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# SPECIAL SOLUTIONS OF A DISCRETE PAINLEVÉ EQUATION FOR QUANTUM MINIMAL SURFACES

**P. A. Clarkson,* A. Dzhamay,† A. N. W. Hone,* and B. Mitchell***

*We consider solutions of a discrete Painlevé equation arising from a construction of quantum minimal surfaces by Arnlind, Hoppe, and Kontsevich, and in earlier work of Cornalba and Taylor on static membranes. While the discrete equation admits a continuum limit to the Painlevé* I *differential equation, we find that it has the same space of initial values as the Painlevé* V *equation with certain specific parameter values. We further explicitly show how each iteration of this discrete Painlevé* I *equation corresponds to a certain composition of Bäcklund transformations for Painlevé* V*, as was first remarked in a work by Tokihiro, Grammaticos, and Ramani. In addition, we show that some explicit special function solutions of Painlevé* V*, written in terms of modified Bessel functions, yield the unique positive solution of the initial value problem required for quantum minimal surfaces.*

**Keywords:** quantum minimal surfaces, discrete Painlevé equations, modified Bessel functions

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## 1. Introduction

Minimal surfaces can be characterized as maps $\mathbf{x}\colon \Sigma \to \mathbb{R}^d$ that extremize the Schild functional

$$S[\mathbf{x}] = \int_\Sigma \sum_{j<k} \{x_j, x_k\}^2\, \omega, \tag{1.1}$$

where $\Sigma$ is a surface with symplectic form $\omega$ and associated Poisson bracket $\{\,\cdot\,,\cdot\,\}$, and $(x_j)_{j=1,\dots,d}$ are coordinates on $\mathbb{R}^d$. The Euler–Lagrange equations obtained from the action $S$ are

$$\sum_{j=1}^{d} \{x_j, \{x_j, x_k\}\} = 0, \qquad k = 1, \dots, d. \tag{1.2}$$

In this context, quantization is achieved by replacing the classical observables $x_j$ with self-adjoint operators $X_j$ that act on a Hilbert space $\mathcal{H}$ and taking the commutator in place of the Poisson bracket. Hence, following [1], one can say that a quantum minimal surface is a collection of such operators satisfying the relations

$$\sum_{j=1}^{d} [X_j, [X_j, X_k]] = 0, \qquad k = 1, \dots, d. \tag{1.3}$$

System (1.3) appeared previously in string theory as a set of matrix equations, as a large-$N$ matrix model [2], or as a static membrane equation [3].

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For the case of minimal surfaces in $\mathbb{R}^4 \cong \mathbb{C}^2$, it is a classical result [4] that an arbitrary analytic function $f$ and the plane curve associated with its graph define a solution of (1.2) by setting

$$z_2 = f(z_1), \qquad z_1 = x_1 + \mathrm{i}x_2, \quad z_2 = x_3 + \mathrm{i}x_4; \tag{1.4}$$

more generally, one can consider a Riemann surface defined by an arbitrary analytic relation $F(z_1, z_2) = 0$. The latter relation between the complex coordinates $z_1$, $z_2$ implies that

$$\{z_1, z_2\} = 0, \tag{1.5}$$

while imposing the requirement of constant curvature gives the equation

$$\{\bar{z}_1, z_1\} + \{\bar{z}_2, z_2\} = \mathrm{i}\kappa, \tag{1.6}$$

where, up to rescaling, $\kappa \in \mathbb{R}$ is the curvature. The real and imaginary parts of (1.5), together with Eq. (1.6), provide three linear relations between the brackets $\{x_j, x_k\}$ for $1 \leqslant j < k \leqslant 4$; these equations constitute a first-order system, which have second-order Euler–Lagrange equations (1.2) as a consequence (thus they are analogous to first-order Bogomol'nyi equations in a field theory). The corresponding solution of Eq. (1.3) has also been considered by Cornalba and Taylor in the context of matrix models [3], taking

$$Z_2 = f(Z_1) \tag{1.7}$$

so that $[Z_1, Z_2] = 0$, with

$$Z_1 = X_1 + \mathrm{i}X_2, \quad Z_2 = X_3 + \mathrm{i}X_4, \qquad [Z_1^\dagger, Z_1] + [Z_2^\dagger, Z_2] = \epsilon\mathbf{1}, \tag{1.8}$$

where $\epsilon \in \mathbb{R}$ is a parameter.

In $d = 4$, the case where (1.7) is the hyperbola $Z_1 Z_2 = c\mathbf{1}$ is the simplest example treated in [5], which admits an elegant operator-valued solution. The next interesting case considered in [3], and by Arnlind and company [5], is the parabola, which (after explicitly parametrizing the curve as $Z_1 = W$, $Z_2 = W^2$) leads to an operator $W$ satisfying

$$[W^\dagger, W] + [(W^\dagger)^2, W^2] = \epsilon\mathbf{1}, \tag{1.9}$$

acting on the Hilbert space $\mathcal{H} = \{|n\rangle \,|\, n = 0, 1, 2, \dots\}$ according to $W|n\rangle = w_n|n+1\rangle$. In terms of the squared amplitude $v_n = |w_n|^2$, applying the expectation $\langle n| \dots |n\rangle$ to both sides of commutator equation (1.9) leads to the third-order difference equation

$$v_n - v_{n-1} + v_{n+1}v_n - v_{n-1}v_{n-2} = \epsilon,$$

which has the form of a total difference. Hence, upon integration (summation) of this discrete equation, we obtain the second-order nonautonomous equation

$$v_n(v_{n+1} + v_{n-1} + 1) = \epsilon n + \zeta, \tag{1.10}$$

for some constant $\zeta$.

Identification of the particular solution of (1.10) required for the quantum minimal surface involves consideration of the semiclassical limit. The classical version of the complex parabola $z_2 = z_1^2$ is parametrized in polar coordinates by $z_1 = re^{\mathrm{i}\varphi}$, $z_2 = r^2 e^{2\mathrm{i}\varphi}$, and hence Poisson bracket equation (1.6) implies that $\tilde{r}$, $\varphi$ are a pair of canonically conjugate (flat) coordinates, where

$$\tilde{r} = r^4 + \frac{r^2}{2} - c. \tag{1.11}$$

Canonical quantization means replacing $\tilde{r} \to -\mathrm{i}\hbar\frac{\partial}{\partial\varphi}$, where the latter is the momentum operator conjugate to $\widehat{U}$, with $e^{\mathrm{i}\widehat{U}}|n\rangle = |n+1\rangle$, and we identify the states $|n\rangle$ for $n \geqslant 0$ with the nonnegative modes $e^{\mathrm{i}n\varphi}$ on the circle. Comparing with (1.11) gives the requirement that $v_n^2 + v_n/2 \sim n\hbar + c$, leading to the approximate solution

$$v_n \approx \frac{1}{4}(\sqrt{1+8(n+1)\epsilon} - 1), \tag{1.12}$$

which agrees with the asymptotic behavior of positive solutions of (1.10), both in the limit $\hbar \to 0$ with $n$ fixed and for $n \to \infty$ with $\hbar$ fixed, provided that the conditions $\zeta = \epsilon = 2\hbar$ and $c = \hbar$ are imposed. Hence, the second-order difference equation is taken as

$$v_{n+1} + v_{n-1} + 1 = \frac{\epsilon(n+1)}{v_n}, \tag{1.13}$$

and one should seek a solution with the initial conditions

$$v_{-1} = 0, \qquad v_0 > 0, \tag{1.14}$$

with the further requirement that $v_n \geqslant 0$ for all $n > 0$, since $v_n$ is a squared amplitude. (Note that approximate form (1.12) also satisfies $v_{-1} = 0$ and $v_n > 0$ for all $n \geqslant 0$.)

Equation (1.10) is an example of a discrete Painlevé equation. It is commonly referred to in the literature as a discrete Painlevé I ($\mathrm{dP_I}$) equation [6], because it has a continuum limit to the continuous Painlevé I equation $\frac{d^2w}{dt^2} = 6w^2 + t$. This $\mathrm{dP_I}$ equation has been obtained as a reduction of a chain of discrete dressing transformations [7], while it is also one among a number of discrete Painlevé equations that were identified by Tokihiro, Grammaticos, and Ramani [8] as arising from compositions of Bäcklund transformations for the Painlevé V equation, that is,

$$\frac{d^2w}{dt^2} = \left(\frac{1}{2w} + \frac{1}{w-1}\right)\left(\frac{dw}{dt}\right)^2 - \frac{1}{t}\frac{dw}{dt} + \frac{(w-1)^2(\alpha w^2+\beta)}{t^2 w} + \frac{\gamma w}{t} + \frac{\delta w(w+1)}{w-1}. \tag{1.15}$$

(For what follows, only the generic case $\delta \neq 0$ is relevant, and thus in that case we can set $\delta = -1/2$.)

For the sake of completeness, and to avoid confusion, we should remark that Eq. (1.10) is not the only discrete equation to be called $\mathrm{dP_I}$. The "standard" version of $\mathrm{dP_I}$ is the equation

$$u_{n+1} + u_n + u_{n-1} = 1 + \frac{\lambda n + \mu}{u_n}, \tag{1.16}$$

with $\lambda$ and $\mu$ constants; see, for example, equation (3.2) in [6]. It is known that Eq. (1.16) is associated with Painlevé IV rather that Painlevé V, cf. [9], [10], and it is also shown in [11], [12] that (1.16) can be derived from Bäcklund transformations of Painlevé IV. (For another approach, via reductions of the Volterra lattice, see [13].)

Note that there are various other inequivalent discrete Painlevé equations referred to as $\mathrm{dP_I}$, or sometimes as alt. $\mathrm{dP_I}$ equations. This is best understood using the Sakai classification scheme for Painlevé equations suggested in the paper [14], which provided a complete classification of possible configuration spaces on which discrete Painlevé dynamics can occur. Such spaces are families of rational algebraic surfaces known as generalized Halphen surfaces (see Fig. 1). For the differential Painlevé equations, these spaces were introduced earlier by Okamoto [15] as the so-called *spaces of initial conditions*, in which case the parameters of the family are essentially the parameters of the differential Painlevé equation, and discrete Painlevé equations are certain compositions of their Bäcklund transformations, as indicated in Fig. 1. The arrows here can be understood, on the one hand, as some parameter degenerations of surface families, and on the other hand, as a result of taking the continuum limit of some particular discrete Painlevé dynamics.

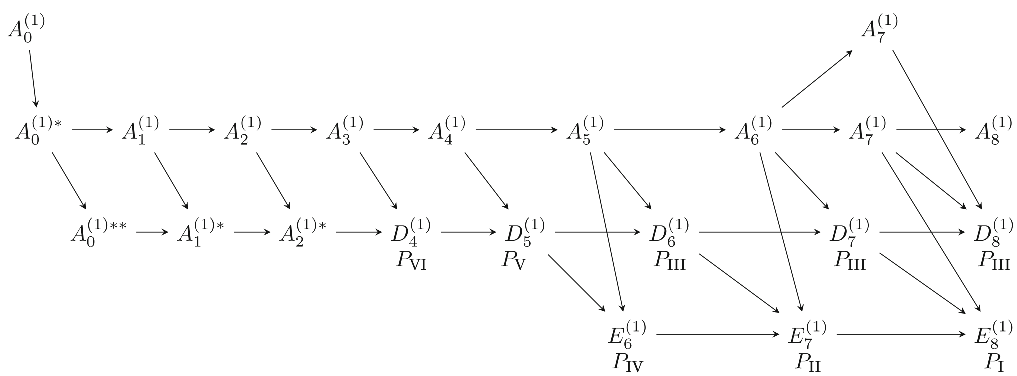



**Fig. 1.** Surface-type classification scheme for Painlevé equations.

As we show in Sec. 3 below, Eq. (1.13) describes a very special dynamics on the $D_5^{(1)}$-surface family. The symmetry group of this family is a fully extended affine Weyl group

$$\widehat{W}(A_3^{(1)}) = W(A_3^{(1)}) \ltimes \mathrm{Aut}(A_3^{(1)}),$$

where $\mathrm{Aut}(A_3^{(1)}) \simeq \mathbb{D}_4$ is the dihedral group of symmetries of the affine $A_3^{(1)}$ Dynkin diagram, i.e., the group of symmetries of a square. This symmetry group describes Bäcklund transformations of the Painlevé V differential equation. Standard examples of discrete Painlevé equations on this surface family correspond to translations in the weight lattice of the usual extended affine Weyl group $\widetilde{W}(A_3^1)$, and are commonly known as $\mathrm{dP_{IV}}$ and $\mathrm{dP_{III}}$ equations. Equation (1.13) is only a quasi-translation, which becomes a translation on a certain sub-locus of the full family with a smaller symmetry group via so-called *projective reduction* [16] (but further discussion of this is outside the scope of the present paper). In contrast, "standard" $\mathrm{dP_I}$ equation (1.16) describes dynamics on the $E_6^{(1)}$ surface family, in alignment with the fact that it is associated with Bäcklund transformations for the Painlevé IV equation.

The purpose of this article is to determine an explicit analytic solution for the initial value problem (1.13), (1.14) associated with a quantum minimal surface. First of all, we consider the existence and uniqueness of a positive solution to the initial value problem (1.14) for the $\mathrm{dP_I}$ equation (thus, $v_n > 0$ for all $n \geqslant 0$). Next, we use the complex geometry of Eq. (1.13), obtained by blowing up $\mathbb{P}^1 \times \mathbb{P}^1$, to show that it corresponds to the same space of initial conditions as Painlevé V equation (1.15) with the specific parameter values

$$(\alpha, \beta, \gamma, \delta) = \left(\frac{(n+1)^2}{18}, -\frac{1}{18}, -\frac{n+1}{3}, -\frac{1}{2}\right).$$

We then proceed to employ some recent results by two of us (Clarkson and Mitchell, obtained in collaboration with Dunning), giving explicit modified Bessel function formulas for families of classical solutions of Painlevé V that were previously considered in the literature [17], [18], and use these to determine an exact analytic expression for the unique solution of the initial value problem (1.13), (1.14) so that $v_n$ remains positive for all $n > 0$. Our main result is as follows.

**Theorem 1.1.** *For each $\epsilon > 0$, the unique positive solution of* dP$_\mathrm{I}$ *equation* (1.13) *subject to initial conditions* (1.14) *is determined by the value of $v_0 = v_0(\epsilon)$, which is given by a ratio of modified Bessel functions, that is,*

$$v_0 = \frac{1}{2}\left(\frac{K_{5/6}(t/2)}{K_{1/6}(t/2)} - 1\right), \quad \textit{where} \quad t = \frac{1}{3\epsilon}. \tag{1.17}$$

*For each $n \geqslant 0$, the corresponding quantities $v_n > 0$ are written explicitly as ratios of Wronskian determinants whose entries are specified in terms of modified Bessel functions.*

## 2. Unique positive solution: cold open

In this section, we present the preliminary steps of the proof that there is a unique solution of dP$_\mathrm{I}$ equation (1.13), subject to initial conditions (1.14), that is nonnegative (in fact, positive) for all $n \geqslant 0$. The precise statement is as follows.

**Theorem 2.1.** *For each value of $\epsilon > 0$ there is a unique value of $v_0 > 0$ such that the solution of second-order difference equation* (1.13) *with initial data* (1.14) *satisfies $v_n > 0$ for all $n \geqslant 0$.*

In our initial approach to proving the above result, we start by considering the set of real sequences $\mathbf{u} = (u_n)_{n\geqslant 0}$, which contains the Banach space

$$\ell^\infty_\epsilon = \big\{\mathbf{u} \,|\, \|\mathbf{u}\| < \infty\big\},$$

where $\|\cdot\|$ denotes the weighted $\ell^\infty$ norm

$$\|\mathbf{u}\| = \sup_{n\geqslant 0} \frac{|u_n|}{\epsilon(n+1)}.$$

Then we can define a transformation $T$, which acts on real nonnegative sequences $\mathbf{u} \geqslant 0$ (that is, $u_n \geqslant 0$ for all $n \geqslant 0$), according to

$$T(u_n) = \begin{cases} \dfrac{\epsilon}{u_{n+1}+1}, & \text{if } n = 0, \\ \dfrac{\epsilon(n+1)}{u_{n+1}+u_{n-1}+1}, & \text{if } n > 0. \end{cases} \tag{2.1}$$

By a convenient abuse of notation, we write $\mathbf{u} \mapsto T\mathbf{u}$ for the action on sequences, while brackets are used to denote the individual terms $T(u_n)$ of a sequence $T\mathbf{u}$. Under the action of $T$, any nonnegative sequence is mapped to a subset of the unit ball in $\ell^\infty_\epsilon$, namely

$$\mathcal{A}^{(0)} = \big\{\mathbf{u} \geqslant 0 \mid \|\mathbf{u}\| \leqslant 1\big\},$$

which is a complete set with respect to this norm.

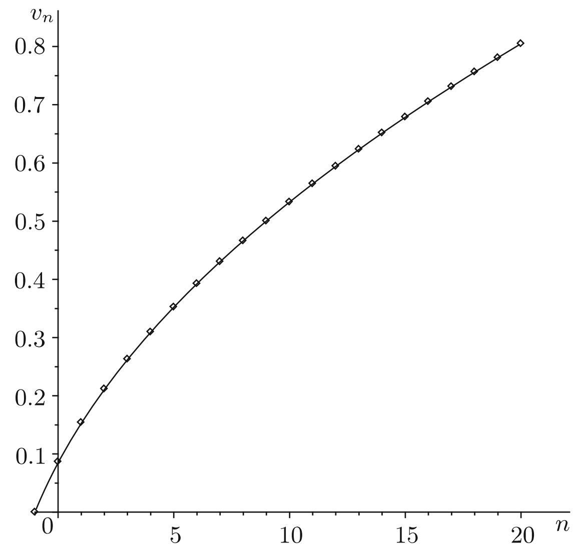



**Fig. 2.** Numerical computation of $v_n$ (dots) with $\epsilon = 0.1$, for $-1 \leqslant n \leqslant 20$, compared with the graph of the approximation (1.12).

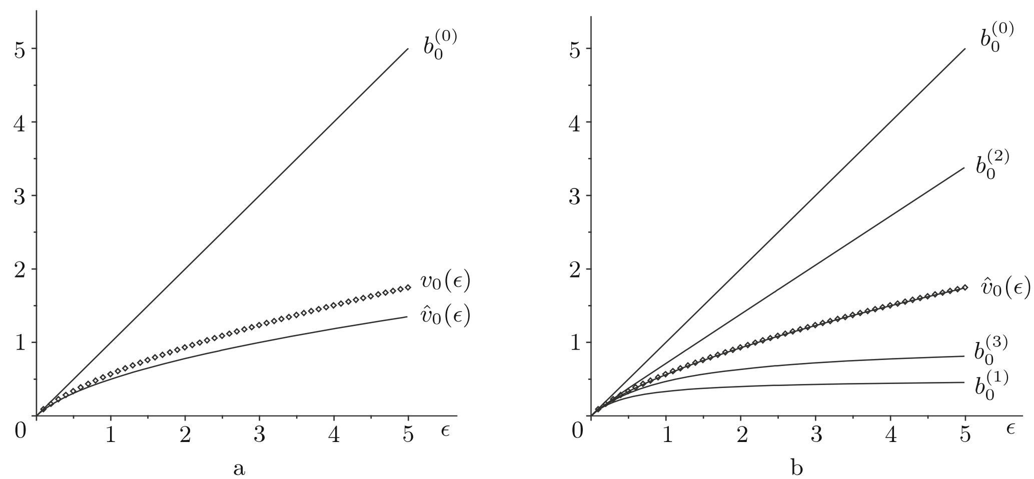



**Fig. 3.** Numerical computation of $v_0(\epsilon)$ (dots) in the range $0 < \epsilon \leqslant 5$, plotted against linear bound $b_0^{(0)} = \epsilon$ and approximation $\hat{v}_0(\epsilon)$ as in (2.33) (a). Same computation, but compared with upper bounds $b_0^{(0)}$, $b_0^{(2)}$, lower bounds $b_0^{(1)}$, $b_0^{(3)}$, and exact formula (1.17) (b).

Numerically, for any fixed $\epsilon > 0$, the repeated application of the mapping $T$ to a (truncated) positive sequence provides a rapid numerical method to obtain the positive solution of the $\mathrm{dP_I}$ equation to any desired precision. (See Figs. 2 and 3, obtained from 100 iterations of $T$ applied to a truncated sequence with $0 \leqslant n \leqslant 20$, where the approximation (1.12) was used to specify the initial conditions and fix the boundary values at $n = -1$ and $n = 21$.)

The set $\mathcal{A}^{(0)}$ is mapped to a subset of itself, and therefore ideally we would want to show that $T$ is a contraction mapping on this set, and hence, by the Banach fixed point theorem, it would follow that it has a unique fixed point $\mathbf{v}$ with $T(\mathbf{v}) = \mathbf{v}$. From (2.1), such a fixed point $\mathbf{v} = (v_n)_{n \geqslant 0}$ is a positive sequence that satisfies $\mathrm{dP_I}$ equation (1.13) with initial condition $v_{-1} = 0$. However, basic estimates and numerical calculations show that $T$ is not a contraction mapping on the whole set $\mathcal{A}^{(0)}$, and in fact the squared mapping $T^2$ behaves better than $T$, and therefore we need to use some more refined bounds to prove the uniqueness of the positive solution $\mathbf{v}$. In particular, we adapt some ideas from [3] and [5], where it was observed that, for each $n \neq 0$, the value of the positive solution $v_n$ should be obtained as the intersection of a sequence of intervals of successively shrinking diameter. Furthermore, at the end of Sec. 4 we proceed to show that there is only one solution of (1.13) satisfying the required bounds.

We define a set of nonnegative sequences $\{\mathbf{b}^{(k)}\}_{k\geqslant 1}$ by successively applying $T$ to the zero sequence $\mathbf{0}$, so that

$$\mathbf{b}^{(-1)} = \mathbf{0}, \qquad \mathbf{b}^{(k)} = T\mathbf{b}^{(k-1)} \quad \text{for} \quad k \geqslant 0. \tag{2.2}$$

The first few steps in the $T$-orbit of $\mathbf{0}$ are specified by a formula for their terms, valid for all $n \geqslant 0$:

$$\mathrm{b}_n^{(0)} = \epsilon(n+1), \qquad \mathrm{b}_n^{(1)} = \frac{\epsilon(n+1)}{1+2\epsilon(n+1)}, \qquad \mathrm{b}_n^{(2)} = \frac{\epsilon(n+1)}{1+\frac{\epsilon n}{1+2\epsilon n}+\frac{\epsilon(n+2)}{1+2\epsilon(n+2)}}. \tag{2.3}$$

Thereafter, for $k \geqslant 3$, there is no longer a uniform expression for the iterate $\mathbf{b}^{(k)}$ as a ratio of polynomials

$$\mathrm{b}_n^{(k)} = \frac{\mathrm{p}_n^{(k)}(\epsilon)}{\mathrm{q}_n^{(k)}(\epsilon)}, \qquad \mathrm{p}_n^{(k)}, \mathrm{q}_n^{(k)} \in \mathbb{Z}[\epsilon],$$

valid for all $n$: due to the fact that Eqs. (2.1) defining $T$ for $n=0$ and $n>0$ are different, the coprime polynomials $\mathrm{p}_n^{(k)}(\epsilon)$ and $\mathrm{q}_n^{(k)}(\epsilon)$ have distinct forms for $n = 0, 1, \ldots, k-3$, while there is another formula for them that is uniformly valid only for $n \geqslant k-2$. For instance, when $k = 3$, we have

$$\mathrm{b}_0^{(3)} = \frac{\epsilon(1+12\epsilon+24\epsilon^2)}{1+14\epsilon+40\epsilon^2+24\epsilon^3}, \qquad \mathrm{b}_n^{(3)} = \frac{\mathrm{p}_n^{(3)}(\epsilon)}{\mathrm{q}_n^{(3)}(\epsilon)} \quad \text{for} \quad n \geqslant 1,$$

where

$$\begin{aligned}
\mathrm{p}_n^{(3)}(\epsilon) &= (n+1)\epsilon\big(1+6n\epsilon+8(n^2-1)\epsilon^2\big)\big(1+6(n+2)\epsilon+8(n+1)(n+3)\epsilon^2\big),\\
\mathrm{q}_n^{(3)}(\epsilon) &= 1+14(n+1)\epsilon+8(9n^2+18n+4)\epsilon^2+8(n+1)(21n^2+42n-11)\epsilon^3+\\
&\quad + 16(n+1)^2(11n^2+22n-20)\epsilon^4+64(n+1)^3(n-1)(n+3)\epsilon^5.
\end{aligned}$$

Nevertheless, for all $n$ there are expressions for $\mathrm{b}_n^{(k)}$ as rational functions of $\epsilon$ and the variable $z = \epsilon(n+1)$, which are described in Lemma 2.3 below.

If we start with a sequence $\mathbf{u} \in \mathcal{A}^{(0)}$ and apply $T$ once, then we find

$$\frac{\epsilon(n+1)}{1+\epsilon n+\epsilon(n+2)} \leqslant \frac{\epsilon(n+1)}{1+u_{n-1}+u_{n+1}} \leqslant \epsilon(n+1),$$

or in other words $\mathrm{b}_n^{(1)} \leqslant T(u_n) \leqslant \mathrm{b}_n^{(0)}$, while another application of $T$ gives

$$\frac{\epsilon(n+1)}{1+\epsilon n+\epsilon(n+2)} \leqslant \frac{\epsilon(n+1)}{1+T(u_{n-1})+T(u_{n+1})} \leqslant \frac{\epsilon(n+1)}{1+\frac{\epsilon n}{1+2\epsilon n}+\frac{\epsilon(n+2)}{1+2\epsilon(n+2)}},$$

so that $\mathrm{b}_n^{(1)} \leqslant T^2(u_n) \leqslant \mathrm{b}_n^{(2)}$. Continuing in this way, by induction we obtain the following result.

**Lemma 2.1.** *For each nonnegative integer $k$, the iterates $T^k\mathbf{u}$ of $\mathbf{u} \in \mathcal{A}^{(0)}$ satisfy the inequalities*

$$\mathrm{b}_n^{(2j-1)} \leqslant T^{2j}(u_n) \leqslant \mathrm{b}_n^{(2j)} \quad \textit{for all} \quad n \geqslant 0 \tag{2.4}$$

*when $k = 2j$ is even, and*

$$\mathrm{b}_n^{(2j+1)} \leqslant T^{2j+1}(u_n) \leqslant \mathrm{b}_n^{(2j)} \quad \textit{for all} \quad n \geqslant 0 \tag{2.5}$$

*when $k = 2j + 1$ is odd. The sequences of lower/upper bounds in* (2.2) *satisfy*

$$0 \leqslant \mathrm{b}_n^{(2j-1)} < \mathrm{b}_n^{(2j+1)} < \mathrm{b}_n^{(2j+2)} < \mathrm{b}_n^{(2j)} \quad \textit{for all} \quad n \geqslant 0, \tag{2.6}$$

*for each $j \in \mathbb{N}$.*

For each $k \geqslant 0$ we have the set $\mathcal{A}^{(k)} = T^k\mathcal{A}^{(0)}$, and the preceding result implies that the next set in the sequence, $\mathcal{A}^{(k+1)} \subset \mathcal{A}^{(k)}$, is a proper subset of the previous one. Furthermore, inequalities (2.6) immediately imply the existence of the limits of upper and lower bounds, that is,

$$\lim_{j\to\infty} \mathrm{b}_n^{(2j-1)} = \limsup_{j\geqslant 0} \mathrm{b}_n^{(2j-1)} \leqslant \liminf_{j\geqslant 0} \mathrm{b}_n^{(2j)} = \lim_{j\to\infty} \mathrm{b}_n^{(2j)} \tag{2.7}$$

for each $n \geqslant 0$. The problem is then how to show the equality of the upper and lower limits above for each $n$, since in that case it immediately follows from (2.4), (2.5) that the iterates $T^k\mathbf{u}$ converge to the unique positive fixed point of $T$.

**Proposition 2.1.** *For all $\epsilon > 0$ there exists* (*at least one*) $v_0 = v_0(\epsilon)$ *such that the solution of* (1.13) *with initial data* (1.14) *is positive and satisfies $v_n > 0$ for all $n \geqslant 0$, as well as*

$$\mathbf{v} = (v_n)_{n\geqslant 0} \in \bigcap_{k\geqslant 0} \mathcal{A}^{(k)},$$

*so that, for all $n \geqslant 0$,*

$$\lim_{j\to\infty} \mathrm{b}_n^{(2j-1)} \leqslant v_n \leqslant \lim_{j\to\infty} \mathrm{b}_n^{(2j)}. \tag{2.8}$$

**Proof.** The existence of a positive solution $\mathbf{v}$ is proved in [5], where it shown that for each $\epsilon > 0$ there is an infinite sequence of open intervals $I_k = I_k(\epsilon) \subset \mathbb{R}$, with $I_1 = (0, \epsilon)$ and $I_k \subset I_{k-1}$, such that $v_0 \in I_k$ implies $v_1, \ldots, v_k > 0$, and $\bigcap_{k\geqslant 0} I_k \neq \varnothing$. Hence, if $v_0 \in \bigcap_{k\geqslant 0} I_k$, then the corresponding sequence $\mathbf{v}$ is a positive solution. Then, because $T\mathbf{v} = \mathbf{v}$, it follows from Lemma 2.1 that $\mathbf{v} \in \mathcal{A}^{(k)}$ for each $k \geqslant 0$, and hence (2.8) holds for each $n \geqslant 0$. ■

It is instructive to compare the upper and lower bounds for different $n$, as well as introduce the rescaled bounds $\rho_n^{(k)}$, which specify the norms:

$$\rho_n^{(k)} = \frac{\mathrm{b}_n^{(k)}}{\epsilon(n+1)}, \qquad \|\mathbf{b}^{(k)}\| = \sup_{n\geqslant 0} \rho_n^{(k)}.$$

Clearly we have $\rho_n^{(-1)} = 0$, while $\mathrm{b}_{n+1}^{(0)} > \mathrm{b}_n^{(0)}$ and $\rho_n^{(0)} = 1$ for all $n$, and from (2.6) we also see that

$$0 < \rho_n^{(k)} < 1 \quad \text{for all} \quad n \geqslant 0, \;\; k \geqslant 1. \tag{2.9}$$

If we now assume for some $k$ that $\mathrm{b}_{n+1}^{(k)} > \mathrm{b}_n^{(k)}$ holds for all $n \geqslant 0$, then from the definition of the map $T$ we may write

$$\frac{1}{\rho_{n+1}^{(k+1)}} - \frac{1}{\rho_n^{(k+1)}} = (1 + \mathrm{b}_n^{(k)} + \mathrm{b}_{n+2}^{(k)}) - (1 + \mathrm{b}_{n-1}^{(k)} + \mathrm{b}_{n+1}^{(k)}) =$$
$$= (\mathrm{b}_n^{(k)} - \mathrm{b}_{n-1}^{(k)}) + (\mathrm{b}_{n+2}^{(k)} - \mathrm{b}_{n+1}^{(k)}) > 0,$$

where we set $\mathrm{b}_{(-1)}^{(k)} = 0$, so that this makes sense when $n = 0$, which implies that $\rho_{n+1}^{(k+1)} < \rho_n^{(k+1)}$. On the other hand,

$$\mathrm{b}_{n+1}^{(k)} > \mathrm{b}_n^{(k)} \quad \Longleftrightarrow \quad (n+2)\rho_{n+1}^{(k)} > (n+1)\rho_n^{(k)},$$

and we can calculate

$$\frac{\mathrm{b}_{n+1}^{(k+1)} - \mathrm{b}_n^{(k+1)}}{\rho_{n+1}^{(k+1)}\rho_n^{(k+1)}} = \epsilon(n+2)(1 + \mathrm{b}_{n-1}^{(k)} + \mathrm{b}_{n+1}^{(k)}) - \epsilon(n+1)(1 + \mathrm{b}_n^{(k)} + \mathrm{b}_{n+2}^{(k)}) =$$
$$= \epsilon\big(1 + (n+2)(n\rho_{n-1}^{(k)} + (n+2)\rho_{n+1}^{(k)}) - (n+1)((n+1)\rho_n^{(k)} + (n+3)\rho_{n+2}^{(k)})\big).$$

If we now assume that $\rho_{n+1}^{(k)} < \rho_n^{(k)}$ holds for all $n \geqslant 0$, then we can replace the term with index $n+2$, and also (for $n > 0$) replace the term with index $n-1$, and thus rearrange the above formula to find the lower bound

$$\frac{\mathrm{b}_{n+1}^{(k+1)} - \mathrm{b}_n^{(k+1)}}{\rho_{n+1}^{(k+1)}\rho_n^{(k+1)}} > \epsilon(1 + \rho_{n+1}^{(k)} - \rho_n^{(k)}) > 0,$$

using (2.9) to obtain the final inequality. Thus, by induction on $k$, we find the following.

**Lemma 2.2.** *For $k \geqslant 0$, the sequences of lower/upper bounds satisfy*

$$\mathrm{b}_{n+1}^{(k)} > \mathrm{b}_n^{(k)} \quad \textit{for all} \quad n \geqslant 0, \tag{2.10}$$

*while for $k \geqslant 1$ the rescaled bounds satisfy*

$$\frac{n+1}{n+2}\rho_n^{(k)} < \rho_{n+1}^{(k)} < \rho_n^{(k)} \quad \textit{for} \quad \textit{all} \quad n \geqslant 0, \tag{2.11}$$

*and hence for each $k$ the norm of $\mathbf{b}^{(k)}$ is $\|\mathbf{b}^{(k)}\| = \rho_0^{(k)}$.*

**Lemma 2.3.** *For each $j \geqslant 0$, the rescaled bounds have the asymptotic behavior*

$$\rho_n^{(2j)} \sim \frac{1}{j+1}, \quad \rho_n^{(2j+1)} \sim \frac{j+1}{2\epsilon(n+1)}, \quad \textit{as} \quad n \to \infty. \tag{2.12}$$

*Moreover, the leading part of the Taylor expansion of each of the rescaled bounds at $\epsilon = 0$ is*

$$\rho_n^{(k)} = 1 - 2\epsilon(n+1) + O(\epsilon^2) \quad \textit{for all} \quad k \geqslant 1, \tag{2.13}$$

*while for all $\epsilon > 0$ the first derivatives with respect to $\epsilon$ satisfy*

$$\frac{d\rho_n^{(k)}}{d\epsilon} < 0, \qquad \frac{d\mathrm{b}_n^{(k)}}{d\epsilon} > 0 \tag{2.14}$$

*for $k \geqslant 1$, $n \geqslant 0$, so that each bound $\rho_n^{(k)}$ and $\mathrm{b}_n^{(k)}$ is monotone decreasing/increasing in $\epsilon$, respectively.*

**Proof.** As noted above, the action of the map $T$ is such that the components $\mathrm{b}_n^{(k)}$ of the sequences $\mathbf{b}^{(k)}$ are rational functions of $\epsilon$ having expression as ratios of coprime polynomials in $\epsilon$ that are uniformly valid for all $n$ when $k < 3$, while for $k \geqslant 3$ these polynomials have a uniform structure for $n \geqslant k-2$ only. However, if we set $z = \epsilon(n+1)$, then for $k = 0, 1, 2$ we can write

$$\begin{aligned} \rho_n^{(0)} &= R^{(0)}(z,\epsilon) := 1, \qquad \rho_n^{(1)} = R^{(1)}(z,\epsilon) := \frac{1}{1+2z}, \\ \rho_n^{(2)} &= R^{(2)}(z,\epsilon) := \frac{1}{1+\frac{z-\epsilon}{1+2(z-\epsilon)}+\frac{z+\epsilon}{1+2(z+\epsilon)}} \end{aligned} \tag{2.15}$$

and for all $n$ we can similarly express $\rho_n^{(k)}$ as $R^{(k)}(z,\epsilon)$, a rational function of $z$ and $\epsilon$ that is determined recursively via the finite difference equation

$$R^{(k+1)}(z,\epsilon) = \frac{1}{1+(z-\epsilon)R^{(k)}(z-\epsilon,\epsilon)+(z+\epsilon)R^{(k)}(z+\epsilon,\epsilon)}. \tag{2.16}$$

Therefore, from the definition of the map $T$, the identity $\rho_n^{(k)} = R^{(k)}(\epsilon(n+1),\epsilon)$ holds for all $n$. The result (2.12) thus corresponds to the asymptotics of the rational functions $R^{(k)}(z,\epsilon)$ as $z \to \infty$, which varies according to the parity of $k$, and we can proceed by induction on $j$. For the base case $j = 0$, we have $R^{(0)}(z,\epsilon) = 1$, $R^{(1)}(z,\epsilon) \sim 1/2z$, and hence the claim for $\rho_n^{(0)}$ is trivially true, while for $\rho_n^{(1)}$ it gives the correct result by substituting $z = \epsilon(n+1)$. Thus, for the induction, for some fixed $j$ we can assume that $R^{(2j)}(z,\epsilon) \sim 1/(j+1)$ as $z \to \infty$, and then by applying (2.16) we immediately obtain

$$R^{(2j+1)}(z,\epsilon) \sim \left(1+\frac{z-\epsilon}{j+1}+\frac{z+\epsilon}{j+1}\right)^{-1} \sim \frac{j+1}{2z}, \qquad z \to \infty,$$

which, upon setting $z = \epsilon(n+1)$ gives the correct leading-order behavior for $\rho_n^{(2j+1)}$ as $n \to \infty$. Applying (2.16) once again gives

$$R^{(2j+2)}(z,\epsilon) \sim \left(1+\frac{(z-\epsilon)(j+1)}{2z}+\frac{(z+\epsilon)(j+1)}{2z}\right)^{-1} \sim \frac{1}{j+2}, \qquad z \to \infty,$$

and this completes the inductive step.

For the leading-order behavior of the scaled bounds at $\epsilon = 0$, it is convenient to write an equivalent version of (2.16) in terms of $\rho_n^{(k)}$, namely

$$\rho_n^{(k+1)} = \left(1+\epsilon n\rho_{n-1}^{(k)}+\epsilon(n+2)\rho_{n+1}^{(k)}\right)^{-1}, \tag{2.17}$$

which is valid for all $n$. When $k = 1$, the leading-order expansion (2.13) is immediately obtained from the geometric series for

$$\rho_n^{(1)} = \frac{1}{1+2\epsilon(n+1)}, \tag{2.18}$$

and the general case easily follows via induction on $k$ by applying (2.17) at each step.

To obtain the monotonicity in $\epsilon$ of $\rho_n^{(k)}$ and $\mathrm{b}_n^{(k)}$, it is clear from (2.18) and from $\mathrm{b}_n^{(1)} = (1-\rho_n^{(1)})/2$ that inequalities (2.14) hold for $k = 1$, and we proceed by induction on $k$. Then, assuming that (for all $n \geqslant 0$) both $d\rho_n^{(k)}/d\epsilon < 0$ and $d\mathrm{b}_n^{(k)}/d\epsilon > 0$ hold for some $k$, differentiating (2.17) yields

$$\frac{d\rho_n^{(k+1)}}{d\epsilon} = -(\rho_n^{(k+1)})^2\left(\frac{d\mathrm{b}_{n-1}^{(k)}}{d\epsilon}+\frac{\mathrm{b}_{n+1}^{(k)}}{d\epsilon}\right) < 0, \tag{2.19}$$

implying that $\rho_n^{(k+1)}$ is monotone decreasing in $\epsilon$. Now differentiating $\mathrm{b}_n^{(k+1)} = \epsilon(n+1)\rho_n^{(k+1)}$ yields

$$\begin{aligned}\frac{d\mathrm{b}_n^{(k+1)}}{d\epsilon} &= (n+1)\left(\rho_n^{(k+1)} + \epsilon\frac{\rho_n^{(k+1)}}{d\epsilon}\right) = \\ &= (n+1)\rho_n^{(k+1)}\left(1 - \epsilon\rho_n^{(k+1)}\left(\frac{d\mathrm{b}_{n-1}^{(k)}}{d\epsilon} + \frac{d\mathrm{b}_{n+1}^{(k)}}{d\epsilon}\right)\right) = \\ &= \mathrm{b}_n^{(k+1)}\left(\epsilon^{-1} - \rho_n^{(k+1)}\left(\frac{d\mathrm{b}_{n-1}^{(k)}}{d\epsilon} + \frac{d\mathrm{b}_{n+1}^{(k)}}{d\epsilon}\right)\right),\end{aligned}$$

where we used (2.19). Then we calculate

$$\begin{aligned}\frac{d\mathrm{b}_{n-1}^{(k)}}{d\epsilon} + \frac{d\mathrm{b}_{n+1}^{(k)}}{d\epsilon} &= n\left(\rho_{n-1}^{(k)} + \epsilon\frac{d\rho_{n-1}^{(k)}}{d\epsilon}\right) + (n+2)\left(\rho_{n+1}^{(k)} + \epsilon\frac{d\rho_{n+1}^{(k)}}{d\epsilon}\right) < \\ &< n\rho_{n-1}^{(k)} + (n+2)\rho_{n+1}^{(k)} = \epsilon^{-1}(\mathrm{b}_{n-1}^{(k)} + \mathrm{b}_{n+1}^{(k)}),\end{aligned}$$

using the inductive hypothesis on $d\rho_n^{(k)}/d\epsilon$, and together with (2.17) this implies that

$$\frac{d\mathrm{b}_n^{(k+1)}}{d\epsilon} > \frac{d\mathrm{b}_n^{(k+1)}}{\epsilon}\left(1 - \rho_n^{(k+1)}(\mathrm{b}_{n-1}^{(k)} + \mathrm{b}_{n+1}^{(k)})\right) = \frac{\mathrm{b}_n^{(k+1)}\rho_n^{(k+1)}}{\epsilon} > 0,$$

as required. ■

**Remark 2.1.** Since the sequence $(\rho_n^{(2j)})_{n\geqslant 0}$ decreases monotonically with $n$, as in Lemma 2.2, and tends to $\frac{1}{j+1}$, it follows that $\rho_n^{(2j)} > 1/(j+1)$ for all $n \geqslant 0$, while the recursion (2.17) with $k = 2j$ shows that $\rho_n^{(2j+1)} < \frac{j+1}{2\epsilon(n+1)}$, which is equivalent to $\mathrm{b}_n^{(2j+1)} < (j+1)/2$.

To further address our main assertion about the uniqueness of the positive solution of (1.13), we introduce the differences

$$\Delta_n^{(k)} = (-1)^k(\rho_n^{(k)} - \rho_n^{(k-1)}), \tag{2.20}$$

where the alternating sign is chosen so that $\Delta_n^{(k)} > 0$ for all $k, n \geqslant 0$, as is seen directly by dividing the inequalities in (2.6) by $\epsilon(n+1)$ for each $n$. Then the coincidence of the lower and upper limits in (2.7), which yields the desired squeezing argument, is equivalent to the statement that

$$\lim_{k\to\infty} \Delta_n^{(k)} = 0 \quad \text{for each} \quad n \geqslant 0. \tag{2.21}$$

To see why the latter result is plausible, we consider the behavior of these differences for small $\epsilon$, which will be needed later.

**Lemma 2.4.** *The leading part of the Taylor expansion of each of the differences* (2.20) *at* $\epsilon = 0$ *is*

$$\Delta_n^{(k)} = c_n^{(k)}\epsilon^k(1 + O(\epsilon)),$$

*where* $c_n^{(0)} = 1$ *for all* $n$*, and the leading coefficient is given recursively by*

$$c_n^{(k+1)} = nc_{n-1}^{(k)} + (n+2)c_{n+1}^{(k)} \quad \textit{for all} \quad k, n \geqslant 0. \tag{2.22}$$

**Proof.** The result is by induction on $k$. For the base case $k = 0$, we have $\Delta_n^{(0)} = 1$, and hence $c_n^{(0)} = 1$, for all $n$. For the inductive step, we write

$$\rho_n^{(k)} - \rho_n^{(k+1)} = \rho_n^{(k)} \rho_n^{(k+1)} \big( (\rho_n^{(k+1)})^{-1} - (\rho_n^{(k)})^{-1} \big),$$

and thus, by using (2.17) and collecting terms inside the round brackets, we obtain the identity

$$\Delta_n^{(k+1)} = \rho_n^{(k)} \rho_n^{(k+1)} \big( \epsilon n \Delta_{n-1}^{(k)} + \epsilon (n+2) \Delta_{n+1}^{(k)} \big). \tag{2.23}$$

Upon using the inductive hypothesis and substituting in the leading-order expansion (2.13) for the two prefactors on the right-hand side of (2.23), we immediately obtain

$$\Delta_n^{(k+1)} = c_n^{(k+1)} \epsilon^{k+1} (1 + O(\epsilon)),$$

where (for each $n$) the leading coefficient $c_n^{(k+1)}$ is given in terms of the coefficients with superscript $k$ by recursion (2.22), as required. ■

Note that we have $c_n^{(1)} = 2(n+1)$, $c_n^{(2)} = 4(n^2 + 2n + 2)$, and it is apparent from (2.22) that these leading coefficients are monotone increasing with $n$, that is, $c_{n+1}^{(k)} > c_n^{(k)}$, for $k \geqslant 1$. This monotonicity is desirable, since it suggests that, for small enough $\epsilon$, we should have $\Delta_{n+1}^{(k)} > \Delta_n^{(k)}$, while from Lemma 2.3 we see that

$$\lim_{n\to\infty} \Delta_n^{(2j)} = \frac{1}{j+1} = \lim_{n\to\infty} \Delta_n^{(2j+1)}; \tag{2.24}$$

thus, if the sequence $(\Delta_n^{(k)})_{n\geqslant 0}$ is increasing with $n$ for each $k \geqslant 1$, then the sought after result (2.21) follows immediately from taking the limit as $j \to \infty$ in (2.24). However, the monotonicity of $c_n^{(k)}$ in $n$ is not enough, because the convergence of the Taylor series (2.13), and hence the result of Lemma 2.4, does not hold uniformly in $n$. For instance, the geometric series for $\rho_n^{(1)}$ has radius of convergence $\frac{1}{2(n+1)}$. Moreover, if we introduce the functions

$$\Delta^{(k)}(z, \epsilon) = (-1)^k \big( R^{(k)}(z, \epsilon) - R^{(k-1)}(z, \epsilon) \big) \tag{2.25}$$

in terms of the $R^{(k)}(z, \epsilon)$ satisfying (2.16), then we might hope to use their behavior in the range $z \geqslant \epsilon$ to determine suitable bounds on the discrete set of values $\Delta_n^{(k)} = \Delta^{(k)}((n+1)\,\epsilon, \epsilon)$. However, this turns out to be tricky for two reasons: first of all, we can show that $\Delta^{(2)}(z, \epsilon)$ and the other functions (2.25) are not monotone in $z$ except for small $\epsilon \lesssim 0.3$; and secondly, for $k \geqslant 3$ these rational functions have poles at certain points in the range $z \geqslant \epsilon$, lying in between the discrete values of interest, so they are unbounded on this range. Figure 4 illustrates these features for $k = 2$ and $k = 4$.

Further numerical investigations suggest that if $\epsilon$ is not too large, then the products $\rho_n^{(2j+1)} \rho_n^{(2j)}$ and $\rho_n^{(2j)} \rho_n^{(2j-1)}$ turn out to be bounded above by their values as $n \to \infty$, as in (2.12). This is in spite of Remark 2.1, which shows that each product consists of an upper bound for one of the factors and a lower bound for the other. It is easy to see that $\rho_n^{(1)} \rho_n^{(0)} < \frac{1}{2\epsilon(n+1)}$, and thus the first case of interest is the product $\rho_n^{(2)} \rho_n^{(1)}$, for which the required bound can be expressed as $R^{(2)}(z, \epsilon) R^{(1)}(z, \epsilon) < \frac{1}{4z}$ for $z \geqslant \epsilon$, and this can be shown to be equivalent to the condition

$$\rho_0^{(2)} \rho_0^{(1)} = \frac{1 + 4\epsilon}{(1 + 2\epsilon)(1 + 6\epsilon)}, < \frac{1}{4\epsilon}$$

which is satisfied whenever $0 < \epsilon < (\sqrt{2} + 1)/2$. This leads to the following.

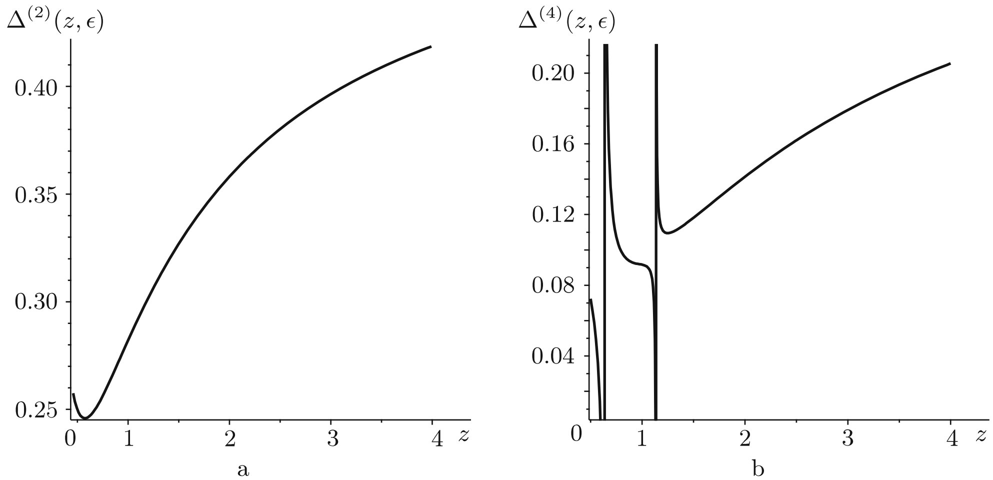



**Fig. 4.** (a) Graph of $\Delta^{(2)}(z,\epsilon)$ against $z$ for $\epsilon = 0.5$ showing the range $0.5 \leqslant z \leqslant 4$, illustrating the local minimum in the interval $\epsilon < z < 2\epsilon$. (b) Graph of $\Delta^{(4)}(z,\epsilon)$ against $z$ for $\epsilon = 0.5$, showing vertical asymptotes at two poles, lying in the intervals $\epsilon < z < 2\epsilon$ and $2\epsilon < z < 3\epsilon$, respectively.

**Conjecture 2.1.** *For each value of $\epsilon$ in the range $0 < \epsilon < \epsilon^\star$, where*

$$\epsilon^* = \frac{1}{2}(\sqrt{2}+1) \approx 1.2071, \tag{2.26}$$

*the rescaled bounds satisfy*

$$\rho_n^{(2j-1)}\rho_n^{(2j-2)} < \frac{1}{2\epsilon(n+1)}, \qquad \rho_n^{(2j)}\rho_n^{(2j-1)} < \frac{j}{2\epsilon(n+1)(j+1)}, \tag{2.27}$$

*for all $n \geqslant 0$ and all $j \geqslant 1$.*

The desirability of the bounds in (2.27), and especially the second one, comes from the fact that it immediately yields an inductive proof that

$$\Delta_n^{(2j+1)} < \Delta_n^{(2j)} < \frac{1}{j+1} \quad \text{for} \quad n \geqslant 0 \tag{2.28}$$

is valid for all $j \geqslant 1$. Indeed, the first inequality in (2.28) is always true (from Lemma 2.1), while for the second one the inductive step is to use (2.23) to obtain

$$\begin{aligned}\Delta_n^{(2j+2)} &= \rho_n^{(2j+2)}\rho_n^{(2j+1)}\big(\epsilon n \Delta_{n-1}^{(2j+1)} + \epsilon(n+2)\Delta_{n+1}^{(2j+1)}\big) < \\ &< \frac{j+1}{2\epsilon(n+1)(j+2)}\left(\frac{\epsilon n}{j+1} + \frac{\epsilon(n+2)}{j+1}\right) = \frac{1}{j+2},\end{aligned}$$

as required (where we used (2.27) and the inductive hypothesis to obtain the inequality). Taking $j \to \infty$ in (2.28), it then follows that

$$\lim_{j\to\infty} \Delta_n^{(2j)} = 0 = \lim_{j\to\infty} \Delta_n^{(2j+1)} \quad \text{for all} \quad n \geqslant 0,$$

yielding the desired squeezing argument, and thus we have the following.

**Corollary 2.1.** *If Conjecture* 2.1 *is valid, then there is a unique positive solution of* (1.13) *with initial data* (1.14) *whenever* $\epsilon$ *lies in the range* (2.26).

We have tried in vain to provide a direct proof of Theorem 2.1, based only on the properties of the mapping $T$. The best we have so far is Corollary 2.1, which relies on an unproven assumption. Fortunately, all is not lost, because in the next section a connection between (1.13) and the space of initial conditions for Painlevé V will be made manifest. Consequently, this will lead to identifying the initial value problem (1.13), (1.14) with certain classical solutions of Painlevé V, and resulting not only in a proof that the positive solution is unique but also in an explicit formula for this solution.

Before switching gears and moving on to consider the geometry of (1.13), we present one more technical result concerning positive solutions.

**Proposition 2.2.** *For each* $k \geqslant 0$, *any positive solution of* (1.13) *satisfies*

$$v_n = \sum_{i=0}^{k} (-1)^i s_{n,i}\, \epsilon^{i+1} + O(\epsilon^{k+2}) \quad \text{as} \quad \epsilon \to 0, \tag{2.29}$$

*where the finite sum coincides with the first* $k+1$ *nonzero terms in the Taylor expansion of the rational function* $\mathrm{b}_n^{(k)}$ *at* $\epsilon = 0$, *that is,*

$$\mathrm{b}_n^{(k)}(\epsilon) = \sum_{i=0}^{k} (-1)^i s_{n,i}\, \epsilon^{i+1} + O(\epsilon^{k+2}), \tag{2.30}$$

*where the coefficients* $s_{n,i}$ *depend only on* $n$. *In particular, for all* $n \geqslant 0$, $s_{n,0} = n+1$, $s_{n,1} = 2(n+1)^2$, $s_{n,2} = 8(n+1)^3 + 4(n+1)$, *independently of* $k$.

**Proof.** By Lemma 2.4, for all $k \geqslant 0$ and for each $n \geqslant 0$, the Taylor expansions of $\rho_n^{(k)}$ and $\rho_n^{(k+1)}$ at $\epsilon = 0$ agree up to and including terms of order $\epsilon^k$, which implies that the corresponding Taylor expansions of $\mathrm{b}_n^{(k)}$ and $\mathrm{b}_n^{(k+1)}$ agree as far as order $\epsilon^{k+1}$, with their first $k+1$ nonzero terms depending only on $n$, as in (2.30). Then, by Proposition 2.1, together with the bounds in Lemma 2.1, a positive solution must satisfy

$$v_n - \sum_{i=0}^{k} (-1)^i s_{n,i} \epsilon^{i+1} = O(\epsilon^{k+2}) \quad \text{as} \quad \epsilon \to 0,$$

for each $k$. From (2.13) it follows that

$$\mathrm{b}_n^{(k)} = (n+1)\epsilon\big(1 - 2(n+1)\epsilon + O(\epsilon^2)\big)$$

for $k \geqslant 1$, giving the stated expressions for $s_{n,0}$ and $s_{n,1}$, while $s_{n,2}$ is obtained from expanding $\mathrm{b}_n^{(k)}$ in (2.3) up to $O(\epsilon^3)$; but for $k \geqslant 3$ these coefficients do not have a uniform expression in $n$. ■

Expansions (2.29) extend to an asymptotic series

$$v_n \sim \sum_{i=0}^{\infty} (-1)^i s_{n,i} \epsilon^{i+1} \tag{2.31}$$

for each $n$. In particular, when $n = 0$, the series is

$$v_0 \sim \epsilon - 2\epsilon^2 + 12\epsilon^3 - 112\epsilon^4 + 1392\epsilon^5 - 21472\epsilon^6 + \cdots \quad \text{as} \quad \epsilon \to 0. \tag{2.32}$$

This should be compared with the Taylor expansion at $\epsilon = 0$ of (1.12) when $n = 0$, that is,

$$\hat{v}_0(\epsilon) = \frac{1}{4}(\sqrt{1+8\epsilon} - 1) = \epsilon - 2\epsilon^2 + 8\epsilon^3 - 40\epsilon^4 + \cdots. \tag{2.33}$$

The bounds $\mathrm{b}_n^{(k)}$ provide a sequence of rational approximations that alternate between upper/lower bounds for even/odd $k$. This is highly reminiscent of the situation for the convergents of Stieltjes-type continued fractions (S-fractions), which also provide successive upper/lower approximations based on a formal series. Indeed, such a fraction can be associated with series (2.31), which, from the stated expressions for $s_{n,0}$, $s_{n,1}$, and $s_{n,2}$, must begin as

$$v_n = \cfrac{(n+1)\epsilon}{1 + \cfrac{2(n+1)\epsilon}{1 + \cfrac{2(n+1+(n+1)^{-1})\epsilon}{1+\cdots}}}. \tag{2.34}$$

In particular, setting $n = 0$ gives the continued fraction for $v_0$, of the form

$$v_0 = \frac{\epsilon}{1+v_1} = \cfrac{\epsilon}{1 + \cfrac{2\epsilon}{1 + \cfrac{4\epsilon}{1 + \cfrac{5\epsilon}{1 + \cfrac{7\epsilon}{1+\cdots}}}}}\ , \tag{2.35}$$

whose coefficients will be described explicitly in due course, towards the end of Sec. 4.

In fact, the continued fraction (2.35) will turn out to provide us with the missing step in the proof of uniqueness of the positive solution. Furthermore, for each $\epsilon > 0$ it will precisely identify this solution as being the one specified by the initial condition

$$v_0 = \frac{1}{2}\left(\frac{K_{5/6}(t/2)}{K_{1/6}(t/2)} - 1\right), \quad \text{where} \quad t = \frac{1}{3\epsilon}.$$

The latter function is plotted against $\epsilon$ in Fig. 3b, with a set of numerical computations using the iterated map $T$ appearing as dots on top of this curve. A key feature of our analysis is to use the fact that this function $v_0$ satisfies the Riccati equation

$$3\epsilon^2 \frac{dv_0}{d\epsilon} = \epsilon(1+2v_0) - v_0 - v_0^2, \tag{2.36}$$

which gives a rapid way to generate the expansion in (2.32) recursively, and similarly $v_1$ satisfies

$$3\epsilon^2 \frac{dv_1}{d\epsilon} = \epsilon(2+v_1) - v_1 - v_1^2. \tag{2.37}$$

These Riccati equations arise as special reductions of the Painlevé V equation, which we proceed to extract from the geometry of discrete equation (1.13) in the next section.

## 3. Complex geometry of the $\mathrm{dP_I}$ equation

In this section, we obtain the space of initial conditions for the $\mathrm{dP_I}$ equation and show that it corresponds to a particular case of the surface type $D_5^{(1)}$, with symmetry type $A_3^{(1)}$, coinciding with that for the Painlevé V equation.

Our goal is to describe the geometry of Eq. (1.13), and then use it to study some of its special solutions. This equation is a special case of the more general $\mathrm{dP_I}$ equation

$$x_{n+1} + x_{n-1} = \frac{\tilde{\alpha} n + \tilde{\beta}}{x_n} + \tilde{\gamma} \tag{3.1}$$

for $\tilde{\alpha} = \tilde{\beta} = \epsilon$ and $\tilde{\gamma} = -1$, and therefore we do the general case and then specialize. Equation (3.1) has been studied in [6]–[8], [19], [20]; see also [21].

We first construct the family of Sakai surfaces regularizing the dynamics (3.1). For that, we begin by rewriting the recurrence as a first-order discrete dynamical system on $\mathbb{C}^1 \times \mathbb{C}^1$ via $y_n := x_{n+1}$. Then Eq. (3.1) can be rewritten as

$$y_n + x_{n-1} = \frac{\tilde{\alpha}n + \tilde{\beta}}{x_n} + \tilde{\gamma}$$

and we obtain the mapping

$$\varphi_n(x_n, y_n) = \left(y_n, \frac{\tilde{\alpha}(n+1) + \tilde{\beta}}{y_n} + \tilde{\gamma} - x_n\right). \tag{3.2}$$

Using the notation $\overline{x}_n = x_{n+1}$, $\underline{x}_n = x_{n-1}$, and same for $y_n$, we can omit the indices and rewrite our mapping and its inverse as

$$\varphi(x, y) = \left(y, \frac{\tilde{\alpha}(n+1) + \tilde{\beta}}{y} + \tilde{\gamma} - x\right), \qquad \varphi^{-1}(x, y) = \left(\frac{\tilde{\alpha}n + \tilde{\beta}}{y} + \tilde{\gamma} - y, x\right). \tag{3.3}$$

Next, we extend the mapping to $\mathbb{P}^1 \times \mathbb{P}^1$ via the introduction of coordinates at infinity, $X = 1/x$ and $Y = 1/y$, and then rewrite mappings (3.3) from the affine chart $(x, y)$ to three other charts, $(X, y)$, $(x, Y)$, and $(X, Y)$. It is easy to see that there are four base points where either rational mapping becomes undefined, which are

$$(X = 0, y = 0), \qquad (X = 0, y = \tilde{\gamma}), \qquad (x = 0, Y = 0), \qquad (x = \tilde{\gamma}, Y = 0). \tag{3.4}$$

As usual, these singularities are resolved by using blowups, where blowing up a point $q_i(x_i, y_i)$ amounts to introducing two new coordinate charts $(\mathrm{u}_i, \mathrm{v}_i)$ and $(U_i, V_i)$ via $x = x_i + \mathrm{u}_i = x_i + U_i V_i$ and $y = y_i + \mathrm{u}_i \mathrm{v}_i = y_i + V_i$. We then extend the mapping to these new charts, check for new base points, resolve them in the same way, and continue this process until the mapping becomes defined everywhere. This process, for discrete Painlevé equations, terminates in a finite number of steps, and in our case, as usual, we obtain eight base points, which are organized into four pairs of infinitely close points, or degeneracy cascades:

$$\begin{aligned}
&q_1\left(X = \frac{1}{x} = 0, y = \tilde{\gamma}\right) \leftarrow q_2\left(\mathrm{u}_1 = X = \frac{1}{x} = 0, \mathrm{v}_1 = x(y - \tilde{\gamma}) = \tilde{\alpha}\right),\\
&q_3\left(X = \frac{1}{x} = 0, y = 0\right) \leftarrow q_4\left(\mathrm{u}_3 = X = \frac{1}{x} = 0, \mathrm{v}_3 = xy = (n+1)\tilde{\alpha} + \tilde{\beta}\right),\\
&q_5\left(x = 0, Y = \frac{1}{y} = 0\right) \leftarrow q_6\left(U_5 = xy = n\tilde{\alpha} + \tilde{\beta}, V_5 = Y = \frac{1}{y} = 0\right),\\
&q_7\left(x = \tilde{\gamma}, Y = \frac{1}{y} = 0\right) \leftarrow q_8\left(U_7 = y(x - \tilde{\gamma}) = -\tilde{\alpha}, V_7 = Y = \frac{1}{y} = 0\right).
\end{aligned} \tag{3.5}$$

This point configuration is easily identified as a point configuration for the $D_5^{(1)}$ Sakai surface family. This family can also be thought of as the Okamoto space of initial conditions for the Painlevé V differential equation in (1.15).

From the point of view of geometry, it is convenient to consider the Hamiltonian version of this equation in the form given in [21], namely

$$\begin{cases}
\dfrac{dq}{dt} = \dfrac{1}{t}\big(q(q-1)(2p+t) - a_1(q-1) - a_3 q\big) = \dfrac{\partial H}{\partial p},\\[2mm]
\dfrac{dp}{dt} = \dfrac{1}{t}\big(p(p+t)(1-2q) + (a_1 + a_3)p - a_2 t\big) = -\dfrac{\partial H}{\partial q},
\end{cases} \tag{3.6}$$

where the Hamiltonian function is

$$H(q,p;t) = \frac{1}{t}\big(q(q-1)p(p+t) - (a_1+a_3)qp + a_1 p + a_2 t q\big), \tag{3.7}$$

and the symplectic structure is given by the standard form $\omega = dp\wedge dq$. The parameters $a_i$ are the so-called *root variables* that are related to the standard Painlevé parameters via

$$\alpha = \frac{1}{2}a_1^2, \quad \beta = -\frac{1}{2}a_3^2, \quad \gamma = a_0 - a_2, \quad \delta = -\frac{1}{2}, \quad \text{where} \quad a_0+a_1+a_2+a_3 = 1. \tag{3.8}$$

System (3.6) is equivalent to (1.15) for the function $w(t) = 1 - 1/q(t)$. The Okamoto space of initial conditions for this system is obtained by removing the irreducible components of the anticanonical divisor (the so-called vertical leaves) from the standard realization of the $D_5^{(1)}$-surface family. This family

$$\mathcal{X} = \mathcal{X}_{\mathbf{a}=(a_0,a_1,a_2,a_3)} = \mathrm{Bl}_{p_1,\dots,p_n}(\mathbb{P}^1\times\mathbb{P}^1)$$

is obtained by blowing up $\mathbb{P}^1\times\mathbb{P}^1$ at the points

$$\begin{aligned} &p_1(\infty,-t) \leftarrow p_2(0,-a_0), \qquad && p_3(\infty,0) \leftarrow p_4(0,-a_2),\\ &p_5(0,\infty) \leftarrow p_6(a_1,0), && p_7(1,\infty) \leftarrow p_8(a_3,0). \end{aligned} \tag{3.9}$$

The latter point configuration and the resulting blowup surface are shown in Fig. 5.

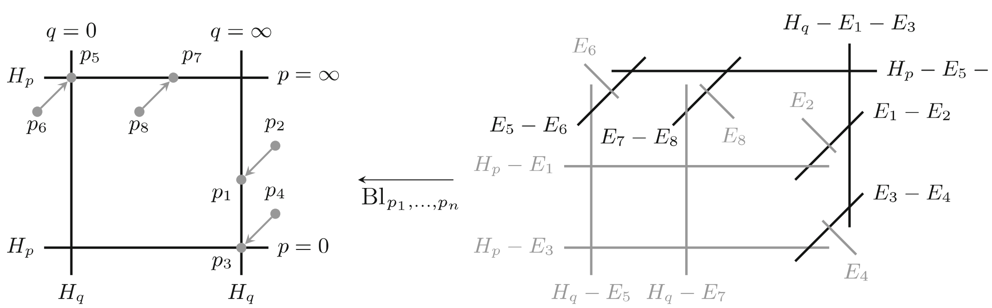



**Fig. 5.** The standard $D_5^{(1)}$ Sakai surface family (the space of initial conditions for system (3.6)).

Note that configurations (3.5) and (3.9) almost match—geometrically they are the same up to the action of rescaling on the axes. Thus, let $q = \lambda x$ and $p = \mu y$. Then the symplectic form is $\omega = dp\wedge dq = \lambda\mu\, dy\wedge dx$, and matching coordinates of the base points results in

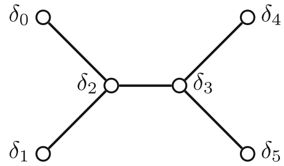
$$\begin{aligned} &\mu\tilde{\gamma} = -t, \qquad \lambda\mu\tilde{\alpha} = -a_0, \qquad \lambda\mu(n\tilde{\alpha}+\tilde{\beta}) = a_1,\\ &\lambda\mu\big((n+1)\tilde{\alpha}+\tilde{\beta}\big) = -a_2, \qquad \lambda\tilde{\gamma} = 1, \quad \lambda\mu(-\tilde{\alpha}) = a_3. \end{aligned}$$

Thus, $\lambda = \tilde{\gamma}^{-1}$ and the root variable normalization $a_0+a_1+a_2+a_3 = -3\tilde{\alpha}\lambda\mu = 1$ gives $\mu = -\tilde{\gamma}(3\tilde{\alpha})^{-1}$. The root variables describing configurations (3.5) then are

$$a_0 = \frac{1}{3}, \qquad a_1 = -\frac{n}{3} - \frac{\tilde{\beta}}{3\tilde{\alpha}}, \qquad a_2 = \frac{n+1}{3} + \frac{\tilde{\beta}}{3\tilde{\alpha}}, \qquad a_3 = \frac{1}{3}. \tag{3.10}$$

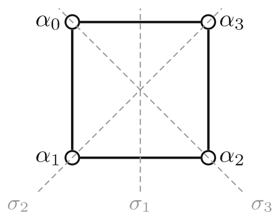
In particular, we see that we obtain an integer translation in the root variables only after three iterations. This can also be seen directly by computing the actions on the Picard lattice $\mathrm{Pic}(\mathcal{X})$ of the mappings $\varphi_*$ and $\varphi^* = (\varphi_*)^{-1}$ induced by mapping (3.3). Specifying the basis as

$$\mathrm{Pic}(\mathcal{X}) = \mathrm{Span}_{\mathbb{Z}}\{\mathcal{H}_x, \mathcal{H}_y, \mathcal{E}_1, \dots, \mathcal{E}_8\}, \tag{3.11}$$




$$\delta_0 = \mathcal{E}_1 - \mathcal{E}_2, \qquad \delta_3 = \mathcal{H}_p - \mathcal{E}_5 - \mathcal{E}_7,$$
$$\delta_1 = \mathcal{E}_3 - \mathcal{E}_4, \qquad \delta_4 = \mathcal{E}_5 - \mathcal{E}_6,$$
$$\delta_2 = \mathcal{H}_q - \mathcal{E}_1 - \mathcal{E}_3, \qquad \delta_5 = \mathcal{E}_7 - \mathcal{E}_8.$$

**Fig. 6.** Surface root basis for a standard realization of a Sakai surface of type $D_5^{(1)}$.




$$\alpha_0 = \mathcal{H}_p - \mathcal{E}_1 - \mathcal{E}_2, \qquad \alpha_2 = \mathcal{H}_p - \mathcal{E}_3 - \mathcal{E}_4,$$
$$\alpha_1 = \mathcal{H}_q - \mathcal{E}_5 - \mathcal{E}_6, \qquad \alpha_3 = \mathcal{H}_q - \mathcal{E}_7 - \mathcal{E}_8.$$
$$\hat{\delta} = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3.$$

**Fig. 7.** Symmetry root basis for the standard $A_3^{(1)}$ symmetry sub-lattice.

the corresponding actions of $\varphi_*$ and $\varphi^*$ are given by

$$\begin{aligned}
&\mathcal{H}_y \xleftarrow{\varphi^*} \mathcal{H}_x \xrightarrow{\varphi_*} \mathcal{H}_x + \mathcal{H}_y - \mathcal{E}_5 - \mathcal{E}_6,\\
&\mathcal{H}_x + \mathcal{H}_y - \mathcal{E}_3 - \mathcal{E}_4 \longleftarrow \mathcal{H}_y \longrightarrow \mathcal{H}_x,\\
&\mathcal{E}_5 \longleftarrow \mathcal{E}_1 \longrightarrow \mathcal{E}_7,\\
&\mathcal{E}_6 \longleftarrow \mathcal{E}_2 \longrightarrow \mathcal{E}_8,\\
&\mathcal{E}_7 \longleftarrow \mathcal{E}_3 \longrightarrow \mathcal{H}_x - \mathcal{E}_6,\\
&\mathcal{E}_8 \longleftarrow \mathcal{E}_4 \longrightarrow \mathcal{H}_x - \mathcal{E}_5,\\
&\mathcal{H}_y - \mathcal{E}_4 \longleftarrow \mathcal{E}_5 \longrightarrow \mathcal{E}_1,\\
&\mathcal{H}_y - \mathcal{E}_3 \longleftarrow \mathcal{E}_6 \longrightarrow \mathcal{E}_2,\\
&\mathcal{E}_1 \longleftarrow \mathcal{E}_7 \longrightarrow \mathcal{E}_3,\\
&\mathcal{E}_2 \longleftarrow \mathcal{E}_8 \longrightarrow \mathcal{E}_4.
\end{aligned} \tag{3.12}$$

We initially use the same choice as in [21] for the surface root basis, shown in Fig. 6, and the symmetry root bases, shown in Fig. 7. Then the action of the mapping $\varphi_*$ on the symmetry root basis is

$$\varphi_*\colon\ \alpha = \langle \alpha_0, \alpha_1, \alpha_2, \alpha_3 \rangle \mapsto \langle \alpha_3, \alpha_0 + \alpha_1, -\alpha_1, \alpha_1 + \alpha_2 \rangle, \tag{3.13}$$

which is *not* a translation on the symmetry sub-lattice. However, it is a *quasi-translation*, since after *three* iterations we obtain a translation, that is,

$$\varphi_*^3\colon\ \alpha = \langle \alpha_0, \alpha_1, \alpha_2, \alpha_3 \rangle \mapsto \alpha + \langle 0, 1, -1, 0 \rangle \hat{\delta}, \qquad \hat{\delta} = -\mathcal{K}_{\mathcal{X}} = \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3. \tag{3.14}$$

There are two standard and nonconjugate examples of discrete Painlevé equations in the $D_5^{(1)}$-family: the equation given in [21, (8.23)], which acts on the symmetry roots as a translation

$$\psi_*\colon\ \alpha = \langle \alpha_0, \alpha_1, \alpha_2, \alpha_3 \rangle \mapsto \alpha + \langle -1, 1, -1, 1 \rangle \hat{\delta}; \tag{3.15}$$

and the equation in [22, (2.33), (2.34)] that acts on the symmetry roots as a translation

$$\phi_*\colon\ \alpha = \langle \alpha_0, \alpha_1, \alpha_2, \alpha_3 \rangle \mapsto \alpha + \langle -1, 0, 0, 1 \rangle \hat{\delta}. \tag{3.16}$$

We see that the cube of our mapping is conjugate, by the half-turn rotation of the Dynkin diagrams, to the dynamics considered by Sakai, also called the $\mathrm{dP_{IV}}$ equation in the original Sakai paper [14].

The dynamical system (3.1) is generated by the birational action of the fully extended affine Weyl group of type $A_3^{(1)}$, that is,

$$\widehat{W}(A_3^{(1)}) := W(A_3^{(1)}) \rtimes \mathrm{Aut}(A_3^{(1)}),$$

where the above semi-direct product structure is given by the action of $\sigma \in \mathrm{Aut}(A_3^{(1)})$ on $W(A_3^{(1)})$ via $w_{\sigma(\alpha_i)} = \sigma w_{\alpha_i} \sigma^{-1}$. The fully extended affine Weyl group acts on point configurations by elementary birational maps on $(q,p)$ and root variables $\boldsymbol{a}$. This is known as a birational representation of this group, and its action by automorphisms of $\mathcal{X}$ is called the *Cremona action* [14]. We describe this birational representation in the following lemma [23, Section A.3].

**Lemma 3.1.** *The birational representation of $\widehat{W}(A_3^{(1)})$, written in the affine $(q,p)$-chart and the root variables $a_i$, is as follows: reflections $w_i$ on $\mathrm{Pic}(\mathcal{X})$ are induced by the elementary birational mappings, also denoted by $w_i$, given by*

$$\begin{aligned}
w_0\colon &\begin{pmatrix} a_0 & a_1 \\ a_2 & a_3 \end{pmatrix}; t; \begin{pmatrix} q \\ p \end{pmatrix} \longmapsto \begin{pmatrix} -a_0 & a_0+a_1 \\ a_2 & a_0+a_3 \end{pmatrix}; t; \begin{pmatrix} q+a_0/(p+t) \\ p \end{pmatrix},\\
w_1\colon &\begin{pmatrix} a_0 & a_1 \\ a_2 & a_3 \end{pmatrix}; t; \begin{pmatrix} q \\ p \end{pmatrix} \longmapsto \begin{pmatrix} a_0+a_1 & -a_1 \\ a_1+a_2 & a_3 \end{pmatrix}; t; \begin{pmatrix} q \\ p-a_1/q \end{pmatrix},\\
w_2\colon &\begin{pmatrix} a_0 & a_1 \\ a_2 & a_3 \end{pmatrix}; t; \begin{pmatrix} q \\ p \end{pmatrix} \longmapsto \begin{pmatrix} a_0 & a_1+a_2 \\ -a_2 & a_2+a_3 \end{pmatrix}; t; \begin{pmatrix} q+a_2/p \\ p \end{pmatrix},\\
w_3\colon &\begin{pmatrix} a_0 & a_1 \\ a_2 & a_3 \end{pmatrix}; t; \begin{pmatrix} q \\ p \end{pmatrix} \longmapsto \begin{pmatrix} a_0+a_3 & a_1 \\ a_2+a_3 & -a_3 \end{pmatrix}; t; \begin{pmatrix} q \\ p-a_3/(q-1) \end{pmatrix}.
\end{aligned} \tag{3.17}$$

*Note that the parameter $t$ can also change when we consider Dynkin diagram automorphisms, and thus it is convenient to include it among the root variables. The actions of the generators $\sigma_1$, $\sigma_2$ of $\mathrm{Aut}(A_3^{(1)})$, shown in Fig. 7, as well as $\sigma_3 = \sigma_1\sigma_2\sigma_1$, are given by the following birational mappings:*

$$\begin{aligned}
\sigma_1\colon &\begin{pmatrix} a_0 & a_1 \\ a_2 & a_3 \end{pmatrix}; t; \begin{pmatrix} q \\ p \end{pmatrix} \longmapsto \begin{pmatrix} a_3 & a_2 \\ a_1 & a_0 \end{pmatrix}; -t; \begin{pmatrix} -p/t \\ qt \end{pmatrix},\\
\sigma_2\colon &\begin{pmatrix} a_0 & a_1 \\ a_2 & a_3 \end{pmatrix}; t; \begin{pmatrix} q \\ p \end{pmatrix} \longmapsto \begin{pmatrix} a_2 & a_1 \\ a_0 & a_3 \end{pmatrix}; -t; \begin{pmatrix} q \\ p+t \end{pmatrix},\\
\sigma_3\colon &\begin{pmatrix} a_0 & a_1 \\ a_2 & a_3 \end{pmatrix}; t; \begin{pmatrix} q \\ p \end{pmatrix} \longmapsto \begin{pmatrix} a_0 & a_3 \\ a_2 & a_1 \end{pmatrix}; -t; \begin{pmatrix} 1-q \\ -p \end{pmatrix}.
\end{aligned} \tag{3.18}$$

Using standard techniques (see, e.g., [24]), mapping (3.1) and its inverse decompose, in terms of generators, as

$$\begin{aligned}
\varphi = \sigma_2\sigma_1 w_2\colon &\begin{pmatrix} a_0 & a_1 \\ a_2 & a_3 \end{pmatrix}; t; \begin{pmatrix} q \\ p \end{pmatrix} \longmapsto \begin{pmatrix} a_1+a_2 & -a_2 \\ a_2+a_3 & a_0 \end{pmatrix}; t; \begin{pmatrix} -p/t \\ t(q+a_2/p-1) \end{pmatrix},\\
\varphi^{-1} = w_2\sigma_1\sigma_2\colon &\begin{pmatrix} a_0 & a_1 \\ a_2 & a_3 \end{pmatrix}; t; \begin{pmatrix} q \\ p \end{pmatrix} \longmapsto \begin{pmatrix} a_3 & a_0+a_1 \\ -a_1 & a_1+a_2 \end{pmatrix}; t; \begin{pmatrix} 1+p/t-a_1/(qt) \\ -qt \end{pmatrix}.
\end{aligned} \tag{3.19}$$

Using $w(t) = 1 - 1/q(t)$ and (3.6), we can rewrite $\varphi$ and $\varphi^{-1}$ as Bäcklund transformations of a solution $w(t)$ of the standard Painlevé V equation (1.15), namely

$$\begin{aligned}
\varphi\colon\ w \mapsto w_+ &= 1 - \frac{1}{\overline{q}} = 1 + \frac{t}{p} = 1 + \frac{2tw}{t\frac{dw}{dt} - a_1 w^2 + (a_1 - a_3 - t)w + a_3},\\
\varphi^{-1}\colon\ w \mapsto w_- &= 1 - \frac{1}{\underline{q}} = 1 - \frac{qt}{qt+qp-a_1} = 1 - \frac{2tw}{t\frac{dw}{dt} + a_1 w^2 - (a_1 + a_3 - t)w + a_3}.
\end{aligned}$$

In the next section, these results are rederived using compositions of classical Bäcklund transformations for Painlevé V.

Note that parameters $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ appearing as coefficients in the Painlevé V equation are

$$\alpha = \frac{(n\tilde{\alpha}+\tilde{\beta})^2}{18\tilde{\alpha}^2}, \qquad \beta = -\frac{1}{18}, \qquad \gamma = -\frac{n\tilde{\alpha}+\tilde{\beta}}{3\tilde{\alpha}}, \qquad \delta = -\frac{1}{2}, \tag{3.20}$$

and further specializing to (1.13) by taking $\tilde{\alpha} = \tilde{\beta} = \epsilon$ and $\tilde{\gamma} = -1$ gives $\lambda = -1$, $\mu = 1/(3\epsilon)$, and

$$\begin{aligned} a_0 = a_3 = \frac{1}{3}, \qquad & a_1 = -\frac{n+1}{3}, \qquad & a_2 = \frac{n+2}{3}, \\ \alpha = \frac{(n+1)^2}{18}, \qquad & \beta = -\frac{1}{18}, \qquad & \gamma = -\frac{n+1}{3}; \end{aligned} \tag{3.21}$$

where the solution $w$ of (1.15) is related to the iterate $v_n = x_n = \lambda^{-1} q_n = \tilde{\gamma} q_n$ of (1.13) via

$$w(t) = 1 + \frac{1}{v_n(\epsilon)} \quad \text{with} \quad t = -\mu\tilde{\gamma} = \frac{1}{3\epsilon}. \tag{3.22}$$

An essential observation to make at this stage is that these values of the root variables, and the corresponding values of $\alpha$, $\beta$, $\gamma$, fall in a particular region of parameter space where Eq. (1.15) is known to admit special solutions in terms of classical special functions: see [25, Sec. 32.10(v)], for instance.

Geometrically, we can see this as follows. For $n = -1$ the root variable $a_1$ vanishes, which corresponds to the appearance of a so-called nodal curve. Indeed, the base point $p_6$ in Eq. (3.9) becomes $p_6(U_5 = V_5 = 0)$, which changes the point configuration and the corresponding blow up picture in Fig. 5 to the one in Fig. 8. The nodal curve is the $(-2)$-curve $H_q - E_5 - E_6$ disjoint from the anticanonical divisor. It is important to note that the existence of nodal curves is preserved by Bäcklund transformations, and such nodal curves define reductions to Riccati equations; see [14], [26].

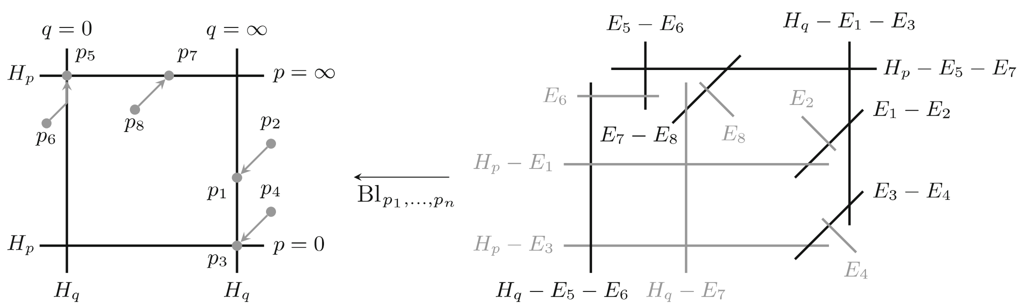



**Fig. 8.** Special $D_5^{(1)}$ Sakai surface with a nodal curve corresponding to $a_1 = 0$.

It is now easy to see that the solution that we are interested in belongs to the Riccati class. Indeed, the initial condition $v_{-1} = 0$ corresponds to $q_{-1} = 0$ and $v_0 = 3\varepsilon p_{-1} = p_{-1}/t$. System (3.6) for $n = -1$ and $a_0 = a_2 = a_3 = 1/3$, $a_1 = 0$ becomes

$$\begin{aligned} \frac{dq_{-1}}{dt} &= \frac{q_{-1}}{t}\left((q_{-1}-1)(2p_{-1}+t) - \frac{1}{3}\right), \\ \frac{dp_{-1}}{dt} &= \frac{1}{t}\left(p_{-1}(p_{-1}+t)(1-2q_{-1}) + \frac{p_{-1}-t}{3}\right). \end{aligned} \tag{3.23}$$

Thus, the flow preserves the nodal curve $q_{-1} = 0$ and is described by the Riccati equation which, when written in terms of $v_0$, becomes

$$\frac{dv_0}{dt} = v_0^2 + \left(1 - \frac{2}{3t}\right)v_0 - \frac{1}{3t}. \tag{3.24}$$

The corresponding reduction of Eq. (1.15) follows from the substitution $w_0 = 1 + 1/v_0$,

$$3t\frac{dw_0}{dt} = w_0^2 - 3tw_0 - 1. \tag{3.25}$$

We revisit this reduction in the next section. Solutions of these Riccati equations can be expressed in terms of classical special functions. This will lead us to the explicit formula for the initial condition, which we now consider.

## 4. Classical solutions of $\mathrm{dP_I}$ from Bäcklund transformations for Painlevé V

In this section we present a large family of special solutions for the Painlevé V equation, connected to one another by Bäcklund transformations, and show how this includes a one-parameter family of solutions for (1.13) corresponding to an explicit orbit of the dynamics whose geometry was revealed in the preceding section. In Sec. 4.1, we construct explicit Wronskian determinant formulas for these solutions, thus providing special solutions of $\tau$-function relations that were obtained in [5]. Finally, in Sec. 4.2, we complete the proof of uniqueness of the positive solution of (1.13), as well as proving Theorem 1.1.

We consider the generic case of Painlevé V (1.15) when $\delta \neq 0$, and as usual we set $\delta = -1/2$, i.e., we take

$$\frac{d^2w}{dt^2} = \left(\frac{1}{2w} + \frac{1}{w-1}\right)\left(\frac{dw}{dt}\right)^2 - \frac{1}{t}\frac{dw}{dt} + \frac{(w-1)^2(\alpha w^2 + \beta)}{t^2 w} + \frac{\gamma w}{t} - \frac{w(w+1)}{2(w-1)}. \tag{4.1}$$

Suppose that $w$ satisfies this Painlevé equation with parameters $(\alpha, \beta, \gamma) = (a^2/2, -b^2/2, c)$; then the Bäcklund transformation $\mathcal{T}_{\varepsilon_1,\varepsilon_2,\varepsilon_3}$ is defined by

$$\mathcal{T}_{\varepsilon_1,\varepsilon_2,\varepsilon_3}(w) = \ 1 - \frac{2\varepsilon_1 tw}{t\frac{dw}{dt} - \varepsilon_2 a w^2 + (\varepsilon_2 a - \varepsilon_3 b + \varepsilon_1 t)w + \varepsilon_3 b}, \tag{4.2}$$

with $\varepsilon_j = \pm 1$ independently for $j = 1, 2, 3$; and for the parameters,

$$\mathcal{T}_{\varepsilon_1,\varepsilon_2,\varepsilon_3}(a,b,c) = \left(\frac{c + \varepsilon_1(1 - \varepsilon_3 b - \varepsilon_2 a)}{2}, \frac{c - \varepsilon_1(1 - \varepsilon_3 b - \varepsilon_2 a)}{2}, \varepsilon_1(\varepsilon_3 b - \varepsilon_2 a)\right), \tag{4.3}$$

see, for example, [27], [28, Sec. 39], or [25, Sec. 32.7(v)]. Note that the Bäcklund transformation $\mathcal{T}_{\varepsilon_1,\varepsilon_2,\varepsilon_3}$ is usually given for the parameters $\alpha$, $\beta$, and $\gamma$ in terms of $a$, $b$, and $c$. However, in order to apply a sequence of Bäcklund transformation, it is better to define the effect of the Bäcklund transformation on the parameters $a$, $b$, and $c$ to avoid any ambiguity in taking a square root. Indeed, the discrete symmetries of Painlevé V, corresponding to the extended affine Weyl group of type $A_3^{(1)}$, act naturally on the root variables, as in (3.8), and the parameters $a$, $b$, $c$ here correspond to $a_1$, $a_3$, $a_0 - a_2$ in the notation of the previous section.

To derive a discrete equation from Bäcklund transformation of a Painlevé equation, we use a Bäcklund transformation $\mathcal{R}$, which relates a solution $w$ to another solution $w_+$, and the inverse transformation $\mathcal{R}^{-1}$, which relates $w$ to a third solution $w_-$. Then eliminating the derivative between the two Bäcklund transformations gives an algebraic equation relating $w$, $w_+$, and $w_-$, which is a discrete equation; cf. [12] for more details of this procedure, see also [11]. Here we consider the Bäcklund transformation $\mathcal{R} = \mathcal{T}_{-1,1,1}$, which has inverse $\mathcal{R}^{-1} = \mathcal{T}_{-1,1,-1} \circ \mathcal{T}_{1,-1,-1} \circ \mathcal{T}_{1,-1,1}$, then

$$w_+ = \mathcal{R}(w; a, b, c) = 1 + \frac{2tw}{t\frac{dw}{dt} - aw^2 + (a - b - t)w + b}, \tag{4.4a}$$

$$w_- = \mathcal{R}^{-1}(w; a, b, c) = 1 - \frac{2tw}{t\frac{dw}{dt} + aw^2 - (a + b - t)w + b} \tag{4.4b}$$

and

$$
\begin{aligned}
(a_+, b_+, c_+) &= \mathcal{R}(a,b,c) = \left(\frac{a+b+c-1}{2}, -\frac{a+b-c-1}{2}, a-b\right),\\
(a_-, b_-, c_-) &= \mathcal{R}^{-1}(a,b,c) = \left(\frac{a-b+c+1}{2}, \frac{a-b-c+1}{2}, a+b\right).
\end{aligned}
$$

Eliminating $dw/dt$ in (4.4a), (4.4b) gives the algebraic equation relating $w$, $w_+$, and $w_-$, and comparing this with (3.22) we set $w = 1 + 1/v$ and $w_\pm = 1 + 1/v_\pm$, which gives

$$
\frac{1}{w_+ - 1} + \frac{1}{w_- - 1} = -\frac{a(w-1)}{t} - 1 \quad \Longrightarrow \quad v_+ + v_- = -1 - \frac{a}{tv}.
$$

**Remark 4.1.** In terms of $w$, transformation (4.4b) is the same as $\mathcal{T}_{1,-1,1}(w)$. However, the parameters $(a, b, c)$ map differently, since

$$
\mathcal{T}_{1,-1,1}(a,b,c) = \left(\frac{a-b+c-1}{2}, \frac{-a+b+c-1}{2}, a+b\right).
$$

Consequently, it is straightforward to show that $\mathcal{T}_{1,-1,1} \circ \mathcal{T}_{-1,1,1}(w) = w$, but $\mathcal{T}_{-1,1,1} \circ \mathcal{T}_{1,-1,1}(w) \neq w$, and therefore $\mathcal{T}_{1,-1,1}$ is not the inverse of $\mathcal{T}_{-1,1,1}$.

Under successive iterations of $\mathcal{R}$ and its inverse, the parameters evolve as

$$
\begin{aligned}
(a_{n+1}, b_{n+1}, c_{n+1}) &= \mathcal{R}(a_n, b_n, c_n) = \left(\frac{a_n+b_n+c_n-1}{2}, -\frac{a_n+b_n-c_n-1}{2}, a_n-b_n\right),\\
(a_{n-1}, b_{n-1}, c_{n-1}) &= \mathcal{R}^{-1}(a_n, b_n, c_n) = \left(\frac{a_n-b_n+c_n+1}{2}, \frac{a_n-b_n-c_n+1}{2}, a_n+b_n\right).
\end{aligned}
$$

From these equations it can be shown that $a_n$ satisfies the third-order difference equation

$$
a_{n+3} - a_n + 1 = 0, \quad \text{with} \quad b_n = a_{n+1} - a_{n+2}, \qquad c_n = a_{n+1} + b_{n+1}, \tag{4.5}
$$

and thus we obtain the solutions

$$
\begin{aligned}
a_n &= \mu \cos\left(\frac{2\pi}{3}n\right) + \kappa \sin\left(\frac{2\pi}{3}n\right) + \lambda - \frac{1}{3}n,\\
b_n &= \sqrt{3}\,\kappa \cos\left(\frac{2\pi}{3}n\right) - \sqrt{3}\,\mu \sin\left(\frac{2\pi}{3}n\right) + \frac{1}{3},\\
c_n &= -2\mu \cos\left(\frac{2\pi}{3}n\right) - 2\kappa \sin\left(\frac{2\pi}{3}n\right) + \lambda - \frac{1}{3}n,
\end{aligned} \tag{4.6}
$$

with $\mu$, $\kappa$, and $\lambda$ arbitrary constants. (Note that $(a_n)_{n\in\mathbb{Z}}$ denotes a sequence of values of the root variable $a$ here, and should not be confused with the indices 0, 1, 2, 3 used in the previous section, as in (3.8), for instance.)

The solution $w_n$ evolves according to

$$
\begin{aligned}
w_{n+1} &= \mathcal{R}(w_n) = 1 + \frac{2tw_n}{t\frac{dw_n}{dt} - a_n w_n^2 + (a_n - b_n - t)w_n + b_n},\\
w_{n-1} &= \mathcal{R}^{-1}(w_n) = 1 - \frac{2tw_n}{t\frac{dw_n}{dt} + a_n w_n^2 - (a_n + b_n - t)w_n + b_n}.
\end{aligned} \tag{4.7}
$$

Eliminating $dw_n/dt$ gives the discrete equation

$$\frac{1}{w_{n+1}-1}+\frac{1}{w_{n-1}-1}=-\frac{a_n(w_n-1)}{t}-1, \tag{4.8}$$

then setting $w_n=1+1/v_n$ gives the discrete equation

$$v_{n+1}+v_{n-1}+1+\frac{a_n}{tv_n}=0, \tag{4.9}$$

which is equivalent to equation (3.23) in [8].

We remark that $v_n(t)$ also satisfies the second-order differential equation

$$\frac{d^2v_n}{dt^2}=\frac{1}{2}\left(\frac{1}{v_n}+\frac{1}{v_n+1}\right)\left(\frac{dv_n}{dt}\right)^2-\frac{1}{t}\frac{dv_n}{dt}-\\ -\frac{a_n^2(v_n+1)^2+b_n^2}{2v_n(v_n+1)t^2}-\frac{c_nv_n(v_n+1)}{t}+\frac{v_n(v_n+1)(2v_n+1)}{2}$$

and the differential-difference equations

$$\begin{aligned} v_{n+1}+\frac{1}{2v_n(v_n+1)}\frac{dv_n}{dt}+\frac{(a_n+b_n)v_n+a_n}{2tv_n(v_n+1)}+\frac{1}{2}=0,\\ v_{n-1}-\frac{1}{2v_n(v_n+1)}\frac{dv_n}{dt}+\frac{(a_n-b_n)v_n+a_n}{2tv_n(v_n+1)}+\frac{1}{2}=0, \end{aligned} \tag{4.10}$$

as well as discrete equation (4.9).

We are interested in the special case of (4.6) when $\mu=\kappa=0$ and $\lambda=-1/3$, i.e.,

$$a_n=-\frac{n+1}{3},\qquad b_n=\frac{1}{3},\qquad c_n=-\frac{n+1}{3}, \tag{4.11}$$

and solutions of Painlevé V (4.1) with

$$(\alpha,\beta,\gamma)=\left(\frac{a_n^2}{2},-\frac{b_n^2}{2},c_n\right)=\left(\frac{(n+1)^2}{18},-\frac{1}{18},-\frac{n+1}{3}\right). \tag{4.12}$$

There are special function solutions of Painlevé V (4.1) with $(\alpha,\beta,\gamma)=(a^2/2,-b^2/2,c)$ if there is some $m\in\mathbb{Z}$ such that either $\varepsilon_1a+\varepsilon_2b+\varepsilon_3c=2m+1$, $a=m$, with $\varepsilon_j=\pm1$, $j=1,2,3$, independently. For parameters (4.11), Painlevé V equation (4.1) has special function solutions, since $a_{3n}-b_{3n}+c_{3n}=-2n-1$, $a_{3n+1}+b_{3n+1}+c_{3n+1}=-2n-1$, and $a_{3n+2}=-n-1$.

**Lemma 4.1.** *The only Riccati equation that is compatible with Painlevé* V *equation* (4.1) *with parameters* $(\alpha,\beta,\gamma)=(a_0^2/2,-b_0^2/2,c_0)=(1/18,-1/18,-1/3)$, *i.e.,*

$$\frac{d^2w_0}{dt^2}=\left(\frac{1}{2w_0}+\frac{1}{w_0-1}\right)\left(\frac{dw_0}{dt}\right)^2-\frac{1}{t}\frac{dw_0}{dt}+\frac{(w_0-1)^2(w_0^2-1)}{18t^2w_0}-\frac{w_0}{3t}-\frac{w_0(w_0+1)}{2(w_0-1)}, \tag{4.13}$$

*is Eq.* (3.25)*, which has solution*

$$w_0(t)=-\frac{C_1\{I_{1/6}(t/2)-I_{-5/6}(t/2)\}+C_2\{K_{1/6}(t/2)+K_{5/6}(t/2)\}}{C_1\{I_{1/6}(t/2)+I_{-5/6}(t/2)\}+C_2\{K_{1/6}(t/2)-K_{5/6}(t/2)\}}, \tag{4.14}$$

*where* $I_\nu(t/2)$ *and* $K_\nu(t/2)$ *are modified Bessel functions, with* $C_1$ *and* $C_2$ *arbitrary constants.*

**Proof.** Using the Riccati equation

$$\frac{dw_0}{dt} = p_2(t)w_0^2 + p_1(t)w_0 + p_0(t),$$

where $p_2(t)$, $p_1(t)$, and $p_0(t)$ are functions to be determined, to remove the derivatives in (4.13) and then equating coefficient of powers of $w_0$ shows that $p_2(t) = 1/3t$, $p_1(t) = -1$, $p_0(t) = -1/3t$; hence we obtain (3.25), as required. Letting $w_0 = -3t\frac{d}{dt}\log\varphi_0$ in (3.25) gives

$$t^2\frac{d^2\varphi_0}{dt^2} + t(t+1)\frac{d\varphi_0}{dt} - \frac{1}{9}\varphi_0 = 0, \tag{4.15}$$

which has solution

$$\varphi_0(t) = \sqrt{t}\left\{C_1\left[I_{1/6}\left(\frac{t}{2}\right) + I_{-5/6}\left(\frac{t}{2}\right)\right] + C_2\left[K_{1/6}\left(\frac{t}{2}\right) - K_{5/6}\left(\frac{t}{2}\right)\right]\right\}e^{-t/2}, \tag{4.16}$$

and thus we obtain solution (4.14), as required. ■

**Remark 4.2.** Special function solutions of the Painlevé V equation (4.1) are usually expressed in terms of the Whittaker functions $M_{\kappa,\mu}(t)$ and $W_{\kappa,\mu}(t)$, or equivalently the Kummer functions $M(a,b,t)$ and $U(a,b,t)$, cf. [25, Sec. 32.10(v)]. However, if $b = 2a+n$, with $n$ an integer, then the Kummer functions $M(a,b,t)$ and $U(a,b,t)$ can be respectively expressed in terms of the modified Bessel functions $I_\nu(t/2)$ and $K_\nu(t/2)$, for example

$$\begin{aligned} M\left(\nu+\frac{1}{2}, 2\nu+1, t\right) &= \Gamma(1+\nu)\left(\frac{t}{4}\right)^{-\nu} I_\nu\left(\frac{t}{2}\right)e^{t/2}, \\ U\left(\nu+\frac{1}{2}, 2\nu+1, t\right) &= \pi^{-1/2}t^{-\nu}K_\nu\left(\frac{t}{2}\right)e^{t/2}, \end{aligned}$$

see [25, Sec. 13.6(iii)]. In terms of Kummer functions, the solution of (4.15) is given by

$$\varphi_0(t) = t^{-1/3}\left\{C_1 M\left(\frac{2}{3}, \frac{1}{3}, t\right) + C_2 U\left(\frac{2}{3}, \frac{1}{3}, t\right)\right\}e^{-t}, \tag{4.17}$$

with $C_1$ and $C_2$ arbitrary constants, which gives the solution of (3.25)

$$w_0(t) = -\frac{3t}{\varphi_0(t)}\frac{d\varphi_0}{dt} = 3(t+1) - \frac{6C_1M(5/3,1/3,t) + 8C_2U(5/3,1/3,t)}{3C_1M(5/3,1/3,t) + 3C_2U(5/3,1/3,t)}. \tag{4.18}$$

The Kummer functions $M(2/3,1/3,t)$, $M(5/3,1/3,t)$, $U(2/3,1/3,t)$, and $U(5/3,1/3,t)$ can be expressed in terms of modified Bessel functions as

$$\begin{aligned} M\left(\frac{2}{3}, \frac{1}{3}, t\right) &= \left(\frac{t}{4}\right)^{5/6}\Gamma\left(\frac{1}{6}\right)\left\{I_{1/6}\left(\frac{t}{2}\right) + I_{-5/6}\left(\frac{t}{2}\right)\right\}e^{t/2}, \\ M\left(\frac{5}{3}, \frac{1}{3}, t\right) &= \frac{1}{2}\left(\frac{t}{4}\right)^{5/6}\Gamma\left(\frac{1}{6}\right)\left\{(3t+4)I_{1/6}\left(\frac{t}{2}\right) + (3t+2)I_{-5/6}\left(\frac{t}{2}\right)\right\}e^{t/2}, \\ U\left(\frac{2}{3}, \frac{1}{3}, t\right) &= \frac{3t^{5/6}}{2\sqrt{\pi}}\left\{K_{5/6}\left(\frac{t}{2}\right) - K_{1/6}\left(\frac{t}{2}\right)\right\}e^{t/2}, \\ U\left(\frac{5}{3}, \frac{1}{3}, t\right) &= \frac{9t^{5/6}}{16\sqrt{\pi}}\left\{(3t+2)K_{5/6}\left(\frac{t}{2}\right) - (3t+4)K_{1/6}\left(\frac{t}{2}\right)\right\}e^{t/2}. \end{aligned}$$

Solutions (4.17) and (4.18) can be expressed in terms of Whittaker functions, since the relationship between the Whittaker functions $M_{\kappa,\nu}(t)$, $W_{\kappa,\nu}(t)$ and the Kummer functions $M(a,b,t)$, $U(a,b,t)$ is given by

$$M_{\kappa,\nu}(t) = t^{\mu+1/2} M\left(\mu - \kappa + \frac{1}{2}, 1 + 2\mu, t\right) e^{-t/2},$$
$$W_{\kappa,\nu}(t) = t^{\mu+1/2} U\left(\mu - \kappa + \frac{1}{2}, 1 + 2\mu, t\right) e^{-t/2},$$

and conversely

$$M(a,b,t) = t^{-1/2} M_{b/2-a,b/2-1/2}(t) e^{t/2},$$
$$U(a,b,t) = t^{-1/2} W_{b/2-a,b/2-1/2}(t) e^{t/2},$$

see [25, Eqs. 13.14.2–13.14.5]

Hence, using (4.16), we obtain

$$w_0(t) = -\frac{3t}{\varphi_0(t)} \frac{d\varphi_0}{dt} = -\frac{C_1\{I_{1/6}(t/2) - I_{-5/6}(t/2)\} + C_2\{K_{1/6}(t/2) + K_{5/6}(t/2)\}}{C_1\{I_{1/6}(t/2) + I_{-5/6}(t/2)\} + C_2\{K_{1/6}(t/2) - K_{5/6}(t/2)\}},$$

which satisfies the Painlevé V equation (4.1) with parameters $(\alpha, \beta, \gamma) = (1/18, -1/18, -1/3)$, and thus $w_1 = \mathcal{R}(w_0; -1/3, 1/3, -1/3)$ is given by

$$w_1(t) = \frac{2\{C_1 I_{1/6}(t/2) + C_2 K_{1/6}(t/2)\}}{(3t+2)\{C_1 I_{1/6}(t/2) + C_2 K_{1/6}(t/2)\} + 3t\{C_1 I_{-5/6}(t/2) - C_2 K_{5/6}(t/2)\}},$$

which satisfies Painlevé V (4.1) with parameters $(\alpha, \beta, \gamma) = (2/9, -1/18, -2/3)$. Therefore, since $v_n = 1/(w_n - 1)$, we have

$$\begin{aligned} v_0(t) &= \frac{1}{w_0(t) - 1} = -\frac{1}{2} - \frac{C_1 I_{-5/6}(t/2) - C_2 K_{5/6}(t/2)}{2\{C_1 I_{1/6}(t/2) + C_2 K_{1/6}(t/2)\}}, \\ v_1(t) &= \frac{1}{w_1(t) - 1} = -1 - \frac{2}{3t} - \\ &\quad - \frac{2\{C_1 I_{-5/6}(t/2) - C_2 K_{5/6}(t/2)\}}{3t\{C_1 I_{1/6}(t/2) + C_2 K_{1/6}(t/2)\} + C_1 I_{-5/6}(t/2) - C_2 K_{5/6}(t/2)}. \end{aligned} \tag{4.19}$$

Furthermore, if we set $a_n = -(n+1)/3$ in (4.9), then we obtain

$$v_n(v_{n+1} + v_{n-1} + 1) = \frac{n+1}{3t}, \tag{4.20}$$

which is (1.13) with $\epsilon = 1/3t$, in agreement with (3.22). Hence, if we put $n = 0$ in (4.20) and use (4.19), then we see that

$$v_{-1} = -v_1 - 1 + \frac{1}{3tv_0} = 0. \tag{4.21}$$

Moreover, if we let $Z_{1/6}(t) = C_1 I_{1/6}(t) + C_2 K_{1/6}(t)$ and $Z_{-5/6}(t) = C_1 I_{-5/6}(t) - C_2 K_{5/6}(t)$, then the first three nonzero iterates of $\mathrm{dP_I}$ equation (4.20) are given by

$$v_0(t) = -\frac{1}{2} - \frac{Z_{-5/6}(t/2)}{2Z_{1/6}(t/2)}, \qquad v_1(t) = -1 - \frac{2}{3t} - \frac{2Z_{-5/6}(t/2)}{3t\{Z_{1/6}(t/2) + Z_{-5/6}(t/2)\}},$$
$$v_2(t) = -\frac{3(t+2)}{2(3t+2)} + \frac{Z_{-5/6}(t/2)}{2Z_{1/6}(t/2)} - \frac{4Z_{-5/6}(t/2)}{(3t+2)\{(3t+2)Z_{1/6}(t/2) + 3tZ_{-5/6}(t/2)\}}.$$

**Remark 4.3.** It was shown in Lemma 4.1 that $w_0(t)$ satisfies the Riccati equation (3.25). It follows that $w_1(t)$ also satisfies a Riccati equation, namely

$$t\frac{dw_1}{dt} = \frac{2}{3}w_1^2 - (t+1)w_1 + \frac{1}{3},$$

and hence $v_0(t)$ and $v_1(t)$ satisfy Riccati equations, namely (3.24) and

$$t\frac{dv_1}{dt} = tv_1^2 + \left(t - \frac{1}{3}\right)v_1 - \frac{2}{3},$$

respectively, which are equivalent to (2.36) and (2.37), after making the change of independent variable $\epsilon = 1/3t$. The solutions $w_n(t)$ and $v_n(t)$ for $n \geqslant 2$ do not satisfy Riccati equations with simple coefficients. However, it can be shown that $v_n$ for $n \geqslant 2$ does satisfy a Riccati equation with coefficients given by combinations of $v_{n-2}$ and lower $v_j$ [29]; for instance, $v_2$ satisfies a Riccati equation that includes $v_0$ among its coefficients (see below).

**4.1. Determinantal representation of the solutions.** In this subsection we show that the solutions of the $\mathrm{dP_I}$ equation (4.20) may be written in terms of determinants involving modified Bessel functions. These determinants can be regarded as particular examples of Painlevé V $\tau$-functions. Moreover, in [5] it is shown that if a $\tau$-function $\upsilon_n$ for (1.13) is introduced via

$$v_n = \frac{\upsilon_n \upsilon_{n-4}}{\upsilon_{n-1}\upsilon_{n-3}}, \tag{4.22}$$

then it satisfies a trilinear (degree 3 homogeneous) equation of order 6. At the end of this subsection, we show that the determinants of modified Bessel functions provide special function solutions of this trilinear equation.

We begin by defining some convenient notation for linear combinations of modified Bessel functions, and associated Wronskian determinants.

**Definition 4.1.** Let $\mathcal{Z}_\nu(t)$ be defined by

$$\mathcal{Z}_\nu(t) = \begin{cases} d_1 I_j(t) + d_2(-1)^j K_j(t), & \nu = j \in \mathbb{Z}, \\ d_1 I_\nu(t) + d_2 I_{-\nu}(t), & \text{otherwise}, \end{cases} \tag{4.23}$$

where $I_\nu(t)$ and $K_\nu(t)$ are modified Bessel functions and $d_1$ and $d_2$ are arbitrary constants. For $m \in \mathbb{Z}$ and $n \in \mathbb{N}$, let $\mathcal{B}_{m,n,\nu}(t)$ be the Wronskian determinant

$$\mathcal{B}_{m,n,\nu}(t) = \mathcal{W}(\{f_{m-\ell,\nu+\ell}(t)\}_{\ell=0}^{n-1}), \tag{4.24}$$

where

$$f_{m,\nu}(t) = \begin{cases} t^{-\nu}\displaystyle\sum_{j=0}^{m}\binom{m}{j}\frac{j+\nu}{(j+2\nu)_{m+1}}(-1)^j\mathcal{Z}_{\nu+j}\left(\frac{t}{2}\right), & m \in \mathbb{N}, \\ t^{|m|-\nu}\displaystyle\sum_{j=0}^{|m|}\binom{|m|}{j}\frac{j-\nu}{(j-2\nu)_{|m|+1}}\mathcal{Z}_{\nu-j}\left(\frac{t}{2}\right), & -m \in \mathbb{N}, \end{cases} \tag{4.25}$$

whenever the denominators in (4.25) are nonzero; let $\mathcal{B}_{m,0,\nu}(t) = 1$.

**Lemma 4.2.** *Let $\mathcal{B}_{m,n,\nu}(t)$ be the determinant of modified Bessel functions given in Definition* 4.1. *We also define the constants $C^{[1]}_{m,n,\nu}$ and $C^{[2]}_{m,n,\nu}$ as*

$$C^{[1]}_{m,n,\nu} = \begin{cases} \frac{1}{2} - m - \nu, & m \geqslant n+2, \\ 1, & m = n+1, \\ -\left(\nu + n + \frac{1}{2}\right)^{-1} & \text{otherwise}, \end{cases} \tag{4.26}$$

$$C^{[2]}_{m,n,\nu} = \begin{cases} -\nu - m - \frac{1}{2}, & m \geqslant n, \\ 1 & \text{otherwise}. \end{cases} \tag{4.27}$$

*Then*

$$w^{[1]}_{m,n,\nu}(t) = C^{[1]}_{m,n,\nu} \frac{\mathcal{B}_{m,n+1,\nu}(t)\mathcal{B}_{m-2,n,\nu+1}(t)}{\mathcal{B}_{m,n,\nu}(t)\mathcal{B}_{m-2,n+1,\nu+1}(t)} \tag{4.28}$$

*is a solution of Painlevé* V *for the parameters*

$$(\alpha,\beta,\gamma) = \left(\frac{(2\nu+2n+1)^2}{8}, -\frac{(2\nu+2m+2n-1)^2}{8}, 2\nu+m-1\right), \tag{4.29}$$

*and*

$$w^{[2]}_{m,n,\nu}(t) = C^{[2]}_{m,n,\nu} \frac{\mathcal{B}_{m,n,\nu+1}(t)\mathcal{B}_{m,n,\nu}(t)}{\mathcal{B}_{m,n-1,\nu+1}(t)\mathcal{B}_{m,n+1,\nu}(t)} \tag{4.30}$$

*is a solution of Painlevé* V *for the parameters*

$$(\alpha,\beta,\gamma) = \left(\frac{n^2}{2}, -\frac{(2\nu+m+n)^2}{2}, m\right). \tag{4.31}$$

**Proof.** The special function solutions of Painlevé V written in terms of the Kummer function $M(a,b,t)$ were derived by Masuda [18]; see also Forrester and Witte [17]. Solutions (4.29), (4.31) may be inferred from the work of Masuda by using [25, Sec. 13.6(iii)]

$$\begin{aligned} M\left(\nu+\frac{1}{2}, 2\nu+1+n, t\right) &= \Gamma(\nu)e^{t/2}\left(\frac{t}{4}\right)^{-\nu} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(2\nu)_k(\nu+k)}{(2\nu+1+n)_k} I_{\nu+k}\left(\frac{t}{2}\right), \\ M\left(\nu+\frac{1}{2}, 2\nu+1-n, t\right) &= \Gamma(\nu-n)e^{t/2}\left(\frac{t}{4}\right)^{n-\nu} \sum_{k=0}^{n} \binom{n}{k} \frac{(2\nu-2n)_k(\nu-n+k)}{(2\nu+1-n)_k} I_{\nu+k-n}\left(\frac{t}{2}\right). \end{aligned} \tag{4.32}$$

If the modified Bessel function $K_\nu(t)$ with $\nu \notin \mathbb{Z}$ is desired in the solution, we use [25, Eq. (10.27.4)]

$$K_\nu(t) = \frac{\pi(I_{-\nu}(t) - I_\nu(t))}{2\sin(\pi\nu)}. \tag{4.33}$$

For $j \in \mathbb{Z}$, the modified Bessel function $K_j(t)$ is given by [25, Eq. (10.27.5)]

$$K_j(t) = \frac{(-1)^{j-1}}{2}\left(\left.\frac{\partial I_\nu(t)}{\partial \nu}\right|_{\nu=j} + \left.\frac{\partial I_\nu(t)}{\partial \nu}\right|_{\nu=-j}\right). \ \blacksquare \tag{4.34}$$

We use the following properties of $f_{m,\nu}(t)$ defined in (4.25) to construct identities for $\mathcal{B}_{m,n,\nu}(t)$.

**Lemma 4.3.** *We have*

$$f_{m,\nu}(t) - f_{m-1,\nu+1}(t) = \left(m + \nu + \frac{1}{2}\right) f_{m+1,\nu}(t), \qquad m \geqslant 1, \tag{4.35a}$$

$$\left(\nu + \frac{1}{2}\right)\big(f_{1,\nu}(t) - f_{-1,\nu+1}(t)\big) = f_{0,\nu}(t), \qquad m = 0, \tag{4.35b}$$

$$f_{m,\nu}(t) + f_{m+1,\nu}(t) = -\left(\nu + \frac{1}{2}\right) f_{m-1,\nu+1}(t), \qquad m \leqslant -1. \tag{4.35c}$$

*The derivatives of $f_{m,\nu}(t)$ are given by*

$$\frac{d}{dt} f_{m,\nu}(t) = \begin{cases} \dfrac{1}{2} f_{m,\nu}(t) - \left(m + \nu + \dfrac{1}{2}\right) f_{m+1,\nu}(t), & m \geqslant 0, \\ \dfrac{1}{2} f_{m,\nu}(t) + f_{m+1,\nu}(t), & m \leqslant -1. \end{cases} \tag{4.36}$$

*Furthermore, if $\nu \notin \mathbb{Z}$, the following symmetry holds:*

$$f_{m,\nu}(t; d_1, d_2) = \begin{cases} t^{-2\nu - m} f_{m,-m-\nu}(t; d_2, d_1), & m \geqslant 0, \\ (-1)^m t^{-2\nu - m} f_{m,-m-\nu}(t; d_2, d_1), & m < 0. \end{cases} \tag{4.37}$$

**Proof.** The properties of $f_{m,\nu}(t)$ are proved using the properties of the modified Bessel functions given in [25, Sec. 10.29]. For example, (4.35a) is given by

$$\begin{aligned} f_{m,\nu}(t) - f_{m-1,\nu+1}(t) = t^{-\nu} \sum_{j=0}^{m} \binom{m}{j} \frac{j+\nu}{(j+2\nu)_{m+1}} (-1)^j \mathcal{Z}_{\nu+j}\left(\frac{t}{2}\right) - \\ - t^{-\nu-1} \sum_{j=0}^{m-1} \binom{m-1}{j} \frac{j+\nu+1}{(j+2\nu+2)_m} (-1)^j \mathcal{Z}_{\nu+1+j}\left(\frac{t}{2}\right). \end{aligned} \tag{4.38}$$

Using [25, Eq. (10.29.1)]

$$\mathcal{Z}_{\nu-1}\left(\frac{t}{2}\right) - \mathcal{Z}_{\nu+1}\left(\frac{t}{2}\right) = \frac{4\nu}{t} \mathcal{Z}_{\nu}\left(\frac{t}{2}\right), \tag{4.39}$$

to rewrite the second sum in (4.38), we obtain

$$\begin{aligned} f_{m,\nu}(t) - f_{m-1,\nu+1}(t) = t^{-\nu} \sum_{j=0}^{m} \binom{m}{j} \frac{j+\nu}{(j+2\nu)_{m+1}} (-1)^j \mathcal{Z}_{\nu+j}\left(\frac{t}{2}\right) + \\ + t^{-\nu} \sum_{j=0}^{m-1} \binom{m-1}{j} \frac{1}{4(j+2\nu+2)_m} (-1)^j \left(\mathcal{Z}_{\nu+2+j}\left(\frac{t}{2}\right) - \mathcal{Z}_{\nu+j}\left(\frac{t}{2}\right)\right). \end{aligned} \tag{4.40}$$

Combining like terms in (4.40) gives (4.35a). The remaining identities are proved similarly. ■

**Lemma 4.4.** *When $\nu \notin \mathbb{Z}$ the Bessel determinant $\mathcal{B}_{m,n,\nu}(t)$ has the following symmetry*:

$$\mathcal{B}_{m,n,\nu}(t;d_1,d_2) = \begin{cases} \dfrac{r_{m,n,\nu}}{r_{m,n,1-m-n-\nu}}\, t^{n(1-m-n-2\nu)}\mathcal{B}_{m,n,1-m-n-\nu}(t;d_2,d_1), & m \geqslant n-1, \\ (-1)^{mn}\dfrac{r_{m,n,\nu}}{r_{m,n,1-m-n-\nu}}\, t^{n(1-m-n-2\nu)}\mathcal{B}_{m,n,1-m-n-\nu}(t;d_2,d_1), & 1 \leqslant m \leqslant n-2, \\ (-1)^{mn}\, t^{n(1-m-n-2\nu)}\mathcal{B}_{m,n,1-m-n-\nu}(t;d_2,d_1), & m \leqslant 0, \end{cases} \tag{4.41}$$

*where $r_{m,n,\nu}$ is the constant*

$$r_{m,n,\nu} = \begin{cases} \displaystyle\prod_{\ell=0}^{n-1}\left(\nu+m+\frac{1}{2}\right)_\ell, & m \geqslant n-1, \\ \displaystyle\prod_{\ell=0}^{m}\left(\nu+n-\frac{1}{2}\right)_\ell, & \text{otherwise.} \end{cases} \tag{4.42}$$

**Proof.** We prove (4.41) when $m \geqslant n-1$. Subtracting column $j+1$ from column $j$ in (4.24) for $j = 1,2,\ldots,k$, $k = n-1, n-2, \ldots, 1$, and using recurrence relation (4.35a), we obtain

$$\mathcal{B}_{m,n,\nu}(t;d_1,d_2) = r_{m,n,\nu}\mathcal{W}(\{f_{m+n-1-2j,\nu+j}(t;d_1,d_2)\}_{j=0}^{n-1}). \tag{4.43}$$

By applying symmetry (4.37) to (4.43), we have

$$\mathcal{B}_{m,n,\nu}(t;d_1,d_2) = r_{m,n,\nu}\mathcal{W}(\{t^{1-m-n-2\nu} f_{m+n-1-2j,1+j-m-n-\nu}(t;d_2,d_1)\}_{j=0}^{n-1}). \tag{4.44}$$

We then use the Wronskian identity [30, Theorem 4.25]

$$\mathcal{W}(g(t)f_1(t), g(t)f_2(t), \ldots, g(t)f_n(t)) = g(t)^n \mathcal{W}(f_1(t), f_2(t), \ldots, f_n(t)), \tag{4.45}$$

in order to remove the powers of $t$ from the Wronskian in (4.44), thus:

$$\begin{aligned} \mathcal{B}_{m,n,\nu}(t;d_1,d_2) &= r_{m,n,\nu}\, t^{n(1-m-n-2\nu)}\mathcal{W}(\{f_{m+n-1-2j,1+j-m-n-\nu}(t;d_2,d_1)\}_{j=0}^{n-1}) = \\ &= \frac{r_{m,n,\nu}}{r_{m,n,1-m-n-\nu}}\, t^{n(1-m-n-2\nu)}\mathcal{B}_{m,n,1-m-n-\nu}(t;d_2,d_1). \end{aligned} \tag{4.46}$$

The proofs of the remaining cases are similar. ■

**Lemma 4.5.** *Let $\mathcal{B}_{m,n,\nu}(t)$ be the determinant of modified Bessel functions defined in Definition* 4.1. *When $m < 0$, $\mathcal{B}_{m,n,\nu}(t)$ satisfies*

$$\mathcal{B}_{m,n-1,\nu+1}(t)\mathcal{B}_{m,n+1,\nu}(t) + \mathcal{B}_{m-1,n,\nu+1}(t)\mathcal{B}_{m+1,n,\nu}(t) = \mathcal{B}_{m,n,\nu+1}(t)\mathcal{B}_{m,n,\nu}(t). \tag{4.47}$$

**Proof.** We prove the Lemma using the Jacobi identity [31], sometimes known as the *Lewis Carroll formula*, for determinants. Let $\mathcal{D}$ be an arbitrary determinant, and $\mathcal{D}\left[\begin{smallmatrix} i \\ j \end{smallmatrix}\right]$ be the determinant with the $i$th row and $j$th column removed from $\mathcal{D}$. Then we have the Jacobi identity:

$$\mathcal{D}\mathcal{D}\begin{bmatrix} i,j \\ k,\ell \end{bmatrix} = \mathcal{D}\begin{bmatrix} i \\ k \end{bmatrix}\mathcal{D}\begin{bmatrix} j \\ \ell \end{bmatrix} - \mathcal{D}\begin{bmatrix} i \\ \ell \end{bmatrix}\mathcal{D}\begin{bmatrix} j \\ k \end{bmatrix}. \tag{4.48}$$

Using the derivative of $f_{m,\nu}(t)$ given in (4.36) we rewrite the Wronskian determinant $\mathcal{B}_{m,n,\nu}(t)$ when $m < 0$ as

$$\mathcal{B}_{m,n,\nu}(t) = \det \left| \sum_{j=0}^{k} \binom{k}{j} 2^{j-k} f_{m+j-\ell,\nu+\ell} \right|_{k,\ell=0}^{n-1}. \tag{4.49}$$

Since we can add a multiple of any row to any other row without changing the determinant in (4.49), we keep the last term in each sum:

$$\mathcal{B}_{m,n,\nu}(t) = \det |f_{m+k-\ell,\nu+\ell}|_{k,\ell=0}^{n-1}. \tag{4.50}$$

We apply the Jacobi identity (4.48) to the determinant in (4.50), choosing $i = 1$, $j = n$ for the rows and $k = 1$, $\ell = n$ for the columns. The relevant minor determinants are

$$\begin{aligned}
\mathcal{B}_{m,n,\nu}\begin{bmatrix} 1, n \\ 1, n \end{bmatrix} &= \det |f_{m+k-\ell,\nu+1}|_{i,j=0}^{n-3} = \mathcal{B}_{m,n-2,\nu+1}, \\
\mathcal{B}_{m,n,\nu}\begin{bmatrix} 1 \\ 1 \end{bmatrix} &= \det |f_{m+k-\ell,\nu+1}|_{i,j=0}^{n-2} = \mathcal{B}_{m,n-1,\nu+1}, \\
\mathcal{B}_{m,n,\nu}\begin{bmatrix} n \\ n \end{bmatrix} &= \det |f_{m+k-\ell,\nu+\ell}|_{i,j=0}^{n-2} = \mathcal{B}_{m,n-1,\nu}, \\
\mathcal{B}_{m,n,\nu}\begin{bmatrix} 1 \\ n \end{bmatrix} &= \det |f_{m+1+k-\ell,\nu+\ell}|_{i,j=0}^{n-2} = \mathcal{B}_{m+1,n-1,\nu}, \\
\mathcal{B}_{m,n,\nu}\begin{bmatrix} n \\ 1 \end{bmatrix} &= \det |f_{m-1+k-\ell,\nu+1+\ell}|_{i,j=0}^{n-1} = \mathcal{B}_{m-1,n-1,\nu+1}.
\end{aligned} \tag{4.51}$$

Substituting (4.51) into (4.48) gives (4.47). ■

**Lemma 4.6.** *Let $\mathcal{B}_{m,n,\nu}(t)$ be the determinant of modified Bessel functions defined in Definition* 4.1. *When $m \leqslant 0$, we have*

$$\begin{aligned}
&\mathcal{B}_{m,n+1,\nu}(t)\mathcal{B}_{m-2,n,\nu+1}(t) + \left(\nu + n + \frac{1}{2}\right)\mathcal{B}_{m-2,n+1,\nu+1}(t)\mathcal{B}_{m,n,\nu}(t) = \\
&= -\mathcal{B}_{m-1,n,\nu+1}(t)\mathcal{B}_{m-1,n+1,\nu}(t).
\end{aligned} \tag{4.52}$$

**Proof.** In Wronskian determinant (4.24), subtracting $\nu + j - 1/2$ times column $j + 1$ from column $j$ for $j = 1, 2, \ldots, k$, where $k$ decreases from $n - 1$ to 1, and using recurrence relation (4.35c), we obtain

$$\mathcal{B}_{m,n,\nu}(t) = (-1)^{n(n-1)/2}\mathcal{W}(\{f_{m+n-1-2j,\nu+j}\}_{j=0}^{n-1}). \tag{4.53}$$

By adding $\nu + j - 1/2$ times column $j + 1$ to column $j$ for $j = 1, \ldots, n - 1$ in (4.53), we have

$$\mathcal{B}_{m+1,n,\nu}(t) = (-1)^{1+n(n+1)/2}\mathcal{W}(\{f_{m+n-1-2j,\nu+j}\}_{j=0}^{n-2}, f_{m+2-n,\nu+n-1}). \tag{4.54}$$

Adding $1/(\nu + n - 3/2)$ times column $n - 1$ to column $n$ in (4.54) gives

$$\mathcal{B}_{m+1,n,\nu}(t) = \frac{1}{\nu + n - 3/2}(-1)^{n(n+1)/2}\mathcal{W}(\{f_{m+n-1-2j,\nu+j}\}_{j=0}^{n-2}, f_{m+4-n,\nu+n-2}). \tag{4.55}$$

Vein and Dale prove three variants of the Jacobi identity (4.48) in [30, Theorem 3.6]. To prove the bilinear relation (4.52), we use

$$\mathcal{A}_n\begin{bmatrix} i \\ p \end{bmatrix}\mathcal{A}_{n+1}\begin{bmatrix} n+1 \\ q \end{bmatrix} - \mathcal{A}_n\begin{bmatrix} i \\ q \end{bmatrix}\mathcal{A}_{n+1}\begin{bmatrix} n+1 \\ p \end{bmatrix} = \mathcal{A}_n\mathcal{A}_{n+1}\begin{bmatrix} i, n+1 \\ p, q \end{bmatrix}, \tag{4.56}$$

which is identity (B) in [30, Theorem 3.6], where we let

$$\mathcal{A}_{n+1} = \mathcal{W}(\{f_{m+n-2j,\nu+j}\}_{j=0}^{n-1}, f_{m+3-n,\nu+n-1}), \qquad \mathcal{A}_n = \mathcal{W}(\{f_{m+n-2j,\nu+j}\}_{j=0}^{n-1}). \tag{4.57}$$

Setting $i = n$, $p = 1$, and $q = n$, we have

$$\begin{aligned}
&\mathcal{A}_n\begin{bmatrix} n \\ 1 \end{bmatrix} = \mathcal{W}(\{f_{m+n-2-2j,\nu+1+j}\}_{j=0}^{n-2}) = (-1)^{(n-1)(n-2)/2}\mathcal{B}_{m,n-1,\nu+1}(t),\\
&\mathcal{A}_{n+1}\begin{bmatrix} n+1 \\ n \end{bmatrix} = \mathcal{W}(\{f_{m+n-2j,\nu+j}\}_{j=0}^{n-2}, f_{m+3-n,\nu+n-1}) = (-1)^{1+n(n+1)/2}\mathcal{B}_{m,n,\nu}(t),\\
&\mathcal{A}_n\begin{bmatrix} n \\ n \end{bmatrix} = \mathcal{W}(\{f_{m+n-2j,\nu+j}\}_{j=0}^{n-2}) = (-1)^{(n-1)(n-2)/2}\mathcal{B}_{m,n,\nu}(t),\\
&\mathcal{A}_{n+1}\begin{bmatrix} n+1 \\ 1 \end{bmatrix} = \mathcal{W}(\{f_{m+n-2-2j,\nu+1+j}\}_{j=0}^{n-2}, f_{m+3-n,\nu+n-1}) = \left(\nu+n-\frac{1}{2}\right)(-1)^{n(n+1)/2}\mathcal{B}_{m,n,\nu}(t),\\
&\mathcal{A}_n = (-1)^{n(n-1)/2}\mathcal{B}_{m,n,\nu}(t),\\
&\mathcal{A}_{n+1}\begin{bmatrix} n, n+1 \\ 1, n \end{bmatrix} = \mathcal{W}(\{f_{m+n-2-2j,\nu+1+j}\}_{j=0}^{n-3}, f_{m+3-n,\nu+n-1}) = (-1)^{1+n(n-1)/2}\mathcal{B}_{m,n,\nu}(t).
\end{aligned}$$

Substituting these equations into Jacobi identity (4.56) gives (4.52). ■

**Theorem 4.1.** *The solutions of* $\mathrm{dP_I}$ (4.20) *are a special case of the modified Bessel function solutions of Painlevé* V *given in Lemma* 4.2, *namely* $v_n(t) = 1/(w_n(t)-1)$, *where* $w_n(t)$ *satisfies*

$$w_n(t) = \begin{cases} w^{[1]}_{1-k,k,-1/6}(t; d_1, d_2), & n = 3k,\ k \in \mathbb{N},\\ w^{[1]}_{-k,k,1/6}(t; d_2, d_1), & n = 3k+1,\ k \in \mathbb{N},\\ w^{[2]}_{-1-k,k+1,-1/6}(t; d_1, d_2), & n = 3k+2,\ k \in \mathbb{N}. \end{cases} \tag{4.58}$$

**Proof.** When $n = 3k$, the parameters of Painlevé V (4.12) become

$$(\alpha, \beta, \gamma) = \left(\frac{(3k+1)^2}{18}, -\frac{1}{18}, -k-\frac{1}{3}\right), \tag{4.59}$$

which is (4.29) with $m = 1-k$, $n = k$, and $\nu = -1/6$. The cases where $n = 3k+1$ and $n = 3k+2$ are obtained similarly. ■

Using the recurrence relations in Lemmas 4.5 and 4.6, the solutions $v_n(t)$ of $\mathrm{dP_I}$ (4.20) may be written as

$$v_n(t) = \begin{cases} 0, & n = -1,\\ \left(k+\frac{1}{3}\right)\dfrac{\mathcal{B}_{1-k,k,-1/6}(t; d_1, d_2)\mathcal{B}_{-1-k,k+1,5/6}(t; d_1, d_2)}{\mathcal{B}_{-k,k,5/6}(t; d_1, d_2)\mathcal{B}_{-k,k+1,-1/6}(t; d_1, d_2)}, & n = 3k,\ k \in \mathbb{N},\\ \left(k+\frac{2}{3}\right)\dfrac{\mathcal{B}_{-k,k,1/6}(t; d_2, d_1)\mathcal{B}_{-2-k,k+1,7/6}(t; d_2, d_1)}{\mathcal{B}_{-1-k,k,7/6}(t; d_2, d_1)\mathcal{B}_{-1-k,k+1,1/6}(t; d_2, d_1)}, & n = 3k+1,\ k \in \mathbb{N},\\ \dfrac{\mathcal{B}_{-1-k,k,5/6}(t; d_1, d_2)\mathcal{B}_{-1-k,k+2,-1/6}(t; d_1, d_2)}{\mathcal{B}_{-2-k,k+1,5/6}(t; d_1, d_2)\mathcal{B}_{-k,k+1,-1/6}(t; d_1, d_2)}, & n = 3k+2,\ k \in \mathbb{N}. \end{cases} \tag{4.60}$$

Furthermore, using the symmetry (4.41), we may rewrite $v_{3k+1}(t)$ as

$$v_{3k+1}(t) = -\frac{3k+2}{3t}\frac{\mathcal{B}_{-k,k,5/6}(t;d_1,d_2)\mathcal{B}_{-2-k,k+1,5/6}(t;d_1,d_2)}{\mathcal{B}_{-1-k,k,5/6}(t;d_1,d_2)\mathcal{B}_{-1-k,k+1,5/6}(t;d_1,d_2)}. \tag{4.61}$$

Substituting (4.60) into (4.20) gives the trilinear equations

$$\begin{aligned}
&\mathcal{B}_{-1-k,k+1,5/6}\left\{\mathcal{B}_{-1-k,k,5/6}\mathcal{B}_{-k,k+1,-1/6} + \left(k+\frac{1}{3}\right)\mathcal{B}_{1-k,k,-1/6}\mathcal{B}_{-2-k,k+1,5/6}\right\} = \\
&\quad = -\mathcal{B}_{-k,k,5/6}\{\mathcal{B}_{-1-k,k,5/6}\mathcal{B}_{-1-k,k+2,-1/6} + \mathcal{B}_{-k,k+1,-1/6}\mathcal{B}_{-2-k,k+1,5/6}\}, \\
&t\mathcal{B}_{-1-k,k+1,5/6}\{\mathcal{B}_{-1-k,k,5/6}\mathcal{B}_{1-k,k+1,-1/6} + \mathcal{B}_{-k,k-1,5/6}\mathcal{B}_{-k,k+1,-1/6}\} = \\
&\quad = \mathcal{B}_{-k,k,5/6}\left\{\mathcal{B}_{-1-k,k,5/6}\mathcal{B}_{-k,k+1,-1/6} + \left(k+\frac{2}{3}\right)\mathcal{B}_{1-k,k,-1/6}\mathcal{B}_{-2-k,k+1,5/6}\right\}, \\
&t\mathcal{B}_{-1-k,k,5/6}\left\{\mathcal{B}_{-1-k,k+1,5/6}\mathcal{B}_{-1-k,k+2,-1/6} + \left(k+\frac{4}{3}\right)\mathcal{B}_{-k,k+1,-1/6}\mathcal{B}_{-2-k,k+2,5/6}\right\} = \\
&\quad = \mathcal{B}_{-2-k,k+1,5/6}\left\{\left(k+\frac{2}{3}\right)\mathcal{B}_{-1-k,k+2,-1/6}\mathcal{B}_{-k,k,5/6} + (k+1)\mathcal{B}_{-k,k+1,-1/6}\mathcal{B}_{-1-k,k+1,5/6}\right\}.
\end{aligned}$$

After a gauge transformation (which depends on $n \mod 3$), to match up the $\tau$-function $v_n$ in (4.22) with an appropriate Wronskian, each of the latter equations is equivalent to the trilinear equation in [5].

**4.2. Unique positive solution: finale.** The preceding results on repeated application of Bäcklund transformations for Painlevé V show that these generate a solution of $\mathrm{dP_I}$ in the case that one initial value $v_{-1} = 0$, while $v_0$ is arbitrary. Indeed, for any choice of $v_0$ there is a value of the ratio $\lambda = C_1/C_2$ in (4.19) that provides a complete solution of the difference equation (4.20) in terms of ratios of modified Bessel functions, and this is equivalent to (1.13) with $t = 1/3\epsilon$. Note that if we rearrange the formula for $v_0$ in (4.19) as

$$2v_0 + 1 = \frac{K_{5/6}(t/2) - \lambda I_{-5/6}(t/2)}{K_{1/6}(t/2) + \lambda I_{1/6}(t/2)}, \qquad t = \frac{1}{3\epsilon}, \tag{4.62}$$

then for any choice of $v_0$ we can invert the Möbius transformation above to find $\lambda$ in terms of $v_0$ and $\epsilon$. Thus, for each fixed $\epsilon$ there is a one-to-one correspondence between the choice of $v_0$ and the choice of parameter $\lambda$. However, we can characterize one particular solution by its distinct asymptotic behavior.

**Proposition 4.1.** *The function*

$$v_0(\epsilon) = \frac{1}{2}\left(\frac{K_{5/6}(1/(6\epsilon))}{K_{1/6}(1/(6\epsilon))} - 1\right) \tag{4.63}$$

*is the unique initial condition for* (1.13) *that has the asymptotic behavior* (2.32) *as* $\epsilon \to 0$.

**Proof.** From the leading-order asymptotics of the modified Bessel functions, that is,

$$K_\nu\left(\frac{t}{2}\right) \sim \sqrt{\frac{\pi}{t}}\, e^{-t/2}, \qquad I_\nu\left(\frac{t}{2}\right) \sim \frac{1}{\sqrt{\pi t}}\, e^{t/2} \quad \text{as} \quad t \to \infty,$$

we see that the right-hand side of (4.62) tends to 1 as $t \to \infty$ when $\lambda = 0$, but otherwise it tends to $-1$. Equivalently, if $\lambda = 0$, then $v_0 \to 0$ as $\epsilon \to 0$, but for all $\lambda \neq 0$ this ratio of modified Bessel functions gives $v_0 \to -1$ as $\epsilon \to 0$. Hence, function (4.63) is the only member of this one-parameter family that is compatible with the asymptotic behavior (2.32) as $\epsilon \to 0$. Since all of the functions $v_0$ given by (4.19) satisfy the Riccati equation (2.36), the latter series (2.32) can be obtained by substituting in

$$v_0 \sim \sum_{i=0}^{\infty} (-1)^i s_{0,i} \epsilon^{i+1}.$$

This immediately yields the recursion

$$s_{0,i+1} = (3i+1)s_{0,i} + \sum_{j=0}^{i} s_{0,i-j} s_{0,j} \quad \text{for} \quad i \geqslant 0, \quad \text{with} \quad s_{0,0} = 1,$$

producing the sequence $1, 2, 12, 112, 1392$, etc. ■

The computation of quotients of modified Bessel functions is an important problem in numerical analysis [32], and continued fraction methods provide effective tools for doing this [33], [34]. For the function (4.63), the continued fraction expansion (2.35) can be calculated directly from the Riccati equation (2.36), which is one among a family that includes many examples first considered in the pioneering works of Euler and Lagrange (see Ch. II in [35]).

If we set

$$\eta_0 = v_0, \qquad \eta_1 = v_1, \qquad \eta_2 = v_0 + v_2, \tag{4.64}$$

then we see that the iteration of (1.13) with $v_{-1} = 0$ is consistent with the recursion

$$\eta_n = \frac{\xi_n \epsilon}{1 + \eta_{n+1}}, \qquad n \geqslant 0, \tag{4.65}$$

and this generates a continued fraction representation for $v_0$ in the form

$$v_0 = \cfrac{\xi_0 \epsilon}{1 + \cfrac{\xi_1 \epsilon}{1 + \cfrac{\xi_2 \epsilon}{1 + \cdots}}}. \tag{4.66}$$

At the same time, given that $v_0$ is a solution of (2.36), it follows by induction that each $\eta_n$ satisfies a Riccati equation,

$$3\epsilon^2 \frac{d\eta_n}{d\epsilon} + \eta_n^2 + (1 - \zeta_n \epsilon)\eta_n - \xi_n \epsilon = 0, \qquad n \geqslant 0, \tag{4.67}$$

provided that $\zeta_{n+1} = 3 - \zeta_n$, $\xi_{n+1} = \xi_n + \zeta_{n+1}$. Then we require $\xi_0 = 1$ and $\zeta_0 = 2$ from (2.36), which implies $\xi_1 = 2$ and $\zeta_1 = 1$, in agreement with (2.37), and hence continued fraction (4.66) and its associated sequence of Riccati equations (4.67) are completely specified by

$$\xi_{2m} = 3m + 1, \quad \zeta_{2m} = 2 \quad \text{and} \qquad \xi_{2m+1} = 3m + 2, \quad \zeta_{2m+1} = 1 \quad \text{for } m \geqslant 0, \tag{4.68}$$

which reveals the pattern in (2.35).

**Remark 4.4.** Upon taking the difference of Eqs. (4.67) for $n = 0$ and $n = 2$, and using $v_2 = \eta_2 - \eta_0$, we find that $v_2$ also satisfies a Riccati equation, that is,

$$3\epsilon^2 \frac{dv_2}{d\epsilon} + v_2^2 + (1 - 2\epsilon + 2v_0)v_2 - 3\epsilon = 0,$$

which has $v_0$ appearing among its coefficients. Similarly, it is possible to use (1.13) to show by induction that all $v_n$ for $n \geqslant 2$ satisfy Riccati equations with $v_j$ for $j \leqslant n-2$ included in their coefficients (cf. [29]).

The continued fraction (4.66) for $\eta_0 = v_0$ thus obtained has a sequence of convergents $\bar{\eta}_0^{(k)} = P^{(k)}/Q^{(k)}$, $k \geqslant 0$, which correctly approximate the first $k+1$ nonzero terms in the series expansion, so

$$\bar{\eta}_0^{(k)} = \sum_{i=0}^{k} (-1)^i s_{0,i} \epsilon^{i+1} + O(\epsilon^{k+2}),$$

and the numerators $P^{(k)}$ and denominators $Q^{(k)}$ are polynomials in $\epsilon$ generated by the same three-term relation

$$P^{(k+1)} = P^{(k)} + \xi_{k+1} \epsilon P^{(k-1)}, \qquad Q^{(k+1)} = Q^{(k)} + \xi_{k+1} \epsilon\, Q^{(k-1)},$$

with initial conditions $P^{(-2)} = 1$, $P^{(-1)} = 0$, $Q^{(-2)} = 0$, $Q^{(-1)} = 1$. Standard theory [35] then implies that with the coefficients $\xi_n$ as above, the continued fraction is convergent for all $\epsilon > 0$, being equal to the alternating sum

$$\eta_0 = \frac{P^{(0)}}{Q^{(0)}} + \sum_{k=1}^{\infty} (-1)^k \frac{\xi_0 \xi_1 \dots \xi_k\, \epsilon^{k+1}}{Q^{(k-1)} Q^{(k)}} = \lim_{k \to \infty} \bar{\eta}_0^{(k)}.$$

However, the continued fraction is formally divergent at $\epsilon = 0$, which corresponds to fact that series (2.32) is divergent. Thus we see that the continued fraction (2.35) represents the function $\eta_0 = v_0$ in (4.63) for all $\epsilon \in (0, \infty)$, and hence this function is positive on the whole positive semi-axis. This provides a much stronger characterization of this function than the asymptotic one in Proposition 4.1, namely the following.

**Corollary 4.1.** *The function* (4.63) *is the unique solution of the Riccati equation* (2.36) *that is positive for all* $\epsilon > 0$.

We finally return to the fixed point method considered in Sec. 2, and the upper/lower bounds on the iterates of the mapping $T$. It turns out that the bound $\mathrm{b}_0^{(k)}$ and the convergent $\bar{\eta}_0^{(k)}$ both approximate asymptotic series (2.32) correctly to the same order $\epsilon^{k+1}$, but the former is a better approximant than the latter in the sense that its coefficient at order $\epsilon^{k+2}$ is closer to the correct value. This leads to a more precise statement in terms of inequalities, as follows.

**Proposition 4.2.** *The convergents of the continued fraction for the function* (4.63) *interlace with the upper/lower bounds obtained for the mapping $T$ in* (2.1), *according to*

$$\mathrm{b}_0^{(2j-1)} < \bar{\eta}_0^{(2j+1)} \leqslant \mathrm{b}_0^{(2j+1)} < \mathrm{b}_0^{(2j+2)} \leqslant \bar{\eta}_0^{(2j+2)} < \mathrm{b}_0^{(2j)} \quad \textit{for all} \quad j \geqslant 0. \tag{4.69}$$

**Proof.** The middle inequality in (4.69) was already shown as part of Lemma 2.1, so the main new content of statement (4.69) can be concisely paraphrased as

$$(-1)^k \mathrm{b}_0^{(k)} \leqslant (-1)^k \bar{\eta}_0^{(k)} < (-1)^k \mathrm{b}_0^{(k-2)} \quad \text{for} \quad k \geqslant 1. \tag{4.70}$$

This is proved by induction on $k$, via a comparison of two different expressions for $v_0$: the first is the standard continued fraction (2.35), which generates the sequence of convergents $\bar{\eta}_0^{(k)}$; while the second is the structure of iteration of (1.13), and the action of the mapping $T$, which generates another sequence of rational approximants $\mathrm{b}_0^{(k)}$, obtained from $v_0$ given as a kind of branched continued fraction:

$$v_0 = \cfrac{\epsilon}{1 + \cfrac{2\epsilon}{1 + \cfrac{\epsilon}{1 + \cfrac{2\epsilon}{1+\cdots}} + \cfrac{3\epsilon}{1 + \cfrac{2\epsilon}{1+\cdots} + \cfrac{4\epsilon}{1+\cdots}}}}\,. \tag{4.71}$$

For the induction, observe that truncation at level $k=0$ in each fraction gives the same approximant $\bar{\eta}_0^{(0)}=\mathrm{b}_0^{(k)}=\epsilon$, and also at levels $k=1$ and $k=2$ we have

$$\bar{\eta}_0^{(1)}=\mathrm{b}_0^{(1)}=\frac{\epsilon}{1+2\epsilon}, \qquad \bar{\eta}_0^{(2)}=\mathrm{b}_0^{(2)}=\frac{\epsilon(1+4\epsilon)}{1+6\epsilon},$$

and therefore by (2.6) with $n=0$ it follows that (4.70) holds for the base cases $k=1,2$; but for $k\geqslant 3$ all of the inequalities in (4.70) become strict. For the induction, we can consider the sequence of convergents $\bar{\eta}_1^{(k)}$ of the standard S-fraction for $v_1$, that is,

$$v_1=\cfrac{2\epsilon}{1+\cfrac{4\epsilon}{1+\cfrac{5\epsilon}{1+\cdots}}},$$

so that we have $\bar{\eta}_0^{(k+1)}=\epsilon/(1+\bar{\eta}_1^{(k)})$, while from the action of $T$ we have

$$T(\mathrm{b}_0^{(k)})=\frac{\epsilon}{1+\mathrm{b}_1^{(k)}}.$$

Hence it follows that (4.70) holds by induction, provided that at the next level we have the analogous inequalities

$$(-1)^k\mathrm{b}_1^{(k)}\leqslant(-1)^k\bar{\eta}_1^{(k)}<(-1)^k\mathrm{b}_1^{(k-2)} \quad \text{for} \quad k\geqslant 1. \tag{4.72}$$

For instance, if (4.72) holds for some even $k=2j$, then

$$\frac{\epsilon}{1+\mathrm{b}_1^{(2j-2)}}<\frac{\epsilon}{1+\bar{\eta}_1^{(2j)}}\leqslant\frac{\epsilon}{1+\mathrm{b}_1^{(2j)}},$$

which is precisely (4.70) for $k=2j+1$; and the reasoning is the same starting from (4.72) with odd $k$, but with the inequalities reversed.

Of course, this begs the question of the validity of (4.72), which must be verified by going down one more level and considering

$$v_1=\frac{2\epsilon}{1+\eta_2}=\frac{2\epsilon}{1+v_0+v_2},$$

for which the leading-order truncation gives $\bar{\eta}_1^{(0)}=\mathrm{b}_1^{(0)}=2\epsilon$, while subsequent truncations require comparison of

$$\eta_2=\cfrac{4\epsilon}{1+\cfrac{5\epsilon}{1+\cdots}} \quad \text{and} \quad v_0+v_2=\cfrac{\epsilon}{1+\cfrac{2\epsilon}{1+\cdots}}+\cfrac{3\epsilon}{1+\cfrac{2\epsilon}{1+\cdots}+\cfrac{4\epsilon}{1+\cdots}} \tag{4.73}$$

at the next stage. It is clear that the leading-order truncation in these equations has $4\epsilon=\epsilon+3\epsilon$, which in turn shows that $\bar{\eta}_1^{(1)}=\mathrm{b}_1^{(1)}$, confirming (4.72) for $k=1$, but for the next order comparison it is required that

$$\frac{4\epsilon}{1+5\epsilon}<\frac{\epsilon}{1+2\epsilon}+\frac{3\epsilon}{1+6\epsilon}. \tag{4.74}$$

The latter is just a particular case of the general inequality

$$\frac{A}{1+B}<\frac{A_1}{1+B_1}+\frac{A_2}{1+B_2}, \quad \text{for} \quad A=A_1+A_2, \quad AB=A_1B_1+A_2B_2, \quad A_1,A_2,B_1,B_2>0,$$

which holds due to the convexity of the function $1/(1+x)$. Then (4.74) implies that $2\epsilon=\mathrm{b}_1^{(0)}>\bar{\eta}_1^{(2)}>\mathrm{b}_1^{(2)}$, establishing (4.72) for $k=2$. Subsequent upper/lower bounds follow similarly, by repeatedly applying the same convexity argument to compare each lower stage of the two continued fractions. ■

We can now present the final steps of the proof of uniqueness of the positive solution, and conclude with the proof of the main theorem.

**Proof of Theorem 2.1.** Taking limits of the middle three inequalities in (4.70) gives

$$\lim_{j\to\infty} \bar{\eta}_0^{(2j+1)} \leqslant \lim_{j\to\infty} \mathrm{b}_0^{(2j+1)} \leqslant \lim_{j\to\infty} \mathrm{b}_0^{(2j+2)} \leqslant \lim_{j\to\infty} \bar{\eta}_0^{(2j+2)}.$$

Then the convergence of the continued fraction (2.35) gives the equality of limits of the upper and lower sequences of convergents, that is,

$$\lim_{j\to\infty} \bar{\eta}_0^{(2j+1)} = v_0 = \lim_{j\to\infty} \bar{\eta}_0^{(2j+2)}, \tag{4.75}$$

which in turn gives

$$\lim_{j\to\infty} \mathrm{b}_0^{(2j+1)} = \lim_{j\to\infty} \mathrm{b}_0^{(2j+2)} \implies \lim_{j\to\infty} \rho_0^{(2j+1)} = \lim_{j\to\infty} \rho_0^{(2j)} \implies$$
$$\implies \lim_{j\to\infty} \Delta_0^{(2j)} = 0 = \lim_{j\to\infty} \Delta_0^{(2j+1)}.$$

Now taking the limit $k \to \infty$ in the $n = 0$ case of (2.23) produces

$$\lim_{k\to\infty} \Delta_0^{(k+1)} = 2\epsilon\Big(\lim_{k\to\infty} \rho_0^{(k)}\Big)^2 \lim_{k\to\infty} \Delta_1^{(k)} \implies \lim_{k\to\infty} \Delta_1^{(k)} = 0,$$

and hence, by induction on $n$, repeated application of (2.23) yields

$$\lim_{k\to\infty} \Delta_n^{(k)} = 0 \quad \text{for all} \quad n \geqslant 0.$$

From the result of Proposition 2.1 we see that, for each $n$, the upper and lower bounds in (2.8) coincide, giving the unique positive solution $\mathbf{v} = (v_n)_{n\geqslant 0}$, as required. ■

**Proof of Theorem 1.1.** It remains to point out that for each $\epsilon > 0$, limit (4.75) obtained from the continued fraction (2.35) is precisely the function $v_0(\epsilon)$ given by (4.63). For the other $v_n$, note from (4.32) that setting $\lambda = C_1/C_2 = 0$ in (4.62) is equivalent to taking $d_1/d_2 = -1$ in (4.60). Thus, without loss of generality, by fixing $d_1 = 1 = -d_2$ in Theorem 4.1, for each $n$ we obtain the explicit expression for $v_n(\epsilon) > 0$ in terms of ratios of Wronskian determinants. ■

## 5. Conclusions

We have shown that the quantum minimal surface obtained from a pair of operators satisfying the equation for a parabola, $Z_2 = Z_1^2$, admits an exact solution in terms of modified Bessel functions, where the positive solution of the associated discrete Painlevé I equation corresponds to a particular sequence of classical solutions of the continuous Painlevé V equation with specific parameter values. The key to finding this exact solution was to use the complex geometry of the discrete Painlevé equation, constructing the associated Sakai surface, which identified the $\mathrm{dP_I}$ equation with the action of a quasi-translation on the space of initial conditions for Painlevé V. Once the appropriate parameters for the Painlevé V equation had been found, this enabled us to compare with known results on classical solutions, and match these up the initial conditions for the $\mathrm{dP_I}$ equation, which identified the unique positive solution. While previous results in the literature have expressed these classical solutions in terms of Whittaker functions (or equivalently, Kummer functions), some current work in progress (by two of us in collaboration with Dunning) has allowed the unique positive solution to be expressed with modified Bessel functions, which are a special case of Kummer functions and Whittaker functions, cf. [25, Secs. 13.6 and 13.18].

It is interesting to note that other instances of classical solutions of Painlevé equations have appeared in the recent literature, providing the unique solutions of discrete Painlevé equations that satisfy positivity or other special initial boundary value problems [36]–[38]. These particular unique solutions seem to arise in specific application areas, such as random matrices and orthogonal polynomials, but it would be worthwhile to see if they can be characterized in some other way (geometrically, for instance). In fact, the asymptotics of oscillatory solutions of certain $dP_I$ equations, including (1.13), were recently considered in [39]. Such solutions are known to arise from a growth problem defined by a normal random matrix ensemble [40], and thus it is natural to wonder whether the unique positive solution of (1.13) has an interpretation in that context.

A recent preprint by Hoppe [29] includes some comments on Whittaker function expressions for $v_0$ and $v_2$, which are equivalent to the ones that we have found. The latter work raises the question of whether similar results should apply for other quantum minimal surfaces from rational curve equations of the form $Z_2^r = Z_1^s$ for positive integers $r < s$ (with $\gcd(r, s) = 1$), following a remark made at the end of [5], where it is suggested that these curves should also give rise to discrete integrable systems. Indeed, condition (1.8) gives rise to a difference equation for $v_n$, which (after integrating) becomes an equation of order $2(s-1)$: this should be a discrete Painlevé equation of higher order. Some more details of the example $(r, s) = (1, 3)$ are considered in [29], where the difference equation in question is

$$v_n(v_{n+1}v_{n+2} + v_{n-1}v_{n+1} + v_{n-2}v_{n-1} + 1) = \epsilon(n + 1).$$

Preliminary investigations show that this equation admits a positive solution with analogous properties to the case of the quantum parabola considered here: since the initial acceptance of this paper, the preprint [41] has appeared, which relates the family of quantum $(1, s)$ curves to orthogonal polynomials with complex densities, recovers our expression for $v_0$ when $s = 2$, and yields formulas for all $v_n$ as a (manifestly positive) ratio of integrals. This and the other $(r, s)$ curves are an interesting subject for further study: while various higher order analogues of discrete Painlevé equations have been considered, there is currently no version of the Sakai theory in dimension greater than two.

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