frac{\ln 2}{2\ln y}} + \left(\frac{1}{y^2 + \frac{y+\sqrt{y^2+4}}{2} - 1}\right)^{\frac{\ln 2}{2\ln y}} =: f(y) + g(y).$$

If we can show that $h(y)$ is increasing on $(4, \infty)$ and $h(4) \geq 1$ then we are done.

A direct computation show that $h(4) \geq 1.15$, so it remains to show that h' is increasing. We establish this by showing that $g' \geq 0$ and $f' \geq 0$ separately.

$$-\frac{f'(y)}{f(y)} = \frac{\ln 2 \ln\left(\frac{2}{y+\sqrt{y^2+4}}\right)}{2y\ln^2 y} + \frac{\ln 2\left(1+\frac{y}{\sqrt{y^2+4}}\right)}{2(y+\sqrt{y^2+4})\ln y}.$$

Note that

$$\ln\left(\frac{2}{y+\sqrt{y^2+4}}\right) = \ln 2 - \ln(y+\sqrt{y^2+4}) \leq \ln 2 - \ln(2y) = -\ln y.$$

Thus

$$\frac{\ln 2 \ln\left(\frac{2}{y+\sqrt{y^2+4}}\right)}{2y\ln^2 y} < -\frac{\ln 2}{2y\ln y}.$$

Also

$$\frac{\ln 2\left(1+\frac{y}{\sqrt{y^2+4}}\right)}{2(y+\sqrt{y^2+4})\ln y} < \frac{2\ln 2}{2(2y)\ln y} = \frac{\ln 2}{2y\ln y}.$$

This implies that

$$-\frac{f'(y)}{f(y)} < \frac{\ln 2}{2y\ln y} - \frac{\ln 2}{2y\ln y} = 0.$$

As $f \geq 0$, we have that $f' > 0$. On the other hand

$$-\frac{g'(y)}{g(y)} = \frac{\ln 2 \ln\left(\frac{1}{y^2+\frac{y+\sqrt{y^2+4}}{2}-1}\right)}{2y\ln^2 y} + \frac{\ln 2\left(2y+\frac{1}{2}+\frac{y}{2\sqrt{y^2+4}}\right)}{2\ln y\left(y^2+\frac{y+\sqrt{y^2+4}}{2}-1\right)}$$

Using

$$\frac{\ln 2 \ln\left(\frac{1}{y^2+\frac{y+\sqrt{y^2+4}}{2}-1}\right)}{2y\ln^2 y} = -\frac{\ln 2 \ln\left(y^2+\frac{y+\sqrt{y^2+4}}{2}-1\right)}{2y\ln^2 y} < -\frac{\ln 2}{y\ln y},$$

together with

$$\frac{\ln 2\left(2y+\frac{1}{2}+\frac{y}{2\sqrt{y^2+4}}\right)}{2\ln y\left(y^2+\frac{y+\sqrt{y^2+4}}{2}-1\right)} < \frac{(2y+1)\ln 2}{\ln y(2y^2+2y-2)} = \frac{(2y+1)\ln 2}{y\ln y\left(2y+2-\frac{2}{y}\right)} < \frac{\ln 2}{y\ln y},$$

we obtain

$$-\frac{g'(y)}{g(y)} < \frac{\ln 2}{y\ln y} - \frac{\ln 2}{y\ln y} = 0.$$

As $g \geq 0$, we have that $g' \geq 0$ so that h is increasing. □

Let us now come back to Theorem 4.6

Proof. We first obtain an upper bound for the $\dim_{\mathcal{H}}(J_A)$. If $u\colon x \mapsto 1$ is the order unit on $C([0,\gamma^{-1}])$ where $\gamma = \min A$ and $s > 0$ then

$$(L_{s,A}u)(x) = \sum_{n\in\mathbb{N}}\left(\frac{1}{2^{2^n}+x}\right)^{2s} \leq \sum_{n\in\mathbb{N}}\left(\frac{1}{2^{2^n}}\right)^{2s} \leq \sum_{n\in\mathbb{N}}\left(\frac{1}{2^{2n}}\right)^{2s} = \frac{1}{2^{4s}-1}.$$

In particular if $s = 1/4$ we have that $L_{s,A}u \leq u$ so $r(L_{1/4,A}) \leq 1$. Thus $\dim_{\mathcal{H}}(J_A) \leq 1/4$.

Note that to prove $\mathcal{DS}(A)$ is nowhere dense, it suffices by Corollary 4.11 to show that for $F \subset A$ finite, with $|F| \geq 2$ we have that

$$\dim_{\mathcal{H}}(J_{F^\sharp}) < \dim_{\mathcal{H}}(J_F).$$

Let $F \subseteq A$ with $|F| \geq 2$, $a_n = \max F$, and set $G = F \setminus \{a_n\}$. Proposition 3.10 implies that

$$\dim_{\mathcal{H}}(J_F) \geq \dim_{\mathcal{H}}(J_{\{a_{n-1},a_n\}}) \geq \frac{1}{2^{n+1}}$$

Fix $\frac{1}{2^{n+1}} \leq s \leq \dim_{\mathcal{H}}(J_A)$ and let v_s be the eigenvector of $L_{s,F}$ with eigenvalue $\lambda_s = r(L_{s,F})$. We have the following

$$\begin{aligned}(L_{s,F^\sharp}v_s)(x) &= L_{s,G}v_s(x) + \sum_{k=1}^{\infty}\left(\frac{1}{a_{n+k}+x}\right)^{2s} v_s\left(\frac{1}{a_{n+k}+x}\right)\\ &\leq (L_{s,G}v_s)(x) + \left(\frac{1}{a_n+x}\right)^{2s} v_s\left(\frac{1}{a_n+x}\right) e^{\frac{2s}{a_n}} \sum_{k=1}^{\infty}\left(\frac{a_n+1}{a_{n+k}+1}\right)^{2s}.\end{aligned}$$

As

$$\frac{a_n+1}{a_{n+k}+1} \leq \frac{a_n+1}{a_{n+k}} = \left(1+\frac{1}{a_n}\right)\frac{a_n}{a_{n+k}},$$

$$e^{\frac{2s}{a_n}} \sum_{k=1}^{\infty}\left(\frac{a_n+1}{a_{n+k}+1}\right)^{2s} \leq e^{\frac{2s}{a_n}}\left(1+\frac{1}{a_n}\right)^{2s} \sum_{k=1}^{\infty}\left(\frac{a_n}{a_{n+k}}\right)^{2s}.$$

Using the bounds $\frac{1}{2^{n+1}} \leq s \leq 1/4$ and the fact that $s \mapsto \sum_{k=1}^{\infty}\left(\frac{a_n}{a_{n+k}}\right)^{2s}$ is decreasing, we have that

$$\begin{aligned}e^{\frac{2s}{a_n}}\left(1+\frac{1}{a_n}\right)^{2s} \sum_{k=1}^{\infty}\left(\frac{a_n}{a_{n+k}}\right)^{2s} &\leq e^{\frac{1}{2a_n}}\left(1+\frac{1}{a_n}\right)^{\frac{1}{2}} \sum_{k=1}^{\infty}\left(\frac{a_n}{a_{n+k}}\right)^{\frac{1}{2^n}}\\ &= e^{\frac{1}{2a_n}}\left(1+\frac{1}{a_n}\right)^{\frac{1}{2}} \sum_{k=1}^{\infty}\left(\frac{2^{2^n}}{2^{2^{n+k}}}\right)^{\frac{1}{2^n}}\\ &= 2e^{\frac{1}{2a_n}}\left(1+\frac{1}{a_n}\right)^{\frac{1}{2}} \sum_{k=1}^{\infty}\frac{1}{2^{2^k}}\\ &\leq 2e^{\frac{1}{2a_n}}\left(1+\frac{1}{a_n}\right)^{\frac{1}{2}} \sum_{k=1}^{\infty}\frac{1}{2^{2k}}\\ &= \frac{2}{3}e^{\frac{1}{2a_n}}\left(1+\frac{1}{a_n}\right)^{\frac{1}{2}}.\end{aligned}$$

The map $a_n \mapsto \frac{2}{3}e^{\frac{1}{2a_n}}\left(1+\frac{1}{a_n}\right)^{\frac{1}{2}}$ is decreasing. Since $|F| \geq 2$, it follows that $\max F = a_n \geq a_2 = 16$. Substituting this we get

$$\frac{2}{3}e^{\frac{1}{2a_n}}\left(1+\frac{1}{a_n}\right)^{\frac{1}{2}} \leq \frac{2}{3}e^{\frac{1}{32}}\left(1+\frac{1}{16}\right)^{\frac{1}{2}} \leq 0.8$$

This implies that for any $\frac{1}{2^{n+1}} \leq s \leq \dim_{\mathcal{H}}(J_A)$ we have that

$$L_{s,F^\sharp}v_s(x) \leq L_{s,G}v_s(x) + 0.8\left(\frac{1}{a_n+x}\right)^{2s} v_s\left(\frac{1}{a_n+x}\right).$$

It then follows from Lemma 2.1 that there exists a $\mu < 1$ such that

$$L_{s,F^\sharp}v_s(x) \leq \mu L_{s,F}v_s(x) = \mu\lambda_s v_s(x).$$

In particular for $1/2^{n+1} \leq s_0 = \dim_{\mathcal{H}}(J_F) \leq 1/4$ we have that $\lambda_{s_0} = 1$ so $L_{s_0,F^\sharp}v_{s_0} \leq \mu v_s$. Thus

$$r(L_{s_0,F^\sharp}) \leq \mu < 1,$$

this implies that $\dim_{\mathcal{H}}(J_{F^\sharp}) < s_0 = \dim_{\mathcal{H}}(J_F)$ as desired. □

Chapter 5

Final Remarks and Open Problems

We conclude this thesis by discussing some open problems and ideas for future work.

5.1 The Set P_q^*

In Chapter 4, we have shown that $[0, \frac{\ln 2}{2\ln q}] \subseteq \mathcal{DS}(P_q^*)$ for $q \geq 2$ and the dimension spectrum is nowhere dense in $(\mu_1, \dim_{\mathcal{H}}(J_{P_q^*}))$, for $q \geq 3$, where $\mu_1 = \dim_{\mathcal{H}}(J_{P_q^* \setminus \{q\}})$. Moreover, the dimension spectrum is nowhere dense in $(\mu_2, \dim_{\mathcal{H}}(J_{P_q^*}))$ for $q = 2$ where $\mu_2 = \dim_{\mathcal{H}}(J_{P_q^* \setminus \{q^2\}})$, see Theorem 4.3 and Theorem 4.4. However, the structure of the dimension spectrum of P_q^* is not well understood in the interval $(\frac{\ln 2}{2\ln q}, \mu_1)$ for $q \geq 3$, nor is it well understood in $(\frac{\ln 2}{2\ln q}, \mu^2)$ for $q = 2$.

Our techniques allow us to obtain some information about the structure of the dimension spectrum in $D_1 := \left(\frac{\ln 2}{2\ln q}, \mu_1\right)$, but unfortunately they do not yield a complete picture. We will discuss these partial results now.

As a first step we observe for $F \subseteq P_q^*$ with $\dim_{\mathcal{H}}(J_F) \in \left(\frac{\ln 2}{2\ln q}, \mu_1\right)$, we must have $1 \in F$ and $q \notin F$. Indeed, if $1 \notin F$, then $F \subseteq P_q$, so that

$$\dim_{\mathcal{H}}(J_F) \leq \dim_{\mathcal{H}}(J_{P_q}) \leq \frac{\ln 2}{2\ln q}.$$

On the other hand, if $q \in F$, then $\{1, q\} \subseteq F$, and it follows from Theorem 4.3 that

$$\mu_1 < \nu_1 = \dim_{\mathcal{H}}(J_{\{1,q\}}) \leq \dim_{\mathcal{H}}(J_F).$$

Thus for any $F \subseteq P_q^*$ such that $\dim_{\mathcal{H}}(J_F) \in D_1$, we have that $F \subseteq \{1, q^2, q^3, \ldots\}$. Therefore we can restrict our alphabet to

$$A_1 = P_q^* \backslash \{q\} = \{1, q^2, q^3, q^4, \ldots\}. \tag{5.1}$$

The following lemma is helpful in giving the structure of the dimension spectrum in D_1

Lemma 5.1. *For $F \subseteq A_1 = P_q^* \setminus \{q\}$ finite such that $\dim_{\mathcal{H}}(J_F) > \frac{\ln 2}{2 \ln q}$, we have that $\dim_{\mathcal{H}}(J_{F^\sharp}) > \frac{\ln 2}{2 \ln q}$ for all $q \geq 3$.*

Proof. Let $s_q = \frac{\ln 2}{2 \ln q}$ and let v_{s_q} be the eigenvector of $L_{s_q,F}$ with eigenvalue λ_{s_q}. As $\dim_{\mathcal{H}}(J_F) > s_q$ we know $\lambda_{s_q} > 1$. Let $q^n = \max F$. Set $G = F \setminus \{q^n\}$. As v_s is decreasing by Theorem 3.2, together with $s_q = \frac{\ln 2}{2 \ln q}$ implies $\frac{1}{q^{2s_q}-1} = 1$, we have

$$\begin{aligned}
(L_{s_q,F^\sharp} v_{s_q})(x) &= (L_{s_q,G} v_{s_q})(x) + \sum_{j \in \mathbb{N}} \left(\frac{1}{q^{n+j} + x} \right)^{2s_q} v_s \left(\frac{1}{q^{n+j} + x} \right) \\
&\geq (L_{s_q,G} v_{s_q})(x) + \left(\frac{1}{q^n + x} \right)^{2s_q} v_{s_q} \left(\frac{1}{q^n + x} \right) \sum_{j \in \mathbb{N}} \left(\frac{1}{q^j} \right)^{2s_q} \\
&= (L_{s_q,G} v_{s_q})(x) + \frac{1}{q^{2s_q} - 1} \left(\frac{1}{q^n + x} \right)^{2s_q} v_{s_q} \left(\frac{1}{q^n + x} \right). \\
&= (L_{s_q,F} v_{s_q})(x) = \lambda_{s_q} v_{s_q}(x)
\end{aligned}$$

As $s_q < \dim_{\mathcal{H}}(J_F)$, it follows that $\lambda_{s_q} > 1$, so $\dim_{\mathcal{H}}(J_{F^\sharp}) > s_q$, thus proving the claim. $\square$

We have the following result.

Theorem 5.2. *(i) For* $q \geq 4$*, we have that*

$$\frac{\ln 2}{2\ln q} < \dim_{\mathcal{H}}\left(J_{A_1\setminus\{q^2\}}\right) < \dim_{\mathcal{H}}(J_{\{1,q^2\}}). \tag{5.2}$$

Moreover, for any $F \subseteq A_1$ *with* $\{1, q^2\} \subseteq F$*, we have that*

$$\dim_{\mathcal{H}}(J_{F^\sharp}) < \dim_{\mathcal{H}}(J_F).$$

(ii) For $q = 3$*, we have that*

$$\frac{\ln 2}{2\ln q} < \dim_{\mathcal{H}}\left(J_{A_1\setminus\{q^3\}}\right) < \dim_{\mathcal{H}}(J_{\{1,q^2,q^3\}}),$$

and if $F \subseteq A_1$ *with* $\{1, q^2, q^3\} \subseteq F$*, then*

$$\dim_{\mathcal{H}}(J_{F^\sharp}) < \dim_{\mathcal{H}}(J_F).$$

Proof. Assume $q \geq 4$ and note that from Lemma 5.1 with $F = \{1, q\}$, if we can show that $\dim_{\mathcal{H}}(J_{\{1,q^2\}}) > \frac{\ln 2}{2\ln q}$, then we know that $\frac{\ln 2}{2\ln q} < \dim_{\mathcal{H}}\left(J_{A_1\setminus\{q^2\}}\right)$. So establish (i) it will remain to show that $\dim_{\mathcal{H}}\left(J_{A_1\setminus\{q^2\}}\right) < \dim_{\mathcal{H}}(J_{\{1,q^2\}})$. So we first establish that $\dim_{\mathcal{H}}(J_{\{1,q^2\}}) > \frac{\ln 2}{2\ln q}$ for $q \geq 4$. Using the same methods as in the proof of Corollary 3.7 we need to find a constant $c > 0$ such that

$$h(4^2) = \left(\frac{1}{\lambda}\right)^{\frac{c}{\ln 4}} + \left(\frac{1}{4^2 + \lambda - 1}\right)^{\frac{c}{\ln 4}} > 1,$$

where $\lambda = (1 + \sqrt{5})/2$. A direct computation show that $c = 0.73$ give $h(4^2) > 1.003$ so

$$\dim_{\mathcal{H}}(J_{\{1,q^2\}}) > \frac{0.73}{\ln q^2} = \frac{0.73}{2\ln q} > \frac{\ln 2}{2\ln q}.$$

Next we show that $\dim_{\mathcal{H}}\left(J_{A_1\setminus\{q^2\}}\right) < \dim_{\mathcal{H}}(J_{\{1,q^2\}})$. Let $G = \{1, q^2\}$. The operator $L_{s,G}$ has eigenvector v_s with spectral radius λ_s. Now

$$\begin{aligned}(L_{s,G^\sharp}v_s)(x) &= \left(\frac{1}{1+x}\right)^{2s} v_s\left(\frac{1}{1+x}\right) + \sum_{j\in\mathbb{N}}\left(\frac{1}{q^{2+j}+x}\right)^{2s} v_s\left(\frac{1}{q^{2+j}+x}\right)\\ &\leq \left(\frac{1}{1+x}\right)^{2s} v_s\left(\frac{1}{1+x}\right) + \left(\frac{1}{q^2+x}\right)^{2s} v_s\left(\frac{1}{q^2+x}\right) e^{\frac{2s}{q^2}}\sum_{j\in\mathbb{N}}\left(\frac{q^2+x}{q^{2+j}+x}\right)^{2s}.\end{aligned}$$

We have that

$$\begin{aligned}\sum_{j\in\mathbb{N}}\left(\frac{q^2+x}{q^{2+j}+x}\right)^{2s} &\leq \sum_{j\in\mathbb{N}}\left(\frac{q^2+1}{q^{2+j}+1}\right)^{2s}\\ &\leq \left(\frac{q^2+1}{q^2}\right)^{2s}\sum_{j\in\mathbb{N}}\left(\frac{1}{q^j}\right)^{2s}\\ &= \frac{\left(1+\frac{1}{q^2}\right)^{2s}}{q^{2s}-1}.\end{aligned}$$

Thus,

$$e^{\frac{2s}{q^2}}\sum_{j\in\mathbb{N}}\left(\frac{q^2+x}{q^{2+j}+x}\right)^{2s} \leq e^{\frac{2s}{q^2}}\frac{\left(1+\frac{1}{q^2}\right)^{2s}}{q^{2s}-1} \leq \frac{e^{\frac{4s}{q^2}}}{q^{2s}-1} =: \gamma(2,q,s),$$

as

$$1 + x \leq e^x \text{ for all } x \geq 0.$$

It follows that

$$(L_{s,G^\sharp}v_s)(x) \leq \left(\frac{1}{1+x}\right)^{2s} v_s\left(\frac{1}{1+x}\right) + \gamma(2,q,s)\left(\frac{1}{q^2+x}\right)^{2s} v_s\left(\frac{1}{q^2+x}\right).$$

For $q \geq 4$, we can show that $\gamma(2,q,s_0) < 1$ with $s_0 = \dim_{\mathcal{H}}(J_G)$. Since $\gamma(2,q,s_0) < 1$, there exists $\mu < 1$ such that

$$L_{s_0,G^\sharp}v_{s_0} \leq \mu L_{s_q,G}v_{s_0} = \mu v_{s_0},$$

hence $r(L_{s_0,G^\sharp}) \leq \mu < 1$. This implies that $\dim_{\mathcal{H}}(J_{G^\sharp}) < s_0 = \dim_{\mathcal{H}}(J_G)$. From above we know that

$$\dim_{\mathcal{H}}(J_{\{1,q^2\}}) > \frac{0.735}{\ln q^2} = \frac{0.735}{2\ln q}.$$

Note that $G \subseteq P_q^* \setminus \{q\}$. So using Theorem 4.3 the bounds for $\dim_{\mathcal{H}}(J_{\{1,4\}})$ in table 3.1 we get that

$$\dim_{\mathcal{H}}(J_G) \leq \dim_{\mathcal{H}}(J_{P_q^*\setminus\{q\}}) = \mu_1 < \nu_1 = \dim_{\mathcal{H}}(J_{\{1,q\}}) \leq \dim_{\mathcal{H}}(J_{\{1,4\}}) \leq \frac{1}{2}.$$

So, for $q \geq 5$

$$q^{2s_q} - 1 \geq q^{\frac{0.735}{\ln q}} - 1 = e^{0.735} - 1 > e^{\frac{4*0.5}{5^2}} > e^{\frac{4s_0}{q^2}},$$

so $\gamma(2, q, s_0) < 1$ as desired.

For $q = 4$, we use the bounds $0.28 < \dim_{\mathcal{H}}(J_{\{1,16}\}) \leq 0.29$ to get $\gamma(2, q, s_0) = \frac{e^{\frac{4s_0}{q^2}}}{q^{2s_0}-1} < 0.92$

For the case $q = 3$, we argue in the same way. Let $G = \{1, q^2\}$ and $H = \{1, q^2, q^3\}$. Let v_s be the eigenvector of $L_{s,H}$, so

$$\begin{aligned}(L_{s,H^\sharp} v_s)(x) &= (L_{s,G} v_s)(x) + \sum_{j\in\mathbb{N}} \left(\frac{1}{q^{3+j}+x}\right)^{2s} v_s\left(\frac{1}{q^{3+j}+x}\right) \\ &\leq (L_{s,G} v_s)(x) + \left(\frac{1}{q^{3+j}+x}\right)^{2s} v_s\left(\frac{1}{q^{3+j}+x}\right) e^{\frac{2s}{q^3}} \sum_{j\in\mathbb{N}} \left(\frac{q^3+x}{q^{3+j}+x}\right)^{2s}.\end{aligned}$$

Arguing as above, we have that

$$e^{\frac{2s}{q^3}} \sum_{j\in\mathbb{N}} \left(\frac{q^3+x}{q^{3+j}+x}\right)^{2s} \leq e^{\frac{2s}{q^3}} \frac{\left(1+\frac{1}{q^3}\right)^{2s}}{q^{2s}-1} \leq \frac{e^{\frac{4s}{q^3}}}{q^{2s}-1} =: \gamma(3, q, s).$$

To show that $\gamma(3, q, s) < 1$ with $s = \dim_{\mathcal{H}}(J_H)$ we can use the numerical method of Falk and Nussbaum to get that $0.380856 \leq \dim_{\mathcal{H}}(J_H) \leq 0.380863$, which gives $\gamma(3,q,s) < 0.81$.

To complete the proof of the Theorem, suppose that $F \subset A_1$ is finite with $\max F = q^k$. Note that we only need to consider the case where $k \geq 3$ as $k = 2$ is already settled in the above inequality. Set $G = F \setminus \max F$. For $s \in \left(\frac{\ln 2}{2\ln q}, \dim_{\mathcal{H}}(J_{A_1})\right)$, let v_s be the positive eigenvector of $L_{s,F}$ with eigenvalue $\lambda_s = r(L_{s,F})$.

Then for $x \in [0,1]$, we have

$$\begin{aligned}(L_{s,F^\sharp}v_s)(x) &= (L_{s,G}v_s)(x) + \sum_{j=1}^{\infty}\left(\frac{1}{q^{k+j}+x}\right)^{2s} v_s\left(\frac{1}{q^{k+j}+x}\right) \\ &\leq (L_{s,G}v_s)(x) + \left(\frac{1}{q^k+x}\right)^{2s} v_s\left(\frac{1}{q^k+x}\right) e^{\frac{2s}{q^k}} \sum_{j=1}^{\infty}\left(\frac{q^k+1}{q^{k+j}+1}\right)^{2s}.\end{aligned}$$

As before,

$$\begin{aligned}e^{\frac{2s}{q^k}} \sum_{j=1}^{\infty}\left(\frac{q^k+1}{q^{k+j}+1}\right)^{2s} &\leq e^{\frac{2s}{q^k}}\left(\frac{q^k+1}{q^k}\right)^{2s} \sum_{j=1}^{\infty}\left(\frac{1}{q^j}\right)^{2s} \\ &\leq \frac{e^{4s/q^k}}{q^{2s}-1} \leq \frac{e^{\frac{2}{q^k}}}{q^{2s}-1} =: \gamma(k,q,s).\end{aligned}$$

Note that if $\gamma(k,q,s) < 1$, then there exists by Lemma 2.1 a $\mu < 1$ such that $L_{s,F^\sharp}v_s \leq \mu L_{s,F}v_s = \mu\lambda_s v_s$. In particular, if this holds for $s = \dim_{\mathcal{H}}(J_F)$, we get that $L_{s,F^\sharp}v_s \leq \mu v_s$. This would imply that $r(L_{s,F^\sharp}) \leq \mu < 1$, hence $\dim_{\mathcal{H}}(J_{F^\sharp}) < s$. We need to show that $\gamma(k,q,s_0) < 1$ where $s_0 = \dim_{\mathcal{H}}(J_F)$. For $q \geq 4$, we have that $\frac{0.735}{2\ln q} \leq s_0 < 0.5$ and

$$q^{2s_0} - 1 \geq q^{\frac{0.735}{\ln q}} - 1 = e^{0.735} - 1 > e^{\frac{4*0.5}{4^3}} > e^{\frac{4s_0}{q^3}} > e^{\frac{4s_0}{q^k}},$$

so $\gamma(k,q,s_0) < 1$ for all $k \geq 3$.

For $q = 3$, we need to show that $\gamma(k,q,s_0) < 1$, for $k \geq 4$ and using the lower bounds from Table 3.1 we know that $\dim_{\mathcal{H}}(J_F) \geq \dim_{\mathcal{H}}(J_{\{1,q^2,q^3\}})$ and

$$\gamma(k,q,s_0) \leq \frac{e^{\frac{2}{q^4}}}{q^{2*0.380856}-1} \leq 0.8 < 1.$$

$\square$

Recall that $A_1 = \{1, q^2, q^3, \ldots\}$ and set

$$I_k = \{1, q^2, \ldots, q^k\} \quad \text{and} \quad T_k = \{q^{k+1}, q^{k+2}, \ldots\}.$$

For $k \geq 2$ let

$$\mu_{k,A_1} = \dim_{\mathcal{H}}(J_{I_{k-1}\cup T_k}) = \dim_{\mathcal{H}}(J_{A_1\setminus\{q^k\}}) \quad \text{and} \quad \nu_{k,A_1} = \dim_{\mathcal{H}}(J_{I_k}).$$

Here we have use the notation μ_{k,A_1} to emphasise that we are restricting out alphabet to A_1. We give the following result, which shows that the dimension spectrum is nowhere dense in the upper end of the interval $D_1 = (\frac{\ln 2}{2\ln q}, \mu_1)$ for $q \geq 4$.

Theorem 5.3. *For $q \geq 4$ and $k \geq 2$,*

(i) $\mu_{k,A_1} < \nu_{k,A_1}$ and $(\mu_{k,A_1}, \nu_{k,A_1}) \cap \mathcal{DS}(P_q^*) = \emptyset$.

(ii) $\mathcal{DS}(P_q^*)$ is nowhere dense in $(\nu_{k,A_1}, \mu_{k+1,A_1})$.

For $q = 3$ the assertion holds for $k \geq 3$.

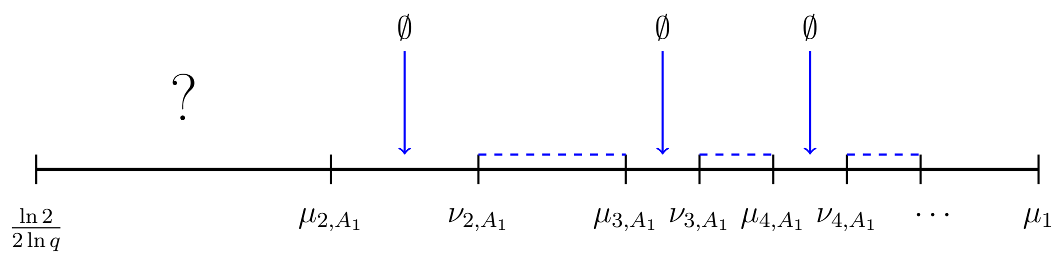
Proof. Suppose that $q \geq 4$ and $k \geq 2$. To prove assertion (i) we first note that we can take $F = I_k = \{1, q^2 \ldots, q^k\}$ in Theorem 5.2 and conclude that $\mu_{k,A_1} < \nu_{k,A_1}$. To see that $(\mu_{k,A_1}, \nu_{k,A_1}) \cap \mathcal{DS}(P_q^*) = \emptyset$ we argue by contradiction. So, suppose that $F \subseteq A_1$ is such that $\mu_{k,A_1} < \dim_{\mathcal{H}}(J_F) < \nu_{k,A_1}$. We claim that $\{1, q^2, \ldots, q^{k-1}\} \subset F$, as otherwise $F \subseteq A_1 \setminus \{q^m\}$ for some $2 \leq m \leq k-1$. In that case by Theorem 5.2, we get that $\dim_{\mathcal{H}}(J_F) < \mu_{m,A_1} < \nu_{m,A_1} \leq \nu_{k-1,A_1} <$

μ_{k,A_1}, which is impossible. As $\{1, q^2, \ldots, q^{k-1}\} \subset F$ and $\dim_{\mathcal{H}}(J_F) < \nu_{k,A_1}$, we know that $q^k \notin F$. Thus, $F \subseteq A_1 \setminus \{q^k\}$, which contradicts the fact that $\mu_{k,A_1} < \dim_{\mathcal{H}}(J_F)$.

To prove assertion (ii) let $F \subset A_1$ be finite with $\nu_{k,A_1} < \dim_{\mathcal{H}}(J_F) < \mu_{k+1,A_1}$. Then $\{1, q^2, \ldots, q^k\} \subset F$, otherwise $F \subset A_1 \backslash \{q^m\}$ for some $m \leq k$, which would imply that $\dim_{\mathcal{H}}(J_F) \leq \mu_{m,A_1} < \nu_{m,A_1} \leq \nu_{k,A_1}$. As $\mu_{k,A_1} < \nu_{k,A_1}$ for all $k \geq 2$, we can combine Lemma 4.9 and Theorem 5.2 and conclude that $\mathcal{DS}(P_q^*)$ is nowhere dense in $(\nu_{k,A_1}, \mu_{k+1,A_1})$ for $k \geq 1$.

The case of $q = 3$ can be derived in the same way. □

Below is an illustration of the structure of the dimension spectrum for P_q^* with $q \geq 4$ in $D_1 = \left(\frac{\ln 2}{2 \ln q}, \mu_1\right)$. For $q = 3$ we have the same structure starting at $\mu_{A_1}^3$.



This means that for $q \geq 4$, the structure is well understood in

$$[\mu_{A_1}^2, \mu_1] = [\dim_{\mathcal{H}}(J_{A_2}), \dim_{\mathcal{H}}(J_{A_1}))],$$

where $A_2 = A_1 \setminus \{q^2\}$. Theorem 5.3 together with Theorem 4.3 and Theorem 4.4, implies that the structure is now well understood in

$$\left[0, \frac{\ln 2}{2 \ln q}\right] \text{ and } \left[\dim_{\mathcal{H}}(J_{A_2}), \dim_{\mathcal{H}}(J_{P_q^*})\right] \quad \text{for } q \geq 4.$$

Ideally, we would like to repeat the process now for the alphabet

$$A_2 = \{1, q^3, q^4, q^5, \ldots\}$$

to get further information in the upper part of the interval

$$D_2 = \left(\frac{\ln 2}{2 \ln q}, \dim_{\mathcal{H}}(J_{A_2}) \right).$$

We will do this for the case $q = 5$ in the next section.

5.2 Additional structure: The case $q = 5$

Throughout this section, $q = 5$. We define $s_q := \frac{\ln 2}{2 \ln q}$. From Theorem 5.2, we have that

$$s_q < \dim_{\mathcal{H}}(J_{A_2}) < \dim_{\mathcal{H}}(J_{\{1,q^2\}}),$$

where $A_2 = A_1 \setminus \{q^2\} = \{1, q^3, q^4, q^5, \ldots\}$.

Set $D_2 = (s_q, \dim_{\mathcal{H}}(J_{A_2}))$. For any $F \subseteq P_q^*$ such that $\dim_{\mathcal{H}}(J_F) \in D_2$, we have that $1 \in F$. Also $q^2 \notin F$ and $q \notin F$. Indeed, if $1 \notin F$, then $F \subseteq P_q = \{q^n : n \in \mathbb{N}\}$ and $\dim_{\mathcal{H}}(J_F) \leq \dim_{\mathcal{H}}(J_{P_q}) < \frac{\ln 2}{2 \ln q}$ which is impossible. Also, if $\{1, q^2\} \subseteq F$ or $\{1, q\} \subseteq F$, then

$$\dim_{\mathcal{H}}(J_F) \geq \dim_{\mathcal{H}}(J_{\{1,q^2\}}) > \dim_{\mathcal{H}}(J_{A_2}).$$

This implies that we can restrict our alphabet to $A_2 = \{1, q^3, q^4, q^5, \ldots\}$ when we consider F with $\dim_{\mathcal{H}}(J_F) \in (s_q, \dim_{\mathcal{H}}(J_{A_2}))$. Using the computational method of Falk and Nussbaum, see Table 3.1, we have that for $q = 5$

$$\dim_{\mathcal{H}}(J_{\{1,q^3\}}) \leq 0.194480 < s_q.$$

Since $\dim_{\mathcal{H}}(J_{A_2}) > s_q$, there exists k_2, such that

$$\dim_{\mathcal{H}}(J_{1,q^3,q^4,\ldots,q^{k_2-1}}) \leq \frac{\ln 2}{2 \ln q} < \dim_{\mathcal{H}}(J_{1,q^3,q^4,\ldots,q^{k_2}}).$$

Set

$$B_2 = \{1, q^3, q^4, \ldots, q^{k_2-1}\}.$$

So k_2 is the first break point for (B_2, s_q) in A_2. In this case $k_2 = 4$ so $B_2 = \{1, q^3\}$, because $\dim_{\mathcal{H}}(J_{\{1,q^3,q^4\}}) \geq 0.223875 > s_q > \dim_{\mathcal{H}}(J_{B_2})$, see Table 3.1. Let $F = \{1, q^3, q^4\}$ so

$$A_3 = F^\sharp = \{1, q^3, q^5, q^6, \ldots\},$$

and let v_s be the eigenvector of $L_{s,F}$ with eigenvalue $\lambda_s = r(L_{s,F})$.

$$\begin{aligned}
(L_{s,A_3}v_s)(x) &= (L_{s,B_2}v_s)(x) + \sum_{j\in\mathbb{N}} \left(\frac{1}{q^{4+j}+x}\right)^{2s} v_s\left(\frac{1}{q^{4+j}+x}\right) \\
&\leq (L_{s,B_2}v_s)(x) + \left(\frac{q^4+1}{q^4}\right)^{2s} e^{\frac{2s}{q^4}} \left(\frac{1}{q^4+x}\right)^{2s} v_s\left(\frac{1}{q^4+x}\right) \sum_{j\in\mathbb{N}} \left(\frac{1}{q^j}\right)^{2s} \\
&\leq (L_{s,B_2}v_s)(x) + \frac{e^{\frac{4s}{q^4}}}{q^{2s}-1} \left(\frac{1}{q^4+x}\right)^{2s} v_s\left(\frac{1}{q^4+x}\right).
\end{aligned}$$

If we can show for $s_0 = \dim_{\mathcal{H}}(J_F)$ that

$$\frac{e^{\frac{4s_0}{q^4}}}{q^{2s_0}-1} < 1,$$

it follows that there exists $\mu < 1$, such that $L_{s_0,A_3}v_{s_0} \leq \mu v_{s_0}$, so that $r(L_{s_0,A_3}) < 1$. Thus, $\dim_{\mathcal{H}}(J_{A_3}) < s_0$. Using the bound in Table 3.1,

$$0.223875 \leq \dim_{\mathcal{H}}(J_F) \leq 0.223878,$$

and this gives $\frac{e^{\frac{4s_0}{q^4}}}{q^{2s_0}-1} \leq 0.95 < 1$. Hence $\dim_{\mathcal{H}}(J_{A_3}) < \dim_{\mathcal{H}}(J_{\{1,q^3,q^4\}})$.

Claim 1: For any $F \subseteq A_2$ finite such that $\{1, q^3, q^4\} \subseteq F$ we have that

$$\dim_{\mathcal{H}}(J_{F^\sharp}) < \dim_{\mathcal{H}}(J_F).$$

Indeed let $q^n = \max F$ and $G = F \setminus \{q^n\}$ and v_s be the eigenvector of $L_{s,F}$ with eigenvalue λ_s, then

$$\begin{aligned}(L_{s,F^\sharp}v_s)(x) &= (L_{s,G}v_s)(x) + \sum_{j\in\mathbb{N}} \left(\frac{1}{q^{n+j}+x}\right)^{2s} v_s\left(\frac{1}{q^{n+j}+x}\right)\\ &\leq (L_{s,G}v_s)(x) + \left(\frac{q^n+1}{q^n}\right)^{2s} e^{\frac{2s}{q^n}} \left(\frac{1}{q^n+x}\right)^{2s} v_s\left(\frac{1}{q^n+x}\right) \sum_{j\in\mathbb{N}}\left(\frac{1}{q^j}\right)^{2s}\\ &\leq (L_{s,G}v_s)(x) + \frac{e^{\frac{4s}{q^n}}}{q^{2s}-1}\left(\frac{1}{q^n+x}\right)^{2s} v_s\left(\frac{1}{q^n+x}\right).\end{aligned}$$

If we can show for $s_0 = \dim_{\mathcal{H}}(J_F)$ that

$$\frac{e^{\frac{4s_0}{q^n}}}{q^{2s_0}-1} < 1,$$

it follows that there exists $\mu < 1$, such that $L_{s_0,F^\sharp}v_{s_0} \leq \mu v_{s_0}$, so that $r(L_{s_0,F^\sharp}) < 1$. Thus, $\dim_{\mathcal{H}}(J_{F^\sharp}) < s_0$. Using the lower bound in Table 3.1, for $\{1, q^3, q^4\}$, and the upper bounds for $\{1, q\}$, we have

$$0.223875 \leq \dim_{\mathcal{H}}(J_F) \leq 0.5,$$

so we obtain

$$\frac{e^{\frac{4s_0}{q^n}}}{q^{2s_0}-1} \leq \frac{e^{\frac{2}{q^4}}}{q^{2s_0}-1} \leq 0.96 < 1.$$

Hence proving Claim 1.

Following the same argument as in the proof of Theorem 5.3, together with Claim 1, we find that the structure of the dimension spectrum is nowhere dense in

$$(\dim_{\mathcal{H}}(J_{\{1,q^3,q^4\}}), \dim_{\mathcal{H}}(J_{A_2})).$$

We know from Lemma 5.1 that if $F \subseteq P_q^* \setminus \{q\}$ finite such that $\dim_{\mathcal{H}}(J_F) > s_q$, we have that $\dim_{\mathcal{H}}(J_{F^\sharp}) > \frac{\ln 2}{2\ln q}$. In particular $\dim_{\mathcal{H}}(A_3) > s_q$.

The next step is the existence of $k_3 \geq 5$, such that if

$$G = \{1, q^3, q^5, q^6, \dots q^{k_2}\}, \text{ then } \dim_{\mathcal{H}}(J_{G \setminus \{q^{k_2}\}}) \leq s_q < \dim_{\mathcal{H}}(J_G).$$

Note that this G exists because $\dim_{\mathcal{H}}(A_2) > s_q$. Pick the first such G. In this case $k_3 = 6$ and

$$B_3 = \{1, q^3, q^5\} = B_2 \cup \{q^5\}$$

Set

$$A_4 = (B_3 \cup \{q^{k_3}\})^\sharp = B_3 \cup \{q^j : j > k_3\} = \{1, q^3, q^5, q^7, q^8 \cdots\}.$$

Using claim 2, we know that $\dim_{\mathcal{H}}(J_{A_4}) > s_q$.

For any $n \geq 3$, assume A_n has been defined and $\dim_{\mathcal{H}}(J_{A_n}) \geq s_q$. Furthermore assume B_{n-1} has been defined with a first break point $q^{k_{n-1}}$ of B_{n-1} in A_{n-1}, then define

$$B_n = B_{n-1} \cup \{q^{k_{n-1}+1}, \dots, q^{k_n - 1}\}$$

where k_n is the first breakpoint in A_n for (B_n, s_q). Note that this k_n exists because $\dim_{\mathcal{H}}(B_{n-1}) < s_q < \dim_{\mathcal{H}}(J_{A_n})$. To complete the inductive step, one would have to show that:

(i) $\dim_{\mathcal{H}}(J_{B_n \cup \{q^{k_n}\}}) > \dim_{\mathcal{H}}(J_{(B_n \cup \{q^{k_n}\})^\sharp})$.

(ii) For all $F \subseteq A_n$ such that $B_n \cup \{q^{k_n}\} \subseteq F$, we have

$$\dim_{\mathcal{H}}(J_{F^\sharp}) < \dim_{\mathcal{H}}(J_F).$$

If these two steps are true, then we set

$$A_{n+1} = (B_n \cup \{q^{k_n}\})^\sharp$$

Using Lemma 5.1, we necessarily have $\dim_{\mathcal{H}}(J_{A_{n+1}}) > s_q$ and we find that the dimension spectrum is nowhere dense in $(\dim_{\mathcal{H}}(J_{A_{n+1}}), \dim_{\mathcal{H}}(J_{A_n}))$.

From this we would generate a sequence of sets (A_n) and (B_n) together with a sequence of integers (k_n) such that q^{k_n} is a first break point for (B_n, s_q) in A_n. It then follows that $\dim_{\mathcal{H}}(J_{B_n}) \uparrow s_q$ and $\dim_{\mathcal{H}}(J_{B_n \cup \{q^{k_n}\}}) \downarrow s_q$.
On the other hand, as $A_{n+1} = (B_n \cup \{q^{k_n}\})^\sharp$, together with

$$\dim_{\mathcal{H}}(J_{B_n \cup \{q^{k_n}\}}) > \dim_{\mathcal{H}}(J_{(B_n \cup \{q^{k_n}\})^\sharp}),$$

we have $s_q < \dim_{\mathcal{H}}(A_{n+1}) < \dim_{\mathcal{H}}(J_{B_n \cup \{q^{k_n}\}})$. Thus $\dim_{\mathcal{H}}(J_{A_n}) \to s_q$. If the inductive step is true, it follows that the dimension spectrum of P_q^* will be nowhere dense in $(\dim_{\mathcal{H}}(J_{A_n+1}), \dim_{\mathcal{H}}(J_{A_n}))$ for all n and this would imply that $\mathcal{DS}(P_q^*)$ is nowhere dense in $[s_q, \dim_{\mathcal{H}}(J_{P_q^*})]$.

We believe this to be the case but have not been able to establish this.

5.3 Conjectures about the structure of the dimension spectrum

In Chapter 4, we analysed the structure of the dimension spectrum for a variety of infinite subsystems of continued fraction.

There were examples where the dimension spectrum was full, a finite union of closed disjoint intervals, nowhere dense everywhere, and a closed interval followed by a nowhere dense part. We will briefly discuss a number of conjectures we believe are true concerning the dimension spectrum both globally and locally. However, these questions lie beyond the scope of our present methods. Developing new techniques to address this question is a key goal for the future work in understanding the structures of the dimension spectrum.

Conjecture 5.4. *Let $A \subseteq \mathbb{N}$ be infinite. Then exactly one of the following is true:*

1. *$\mathcal{DS}(A)$ is a disjoint union of closed intervals.*

2. *There exists a unique value* $\tau > 0$ *such that* $[0, \tau] \subseteq \mathcal{DS}(A)$ *and* $\mathcal{DS}(A)$ *is nowhere dense in* $[\tau, \dim_{\mathcal{H}}(J_A)]$.

3. $\mathcal{DS}(A)$ *is a nowhere dense set in* $[0, \dim_{\mathcal{H}}(J_A)]$.

We already know for the case M_q that the dimension spectrum has finitely many intervals, but one can then ask a question if there exists an $A \subseteq \mathbb{N}$ whose dimension spectrum consist of infinitely many disjoint closed intervals. We have no example of this structure at present.

To establish Conjecture 5.4 it would be useful to have some more basic facts concerning the structure locally of the dimension spectrum of continued fractions. We believe the following too holds.

Let $A \subseteq \mathbb{N}$ be infinite and consider $\mathcal{DS}(A)$, the dimension spectrum of A. We believe the following to be true.

Conjecture 5.5. *Let* $A = \{a_1, a_2, \ldots\}$ *and* $0 \le a < b < c < d$ *be such that* $[a, b] \subset \mathcal{DS}(A)$ *and* $[c, d] \subset \mathcal{DS}(A)$ *then exactly one of the following is true*

1. $\mathcal{DS}(A) \cap (b, c) = \emptyset$.

2. *If* $\mathcal{DS}(A) \cap (b, c) \neq \emptyset$*, then for all* $t \in \mathcal{DS}(A) \cap (b, c)$*, there exists a* $\delta > 0$ *such that* $[t - \delta, t] \subset \mathcal{DS}(A)$ *or* $[t, t + \delta] \subset \mathcal{DS}(A)$.

This observations implies that between two solid intervals which lies in the dimension spectrum, the dimension spectrum can either be an empty set or a union of disjoint intervals.

We also believe the following two assertions to be true.

Conjecture 5.6. *Let* $A = \{a_1, a_2, \ldots\}$ *and* $0 < a < b$ *be such that* $[a, b] \subset \mathcal{DS}(A)$*. Then there exists* $0 < \delta < a$ *such that*

$$[0, \delta] \subset \mathcal{DS}(A).$$

Conjecture 5.7. *Let* $A = \{a_1, a_2, \ldots\}$ *and* $0 < a < b < \dim_{\mathcal{H}}(J_A)$ *be such that* $[a, b] \cap \mathcal{DS}(A) \neq \emptyset$. *If* $\mathcal{DS}(A)$ *is nowhere dense in* $[a, b]$ *then* $\mathcal{DS}(A)$ *is nowhere dense in* $[a, \dim_{\mathcal{H}}(J_A)]$.

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