# Further Studies of Scale Elasticity in Multi-input Multi-output Production Technologies: Network DEA Modelling and Applications




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# Declaration of Authorship

I, Guanjiang Wan, declare that this thesis titled, "Further Studies of Scale Elasticity in Multi-input Multi-output Production Technologies: Network DEA Modelling and Applications" and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University,
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated,
- Where I have consulted the published work of others, this is always clearly attributed,
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work,
- I have acknowledged all main sources of help,
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

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Date:

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# Abstract

Scale properties, including scale elasticity (SE) and returns to scale (RTS), are important characteristics of the production frontier and play an essential role in analyzing the productivity of strongly efficient decision-making units (DMU). For multi-input multi-output production technologies, scale elasticity (SE) has been defined as the largest proportional change in outputs in response to a marginal proportional change of inputs. Given the network data envelopment analysis (DEA), existing research focused on estimating SE for each sub-technology to locate the sources of scale economies in network technologies. However, the relationship between the overall SE and the SE in each sub-technology is unclear since the existing methods of estimating SE in sub-technologies rely on separately extracting the transformation function of each sub-technology. Moreover, the existing research on estimating SE in network production technologies ignored the non-proportional changes and undesirable outputs. This thesis, therefore, will develop methods of estimating and decomposing SE based on the response function defined by various network production technologies. The main contributions of this thesis are as follows.

- For multi-activity and multi-stage technologies, two DEA-based methods of decomposing SE are proposed, respectively. The first method automatically allocates weights of SE in each activity, and the second method can yield a unique solution for each sub-stage while reflecting the special scale status of the points that cannot further increase the intermediate products.

- It develops directional scale elasticity (DSE) in multi-activity and multi-stage technologies, respectively, followed by exploring the potential directions to maximise the value of DSE by using heuristic searching.

- Considering the assumptions of strong and selective weak disposability in undesirable outputs, it proposes novel multi-activity and two-stage production technologies with undesirable measures, and develops the corresponding methods to estimated SE.

- It develops a novel method of estimating and decomposing SE in a proposed network technology with two-stage parallel structures. A case study in the context of the sustainability evaluation of commercial banks illustrates the method.

# Contents

**Declaration** i

**Acknowledgements** ii

**Abstract** iv

**List of Figures** ix

**List of Tables** x

**List of Abbreviations** xii

**1 Introduction** 1
- 1.1 Background . . . 1
- 1.2 Research questions . . . 2
- 1.3 Aim and objectives . . . 3
- 1.4 Thesis structure . . . 4

**2 Literature Review** 6
- 2.1 Introduction . . . 6
- 2.2 SE Estimation and related theories . . . 6
  - 2.2.1 Estimate SE for known production functions . . . 6
  - 2.2.2 Estimate SE for unknown production functions . . . 9
- 2.3 Structures and assumptions of production technologies . . . 19
  - 2.3.1 Network DEA . . . 19
  - 2.3.2 Undesirable outputs in DEA . . . 21
- 2.4 Sustainability . . . 22
  - 2.4.1 Concepts of sustainability and SD . . . 22
  - 2.4.2 Triple bottom line . . . 27
  - 2.4.3 Stakeholder theory . . . 28
- 2.5 Summary . . . 29

**3 Estimating SE for Multi-input Multi-output Production Technologies with Parallel and Series Structures 31**
3.1 Introduction and general preliminaries . . . 31
3.1.1 Introduction . . . 31
3.1.2 Preliminaries for general production technologies . . . 32
3.2 Estimating SE in multi-activity technologies . . . 33
3.2.1 Preliminaries . . . 33
3.2.2 SE estimation under known transformation functions . 35
3.2.3 SE estimation and decomposition under the DEA framework . . . 38
3.2.4 Case study . . . 46
3.3 Estimating SE in multi-stage technologies . . . 50
3.3.1 Preliminaries . . . 50
3.3.2 SE estimation under known transformation functions . 51
3.3.3 SE estimation and decomposition under the DEA framework . . . 53
3.3.4 Case study . . . 61
3.4 Chapter summary . . . 63

**4 Development of Directional SE for Network DEA 64**
4.1 Introduction . . . 64
4.2 Development of DSE in network production technologies . . . 65
4.2.1 DSE in parallel production systems . . . 65
4.2.2 DSE in series production systems . . . 71
4.3 Exploring optimal directions for increasing the scale . . . 74
4.3.1 DSE with the optimal input direction in multi-activity production technologies . . . 74
4.3.2 DSE with the optimal of inputs and intermediate products in multi-stage production technologies . . . 77
4.4 Case studies . . . 79
4.4.1 Case study of UK universities . . . 79
4.4.2 Case study of Chinese cement companies . . . 85
4.5 Chapter summary . . . 87

**5 Estimating SE in the Presence of Undesirable Outputs 89**
5.1 Introduction . . . 89
5.2 Estimating DSE in multi-activity technologies with undesirable outputs . . . 90
5.2.1 Expressions of multi-activity technologies with undesirable outputs . . . 90
5.2.2 SE estimation . . . 94

5.2.3 Case study . . . . . 96
5.3 Estimating DSE in multi-stage technologies with undesirable outputs . . . . . 102
5.3.1 Expressions of multi-stage technologies with undesirable outputs . . . . . 102
5.3.2 DSE estimation . . . . . 106
5.3.3 Case study . . . . . 111
5.4 Chapter summary . . . . . 114

**6 SE Estimation and Decomposition for Production Technologies with Closed Network Structures 116**
6.1 Introduction . . . . . 116
6.2 Two-stage parallel production technologies . . . . . 117
6.2.1 Notations of the two-stage parallel system . . . . . 117
6.2.2 Two-stage parallel DEA models . . . . . 119
6.3 SE estimation and decomposition in network production technologies . . . . . 121
6.4 Numerical study . . . . . 127
6.5 Chapter summary . . . . . 133

**7 Conclusions and Further Research 134**
7.1 Conclusions . . . . . 134
7.2 Limitations and Directions for Future Research . . . . . 135

**A Original Data 150**

**B Projection Data 153**

# List of Figures

1.1 Thesis structure . . . . . 5

2.1 Frontier of congestion with one input and one output. . . . . 15

3.1 Multi-activity production system. . . . . 34
3.2 Different projections of inefficient DMU. . . . . 40
3.3 Multi-stage production system. . . . . 51
3.4 Two-stage production system. . . . . 53
3.5 Different projections of point G. . . . . 55

4.1 Relationship between DSE and input directions. . . . . 84
4.2 Relationship between DSE and directions of intermediate products. . . . . 86

6.1 Network production system. . . . . 117
6.2 Two-stage network structure of 18 commercial banks. . . . . 128
6.3 Relationship between DSE and directions of green and non-green loans. . . . . 132

# List of Tables


3.1 SE and RTS results of the example . . . . . 38
3.2 Results from the DDO method and FDM based on the projections in Yang & Liu (2017). . . . . 42
3.3 Descriptive statistics for the data of 43 UK universities . . . . 47
3.4 Right-hand and left-hand SE for 43 UK universities . . . . . 48
3.5 Right-hand SE decomposition . . . . . 48
3.6 Left-hand SE decomposition . . . . . 49
3.7 Original data and projections of example data . . . . . 55
3.8 Right-hand and left-hand SE for the overall system, sub-the first stage and sub-the second stage . . . . . 59
3.9 RTS for the overall system, the first stage and the second stage 60
3.10 Descriptive statistics for the data of 15 Chinese cement companies . . . . . 61
3.11 SE results and decomposition of cement companies . . . . . 62

4.1 Right-hand and left-hand SE for 43 UK universities in case 1 . 80
4.2 Right-hand and left-hand SE for 43 UK universities in case 2 . 81
4.3 Right-hand and left-hand SE for 43 UK universities in case 3 . 81
4.4 RTS of $DMU_{12}$, $DMU_{27}$ and $DMU_{38}$ . . . . . 82
4.5 Optimal input directions and corresponding DSE . . . . . 83
4.6 The range of $\omega^1$ making increasing or constant RTS prevails on the right-hand side . . . . . 85
4.7 Optimal production structures and corresponding DSE . . . . 86

5.1 Retraction data of 43 UK universities . . . . . 97
5.2 Right-hand and left-hand SE for 43 UK universities with undesirable outputs (selective weakly disposability) . . . . . 99
5.3 Right-hand and left-hand SE for 43 UK universities with undesirable outputs (strongly disposability) . . . . . 102
5.4 Undesirable outputs of 15 cement companies . . . . . 111
5.5 SE results and decomposition of cement companies with undesirable outputs . . . . . 114

6.1 Data description of 18 commercial banks . . . . . 127
6.2 Right-hand SE results and decomposition . . . . . 131
6.3 Optimal loan structures and corresponding DSE . . . . . 131

A.1 Original data of 15 Chinese cement companies . . . . . 150
A.2 QS top 500 universities in UK . . . . . 151
A.3 18 Chinese commercial banks . . . . . 152

B.1 Projections and efficiencies of 15 Chinese cement companies . 153
B.2 Projections and efficiencies of 43 UK universities . . . . . 154
B.3 Projections and efficiencies of 18 Chinese commercial banks . . 155
B.4 Projections and efficiencies of 43 UK universities with retraction (selective weak disposability) . . . . . 156
B.5 Projections and efficiencies of 43 UK universities with retraction (strong disposability) . . . . . 157
B.6 Projections and efficiencies of 15 Chinese cement companies with undesirable outputs . . . . . 158

# List of Abbreviations

**BCC** Banker, Charnes and Cooper

**CCR** Charnes, Cooper and Rhodes

**CRS** Constant Returns to Scale

**DDO** Directional Derivative of Optimal-value-function

**DEA** Data Envelopment Analysis

**DM** Decision Maker

**DMU** Decision Making Unit

**DRS** Decreasing Returns to Scale

**FDM** Finite Difference Method

**IRS** Increasing Returns to Scale

**LRTS** Left-Hand Returns to Scale

**LSE** Left-Hand Scale Elasticity

**NDEA** Network Data Envelopment Analysis

**RRTS** Right-Hand Returns to Scale

**RSE** Right-Hand Scale Elasticity

**RTS** Returns to Scale

**SBM** Slack-Based Measure

**SE** Scale Elasticity

**VRS** Variable Returns to Scale

# Chapter 1

# Introduction

## 1.1 Background

Scale elasticity (SE) and its qualitative measure, returns to scale (RTS), play important roles in decision makers' (DM) determinations on the strategies of rescaling. The two concepts have been well defined under the neoclassical economics framework for multi-input multi-output productions. Since the studies in the 1970s (Hanoch 1970, Panzar & Willig 1977, Starrett 1977), SE is defined as a proportional change in all outputs in response to a proportional change in all inputs for known production functions with assuming sufficiently smooth. For example, the conventional SE equaling to 2 means that a 1% proportional change in all inputs leads to a 2% proportional increase in all outputs.

Later, the theory of SE and RTS was extended to the non-parametric method of data envelopment analysis (DEA), which was first proposed by Charnes et al. (1978). DEA is an effective tool that measures the relative efficiency of DMUs for multi-input multi-output production processes without known production functions. SE estimation in the DEA framework has been applied to many areas (e.g., research institutes (Yang & Liu 2017, Papaioannou & Podinovski 2023), agriculture (Sarac et al. 2022) and bank industry (Amirteimoori et al. 2024)). Due to the piecewise linear property of DEA frontiers, SE is initially evaluated on the appropriated supporting hyperplane obtained by analyzing the shadow prices (Banker et al. 1984). In this way, achieving SE estimation in the DEA framework needs to locate a supporting hyperplane that is regarded as a production function (e.g., see Banker & Thrall (1992), Førsund & Hjalmarsson (2004), Sueyoshi & Sekitani (2007)). Therefore, several specific methods of SE estimation and decomposition in network data envelopment analysis (NDEA) technologies

were suggested along this line of thinking (Sahoo et al. 2014, Ahangary et al. 2023, Amirteimoori et al. 2024). Podinovski et al. (2016) proposed a general approach to calculating SE in any polyhedral technologies, such as multi-component technologies (Podinovski 2022) and technologies with undesirable outputs (Podinovski 2019). The limitation of analyzing conventional SE is assuming the changes of inputs (outputs) are in the same proportion. However, unlike the definition of conventional SE that implies the translation from proportional increases in inputs to proportional increases in outputs, the notion of directional scale elasticity (DSE) proposed by Yang, Liu, Shen & Li (2014) and Balk et al. (2015) in DEA framework and transformation functions allows the translation of increases in inputs to increases in outputs along any directions. The notion of DSE has not been employed in any NDEA frameworks (i.e., series, parallel and mixture structures (Kao 2014)).

Consequently, this thesis further extends the theory of SE in multi-input multi-output production technologies with network structures. The thesis advances methods to estimate and decompose (directional) SE in network production technologies. Additionally, this thesis examines the applications of SE in various production systems (i.e., higher education, cement and bank industry).

## 1.2 Research questions

Appropriately estimating and reasonably analyzing SE are vital for exploring the directions of rescaling in which productivity increases. Given the limitations of existing studies, this thesis answers the following methodological research questions.

Existing research related to estimating SE for DMUs in network production structures focuses the decomposition methods by separately extracting transformation functions of each sub-system in the dual (multiplier) DEA model with additional assumptions (Sahoo et al. 2014, Fakhr Mousavi et al. 2023, Amirteimoori et al. 2024), such as uniqueness of the dual optimal solution and strong complementary slackness conditions. Due to these assumptions, the results of SE decomposition tend to be unstable and often require handling several special cases individually. Therefore, relaxing these assumptions can lead to more stable SE decomposition results and simplify the projection method for inefficient DMUs, thereby enhancing the applicability of SE decomposition. This leads to the first research question:

- Q1 How to estimate and decompose the SE of DMUs in network production technologies without additional assumptions?

The studies related to estimating DSE focused on production technologies with the black-box structure (Yang, Liu, Shen & Li 2014, Balk et al. 2015, Yang & Liu 2017). However, the influence of adjusting the internal structure of network production processes on productivity needs to be examined. Therefore, the second research question is:

- Q2 How to measure marginal changes along various directions of DMUs in network production technologies?

Podinovski (2019) studied SE estimation in non-network, non-parametric technologies with weak or strong disposability assumptions for undesirable outputs. There is a lack of studies estimating the SE of DMUs in network production technologies with multiple disposability assumptions in undesirable outputs. Then we have the third research question:

- Q3 How to estimate SE in network production technologies with multiple undesirable outputs under different disposability assumptions?

To our best knowledge, there is little existing studies estimated and decomposed SE in production technologies with a two-stage parallel structure. Therefore, the research question is:

- Q4 How to estimate and decompose the SE of DMUs in two-stage parallel production technologies?

## 1.3 Aim and objectives

The thesis aims to develop DEA-based methods to estimate and decompose SE in network production technologies while allowing non-proportional changes and considering undesirable outputs. The developed methods provide empirical results and analysis in the context of higher education, cement enterprise and bank industry. The objectives are

- to propose novel methods for estimating and decomposing SE in multi-activity and multi-stage production technologies defined by the non-parametric methodology of DEA, respectively.
- to extend the notion of DSE in network production technologies, thereby providing DMs with the optimal directions for adjusting the scale.
- to propose novel network production technologies containing undesirable measures with various disposability assumptions and developing the corresponding methods to estimate SE.
- to propose a production technology with a two-stage parallel structure and to develop a method for estimating and decomposing SE.

## 1.4 Thesis structure

The remainder of this thesis is structured as follows.

Chapter 2 reviews the existing literature on SE estimation, especially on the DEA frontier. This chapter provides a general review of related definitions, formulas and models of SE and its estimation, followed by a brief overview of related concepts, such as RTS and congestion. In addition, the following section reviews the basic models of NDEA and different treatments for undesirable outputs in DEA models, followed by the existing methods of SE estimation in the context of NDEA. The chapter ends with DEA applications in sustainability assessments and evaluations of university performance, which are the background of the empirical studies of this thesis.

Chapter 3 proposes the general methods for SE estimation and decomposition for DMUs in production technologies with the parallel and series structures, in which the developed methods under the DEA framework simplify the assumptions existing in the existing studies, followed by two empirical studies are carried out in the context of higher education and cement companies, corresponding to parallel and series structure, respectively.

Chapter 4 first introduces DSE into production technologies with parallel and series structures. Moreover, two novel concepts, relative/absolute decreasing returns to scale (DRS), are proposed to indicate if there are pathways to follow so that the DMU can avoid the situation of DRS. The chapter ends with two case studies based on the same data set in Chapter 3.

Chapter 5 proposes novel multi-activity and two-stage production technologies under different assumptions of disposability for undesirable outputs, including strong and selective weak disposability. Thereafter, the corresponding methods of estimating SE are developed. In empirical studies, paper retraction and various pollutions are treated as undesirable outputs for the case of higher education and cement companies, respectively.

In Chapter 6, a combined method of estimating and decomposing SE is proposed in general production technology with a mixture network structure, followed by applying the proposed method to commercial banks considering the carbon reduction of green loans.

The final chapter concludes with main contributions in methodological and empirical aspects. The possible directions for future research are also proposed. The interrelationships of each chapter in the thesis are exhibited in Fig.1.1.

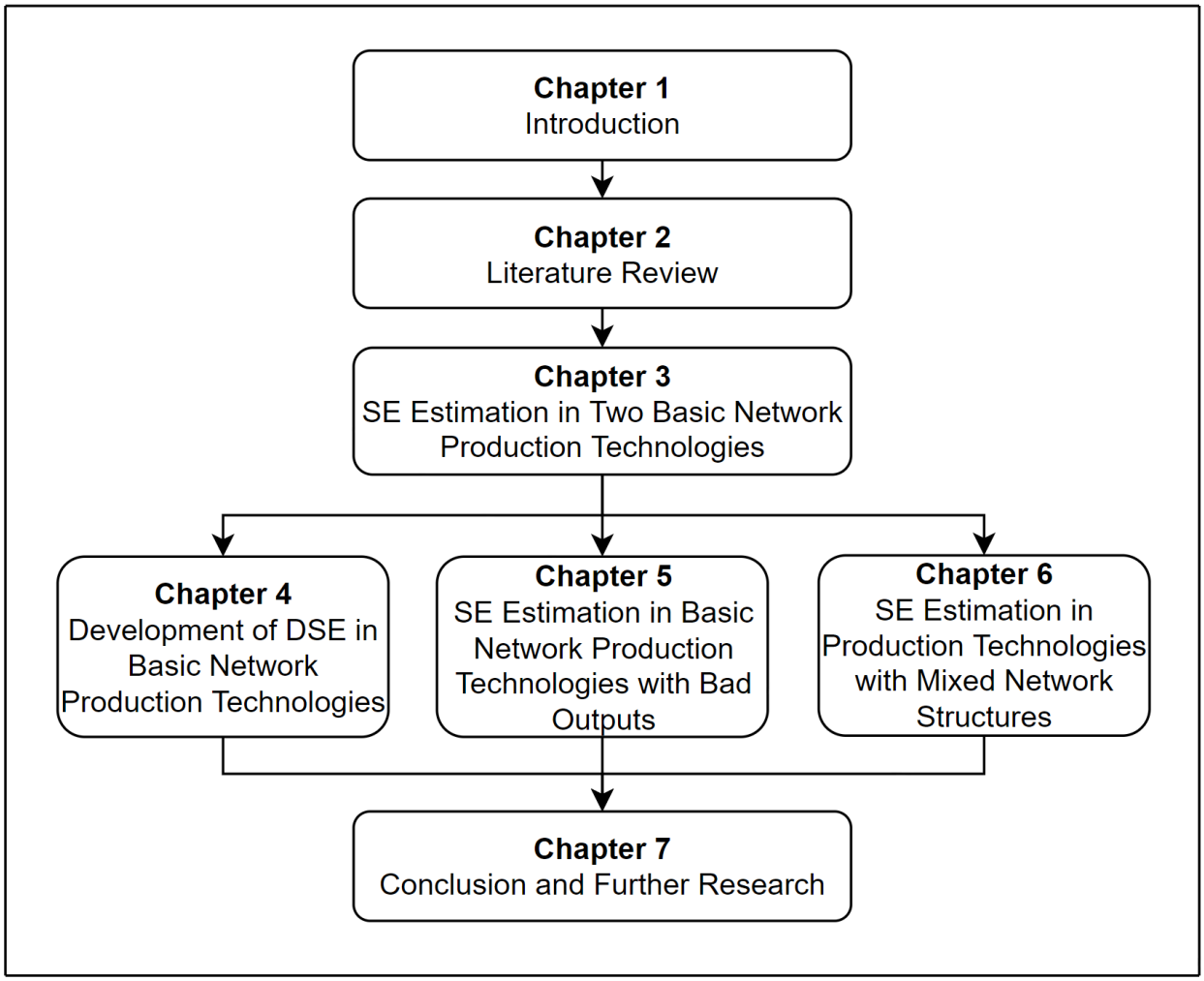



Figure 1.1: Thesis structure

# Chapter 2

# Literature Review

## 2.1 Introduction

Scale properties of production technologies, such as scale elasticity (SE) or returns to scale (RTS) play an important role in decision-making related to adjusting the scale of DMUs. This chapter aims to clarify the scope of this thesis by reviewing the basic concepts related to SE. Moreover, knowledge gaps about existing methods of estimating SE need to be identified in this chapter. Furthermore, reviewing related concepts of sustainability is necessary to establish the empirical context of the thesis.

## 2.2 SE Estimation and related theories

### 2.2.1 Estimate SE for known production functions

The scale properties of production functions are critical mathematical concepts in the neoclassical production theory (Frisch 1964). Scale elasticity (SE) and returns to scale (RTS) are two essential notions of scale property derived by classical and neoclassical economists in the 19th century (Podinovski & Førsund 2022). RTS is used to characterize the rate at which the output increases when the inputs are increased (Robert & Daniel 2018). There are three types of RTS: Increasing Returns to Scale (IRS), Constant Returns to Scale (CRS) and Decreasing Returns to Scale (DRS). For example, the IRS indicates a situation where output is more than double when all inputs are doubled. SE, or Passus coefficient (Frisch 1964), is the relative increase in output by a proportional increase in all inputs (Førsund 1996). In essence, SE is a measure for quantifying RTS (Førsund & Hjalmarsson 2004), which means the value of SE indicates the types of RTS. Initially, an-

alyzing SE and RTS focused on the production process with a single output (also called single production), which was appropriate for many cases (Frisch 1964). If there is a standard measure or an index for integrating production output, a single production assumption can also be set up in macroscopic analysis. Some early works exist for these cases (Frisch 1964, Hanoch 1970, Panzar & Willig 1977). Moreover, these works are based on a specifically explicit production function with a single output, as shown in the following equation:

$$y = f(X), \tag{2.1}$$

where $y$ represents the single output and $X = (x_1, \ldots, x_m)^\top \in \mathbb{R}_+^m$ represents the input vector.

We assume the production function (2.1) satisfies $f(\mathbf{0}) = 0$, monotonicity and continuous differentiability. Considering a $\mathrm{DMU}_o(X_o, y_o)$ satisfy $y_o = f(X_o)$, let $t \geq 0$ denote an arbitrary proportional change in each input, and $\beta(t)$ represent the correspondingly proportional change in the output. We have:

$$\beta(t) y_o = f(tX_o), \tag{2.2}$$

where $\beta(t)$ is continuously differentiable and satisfies $\beta(1) = 1$.

Let $\varepsilon(X_o, y_o)$ represent the SE evaluated at $\mathrm{DMU}_o(X_o, y_o)$. Then, $\varepsilon(X_o, y_o)$ is defined as the derivative of $\beta(t)$ evaluated at $t = 1$ (Frisch 1964):

$$\varepsilon\left(X_o, y_o\right) = \beta'(1) = \lim_{t \to 1} \frac{\beta(t) - 1}{t - 1}\bigg|_{(X_o, y_o)}$$

By differentiating Eq. (2.2) by $t$ and assuming $y_o \neq 0$, we have:

$$\varepsilon\left(X_o, y_o\right) = \left( \sum_{i=1}^{m} \frac{\partial f\left(x_{10}, \ldots, x_{m0}\right)}{\partial x_i} x_{i0} \right) / y_o \tag{2.3}$$

Equation (2.3) is the explicit formula of SE, which denotes a multiplier by which a marginally proportional change $t$ of input vector $X_o$ transforms into the proportional change $\beta(t)$ of the single output $y_o$ that accords with the production function (2.1).

The multi-output production was based on a transformation function (assuming it is sufficiently smooth) expressed as an implicit form:

$$F(X, Y) = 0, \tag{2.4}$$

where $X = (x_1, \ldots, x_m)^\top \in \mathbb{R}_+^m$ represents the input vector and $Y = (y_1, \ldots, y_s)^\top \in \mathbb{R}_+^s$ represents the output vector.

Considering a $\mathrm{DMU}_o(X_o, Y_o)$ satisfies $F(X_o, Y_o) = 0$, $X_o \neq \mathbf{0}$ and $Y_o \neq \mathbf{0}$, let $t \geq 0$ denote an arbitrary proportional change in each input, $\beta(t)$ represent the correspondingly proportional change in each output. We have:

$$F(tX_o, \beta(t)Y_o) = 0, \tag{2.5}$$

where $\beta(t)$ is continuously differentiable and satisfies $\beta(1) = 1$.

In this case, $\beta(t)$ represents the maximum proportional change in outputs permitted by the proportional change $t$ in inputs Starrett (1977). Then, SE represents the maximum proportionate change in all outputs allowed by an equal proportionate change in all inputs (Färe et al. 1985). The specific formulation is as follows:

$$\varepsilon\left(X_o, Y_o\right) = \beta'(1) = \lim_{t\to 1} \frac{\beta(t) - 1}{t - 1}\bigg|_{(X_o, Y_o)} = \lim_{t\to 1} \frac{\max\left\{\beta \mid F\left(tX_o, \beta(t)Y_o\right) = 0\right\} - 1}{t - 1}, \tag{2.6}$$

where $\beta(t) = \max\left\{\beta \mid F\left(tX_o, \beta(t)Y_o\right) = 0\right\}$ holds.

Then, the SE evaluated at $(X_o, Y_o)$ is the derivative of $\beta(t)$ evaluated at $t = 1$:

$$\varepsilon\left(X_o, Y_o\right) = -\left(\sum_{i=1}^{m} \frac{\partial F\left(X_o, Y_o\right)}{\partial x_{io}} x_{io}\right)\left(\sum_{r=1}^{s} \frac{\partial F\left(X_o, Y_o\right)}{\partial y_{ro}} y_{ro}\right)^{-1}. \tag{2.7}$$

In this case, SE is the equal proportional change in all outputs in response to the proportionate change in all inputs since the research in 1970s (Hanoch 1970, Panzar & Willig 1977, Starrett 1977). Compare with the related studies before 1970s, SE is evaluated in single-output production process only, and is used to define the production process satisfying Regular Ultra-passum Law proposed by Frisch (1964). Specifically, the law is a special pattern of production process: increasing RTS is shown initially and decreasing RTS is shown finally. Obviously, the case of production function with a single output is a special case of a transformation function. With the definition of SE developed from single-output to multi-output, the change in a single output is switched to an equal proportional change in all outputs.

In particular, Yang, Liu, Shen & Li (2014) discussed the situation of a implicit-function group that satisfies the implicit function theorem. That is, if a multi-variable implicit system $(F_1(X, Y) = 0, \ldots, F_r(X, Y) = 0)$ is continuously differentiable and the Jacobian matrix with respect to $y_{ro}$ is invertible at a point, then locally $y_r$ can be expressed as a differentiable function of $X$. By introducing a vector of higher order infinitesimal $\boldsymbol{\epsilon}(t) = (\epsilon_1, \ldots, \epsilon_s)^\top$, where $\lim_{t\to 0} \epsilon_r(t) = 0$ and $\lim_{t\to 0} \epsilon_r'(t) = 0$, they construct the output

scaling factor $\beta(t)$ by supposing $\beta_r(t) = \delta_r\beta(t) + \epsilon_r(t)(\forall r)$, where $\sum_{r=1}^{s}\delta_r = s(\delta_r \geq 0)$ is the fixed numbers representing the change direction of outputs in response to the change of inputs. Note that we may assume $\sum_{r=1}^{s}\delta_r = A(\delta_r \geq 0)$, where $A$ is an arbitrary positive number. Without loss of generality and for simplicity, $\sum_{r=1}^{s}\delta_r = s(\delta_r \geq 0)$ is assumed to make sure the SE result is the same as traditional SE in the diagonal direction.

### 2.2.2 Estimate SE for unknown production functions

#### Basic DEA models

The idea of DEA originated from the research of the expansion from productivity to efficiency (Farrell 1957). DEA was then formally introduced by (Charnes et al. 1978), and the resulting model is known as the CCR model, which was subsequently extended by (Banker et al. 1984) (aka the BCC model). DEA does not require explicit specification of the functions that relate outputs to inputs. That is, cross-comparisons to weigh or evaluate the relative importance of these different inputs and outputs are not needed to evaluate technical efficiency in DEA (Banker et al. 1989). Therefore, DEA become a popular tool for estimating efficiency scores for decision-making units (DMUs) (e.g., (Yang et al. 2009), (Chang et al. 2016), (Cook et al. 2019) and (Ganji et al. 2023)). DEA can exhibit the results of the relative efficiency score and be used to conduct process benchmarking analysis, which can provide the direction for further improvement. The objective of DEA is to analyse the homogenous DMUs that conform to three standards in terms of the same tasks, the same external environment and the same input-output indicators. The concrete DEA model is determined by specific research questions.

Regarding the basic principle of traditional DEA model, total $n$ DMUs are ready to be evaluated with $m$ inputs and $s$ outputs. $x_{ij}$ represents the i-th input of the j-th DMU, $y_{rj}$ represents the r-th output of the j-th DMU. The input vector and output vector of the j-th DMU are referred to as $X_j = (x_{1j}, x_{2j}, \ldots, x_{mj})^\top$ and $Y_j = (y_{1j}, y_{2j}, \ldots, y_{sj})^\top$, respectively. Assuming $x_{ij} \geq 0$ and $y_{rj} \geq 0$, let $\boldsymbol{w} = (w_1, w_2, \ldots, w_m)^\top$ and $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_s)^\top$ be the weights vector of inputs and outputs, respectively. Then, each DMU has a corresponding efficiency score, as shown in the formula (2.8):

$$E_j = \frac{\boldsymbol{\mu}^T Y_j}{\boldsymbol{w}^T X_j} = \frac{\sum_{r=1}^{s}\mu_r y_{rj}}{\sum_{i=1}^{m} w_i x_{ij}}, j = 1, 2, \cdots, n. \tag{2.8}$$

There is always combination of weigh $\boldsymbol{w}$ and $\boldsymbol{\mu}$ that make $E_j \leq 1$. If the evaluation objective is $\mathrm{DMU}_o$, the original input and output are represented

as $X_o$ and $Y_o$, respectively, therefore the relative efficiency of $\mathrm{DMU}_o$ can be calculated as formula (2.9), which is called CCR-DEA model (Charnes et al. 1978):

$$E_0 = \max_{w_i,\mu_r} \frac{\sum_{r=1}^{s} \mu_r y_{r0}}{\sum_{i=1}^{m} w_i x_{i0}}$$
$$\text{s.t.} \begin{cases} \frac{\sum_{r=1}^{s} \mu_r y_{rj}}{\sum_{i=1}^{m} w_i x_{ij}} \leq 1, \\ w_i, \mu_r \geq 0, \\ j = 1, \ldots, n, i = 1, \ldots, m, r = 1, \ldots, s. \end{cases} \tag{2.9}$$

Let $t = \frac{1}{\boldsymbol{w}^T X_o}$, $\boldsymbol{v} = t\boldsymbol{w}$, and $\boldsymbol{u} = t\boldsymbol{\mu}$ hold, then $\boldsymbol{w}^T X_o = 1$. Thus, model (2.9) is converted to a linear programming model as model (2.10):

$$E_0 = \max_{v_i,u_r} \sum_{r=1}^{s} u_r y_{r0}$$
$$\text{s.t.} \begin{cases} \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0, \\ \sum_{i=1}^{m} v_i x_{ij} = 1, \\ v_i, u_r \geq 0, \\ j = 1, \ldots, n, i = 1, \ldots, m, r = 1, \ldots, s. \end{cases} \tag{2.10}$$

Then the corresponding dual model can be obtained, and the model possesses more explicitly economic significance:

$$E_0^{CCR-I} = \min_{\theta,\lambda_j} \theta$$
$$\text{s.t.} \begin{cases} \sum_{j=1}^{n} \lambda_j x_{ij} \leq \theta x_{i0} \\ \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{r0}, \\ \sum_{j=1}^{n} \lambda_j = 1, \\ \lambda_j \geq 0, \theta \text{ signfree}, \\ j = 1, \ldots, n, i = 1, \ldots, m, r = 1, \ldots, s. \end{cases} \tag{2.11}$$

Usually, researchers consider the following three aspects when formulating a DEA model: 1) PPS (Production Possibility Set), which can be assumed to be CRS (Constant Returns to Scale), VRS (Variable Returns to Scale), NIRS (Nonincreasing Returns to Scale), NDRS (Nondecreasing Returns to Scale) and FDH (Free Disposal Hull). 2) Preference, which can be set as Pareto preference, average preference, matrix preference and so on. 3) Measure,

which represents the specific measurement scale based on preference (Liu et al. 2009).

Indeed, with the increasing complexity of problems, DEA models have contained more and more elements that should be considered and reformulated, such as the type of variables, the boundary of problems and the attribute of data. Meanwhile, to consider the internal structure - also called the black box, some special models, such as the network DEA model (Kao 2017), the parallel DEA model (Wu et al. 2016) and the non-homogenous DEA model (Li et al. 2016), were developed and applied in various area.

Preference is a kind of comparison method stipulated by decision makers for all DMUs to be evaluated in the sense that preference is a way to reflect the inclination level of decision makers for each DMU. In daily economic life, Pareto preference is a well-known selection principle, which originates from the concept of "Pareto Optimality" proposed by Vilfredo - a famous economist. The intrinsic implication of Pareto Optimality represents an ideal state of resource allocation – assuming that there are a group of people and specific resources, changing in resource allocation state makes one person better off at least. According to the Pareto Optimality, if two elements exist in $\mathbb{R}^n$, $y = (y_1, \ldots, y_n)^\top$, $z = (z_1, \ldots, z_n)^\top$, then $y$ is strictly worse than $z$ under Pareto preference, which can be represented as $y \prec z$, if and only if: (a) $y_i \leq z_i, i = 1, \ldots, n$; (b) and for some $j = 1, \ldots, n, y_j < z_j$. In the process of applying the DEA method to evaluation, what should be emphasised is that different issues correspond to various preferences. For example, lexicographic preference is used to rank the Olympic medal, and matrix preference is applied to analyse the problems with multilayer indicators. Inappropriate preference can cause the inaccuracy of results or misinterpretation of the problem, so the benchmarking will lead in the wrong direction.

After determining the preference of decision-makers, the measure function should be set up to measure the distance between $\mathrm{DMU}_o$ and its corresponding benchmark. The measure function is a tool to calculate the extent of good or bad. Concerning the primary form of measure function, Farrell decomposed the production efficiency into technical efficiency and allocation efficiency and proposed to measure the technical efficiency of $\mathrm{DMU}_o$ with the radial measure in the sense that the value of all input or output indicators increases or decreases in the same proportion (Farrell 1957). Therefore, the distance between $\mathrm{DMU}_o$ and the benchmark $\boldsymbol{B} = (b_1, \ldots, b_m)^\top$ can be defined as follows:

$$m_D\left(X_o, \boldsymbol{B}\right) = \max\left\{\theta \mid \theta X_o \leq \boldsymbol{B}\right\}. \tag{2.12}$$

Although the form of radial measure is straightforward and clear, the radial measure is not strictly monotonous. Hence, the efficiency results will

be biased. Therefore, the slacks are introduced to solve the problem, and the Almost Radial measure is introduced as follows:

$$m_{DS}(X_o, B) = \max\left\{\theta + \varepsilon \sum_{i=1}^{m} s_i^+ \mid \theta X_o + s^+ \leq B, s_i^+ \geq 0\right\}, \tag{2.13}$$

where $\varepsilon$ represents the Non-Archimedean Infinitesimal, $s^+ = (s_1^+, \ldots, s_m^+)^\top$.

It can be found that the above measure consists of two parts (i.e., $\theta$ and $\varepsilon \sum_{i=1}^{m} s_i^+$). When the results of the efficiency score obtained by the Radial measure possess the weak ability of discrimination, using the almost radial measure is a relatively effective way to reinforce the discrimination ability. However, the units of measurement of the obtained slacks of DUMs are usually inconsistent because the value of slacks represents the measure of absolute value. To solve this problem, Fare and Lovell proposed the following measure (Charnes et al. 1978):

$$m_{FL}(X_o, B) = \max\left\{\frac{1}{m}\sum_{i=1}^{m}\theta_i \,\middle|\, \Theta X_o \leq B, \theta_i \geq 1\right\}, \tag{2.14}$$

where $\Theta$ represents the diagonal matrix $diag(\theta_1, \ldots, \theta_m)$.

According to model (2.14), the core of this measure is the maximum arithmetic mean of the increasing and decreasing ratio of the measurement unit along the direction of each indicator. The measure is named the Russell measure (Russell 1985, 1990). The Russell measure allows input or output indicators to compensate for each other, which can be utilized in weak sustainability evaluation.

PPS is the notion that indicates the set of all feasible input-output combinations that can be used by a production unit. If the PPS possesses the properties of convexity, conicity, inefficiency and minimum extrapolation at the same time and satisfies the Pareto preference, then the production possibility set with constant returns to scale is shown in model (2.15):

$$PPS_{CRS} = \left\{(X, Y) \in \mathbb{R}_+^{m+s} \mid x_i \geq \sum_{j=1}^{n} \lambda_j x_{ij}, y_r \leq \sum_{j=1}^{n} \lambda_j y_{rj}, \lambda_j \geq 0\right\}. \tag{2.15}$$

If the property of conicity is eliminated from the postulate set only, then the PPS under variable returns to scale is shown in model (2.16):

$$PPS_{VRS} = \left\{(X, Y) \in \mathbb{R}_+^{m+s} \mid x_i \geq \sum_{j=1}^{n} \lambda_j x_{ij}, y_r \leq \sum_{j=1}^{n} \lambda_j y_{rj}, \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0\right\}. \tag{2.16}$$

The PPS is the base for formulating the DEA models because all of the postulates can be found through it in terms of preference, measure and so on. Take the following input-oriented BCC model as an example to find projections for inefficient DMUs.

$$\theta^*_{BCC-I} = \min_{\theta,\lambda_j,s_i^-,s_r^+} \theta + \epsilon(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+)$$
$$\text{s.t.} \begin{cases} \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = \theta x_{io}\ (\forall i), \\ \sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = y_{ro}\ (\forall r), \\ \sum_{j=1}^{n} \lambda_j = 1, \\ \lambda_j, s_i^-, s_r^+ \geq 0\ (\forall j, i, r), \theta \text{ signfree.} \end{cases} \tag{2.17}$$

where $\epsilon$ is a non-Archimedean small positive number.

Let $(\theta^*, s_i^{-*}, s_r^{+*})$ be the vector of the optimal solutions of model (2.17). $\text{DMU}_o(X_o, Y_o)$ is defined as strongly efficient DMU if and only if the optimal solution of the acquired from model (2.17) satisfies $\theta^*_{BCC-I} = 1$ and slacks for all inputs and outputs are equal to zero (i.e., $s_i^{-*} = 0(\forall i)$ and $s_r^{+*} = 0(\forall r)$). If not, the inefficient DMUs can be projected onto the strongly efficient frontier by applying $x^*_{io} = x_{io} - s_i^{-*}(\forall i)$ and $y^*_{ro} = y_{ro} + s_r^{+*}(\forall r)$. Then the projection point for $\text{DMU}_o(X_o, Y_o)$ is $\text{DMU}^*_o(X^*_o, Y^*_o)$. Especially, if $\text{DMU}_o(X_o, Y_o)$ is strongly efficient, then it is also represented by $\text{DMU}^*_o(X^*_o, Y^*_o)$.

### SE estimation

There is growing concern about the production processes with multi-outputs (Cherchye et al. 2015, Wang 2020, Saelens 2024) since modern production processes become increasingly complex. For example, multi-input multi-output production activities must be analyzed in some non-profit sectors, such as museums (Basso et al. 2018) and research institutions (Yang & Liu 2017). In the case of multi-input multi-output production processes, the classical method to calculate SE assumes a consistent response function for each output factor based on a known transformation or production function. In this sense, SE is the response of the output bundle to marginal changes in the input bundle at any point of a production frontier (Podinovski & Førsund 2010). In terms of the cases of unknown production functions, like research institutions (Yang & Liu 2017), DEA, as a non-parameter method, can depict the production relations by constructing a PPS or technology set. The notion of SE has been well defined in smooth and implicit transformation functions or production frontiers, and Podinovski et al. (2016) illustrated that it can be generalizable in any polyhedral technologies.

A amount of research has been related to estimating SE by DEA models since Banker et al. (1984). For some production processes that cannot be described by a single transformation (or production) function, DEA can depict production relations by constructing a technology set. Due to the piecewise linear property of frontiers constructed by DEA, SE is initially evaluated on one or two appropriate supporting hyperplanes, which are located by analyzing the shadow prices (Banker et al. 1984). Therefore, evaluating SE under the DEA framework is simplified to evaluate SE for transformation functions (e.g., see Banker & Thrall (1992), Førsund & Hjalmarsson (2004), Sueyoshi & Sekitani (2007)). For the $DMU_o$ that is on the strongly efficient frontier of single-stage VRS technology, a common method to estimate SE is to locate the two special supporting hyperplanes that can achieve the maximum/minimum value of the shadow prices of constrain $e^T\lambda = 1$(e.g., see Banker et al. (1984)). Banker & Thrall (1992) indicate the range of shadow price of convex constrain in the BCC-DEA model determines the upper and lower bounds of SE. Specifically, because multiple hyperplanes usually exist on an enormously efficient point, measuring the upper and lower bounds of an intercept of the supporting hyperplane(s) is a standard method to determine RTS (Sueyoshi & Sekitani 2007) and further estimate SE. The recent research includes (Amirteimoori et al. 2022, Fakhr Mousavi et al. 2023, Amirteimoori et al. 2023).

Furthermore, estimating SE and determining RTS based on DEA models have been studied widely. This area introduces and discusses non-radial measures (Banker et al. 2004, Sueyoshi & Sekitani 2007, Allahyar et al. 2018). Some researchers studied the estimation of SE within the network-DEA framework (Sahoo et al. 2014, Podinovski et al. 2016, Fakhr Mousavi et al. 2023). Undesirable outputs are also considered in calculating SE and determining RTS (Sueyoshi & Goto 2013, Podinovski et al. 2016, Amirmohammadi et al. 2021). There were also some works estimating SE under other nonparametric frameworks, such as stochastic technology (Amirteimoori et al. 2022), free disposal hull (FDH) technology (Mostafaee & Soleimani-Damaneh 2020) and so on. Many kinds of DEA-based models have been used into SE estimation to deal with different situations, such as considering directional changing in inputs and outputs (Balk et al. 2015, Yang, Rousseau, Yang & Liu 2014) , network structure (Fakhr Mousavi et al. 2023, Podinovski et al. 2016, Sahoo et al. 2014) and undesirable outputs (Amirmohammadi et al. 2021, Podinovski et al. 2016, Sueyoshi & Goto 2013). According to these research, many factors can influence the results of SE estimation, such as the way to projecting inefficient DMUs, the choice of management perspectives and the different description of the production process.

### Congestion

The notion of congestion describes the problem of excessive inputs. In the economic sense, the notion of congestion is defined as follows:

**Definition 2.1.** *(Wei & Yan 2004) (Congestion) Decreasing one or more inputs can cause increases in one or more outputs, without reducing any other inputs/outputs, or, increasing one or more inputs can cause decreases in one or more outputs, without increasing any other inputs/outputs.*

Färe & Svensson (1980) formally introduced the concept of congestion with a solid theoretical basis, where three forms of congestion are defined for a production function with a single output. This congestion phenomenon is observed when a decrease in one or more inputs will cause the maximum possible increase of one or more outputs, without worsening any other input or output (Cooper et al. 2000). Proceeding in reverse, congestion occurs when increases in one or more inputs can be associated with decreases in one or more outputs without improving any other input or output. The technology of measuring the congestion under VRS assumption is constructed as below:

$$T_{\text{convex}} = \left\{ (X,Y) \in \mathbb{R}_+^{m+s} \mid \sum_{j=1}^{n} \lambda_j x_{ij} = x_i, \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_r, \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0, \forall j \right\} \tag{2.18}$$

A production process is shown as an example to show the characteristics of the frontier of $T_{\text{convex}}$.

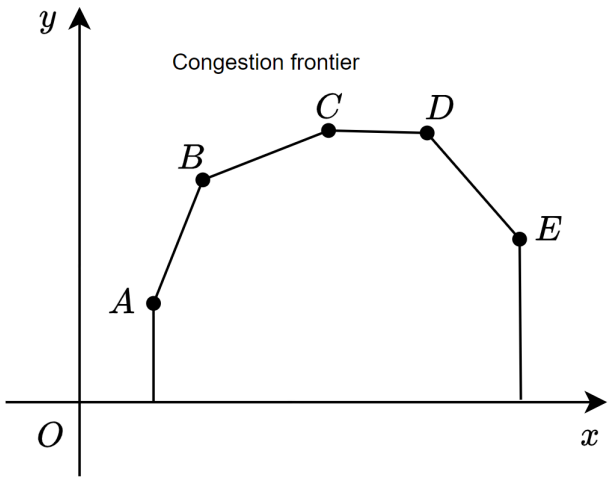



Figure 2.1: Frontier of congestion with one input and one output.

It is obvious that $\mathrm{DMU}_D$ is of congestion based on Definition 2.1.

According to technology (2.18), the consistent DEA model is formulated as follows:

$$\theta_C^* = \max_{\theta,\lambda_j,s_r^+} \theta + \epsilon \sum_{r=1}^{s} s_r^+$$
$$\text{s.t.} \begin{cases} \sum_{j=1}^{n} \lambda_j x_{ij} = x_{io}\ (\forall i), \\ \sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = \theta y_{ro}\ (\forall r), \\ \sum_{j=1}^{n} \lambda_j = 1, \\ \lambda_j, s_r^+ \geq 0\ (\forall j, r), \theta \text{ signfree.} \end{cases} \tag{2.19}$$

where $\epsilon$ is a non-Archimedean small positive number.

Let $(\theta^*, s_r^{+*})$ be the vector of the optimal solutions of the model (2.19). $\mathrm{DMU}_o(X_o, Y_o)$ is defined as strongly efficient DMU if and only if the optimal solution of the acquired from model (2.17) meet $\theta_C^* = 1$ and slacks for all outputs are equal to zero (i.e., $s_r^{+*} = 0(\forall r)$). If not, the inefficient DMUs can be projected onto the strongly efficient frontier by applying $x_{io}^* = x_{io}(\forall i)$ and $y_{ro}^* = y_{ro} + s_r^{+*}(\forall r)$. Then the projection point for $\mathrm{DMU}_o(X_o, Y_o)$ is $\mathrm{DMU}_o^*(X_o^*, Y_o^*)$.

Up to now, there are mainly three approaches to measuring and determining congestion by the DEA method. The first step of these approaches is to calculate the efficiency of DMUs to be evaluated by following the output-oriented BCC-DEA model (2.20):

$$E_0^{BCC-O} = \min_{\theta,\lambda_j,s_i^-,s_r^+} \theta + \varepsilon \left( \sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+ \right)$$
$$\text{s.t.} \begin{cases} \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = x_{io} \\ \sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = \theta y_{ro}, \\ \sum_{j=1}^{n} \lambda_j = 1, \\ \lambda_j, s_i^-, s_r^+ \geq 0, \theta \text{ signfree,} \\ j = 1, \ldots, n, i = 1, \ldots, m, r = 1, \ldots, s. \end{cases} \tag{2.20}$$

The first approach is proposed by Färe et al. (1985), which developed congestion analysis from the classical economic framework to the DEA framework. They introduced a model with the assumption of weak input disposability, meaning inputs cannot be increased without limit. The model is expressed as follows:

$$\beta^* = \min_{\beta,\lambda_j} \theta$$
$$\text{s.t.} \begin{cases} \sum_{j=1}^{n} \lambda_j x_{ij} = \tau x_{io} \\ \sum_{j=1}^{n} \lambda_j y_{rj} \geq \beta y_{ro}, \\ \sum_{j=1}^{n} \lambda_j = 1, \\ \lambda_j \geq 0, 1 \geq \tau \geq 0, \beta \text{ signfree}, \\ j = 1, \ldots, n, i = 1, \ldots, m, r = 1, \ldots, s. \end{cases} \tag{2.21}$$

Let $E_0^{BCC-O*}$ and $\beta^*$ represent the optimal value of models (2.20) and (2.21). Then the value of $\beta^*/E_0^{BCC-O*}$ represents the measurement to gauge the extent of congestion. It is worth noting that there is no congestion in the FGL approach when there is only one input, because in this case model (2.20) is equivalent to model (2.21) (Yang et al. 2020).

The second approach is proposed by Cooper et al. (1996). Different from the first approach, this approach utilizes a slack-based model with strong disposability assumption of inputs. After conducting model (2.20), there are two other steps of this approach. Firstly, conducting projection for all DMUs by model (2.20) using the following equation: $x_{io}^* = x_{io} - s_i^{-*}$, $y_{ro}^* = y_{ro} - s_r^{+*}$. Then solve the following model to calculate the maximum number of inputs that can be reduced:

$$\beta^* = \min_{\delta_i^-,\lambda_j} \sum_{i=1}^{m} \delta_i^-$$
$$\text{s.t.} \begin{cases} \sum_{j=1}^{n} \lambda_j x_{ij} - \delta_i^- = x_{i0}^* \\ \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{r0}^*, \\ \sum_{j=1}^{n} \lambda_j = 1, \\ \lambda_j \geq 0, \delta_i^- \leq s_i^{-*}, \\ j = 1, \ldots, n, i = 1, \ldots, m, r = 1, \ldots, s. \end{cases} \tag{2.22}$$

Let the optimal solutions of model (2.22) be $\delta_i^{-*}$, which represents the maximum amount in the i-th input that can be decreased. Then, the extent of input congestion related to input $x_{i0}$ can be described as $s_i^c = s_i^{-*} - \delta_i^{-*}$. It is worth noting that unit consistency should be considered when using this method.

The third approach is called the WY-TS model (Tone & Sahoo 2004, Wei & Yan 2004, 2009) . They firstly proposed a PPS $P_{\text{convex}}$ that excludes the

disposability assumption of inputs:

$$T_{\text{convex}} = \left\{ (X,Y) \in \mathbb{R}_+^{m+s} \mid \sum_{j=1}^{n} \lambda_j x_{ij} = x_i, \sum_{j=1}^{n} \lambda_j y_{rj} \geq y_r, \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0, \forall j \right\} \tag{2.23}$$

We can observe that technology (2.23) allows the frontier to bend downward, on which the SE of the DMU is negative, that is, the congestion effect occurs (Yang et al. 2020). In other words, this approach is on the basis of the negative returns to scale of congestion. According to $T_{\text{convex}}$ as shown in Eq. (2.23), the consistent DEA model is formulated as follows.

$$\begin{aligned} \pi^* = \max_{\pi,\lambda_j} \pi \\ \text{s.t.} \begin{cases} \sum_{j=1}^{n} \lambda_j x_{ij} = x_{io} \\ \sum_{j=1}^{n} \lambda_j y_{rj} \geq \pi y_{ro}, \\ \sum_{j=1}^{n} \lambda_j = 1, \\ \lambda_j \geq 0, \pi \text{ signfree}, \\ j = 1,\ldots,n, i = 1,\ldots,m, r = 1,\ldots,s. \end{cases} \end{aligned} \tag{2.24}$$

For the optimal value $\pi^*$ for model (2.24), the ratio $\pi^*/\theta^*$ can be utilized to measure the degree of the congestion effect. To make comparison of the classical BCC model (Banker et al. 1984), we have the dual model of model (2.24) as follows:

$$\begin{aligned} \pi^* = \min_{v_i,u_r,\mu_0} \sum_{i=1}^{m} v_i x_{io} - \mu_0 \\ \text{s.t.} \begin{cases} -\sum_{r=1}^{s} u_r y_{rj} + \sum_{i=1}^{m} v_i x_{ij} - \mu_0 \leq 0, \\ \sum_{r=1}^{s} u_r y_{rj} = 1, \\ v_i \geq 0, v_i, \mu_0 \text{ signfree}, \\ j = 1,\ldots,n, i = 1,\ldots,m, r = 1,\ldots,s. \end{cases} \end{aligned} \tag{2.25}$$

Obviously, parameters $v_i$ are sign free rather than non-negative in classical BCC model, which implies that the frontier of PPS (2.23) can be downbent. Besides, Tone & Sahoo (2004) proposed the definitions of strong congestion and weak congestion as follows.

**Definition 2.2.** *(Strong congestion)(Tone & Sahoo 2004): DMU*$(X_o, Y_o)$ *is strong congestion (SC) if there exists an activity* $\left(\overline{X}_o, \overline{Y}_o\right) \in P_{convex}$ *such that*

$\overline{X}_o = \alpha X_o (0 < \alpha < 1)$ *and* $\overline{Y}_o \geq \beta Y_o (\beta > 1)$*, where* $(X_o, Y_o) \in P_{convex}$ *and* $DMU_o(X_o, Y_o)$ *is strongly efficient.*

**Definition 2.3.** *(Weak congestion)(Tone & Sahoo 2004):* $DMU(X_o, Y_o)$ *is (weak) congestion ((W)C) if it is strongly efficient with respect to* $P_{convex}$ *and there exists an activity in* $P_{convex}$ *that uses fewer resources in one or more inputs for making more products in one or more outputs.*

The way to determine SC and WC is introduced by Tone & Sahoo (2004) as following:

Step 1: Solve model (2.20) and we have:

(a) If $\theta^* = 1$, $s_r^{+*} = 0$ and $s_i^{-*} = 0$, then DMU$(X_o, Y_o)$ is BCC-efficient and not congested.

(b) If $\theta^* = 1$, $s_r^{+*} = 0$ and $s_i^{-*} \neq 0$, then DMU$(X_o, Y_o)$ is BCC-inefficient.

(c) If ($\theta^* = 1$, $s_r^{+*} \neq 0$) or $\theta^* > 1$, DMU$(X_o, Y_o)$ displays congestion, go to step 2.

Step 2: Solve the following model (2.26):

$$\bar{\omega} = \max_{v_i, u_r, \mu_0} \mu_0$$
$$\text{s.t.} \begin{cases} \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} + \mu_0 \leq 0, \\ \sum_{r=1}^{s} u_r y_{ro} - \sum_{i=1}^{m} v_i x_{io} + \mu_0 = 0, \\ \sum_{r=1}^{s} u_r y_{ro} = 1, \\ u_r \geq 0, v_i, \mu_0 \text{ signfree}, \\ j = 1, \ldots, n, i = 1, \ldots, m, r = 1, \ldots, s. \end{cases} \tag{2.26}$$

Let $\bar{\rho} = \bar{\omega} + 1$ hold, if $\bar{\rho} < 0$, DMU$(X_o, Y_o)$ is SC, if $\bar{\rho} \geq 0$, DMU$(X_o, Y_o)$ is WC.

## 2.3 Structures and assumptions of production technologies

### 2.3.1 Network DEA

To locate the inefficiency for DMUs, considering the internal structure has been demonstrated to be necessary only if the data is available (Kao & Hwang 2010). Moreover, systematic thinking is increasingly important in sustainability evaluation (Lobos & Partidario 2014), the production process will be extended into a broader sense by introducing network structures. Since the term network DEA was first presented by Rolf et al. (2000), there is plenty

of research related to measuring efficiencies for different production structures. Whereas the existing research on estimating SE or determining RTS for network structure is relatively rare.

The initial idea of considering network structure in a DEA model is to obtain more meaningful results of efficiency (Kao 2017). In other words, when a production activity is regarded as a black box, then the efficiency of many DMUs might be overestimated due to the fact that sub-activities of production are ignored. In addition, breaking the black box or considering the internal structure of the production process helps more clearly and definitely indicates the improvement direction for non-efficient DMUs. For example, Yang et al. (2017) employed the evidential-reasoning approach to integrate the efficiencies of the two individual stages into a unified measure of overall efficiency in open two-stage production systems. Lee (2021) proposed a model that can decompose the efficiency and loss of network DEA based on the VRS assumption. Chen & Wang (2025) identified the pitfalls of efficiency evaluation in two-stage DEA and proposed approaches for efficiency decomposition and frontier projection based on an equivalence relationship. In terms of determining RTS and estimating SE, consideration of network or multi-stage structure is also necessary. Specifically, the determination of RTS for the whole production activity cannot provide distinctive guidance for each production stage. However, it is possible that different production stages possess different scale properties. Then, the adjustment for scale based on the SE or RTS of the whole production process will lead to inaccurate management effects. In terms of DEA modelling, Sahoo et al. (2014) proposed a calculation method in line with the simplest two-stage production activity (all indicators conform to the assumption of strong disposability, all intermediate products are inputs to the second stage, and no extra inputs are needed in the second stage). Besides, based on the different assumptions for the two sub-technologies (whether the two sub-technologies are independent or not), they studied the differences of SE and TE calculation in terms of the two conditions. Khaleghi et al. (2012) regarded SE of the whole production process as the product of SE of two sub-stages. Based on the work of Sueyoshi & Sekitani (2007), Fakhr Mousavi et al. (2023) utilized RAM (Range Adjusted Measure) and SCSC (strong complementary slackness conditions) into two-stage DEA model and explored the lower and upper bounds of the intercept of supporting hyperplane or so-called shadow price of convexity restriction in DEA model, whence the problem of multiple hyperplanes is solved in their specification of DEA models.

### 2.3.2 Undesirable outputs in DEA

One of the features of sustainability analysis is the presence of undesirable outputs, which are usually produced with one or more desirable outputs. Under the DEA framework, several approaches deal with undesirable outputs. Transformation approaches are intuitive and easy to use. For example, one of the available methods is $f(U) = -U$, which is derived from Koopmans (1951). In this way, the undesirable outputs/inputs are converted to desirable outputs/inputs. In contrast, the problem of dealing with negative data will arise due to the assumption of positive data for the basic DEA model. Therefore, $f(U) = -U + b$ is regarded as another transformation to be used (Ali & Seiford 1990, Seiford & Zhu 2002), where $b$ is a positive constant to guarantee $f(U)$ to be positive. However, the value of $b$ significantly impacts the ranking after evaluation by carrying out the DEA model (Liu et al. 2010). Besides, the multiplicative inverse ($f(U) = 1/U$) is also a common way of transformation (Lovell et al. 1995). However, the nonlinear transformation might cause more complex cases (Liu & Sharp 1999). A simple method can avoid problems caused by transformation methods, i.e., simply regarding undesirable outputs/inputs as desirable inputs/outputs. However, this cannot describe the relation of real production. In addition, as undesirable outputs, pollutants and greenhouse gases always appear in the production relationship as by-products of desirable outputs (e.g., Carbon dioxide is always produced along with electricity generation). Consequently, the assumption of disposability needs to be reconsidered. In other words, when we formulate production technologies including undesirable outputs, it is crucial to determine the relationship between the undesirable outputs and each desirable output in the production process. Weak disposability, which specifies the proportionality between desirable and undesirable outputs and implies that undesirable outputs cannot decrease freely, is firstly utilized to explain that the undesirable outputs are not free disposal (Färe et al. 1996), which lays the foundation for sustainability evaluation using DEA (Zhou et al. 2008, Song et al. 2012, Jiang et al. 2021). In terms of modelling undesirable outputs in the DEA model, Kuosmanen (2005) proposed a linearized model corresponding to a weak disposability assumption for undesirable outputs. Liu et al. (2010) discussed and concluded the different measurements used in the DEA model when dealing with undesirable outputs. Moreover, there are other types of disposability. For example, natural and managerial disposability was proposed from the perspective of corporate strategy (Sueyoshi & Goto 2012). Chen et al. (2017) proposed the assumption of semi-disposability, which allows decision-makers to manage undesirable outputs freely within a certain level of technology. Otherwise, reducing undesirable outputs needs a

proportional reduction in desirable outputs. Mehdiloo & Podinovski (2019) proposed the axioms called selective strong and weak disposability, which allows some desirable outputs to be individually strongly disposable while other desirable outputs and undesirable outputs are jointly weakly disposable.

## 2.4 Sustainability

Specifically, in terms of application into sustainability evaluation and RTS analysis, exploration and application of network structure will provide a system perspective, which is one of the major directions of developments in the literature on sustainability (Williams et al. 2017). However, no existing literature was involved in studying scale characteristics for a production process with a complex structure under a sustainability background. Note that the model of measuring the performance of a network system does not have a universal expression due to the various possibilities of network structures. Therefore, we need to construct models based on the specific sustainability features.

### 2.4.1 Concepts of sustainability and SD

Research on sustainability evaluation for different corporations focuses on various aspects, such as performance involving policies, decisions, and actions that create economic, social and environmental results (Büyüközkan & Karabulut 2018), intensification involving economic returns and integrity of ecosystems (Cassman & Grassini 2020), readiness involving the uniformity of strategy and operations (Barletta et al. 2021), processes involving interior and exterior (Ayele et al. 2021), and dynamic capabilities involving reconfiguring resource bases (Strauss et al. 2017). Utilising DEA in the corporate sustainability evaluation is gaining increasing attention (Chen et al. 2021).

Sustainability and SD (Sustainable Development) are formally introduced in a report called *World Conservation Strategy* (World Wildlife Fund 1980). However, it is limited only to forestry and fisheries when considering the approaches to achieving sustainable development. Whereafter in order to expand the scope of SD to more general fields and human activities, *Our Common Future* was published in 1987. Based on the report from Brundtland, "sustainable development is a development that meets the needs of the present without compromising the ability of future generations to meet their own needs", which is the most well-known and acceptable definition of SD was introduced (Imperatives 1987).

Besides, the concepts of SD and sustainability have been introduced in

various fields, such as economics, politics, and sociology. Meanwhile, because of the flexibility of the concepts of SD and sustainability, plenty of definitions are described from diverse perspectives (Ciegis et al. 2009), which causes the concepts of SD and sustainability to be vague. That is because SD or sustainability is a complicated and multidimensional issue, while the different contexts hold different standards (Neumayer 2003). For instance, in the early stage, around 60 concepts of sustainability, which were proposed from 1979 to 1988, were reviewed by the World Bank on the basis of the conventional neoclassical theory of economics (Pezzey 1989). Pezzey (1989) also indicated that most of the sustainability criteria have the characteristic of being long-term, which should be expressed in terms of constraints instead of maximising. Therefore, non-declining per capita utility is considered appropriate for the SD definition. In addition, Ciegis et al. (2009) indicated that there are over 100 definitions in the economic literature. Similarly, from an economic perspective, the definition of SD can be stated in terms of utility, and it might be supported by most people as follows: "Development is defined here to be sustainable if it does not decrease the capacity to provide non-declining per capita utility for infinity" (Neumayer 2003). In conclusion, most explanations of SD and sustainability imply no decline exists in the quality of human life, instead of just satisfying and sustaining the existence of life.

Besides, what should be accepted is that the concepts of SD and sustainability are still vague and somewhat contestable. So, sometimes, the two concepts are regarded as having synonymous meanings (Boscardin & Bossert 2015). Nevertheless, the two standpoints are mainly related to the difference between the two concepts. Apart from the mentioned opinion that considers sustainability as a part of SD (Robert et al. 2005, Ciegis et al. 2009), another view regards SD as a process and considers sustainability as the ultimate aim of SD, which means that SD is a dynamic process while sustainability is a quality issue (Hjorth & Madani 2014). Nonetheless, sustainability requirements can be changed due to the diverse contexts.

Based on the definitions of broad sustainability, we can find that sustainability is something that improves the quality of human life while living within the carrying capacity of supporting ecosystems (IUCN & WWF 1991). Corporate sustainability refers to a company's voluntary activities that demonstrate the inclusion of social and environmental concerns in business operations and in interactions with stakeholders (Van Marrewijk & Werre 2003). In particular, the dimensions of nature, life support and community should be sustainable. Meanwhile, the dimensions of people, economy and society should be developed (Robert et al. 2005). This phenomenon also implies the strong extensibility of the SD concept so that SD can be utilised

and adopted in various conditions and backgrounds. Naturally, to be consistent with the specific context and issue, the decision-makers or individuals at different levels can reformulate the details of the SD concept. For example, it can be adjusted to adapt the situations from local to global, or it can be reinterpreted to fit a series of activities within certain institutions or organisations, such as government, colleges, profit corporations and so on. Furthermore, observing the diversely emphasised aspects of SD concepts is also a distinct pathway to distinguish the essential differences among SD concepts. For example, some are anthropocentric, and others are ecocentric, which is also a significant point in discriminating between weak and strong sustainability.

Since the report from Brundtland in 1987, an increasing number of policies and guidelines have been published globally. Based on different subjects that desire to achieve specific sustainability goals, the current longitudinal classification of sustainability agenda will be exhibited at the global, country, industry, and organisational levels. Systematically speaking, the policies and agendas at a high level will guide the direction of strategies at a low level. At the global level, the most widespread agenda recently is the seventeen SDGs (Sustainable Development Goals) proposed by the United Nations General Assembly in 2015, which are intended to be achieved by 2030. These 17 goals are constructed as an interlinked and complex system (United Nations 2015). In order to make the goals more executable, the United Nations General Assembly propounded the corresponding 169 targets along with the specific indicators and measurements in 2017.

Furthermore, relying on official statistics from the UN and other international organisations, 232 indicators have been decided and revealed by SDG-Tracker in 2018, which facilitates tracking and visualising progress towards the goals. Tools like SDG-Tracker can guarantee the accuracy and availability of the data. Besides, the indicators will be altered with the deepening of sustainability research. The United Nations Statistical Commission reviewed the indicator system in 2020 and offered 36 suggestions for amendments to existing indicators in terms of replacing, revising, and deleting. Moreover, the next reviewing session for indicators will be scheduled for 2025. The SDGs' ability to grasp global sustainability will be more powerful, along with the planned and reflective sessions. However, managing the increasingly comprehensive indicator framework is a complicated task, and the United Nations has played a good leadership role. There are also other global sustainability initiatives, such as "Green Economy Roadmap", published by the International Chamber of Commerce in 2010, "Green Growth Indicators", published by the Organisation for Economic Co-operation and Development, and *Inclusive Green Growth: The Pathway to Sustainable De-*

*velopment* published by World Bank in 2012. Compared with those agendas, the SDGs are more comprehensive and progressive. However, the sustainability issues are too complicated to grasp overall since tracking all the ecological features is incredible for current human beings.

At the country level, under diverse contexts in terms of culture, development stage and natural environment, different countries will take significant account of sustainability or SD issues when setting up specific national public policies (Happaerts 2012). After a conference called the World Summit on Sustainable Development (WSSD), held in 2012, most countries revised their National Sustainable Development Strategy (NSDS), which was first proposed in Agenda 21 in 1992. United Nations Conference on Sustainable Development (UNCSD) interpreted the NSDS as a coordinated, participatory and iterative process of thoughts and actions to achieve economic, environmental and social objectives in a balanced and integrated manner. Besides, the NSDS was initially required to be developed through the broadest possible participation and based on a thorough assessment of the current situation and initiatives, which means that formal SD strategy processes should be tightly connected to the whole system rather than treated as an independent strategy and separated from other mechanisms of strategic decision-making (Meadowcroft 2007). Consequently, most countries have proposed and continued to review NSDS and the corresponding targets and policies based on Agenda 21. Concerning the ranking study of sustainability at the country level, except for a few countries with missing data, most worldwide countries have been ranked based on the SDG Index and Dashboards (Sachs et al. 2021).

At the industrial level, global industrialisation pushed by several industrial revolutions has profoundly affected modern society's construction and economic growth over more than 200 years. However, the non-stop process of industrialisation causes increasing environmental pollution and energy consumption. Specifically, industry is responsible for around 40% of carbon dioxide (CO2) emissions and one-third of energy consumption. However, because of the different functions of diverse industries, sustainability differences between industries can be very significant. Moreover, it is complex and difficult to examine the influence of individual industries nowadays due to the various standards of industry division and the cross-industry business trends, despite the fact that sustainability reports are still important to observe the overall condition in traditional industries. Currently, some investors who consider ESG (Environmental, Social and Governance) factors tend to pay attention to the sustainability of different industries, which will influence their investment preference. Concerning the sustainability reporting rates of industry, the report published by KPMG (Klynveld Peat Marwick Goerdeler) in 2020

indicated that the high-pollution industries, such as oil, manufacturing and automotive, possess high sustainability reporting rates – 100%, 97%, 95%, respectively. The sustainability reporting rate in manufacturing increased from 88% to 97% in 3 years, meaning sustainability has been a concern for plentiful industries, especially traditional ones. This shows that corporates in the same industry tend to hold a consensus on sustainability, which undisputedly promotes the process of industry sustainability. Although the consensus at the horizontal level can help with grasping the integral sustainability situation, ignoring context characteristics will bring little assistance to sustainability management.

At the organisational level, as the rational response to the higher level mentioned above, the requirements of the business model need to be shifted into higher eco-efficiency to meet the needs of sustainability. At the organisational level, sustainability is mainly exhibited in the diverse frameworks of reports. The first-round trend of reports related to sustainability factors occurred in the 1970s, and most of them were published by large multinational companies in the US and Western Europe (Kolk 2010). According to the report of KPMG (2020), most companies that hold large-scale have published sustainability reports. Currently, several international organisations have formulated different standards for sustainability reports, such as the Global Reporting Initiative Guidelines, the United Nations Global Compact and the International Standard Organization's Guidance on Social Responsibility (ISO26000) (Fortanier et al. 2011). Although no unified criterion existed in the sustainability report, at least increasing attention to sustainability is better than no concern (Elkington 2020).

World Business Council for Sustainable Development (WBCSD), launched at the Rio de Janeiro Earth Summit of 1992, is an organisation of over 200 international companies led by CEOs. The report of WBCSD in 2000 proposed "the simultaneous pursuit of economic prosperity, environmental quality and social equity". Companies aiming for sustainability need to perform not against a single financial bottom line but against the triple bottom line. Besides, consideration of sustainability at the organisational level usually refers to the triple bottom line approach, through the way that the enterprises not only seek to better financial performance but also perform well from an environmental and social point of view (Carroll & Brown 2022).

Consequently, the current research on corporate sustainability evaluation mainly focuses on the units to be evaluated. For instance, diverse kinds of frameworks of sustainability reports, such as the Global Reporting Initiative (GRI), Carbon Disclosure Project (CDP) and Greenhouse Gas Protocol (GHG Protocol) (Siew 2015), only focus on appraisal directly related to the company under evaluation. In the academic area, (Zenya & Nystad 2018)

proposed to utilise an Enterprise Sustainability Evaluation Tool (E-SET) to attain a score for a company by a set of indicators of its condition. At the industry level, the report published by KPMG (Klynveld Peat Marwick Goerdeler 2020) indicated that sustainability has been a concern for plentiful industries, mainly traditional industries. However, assessing sustainability across industries remains challenging due to the different contexts that hold different standards (Neumayer 2003).

### 2.4.2 Triple bottom line

Although plentiful discussions of sustainability existed in the literature, most were under one perspective, such as the environment or society. When considering the analysis towards SD, we should regard the basis as the system that can involve the dimensions rather than the dimensions independently (Ciegis et al. 2009). Especially the majority of literature related to sustainability focuses on the environmental aspect while paying little attention to the social and economic aspects (Lozano 2008, Büyüközkan & Karabulut 2018). The lack of comprehensive analysis also exhibits the various standards of sustainability. A well-known framework of sustainability, which synthetically considers economic, social and environmental aspects, is called "Triple Bottom Line (TBL)" (Elkington 1998), which is also interpreted into people, planet, and profit (the three Ps). The three dimensions of TBL are economy, society, and environment, respectively, which suggests that grasping the new challenge requires rethinking the present performance and the goal for the future. There are two basic views of the relationships of the three elements. One is that the three aspects of TBL are regarded as equally important, whereas the other represents that the environment is treated as the foundation of society; meanwhile, society is taken as the basis of the economy in the sense that the society is a sub-system of environment and the economy is a part of society (Lozano 2008). What should be pointed out is if any individuals and societies have not met their basic survival needs, protecting the environment becomes relatively unimportant for them, at least in the short term. Therefore, the perspective of society as a sub-system of the environment is not definite, at least in the short term. Besides, the social system is also difficult to analyse when linking it with the condition of the natural environment (Salzmann et al. 2002).

Besides, based on the different perspectives of the relationship between natural capital and man-made capital, the degree of sustainability goal is primarily divided into two levels in terms of weak sustainability (WS) and strong sustainability (SS) (Gutés 1996). Neumayer stated that SS encompasses WS but possesses some further requirements compared with WS. He

regarded SS as a non-substitutable paradigm, which means natural capital cannot be required to be substituted by other forms of capital (Neumayer 2003). Furthermore, a more detailed classification of sustainability was proposed in terms of very weak, weak, strong and strong categories (Gallopín 2003). Very strong sustainability represents valuing ecological sustainability beyond economic and social sustainability, while very weak sustainability represents the complete substitutability of different types of capital. For example, the mainstream of SD definitions in international reports can be seen as weak sustainability (Imperatives 1987). Specifically, although the 17 goals definitely can be allotted into three pillars of SD or sustainability, and they are linked with each other, the set of these goals is a picture that shows the important aspects that humans have explored to achieve peace and prosperity for now and in the future.

### 2.4.3 Stakeholder theory

Many stakeholders are involved in a sustainability system because the economic benefits are often owned locally, and the corresponding costs are frequently borne globally (Büyüközkan & Karabulut 2018). Therefore, the theories related to stakeholders are essential elements for sustainability research. The definition of "stakeholder" was first proposed by a Stanford memo in 1963, which referred to "those groups without whose support the organisation would cease to exist" (Mitchell et al. 1997).

Therefore, an enterprise is a complex system of multiple subjects based on the definitions of stakeholders. Specifically, besides some of the most obvious stakeholders such as shareholders, employees, and customers, others like competitors, suppliers, communities, the media, the whole society, and the general public are also stakeholders (Carroll & Brown 2022). However, the subjects that have interest relations with corporations are not only people, and the communities of people, but also life species and ecosystems other than human beings, specifically, the natural environment, non-human species of life and future generations should all be important stakeholders of the enterprise (Starik 1993). Because the environment has no voice, it is often ignored at the macro-level and decision-making of the enterprise, and it should be clear that the environment is also an important stakeholder (Jacobs 1997). It is worth noting that the stakeholder relationship with the enterprise is formed directly through human participation. In contrast, some stakeholders do not possess sociality because they do not have relationships with the enterprise through real people, such as the natural environment, human descendants, and non-human species (Wheeler et al. 1998). Sustainability and its evaluation is a global issue (Büyüközkan & Karabulut 2018),

but the current evaluation subject is very often local. Therefore, the results of the direct evaluation may not be valid for sustainability. From the stakeholders' perspective, many stakeholders would not be considered when the objective of sustainability evaluation is a company-centric system (Wheeler et al. 1998).

Furthermore, soft systems methodology (SSM) has proved to be very useful in identifying stakeholders (Wang et al. 2015). SSM was first proposed to solve "hard" problems (with clearly defined goals) in the area of biology and engineering (Checkland 1972). Based on the idea of critical systems heuristics (Vos 2003) and soft system methodology (based on the assumption of an open world), Wang et al. (2015) have identified the groups that have direct relationships with the system (owners, customers, actors and environment groups), and the groups that have indirect relationships with the system (external groups). Meanwhile, the key activities and roles of stakeholder groups for attaining the organisational goals also can be grasped (Wang et al. 2015).

## 2.5 Summary

From the foregoing literature review, we can identify the following research gaps.

- Existing research on DEA has primarily focused on efficiency evaluation in production technologies with complex structures, while relatively little attention has been paid to scale elasticity. Due to the fact that SE is an essential indication related to resource allocation and strategic expansion, examining SE in sub-technologies is important for decision-makers to locate the sources of increasing returns. Chapter 3 and Chapter 6 will develop the methods of estimating and decomposing SE in network production technologies.

- Considering that decision-makers can adjust the direction of changes in inputs or outputs within a sub-technology, expanding the notion of DSE within network production technologies can provide decision-makers with a broader scope of feasible strategies for resource allocation. The existing research on directional scale elasticity ignored network production technologies. Chapter 4 will develop the notion of directional scale elasticity in network production technologies.

- Given the growing emphasis on SD, decision-makers have become increasingly concerned with undesirable outputs related to environmental

and social dimensions. Estimating SE for network production technologies that include undesirable outputs can provide DMUs with scale adjustment suggestions that are aligned with SD objectives. Therefore, deeply exploring the scale proprieties for DMUs in network production technologies is essential. There is a lack of investigation into scale properties within the context of sustainability. Chapter 5 will develop the methods of estimating SE in network production technologies with undesirable outputs.

# Chapter 3

# Estimating SE for Multi-input Multi-output Production Technologies with Parallel and Series Structures

## 3.1 Introduction and general preliminaries

### 3.1.1 Introduction

One of the most distinctive characteristics of network production systems is the existence of sub-systems. Parallel and series structures are two basic network structures (the mixture of them will be discussed in Chapter 6). A core concern in the growing research related to DEA-based methods of SE estimation in production technologies with network structures is the decomposition of overall SE into each sub-system's SEs, which can help locate the sources of scale economies. Sahoo et al. (2014) suggested a method to decompose the overall SE of basic two-stage technologies with the assumption of the additive network technology. However, their method relies on a specific linear program that produces alternate optima, thereby, the results of decomposition are unstable. Based on the theorem of marginal values in linear programming, Podinovski et al. (2016) proposed a general method to estimate SE in a simple two-stage technology without considering the decomposition of the overall SE. Podinovski (2022) developed a multi-component technology with the assumption of the worst-case and mentioned that the overall SE can be estimated in the developed technology. However, it did not decompose the overall SE. In addition, Fakhr Mousavi et al. (2023) and

Amirteimoori et al. (2024) extracted transformation functions of each sub-technology to separately calculate SEs. Amirteimoori et al. (2024) proposed a computational method to calculate the SE of each sub-system in a parallel-series production process, in which the same weights are allocated into the SE of each parallel activity. These methods aggregated each sub-stage SE into the overall SE instead of decomposing the overall SE. The above literature review shows that there is little research related to estimating SE in parallel production systems. This chapter focuses on estimating and decomposing SE in VRS(variable returns to scale) technology with parallel and series structures.

The remainder of this chapter is divided into the following sections: In Section 3.2 and 3.3, methods for estimating SE in parallel and series production techniques are developed, respectively, and case studies are carried out in UK higher education and Chinese cement companies, respectively. Section 3.4 concludes the merits of developed methods.

### 3.1.2 Preliminaries for general production technologies

Let $\mathbb{R}_+^d$ denote the non-negative orthant of the $d$-dimensional Euclidean space. We consider an arbitrary production technology $T \subset \mathbb{R}_+^{m+s}$. The set of inputs and outputs are denoted by $I \subset \mathbb{R}_+^m$ and $O \subset \mathbb{R}_+^s$, respectively. For the $n(n \geq 1)$ observable DMUs in $T$, each DMU$(X, Y)$, where $X = (x_1, \ldots, x_m)^\top$ and $Y = (y_1, \ldots, y_s)^\top$, is assumed to have at least one nonzero input and one nonzero output. Note that the vectors in this chapter are all column vectors and are denoted by uppercase letters. The bold notations $\boldsymbol{e}$ and $\mathbf{0}$ represent the vectors that all components are equal to 1 and 0, respectively. Their dimension depends on the context in which they are used. Let technology $T$ include all possible DMU$_j(X_j, Y_j)$:

$$T = \left\{(X, Y) \in \mathbb{R}_+^m \times \mathbb{R}_+^s \mid X \text{ can produce } Y\right\}. \tag{3.1}$$

For vectors $A, B \in \mathbb{R}_+^n$, $A \leq B$ indicates $A_j \leq B_j$ $(\forall j, j \in N = \{1, \ldots, n\}(n \geq 1))$. We assume $T$ is free disposable in all inputs and outputs, i.e., if $(X, Y) \in T$, $X' \geq X$ and $0 \leq Y' \leq Y$, then $(X', Y') \in T$.

## 3.2 Estimating SE in multi-activity technologies

### 3.2.1 Preliminaries

Specific to a production with $K$ activities, each activity $k \in H = \{1, \ldots, K\} (K \geq 2)$ has its own specific input set and output set, which are represented by $I^k$ and $O^k$, respectively. In addition, let $I^S$ and $O^S$ represent the shared input set and output set in which the inputs and outputs are allocated into each activities with known proportions. We assume that the input set $I$ and output set $O$ are decomposed as follows, respectively:

$$I = I^S \cup \left( \bigcup_{k \in H} I^k \right), O = O^S \cup \left( \bigcup_{k \in H} O^k \right). \tag{3.2}$$

In the case of $K = 1$, we have $I^k = I^S$ and $O^k = O^S$, the technology represents a single-activity production process. For $K \geq 2$, we consider the production system in which all sub-technologies are non-joint in both inputs and outputs. Let $|A|$ represent the cardinal number of set $A$. For any $(X_j, Y_j)$ in the technology (3.1), corresponding to the decomposition of input and output sets as shown in (3.2), we denote:

$$X = (X^1; \ldots; X^K; X^S), Y = (Y^1; \ldots; Y^K; Y^S), \tag{3.3}$$

where $X^k = (x_1^k, \ldots, x_{|I^k|}^k)^\top (\forall k)$, $X^S = (x_1^S, \ldots, x_{|I^S|}^S)^\top$, $Y^k = (y_1^k, \ldots, y_{|O^k|}^k)^\top (\forall k)$ and $Y^S = (y_1^S, \ldots, y_{|O^S|}^S)^\top$.

Furthermore, we assume that the input and output sets of the $k$th activity are decomposed as follows:

$$I^{Ak} = I^k \cup I^S, O^{Ak} = O^k \cup O^S. \tag{3.4}$$

We assume that each shared input $x_i^S (i \in I^S)$ and each shared output $y_r^S (r \in O^S)$ are allocated into each activity by known proportion vectors $Q_i = (q_i^1, \ldots, q_i^K)^\top (i \in I^S)$ and $P_r = (p_r^1, \ldots, p_r^K)^\top (r \in O^S)$, respectively, where $\sum_{k \in H} q_i^k = 1$ and $\sum_{k \in H} p_r^k = 1$. Let $x_i^{Sk} = q_i^k x_i^S$ and $y_r^{Sk} = p_r^k y_r^S$ represent the shared inputs and outputs allocated to activity $k$. For activity $k$ of $(X, Y)$, corresponding to the decomposition of input and output sets as shown in (3.4), we denote:

$$X^{Ak} = (X^k; X^{Sk}), Y^{Ak} = (Y^k; Y^{Sk}), \tag{3.5}$$

where $X^{Sk} = (x_1^{Sk}, \ldots, x_{|I^S|}^{Sk})^\top (\forall k)$ and $Y^{Sk} = (y_1^{Sk}, \ldots, y_{|O^S|}^{Sk})^\top (\forall k)$.

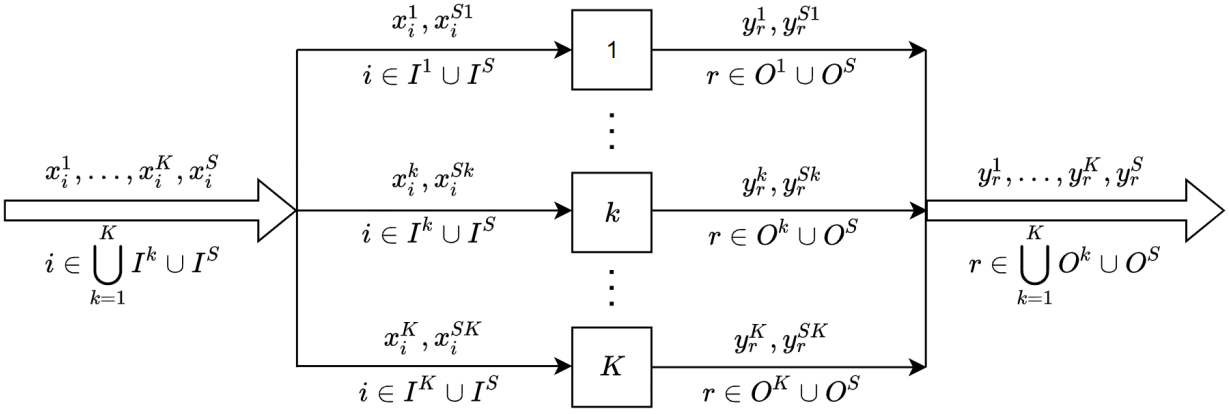



Figure 3.1: Multi-activity production system.

The production system including input and output exhibited in (3.3) and (3.5) is shown in Fig. 3.1.

Let $\mathrm{DMU}_o(X_o, Y_o)$ denote the DMU to be evaluated on the frontier of technology $T$, the rest of our discussion is all about $\mathrm{DMU}_o(X_o, Y_o)$ that is on the frontier of multi-activity technology $T$.

**Remark 3.1.** *Whether the inputs or outputs are treated as shared measures primarily depends on the consistency of production structure in each activity. For example, for different departments in a university, staff and expenses can be regarded as inputs, while student enrollments and scholarly publications can be regarded as outputs. Therefore, we can treat all these factors with the same types and units as the shared measures technically. However, if decision makers think some of these factors are non-compensable across all activities, we need to treat them as activity-specific measures.*

For a general multi-activity technology $T$ defined by (3.1), we consider that all inputs are changed by a factor $t$. We define the corresponding change $\beta(t)$ of all outputs:

$$\beta(t) = \max\{\beta \mid ((1+t)X_o, (1+\beta)Y_o) \in T, \beta \in \mathbb{R}\}, \tag{3.6}$$

where $\odot$ represents the Hadamard product operator, and the same notation applies hereafter.

Note that $\beta(t)$ is actually not always differentiable because function (3.6) is a maximum function. We assume $\beta(t)$ defined by function (3.6) is right and left differentiable, and $\beta(0) = 0$ is satisfied. Then we can calculate SE by the right and left derivative of $\beta(0) = 0$ evaluated at $t = 0$:

$$\varepsilon^+(X_o, Y_o) = \left.\frac{\partial \beta(t)}{\partial t}\right|_{t=0^+}, \varepsilon^-(X_o, Y_o) = \left.\frac{\partial \beta(t)}{\partial t}\right|_{t=0^-}. \tag{3.7}$$

Let $\varepsilon^+(X_o, Y_o)$ and $\varepsilon^-(X_o, Y_o)$ denote the right-hand and left-hand SE, respectively. Specifically, $\varepsilon^+(X_o, Y_o)$ and $\varepsilon^-(X_o, Y_o)$ represent the largest proportional change of all outputs $Y_o$ in the technology $T$ if all inputs $X_o$ in the technology $T$ are marginally increased by a small factor $t$. For example, if $\varepsilon^+(X_o, Y_o) = 2$, a 1% increase in all inputs leads to a maximum 2% proportional increase in all outputs. Similarly, if $\varepsilon^-(X_o, Y_o) = 2$, a 1% decrease in all inputs leads to a maximum 2% proportional decrease in all outputs.

In line with the specific production structure as shown in Fig. 3.1, for the $\mathrm{DMU}_o \in T$ in each activity $k \in H$, all inputs and outputs after proportional changes are represented by:

$$\tilde{X}_o^{Ak} = ((1+t) \odot X_o^k; (1+t) \odot X_o^{Sk})(\forall k),$$
$$\tilde{Y}_o^{Ak} = ((1+\beta) \odot Y_o^k; (1+\beta) \odot Y_o^{Sk})(\forall k),$$

respectively. Let $\tilde{X}_o^{Sk} = (1+t) \odot X_o^{Sk}(\forall k)$ and $\tilde{Y}_o^{Sk} = (1+\beta) \odot Y_o^{Sk}(\forall k)$ hold, the overall shared inputs and outputs after proportional changes are represented by:

$$\tilde{X}_o^S = \tilde{X}_o^{S1} + \cdots + \tilde{X}_o^{SK},$$
$$\tilde{Y}_o^S = \tilde{Y}_o^{S1} + \cdots + \tilde{Y}_o^{SK}.$$

### 3.2.2 SE estimation under known transformation functions

For the technologies defined by a known transformation function of each activity, we propose a new estimating method of SE. In a multi-activity production system, we assume that each activity is defined by an implicit transformation function to represent the transformation relationship between inputs and outputs:

$$f^k(X^{Ak}, Y^{Ak}) = 0(\forall k). \tag{3.8}$$

**Assumption 3.1.** *All transformation functions in* (3.8) *satisfy continuous differentiability in all its variables, strictly decreasing in all inputs and strictly increasing in all outputs, that is:*

$$\frac{\partial f^k(X^{Ak}, Y^{Ak})}{\partial x_i} < 0 \ (\forall k, i \in I^{Ak}),$$
$$\frac{\partial f^k(X^{Ak}, Y^{Ak})}{\partial y_r} > 0 \ (\forall k, r \in O^{Ak}).$$

Then the technology $T_P$ is defined as follows:

$$T_P = \left\{ (X, Y) \in \mathbb{R}_+^{|I|} \times \mathbb{R}_+^{|O|} \mid f^k(X^{Ak}, Y^{Ak}) \le 0(\forall k) \right\}. \tag{3.9}$$

For a $\mathrm{DMU}_o(X_o, Y_o)$ that satisfies $f^k(X^{Ak}, Y^{Ak}) = 0(\forall k)$, the output response functions are inconsistent in different activities since technology $T_P$ in (3.9) is based on multiple transformation functions $f^k(X^k, X^{Sk}, Y^k, Y^{Sk}) = 0(\forall k)$. To unify the response functions, we first assume that there is a factor of the output response function in each separate activity:

$$\beta^k(t) = \max\left\{\beta^k \mid f^k\left((1+t)X_o^{Ak}, (1+\beta^k)Y_o^{Ak}\right) \le 0, \beta^k \in \mathbb{R}\right\}(\forall k). \quad (3.10)$$

We assume $\beta^k(t)(\forall k)$ is differentiable at $t = 0$ and $\beta^k(0) = 0(\forall k)$. Since each activity is expressed by a continuously differentiable transformation function $f^k$, let $\tilde{X}_o^{Ak} = (1+t)X_o^{Ak}$ and $\tilde{Y}_o^{Ak} = (1+\beta^k(t))Y_o^{Ak}$ hold, we have $f^k(\tilde{X}_o^{Ak}, \tilde{Y}_o^{Ak}) = 0(\forall k)$. Then we can calculate $\varepsilon^k(X_o^{Ak}, Y_o^{Ak})$ for each activity, like the method for obtaining the standard formula of SE (Hanoch 1970, Panzar & Willig 1977, Starrett 1977), differentiating $f^k(\tilde{X}_o^{Ak}, \tilde{Y}_o^{Ak}) = 0(\forall k)$ by $t$, we have:

$$\sum_{i\in I^k}\frac{\partial f^k}{\partial x_i^k}x_{io}^k + \sum_{i\in I^S}\frac{\partial f^k}{\partial x_i^{Sk}}x_{io}^{Sk} + \frac{\partial \beta^k}{\partial t}\left(\sum_{r\in O^k}\frac{\partial f^k}{\partial y_r^k}y_{ro}^k + \sum_{r\in O^S}\frac{\partial f^k}{\partial y_r^{Sk}}y_{ro}^{Sk}\right) = 0(\forall k).$$

Note that we have $\frac{\partial f^k}{\partial x_i^S} = \frac{\partial f^k}{\partial x_i^{Sk}}\frac{\partial x_i^{Sk}}{\partial x_i^S}(\forall k)$ and $\frac{\partial f^k}{\partial y_r^S} = \frac{\partial f^k}{\partial y_r^{Sk}}\frac{\partial y_r^{Sk}}{\partial y_r^S}(\forall k)$ based on the chain rule of composite functions. Consider $x_i^{Sk} = q_i^k x_i^S(\forall k)$ and $y_r^{Sk} = p_r^k y_r^S(\forall k)$, we can calculate the derivatives of $\beta^k(t)(\forall k)$ at $t = 0$ for each activity by the following equation:

$$\begin{aligned}\varepsilon^k(X_o^{Ak}, Y_o^{Ak}) &= \left.\frac{\partial \beta^k(t)}{\partial t}\right|_{t=0} \\ &= -\frac{\sum_{i\in I^k}\frac{\partial f^k}{\partial x_i^k}x_{io}^k + \sum_{i\in I^S}\frac{\partial f^k}{\partial x_i^S}x_{io}^S}{\sum_{r\in O^k}\frac{\partial f^k}{\partial y_r^k}y_{ro}^k + \sum_{r\in O^S}\frac{\partial f^k}{\partial y_r^S}y_{ro}^S}(\forall k).\end{aligned} \quad (3.11)$$

Based on the disposability assumption of technology $T_P$, we propose the following Theorem to estimate SE:

**Theorem 1.** *Let $(X_o, Y_o) \in T_P$ satisfy Assumption 3.1 and $f^k(X_o^{Ak}, Y_o^{Ak}) = 0(\forall k)$, then the right-hand and left-hand SE can be obtained by solving the following problems:*

$$\varepsilon^+(X_o, Y_o) = \min_{1\le k\le K}\varepsilon^k(X_o^{Ak}, Y_o^{Ak}), \quad (3.12)$$

$$\varepsilon^-(X_o, Y_o) = \max_{1\le k\le K}\varepsilon^k(X_o^{Ak}, Y_o^{Ak}). \quad (3.13)$$

*Proof of Theorem 1.* For any given $(X_o, Y_o) \in T_P$ defined by (3.9), assume the input scaling factor $t$ and the corresponding output response function $\beta^k(t)$ is determined by (3.10) and satisfies $f^k(\tilde{X}_o^{Ak}, \tilde{Y}_o^{Ak}) = 0(\forall k)$, where $\tilde{X}_o^{Ak} = (1+t)X_o^{Ak}$ and $\tilde{Y}_o^{Ak} = (1+\beta^k(t))Y_o^{Ak}$.

Furthermore, we assume $0 \leq e^+ = \beta^{1\prime}(0) = \cdots = \beta^{w\prime}(0) < \beta^{w+1\prime}(0) \leq \cdots \leq \beta^{u\prime}(0) < \beta^{u+1\prime}(0) = \cdots = \beta^{K\prime}(0) = e^-$. Note that $\beta^k(0) = 0$ for all $k = 1, \ldots, K$.

Then, there exists $\tau_1 > 0$, such that $\beta^k(t^+) > \beta^{\hat{k}}(t^+)$ holds for all $k = w+1, \ldots, K$, $\hat{k} = 1, \ldots, w$ and $0 < t^+ < \tau_1$. Note that $\beta^{\hat{k}}(t^+) = \min\limits_{1\leq k\leq K} \beta^k(t^+)$. Thus, $f^k\left((1+t^+)X_o^{Ak}, (1+\beta^{\hat{k}})Y_o^{Ak}\right) \leq 0(k = 1, \ldots, K)$ holds, that is, $((1+t^+)X_o, (1+\beta^{\hat{k}})Y_o) \in T_P$ holds. It is obvious that $((1+t^+)X_o, (1+\beta^k)Y_o) \notin T_P$ holds for all $k = w+1, \ldots, K$.

Hence, we have:

$\varepsilon^+ = \beta^{\hat{k}\prime}(0) = \min\limits_{1\leq k\leq K} \beta^{k\prime}(0) = \min\limits_{1\leq k\leq K} \varepsilon^k(X_o^{Ak}, Y_o^{Ak})$.

This completes the first half of the proof.

There exists $\tau_2 > 0$, such that $\beta^k(t^-) > \beta^{\hat{k}}(t^-)$ holds for all $k = 1, \ldots, u$, $\hat{k} = u+1, \ldots, K$ and $-\tau_2 < t^- < 0$. Note that $\beta^{\hat{k}}(t^-) = \min\limits_{1\leq k\leq K} \beta^k(t^-)$. Thus, $f^k\left((1+t^-)X_o^{Ak}, (1+\beta^{\hat{k}})Y_o^{Ak}\right) \leq 0(k = 1, \ldots, K)$ holds, that is, $((1+t^-)X_o, (1+\beta^{\hat{k}})Y_o) \in T_P$ holds. It is obvious that $((1+t^-)X_o, (1+\beta^k)Y_o) \notin T_P$ holds for all $k = 1, \ldots, u$. Hence, we have:

$\varepsilon^- = \beta^{\hat{k}\prime}(0) = \max\limits_{1\leq k\leq K} \beta^{k\prime}(0) = \max\limits_{1\leq k\leq K} \varepsilon^k(X_o^{Ak}, Y_o^{Ak})$.

This completes the second half of the proof. □

It is worth noting that Eqs. (3.12) and (3.13) are equivalent when $K = 1$. Accordingly, RTS is defined based on the value of $\varepsilon^+$ and $\varepsilon^-$ on the $\mathrm{DMU}_o(X_o, Y_o)$:

**Definition 3.1.** *(Yang, Rousseau, Yang & Liu 2014)$DMU_o(X_o, Y_o)$ exhibits*

- *Increasing RTS on the right-hand(or left-hand) side if $\varepsilon^+(X_o, Y_o) > 0$(or $\varepsilon^-(X_o, Y_o) > 0$).*
- *Decreasing RTS on the right-hand(or left-hand) side if $\varepsilon^+(X_o, Y_o) < 0$(or $\varepsilon^-(X_o, Y_o) < 0$).*
- *Constant RTS on the right-hand(or left-hand) side if $\varepsilon^+(X_o, Y_o) = 0$(or $\varepsilon^-(X_o, Y_o) = 0$).*

To illustrate the SE and RTS defined in the multi-activity production technology (3.9), we consider a technology $T_{P1}$ with two activities defined as follows:

$$T_{P1} = \left\{ \left(x^1, x^2, x^S, y^1, y^2, y^S\right) \mid 3y^1 + 0.8y^S - 2\left[x^1\left(0.5x^S\right)\right]^{0.5} \leq 0, \right. \\ \left. 2y^2 + 0.2y^S - \left[x^2\left(0.5x^S\right)\right]^{0.5} \leq 0 \right\}.$$

Obviously, we have $x^{S1} = 0.5x^S$, $x^{S2} = 0.5x^S$, $y^{S1} = 0.8y^S$ and $y^{S2} = 0.2y^S$. We consider an efficient $\mathrm{DMU}_1(2, 2, 4, 0.8, 0.8, 2)$, $\mathrm{DMU}_2(1, 3, 4, (2\sqrt{2}-0.8)/3, (\sqrt{6}-0.2)/2, 2)$ and $\mathrm{DMU}_3(3, 1, 4, 2\sqrt{6}/3 - 0.8, \sqrt{2}/2 - 0.3, 3)$ in technology $T_{p1}$. Then we can straightforwardly calculate $\varepsilon^1$ and $\varepsilon^2$ using (3.11). Consequently, based on Theorem 1 and Definition 3.1, we can straightforwardly obtain SE and determine RTS as shown in Table 3.1.

Table 3.1: SE and RTS results of the example

| DMU | $\varepsilon^1$ | $\varepsilon^2$ | $\varepsilon^+$ | $\varepsilon^-$ | R-RTS | L-RTS |
|---|---|---|---|---|---|---|
| $\mathrm{DMU}_1$ | 1.00 | 1.00 | 1.00 | 1.00 | C | C |
| $\mathrm{DMU}_2$ | 0.97 | 0.58 | 0.58 | 0.97 | D | D |
| $\mathrm{DMU}_3$ | 0.58 | 1.25 | 0.58 | 1.25 | D | I |

L-RTS and R-RTS mean the left-hand RTS and right-hand RTS, respectively. "I", "D" and "C" refer to increasing, decreasing and constant RTS, respectively.

### 3.2.3 SE estimation and decomposition under the DEA framework

In the case that a transformation function of each production activity is unknown, we consider the technology $T_P$ under the DEA framework. Following the "minimum extrapolation" postulate of DEA developed by Banker et al. (1984), we assume that $T_P$ satisfies several standard axioms as follows:

- **A1** Strong disposability of inputs and outputs: If $(X, Y) \in T_P$, we have $(X^*, Y^*) \in T_P (\forall X^*$ and $Y^*$ such that $X^* \geq X$ and $0 \leq Y^* \leq Y)$.
- **A2** Convexity: Technology $T_P$ is a convex set.
- **A3** Feasibility of all observed DMUs: For any $j \in N$, $\left(X_j, Y_j\right) \in T_P$.

By aggregating all shared inputs $X^{Sk}$ and outputs $Y^{Sk}$ allocated into each activity to one shared inputs $X^S$ and outputs $Y^S$, respectively, the technology $T_P$ is expressed as the following form:

$$T_P = \left\{ (X,Y) \in \mathbb{R}_+^{|I|+|O|} \;\middle|\; \begin{array}{l} \sum_{j=1}^{n} \lambda_j^k x_{ij}^k \le x_i^k \ (\forall k, i \in I^k), \\ \sum_{k=1}^{K} \sum_{j=1}^{n} \lambda_j^k x_{ij}^{Sk} \le x_i^S \ (i \in I^S), \\ \sum_{j=1}^{n} \lambda_j^k y_{rj}^k \ge y_r^k \ (\forall k, r \in O^k), \\ \sum_{k=1}^{K} \sum_{j=1}^{n} \lambda_j^k y_{rj}^{Sk} \ge y_r^S \ (r \in O^S), \\ \sum_{j=1}^{n} \lambda_j^k = 1 \ (\forall k), \lambda_j^k \ge 0 \ (\forall j, k). \end{array} \right\} \tag{3.14}$$

It is worth noting that the target point discussed above are on the efficient frontier of technology sets(i.e., the DMU$_o$ satisfies $f^k(X_o^{Ak}, Y_o^{Ak}) = 0(\forall k)$) since the notion of SE is a local property. Generally, the SE of inefficient DMUs are discussed at the projections with respect to the efficient frontier. The key of this subsection is to study the models to estimate SE in multi-activity production technologies. Although the value of SE might depend on the efficiency measures, this subsection will carry out projection in one way when estimating SE (regarding dealing with the problem of multiple projections). Specifically, the following output-oriented and input-oriented models are used to find projections for inefficient DMUs:

$$\frac{1}{E_0^{P-O}} = \max_{\theta, \lambda_j^k} \theta$$
$$\text{s.t.} \left\{ \begin{array}{l} \sum_{j=1}^{n} \lambda_j^k x_{ij}^k \le x_{io}^k \ (\forall k, i \in I^k), \\ \sum_{k=1}^{K} \sum_{j=1}^{n} \lambda_j^k x_{ij}^{Sk} \le x_{io}^S \ (i \in I^S), \\ \sum_{j=1}^{n} \lambda_j^k y_{rj}^k \ge \theta y_{ro}^k \ (\forall k, r \in O^k), \\ \sum_{k=1}^{K} \sum_{j=1}^{n} \lambda_j^k y_{rj}^{Sk} \ge \theta y_{ro}^S \ (r \in O^S), \\ \sum_{j=1}^{n} \lambda_j^k = 1 \ (\forall k), \\ \lambda_j^k \ge 0 \ (\forall j, k), \theta \text{ signfree.} \end{array} \right. \tag{3.15}$$

$$E_0^{P-I} = \min_{\theta, \lambda_j^k} \theta$$
$$\text{s.t.} \left\{ \begin{array}{l} \sum_{j=1}^{n} \lambda_j^k x_{ij}^k \le \theta x_{io}^k \ (\forall k, i \in I^k), \\ \sum_{k=1}^{K} \sum_{j=1}^{n} \lambda_j^k x_{ij}^{Sk} \le \theta x_{io}^S \ (i \in I^S), \\ \sum_{j=1}^{n} \lambda_j^k y_{rj}^k \ge y_{ro}^k \ (\forall k, r \in O^k), \\ \sum_{k=1}^{K} \sum_{j=1}^{n} \lambda_j^k y_{rj}^{Sk} \ge y_{ro}^S \ (r \in O^S), \\ \sum_{j=1}^{n} \lambda_j^k = 1 \ (\forall k), \\ \lambda_j^k \ge 0 \ (\forall j, k), \theta \text{ signfree.} \end{array} \right. \tag{3.16}$$

It is worth noting that different ways of projection can cause different SE results, Fig. 3.2 can straightforwardly illustrate this problem given the basic VRS technology with one input and one output.

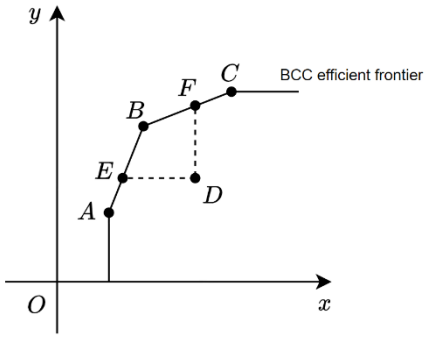



Figure 3.2: Different projections of inefficient DMU.

In Fig. 3.2, for inefficient DMU D, different projections E and F are on the different support hyperplane, thus different projections can cause different SE. We will carry out projection in one way when estimating SE (regarding dealing with the problem of multiple projections, see Førsund (1996) and Sueyoshi & Sekitani (2007) in detail).

**Remark 3.2.** *Due to the characteristic of the DEA frontier, inevitably there are some points whose inputs(outputs) cannot be further reduced(increased). These two types of special points, called smallest scale size(SSS) and largest scale size(LSS) points, respectively, which are first introduced by Banker & Thrall (1992). Specifically, the value of the left-hand SE is* $+\infty$ *for SSS points (e.g., DMU A), and the value of the right-hand SE is* 0 *for LSS points (e.g., DMU C). The the same situation occurs in the results of SE since the classic SE is one of the special SE. Yang & Liu (2017) defined directional SSS and LSS in for the efficient points in the classic VRS technology.*

Assume we use the output-oriented model (3.15) to project inefficient DMUs. Let $(\theta^*, \lambda_j^{k*})(\forall k, j)$ represent the optimal value of model (3.15). $\text{DMU}_o$ is defined as the output radial efficient DMU if and only if the optimal solution of the acquired from model (3.15) satisfies the condition $\theta^* = 1$. If not, the inefficient DMUs can be projected onto the efficient frontier by applying $x_{io}^{k*} = \sum_{j=1}^{n} \lambda_j^{k*} x_{ij}^k (\forall k, i \in I^k)$, $y_{ro}^{k*} = \sum_{j=1}^{n} \lambda_j^{k*} y_{rj}^k (\forall k, r \in O^k)$, $x_{io}^{Sk*} = \sum_{j=1}^{n} \lambda_j^{k*} x_{ij}^{Sk} (\forall k, i \in I^S)$ and $y_{ro}^{Sk*} = \sum_{j=1}^{n} \lambda_j^{k*} y_{rj}^{Sk} (\forall k, r \in O^S)$. Note that the proportions of shared inputs and outputs change after projec-

tion for inefficient DMUs[1]. Then the projection point for $\mathrm{DMU}_o(X_o, Y_o)$ is $\mathrm{DMU}_o^*(X_o^*, Y_o^*)$. Especially, even if $\mathrm{DMU}_o(X_o, Y_o)$ is efficient, it is also represented by $\mathrm{DMU}_o^*(X_o^*, Y_o^*)$. Then, the output response function is defined as follows:

$$\beta(t) = \max\left\{\beta \mid \left((1+t)X_o^*, (1+\beta)Y_o^*\right) \in T_P, \beta \in \mathbb{R}\right\} \tag{3.17}$$

where $\beta(0) = 0$ is assumed to indicate that $Y_o^*$ cannot be changed within $T_P$ on condition that $X_o^*$ keeps unchanged.

In line with the specific expression of $T_P$ as shown in (3.14), the function (3.17) becomes a linear program as follows:

$$\begin{aligned}
\beta(t) = \max_{\beta, \lambda_j^k} & \beta \\
\text{s.t.} & \begin{cases}
\sum_{j=1}^{n} \lambda_j^k x_{ij}^{k*} \le (1+t) x_{io}^{k*} \ (\forall k, i \in I^k), \\
\sum_{k=1}^{K} \sum_{j=1}^{n} \lambda_j^k x_{ij}^{Sk*} \le (1+t) x_{io}^{S*} \ (i \in I^S), \\
\sum_{j=1}^{n} \lambda_j^k y_{rj}^{k*} \ge (1+\beta) y_{ro}^{k*} \ (\forall k, r \in O^k), \\
\sum_{k=1}^{K} \sum_{j=1}^{n} \lambda_j^k y_{rj}^{Sk*} \ge (1+\beta) y_{ro}^{S*} \ (r \in O^S), \\
\sum_{j=1}^{n} \lambda_j^k = 1 \ (\forall k), \\
\lambda_j^k \ge 0 \ (\forall j, k), \beta \text{ signfree.}
\end{cases}
\end{aligned} \tag{3.18}$$

To estimate the right-hand and left-hand SE (i.e., solve (3.7)) based on the output response function (3.18), there are two typical ways: one is finite difference method (FDM) proposed by Rosen et al. (1998) and applied into estimating SE by Yang & Liu (2017), the other is the directional derivatives of the optimal-value-function (DDO) method proposed by Shapiro (1979) and applied into estimating SE by Podinovski et al. (2009). Because of the necessity of pre-set parameters when using FDM to estimate, FDM might even cause some infeasible results empirically (Ren et al. 2021). Below a numerical case is carried out to illustrate the practical difference between FDM and DDO. We first set both $t^-$ and $t^+$ equal to $1E^{-6}$ and $1E^{-9}$ for FDM. Based on the dataset of projections of 16 DMUs from Yang & Liu (2017), we pick out three DMUs as the examples, as shown in Table 3.2. It is obvious that different values of $t^-$ and $t^+$ can result in SE. Besides, the characteristics of RTS are impacted due to adjusting values of $t^-$ and $t^+$. However, the method of DDO does not require any pre-set value for any parameters, and the results obtained from DDO hold uniqueness. Table 3.2

[1]It is obviously that $\mathrm{DMU}_o^*(X_o^*, Y_o^*)$ is output radial efficient. Podinovski et al. (2016) has demonstrated that the output radial efficient DMUs are sufficient for estimating SE since the output-oriented radial efficient $\mathrm{DMU}_o^*$ ensures the validity of the assumption $\beta(0) = 0$ for the DMU.

illustrates that the DDO method is more stable than FDM in calculating the final results of SE.

Table 3.2: Results from the DDO method and FDM based on the projections in Yang & Liu (2017).

| Institutes | DDO method | | | FDM($t^{-/+} = 1E^{-6}$) | | | FDM($t^{-/+} = 1E^{-9}$) | | |
|---|---|---|---|---|---|---|---|---|---|
| | LSE | RSE | L/RRTS | LSE | RSE | L/RRTS | LSE | RSE | L/RRTS |
| $DMU_4$ | 1.02 | 0.38 | I/D | 1.02 | 0.38 | I/D | 1.02 | 1.02 | I/I |
| $DMU_6$ | 1.05 | 0.88 | I/D | 1.05 | 0.88 | I/D | 0.88 | 0.88 | D/D |
| $DMU_7$ | 1.04 | 0.89 | I/D | 1.04 | 0.89 | I/D | 1.04 | 1.04 | I/I |

Note: LRTS and RRTS mean the left-hand RTS and right-hand RTS, respectively. "I", "D" and "C" refer to increasing RTS, decreasing RTS and constant RTS, respectively.

Therefore, we use the DDO method to estimate SE and propose the following Theorem:

**Theorem 2.** *Let $(X_o^*, Y_o^*) \in T_P$ be defined by* (3.14) *and $(X_o^*, Y_o^*)$ be an output-radial efficient point, then the right-hand and left-hand SE can be obtained by solving the following problems:*

$$\varepsilon^+ = \min_{v_i^k, v_i^S, u_r^k, u_r^S, u_0^k} \sum_{k\in H}\sum_{i\in I^k} v_i^k x_{io}^{k*} + \sum_{i\in I^S} v_i^S x_{io}^{S*}$$

$$s.t. \begin{cases} \sum_{k\in H}\sum_{i\in I^k} v_i^k x_{io}^{k*} + \sum_{i\in I^S} v_i^S x_{io}^{S*} - \sum_{k\in H}\sum_{r\in O^k} u_r^k y_{ro}^{k*} \\ - \sum_{r\in O^S} u_r^S y_{ro}^{S*} + \sum_{k\in H} u_0^k = 0, \\ \sum_{k\in H}\sum_{r\in O^k} u_r^k y_{ro}^{k*} + \sum_{r\in O^S} u_r^S y_{ro}^{S*} = 1, \\ - \sum_{i\in I^k} v_i^k x_{ij}^{k*} - \sum_{i\in I^S} v_i^S x_{ij}^{Sk*} + \sum_{r\in O^k} u_r^k y_{rj}^{k*} \\ + \sum_{r\in O^S} u_r^S y_{rj}^{Sk*} - u_0^k \le 0\ (\forall j, k), \\ v_i^k, v_i^S, u_i^k, u_i^S \ge 0\ (\forall k), u_0^k \ \textit{signfree}\ (\forall k). \end{cases} \tag{3.19}$$

$$\varepsilon^- = \max_{v_i^k, v_i^S, u_i^k, u_i^S, u_0^k} \sum_{k\in H}\sum_{i\in I^k} v_i^k x_{io}^{k*} + \sum_{i\in I^S} v_i^S x_{io}^{S*} \tag{3.20}$$
$$s.t.\ \textit{All constrains in model}\ (3.19).$$

*Proof of Theorem 2.* Based on the concept of DDO, the Proposition formulated by Podinovski et al. (2009) exhibits:

**Proposition 3.1.** *Let $\eta(\boldsymbol{b})$ be the function of $\boldsymbol{b} = (\boldsymbol{b}_1; \boldsymbol{b}_2)$, which is defined as the optimal value of the following linear program in terms of the vector $x \in$*

$\mathbb{R}^n$ *(assuming it has an optimal solution), where* $\boldsymbol{J}$ *is a subset of* $\mathbb{Z} = 1, \ldots, \boldsymbol{n}$*, which is the set of integers:*

$$\eta(b) = \max_{x_1,\ldots,x_n} \left\{ c^T X \mid \overline{A}_1 X \leq \boldsymbol{b}_1, \overline{A}_2 X = \boldsymbol{b}_2, x_j \geq 0, j \in \boldsymbol{J} \right\}, \tag{3.21}$$

*where* $c = (c_1, \ldots, c_n)^\top$*,* $X = (x_1, \ldots, x_n)^\top$*,* $\overline{A}_1$ *and* $\overline{A}_2$ *represent the matrices with* $\boldsymbol{n}$ *columns.*

*Assume a non-zero vector* $\boldsymbol{d}$ *is a feasible direction at* $\boldsymbol{b}$*, which means Eq.* (3.21) *has an optimal solution with vector* $\boldsymbol{b'} = \boldsymbol{b} + \phi \boldsymbol{d}$ *on its right-hand side for all sufficiently small* $\phi > 0$*. Under those conditions, the directional derivative* $\eta'(\boldsymbol{b}; \boldsymbol{d})$ *of* $\phi$ *at* $\boldsymbol{b}$ *in the direction* $\boldsymbol{d}$ *exists and is represented as:*

$$\eta'(\boldsymbol{b}; \boldsymbol{d}) = \min_{w_1,\ldots,w_n} \left\{ w^T \boldsymbol{d} \mid w \textit{optimal in the dual to model } (3.21) \right\}. \tag{3.22}$$

According to model (3.22), $\eta'(\boldsymbol{b}; \boldsymbol{d})$ refers to the relative variation of $\boldsymbol{b}$ in the direction $\boldsymbol{d}$. Note that model (3.18) is a special case of model (3.21), based on the Eq. (3.7), $\varepsilon^+$ is regarded as the directional derivative of $\eta(\boldsymbol{b})$ with $\boldsymbol{b} = (X_o^{1*}, \ldots, X_o^{K*}, X_o^{S*}, -Y_o^{1*}, \ldots, Y_o^{K*}, -Y_o^{S*}, \mathbf{1})$ in the direction $\boldsymbol{d} = (X_o^{1*}, \ldots, X_o^{K*}, X_o^{S*}, \mathbf{0}, \ldots, \mathbf{0}, \mathbf{0}, \mathbf{0})$:

$$\begin{aligned} \varepsilon^+ &= \eta'(\boldsymbol{b}; \boldsymbol{d}) \\ &= \eta'((X_o^{1*}, \ldots, X_o^{K*}, X_o^{S*}, -Y_o^{1*}, \ldots, Y_o^{K*}, -Y_o^{S*}, \mathbf{1}); \\ &\quad (X_o^{1*}, \ldots, X_o^{K*}, X_o^{S*}, \mathbf{0}, \ldots, \mathbf{0}, \mathbf{0}, \mathbf{0})) \end{aligned} \tag{3.23}$$

Considering the dual model of model (3.18) with $t = 0$, we have:

$$\begin{aligned} \beta(0) = \min_{v_i^k, v_i^S, u_i^k, u_i^S, u_0^k} & \sum_{k \in H} \sum_{i \in I^k} v_i^k x_{io}^{k*} + \sum_{i \in I^S} v_i^S x_{io}^{S*} - \sum_{k \in H} \sum_{r \in O^k} u_r^k y_{ro}^{k*} - \sum_{r \in O^S} u_r^S y_{ro}^{S*} + \sum_{k \in H} u_0^k \\ \text{s.t.} & \begin{cases} \sum_{k \in H} \sum_{r \in O^k} u_r^k y_{ro}^{k*} + \sum_{r \in O^S} u_r^S y_{ro}^{S*} = 1, \\ -\sum_{i \in I^k} v_i^k x_{ij}^{k*} - \sum_{i \in I^S} v_i^S x_{ij}^{Sk*} + \sum_{r \in O^k} u_r^k y_{rj}^{k*} \\ + \sum_{r \in O^S} u_r^S y_{rj}^{Sk*} - u_0^k \leq 0 \, (\forall j, k), \\ v_i^k, v_i^S, u_i^k, u_i^S \geq 0 \, (\forall k), u_0^k \text{ signfree } (\forall k). \end{cases} \end{aligned} \tag{3.24}$$

Because $\text{DMU}_o^*(X_o^*, Y_o^*)$ is efficient, we have $\beta(0) = 0$. That is, $\sum_{k \in H} \sum_{i \in I^k} v_i^k x_{io}^{k*} + \sum_{i \in I^S} v_i^S x_{io}^{S*} - \sum_{k \in H} \sum_{r \in O^k} u_r^k y_{ro}^{k*} - \sum_{r \in O^S} u_r^S y_{ro}^{S*} + \sum_{k \in H} u_0^k = 0$. Let $\boldsymbol{\Omega}$ represent the set of all optimal solutions of model (3.24), based on model (3.24),

Eq. (3.23) is expressed by the following program:

$$\begin{aligned}
&\left.\frac{\partial \beta(t)}{\partial t}\right|_{t=0^+} \\
&= \min_{(v_i^k, v_i^S, u_i^k, u_i^S, u_0^k) \subset \Omega} \left\langle \left(v_i^k, v_i^S, u_i^k, u_i^S, u_0^k\right), \left(X_o^{1*}, \ldots, X_o^{K*}, X_o^{S*}, \mathbf{0}, \ldots, \mathbf{0}, \mathbf{0}, \mathbf{0}\right) \right\rangle \\
&= \min_{(v_i^k, v_i^S, u_i^k, u_i^S, u_0^k) \subset \Omega} \sum_{k \in H} \sum_{i \in I^k} v_i^k x_{io}^{k*} + \sum_{i \in I^S} v_i^S x_{io}^{S*}.
\end{aligned} \tag{3.25}$$

Consequently, the right-hand SE is estimated by solving the following problem:

$$\begin{aligned}
\varepsilon^+ = &\min_{v_i^k, v_i^S, u_i^k, u_i^S, u_0^k} \sum_{k \in H} \sum_{i \in I^k} v_i^k x_{io}^{k*} + \sum_{i \in I^S} v_i^S x_{io}^{S*} \\
&\text{s.t.} \begin{cases} \sum_{k \in H} \sum_{i \in I^k} v_i^k x_{io}^{k*} + \sum_{i \in I^S} v_i^S x_{io}^{S*} - \sum_{k \in H} \sum_{r \in O^k} u_r^k y_{ro}^{k*} \\ - \sum_{r \in O^S} u_r^S y_{ro}^{S*} + \sum_{k \in H} u_0^k = 0, \\ \text{All constrains in model (3.24).} \end{cases}
\end{aligned} \tag{3.26}$$

Similarly, the left-hand SE is estimated by solving the following model:

$$\begin{aligned}
\varepsilon^- = &\max_{v_i^k, v_i^S, u_i^k, u_i^S, u_0^k} \sum_{k \in H} \sum_{i \in I^k} v_i^k x_{io}^{k*} + \sum_{i \in I^S} v_i^S x_{io}^{S*} \\
&\text{s.t. All constrains in model (3.26).}
\end{aligned} \tag{3.27}$$

The proof is completed. □

The definition of RTS is the same as the Definition 3.1.

Due to the fact that the scaling factor $\beta$ in each activity $k$ is the same as shown in model (3.18), the overall SE cannot be decomposed directly.

For an efficient DMU, we assume that the following transformation function holds:

$$F(X_o^*, Y_o^*) = \sum_{k \in H} \sum_{i \in I^k} v_i^{k*} x_{io}^{k*} + \sum_{i \in I^S} v_i^{S*} x_{io}^{S*} - \sum_{k \in H} \sum_{r \in O^k} u_r^{k*} y_{ro}^{k*} - \sum_{r \in O^S} u_r^{S*} y_{ro}^{S*} + \sum_{k \in H} u_0^{k*} = 0, \tag{3.28}$$

where $v_i^{k*}, v_i^{S*}, u_r^{k*}, u_r^{S*}$ and $u_0^{k*}$ are assumed as the unique optimal values obtained by solving model (3.26) or (3.27).

We further assume that the overall system is additive, in which case the transformation function of each activity is expressed as follows:

$$f^k(X_o^{Ak*}, Y_o^{Ak*}) = \sum_{i \in I^k} v_i^{k*} x_{io}^{k*} + \sum_{i \in I^S} v_i^{Sk*} x_{io}^{S*} - \sum_{r \in O^k} u_r^{k*} y_{ro}^{k*} - \sum_{r \in O^S} u_r^{S*} y_{ro}^{Sk*} + u_0^{k*} = 0. \tag{3.29}$$

By applying the standard formula of SE (Hanoch 1970, Panzar & Willig 1977, Starrett 1977), the SE of the whole production system and each activity are expressed as:

$$\begin{aligned}\varepsilon(X_o^*, Y_o^*) &= \frac{\sum_{k\in H}\sum_{i\in I^k} v_i^{k*} x_{io}^{k*} + \sum_{i\in I^S} v_i^{S*} x_{io}^{S*}}{\sum_{k\in H}\sum_{r\in O^k} u_r^{k*} y_{ro}^{k*} + \sum_{r\in O^S} u_r^{S*} y_{ro}^{S*}} \\ &= 1 - \sum_{k\in H} u_0^{k*},\end{aligned} \tag{3.30}$$

$$\begin{aligned}\varepsilon^k(X_o^{Ak*}, Y_o^{Ak*}) &= \frac{\sum_{i\in I^k} v_i^{k*} x_{io}^{k*} + \sum_{i\in I^S} v_i^{S*} x_{io}^{Sk*}}{\sum_{r\in O^k} u_r^{k*} y_{ro}^{k*} + \sum_{r\in O^S} u_r^{S*} y_{ro}^{Sk*}}(\forall k) \\ &= 1 - \frac{u_0^{k*}}{\sum_{r\in O^k} u_r^{k*} y_{ro}^{k*} + \sum_{r\in O^S} u_r^{S*} y_{ro}^{Sk*}}(\forall k).\end{aligned} \tag{3.31}$$

Let $\sum_{r\in O^k} u_r^{k*} y_{ro}^{k*} + \sum_{r\in O^S} u_r^{S*} y_{ro}^{Sk*} = R^k(\forall k)$ hold, the overall SE $\varepsilon(X_o^*, Y_o^*)$ is the weighted sum of $\varepsilon^k(X_o^{Ak*}, Y_o^{Ak*})$, that is:

$$\varepsilon(X_o^*, Y_o^*) = \sum_{k\in H} R^k \varepsilon^k(X_o^{Ak*}, Y_o^{Ak*}). \tag{3.32}$$

It is obvious that the weights allocated to $k^{\text{th}}$ activity (i.e., $R^k$) is positively related to the projections of outputs and their shadow prices in this activity. Specifically, $R^k$ is calculated as the weighted sum of its activity-specific outputs ($y_{ro}^{k*}$) and the allocated share of shared outputs ($y_{ro}^{Sk*}$), using the corresponding shadow prices as weights. Therefore, $R^k$ can be regarded as the contribution of aggregated output that the $k^{\text{th}}$ activity has produced in total aggregated output produced by the system. In other words, for each efficient $\text{DMU}_o(X_o^*, Y_o^*)$, activities that contribute more to the total aggregated output are assigned greater weights in the SE calculation.

Note that $\sum_{k\in H} R^k = 1$ holds based on the first constrain in model (3.24). Consequently, the overall SE $\varepsilon(X_o^*, Y_o^*)$ is expressed as the weighted sum of all SE $\varepsilon^k(X_o^{Ak*}, Y_o^{Ak*})(\forall k)$ in each activity. The weights are allocated by the model.

Considering the right-hand and left-hand SE, we propose a method of decomposing SE in multi-activity technologies expressed as (3.14) as follows.

- For the right-hand SE of an efficient $\mathrm{DMU}_o(X_o^*, Y_o^*)$, we have:

$$\begin{aligned} \varepsilon^{k+} &= 1 - \frac{u_0^{k*}}{\sum_{r\in O^k} u_r^{k*} y_{ro}^{k*} + \sum_{r\in O^S} u_r^{S*} y_{ro}^{Sk*}} (\forall k), \\ \varepsilon^{+} &= 1 - \sum_{k\in H} u_0^{k*} \\ &= \sum_{k\in H} \varepsilon^{k+} \Big( \sum_{r\in O^k} u_r^{k*} y_{ro}^{k*} + \sum_{r\in O^S} u_r^{S*} y_{ro}^{Sk*} \Big), \end{aligned} \tag{3.33}$$

  where $u_r^{k*}, u_r^{S*}$ and $u_0^{k*}$ are optimal values obtained by model (3.26).

- For the left-hand SE, we have:

$$\begin{aligned} \varepsilon^{k-} &= 1 - \frac{u_0^{k*}}{\sum_{r\in O^k} u_r^{k*} y_{ro}^{k*} + \sum_{r\in O^S} u_r^{S*} y_{ro}^{Sk*}} (\forall k), \\ \varepsilon^{-} &= 1 - \sum_{k\in H} u_0^{k*} \\ &= \sum_{k\in H} \varepsilon^{k-} \Big( \sum_{r\in O^k} u_r^{k*} y_{ro}^{k*} + \sum_{r\in O^S} u_r^{S*} y_{ro}^{Sk*} \Big), \end{aligned} \tag{3.34}$$

where $u_r^{k*}, u_r^{S*}$ and $u_0^{k*}$ are optimal values obtained by model (3.27).

It is worth noting that the values of $\varepsilon^{k+}$ and $\varepsilon^{k-}$ are undefined when $\sum_{r\in O^k} u_r^{k*} y_{ro}^{k*} + \sum_{r\in O^S} u_r^{S*} y_{ro}^{Sk*} = 0$ (i.e., $R^k = 0$). Therefore, the allocated weights of such activities are 0, and thereby the results of $\varepsilon^+$ and $\varepsilon^-$ cannot be influenced.

### 3.2.4 Case study

#### Data

In this subsection, we carry out an illustrative example of SE estimation and decomposition of UK universities that are top 500 from the newest QS World University Rankings. The data related to students, staff and finance in 2022/2023 academic year is collected from Higher Education Statistics Agency (HESA), which is a part of Joint Information Systems Committee (JISC). The scholarly publication data in 2023 is collected form Web of Science (WOS). All data used in this thesis is open and accessible online.

Due to missing data, 5 universities are not considered. Consequently, we have 43 DMUs in total. Each university is viewed as a DMU that is consisted of departments of natural science (NS) and social science (SS). Specifically, the first activity (i.e., NS) includes bio-science, mathematics, physics and

so on, while the second activity (i.e., SS) includes humanities, law, business and so on. Although this separation is not immaculate, it can recognize the distinctiveness of teaching and research between the different disciplines. As the discussion in Remark 3.1, here we assume the number of staff and students are non-compensable across the two activities. Table 3.3 exhibits the descriptive statistics of the data that we use in this case study.

Table 3.3: Descriptive statistics for the data of 43 UK universities

| Measure | Mean | SD | Min. | Max. |
|---|---|---|---|---|
| Activity-specific inputs | | | | |
| Staff (NS) | 2372.44 | 2050.52 | 225 | 8270 |
| Staff (SS) | 1005.47 | 579.20 | 285 | 2795 |
| Shared inputs | | | | |
| Expenditure | 223.41 | 160.33 | 38.7 | 701.27 |
| Activity-specific outputs | | | | |
| Undergraduates (NS) | 8056.05 | 3742.81 | 790 | 18330 |
| Undergraduates (SS) | 8182.44 | 6596.86 | 70 | 45560 |
| Postgraduates (NS) | 2798.26 | 2184.40 | 600 | 12680 |
| Postgraduates (SS) | 3245.81 | 1848.52 | 1120 | 9295 |
| Shared outputs | | | | |
| Scholarly publications | 3329.93 | 3089.48 | 472 | 13583 |

As shown in Table 3.3, the number of staff in the tow activities are regarded as activity-specific inputs (i.e., $x^1$ and $x^2$). The total expenditure is the shared input (i.e., $x^S$) that is allocated into the tow activities with known proportions, which is determined according to the HESA cost centres. Regarding outputs, the number of undergraduates and postgraduates are two activity-specific outputs (i.e., $y_1^1$, $y_1^2$, $y_2^1$and $y_2^2$), and the number of scholarly publications is the shared output (i.e., $y^S$) with known proportions, which is determined according to the WOS Citation Topics.

**Results**

As discussed previously, SE is only analyzed for efficient DMUs of the technology $T_P$. Here we use output-oriented model (i.e., model (3.15)) to project all inefficient DMUs to the efficient frontier. The results of projections and efficient scores are shown in Table B.2. Then we calculate the right-hand and left-hand SE by using proposed models (3.26) and (3.27). Specifically, the overall SE is decomposed into two activities by applying the proposed

method shown in (3.33) and (3.34). From the results in Table 3.4, apart form the projected $DMU_{36}$, $DMU_{40}$ and $DMU_{43}$, the other DMUs need to decrease or keep the scale. In addition, the results in Tables 3.5 and 3.6 show that some DMUs can further increase the scale of SS activity even the overall right-hand SE is less than 1 (e.g., $DMU_3$, $DMU_9$ and $DMU_{10}$).

Table 3.4: Right-hand and left-hand SE for 43 UK universities

| DMU | $\varepsilon^+$ | $\varepsilon^-$ | DMU | $\varepsilon^+$ | $\varepsilon^-$ | DMU | $\varepsilon^+$ | $\varepsilon^-$ |
|---|---|---|---|---|---|---|---|---|
| $DMU_1$ | 0.56 | 0.97 | $DMU_{16}$ | 0.62 | 0.84 | $DMU_{31}$ | 0.73 | 0.78 |
| $DMU_2$ | 0.57 | 0.82 | $DMU_{17}$ | 0.69 | 0.84 | $DMU_{32}$ | 0.35 | 0.37 |
| $DMU_3$ | 0.65 | 0.98 | $DMU_{18}$ | 0.50 | 0.50 | $DMU_{33}$ | 0.72 | 0.74 |
| $DMU_4$ | 0.00 | 0.98 | $DMU_{19}$ | 0.73 | 0.78 | $DMU_{34}$ | 0.00 | 2.70 |
| $DMU_5$ | 0.00 | 0.79 | $DMU_{20}$ | 0.28 | 0.81 | $DMU_{35}$ | 0.68 | 1.25 |
| $DMU_6$ | 0.00 | 0.69 | $DMU_{21}$ | 0.55 | 0.57 | $DMU_{36}$ | 1.53 | $+\infty$ |
| $DMU_7$ | 0.46 | 0.61 | $DMU_{22}$ | 0.80 | 0.83 | $DMU_{37}$ | 0.50 | 0.63 |
| $DMU_8$ | 0.80 | 1.10 | $DMU_{23}$ | 0.70 | 0.79 | $DMU_{38}$ | 0.60 | 0.72 |
| $DMU_9$ | 0.58 | 0.65 | $DMU_{24}$ | 0.48 | 0.51 | $DMU_{39}$ | 0.55 | 0.68 |
| $DMU_{10}$ | 0.51 | 0.60 | $DMU_{25}$ | 0.48 | 12.96 | $DMU_{40}$ | 1.32 | 1.53 |
| $DMU_{11}$ | 0.39 | 0.85 | $DMU_{26}$ | 0.30 | 0.65 | $DMU_{41}$ | 0.72 | 1.01 |
| $DMU_{12}$ | 0.74 | 0.88 | $DMU_{27}$ | 0.73 | 1.09 | $DMU_{42}$ | 0.82 | $+\infty$ |
| $DMU_{13}$ | 0.72 | 0.73 | $DMU_{28}$ | 0.70 | 0.94 | $DMU_{43}$ | 1.39 | 1.40 |
| $DMU_{14}$ | 0.74 | 0.78 | $DMU_{29}$ | 0.47 | 0.85 | | | |
| $DMU_{15}$ | 0.71 | 1.11 | $DMU_{30}$ | 0.72 | 0.76 | | | |

Table 3.5: Right-hand SE decomposition

| DMU | $\varepsilon^{1+}$ | $\varepsilon^{2+}$ | DMU | $\varepsilon^{1+}$ | $\varepsilon^{2+}$ | DMU | $\varepsilon^{1+}$ | $\varepsilon^{2+}$ |
|---|---|---|---|---|---|---|---|---|
| $DMU_1$ | 0.55 | 0.87 | $DMU_{16}$ | 0.60 | 0.80 | $DMU_{31}$ | 0.68 | 1.15 |
| $DMU_2$ | 0.65 | 0.00 | $DMU_{17}$ | 0.66 | 0.98 | $DMU_{32}$ | 0.35 | undef. |
| $DMU_3$ | 0.63 | 1.08 | $DMU_{18}$ | 0.50 | undef. | $DMU_{33}$ | 0.68 | 1.15 |
| $DMU_4$ | 0.00 | undef. | $DMU_{19}$ | 0.65 | 1.09 | $DMU_{34}$ | undef. | 0.00 |
| $DMU_5$ | undef. | 0.00 | $DMU_{20}$ | 0.28 | undef. | $DMU_{35}$ | 0.69 | 0.68 |
| $DMU_6$ | 0.00 | undef. | $DMU_{21}$ | 0.55 | 0.94 | $DMU_{36}$ | 1.54 | 1.50 |
| $DMU_7$ | 0.46 | undef. | $DMU_{22}$ | 0.77 | 1.03 | $DMU_{37}$ | 0.50 | undef. |
| $DMU_8$ | 0.88 | 0.78 | $DMU_{23}$ | 0.62 | 1.07 | $DMU_{38}$ | undef. | 0.60 |
| $DMU_9$ | 0.57 | 1.05 | $DMU_{24}$ | 0.48 | undef. | $DMU_{39}$ | 0.55 | undef. |
| $DMU_{10}$ | 0.50 | 1.05 | $DMU_{25}$ | 0.48 | 0.63 | $DMU_{40}$ | 1.32 | undef. |
| $DMU_{11}$ | undef. | 0.39 | $DMU_{26}$ | 0.30 | undef. | $DMU_{41}$ | 0.65 | 0.85 |

| DMU | $\varepsilon^{1+}$ | $\varepsilon^{2+}$ | DMU | $\varepsilon^{1+}$ | $\varepsilon^{2+}$ | DMU | $\varepsilon^{1+}$ | $\varepsilon^{2+}$ |
|---|---|---|---|---|---|---|---|---|
| $DMU_{12}$ | undef. | 0.74 | $DMU_{27}$ | 0.69 | 1.21 | $DMU_{42}$ | undef. | 0.82 |
| $DMU_{13}$ | 0.71 | 0.75 | $DMU_{28}$ | 0.64 | 0.99 | $DMU_{43}$ | 1.44 | 0.99 |
| $DMU_{14}$ | 0.69 | 1.03 | $DMU_{29}$ | 0.45 | 0.81 | | | |
| $DMU_{15}$ | 0.64 | 1.62 | $DMU_{30}$ | 0.70 | 1.00 | | | |

Note: "undef." is an abbreviation for "undefined", and the same notation applies hereafter.

Table 3.6: Left-hand SE decomposition

| DMU | $\varepsilon^{1-}$ | $\varepsilon^{2-}$ | DMU | $\varepsilon^{1-}$ | $\varepsilon^{2-}$ | DMU | $\varepsilon^{1-}$ | $\varepsilon^{2-}$ |
|---|---|---|---|---|---|---|---|---|
| $DMU_{1}$ | 0.95 | 1.34 | $DMU_{16}$ | undef. | 0.84 | $DMU_{31}$ | 0.72 | 1.33 |
| $DMU_{2}$ | 0.90 | 0.15 | $DMU_{17}$ | undef. | 0.84 | $DMU_{32}$ | 0.37 | undef. |
| $DMU_{3}$ | 0.97 | 1.08 | $DMU_{18}$ | 0.50 | undef. | $DMU_{33}$ | 0.69 | 1.26 |
| $DMU_{4}$ | 0.98 | 1.01 | $DMU_{19}$ | 0.71 | 1.09 | $DMU_{34}$ | undef. | 2.70 |
| $DMU_{5}$ | 0.77 | 0.86 | $DMU_{20}$ | 0.77 | 1.32 | $DMU_{35}$ | 1.30 | 1.02 |
| $DMU_{6}$ | 0.69 | undef. | $DMU_{21}$ | 0.55 | 1.09 | $DMU_{36}$ | inf. | inf. |
| $DMU_{7}$ | 0.61 | undef. | $DMU_{22}$ | 0.79 | 1.09 | $DMU_{37}$ | 0.63 | undef. |
| $DMU_{8}$ | 1.62 | 0.81 | $DMU_{23}$ | 0.72 | 1.11 | $DMU_{38}$ | undef. | 0.72 |
| $DMU_{9}$ | 0.62 | 1.05 | $DMU_{24}$ | 0.51 | undef. | $DMU_{39}$ | 0.68 | undef. |
| $DMU_{10}$ | 0.56 | 1.05 | $DMU_{25}$ | undef. | 12.96 | $DMU_{40}$ | 1.53 | undef. |
| $DMU_{11}$ | 0.90 | 0.78 | $DMU_{26}$ | 0.65 | undef. | $DMU_{41}$ | 1.00 | 1.05 |
| $DMU_{12}$ | 0.86 | 0.93 | $DMU_{27}$ | 1.02 | 1.70 | $DMU_{42}$ | inf. | inf. |
| $DMU_{13}$ | 0.72 | 0.79 | $DMU_{28}$ | 0.87 | 1.10 | $DMU_{43}$ | 1.43 | 1.18 |
| $DMU_{14}$ | 0.72 | 1.13 | $DMU_{29}$ | undef. | 0.85 | | | |
| $DMU_{15}$ | 1.02 | 1.62 | $DMU_{30}$ | 0.73 | 1.05 | | | |

Note: "inf." is an abbreviation for "infeasible", and the same notation applies hereafter.

Based on the results in Table 3.4, it is worth noting that the right-hand SE (i.e., $\varepsilon^{+}$) and left-hand SE (i.e., $\varepsilon^{-}$) differ substantially for some DMUs, such as $DMU_{25}$ and $DMU_{35}$, while for some DMUs, the two values are close, even equal (e.g., $DMU_{18}$). This observation raises the question of why such discrepancies occur and, more importantly, how these right- and left-hand SE should be appropriately interpreted and applied in practice. Specifically, due to the piecewise linear property of DEA frontiers, the DMU can be regarded as operating in a relatively stable region of the frontier, such as the point E or F as shown in Fig. 3.2. When the right- and left-hand SE differ, the DMU is located at a corner point of the frontier, such as the point B as shown in Fig. 3.2. As we discussed in Section 3.2.1, the right-hand SE reflects

the proportional change of outputs when the inputs are slightly increased in proportion, while the left-hand SE counterpart measures the proportional change of outputs when the inputs are slightly reduced in proportion.

From a practical perspective, to achieve the goal of optimising resource allocation, decision makers should rely on the state of right-hand RTS and left-hand RTS according to Definition 3.1. That is, if increasing (or decreasing) RTS prevails on both sides, decision makers are advised to enlarge (or reduce) the scale, and thereby focus on the right-hand (or left-hand) SE. Otherwise, when $\varepsilon^{+} \leq 1$ and $\varepsilon^{-} \geq 1$ (note that $\varepsilon^{-}\left(X_o^*, Y_o^*\right) \geq \varepsilon^{+}\left(X_o^*, Y_o^*\right)$ always holds, see Podinovski & Førsund (2022)), decision makers are advised to maintain the scale. However, in the real world, due to some external factors, such as policy requirements or market competition, decision makers have to adjust the scale of the DMU regardless of the impact of productivity. In this case, the right- and left-hand SE provide decision makers with information about the potential impacts that adjusting the scale might bring. For example, for $\mathrm{DMU}_{25}$, the negative impact on productivity caused by reducing its scale is significantly greater than that resulting from expanding its scale.

In summary, when both sides indicate increasing (or decreasing) RTS, it is advisable to expand (or reduce) the scale. When the case of $\varepsilon^{+} \leq 1$ and $\varepsilon^{-} \geq 1$ arises, the current scale should be maintained.If external factors necessitate scale adjustments, the right- and left-hand SE can help decision makers understand how expansion or reduction will affect productivity and prepare a plan to mitigate any possible negative impact.

## 3.3 Estimating SE in multi-stage technologies

### 3.3.1 Preliminaries

Consider that the general production technology $T$ has $L$-stage series structures, we reformulate a $L$-stage multi-stage technology $T_S \subset \mathbb{R}_{+}^{m+h^{(1)}+\cdots+h^{(L-1)}+s}$ in which the set of initial inputs, intermediate products and final outputs are denoted by $I \subset \mathbb{R}_{+}^{m}$, $Z \subset \mathbb{R}_{+}^{h^{(l)}}(l = 1, \ldots, L-1)$ and $O \subset \mathbb{R}_{+}^{m}$, respectively. For all DMUs that can be observed in $T_S$, we denote $n$ as the number of all $\mathrm{DMU}_j(X_j, Z_j^{(1)}, \ldots, Z_j^{(L-1)}, Y_j)(j \in N = \{1, \ldots, n\}(n \geq 1))$, where $X_j = (x_{1j}, \ldots, x_{mj})^{\top}$, $Z_j^{(l)} = (z_{1j}^{(l)}, \ldots, z_{h^{(l)}j}^{(l)})^{\top}(l = 1, \ldots, L-1)$ and $Y_j = (y_{1j}, \ldots, y_{sj})^{\top}$, and assume $X_j, Z_j^{(l)}, Y_j \neq \mathbf{0}(\forall l, j)$. Let technology $T_S$

include all possible $\mathrm{DMU}_j(X_j, Z_j^{(1)}, \ldots, Z_j^{(L-1)}, Y_j)$:

$$T_S = \left\{ \left( X, Z^{(1)}, \ldots, Z^{(L-1)}, Y \right) \in \mathbb{R}_+^{m+h^{(1)}+\cdots+h^{(L-1)}+s} \mid X \text{ can produce } Z^{(l)}, \right. \\ \left. \cdots \right. \\ \left. Z^{(L-1)} \text{ can produce } Y \right\}. \tag{3.35}$$

The relationship among each stage $l \in G = 1, \ldots, L (L \geq 2)$ is shown in Fig. 3.3. Specifically, the initial inputs $X$ are consumed to produce the intermediate products $Z^{(1)}$ in the stage 1. Then, all intermediate products $Z^{(1)}$ are used to produce the intermediate products $Z^{(2)}$ in stage 2. The process is ended until the final outputs $Y$ are produced by consuming all the intermediate products $Z^{(L-1)}$ in the stage $L$. We assume that the intermediate products produced in stage $l$ must be consumed completely in stage $l+1$, as shown in Fig. 3.3.

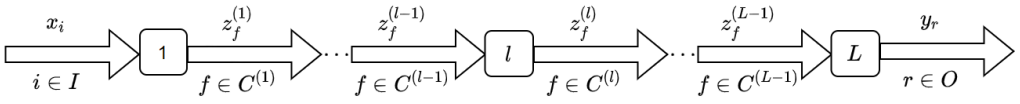



Figure 3.3: Multi-stage production system.

### 3.3.2 SE estimation under known transformation functions

We assume that the whole two-stage production system is defined by the following transformation function group:

$$\begin{cases} f^{(1)}\left(X, Z^{(1)}\right) = 0, \\ \cdots \\ f^{(l)}\left(Z^{(l-1)}, Z^{(l)}\right) = 0, \\ \cdots \\ f^{(L)}\left(Z^{(L-1)}, Y\right) = 0. \end{cases} \tag{3.36}$$

Similar to Assumption 3.1, we assume continuous differentiability, strictly decreasing in all inputs and strictly increasing in all outputs for each transformation function.

**Assumption 3.2.** *All transformation functions in* (3.36) *satisfy:*

$$\frac{\partial f^{(l)}(Z^{(l-1)}, Z^{(l)})}{\partial z_f^{(l-1)}} < 0 \ (\forall l, f \in C^{(l-1)}),$$

$$\frac{\partial f^{(l)}(Z^{(l-1)}, Z^{(l)})}{\partial z_f^{(l)}} > 0 \ (\forall l, f \in C^{(l)}),$$

*where* $Z^{(0)}$ *is regarded as an initial input of* $X$ *when* $l = 1$, $Z^{(L)}$ *is regarded as the final output of* $Y$ *when* $l = L$.

Then the technology $T_S$ is defined as follows:

$$T_S = \left\{ (X, \ldots, Z^{(l)}, \ldots, Y) \in \mathbb{R}_+^{|I|+\cdots+|Z^{(l)}|+\ldots|O|} \left| \begin{array}{l} f^{(1)}\left(X, Z^{(1)}\right) \le 0, \\ \ldots, \\ f^{(l)}\left(Z^{(l-1)}, Z^{(l)}\right) \le 0, \\ \ldots, \\ f^{(L)}\left(Z^{(L-1)}, Y\right) \le 0. \end{array} \right. \right\} \tag{3.37}$$

For a $\mathrm{DMU}_o(X_o, \ldots, Z_o^{(l)}, \ldots, Y_o)$ that satisfies (3.36), $\varphi^{(l)}(\varphi^{(l-1)})$ represents the response factor of each intermediate product. By introducing the scaling factor $t$ to initial inputs, we assume the following equations are satisfied:

$$\left\{ \begin{array}{l} f^{(1)}\left(tX, \varphi Z^{(1)}\right) = 0, \\ \cdots \\ f^{(l)}\left(\varphi^{(l-1)} Z^{(l-1)}, \varphi^{(l)} Z^{(l)}\right) = 0, \\ \cdots \\ f^{(L)}\left(\varphi^{(L-1)} Z^{(L-1)}, \beta Y\right) = 0. \end{array} \right. \tag{3.38}$$

Especially, $\varphi^{(0)}$ and $\varphi^{(L)}$ are regarded as the scaling factor of initial inputs and final outputs, i.e., $\varphi^{(0)} = t$ and $\varphi^{(L)} = \beta$.

Therefore, for any stage $l (l \in G)$, the SE of sub-system is expressed as:

$$\left\{ \begin{array}{l} \varepsilon^{(S1)} = \left.\frac{\partial \varphi^{(1)}(t)}{\partial t}\right|_{t=0} = -\sum_{i \in I} \frac{\partial f^{(1)}}{\partial x_i} x_{io} / \sum_{f \in C^{(1)}} \frac{\partial f^{(1)}}{\partial z_f^{(1)}} z_{fo}^{(1)}, \\ \cdots \\ \varepsilon^{(l)} = \left.\frac{\partial \varphi^{(l)}(\varphi^{(l-1)})}{\partial \varphi^{(l-1)}}\right|_{\varphi^{(l-1)}=0} = -\sum_{f \in C^{(l-1)}} \frac{\partial f^{(l)}}{\partial z_f^{(l-1)}} z_{fo}^{(l-1)} / \sum_{f \in C^{(l)}} \frac{\partial f^{(l)}}{\partial z_f^{(l)}} z_{fo}^{(l)}, \\ \cdots \\ \varepsilon^{(L)} = \left.\frac{\partial \varphi^{(L)}(\varphi^{(L-1)})}{\partial \varphi^{(L-1)}}\right|_{\varphi^{(L-1)}=0} = -\sum_{f \in C^{(L-1)}} \frac{\partial f^{(L)}}{\partial z_f^{(L-1)}} z_{fo}^{(L-1)} / \sum_{r \in O} \frac{\partial f^{(L)}}{\partial y_r} y_{ro}. \end{array} \right. \tag{3.39}$$

Note that $\frac{\partial f^{(l)}(Z^{(l-1)}, Z^{(l)})}{\partial z_f^{(l)}} = -\frac{\partial f^{(l+1)}(Z^{(l)}, Z^{(l+1)})}{\partial z_f^{(l)}}$ $(\forall l, f \in C^{(l-1)})$ is satisfied since we assume all intermediate products are consumed completely. Therefore, the overall SE $\varepsilon^N$ of the whole series production system is the product

of SE in each sub-systems:

$$\varepsilon^N = \left.\frac{\partial \beta(t)}{\partial t}\right|_{t=0} = \varepsilon^{(S1)} * \cdots * \varepsilon^{(l)} * \cdots * \varepsilon^{(L)}. \tag{3.40}$$

### 3.3.3 SE estimation and decomposition under the DEA framework

To simplify the situation in DEA, we consider a two-stage series production system as shown in Fig. 3.4.

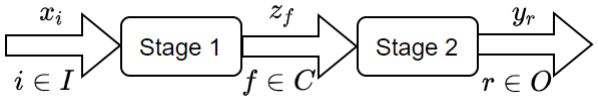



Figure 3.4: Two-stage production system.

This subsection proposes a general approach to discomposing SE for a two-stage production system as shown in Fig. 3.4. We assume that $T_S$ satisfies several standard axioms as follows:

- **A1** Strong disposability of inputs and outputs: If $(X, Z, Y) \in T_S$, we have $(X^*, Z, Y^*) \in T_S (\forall X^*$ and $Y^*$ such that $X^* \geq X$ and $0 \leq Y^* \leq Y)$.
- **A2** Convexity: Technology $T_S$ is a convex set.
- **A3** Feasibility of all observed DMUs: For any $j \in N$, $\left(X_j, Z_j, Y_j\right) \in T_S$.

Assume all intermediate products $Z$ are completely used, the technology $T_S$ is expressed as the following form:

$$T_S = \left\{ (X, Z, Y) \in \mathbb{R}_+^{|I|+|C|+|O|} \;\middle|\; \begin{array}{l} \sum_{j=1}^n \lambda_j^{(1)} x_{ij} \leq x_i (\forall i), \sum_{j=1}^n \lambda_j^{(1)} z_{fj} \geq z_f (\forall f), \\ \sum_{j=1}^n \lambda_j^{(2)} z_{fj} \leq z_f (\forall f), \sum_{j=1}^n \lambda_j^{(2)} y_{rj} \geq y_r (\forall r), \\ \sum_{j=1}^n \lambda_j^{(1)} = 1, \sum_{j=1}^n \lambda_j^{(2)} = 1, \lambda_j^{(1)}, \lambda_j^{(2)} \geq 0 (\forall j). \end{array} \right\} \tag{3.41}$$

To determine the efficient DMUs and projections for inefficient DMUs, the

two-stage DEA model are constructed in output-oriented manner as follows:

$$\frac{1}{E_0^{S-O}} = \max_{\theta,\lambda_j^{(1)},\lambda_j^{(2)},z_f} \theta$$
$$\text{s.t.} \begin{cases} \sum_{j=1}^n \lambda_j^{(1)} x_{ij} \le x_{io}\ (\forall i), \sum_{j=1}^n \lambda_j^{(1)} z_{fj} \ge z_f\ (\forall f), \\ \sum_{j=1}^n \lambda_j^{(2)} z_{fj} \le z_f\ (\forall f), \sum_{j=1}^n \lambda_j^{(2)} y_{rj} \ge \theta y_{ro}\ (\forall r), \\ \sum_{j=1}^n \lambda_j^{(1)} = 1, \sum_{j=1}^n \lambda_j^{(2)} = 1, \\ \lambda_j^{(1)}, \lambda_j^{(2)} \ge 0\ (\forall j), \theta, z_f \text{ signfree } (\forall f). \end{cases} \tag{3.42}$$

Let $(\theta^*, \lambda_j^{(1)*}, \lambda_j^{(2)*}, z_f^*)(\forall j, f)$ represent the optimal value of model (3.42). $\text{DMU}_o$ is defined as efficient DMU if and only if the optimal solution of the acquired from model (3.42) satisfy $\theta_{N-O}^* = 1$ and slacks for all inputs and outputs is equal to zero. If not, the inefficient DMUs can be projected onto the efficient frontier by applying $x_{io}^* = \sum_{j=1}^n \lambda_j^{(1)*} x_{ij}$, $z_{fo}^* = z_f^*$ and $y_{ro}^* = \sum_{j=1}^n \lambda_j^{(2)*} y_{rj}$. Then the projection point for $\text{DMU}_o(X_o, Z_o, Y_o)$ is $\text{DMU}_o^*(X_o^*, Z_o^*, Y_o^*)$. Especially, if $\text{DMU}_o(X_o, Z_o, Y_o)$ is efficient, then it is also represented by $\text{DMU}_o^*(X_o^*, Z_o^*, Y_o^*)$. In addition, the input-oriented model is:

$$E_0^{S-I} = \min_{\theta,\lambda_j^{(1)},\lambda_j^{(2)},z_f} \theta$$
$$\text{s.t.} \begin{cases} \sum_{j=1}^n \lambda_j^{(1)} x_{ij} \le \theta x_{io}\ (\forall i), \sum_{j=1}^n \lambda_j^{(1)} z_{fj} \ge z_f\ (\forall f), \\ \sum_{j=1}^n \lambda_j^{(2)} z_{fj} \le z_f\ (\forall f), \sum_{j=1}^n \lambda_j^{(2)} y_{rj} \ge y_{ro}\ (\forall r), \\ \sum_{j=1}^n \lambda_j^{(1)} = 1, \sum_{j=1}^n \lambda_j^{(2)} = 1, \\ \lambda_j^{(1)}, \lambda_j^{(2)} \ge 0\ (\forall j), \theta, z_f \text{ signfree } (\forall f). \end{cases} \tag{3.43}$$

To illustrate the different ways of projecting in a basic two-stage production, an example data set form Sahoo et al. (2014) and its projections in different ways are given in Table 3.7.

Table 3.7: Original data and projections of example data

| DMU | Original data | | | Input-oriented | | | Output-oriented | | |
|---|---|---|---|---|---|---|---|---|---|
| | X | Z | Y | X | Z | Y | X | Z | Y |
| A | 1.5 | 1 | 1 | 1.5 | 1 | 1 | 1.5 | 1 | 1 |
| B | 4 | 6 | 4 | 2.17 | 2.8 | 4 | 4 | 6 | 6.67 |
| C | 6 | 7 | 7 | 6 | 7 | 7 | 6 | 7 | 7 |
| D | 2 | 2.5 | 3 | 1.9 | 2.2 | 3 | 2 | 2.5 | 3.5 |
| E | 5 | 4 | 6 | 2.86 | 4 | 6 | 5 | 6.5 | 6.83 |
| F | 4 | 3.5 | 2 | 1.7 | 1.6 | 2 | 4 | 6 | 6.67 |
| G | 5.5 | 5 | 5 | 2.51 | 3.4 | 5 | 5.5 | 6.75 | 6.92 |
| H | 4.5 | 6 | 6 | 2.86 | 4 | 6 | 4.5 | 6.25 | 6.75 |
| I | 7 | 6.5 | 6.5 | 3.71 | 5.5 | 6.5 | 6 | 7 | 7 |

Note that the intermediate products can increase or decrease after conducting projection. In specific, based on the basic two-stage structure as describing in model (3.43) and (3.42), different ways of projecting point $G$ are illustrated as Fig. 3.5.

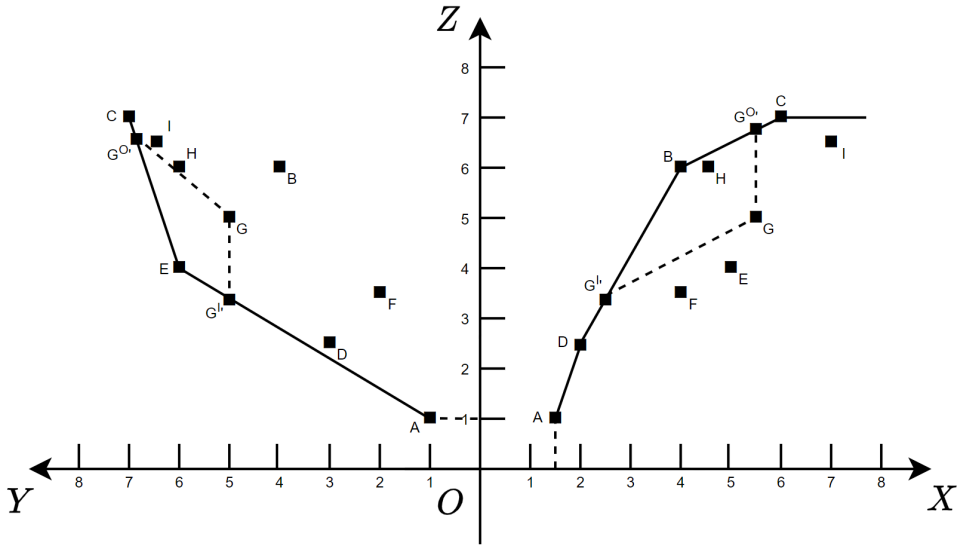



Figure 3.5: Different projections of point G.

Obviously, when the output-oriented projection is conducted, the output and intermediate of $G^{O'}$ are larger than the inefficient DMU $G$, while the input remains constant. When the input-oriented projection is conducted,

the input and intermediate of $G^{I'}$ are smaller than the inefficient DMU $G$, while the output remains constant.

Then, for all efficient $\text{DMU}_o^*$, the response function of final outputs is defined as follows:

$$\beta(\varphi(t)) = \max\left\{\beta \mid \left((1+t)X_o^*, (1+\varphi)Z_o^*, (1+\beta)Y_o^*\right) \in T_S, \varphi, \beta \in \mathbb{R}\right\}, \quad (3.44)$$

where we assume $\beta = \varphi = 0$ when $t = 0$ to indicate that $Y_o^*$ and $Z_o^*$ cannot be changed within $T_S$ on condition that $X_o^*$ keeps unchanged.

Different form the single stage production, the changes in inputs cannot influence outputs directly in a two-stage production process. Therefore, the changes in intermediate products are essential to be considered for calculating SE of overall system and sub-stages. Assume the derivatives of $\beta(\varphi(t))$ exist when $t = 0$, for the overall system, the first stage and the second stage, the right-hand and left-hand SE are defined as follows:

$$\begin{aligned}
\varepsilon^{(S)+}(X_o^*, Z_o^*, Y_o^*) &= \left.\frac{\partial\beta}{\partial t}\right|_{t=0^+}, \quad \varepsilon^{(S)-}(X_o^*, Z_o^*, Y_o^*) = \left.\frac{\partial\beta}{\partial t}\right|_{t=0^-}, \\
\varepsilon^{(S1)+}(X_o^*, Z_o^*, Y_o^*) &= \left.\frac{\partial\varphi}{\partial t}\right|_{t=0^+}, \quad \varepsilon^{(S1)-}(X_o^*, Z_o^*, Y_o^*) = \left.\frac{\partial\varphi}{\partial t}\right|_{t=0^-}, \\
\varepsilon^{(S2)+}(X_o^*, Z_o^*, Y_o^*) &= \left.\frac{\partial\beta}{\partial\varphi}\right|_{\varphi=0^+}, \quad \varepsilon^{(S2)-}(X_o^*, Z_o^*, Y_o^*) = \left.\frac{\partial\beta}{\partial\varphi}\right|_{\varphi=0^-}.
\end{aligned} \quad (3.45)$$

Thus, for the composite function $\beta(\varphi(t))$ as shown in Eq. (3.44), $\varepsilon^{(S1)+}$ and $\varepsilon^{(S1)-}$ are the right and left derivatives of the inside function $\varphi(t)$ at $t = 0$, respectively, while $\varepsilon^{(S2)+}$ and $\varepsilon^{(S2)-}$ are the right and left derivatives of the outside function $\beta(\varphi)$ at $t = 0$, respectively. Due to the assumption of $\varphi(0) = 0$, the right and left derivatives of the composite function $\beta(\varphi(t))$ at $t = 0$ (i.e., $\varepsilon^{(S)+}$ and $\varepsilon^{(S)-}$, respectively), are the product of $\varepsilon^{(S1)+}$ and $\varepsilon^{(S2)+}$, $\varepsilon^{(S1)-}$ and $\varepsilon^{(S2)-}$, respectively:

$$\begin{aligned}
\varepsilon^{(S)+}(X_o^*, Z_o^*, Y_o^*) &= \varepsilon^{(S1)+}(X_o^*, Z_o^*, Y_o^*) * \varepsilon^{(S2)+}(X_o^*, Z_o^*, Y_o^*), \\
\varepsilon^{(S)-}(X_o^*, Z_o^*, Y_o^*) &= \varepsilon^{(S1)-}(X_o^*, Z_o^*, Y_o^*) * \varepsilon^{(S2)-}(X_o^*, Z_o^*, Y_o^*).
\end{aligned} \quad (3.46)$$

To estimate SE for efficient $\text{DMU}_o^*(X_o^*, Z_o^*, Y_o^*) \in T_S$ in the overall system, the first stage and the second stage, $\beta(t)$, $\varphi(t)$ and $\beta(\varphi)$ are expressed as

following linear programs, respectively:

$$\beta(t) = \max_{\varphi,\beta,\lambda_j^{(1)},\lambda_j^{(2)}} \beta$$
$$\text{s.t.} \begin{cases} \sum_{j=1}^{n} \lambda_j^{(1)} x_{ij}^* \le (1+t) x_{io}^* \ (\forall i), \\ \sum_{j=1}^{n} \lambda_j^{(1)} z_{fj}^* \ge (1+\varphi) z_{fo}^* \ (\forall f), \\ \sum_{j=1}^{n} \lambda_j^{(2)} z_{fj}^* \le (1+\varphi) z_{fo}^* \ (\forall f), \\ \sum_{j=1}^{n} \lambda_j^{(2)} y_{rj}^* \ge (1+\beta) y_{ro}^* \ (\forall r), \\ \sum_{j=1}^{n} \lambda_j^{(1)} = 1, \sum_{j=1}^{n} \lambda_j^{(2)} = 1, \\ \lambda_j^{(1)}, \lambda_j^{(2)} \ge 0 \ (\forall j), \varphi, \beta \text{ signfree.} \end{cases} \tag{3.47}$$

$$\varphi(t) = \max_{\varphi,\beta,\lambda_j^{(1)},\lambda_j^{(2)}} \varphi \quad \text{s.t. all constrains in model (3.47).} \tag{3.48}$$

$$\beta(\varphi) = \max_{t,\beta,\lambda_j^{(1)},\lambda_j^{(2)}} \beta$$
s.t. all constraints in model (3.47), where $\varphi$ is converted into a scalar, $t$ is converted into an unknown variable satisfying $t \ge 0$. (3.49)

Then, we use the DDO method to estimate SE for each sub-stage and the overall system. The following Theorem is proposed:

**Theorem 3.** *Let $(X_o^*, Z_o^*, Y_o^*) \in T_S$ be defined by (3.41) and $(X_o^*, Z_o^*, Y_o^*)$ be an efficient point, then the right-hand SE in the overall system, the first stage and the second stage are estimated by the following problems, respectively:*

$$\varepsilon^{(S)+} = \min_{v_i,\tau_f^{(1)},\tau_f^{(2)},u_r,u_{10},u_{20}} \sum_{i=1}^{m} v_i x_{io}^*$$
$$s.t. \begin{cases} \sum_{i=1}^{m} v_i x_{io}^* - \sum_{f=1}^{h} \tau_f^{(1)} z_{fo}^* + \sum_{f=1}^{h} \tau_f^{(2)} z_{fo}^* \\ - \sum_{r=1}^{s} u_r y_{ro}^* + u_{10} + u_{20} = 0, \\ \sum_{i=1}^{m} v_i x_{ij}^* - \sum_{f=1}^{h} \tau_f^{(1)} z_{fj}^* + u_{10} \ge 0 \ (\forall j), \\ \sum_{f=1}^{h} \tau_f^{(2)} z_{fj}^* - \sum_{r=1}^{s} u_r y_{rj}^* + u_{20} \ge 0 \ (\forall j), \\ \sum_{f=1}^{h} \tau_f^{(1)} z_{fo}^* - \sum_{f=1}^{h} \tau_f^{(2)} z_{fo}^* = 0, \\ \sum_{r=1}^{s} u_r y_{ro}^* = 1, \\ v_i, u_r, \tau_f^{(1)}, \tau_f^{(2)} \ge 0 \ (\forall i, r, f), u_{10}, u_{20} \ \textit{signfree,} \end{cases} \tag{3.50}$$

$$\varepsilon^{(S1)+} = \min_{v_i,\tau_f^{(1)},\tau_f^{(2)},u_r,u_{10},u_{20}} \sum_{i=1}^{m} v_i x_{io}^*$$
$$s.t. \begin{cases} \text{The first } 1+2*n \text{ constrains in model (3.50)}, \\ \sum_{f=1}^{h} \tau_f^{(1)} z_{fo}^* - \sum_{f=1}^{h} \tau_f^{(2)} z_{fo}^* = 1, \\ \sum_{r=1}^{s} u_r y_{ro}^* = 0, \\ v_i, u_r, \tau_f^{(1)}, \tau_f^{(2)} \geq 0\,(\forall i, r, f), u_{10}, u_{20} \text{ signfree}, \end{cases} \tag{3.51}$$

$$\varepsilon^{(S2)+} = \min_{v_i,\tau_f^{(1)},\tau_f^{(2)},u_r,u_{10},u_{20}} -\sum_{f=1}^{h} \tau_f^{(1)} z_{fo}^* + \sum_{f=1}^{h} \tau_f^{(2)} z_{fo}^*$$
$$s.t. \begin{cases} \text{The first } 1+2*n \text{ constrains in model (3.50)}, \\ -\sum_{i=1}^{m} v_i x_{io}^* = 0, \\ \sum_{r=1}^{s} u_r y_{ro}^* = 1, \\ v_i, u_r, \tau_f^{(1)}, \tau_f^{(2)} \geq 0\,(\forall i, r, f), u_{10}, u_{20} \text{ signfree}. \end{cases} \tag{3.52}$$

*The left-hand SE in the overall system, the first stage and the second stage are estimated by the following models, respectively:*

$$\varepsilon^{(S)-} = \max_{v_i,\tau_f^{(1)},\tau_f^{(2)},u_r,u_{10},u_{20}} \sum_{i=1}^{m} v_i x_{io}^* \tag{3.53}$$
$$s.t. \text{ all constrains in model (3.50)},$$

$$\varepsilon^{(S1)-} = \max_{v_i,\tau_f^{(1)},\tau_f^{(2)},u_r,u_{10},u_{20}} \sum_{i=1}^{m} v_i x_{io}^* \tag{3.54}$$
$$s.t. \text{ all constrains in model (3.51)},$$

$$\varepsilon^{(S2)-} = \max_{v_i,\tau_f^{(1)},\tau_f^{(2)},u_r,u_{10},u_{20}} -\sum_{f=1}^{h} \tau_f^{(1)} z_{fo}^* + \sum_{f=1}^{h} \tau_f^{(2)} z_{fo}^* \tag{3.55}$$
$$s.t. \text{ all constrains in model (3.52)}.$$

The proof of Theorem 3 is similar with the proof of Theorem 2.

Accordingly, RTS of the overall system and each sub-system is defined as follows:

**Definition 3.2.** *In the overall production system, $DMU_o(X_o^*, Z_o^*, Y_o^*)$ exhibits*

- *Increasing RTS on the right-hand(or left-hand) side if $\varepsilon^{(S)+} > 0$ (or $\varepsilon^{(S)-} > 0$).*

- *Decreasing RTS on the right-hand(or left-hand) side if* $\varepsilon^{(S)+} < 0$ *(or* $\varepsilon^{(S)-} < 0$*).*
- *Constant RTS on the right-hand(or left-hand) side if* $\varepsilon^{(S)+} = 0$ *(or* $\varepsilon^{(S)-} = 0$*).*

*In the first stage production system,* $DMU_o(X_o^*, Z_o^*, Y_o^*)$ *exhibits*

- *Increasing RTS on the right-hand(or left-hand) side if* $\varepsilon^{(S1)+} > 0$ *(or* $\varepsilon^{(S1)-} > 0$*).*
- *Decreasing RTS on the right-hand(or left-hand) side if* $\varepsilon^{(S1)+} < 0$ *(or* $\varepsilon^{(S1)-} < 0$*).*
- *Constant RTS on the right-hand(or left-hand) side if* $\varepsilon^{(S1)+} = 0$ *(or* $\varepsilon^{(S1)-} = 0$*).*

*In the second stage production system,* $DMU_o(X_o^*, Z_o^*, Y_o^*)$ *exhibits*

- *Increasing RTS on the right-hand(or left-hand) side if* $\varepsilon^{(S2)+} > 0$ *(or* $\varepsilon^{(S2)-} > 0$*).*
- *Decreasing RTS on the right-hand(or left-hand) side if* $\varepsilon^{(S2)+} < 0$ *(or* $\varepsilon^{(S2)-} < 0$*).*
- *Constant RTS on the right-hand(or left-hand) side if* $\varepsilon^{(S2)+} = 0$ *(or* $\varepsilon^{(S2)-} = 0$*).*

To illustrate $\varepsilon^{(S)+} = \varepsilon^{(S1)+} * \varepsilon^{(S2)+}$ and $\varepsilon^{(S)-} = \varepsilon^{(S1)-} * \varepsilon^{(S2)-}$, here we use the input-oriented projections as shown in Table 3.7, the right-hand and left-hand SE for overall system and each sub-stage are calculated based on Theorem 3, as shown in Table 3.8.

Table 3.8: Right-hand and left-hand SE for the overall system, sub-the first stage and sub-the second stage

| DMU | $\varepsilon^{(S)+}$ | $\varepsilon^{(S1)+}$ | $\varepsilon^{(S2)+}$ | $\varepsilon^{(S)-}$ | $\varepsilon^{(S1)-}$ | $\varepsilon^{(S2)-}$ |
|---|---|---|---|---|---|---|
| A | 7.5 | 4.5 | 1.67 | $+\infty$ | $+\infty$ | $+\infty$ |
| B | 1.58 | 1.36 | 1.17 | 2 | 1.71 | 1.17 |
| C | 0 | 0 | inf. | 0.19 | 0.56 | 0.33 |
| D | 2.33 | 1.91 | 1.22 | 3.17 | 2.59 | 1.22 |
| E | 0.28 | 1.25 | 0.22 | 1.39 | 1.25 | 1.11 |
| F | 4.25 | 3.19 | 1.33 | 4.25 | 3.19 | 1.33 |
| G | 1.47 | 1.29 | 1.13 | 1.47 | 1.29 | 1.13 |
| H | 0.28 | 1.25 | 0.22 | 1.39 | 1.25 | 1.11 |
| I | 0.13 | 0.44 | 0.28 | 0.33 | 1.18 | 0.28 |

The results exhibited in Table 3.8 illustrate the relationship among sub-stages and the overall system (i.e., $\varepsilon^{(S)+} = \varepsilon^{(S1)+} * \varepsilon^{(S2)+}$ and $\varepsilon^{(S)-} = \varepsilon^{(S1)-} * \varepsilon^{(S2)-}$). Note that DMU C and DMU A are two special points in Fig. 3.5 in which DMU C cannot further increase intermediate products, and DMU A cannot further decrease intermediate products (regarding the discussion of these special points, see Remark 3.2). Consequently, based on the Difination 3.2, RTS of each DMU can be determined in Table 3.9.

Table 3.9: RTS for the overall system, the first stage and the second stage

| DMU | RTSN | RTS1 | RTS2 |
|---|---|---|---|
| A | I | I | I |
| B | I | I | I |
| C | D | D | U |
| D | I | I | I |
| E | C | I | C |
| F | I | I | I |
| G | I | I | I |
| H | C | I | C |
| I | D | C | D |

Note: "I", "D", "C" and "U" refer to increasing RTS, decreasing RTS, constant RTS and undefined, respectively.

According to Table 3.9, RTS situations in the overall system and each sub-stage are inconsistent for DMU E, H and I. Therefore, the recommendations related to adjusting the scale of the input can be specific to each sub-stage. Comparing with the decomposition method proposed by Sahoo et al. (2014), there are two merits of the method we have developed. First, the developed method can identify the points that the intermediate products cannot further increase. Take the efficient point C in Fig. 3.5 as an example, it is obvious that its intermediate products is the largest of all projected DMUs. This means that any changes in its inputs cannot increase the intermediate products, that is, the point whose intermediate product is larger than point C's is undefined in the production technology. Therefore, $\varepsilon^{(S2)+}$ of point C is infeasible as shown in Table 3.8 while it is evaluated as 0 by using Sahoo's method. Second, Sahoo's method relies on the optimal solution of a constructed linear programming, in which alternate optima exists. Therefore, the results of decomposing SE might be non-unique. However, the developed methods in this section estimate SE in overall stage and sub-stages by using three different linear programs, thereby the results are unique.

### 3.3.4 Case study

#### Data

In this subsection, we conduct an illustrative example of SE estimation and decomposition of Chinese listed cement companies. All of the data are collected from open report (i.e., annual financial report and ESG report) in 2023. Total 15 companies are considered as DMUs in this case study. Each company are viewed as a two-stage production including manufacturing and marketing, the intermediate products are the products that are traded in a market. Specifically, in the first stage, the companies use energy and human resources to produce cements (or clinkers) and aggregates, which are the two main raw materials for making commercial concretes. In the second stage, the two products are sold and produce profits and taxes, which represent the impact on the shareholders and society, respectively. The original data is exhibited in Table A.1, and Table 3.10 presents the descriptive statistics for the 15 cement companies.

Table 3.10: Descriptive statistics for the data of 15 Chinese cement companies

| Measure | Observations | Mean | SD | Min. | Max. |
|---|---|---|---|---|---|
| | | Initial inputs | | | |
| Employees | 15 | 17543 | 19079 | 1241 | 61662 |
| Energy | 15 | 7.38 | 9.08 | 0.20 | 32.49 |
| | | Intermediate products | | | |
| Clinker & Cement | 15 | 66.95 | 80.52 | 3.92 | 272.26 |
| Aggregate | 15 | 44.28 | 61.59 | 0.00 | 188.14 |
| | | Final outputs | | | |
| Profit | 15 | 1001.87 | 2715.43 | -1498.37 | 10430.14 |
| Tax | 15 | 482.40 | 643.82 | 6.80 | 2297.51 |

#### Results

We use the output-oriented model (i.e., model (3.42)) to project all inefficient DMUs to the efficient frontier of production technology $T_S$. The results of projections and efficient scores are shown in the Table B.1. Then we calculate the right-hand and left-hand SE for overall stage and each sub-stage based on Theorem 3, as shown in Table 3.11.

Table 3.11: SE results and decomposition of cement companies

| DMU | $\varepsilon^{(S)+}$ | $\varepsilon^{(S1)+}$ | $\varepsilon^{(S2)+}$ | $\varepsilon^{(S)-}$ | $\varepsilon^{(S1)-}$ | $\varepsilon^{(S2)-}$ |
|---|---|---|---|---|---|---|
| $DMU_1$ | 0.00 | 0.00 | inf. | 0.97 | 1.00 | 0.97 |
| $DMU_2$ | 0.00 | 0.00 | inf. | 0.97 | 1.00 | 0.97 |
| $DMU_3$ | 0.63 | 0.97 | 0.65 | 0.77 | 0.99 | 0.78 |
| $DMU_4$ | 0.00 | 0.00 | inf. | 0.83 | 1.06 | 0.78 |
| $DMU_5$ | 0.50 | 0.89 | 0.56 | 0.86 | 0.91 | 0.94 |
| $DMU_6$ | 0.58 | 0.98 | 0.59 | 0.93 | 0.98 | 0.95 |
| $DMU_7$ | 0.43 | 0.92 | 0.46 | 0.46 | 0.95 | 0.49 |
| $DMU_8$ | 0.06 | 0.67 | 0.08 | 0.57 | 0.87 | 0.65 |
| $DMU_9$ | 0.00 | 0.61 | 0.00 | 1.04 | 1.98 | 0.53 |
| $DMU_{10}$ | 0.27 | 0.97 | 0.28 | 1.00 | 0.97 | 1.04 |
| $DMU_{11}$ | 0.24 | 0.82 | 0.29 | 0.76 | 1.75 | 0.43 |
| $DMU_{12}$ | 0.20 | 1.08 | 0.18 | 0.21 | 1.13 | 0.19 |
| $DMU_{13}$ | 0.19 | 0.81 | 0.23 | 0.25 | 1.10 | 0.23 |
| $DMU_{14}$ | 0.05 | 0.64 | 0.08 | $+\infty$ | $+\infty$ | $+\infty$ |
| $DMU_{15}$ | 0.56 | 0.96 | 0.58 | 0.72 | 0.99 | 0.73 |

From the SE results shown in Table 3.11, we find that all of the cement companies selected in this case study cannot further increase the scale by proportionally increasing inputs since the right-hand overall SE of all DMUs are less than 1, which means decreasing RTS prevails on the right-hand side for all DMUs in the overall stage. In particular, all DMUs are advised to decrease their overall scale, except for DMU9, DMU10, and $DMU_{14}$, which are recommended to maintain their current scale since $\varepsilon^{(S)-} \geq 1$ for these DMUs.

Furthermore, apart from $DMU_1$, $DMU_2$ and $DMU_4$, which cannot further increase intermediate products proportionally, the right-hand SE of other DMUs in stage 1 is always larger than the right-hand SE in stage 2. Given that the SE of the overall stage is the product of SE of each sub-stage, decreasing RTS of the overall stage on the right-hand side results from the second stage (i.e., marketing stage). In other words, if these cement companies choose to expand their scale, under the existing production and marketing technologies, the decline in productivity at the marketing stage would exceed that at the production stage. Therefore, to achieve an increase in production scale with less productivity loss, cement companies must first implement reforms in the marketing stage.

## 3.4 Chapter summary

For the production technologies with two basic network structures (i.e., parallel and series structures), this chapter developed DEA-based methods to estimate and decompose the SE of the whole technology into each sub-technology. In terms of production technologies with general parallel structures, this chapter developed the models to estimate the overall SE and proposed a decomposition method. The overall SE is demonstrated to be a weighted sum of SE in each sub-activity, in which the weight allocated to each activity is automatically generated by the model. For two-stage production technologies, this chapter proposed a method of decomposing overall SE into each stage, in which the decomposition results obtained by the proposed method are unique. In addition, two case studies were carried out to illustrate the proposed method.

The contributions of this chapter are summarized below:

- This chapter developed the method of SE estimation in multi-activity technologies with known proportions of shared indicators and proposing a method of decomposing the SE of the whole production system under the DEA framework.
- This chapter developed the method of SE estimation in multi-stage technologies and proposing a novel method of decomposing SE into different stages under the DEA framework.

# Chapter 4

# Development of Directional SE for Network DEA

## 4.1 Introduction

Considering estimating SE along any directions, DSE was introduced in transformation functions (Balk et al. 2015) and VRS technologies (Yang, Liu, Shen & Li 2014). Generally, to be consistent with classical SE, the sum of directions of changing in inputs and outputs are equal to the number of dimensions, respectively. In particular, the directional vector or inputs means the various options of investing or allocating resources (e.g., see Yang & Liu (2017)), and the directional vector of outputs means various management preferences, which reflect the expected proportions of changes in outputs (e.g., see Ren et al. (2021)). However, the existing research related to DSE definition and estimation was limited to joint production technologies. This chapter develops the notion of DSE into multi-activity and multi-stage production technologies, respectively.

Based on the notion of DSE, the direction vector of inputs can usually be set by DMs, which means that setting the vector of inputs is an essential part of development since different direction settings can cause different results of SE and RTS. Undoubtedly, managers prefer to adjust the scale in a direction that leads to IRS or maximizes SE. Thus, finding the optimal input direction for DMUs can provide important suggestions for DMs. Besides, there is a particular case for the possible results: the value of SE (here, we take the right-hand SE as an example) in all input directions is less than the one for a DMU. That is, there is no direction worth following, or this DMU has no alternative but to reduce or maintain the same scale. In this chapter, the case where the elasticity of scale in the optimal input direction is less

than one will be first defined as the absolute decreasing RTS. Conversely, the least worthwhile direction to follow for a DMU can also be obtained by adjusting the optimal direction. If the value of SE is less than one, what is certain is that at least one direction is unworthy to follow. Therefore, the case where the elasticity of scale in the least worthwhile input direction is less than one will be first defined as the relative decreasing RTS. This chapter will propose precise directions for relative and absolute decreasing RTS through specific DEA models. The definitions can provide the optimal direction of scale change for DMs and evaluate the corporate capabilities of effectively expanding scale to satisfy sustainability. Based on pre-setting specific directions for changes in all inputs, the value of directional SE can be obtained to represent the scale properties in this direction. However, there are innumerable direction portfolios that DMs can discretionarily adjust. Thus, analyzing the characteristics of SE in all possible input directions can not only provide managers with the optimal path of scale change but also indicate whether it is a suitable occasion for initiating and implementing reforms.

The remainder of this chapter is divided into the following sections: Section 4.2 develops the notion of DSE into multi-activity and multi-stage production technologies, respectively. Section 4.3 proposes models to explore optimal directions for increasing the scale of projected DMUs. Section 4.4 carries out two case studies based on the data set in Chapter 3. Section 4.5 is the chapter summary.

## 4.2 Development of DSE in network production technologies

### 4.2.1 DSE in parallel production systems

For general multi-activity technologies $T_p$ defined by (3.14), we consider that all inputs are changed by a factor $t$ along the direction vector $\boldsymbol{\omega}(\boldsymbol{\omega} \geq \mathbf{0})$. We define the corresponding change $\beta(t)$ of all outputs along the direction vector $\boldsymbol{\delta}(\boldsymbol{\delta} \geq \mathbf{0})$:

$$\beta(t) = \max \{\beta \mid ((\boldsymbol{e} + \boldsymbol{\omega} t) \odot X_o, (\boldsymbol{e} + \boldsymbol{\delta}\beta) \odot Y_o) \in T_P, \beta \in \mathbb{R}\}. \tag{4.1}$$

We assume $\beta(t)$ defined by (4.1) is right and left differentiable, and $\beta(0) = 0$ is satisfied. Then we can calculate DSE by the right and left derivative of $\beta(0) = 0$ evaluated at $t = 0$:

$$\varepsilon^+(X_o, Y_o) = \left.\frac{\partial \beta(t)}{\partial t}\right|_{t=0^+}, \varepsilon^-(X_o, Y_o) = \left.\frac{\partial \beta(t)}{\partial t}\right|_{t=0^-}. \tag{4.2}$$

Let $\varepsilon^+(X_o, Y_o)$ and $\varepsilon^-(X_o, Y_o)$ denote the right-hand and left-hand DSE, respectively. Specifically, $\varepsilon^+(X_o, Y_o)$ and $\varepsilon^-(X_o, Y_o)$ represent the largest change of all outputs $Y_o$ in the technology $T_P$ along the directional vector $\boldsymbol{\delta}$ if all inputs $X_o$ in the technology $T_P$ are marginally increased by a small factor $t$ along the directional vector $\boldsymbol{\omega}$. For example, if $\varepsilon^+(X_o, Y_o) = 2$, a 1% increase in all inputs along a specific direction vector $\boldsymbol{\omega}$ leads to a maximum 2% increase in all outputs along a specific direction vector $\boldsymbol{\delta}$. Similarly, if $\varepsilon^-(X_o, Y_o) = 2$, a 1% decrease in all inputs along a specific direction vector $\boldsymbol{\omega}$ leads to a maximum 2% decrease in all outputs along a specific direction vector $\boldsymbol{\delta}$. To be consistent with classic SE, DSE evaluating in single-activity production (i.e., $K = 1$) follows $\sum_{i\in I} \omega_i = |I|$ and $\sum_{r\in O} \delta_r = |O|$ for standardization (e.g., see Yang & Liu (2017)), which implies that the changes in inputs or outputs allow for mutual compensation.

However, in a parallel production system with $K(K \geq 2)$ activities (as shown in Fig. 3.1), the standardization need to be adjusted since the change directions of shared measures allocated to each activity can be different. In line with the specific production structure as shown in Fig. 3.1, we consider that each factor in activity $k \in H$ is assigned a direction for a $\mathrm{DMU}_o \in T_P$. Therefore, what managers need to determine is the change directions of $X_o^{Sk}$, $Y_o^{Sk}$, $X_o^k$ and $Y_o^k$. That is, for a activity $k$ in the technology $T_P$, let $\omega_i^k (i \in I^k)$ and $\delta_r^k (r \in O^k)$ denote the change directions of activity-specific inputs $x_{io}^k$ and outputs $y_{ro}^k$, respectively, let $\omega_i^{Sk} (i \in I^S)$ and $\delta_r^{Sk} (r \in O^S)$ denote the change directions of shared inputs $x_{io}^{Sk}$ and outputs $y_{ro}^{Sk}$ allocated to each activity, respectively. Thus, for the $\mathrm{DMU}_o \in T_P$ in each activity $k \in H$, all inputs and outputs after directional changes are represented by:

$$\tilde{X}_o^{Ak} = ((\boldsymbol{e} + \boldsymbol{\omega}^k t) \odot X_o^k; (\boldsymbol{e} + \boldsymbol{\omega}^{Sk} t) \odot X_o^{Sk})(\forall k),$$
$$\tilde{Y}_o^{Ak} = ((\boldsymbol{e} + \boldsymbol{\delta}^k \beta) \odot Y_o^k; (\boldsymbol{e} + \boldsymbol{\delta}^{Sk} \beta) \odot Y_o^{Sk})(\forall k),$$

where $\boldsymbol{\omega}^k = (\omega_1^k, \ldots, \omega_{|I^k|}^k)^\top (\forall k)$, $\boldsymbol{\omega}^{Sk} = (\omega_1^{Sk}, \ldots, \omega_{|I^S|}^{Sk})^\top (\forall k)$, $\boldsymbol{\delta}^k = (\delta_1^k, \ldots, \delta_{|O^k|}^k)^\top (\forall k)$ and $\boldsymbol{\delta}^{Sk} = (\delta_1^{Sk}, \ldots, \delta_{|O^S|}^{Sk})^\top (\forall k)$.

Let $\tilde{X}_o^{Sk} = (\boldsymbol{e} + \boldsymbol{\omega}^{Sk} t) \odot X_o^{Sk} (\forall k)$ and $\tilde{Y}_o^{Sk} = (\boldsymbol{e} + \boldsymbol{\delta}^{Sk} \beta) \odot Y_o^{Sk} (\forall k)$ hold, the overall shared inputs and outputs after directional changes are represented by:

$$\tilde{X}_o^S = \tilde{X}_o^{S1} + \cdots + \tilde{X}_o^{SK},$$
$$\tilde{Y}_o^S = \tilde{Y}_o^{S1} + \cdots + \tilde{Y}_o^{SK},$$

respectively. With the assumption of known proportion vectors $Q_{io} (\forall i \in I^S)$ and $P_{ro} (\forall r \in O^S)$, the change direction vectors of overall shared inputs and outputs are $\boldsymbol{\omega}^S = (\sum_{k\in H} q_{1o}^k \omega_1^{Sk}, \ldots, \sum_{k\in H} q_{|I^S|o}^k \omega_{|I^S|}^{Sk})^\top$ and $\boldsymbol{\delta}^S =$

$(\sum_{k\in H} p^k_{1o}\delta^{Sk}_1, \ldots, \sum_{k\in H} p^k_{|O^S|o}\delta^{Sk}_{|O^S|})^\top$, respectively. Therefore, the change direction of each shared measure is regarded as the aggregation of the change directions in the parts allocated to each activity.

For the described setting above, we consider three management perspectives, which represents different situations of mutual compensation among changes in inputs or outputs.

- Management perspective 1: the directions of changes in inputs and outputs are completely mutual compensation, respectively. Accordingly, the standardization of directions satisfies:

$$\begin{aligned}\sum_{k\in H}\sum_{i\in I^k}\omega^k_i + \sum_{i\in I^S}\sum_{k\in H}\omega^{Sk}_i &= \sum_{k\in H}|I^k| + K|I^S|,\\ \sum_{k\in H}\sum_{r\in O^k}\delta^k_r + \sum_{r\in O^S}\sum_{k\in H}\delta^{Sk}_r &= \sum_{k\in H}|O^k| + K|O^S|.\end{aligned} \tag{4.3}$$

- Management perspective 2: the direction vectors of each activity are determined independently, that is, the directions of changes in inputs and outputs are mutual compensation only within each activity. Accordingly, the standardization of directions satisfies:

$$\begin{aligned}\sum_{i\in I^k}\omega^k_i + \sum_{i\in I^S}\omega^{Sk}_i &= |I^k| + |I^S|(\forall k),\\ \sum_{r\in O^k}\delta^k_r + \sum_{r\in O^S}\delta^{Sk}_r &= |O^k| + |O^S|(\forall k).\end{aligned} \tag{4.4}$$

  Note that the standardization condition (4.4) is a special case of (4.3). In other words, if condition (4.4) is satisfied, (4.3) must also be satisfied.

- Management perspective 3: the directions of changes in inputs and outputs are mutual compensation for each activity while the directions are the same within each activity. Accordingly, in addition to satisfying the standardization condition (4.3), it is also necessary to satisfy:

$$\omega^k_{i\in I^k} = \omega^{Sk}_{i\in I^S}(\forall k), \delta^k_{r\in O^k} = \delta^{Sk}_{r\in O^S}(\forall k). \tag{4.5}$$

  It is worth noting that if standardization condition (4.4) and (4.5) are satisfied simultaneously, then all directions are equal to 1, which is the same as classic SE.

For illustration, we consider there are one shared input($x^S$) in a parallel production systems including two activities (i.e., $k = 2$), and the two activities also have their own activity-specific inputs ($x^1$ and $x^2$). From the

management perspective 1, 2 and 3, the direction vector $(\omega^1, \omega^2, \omega^{S1}, \omega^{S2})$ can be set as $(1.2, 0.8, 1.6, 0.4)$, $(1.2, 1.6, 0.8, 0.4)$ and $(1.2, 0.8, 1.2, 0.8)$, respectively.

Consequently, we have the direction vectors $\boldsymbol{\omega} = (\boldsymbol{\omega}^1; \ldots; \boldsymbol{\omega}^K; \boldsymbol{\omega}^{S1}; \ldots; \boldsymbol{\omega}^{SK})$ and $\boldsymbol{\delta} = (\boldsymbol{\delta}^1; \ldots; \boldsymbol{\delta}^K; \boldsymbol{\delta}^{S1}; \ldots; \boldsymbol{\delta}^{SK})$ in which the standardization (4.3), or (4.4), or (4.3) with (4.5) is assumed. In line with the specific expression of $T_P$ as shown in (3.14), the directional response function (4.1) becomes a linear program as follows:

$$\begin{aligned}
\beta(t) = \max_{\beta, \lambda_j^k} & \beta \\
\text{s.t.} & \begin{cases}
\sum_{j=1}^n \lambda_j^k x_{ij}^{k*} \le (1 + \omega_i^k t) x_{io}^{k*} \, (\forall k, i \in I^k), \\
\sum_{k=1}^K \sum_{j=1}^n \lambda_j^k x_{ij}^{Sk*} \le (1 + \omega_i^S t) x_{io}^{S*} \, (i \in I^S), \\
\sum_{j=1}^n \lambda_j^k y_{rj}^{k*} \ge (1 + \delta_r^k \beta) y_{ro}^{k*} \, (\forall k, r \in O^k), \\
\sum_{k=1}^K \sum_{j=1}^n \lambda_j^k y_{rj}^{Sk*} \ge (1 + \delta_r^S \beta) y_{ro}^{S*} \, (r \in O^S), \\
\sum_{j=1}^n \lambda_j^k = 1 \, (\forall k), \\
\lambda_j^k \ge 0 \, (\forall j, k), \beta \text{ signfree},
\end{cases}
\end{aligned} \tag{4.6}$$

where $\omega_i^S = \sum_{k \in H} q_{io}^{k*} \omega_i^{Sk} (\forall i \in I^S)$ and $\delta_r^S = \sum_{k \in H} p_{ro}^{k*} \delta_r^{Sk} (\forall r \in O^S)$ hold.

Therefore, similar to Theorem 2, we use the DDO method to estimate DSE and propose the following Theorem:

**Theorem 4.** *Let $(X_o^*, Y_o^*) \in T_P$ be defined by (3.14) and $(X_o^*, Y_o^*)$ be an output-radial efficient point, then the right-hand and left-hand DSE can be obtained by solving the following problems:*

$$\begin{aligned}
\varepsilon^+ = \min_{v_i^k, v_i^S, u_i^k, u_i^S, u_0^k} & \sum_{k \in H} \sum_{i \in I^k} v_i^k \omega_i^k x_{io}^{k*} + \sum_{i \in I^S} v_i^S \omega_i^S x_{io}^{S*} \\
s.t. & \begin{cases}
\sum_{k \in H} \sum_{i \in I^k} v_i^k x_{io}^{k*} + \sum_{i \in I^S} v_i^S x_{io}^{S*} - \sum_{k \in H} \sum_{r \in O^k} u_r^k y_{ro}^{k*} \\
- \sum_{r \in O^S} u_r^S y_{ro}^{S*} + \sum_{k \in H} u_0^k = 0, \\
\sum_{k \in H} \sum_{r \in O^k} u_r^k \delta_r^k y_{ro}^{k*} + \sum_{r \in O^S} u_r^S \delta_r^S y_{ro}^{S*} = 1, \\
- \sum_{i \in I^k} v_i^k x_{ij}^{k*} - \sum_{i \in I^S} v_i^S x_{ij}^{S*} + \sum_{r \in O^k} u_r^k y_{rj}^{k*} \\
+ \sum_{r \in O^S} u_r^S y_{rj}^{S*} - u_0^k \le 0 \, (\forall j, k), \\
v_i^k, v_i^S, u_i^k, u_i^S \ge 0 \, (\forall k), u_0^k \text{ signfree } (\forall k),
\end{cases}
\end{aligned} \tag{4.7}$$

$$\begin{aligned}
\varepsilon^- = \max_{v_i^k, v_i^S, u_i^k, u_i^S, u_0^k} & \sum_{k \in H} \sum_{i \in I^k} v_i^k \omega_i^k x_{io}^{k*} + \sum_{i \in I^S} v_i^S \omega_i^S x_{io}^{S*} \\
& \text{s.t. All constrains in model (4.7).}
\end{aligned} \tag{4.8}$$

The proof of Theorem 4 is similar with the proof of Theorem 2. Accordingly, directional RTS is defined based on the value of $\boldsymbol{\varepsilon}^+$ and $\boldsymbol{\varepsilon}^-$ on the $\mathrm{DMU}_o^*(X_o^*, Y_o^*)$:

**Definition 4.1.** *$DMU_o^*(X_o^*, Y_o^*)$ exhibits*

- *Increasing RTS on the right-hand(or left-hand) side if $\boldsymbol{\varepsilon}^+(X_o^*, Y_o^*) > 0$(or $\boldsymbol{\varepsilon}^-(X_o^*, Y_o^*) > 0$) in the direction of $\boldsymbol{\omega}$ and $\boldsymbol{\delta}$.*
- *Decreasing RTS on the right-hand(or left-hand) side if $\boldsymbol{\varepsilon}^+(X_o^*, Y_o^*) < 0$(or $\boldsymbol{\varepsilon}^-(X_o^*, Y_o^*) < 0$) in the direction of $\boldsymbol{\omega}$ and $\boldsymbol{\delta}$.*
- *Constant RTS on the right-hand(or left-hand) side if $\boldsymbol{\varepsilon}^+(X_o^*, Y_o^*) = 0$(or $\boldsymbol{\varepsilon}^-(X_o^*, Y_o^*) = 0$) in the direction of $\boldsymbol{\omega}$ and $\boldsymbol{\delta}$.*

Similar to the method of decomposing DSE proposed in section 3.2.3, DSE can also be decomposed in multi-activity technologies (3.14) as follows.

- For the right-hand DSE of an efficient $\mathrm{DMU}_o(X_o^*, Y_o^*)$, we have:

$$\begin{aligned} \varepsilon^{k+} &= \frac{\sum_{i\in I^k} v_i^k \omega_i^k x_{io}^{k*} + \sum_{i\in I^S} v_i^S \omega_i^{Sk} x_{io}^{Sk*}}{\sum_{r\in O^k} u_r^k \delta_r^k y_{ro}^{k*} + \sum_{r\in O^S} u_r^S \delta_r^{Sk} y_{ro}^{Sk*}} (\forall k), \\ \varepsilon^{+} &= \sum_{k\in H} \varepsilon^{k+} \Big( \sum_{r\in O^k} u_r^{k*} \delta_r^k y_{ro}^{k*} + \sum_{r\in O^S} u_r^{S*} \delta_r^S y_{ro}^{Sk*} \Big), \end{aligned} \tag{4.9}$$

  where $u_r^{k*}, u_r^{S*}$ and $u_0^{k*}$ are optimal values obtained by model (4.7).

- For the left-hand DSE, we have:

$$\begin{aligned} \varepsilon^{k-} &= \frac{\sum_{i\in I^k} v_i^k \omega_i^k x_{io}^{k*} + \sum_{i\in I^S} v_i^S \omega_i^{Sk} x_{io}^{Sk*}}{\sum_{r\in O^k} u_r^k \delta_r^k y_{ro}^{k*} + \sum_{r\in O^S} u_r^S \delta_r^{Sk} y_{ro}^{Sk*}} (\forall k), \\ \varepsilon^{-} &= \sum_{k\in H} \varepsilon^{k-} \Big( \sum_{r\in O^k} u_r^{k*} \delta_r^k y_{ro}^{k*} + \sum_{r\in O^S} u_r^{S*} \delta_r^S y_{ro}^{Sk*} \Big), \end{aligned} \tag{4.10}$$

where $u_r^{k*}, u_r^{S*}$ and $u_0^{k*}$ are optimal values obtained by model (4.8).

In particular, assume that only the scale of one activity (i.e., $k = k_0$) can be changed in the short term, while the scale of the other activities only can remain the same in the short term. The notion of partial DSE is developed into multi-activity production technologies. Partial DSE represents the response of $Y^{k_0}$ and $Y^{Sk_0}$ along the direction $(\boldsymbol{\delta}^{k_0}, \boldsymbol{\delta}^{Sk_0})^\top$ to marginal changes of $X^{k_0}$ and $X^{Sk_0}$ along the direction $(\boldsymbol{\omega}^{k_0}, \boldsymbol{\omega}^{Sk_0})^\top$, while keeping $X^k$, $X^{Sk}$, $Y^k$

and $Y^{Sk}$ unchanged for $k \neq k_0$. We first define the partial response function in the direction as follows:

$$\beta^{k_0}(t) = \max\left\{\beta \mid \left((\boldsymbol{e} + \boldsymbol{\omega}^{Ak_0} t) \odot X_o, (\boldsymbol{e} + \boldsymbol{\delta}^{Ak_0}\beta) \odot Y_o\right) \in T_P, \beta \in \mathbb{R}\right\}, \quad (4.11)$$

where $\beta^{k_0}(0) = 0$ is assumed, $\boldsymbol{\omega}^{Ak_0} = (\mathbf{0}; \ldots; \mathbf{0}; \boldsymbol{\omega}^{k_0}; \mathbf{0}; \ldots; \mathbf{0}; \mathbf{0}; \ldots; \mathbf{0}; \boldsymbol{\omega}^{Sk_0}; \mathbf{0}; \ldots; \mathbf{0})^\top$ and $\boldsymbol{\delta}^{Ak_0} = (\mathbf{0}; \ldots; \mathbf{0}; \boldsymbol{\delta}^{k_0}; \mathbf{0}; \ldots; \mathbf{0}; \mathbf{0}; \ldots; \mathbf{0}; \boldsymbol{\delta}^{Sk_0}; \mathbf{0}; \ldots; \mathbf{0})^\top$ hold.

For the purpose of standardization, we have $\sum_{i\in I^{k_0}} \omega_i^{k_0} + \sum_{i\in I^S} \omega_i^{Sk_0} = |I^{k_0}| + |I^S|$ and $\sum_{r\in O^{k_0}} \delta_r^{k_0} + \sum_{r\in O^S} \delta_r^{Sk_0} = |O^{k_0}| + |O^S|$. In line with the specific expression of $T_P$ as shown in (3.14), the directional response function (4.1) becomes a linear program as follows:

$$\begin{aligned}
\beta^{k_0}(t) = \max_{\beta^{k_0}, \lambda_j^k} \; & \beta^{k_0} \\
\text{s.t.} \; & \begin{cases}
\sum_{j=1}^n \lambda_j^k x_{ij}^{k*} \leq (1 + \omega_i^{k_0} t) x_{io}^{k_0*} \; (i \in I^{k_0}), \\
\sum_{j=1}^n \lambda_j^k x_{ij}^{k*} \leq x_{io}^{k*} \; (k \neq k_0, i \in I^k), \\
\sum_{k=1}^K \sum_{j=1}^n \lambda_j^k x_{ij}^{Sk*} \leq (1 + \omega_i^{Sk_0} q_{io}^{k_0} t) x_{io}^{S*} \; (i \in I^S), \\
\sum_{j=1}^n \lambda_j^k y_{rj}^{k*} \geq (1 + \delta_r^{k_0} \beta^{k_0}) y_{ro}^{k_0*} \; (r \in O^{k_0}), \\
\sum_{j=1}^n \lambda_j^k y_{rj}^{k*} \geq y_{ro}^{k*} \; (k \neq k_0, r \in O^k), \\
\sum_{k=1}^K \sum_{j=1}^n \lambda_j^k y_{rj}^{Sk*} \geq (1 + \delta_r^{Sk_0} p_{io}^{k_0} \beta^{k_0}) y_{ro}^{S*} \; (r \in O^S), \\
\sum_{j=1}^n \lambda_j^k = 1 \; (\forall k), \\
\lambda_j^k \geq 0 \; (\forall j, k), \beta^{k_0} \text{ signfree.}
\end{cases}
\end{aligned} \quad (4.12)$$

Therefore, we use the DDO method to estimate DSE and propose the following Theorem:

**Theorem 5.** *Let $(X_o^*, Y_o^*) \in T_P$ be defined by (3.14) and $(X_o^*, Y_o^*)$ be an output-radial efficient point, then the right-hand and left-hand partial DSE can be obtained by solving the following problems:*

$$\begin{aligned}
\varepsilon_{k_0}^+ = \min_{v_i^k, v_i^S, u_i^k, u_i^S, u_0^k} \; & \sum_{i\in I^k} v_i^k \omega_i^{k_0} x_{io}^{k_0*} + \sum_{i\in I^S} v_i^S \omega_i^{Sk_0} x_{io}^{Sk_0*} \\
s.t. \; & \begin{cases}
\sum_{k\in H} \sum_{i\in I^k} v_i^k x_{io}^{k*} + \sum_{i\in I^S} v_i^S x_{io}^{S*} - \sum_{k\in H} \sum_{r\in O^k} u_r^k y_{ro}^{k*} \\
- \sum_{r\in O^S} u_r^S y_{ro}^{S*} + \sum_{k\in H} u_0^k = 0, \\
\sum_{r\in O^k} u_r^k \delta_r^{k_0} y_{ro}^{k_0*} + \sum_{r\in O^S} u_r^S \delta_r^{Sk_0} y_{ro}^{Sk_0*} = 1, \\
- \sum_{i\in I^k} v_i^k x_{ij}^{k*} - \sum_{i\in I^S} v_i^S x_{ij}^{S*} + \sum_{r\in O^k} u_r^k y_{rj}^{k*} \\
+ \sum_{r\in O^S} u_r^S y_{rj}^{S*} - u_0^k \leq 0 \; (\forall j, k), \\
v_i^k, v_i^S, u_i^k, u_i^S \geq 0 \; (\forall k), u_0^k \; \textit{signfree} \; (\forall k),
\end{cases}
\end{aligned} \quad (4.13)$$

$$\varepsilon_{k_0}^{-} = \max_{v_i^k, v_i^S, u_i^k, u_i^S, u_0^k} \sum_{i \in I^k} v_i^k \omega_i^{k_0} x_{io}^{k_0 *} + \sum_{i \in I^S} v_i^S \omega_i^{Sk_0} x_{io}^{Sk_0 *} \tag{4.14}$$
$$s.t.\ \textit{All constrains in model}\ (4.13).$$

The proof of Theorem 5 is similar with the proof of Theorem 2. It is worth noting that the purpose of estimating the partial DSE is different from decomposing DSE, in which the partial DSE shows the directional changes in the outputs of a certain activity in response to the marginal directional changes in inputs of this activity, while keeping inputs and outputs in the other activities unchanged, and the decomposition of DSE explores the DSE of each activity on condition that marginal directional changes in all the inputs result in directional changes in all the outputs.

### 4.2.2 DSE in series production systems

For the two-stage series production technologies $T_S$ defined by (3.41), we define the change directions of inputs, intermediate products and outputs as $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_{|I|})^\top (\boldsymbol{\omega} > \mathbf{0})$, $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_{|C|})^\top (\boldsymbol{\alpha} > \mathbf{0})$ and $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_{|O|})^\top$ $(\boldsymbol{\delta} > \mathbf{0})$, respectively. In addition, we let $\sum_{i \in |I|} \omega_i = |I|$, $\sum_{i \in |I|} \alpha_f = |C|$ and $\sum_{r \in |O|} \delta_r = |O|$ hold for standardization. Especially, the directions of intermediate products are first considered and represent adjustable production structures. To be consistent with the models without direction vector for intermediate products, it is necessary to ensure all the increments of intermediate output in the first stage are invested in the second stage, here we assume that the direction vector of the intermediate products in the first stage is the same as the direction vector in the second stage. Then, for all efficient $\mathrm{DMU}_o^*$, the directional response function of final outputs is defined as follows:

$$\begin{aligned} &\beta(\varphi(t)) \\ &= \max \left\{ \beta \mid \left( (\boldsymbol{e} + \boldsymbol{\omega} t) \odot X_o^*, (\boldsymbol{e} + \boldsymbol{\alpha}\varphi) \odot Z_o^*, (\boldsymbol{e} + \boldsymbol{\delta}\beta) \odot Y_o^* \right) \in T_S, \varphi, \beta \in \mathbb{R} \right\}, \end{aligned} \tag{4.15}$$

where $\beta(\varphi(0)) = 0$ are assumed to indicate that $Y_o^*$ cannot be changed within $T_S$ on condition that $X_o^*$ keeps unchanged.

Apparently, introducing direction vectors does not affect the Eq. (3.46). To estimate DSE for efficient $\mathrm{DMU}_o^*(X_o^*, Z_o^*, Y_o^*) \in T_S$ in the overall system, the first stage and the second stage, $\beta(t)$, $\varphi(t)$ and $\beta(\varphi)$ are expressed as the

following linear programs, respectively:

$$\beta(t) = \max_{\varphi,\beta,\lambda_j^{(1)},\lambda_j^{(2)}} \beta$$

$$\text{s.t.} \begin{cases} \sum_{j=1}^{n} \lambda_j^{(1)} x_{ij}^* \le (1+\omega_i t) x_{io}^* \ (\forall i), \\ \sum_{j=1}^{n} \lambda_j^{(1)} z_{fj}^* \ge (1+\alpha_f \varphi) z_{fo}^* \ (\forall f), \\ \sum_{j=1}^{n} \lambda_j^{(2)} z_{fj}^* \le (1+\alpha_f \varphi) z_{fo}^* \ (\forall f), \\ \sum_{j=1}^{n} \lambda_j^{(2)} y_{rj}^* \ge (1+\delta_r \beta) y_{ro}^* \ (\forall r), \\ \sum_{j=1}^{n} \lambda_j^{(1)} = 1, \sum_{j=1}^{n} \lambda_j^{(2)} = 1, \\ \lambda_j^{(1)}, \lambda_j^{(2)} \ge 0 \ (\forall j), \varphi, \beta \text{ signfree}, \end{cases} \tag{4.16}$$

$$\varphi(t) = \max_{\varphi,\beta,\lambda_j^{(1)},\lambda_j^{(2)}} \varphi \quad \text{s.t. all constrains in model (4.16)}, \tag{4.17}$$

$$\beta(\varphi) = \max_{t,\beta,\lambda_j^{(1)},\lambda_j^{(2)}} \beta \quad \text{s.t. all constraints in model (4.16), where } \varphi \text{ is converted into a scalar, } t \text{ is converted into an unknown variable satisfying } t \ge 0. \tag{4.18}$$

Similarly, we use the DDO method to estimate DSE for each sub-stage and the overall system in the following Theorem:

**Theorem 6.** *Let $(X_o^*, Z_o^*, Y_o^*) \in T_S$ be defined by (3.41) and $(X_o^*, Z_o^*, Y_o^*)$ be an output-radial efficient point, then the right-hand DSE in the overall system, the first stage and the second stage can be obtained by solving the following problems, respectively:*

$$\varepsilon^{(S)+} = \min_{v_i,\tau_f^{(1)},\tau_f^{(2)},u_r,u_{10},u_{20}} \sum_{i=1}^{m} v_i \omega_i x_{io}^*$$

$$s.t. \begin{cases} \sum_{i=1}^{m} v_i x_{io}^* - \sum_{f=1}^{h} \tau_f^{(1)} z_{fo}^* + \sum_{f=1}^{h} \tau_f^{(2)} z_{fo}^* \\ - \sum_{r=1}^{s} u_r y_{ro}^* + u_{10} + u_{20} = 0, \\ \sum_{i=1}^{m} v_i x_{ij}^* - \sum_{f=1}^{h} \tau_f^{(1)} z_{fj}^* + u_{10} \ge 0 \ (\forall j), \\ \sum_{f=1}^{h} \tau_f^{(2)} z_{fj}^* - \sum_{r=1}^{s} u_r y_{rj}^* + u_{20} \ge 0 \ (\forall j), \\ \sum_{f=1}^{h} \tau_f^{(1)} \alpha_f z_{fo}^* - \sum_{f=1}^{h} \tau_f^{(2)} \alpha_f z_{fo}^* = 0, \\ \sum_{r=1}^{s} u_r \delta_r y_{ro}^* = 1, \\ v_i, u_r, \tau_f^{(1)}, \tau_f^{(2)} \ge 0 \ (\forall i, r, k), u_{10}, u_{20} \ \textit{signfree}, \end{cases} \tag{4.19}$$

$$\varepsilon^{(S1)+} = \min_{v_i,\tau_f^{(1)},\tau_f^{(2)},u_r,u_{10},u_{20}} \sum_{i=1}^{m} v_i \omega_i x_{io}^*$$
$$s.t. \begin{cases} \text{The first } 1+2*n \text{ constrains in model (4.19)}, \\ \sum_{f=1}^{h} \tau_f^{(1)} \alpha_f z_{fo}^* - \sum_{f=1}^{h} \tau_f^{(2)} \alpha_f z_{fo}^* = 1, \\ \sum_{r=1}^{s} u_r \delta_r y_{ro}^* = 0, \\ v_i, u_r, \tau_f^{(1)}, \tau_f^{(2)} \geq 0\,(\forall i, r, k), u_{10}, u_{20} \text{ signfree}, \end{cases} \tag{4.20}$$

$$\varepsilon^{(S2)+} = \min_{v_i,\tau_f^{(1)},\tau_f^{(2)},u_r,u_{10},u_{20}} -\sum_{f=1}^{h} \tau_f^{(1)} \alpha_f z_{fo}^* + \sum_{f=1}^{h} \tau_f^{(2)} \alpha_f z_{fo}^*$$
$$s.t. \begin{cases} \text{The first } 1+2*n \text{ constrains in model (4.19)}, \\ -\sum_{i=1}^{m} v_i \omega_i x_{io}^* = 0, \\ \sum_{r=1}^{s} u_r \delta_r y_{ro}^* = 1, \\ v_i, u_r, \tau_f^{(1)}, \tau_f^{(2)} \geq 0\,(\forall i, r, k), u_{10}, u_{20} \text{ signfree}. \end{cases} \tag{4.21}$$

*The left-hand DSE in the overall system, the first stage and the second stage are estimated by the following models, respectively:*

$$\varepsilon^{(S)-} = \max_{v_i,\tau_f^{(1)},\tau_f^{(2)},u_r,u_{10},u_{20}} \sum_{i=1}^{m} \omega_i v_i x_{io}^* \tag{4.22}$$
$$s.t. \text{ all constrains in model (4.19)},$$

$$\varepsilon^{(S1)-} = \max_{v_i,\tau_f^{(1)},\tau_f^{(2)},u_r,u_{10},u_{20}} \sum_{i=1}^{m} \omega_i v_i x_{io}^* \tag{4.23}$$
$$s.t. \text{ all constrains in model (4.20)},$$

$$\varepsilon^{(S2)-} = \max_{v_i,\tau_f^{(1)},\tau_f^{(2)},u_r,u_{10},u_{20}} -\sum_{f=1}^{h} \tau_f^{(1)} \alpha_f z_{fo}^* + \sum_{f=1}^{h} \tau_f^{(2)} \alpha_f z_{fo}^* \tag{4.24}$$
$$s.t. \text{ all constrains in model (4.21)}.$$

The proof of Theorem 6 is similar with the proof of Theorem 2. Accordingly, RTS of the overall system and each sub-system is defined as follows:

**Definition 4.2.** *In the overall production system, $DMU_o(X_o^*, Z_o^*, Y_o^*)$ exhibits*

- *Increasing RTS on the right-hand(or left-hand) side if $\varepsilon^{(S)+} > 0$ (or $\varepsilon^{(S)-} > 0$) in the direction of $\boldsymbol{\omega}$, $\boldsymbol{\alpha}$ and $\boldsymbol{\delta}$.*

- *Decreasing RTS on the right-hand(or left-hand) side if* $\varepsilon^{(S)+} < 0$ *(or* $\varepsilon^{(S)-} < 0$*) in the direction of* $\boldsymbol{\omega}$, $\boldsymbol{\alpha}$ *and* $\boldsymbol{\delta}$.

- *Constant RTS on the right-hand(or left-hand) side if* $\varepsilon^{(S)+} = 0$ *(or* $\varepsilon^{(S)-} = 0$*) in the direction of* $\boldsymbol{\omega}$, $\boldsymbol{\alpha}$ *and* $\boldsymbol{\delta}$.

*In the first stage production system,* $DMU_o(X_o^*, Z_o^*, Y_o^*)$ *exhibits*

- *Increasing RTS on the right-hand(or left-hand) side if* $\varepsilon^{(S1)+} > 0$ *(or* $\varepsilon^{(S1)-} > 0$*) in the direction of* $\boldsymbol{\omega}$, $\boldsymbol{\alpha}$ *and* $\boldsymbol{\delta}$.

- *Decreasing RTS on the right-hand(or left-hand) side if* $\varepsilon^{(S1)+} < 0$ *(or* $\varepsilon^{(S1)-} < 0$*) in the direction of* $\boldsymbol{\omega}$, $\boldsymbol{\alpha}$ *and* $\boldsymbol{\delta}$.

- *Constant RTS on the right-hand(or left-hand) side if* $\varepsilon^{(S1)+} = 0$ *(or* $\varepsilon^{(S1)-} = 0$*) in the direction of* $\boldsymbol{\omega}$, $\boldsymbol{\alpha}$ *and* $\boldsymbol{\delta}$.

*In the second stage production system,* $DMU_o(X_o^*, Z_o^*, Y_o^*)$ *exhibits*

- *Increasing RTS on the right-hand(or left-hand) side if* $\varepsilon^{(S2)+} > 0$ *(or* $\varepsilon^{(S2)-} > 0$*) in the direction of* $\boldsymbol{\omega}$, $\boldsymbol{\alpha}$ *and* $\boldsymbol{\delta}$.

- *Decreasing RTS on the right-hand(or left-hand) side if* $\varepsilon^{(S2)+} < 0$ *(or* $\varepsilon^{(S2)-} < 0$*) in the direction of* $\boldsymbol{\omega}$, $\boldsymbol{\alpha}$ *and* $\boldsymbol{\delta}$.

- *Constant RTS on the right-hand(or left-hand) side if* $\varepsilon^{(S2)+} = 0$ *(or* $\varepsilon^{(S2)-} = 0$*) in the direction of* $\boldsymbol{\omega}$, $\boldsymbol{\alpha}$ *and* $\boldsymbol{\delta}$.

## 4.3 Exploring optimal directions for increasing the scale

### 4.3.1 DSE with the optimal input direction in multi-activity production technologies

Scale reduction is usually an unwilling and situation-based response to adverse conditions (Datta et al. 2010). Managers are often motivated to pursue scale expansion due to the competitive advantages it provides. Additionally, managerial incentives encourage scale expansion because larger scale often means greater personal benefits and influence. Therefore, we consider a situation that managers are concerned about adjusting the proportions of increases in inputs to increase the scale in the most effective way. In specific, under a certain management preference (i.e., the change directions of outputs are determined, or $\boldsymbol{\delta}^O$ is decided), the managers need to explore the optimal

input directions from one of the three management perspectives so that the right-hand DSE (i.e., $\varepsilon^{+}(X_o^*, Y_o^*)$) reaches its maximum. Therefore, the input directions $\omega_i^k$ and $\omega_i^{Sk}$ become variables in model (4.7). For example, from the management perspective 1 as shown in Eq. (4.3), a bi-level programming model is formulated to find the optimal input directions:

$$\begin{aligned}
\varepsilon^{+}_{\max}(X_o^*, Y_o^*) = & \max_{\omega_i^k, \omega_i^{Sk}, v_i^k, v_i^S, u_i^k, u_i^S, u_0^k} \sum_{k\in H}\sum_{i\in I^k} v_i^k \omega_i^k x_{io}^{k*} + \sum_{i\in I^S} v_i^S \omega_i^S x_{io}^{S*} \\
\text{s.t.} \quad & \sum_{k\in H}\sum_{i\in I^k} \omega_i^k + \sum_{i\in I^S}\sum_{k\in H} \omega_i^{Sk} = \sum_{k\in H} |I^k| + K|I^S|, \omega_{i\in I^k}^k, \omega_{i\in I^S}^{Sk} \geq 0 (\forall k). \\
& \min_{v_i^k, v_i^S, u_i^k, u_i^S, u_0^k} \sum_{k\in H}\sum_{i\in I^k} v_i^k \omega_i^k x_{io}^{k*} + \sum_{i\in I^S} v_i^S \omega_i^S x_{io}^{S*} \\
& \text{s.t. All constrains in model (4.7).}
\end{aligned} \tag{4.25}$$

Note that the constrains in upper level is the input standardization condition (4.3). It is difficult to solve model (4.25) since it is a nonlinear and non-convex model. Nevertheless, we can solve a series of linear programs by varying $\boldsymbol{\delta}^k$ and $\boldsymbol{\delta}^{Sk}$ based on the standardization conditions and a fixed step-length (e.g., 0.1). Therefore, solving model (4.25) is converted to solving linear model (4.7) with a series of feasible directions satisfying Eq. (4.3). Then, by comparing all the optimal solutions of model (4.7), the largest one is the global optimal solution of model (4.25). Similarly, the constrains in the upper level can be replaced with input standardization condition (4.4) or (4.5) with (4.3) under the management perspective 2 or 3, respectively. If there is at least one input direction vector that makes $\varepsilon^{+}(X_o^*, Y_o^*) \geq 1$, we can say the $\text{DMU}_o(X_o, Y_o)$ have at least one path to further increase the scale at its projection. If not, we claim that the evaluated DMUs need to carry out reforms if the managers of them pursue scale expansion. Let the optimal solution of model (4.25) be $\varepsilon^{+*}_{\max}$, we have the following definition of absolute decreasing RTS to define the projected DMUs that are not suggested to further increase the scale.

**Definition 4.3.** *If $\varepsilon^{+*}_{max} < 1$ holds, $DMU_o^*(X_o^*, Y_o^*)$ is of absolute decreasing RTS on the right-hand side.*

**Remark 4.1.** *Based on the application of FDM in calculating SE, the optimal input directions and corresponding DSE can be calculated by the following*

*model*[1]*:*

$$\varepsilon^{+}_{max}(X^*_o, Y^*_o) = \max_{\omega^k_i, \omega^{Sk}_i} \max_{\beta, \lambda^k_j} \beta/t$$

$$s.t. \begin{cases} \sum_{j=1}^{n} \lambda^k_j x^{k*}_{ij} \le (1+\omega^k_i t) x^{k*}_{io} \, (\forall k, i \in I^k), \\ \sum_{k=1}^{K} \sum_{j=1}^{n} \lambda^k_j x^{Sk*}_{ij} \le (1+\sum_{k\in H} q^{k*}_{io} \omega^{Sk}_i t) x^{S*}_{io} \, (i \in I^S), \\ \sum_{j=1}^{n} \lambda^k_j y^{k*}_{rj} \ge (1+\delta^k_r \beta) y^{k*}_{ro} \, (\forall k, r \in O^k), \\ \sum_{k=1}^{K} \sum_{j=1}^{n} \lambda^k_j y^{Sk*}_{rj} \ge (1+\sum_{k\in H} p^{k*}_{ro} \delta^{Sk}_r \beta) y^{S*}_{ro} \, (r \in O^S), \\ \sum_{j=1}^{n} \lambda^k_j = 1 \, (\forall k), \\ \sum_{k\in H} \sum_{i\in I^k} \omega^k_i + \sum_{i\in I^S} \sum_{k\in H} \omega^{Sk}_i = \sum_{k\in H} |I^k| + K|I^S|, \\ \lambda^k_j, \omega^k_{i\in I^k}, \omega^{Sk}_{i\in I^S} \ge 0 \, (\forall j, k), \beta \ \textit{signfree}, \end{cases} \tag{4.26}$$

*where $t$ is a sufficiently small number.*

*Note that model* (4.26) *is a linear model, thereby the optimal solution $(\omega^k_i, \omega^{Sk}_i)$ and $\beta^*/t$ are the optimal input direction vector and corresponding DSE. However, for keeping the accuracy of calculating DSE, we only discuss the DSE results calculated by the DDO method in this chapter.*

As an extension, to obtain the minimum value of DSE on the right-hand side for $\mathrm{DMU}^*_o(X^*_o, Y^*_o)$ in a certain input direction, similarly with model (4.25), the bi-level programming model is formulated as follows:

$$\begin{aligned} \varepsilon^{+}_{\min}(X^*_o, Y^*_o) = & \min_{\omega^k_i, \omega^{Sk}_i, v^k_i, v^S_i, u^k_i, u^S_i, u^k_0} \sum_{k\in H} \sum_{i\in I^k} v^k_i \omega^k_i x^{k*}_{io} + \sum_{i\in I^S} v^S_i \omega^S_i x^{S*}_{io} \\ \text{s.t.} \ & \sum_{k\in H} \sum_{i\in I^k} \omega^k_i + \sum_{i\in I^S} \sum_{k\in H} \omega^{Sk}_i = \sum_{k\in H} |I^k| + K|I^S|, \omega^k_{i\in I^k}, \omega^{Sk}_{i\in I^S} \ge 0 (\forall k). \\ & \min_{v^k_i, v^S_i, u^k_i, u^S_i, u^k_0} \sum_{k\in H} \sum_{i\in I^k} v^k_i \omega^k_i x^{k*}_{io} + \sum_{i\in I^S} v^S_i \omega^S_i x^{S*}_{io} \\ & \text{s.t. All constrains in model (4.7).} \end{aligned} \tag{4.27}$$

Obviously, the objective functions of upper-level and lower-level programming are the same as the one in model (4.27), thus, it can be reformulated

[1]Regarding the proof of $\max_x \max_y f(x,y) = \max_{(x,y)} f(x,y)$, see Lemma 3.1 in Shen et al. (2018)

as a single-level programming directly as the following model:

$$\varepsilon^{+}_{\min}(X^*_o, Y^*_o) = \min_{\omega^k_i,\omega^{Sk}_i,v^k_i,v^S_i,u^k_i,u^S_i,u^k_0} \sum_{k\in H}\sum_{i\in I^k} v^k_i \omega^k_i x^{k*}_{io} + \sum_{i\in I^S} v^S_i \omega^S_i x^{S*}_{io}$$

$$\text{s.t.} \begin{cases} \sum_{k\in H}\sum_{i\in I^k} \omega^k_i + \sum_{i\in I^S}\sum_{k\in H} \omega^{Sk}_i = \sum_{k\in H}|I^k| + K|I^S|, \\ \omega^k_{i\in I^k}, \omega^{Sk}_{i\in I^S} \geq 0(\forall k), \\ \text{All constrains in model (4.7).} \end{cases} \tag{4.28}$$

Hereafter, let the optimal solution of model (4.28) be $\varepsilon^{+*}_{\min}$. Evidently, if $\varepsilon^{+*}_{\min} < 1$, then there is at least one direction causing decreasing RTS for $\text{DMU}^*_o(X^*_o, Y^*_o)$ on the right-hand. Therefore, we have the following definition to define relative decreasing RTS.

**Definition 4.4.** *If $\varepsilon^{+*}_{min} < 1$ holds, $DMU^*_o(X^*_o, Y^*_o)$ is of relative decreasing RTS on the right-hand side.*

It is worth noting that absolute decreasing RTS is a special situation of relative decreasing RTS, because absolute decreasing RTS indicates that the scale status of $\text{DMU}^*_o(X^*_o, Y^*_o)$ for all possible input directions is decreasing RTS while relative decreasing RTS only indicates the possibility of existence of an input direction to cause decreasing RTS. That is, if $\text{DMU}^*_o(X^*_o, Y^*_o)$ is of absolute decreasing RTS, then it is of relative decreasing RTS. By observing the number of DMUs in the status of relative decreasing RTS and absolute decreasing RTS, the RTS situation of the whole DMU set to be evaluated can be grasped. For example, if most of the DMUs are determined as absolute decreasing RTS on the right-hand, then, for this DMU set (e.g., a certain industry), it is hard to increase the overall scale.

### 4.3.2 DSE with the optimal of inputs and intermediate products in multi-stage production technologies

In the two-stage series production technologies $T_S$, the directions of intermediate products are first introduced to represent the adjustment of production structures of the intermediate products in the subsection 4.2.2. Therefore, this subsection focuses on exploring the optimal directions of inputs and intermediate products under a fixed management preference (i.e., $\boldsymbol{\delta}$ is fixed). Given the models (4.19), (4.20) and (4.21), we have the following bi-level programs to find the optimal directions that make $\varepsilon^{(S)+}$, $\varepsilon^{(S1)+}$ and $\varepsilon^{(S2)+}$

reach their maximum values, respectively:

$$\begin{aligned}
\varepsilon_{\max}^{(N)+}(X_o^*, Z_o^*, Y_o^*) = & \max_{v_i,\tau_f^{(1)},\tau_f^{(2)},u_r,u_{10},u_{20},\omega_i,\alpha_f} \sum_{i=1}^{m} v_i\omega_i x_{io}^* \\
\text{s.t.} \quad & \sum_{i\in I}\omega_i = |I|, \sum_{f\in C}\alpha_f = |C|, \omega_{i\in I}, \alpha_{f\in C} \geq 0. \\
& \min_{v_i,\tau_f^{(1)},\tau_f^{(2)},u_r,u_{10},u_{20}} \sum_{i=1}^{m} v_i\omega_i x_{io}^* \\
& \text{s.t. All constrains in model (4.19),}
\end{aligned} \tag{4.29}$$

$$\begin{aligned}
\varepsilon_{\max}^{(1)+}(X_o^*, Z_o^*, Y_o^*) = & \max_{v_i,\tau_f^{(1)},\tau_f^{(2)},u_r,u_{10},u_{20},\omega_i,\alpha_f} \sum_{i=1}^{m} v_i\omega_i x_{io}^* \\
\text{s.t.} \quad & \sum_{i\in I}\omega_i = |I|, \sum_{f\in C}\alpha_f = |C|, \omega_{i\in I}, \alpha_{f\in C} \geq 0. \\
& \min_{v_i,\tau_f^{(1)},\tau_f^{(2)},u_r,u_{10},u_{20}} \sum_{i=1}^{m} v_i\omega_i x_{io}^* \\
& \text{s.t. All constrains in model (4.20),}
\end{aligned} \tag{4.30}$$

$$\begin{aligned}
\varepsilon_{\max}^{(2)+}(X_o^*, Z_o^*, Y_o^*) = & \min_{v_i,\tau_f^{(1)},\tau_f^{(2)},u_r,u_{10},u_{20}} -\sum_{f=1}^{h}\tau_f^{(1)}\alpha_f z_{fo}^* + \sum_{f=1}^{h}\tau_f^{(2)}\alpha_f z_{fo}^* \\
\text{s.t.} \quad & \sum_{i\in I}\omega_i = |I|, \sum_{f\in C}\alpha_f = |C|, \omega_{i\in I}, \alpha_{f\in C} \geq 0. \\
& \min_{v_i,\tau_f^{(1)},\tau_f^{(2)},u_r,u_{10},u_{20}} -\sum_{f=1}^{h}\tau_f^{(1)}\alpha_f z_{fo}^* + \sum_{f=1}^{h}\tau_f^{(2)}\alpha_f z_{fo}^* \\
& \text{s.t. All constrains in model (4.21).}
\end{aligned} \tag{4.31}$$

Due to the non-convexity and non-linearity of model (4.29), (4.30) and (4.31), we still use heuristic searching method mentioned in the previous subsection to solve these models. Specifically, we solve model (4.19), (4.20) and (4.21) with different feasible directions varying by a fixed length, in which the directions satisfy the standardization conditions as shown in constrains of the upper level of the three models above. Then, by separately comparing all the optimal solutions of model (4.19), (4.20) and (4.21), the three largest value are the optimal values of model (4.29), (4.30) and (4.31), respectively. Let the optimal values be $\varepsilon_{\max}^{(N)+*}(X_o^*, Z_o^*, Y_o^*)$, $\varepsilon_{\max}^{(1)+*}(X_o^*, Z_o^*, Y_o^*)$ and

$\varepsilon_{\max}^{(2)+*}(X_o^*, Z_o^*, Y_o^*)$. Obviously, their corresponding directions might not be the same. For two-stage series production technologies, the decision-makers are more concerned about if final outputs can increase effectively. Therefore, we define the projected DMUs that are of absolute decreasing RTS in the overall stage of $T_S$ as follows:

**Definition 4.5.** *If $\varepsilon_{max}^{(N)+*}(X_o^*, Z_o^*, Y_o^*) < 1$ holds, $DMU_o^*(X_o^*, Z_o^*, Y_o^*)$ is of absolute decreasing RTS on the right-hand side in the overall stage of $T_S$.*

If a $\text{DMU}_o^*(X_o^*, Z_o^*, Y_o^*)$ is of absolute decreasing RTS, it is not suggested to further increase the scale in any input directions with any production structures of intermediate products. Then, we can find the minimum of $\varepsilon^{(S)+}(X_o^*, Z_o^*, Y_o^*)$ by the following model:

$$\begin{aligned}\varepsilon_{\min}^{(N)+}(X_o^*, Z_o^*, Y_o^*) = & \min_{v_i, \tau_f^{(1)}, \tau_f^{(2)}, u_r, u_{10}, u_{20}, \omega_i, \alpha_f} \sum_{i=1}^{m} v_i \omega_i x_{io}^* \\ & \text{s.t.} \begin{cases} \sum_{i\in I} \omega_i = |I|, \sum_{f\in C} \alpha_f = |C|, \\ \omega_{i\in I}, \alpha_{f\in C} \geq 0, \\ \text{All constrains in model (4.19).} \end{cases}\end{aligned} \tag{4.32}$$

Hereafter, let the optimal solution of model (4.32) be $\varepsilon_{\min}^{(N)+*}$. Evidently, if $\varepsilon_{\min}^{(N)+*} < 1$, then there are at least one direction causing decreasing RTS for $\text{DMU}_o^*(X_o^*, Z_o^*, Y_o^*)$ on the right-hand. Therefore, we have the following definition to define relative decreasing RTS of the overall stage.

**Definition 4.6.** *If $\varepsilon_{min}^{(N)+} < 1$ holds, $DMU_o^*(X_o^*, Z_o^*, Y_o^*)$ is of relative decreasing RTS on the right-hand side.*

## 4.4 Case studies

### 4.4.1 Case study of UK universities

To simplify the situations in comparing the DSE, all output directions are set to 1 (i.e., $\delta_1^1 = \delta_1^2 = \delta_2^1 = \delta_2^2 = \delta^{S1} = \delta^{S2} = 1$), and we set four different input directions:

- Case 1: $(\omega^1, \omega^2, \omega^{S1}, \omega^{S2}) = (0.1, 0.1, 1.9, 1.9)$.
- Case 2: $(\omega^1, \omega^2, \omega^{S1}, \omega^{S2}) = (0.1, 1.9, 1.9, 0.1)$.

- Case 3: $(\omega^1, \omega^2, \omega^{S1}, \omega^{S2}) = (0.1, 1.9, 0.1, 1.9)$.
- Case 4: $(\omega^1, \omega^2, \omega^{S1}, \omega^{S2}) = (1, 1, 1, 1)$.

The directions set in cases 1, 2 and 3 represent management perspective 1, 2 and 3, respectively. Case 4 represents the situation of the radial direction in inputs. Specifically, in case 1, the proportion of increases in total expenditure is larger than that in staff within each activity (i.e., $\omega^1 = \omega^2 < \omega^{S1} = \omega^{S2}$) since the standardization condition (4.3) allows the directions of changes in inputs are completely mutual compensation. In case 2, given that the standardization condition (4.4) requires the directions of changes in inputs are mutual compensation only within each activity, the proportion of increases in NS staff is smaller than that in total expenditure allocated in NS (i.e., $\omega^1 < \omega^{S1}$) while the situation is opposite in SS activity (i.e., $\omega^2 < \omega^{S2}$). In case 3, the proportion of input increases in SS activity is larger than that in NS activity (i.e., $\omega^1 = \omega^{S1} < \omega^2 = \omega^{S2}$) based on the standardization conditions (4.5) and (4.3). Then, by applying model (4.7) and (4.8), the right-hand and left-hand DSE of each projected DMU are calculated in the four cases. Table 4.1, 4.2, 4.3 and 3.4 display the right-hand and left-hand DSE results of DMUs projected by model (3.15) in the first three cases, respectively.

Table 4.1: Right-hand and left-hand SE for 43 UK universities in case 1

| DMU | $\varepsilon^+$ | $\varepsilon^-$ | DMU | $\varepsilon^+$ | $\varepsilon^-$ | DMU | $\varepsilon^+$ | $\varepsilon^-$ |
|---|---|---|---|---|---|---|---|---|
| $DMU_1$ | 0.06 | 0.32 | $DMU_{16}$ | 0.08 | 0.77 | $DMU_{31}$ | 0.69 | 0.92 |
| $DMU_2$ | 0.06 | 0.18 | $DMU_{17}$ | 0.89 | 0.93 | $DMU_{32}$ | 0.03 | 0.04 |
| $DMU_3$ | 0.40 | 1.14 | $DMU_{18}$ | 0.05 | 0.05 | $DMU_{33}$ | 0.43 | 0.80 |
| $DMU_4$ | 0.00 | 1.77 | $DMU_{19}$ | 0.71 | 0.90 | $DMU_{34}$ | 0.00 | 2.53 |
| $DMU_5$ | 0.00 | 1.06 | $DMU_{20}$ | 0.03 | 0.46 | $DMU_{35}$ | 0.07 | 1.40 |
| $DMU_6$ | 0.00 | 0.07 | $DMU_{21}$ | 0.06 | 0.11 | $DMU_{36}$ | 0.20 | $+\infty$ |
| $DMU_7$ | 0.05 | 0.06 | $DMU_{22}$ | 0.48 | 0.93 | $DMU_{37}$ | 0.05 | 0.06 |
| $DMU_8$ | 0.08 | 1.56 | $DMU_{23}$ | 0.42 | 0.48 | $DMU_{38}$ | 0.06 | 0.07 |
| $DMU_9$ | 0.11 | 0.48 | $DMU_{24}$ | 0.05 | 0.05 | $DMU_{39}$ | 0.05 | 0.07 |
| $DMU_{10}$ | 0.11 | 0.51 | $DMU_{25}$ | 0.08 | 1.30 | $DMU_{40}$ | 0.13 | 0.15 |
| $DMU_{11}$ | 0.04 | 1.62 | $DMU_{26}$ | 0.03 | 0.45 | $DMU_{41}$ | 1.01 | 1.77 |
| $DMU_{12}$ | 0.07 | 1.68 | $DMU_{27}$ | 0.62 | 1.43 | $DMU_{42}$ | 0.08 | $+\infty$ |
| $DMU_{13}$ | 0.86 | 0.95 | $DMU_{28}$ | 0.14 | 0.39 | $DMU_{43}$ | 0.18 | 0.26 |
| $DMU_{14}$ | 0.46 | 0.90 | $DMU_{29}$ | 0.08 | 1.17 | | | |
| $DMU_{15}$ | 0.60 | 1.48 | $DMU_{30}$ | 0.86 | 1.18 | | | |

Table 4.2: Right-hand and left-hand SE for 43 UK universities in case 2

| DMU | $\varepsilon^+$ | $\varepsilon^-$ | DMU | $\varepsilon^+$ | $\varepsilon^-$ | DMU | $\varepsilon^+$ | $\varepsilon^-$ |
|---|---|---|---|---|---|---|---|---|
| $DMU_1$ | 0.11 | 0.38 | $DMU_{16}$ | 0.52 | 1.59 | $DMU_{31}$ | 0.56 | 0.67 |
| $DMU_2$ | 0.06 | 0.14 | $DMU_{17}$ | 0.73 | 1.59 | $DMU_{32}$ | 0.03 | 0.04 |
| $DMU_3$ | 0.38 | 1.00 | $DMU_{18}$ | 0.05 | 0.05 | $DMU_{33}$ | 0.40 | 0.59 |
| $DMU_4$ | 0.00 | 1.31 | $DMU_{19}$ | 0.44 | 0.54 | $DMU_{34}$ | 0.00 | 5.14 |
| $DMU_5$ | 0.00 | 0.87 | $DMU_{20}$ | 0.03 | 0.41 | $DMU_{35}$ | 0.32 | 1.03 |
| $DMU_6$ | 0.00 | 0.07 | $DMU_{21}$ | 0.07 | 0.14 | $DMU_{36}$ | 0.22 | $+\infty$ |
| $DMU_7$ | 0.05 | 0.06 | $DMU_{22}$ | 0.43 | 0.67 | $DMU_{37}$ | 0.05 | 0.06 |
| $DMU_8$ | 0.30 | 1.37 | $DMU_{23}$ | 0.33 | 0.43 | $DMU_{38}$ | 1.14 | 1.36 |
| $DMU_9$ | 0.11 | 0.34 | $DMU_{24}$ | 0.05 | 0.05 | $DMU_{39}$ | 0.05 | 0.07 |
| $DMU_{10}$ | 0.10 | 0.36 | $DMU_{25}$ | 0.19 | 24.63 | $DMU_{40}$ | 0.13 | 0.15 |
| $DMU_{11}$ | 0.38 | 1.55 | $DMU_{26}$ | 0.03 | 0.33 | $DMU_{41}$ | 0.64 | 1.23 |
| $DMU_{12}$ | 1.02 | 1.57 | $DMU_{27}$ | 0.50 | 1.10 | $DMU_{42}$ | 1.57 | $+\infty$ |
| $DMU_{13}$ | 0.69 | 0.72 | $DMU_{28}$ | 0.37 | 0.63 | $DMU_{43}$ | 0.33 | 0.39 |
| $DMU_{14}$ | 0.40 | 0.57 | $DMU_{29}$ | 0.25 | 1.61 | | | |
| $DMU_{15}$ | 0.47 | 1.06 | $DMU_{30}$ | 0.65 | 0.98 | | | |

Table 4.3: Right-hand and left-hand SE for 43 UK universities in case 3

| DMU | $\varepsilon^+$ | $\varepsilon^-$ | DMU | $\varepsilon^+$ | $\varepsilon^-$ | DMU | $\varepsilon^+$ | $\varepsilon^-$ |
|---|---|---|---|---|---|---|---|---|
| $DMU_1$ | 0.11 | 0.27 | $DMU_{16}$ | 0.16 | 1.59 | $DMU_{31}$ | 0.28 | 0.33 |
| $DMU_2$ | 0.06 | 0.13 | $DMU_{17}$ | 0.24 | 1.59 | $DMU_{32}$ | 0.03 | 0.04 |
| $DMU_3$ | 0.14 | 0.23 | $DMU_{18}$ | 0.05 | 0.05 | $DMU_{33}$ | 0.25 | 0.28 |
| $DMU_4$ | 0.00 | 0.97 | $DMU_{19}$ | 0.35 | 0.43 | $DMU_{34}$ | 0.00 | 5.14 |
| $DMU_5$ | 0.00 | 0.87 | $DMU_{20}$ | 0.03 | 0.32 | $DMU_{35}$ | 0.39 | 1.03 |
| $DMU_6$ | 0.00 | 0.07 | $DMU_{21}$ | 0.07 | 0.12 | $DMU_{36}$ | 0.22 | $+\infty$ |
| $DMU_7$ | 0.05 | 0.06 | $DMU_{22}$ | 0.27 | 0.35 | $DMU_{37}$ | 0.05 | 0.06 |
| $DMU_8$ | 0.93 | 1.37 | $DMU_{23}$ | 0.36 | 0.46 | $DMU_{38}$ | 1.14 | 1.36 |
| $DMU_9$ | 0.09 | 0.20 | $DMU_{24}$ | 0.05 | 0.05 | $DMU_{39}$ | 0.05 | 0.07 |
| $DMU_{10}$ | 0.08 | 0.21 | $DMU_{25}$ | 0.07 | 24.63 | $DMU_{40}$ | 0.13 | 0.15 |
| $DMU_{11}$ | 0.26 | 1.55 | $DMU_{26}$ | 0.03 | 0.18 | $DMU_{41}$ | 0.50 | 1.07 |
| $DMU_{12}$ | 0.49 | 1.57 | $DMU_{27}$ | 0.24 | 0.44 | $DMU_{42}$ | 1.57 | $+\infty$ |
| $DMU_{13}$ | 0.32 | 0.39 | $DMU_{28}$ | 0.38 | 0.66 | $DMU_{43}$ | 0.34 | 0.42 |
| $DMU_{14}$ | 0.31 | 0.42 | $DMU_{29}$ | 0.14 | 1.61 | | | |
| $DMU_{15}$ | 0.30 | 0.53 | $DMU_{30}$ | 0.22 | 0.37 | | | |

Obviously, the results of DSE are different due to the differences of input

directions set by different management perspectives. Take $DMU_{12}$, $DMU_{27}$ and $DMU_{38}$ as examples, we exhibit the influence of different directions to RTS in Table 4.4.

Table 4.4: RTS of $DMU_{12}$, $DMU_{27}$ and $DMU_{38}$

| DMU | Case1 | | Case2 | | Case3 | | Case4 | |
|---|---|---|---|---|---|---|---|---|
| | RRTS | LRTS | RRTS | LRTS | RRTS | LRTS | RRTS | LRTS |
| $DMU_{12}$ | D | I | I | I | D | I | D | D |
| $DMU_{27}$ | D | I | D | I | D | D | D | I |
| $DMU_{38}$ | D | D | I | I | I | I | D | D |

Note: RRTS and LRTS mean the right-hand RTS and left-hand RTS, respectively. "I", "D" and "C" refer to increasing, decreasing and constant RTS, respectively.

Specifically, for $DMU_{12}$, it is suggested to decrease the scale along the redial direction (i.e., case 4) since decreasing RTS prevails on the right-hand and left-hand side simultaneously. However, the scale of $DMU_{12}$ needs to keep its scale in cases 1 and 3 since decreasing RTS prevails on the right-hand side and increasing RTS prevails on the left-hand side, while $DMU_{12}$ can increase its scale in case 2 since it exhibits increasing RTS on the right-hand and left-hand side simultaneously. The scale of $DMU_{27}$ is suggested to keep its scale in cases 1, 2 and 4 while it needs to decrease its scale in case 3. $DMU_{38}$ is suggested to reduce its scale in cases 1 and 4 while the scale can be further increased in cases 2 and 3. Consequently, the above results verify that the SE and RTS on the right-hand and left-hand side are influenced by different directions under different management perspectives.

In the following, we focus on analyzing the optimal directions of inputs under the management perspective 3 (i.e., the change directions of inputs satisfy standardization condition (4.3) and (4.5)). Starting by adding Eq. (4.5) as a constrain in the upper level of model (4.25), we have the change directions of staff is equal to that of expenditures in both NS and SS activities (i.e., $\omega^1 = \omega^{S1}$ and $\omega^2 = \omega^{S2}$ hold). Then, we vary $\omega^1$ and $\omega^2$ by a fixed step-length of 0.01, while keeping $\omega^1 + \omega^2 = 2$. After calculating all results of DSE, we pick up the largest DSE and find the corresponding directions. The results are shown in Table 4.5.

Table 4.5: Optimal input directions and corresponding DSE

| DMU | $\varepsilon^+_{\max}$ | $(\omega^1, \omega^2)$ | DMU | $\varepsilon^+_{\max}$ | $(\omega^1, \omega^2)$ | DMU | $\varepsilon^+_{\max}$ | $(\omega^1, \omega^2)$ |
|---|---|---|---|---|---|---|---|---|
| $DMU_1$ | 1.05 | (2,0) | $DMU_{16}$ | 1.00 | (2,0) | $DMU_{31}$ | 1.22 | (2,0) |
| $DMU_2$ | 1.15 | (2,0) | $DMU_{17}$ | 0.76 | (1.13,0.87) | $DMU_{32}$ | 0.69 | (2,0) |
| $DMU_3$ | 1.22 | (2,0) | $DMU_{18}$ | 1.00 | (2,0) | $DMU_{33}$ | 1.23 | (2,0) |
| $DMU_4$ | 0.00 | inf | $DMU_{19}$ | 1.09 | (1.99,0.01) | $DMU_{34}$ | 0.00 | inf |
| $DMU_5$ | 0.00 | inf | $DMU_{20}$ | 0.56 | (2,0) | $DMU_{35}$ | 0.69 | (1.14,0.86) |
| $DMU_6$ | 0.00 | inf | $DMU_{21}$ | 1.05 | (2,0) | $DMU_{36}$ | 1.83 | (1.97,0.03) |
| $DMU_7$ | 0.91 | (2,0) | $DMU_{22}$ | 1.35 | (2,0) | $DMU_{37}$ | 0.99 | (2,0) |
| $DMU_8$ | 1.00 | (0.53,1.47) | $DMU_{23}$ | 1.00 | (2,0) | $DMU_{38}$ | 1.20 | (0,2) |
| $DMU_9$ | 1.11 | (2,0) | $DMU_{24}$ | 0.97 | (2,0) | $DMU_{39}$ | 1.09 | (2,0) |
| $DMU_{10}$ | 0.97 | (2,0) | $DMU_{25}$ | 0.60 | (1.27,0.73) | $DMU_{40}$ | 2.64 | (2,0) |
| $DMU_{11}$ | 0.41 | (0.96,1.04) | $DMU_{26}$ | 0.60 | (2,0) | $DMU_{41}$ | 0.73 | (1.09,0.91) |
| $DMU_{12}$ | 0.77 | (0.95,1.05) | $DMU_{27}$ | 1.26 | (2,0) | $DMU_{42}$ | 1.65 | (0,2) |
| $DMU_{13}$ | 1.09 | (2,0) | $DMU_{28}$ | 0.99 | (2,0) | $DMU_{43}$ | 2.49 | (1.99,0.01) |
| $DMU_{14}$ | 1.16 | (2,0) | $DMU_{29}$ | 0.62 | (2,0) | | | |
| $DMU_{15}$ | 1.17 | (2,0) | $DMU_{30}$ | 1.17 | (2,0) | | | |

According to Table 4.5, without considering the 4 SSS points (i.e., projections of $DMU_4$, $DMU_5$, $DMU_6$ and $DMU_{34}$), 24 universities on their projections can further increase the scale along their optimal input directions, while 15 universities cannot further increase their scale along any input directions. Based on the definition 4.3, these 15 universities are of absolute decreasing RTS. In these 24 DMUs that can further increase the scale, only the optimal input directions of $DMU_8$, $DMU_{38}$ and $DMU_{42}$ satisfy $\omega^1 < \omega^2$. That is, allocating more staff and expenditures into the NS activity is beneficial to further scale expansion for most of universities. To carry out an in-depth analysis for the right-hand RTS of each DMU, we exhibit the relationship between $\varepsilon^+$ and $\omega^1$ for all DMUs, as shown in Fig. 4.1.

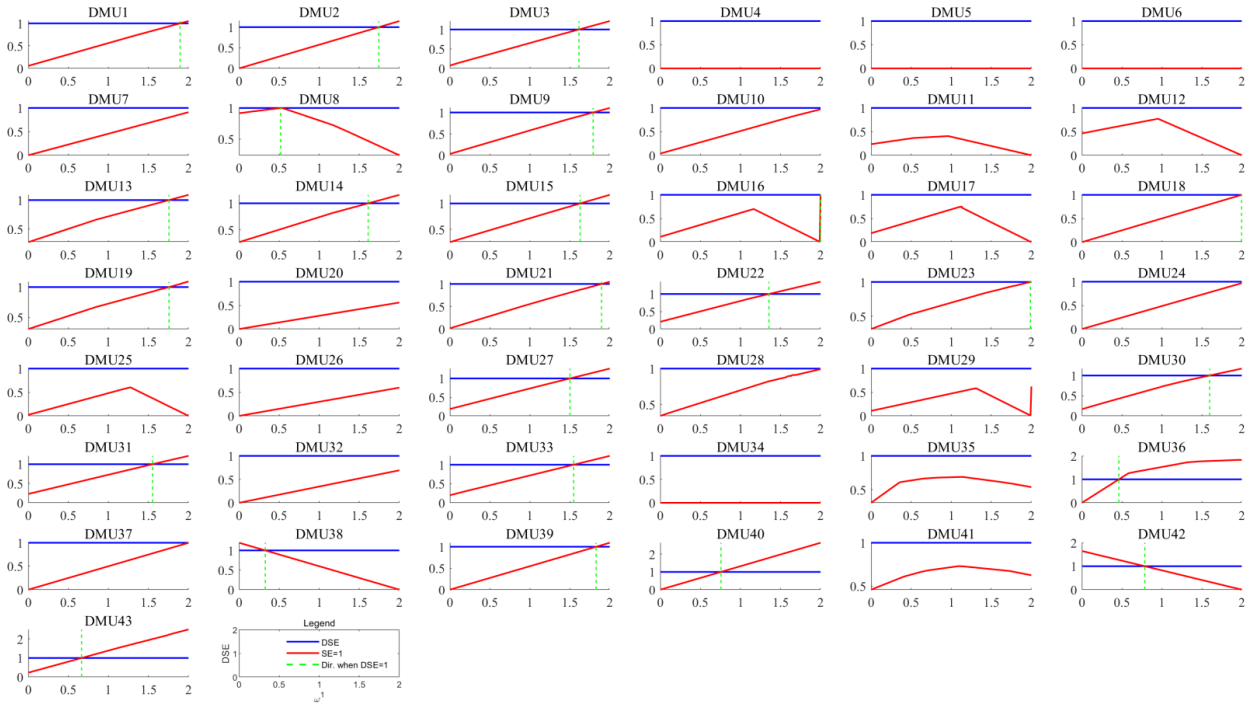



Figure 4.1: Relationship between DSE and input directions.

As shown in the last sub-figure in Fig. 4.1, the X-axis and Y-axis represent the change of $\omega^1$ (i.e., the change direction of inputs in NS activity) and $\varepsilon^+$, respectively. Therefore, we can intuitively observe the relationships between $\omega^1$ and $\varepsilon^+$ for each projected DMUs by solid red lines in each sub-figure. Moreover, the solid blue lines representing $\varepsilon^+ = 1$ provide benchmarks for investigating if increasing RTS prevails based on a certain direction. The dotted green line in each sub-figure represents the value of $\omega^1$ at the intersection (if exist) of the red line and the blue line. The results from Fig. 4.1 indicate that for most DMUs (e.g., $\text{DMU}_1$, $\text{DMU}_2$ and $\text{DMU}_3$), $\varepsilon^+$ increases as the proportion of changes in inputs of NS activities increases. Combining the results in Table 4.5 and Fig. 4.1, we can easily find that not all projected DMUs have an available way to increase the scale effectively. For the projected DMUs with feasible change directions that make increasing RTS prevails, the ranges of change directions are also different. Therefore, we display the range of $\omega^1$ that the increasing or constant RTS prevails on the right-hand side, as shown in Table 4.6.

Table 4.6: The range of $\omega^1$ making increasing or constant RTS prevails on the right-hand side

| DMU | $\omega^1$ | DMU | $\omega^1$ | DMU | $\omega^1$ |
|---|---|---|---|---|---|
| $DMU_1$ | [1.9,2] | $DMU_{16}$ | 2.00 | $DMU_{31}$ | [1.56,2] |
| $DMU_2$ | [1.75,2] | $DMU_{17}$ | inf. | $DMU_{32}$ | inf. |
| $DMU_3$ | [1.62,2] | $DMU_{18}$ | 2.00 | $DMU_{33}$ | [1.55,2] |
| $DMU_4$ | inf. | $DMU_{19}$ | [1.76,2] | $DMU_{34}$ | inf. |
| $DMU_5$ | inf. | $DMU_{20}$ | inf. | $DMU_{35}$ | inf. |
| $DMU_6$ | inf. | $DMU_{21}$ | [1.9,2] | $DMU_{36}$ | [0.47,2] |
| $DMU_7$ | inf. | $DMU_{22}$ | [1.36,2] | $DMU_{37}$ | inf. |
| $DMU_8$ | [0.52,0.53] | $DMU_{23}$ | [1.99,2] | $DMU_{38}$ | [0,0.31] |
| $DMU_9$ | [1.8,2] | $DMU_{24}$ | inf. | $DMU_{39}$ | [1.83,2] |
| $DMU_{10}$ | inf. | $DMU_{25}$ | inf. | $DMU_{40}$ | [0.76,2] |
| $DMU_{11}$ | inf. | $DMU_{26}$ | inf. | $DMU_{41}$ | inf. |
| $DMU_{12}$ | inf. | $DMU_{27}$ | [1.51,2] | $DMU_{42}$ | [0,0.78] |
| $DMU_{13}$ | [1.76,2] | $DMU_{28}$ | inf. | $DMU_{43}$ | [0.67,2] |
| $DMU_{14}$ | [1.62,2] | $DMU_{29}$ | inf. | | |
| $DMU_{15}$ | [1.63,2] | $DMU_{30}$ | [1.6,2] | | |

Obviously, $\omega^1 = 1$ represents the inputs change along the redial direction, and $\omega^1 \in (1, 2]$ means that more resources are allocated to the NS activity while $\omega^1 \in [0, 1)$ means that more resources are allocated to the SS activity. From the results of Table 4.6, we can find that 24 DMUs have the ability to further increase the scale of inputs. Specifically, 17 of them can only find the feasible ways by allocating more resources to the NS activity, while 3 of them (i.e.,$DMU_8$, $DMU_{38}$ and $DMU_{42}$) can only find feasible ways by allocating more resources to the SS activity. In particular, for the projected $DMU_{36}$, $DMU_{40}$ and $DMU_{43}$, whichever activity they allocate more resources to has a path that makes increasing RTS prevail.

### 4.4.2 Case study of Chinese cement companies

To simplify the situations, all directions of original inputs and final outputs are set to 1 (i.e., $\omega_1 = \omega_2 = \delta_1 = \delta_2 = 1$). With this setting, we assume that the decision-makers are more concerned about the influence of production structures to $\varepsilon^{(S)+}$. Specifically, for the two-stage production process as shown in Table 3.10, assuming that the change directions of inputs and outputs are unchanged, we aim to investigate the influence of varying the proportions of increasing clinker & cement and aggregate on $\varepsilon^{(S)+}$. Consider

model (4.32) with $\omega_1 = \omega_2 = \delta_1 = \delta_2 = 1$, we vary $\alpha_1$ and $\alpha_2$ by a fixed step-length of 0.01, while keeping $\alpha_1 + \alpha_2 = 2$. After calculating all results of DSE, the largest DSE and the corresponding directions are shown in Table 4.7.

Table 4.7: Optimal production structures and corresponding DSE

| DMU | $\varepsilon^{N+}_{\max}$ | $(\alpha^1, \alpha^2)$ | DMU | $\varepsilon^{N+}_{\max}$ | $(\alpha^1, \alpha^2)$ |
|---|---|---|---|---|---|
| $DMU_1$ | 0.00 | inf. | $DMU_9$ | 0.00 | inf. |
| $DMU_2$ | 0.00 | inf. | $DMU_{10}$ | 0.31 | (1.71,0.29) |
| $DMU_3$ | 0.64 | (0.92,1.08) | $DMU_{11}$ | 0.28 | (1.88,0.12) |
| $DMU_4$ | 0.00 | inf. | $DMU_{12}$ | 0.20 | (0.01,1.99) |
| $DMU_5$ | 0.50 | (1.44,0.56) | $DMU_{13}$ | 0.20 | (0.38,1.62) |
| $DMU_6$ | 0.58 | (0.42,1.58) | $DMU_{14}$ | 0.05 | (0.01,1.99) |
| $DMU_7$ | 0.44 | (0.84,1.16) | $DMU_{15}$ | 0.56 | (0.93,1.07) |
| $DMU_8$ | 0.24 | (0.37,1.63) | | | |

According to Table 4.7, without considering the 4 SSS points (i.e., projections of $DMU_1$, $DMU_2$, $DMU_4$ and $DMU_9$), all the other cement companies cannot further increase their scale along any input directions. In other words, all the cement companies are of absolute decreasing RTS based on the definition 4.5. To carry out in-depth analysis for the right-hand DSE of each DMU, we exhibit the relationship between $\varepsilon^{N+}$ and $\alpha^1$ for all DMUs, as shown in Fig. 4.2.

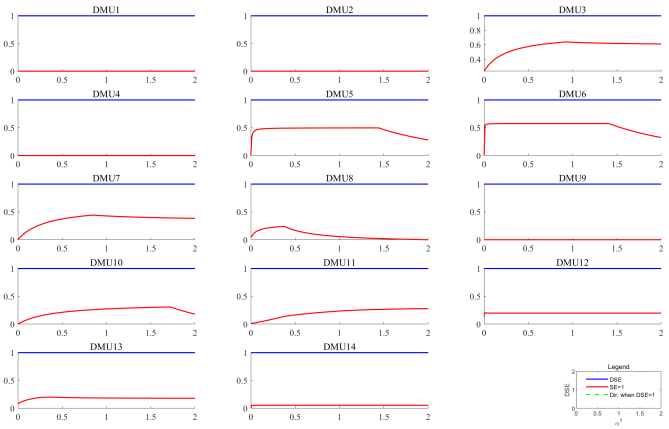



Figure 4.2: Relationship between DSE and directions of intermediate products.

As shown in the last sub-figure in Fig. 4.2, the X-axis and Y-axis represent the change of $\alpha^1$ (i.e., the change direction of Clinker & Cement) and $\varepsilon^{N+}$, respectively. Intuitively, the relationships between $\alpha^1$ and $\varepsilon^{N+}$ for each projected DMUs by solid red lines in each sub-figure. The blue lines representing $\varepsilon^{N+} = 1$ provide benchmarks for investigating if increasing RTS prevails based on a certain direction. The results form Fig. 4.2 indicate that adjusting directions of intermediate products can influence $\varepsilon^{N+}$ although the adjustment of production structures cannot make increasing RTS prevails in any cement companies.

## 4.5 Chapter summary

Due to the lack of research on defining and estimating DSE in network production systems, this chapter developed the notion of DSE in multi-activity and multi-stage production technologies. Given the existence of activity-specific and shared inputs and outputs, this chapter considered that the standardization conditions of change directions need to be modified from different management perspectives, which reflect the mutual compensation differences in the specified direction of changes in inputs. Our case studies showed that the results of DSE are different under different standardization conditions determined by the three proposed management perspectives. In addition, changes in different directions of intermediate products, as the outputs of the first stage and the inputs of the second stage in the producing-marketing two-stage production technologies, represent different production structures. This chapter first developed DSE into two-stage production technologies by introducing a direction vector for intermediate products.

Furthermore, based on the developed notion of DSE in multi-activity production technologies, a bi-level model was proposed to find the optimal input directions that maximize output changes. By using heuristic searching, DSE corresponding to different input directions can be presented. For similar purposes, another bi-level model was proposed in multi-activity production technologies. In addition, the definition of absolute decreasing RTS was proposed to describe the DMUs with no possible direction to achieve IRS. In addition, the definition of relative decreasing RTS was proposed to describe the DMUs with at least one direction to let them be in the status of decreasing RTS. Based on the data set in Chapter3, by applying the bi-level model, we found that most universities should increase the proportions of inputs in the NS activity to expand the scale of inputs, and adjusting production structures cannot make increasing RTS prevail in any cement companies.

The contributions of this chapter are summarized below:

- This chapter developed the notion of DSE in multi-activity and multi-stage production technologies, respectively, in which the standardization directions corresponding to the three management perspectives of input and output changes were proposed in parallel production systems, and the directions of intermediate products were first introduced in series production systems.

- This chapter proposed bi-level models to calculate the optimal input directions (including intermediate product directions in multi-stage production technologies) that can maximize the output increases under standardization conditions and uses heuristic searching to solve the model, followed by defining the notion of absolute and relative decreasing RTS.

# Chapter 5

# Estimating SE in the Presence of Undesirable Outputs

## 5.1 Introduction

In previous chapters, both multi-activity and multi-stage production technologies assumed strongly disposability of inputs and outputs (see Axiom 1 in Chapter 3). Based on this assumption, each output can be decreased independently while keeping other measures constant. However, in practice, some outputs might be related to each other. For example, in a power plant, a 10% reduction in $CO_2$ emissions is possible if accompanied by a 10% reduction in electricity, holding all inputs constant. In this case, we need to assume weakly disposability in electricity and $CO_2$ emissions. This assumption means that bad and good outputs can be reduced simultaneously by the same abatement factors (Kuosmanen 2005). Mehdiloo & Podinovski (2019) generalized the assumption and proposed the notion of selective strong and weakly disposability, in which the generalized assumption means that several outputs (including good and bad outputs) are jointly weakly disposable as a group. In practice, undesirable outputs are inevitably taken into account in the process of sustainability evaluation. Generally, undesirable outputs are environmental factors, such as carbon emission, sewage discharge or other pollutions that come with producing desirable outputs. Given the real technical features in production activities, different assumptions of disposability need to be distinguished to match relations between desirable and undesirable outputs (Yang & Pollitt 2010).

For existing research about estimating SE in production technologies with undesirable outputs, some of them assumed a consistent response of desirable and undesirable outputs (e.g., Podinovski (2019), Amirmohammadi

et al. (2021), Shen et al. (2024)), while the others calculated the marginal response of desirable and undesirable outputs, respectively (e.g., Sueyoshi & Goto (2011, 2013)). However, there is a lack of research concerning different disposability assumptions simultaneously in multi-activity or multi-stage production technologies.

The remainder of this chapter is divided into the following sections: Section 5.2 and 5.3 propose methods of estimating SE in novel multi-activity and multi-stage technologies with undesirable outputs, respectively. Considering the retraction of scholarly publications in UK universities and pollution emissions in Chinese cement companies, two case studies are carried out to illustrate the proposed methods. Section 5.4 is the summary of this chapter.

## 5.2 Estimating DSE in multi-activity technologies with undesirable outputs

### 5.2.1 Expressions of multi-activity technologies with undesirable outputs

Considering the existence of undesirable outputs, the set of all outputs $O$ in the production technology $T_P$ is restated as follows:

$$O = O^{DS} \cup O^{US} \cup O^{DWS} \cup O^{UWS} \cup \left( \bigcup_{k \in H} (O^{Dk} \cup O^{Uk} \cup O^{DWk} \cup O^{UWk}) \right). \quad (5.1)$$

Note that the set of all inputs $I$ remain the same as shown in Eq. (3.2). We assume that the production processes of each activity are mutually independent. Therefore, the undesirable output produced within an activity cannot be related to the desirable output from other activities. In activity $k$, the union of $O^{Dk}$, $O^{DWk}$, $O^{DS}$ and $O^{DWS}$ represents the set of desirable outputs, in which the desirable outputs in $O^{DWk} \cup O^{DWS}$ and undesirable outputs in $O^{UWk} \cup O^{UWS}$ are jointly weakly disposable, that is, they can be reduced by the same abatement factor $\pi^k \in [0, 1]$. In addition, the undesirable outputs in the union of $O^{Uk}$ and $O^{US}$ are assumed as strongly disposability, i.e., they can be freely reduced.

Corresponding to the decomposition of the output set as shown in (5.1), the output vector can be expressed as follows:

$$\begin{aligned} Y = (&Y^{D1}; \ldots; Y^{DK}; Y^{U1}; \ldots; Y^{UK}; Y^{DW1}; \ldots; Y^{DWK}; \\ &Y^{UW1}; \ldots; Y^{UWK}; Y^{DS}; Y^{US}; Y^{DWS}; Y^{UWS}), \end{aligned} \quad (5.2)$$

where $Y^{Dk} = (y_1^{Dk}, \ldots, y_{|O^{Dk}|}^{Dk})^\top (\forall k)$, $Y^{Uk} = (y_1^{Uk}, \ldots, y_{|O^{Uk}|}^{Uk})^\top (\forall k)$, $Y^{DWk} = (y_1^{DWk}, \ldots, y_{|O^{DWk}|}^{DWk})^\top (\forall k)$, $Y^{UWk} = (y_1^{UWk}, \ldots, y_{|O^{UWk}|}^{UWk})^\top (\forall k)$, $Y^{DS} = (y_1^{DS}, \ldots, y_{|O^{DS}|}^{DS})^\top$, $Y^{US} = (y_1^{US}, \ldots, y_{|O^{US}|}^{US})^\top$, $Y^{DWS} = (y_1^{DWS}, \ldots, y_{|O^{DWS}|}^{DWS})^\top$ and $Y^{UWS} = (y_1^{UWS}, \ldots, y_{|O^{UWS}|}^{UWS})^\top$.

Furthermore, we assume that the output sets of the $k$ activity are decomposed as follows:

$$O^{Ak} = O^{Dk} \cup O^{Uk} \cup O^{DWk} \cup O^{UWk} \cup O^{DS} \cup O^{US} \cup O^{DWS} \cup O^{UWS}. \quad (5.3)$$

We assume that each shared desirable output $y_r^{DS} (r \in O^{DS})$ and $y_r^{DWS} (r \in O^{DWS})$ are allocated into each activity by known proportion vectors $P_r^{DS} = (p_r^{DS1}, \ldots, p_r^{DSK})^\top (r \in O^{DS})$ and $P_r^{DWS} = (p_r^{DWS1}, \ldots, p_r^{DWSK})^\top (r \in O^{DWS})$, respectively, such that $\sum_{k \in H} p_r^{DSk} = 1$ and $\sum_{k \in H} p_r^{DWSk} = 1$. Let $y_r^{DSk} = p_r^{DSk} y_r^{DS}$ and $y_r^{DWSk} = p_r^{DWSk} y_r^{DWS}$ represent the part of shared desirable outputs allocated to activity $k$. Similarly, we assume that each shared undesirable output $y_r^{US} (r \in O^{US})$ and $y_r^{UWS} (r \in O^{UWS})$ are allocated into each activity by known proportion vectors $P_r^{US} = (p_r^{US1}, \ldots, p_r^{USK})^\top (r \in O^{US})$ and $P_r^{UWS} = (p_r^{UWS1}, \ldots, p_r^{UWSK})^\top (r \in O^{UWS})$, respectively, such that $\sum_{k \in H} p_r^{USk} = 1$ and $\sum_{k \in H} p_r^{UWSk} = 1$. Let $y_r^{USk} = p_r^{USk} y_r^{US}$ and $y_r^{UWSk} = p_r^{UWSk} y_r^{UWS}$ represent the part of shared undesirable outputs allocated to activity $k$. For activity $k$ of $(X, Y)$, corresponding to the decomposition of output set as shown in (5.3), we have:

$$Y^{Ak} = (Y^{Dk}; Y^{Uk}; Y^{DWk}; Y^{UWk}; Y^{DSk}; Y^{USk}; Y^{DWSk}; Y^{UWSk}), \quad (5.4)$$

where $Y^{DSk} = (y_1^{DSk}, \ldots, y_{|O^{DS}|}^{DSk})^\top (\forall k)$, $Y^{USk} = (y_1^{USk}, \ldots, y_{|O^{US}|}^{USk})^\top (\forall k)$, $Y^{DWSk} = (y_1^{DWSk}, \ldots, y_{|O^{DWS}|}^{DWSk})^\top (\forall k)$ and $Y^{UWSk} = (y_1^{UWSk}, \ldots, y_{|O^{UWS}|}^{UWSk})^\top (\forall k)$.

In Section 3.2.3, strongly disposability of inputs and outputs is assumed for the production technology $T_P$ (as shown in axiom 1). Let $T_P^U$ represent the multi-activity production technology with multiple undesirable outputs. Including the axioms of convexity and feasibility, we assume four axioms for $T_P^U$ as follows:

- **A1*** Selective strongly disposability: If $(X, Y) \in T_P^U$, we have $(X^{k*}(\forall k), X^{S*}, Y^{Dk*}(\forall k), Y^{Uk*}(\forall k), Y^{DWk*}(\forall k), Y^{UWk}(\forall k), Y^{DS*}, Y^{US*}, Y^{DWS*}, Y^{UWS}) \in T_P^U$ (for all $X^{k*} \geq X^k (\forall k), X^{S*} \geq X^S, 0 \leq Y^{Dk*} \leq Y^{Dk} (\forall k), Y^{Uk*} \geq Y^{Uk} (\forall k), 0 \leq Y^{DWk*} \leq Y^{DWk} (\forall k), 0 \leq Y^{DS*} \leq Y^{DS}, Y^{US*} \geq Y^{US}$ and $0 \leq Y^{DWS*} \leq Y^{DWS}$).

- **A2** Convexity: Technology $T_P^U$ is a convex set.

- **A3** Feasibility of all observed DMUs: For any $j \in N$, $\left(X_j, Y_j\right) \in T_P^U$.

- **A4** Selective weakly disposability: If $(X,Y) \in T_P^U$, we have $(X^k(\forall k), X^S, Y^{Dk}(\forall k), \pi^k Y^{Uk}(\forall k), \pi^k Y^{DWk}(\forall k), Y^{UWk}(\forall k), Y^{DS}, Y^{US}, \pi^k Y^{DWSk}(\forall k), \pi^k Y^{UWSk}(\forall k)) \in T_P^U$ (for all $0 \leq \pi^k \leq 1$).

By employing axiom 1*, 2, 3 and 4 above, the production technology $T_P^U$ can be stated as:

$$T_P^U = \left\{ (X,Y) \in \mathbb{R}_+^{|I|+|O|} \;\middle|\; \begin{array}{l} \sum_{j=1}^n \lambda_j^k x_{ij}^k \leq x_i^k \; (\forall k, i \in I^k), \\ \sum_{k=1}^K \sum_{j=1}^n \lambda_j^k x_{ij}^{Sk} \leq x_i^S \; (i \in I^S), \\ \sum_{j=1}^n \lambda_j^k y_{rj}^{Dk} \geq y_r^{Dk} \; (\forall k, r \in O^{Dk}), \\ \sum_{j=1}^n \lambda_j^k y_{rj}^{Uk} \leq y_r^{Uk} \; (\forall k, r \in O^{Uk}), \\ \sum_{j=1}^n \lambda_j^k \pi_j^k y_{rj}^{DWk} \geq y_r^{DWk} \; (\forall k, r \in O^{DWk}), \\ \sum_{j=1}^n \lambda_j^k \pi_j^k y_{rj}^{UWk} = y_r^{UWk} \; (\forall k, r \in O^{UWk}), \\ \sum_{k=1}^K \sum_{j=1}^n \lambda_j^k y_{rj}^{DSk} \geq y_r^{DS} \; (r \in O^{DS}), \\ \sum_{k=1}^K \sum_{j=1}^n \lambda_j^k y_{rj}^{USk} \leq y_r^{US} \; (r \in O^{US}), \\ \sum_{k=1}^K \sum_{j=1}^n \lambda_j^k \pi_j^k y_{rj}^{DWSk} \geq y_r^{DWS} \; (r \in O^{DWS}), \\ \sum_{k=1}^K \sum_{j=1}^n \lambda_j^k \pi_j^k y_{rj}^{UWSk} = y_r^{UWS} \; (r \in O^{UWS}), \\ \sum_{j=1}^n \lambda_j^k = 1 \; (\forall k), \lambda_j^k \geq 0 \; (\forall k), \\ \pi_j^k \in [0,1] \; (\forall k, j). \end{array} \right\} \tag{5.5}$$

Due to the introduction of abatement factors $\pi_j^k$, nonlinear terms $\lambda_j^k \pi_j^k (\forall k)$ exist in $T_P^U$. Let $\lambda_j^k \pi_j^k = \lambda_j^k - \mu_j^k (\forall k)$ hold, technology (5.5) can be restated

as:

$$T_P^U = \left\{ (X,Y) \in \mathbb{R}_+^{|I|+|O|} \;\middle|\; \begin{array}{l} \sum_{j=1}^{n} \lambda_j^k x_{ij}^k \le x_i^k \; (\forall k, i \in I^k), \\ \sum_{k=1}^{K} \sum_{j=1}^{n} \lambda_j^k x_{ij}^{Sk} \le x_i^S \; (i \in I^S), \\ \sum_{j=1}^{n} \lambda_j^k y_{rj}^{Dk} \ge y_r^{Dk} \; (\forall k, r \in O^{Dk}), \\ \sum_{j=1}^{n} \lambda_j^k y_{rj}^{Uk} \le y_r^{Uk} \; (\forall k, r \in O^{Uk}), \\ \sum_{j=1}^{n} (\lambda_j^k - \mu_j^k) y_{rj}^{DWk} \ge y_r^{DWk} \; (\forall k, r \in O^{DWk}), \\ \sum_{j=1}^{n} (\lambda_j^k - \mu_j^k) y_{rj}^{UWk} = y_r^{UWk} \; (\forall k, r \in O^{UWk}), \\ \sum_{k=1}^{K} \sum_{j=1}^{n} \lambda_j^k y_{rj}^{DSk} \ge y_r^{DS} \; (r \in O^{DS}), \\ \sum_{k=1}^{K} \sum_{j=1}^{n} \lambda_j^k y_{rj}^{USk} \le y_r^{US} \; (r \in O^{US}), \\ \sum_{k=1}^{K} \sum_{j=1}^{n} (\lambda_j^k - \mu_j^k) y_{rj}^{DWSk} \ge y_r^{DWS} \; (r \in O^{DWS}), \\ \sum_{k=1}^{K} \sum_{j=1}^{n} (\lambda_j^k - \mu_j^k) y_{rj}^{UWSk} = y_r^{UWS} \; (r \in O^{UWS}), \\ \sum_{j=1}^{n} \lambda_j^k = 1 \; (\forall k), \lambda_j^k - \mu_j^k \ge 0 \; (\forall j, k), \\ \lambda_j^k, \mu_j^k \ge 0 \; (\forall j, k). \end{array} \right\} \tag{5.6}$$

Furthermore, the following section will present the model of SE estimation on the frontier of technology (5.5).

### 5.2.2 SE estimation

Regarding the production technology $T_P^U$ as shown in (5.6), the output-oriented DEA model is stated as:

$$\frac{1}{E_0^{UP-O}} = \max_{\theta,\lambda_j^k,\mu_j^k} 1+\theta$$

$$\text{s.t.} \begin{cases} \sum_{j=1}^{n} \lambda_j^k x_{ij}^k \le x_i^k \ (\forall k, i \in I^k), \\ \sum_{k=1}^{K}\sum_{j=1}^{n} \lambda_j^k x_{ij}^{Sk} \le x_i^S \ (i \in I^S), \\ \sum_{j=1}^{n} \lambda_j^k y_{rj}^{Dk} \ge (1+\theta) y_r^{Dk} \ (\forall k, r \in O^{Dk}), \\ \sum_{j=1}^{n} \lambda_j^k y_{rj}^{Uk} \le (1-\theta) y_r^{Uk} \ (\forall k, r \in O^{Uk}), \\ \sum_{j=1}^{n} (\lambda_j^k - \mu_j^k) y_{rj}^{DWk} \ge (1+\theta) y_r^{DWk} \ (\forall k, r \in O^{DWk}), \\ \sum_{j=1}^{n} (\lambda_j^k - \mu_j^k) y_{rj}^{UWk} = (1+\theta) y_r^{UWk} \ (\forall k, r \in O^{UWk}), \\ \sum_{k=1}^{K}\sum_{j=1}^{n} \lambda_j^k y_{rj}^{DSk} \ge (1+\theta) y_r^{DS} \ (r \in O^{DS}), \\ \sum_{k=1}^{K}\sum_{j=1}^{n} \lambda_j^k y_{rj}^{USk} \le (1-\theta) y_r^{US} \ (r \in O^{US}), \\ \sum_{k=1}^{K}\sum_{j=1}^{n} (\lambda_j^k - \mu_j^k) y_{rj}^{DWSk} \ge (1+\theta) y_r^{DWS} \ (r \in O^{DWS}), \\ \sum_{k=1}^{K}\sum_{j=1}^{n} (\lambda_j^k - \mu_j^k) y_{rj}^{UWSk} = (1+\theta) y_r^{UWS} \ (r \in O^{UWS}), \\ \sum_{j=1}^{n} \lambda_j^k = 1 \ (\forall k), \lambda_j^k - \mu_j^k \ge 0 \ (\forall j, k), \\ \lambda_j^k, \mu_j^k \ge 0 \ (\forall k), \theta \text{ signfree.} \end{cases} \tag{5.7}$$

Let $(\theta^*, \lambda_j^{k*}, \mu_j^{k*})(\forall j, k)$ represent the optimal value of model (5.7). $\text{DMU}_o$ is defined as the output radial efficient DMU if and only if the optimal solution of the acquired from model (5.7) satisfy $\theta^* = 1$. If not, the inefficient DMUs can be projected onto the efficient frontier by applying the following equations:

$$\begin{aligned} x_{io}^{k*} &= \textstyle\sum_{j=1}^{n} \lambda_j^{k*} x_{ij}^k (\forall k, i \in I^k), x_{io}^{Sk*} = \sum_{j=1}^{n} \lambda_j^{k*} x_{ij}^{Sk}, (\forall k, i \in I^S), \\ y_{ro}^{Dk*} &= \textstyle\sum_{j=1}^{n} \lambda_j^{k*} y_{rj}^{Dk} (\forall k, r \in O^{Dk}), y_{ro}^{Uk*} = \sum_{j=1}^{n} \lambda_j^{k*} y_{rj}^{Uk} (\forall k, r \in O^{Uk}), \\ y_{ro}^{DWk*} &= \textstyle\sum_{j=1}^{n} (\lambda_j^{k*} - \mu_j^{k*}) y_{rj}^{DWk} (\forall k, r \in O^{DWk}), \\ y_{ro}^{UWk*} &= \textstyle\sum_{j=1}^{n} (\lambda_j^{k*} - \mu_j^{k*}) y_{rj}^{UWk} (\forall k, r \in O^{UWk}), \\ y_{ro}^{DSk*} &= \textstyle\sum_{j=1}^{n} \lambda_j^{k*} y_{rj}^{DSk} (\forall k, r \in O^{DS}), y_{ro}^{USk*} = \sum_{j=1}^{n} \lambda_j^{k*} y_{rj}^{USk} (\forall k, r \in O^{US}), \\ y_{ro}^{DWSk*} &= \textstyle\sum_{j=1}^{n} (\lambda_j^{k*} - \mu_j^{k*}) y_{rj}^{DWSk} (\forall k, r \in O^{DWS}), \\ y_{ro}^{UWSk*} &= \textstyle\sum_{j=1}^{n} (\lambda_j^{k*} - \mu_j^{k*}) y_{rj}^{UWSk} (\forall k, r \in O^{UWS}). \end{aligned} \tag{5.8}$$

Then, projected $\mathrm{DMU}_o(X_o^*, Y_o^*)$ can be obtained by applying Eq. (5.8). For all output-radial efficient $\mathrm{DMU}_o^*$ in the technology $T_P^U$, the response function of outputs is defined as follows:

$$\beta(t) = \max\left\{\beta \mid \left((1+t)X_o^*, (1-\beta)Y_o^{U*}, (1+\beta)Y_o^{D*}, (1+\beta)Y_o^{W*}\right) \in T_P^U, \beta \in \mathbb{R}\right\}, \tag{5.9}$$

where we assume $\beta = 0$ when $t = 0$ indicating that $Y_o^*$ cannot be changed within $T_P^U$ on the condition that $X_o^*$ keeps unchanged, the output vectors $Y_o^{U*}$, $Y_o^{D*}$ and $Y_o^{W*}$ include all outputs in the subset $\bigcup_{k\in H} O^{Uk} \cup O^{US}$, $\bigcup_{k\in H} O^{DWk} \cup O^{DS}$ and $\bigcup_{k\in H}(O^{DWk} \cup O^{UWk}) \cup O^{DWS} \cup O^{UWS}$, respectively.

To estimate SE for efficient $\mathrm{DMU}_o^*(X_o^*, Y_o^*) \in T_P^U$, $\beta(t)$ is stated as the following linear program:

$$\begin{aligned}
\beta(t) = \max_{\beta,\lambda_j^k,\mu_j^k} \ & \beta \\
\text{s.t.} \ & \begin{cases}
\sum_{j=1}^n \lambda_j^k x_{ij}^k \le (1+t)x_i^k \ (\forall k, i \in I^k), \\
\sum_{k=1}^K \sum_{j=1}^n \lambda_j^k x_{ij}^{Sk} \le (1+t)x_i^S \ (i \in I^S), \\
\sum_{j=1}^n \lambda_j^k y_{rj}^{Dk} \ge (1+\beta)y_r^{Dk} \ (\forall k, r \in O^{Dk}), \\
\sum_{j=1}^n \lambda_j^k y_{rj}^{Uk} \le (1-\beta)y_r^{Uk} \ (\forall k, r \in O^{Uk}), \\
\sum_{j=1}^n (\lambda_j^k - \mu_j^k) y_{rj}^{DWk} \ge (1+\beta)y_r^{DWk} \ (\forall k, r \in O^{DWk}), \\
\sum_{j=1}^n (\lambda_j^k - \mu_j^k) y_{rj}^{UWk} = (1+\beta)y_r^{UWk} \ (\forall k, r \in O^{UWk}), \\
\sum_{k=1}^K \sum_{j=1}^n \lambda_j^k y_{rj}^{DSk} \ge (1+\beta)y_r^{DS} \ (r \in O^{DS}), \\
\sum_{k=1}^K \sum_{j=1}^n \lambda_j^k y_{rj}^{USk} \le (1-\beta)y_r^{US} \ (r \in O^{US}), \\
\sum_{k=1}^K \sum_{j=1}^n (\lambda_j^k - \mu_j^k) y_{rj}^{DWSk} \ge (1+\beta)y_r^{DWS} \ (r \in O^{DWS}), \\
\sum_{k=1}^K \sum_{j=1}^n (\lambda_j^k - \mu_j^k) y_{rj}^{UWSk} = (1+\beta)y_r^{UWS} \ (r \in O^{UWS}), \\
\sum_{j=1}^n \lambda_j^k = 1 \ (\forall k), \lambda_j^k - \mu_j^k \ge 0 \ (\forall j, k), \\
\lambda_j^k, \mu_j^k \ge 0 \ (\forall k), \beta \text{ signfree}.
\end{cases}
\end{aligned} \tag{5.10}$$

Then, based on the DDO method, the methods of estimating SE in the production technology (5.6) is proposed in the following theorem:

**Theorem 7.** *Let $(X_o^*, Y_o^*) \in T_P^U$ be defined by (5.6) and $(X_o^*, Y_o^*)$ be an output-radial efficient point, then the right-hand and left-hand SE can be obtained*

*by solving the following problems:*

$$\varepsilon^{+} = \min_{\gamma_j^k, v_i^k, v_i^S, u_r^{Dk}, u_r^{Uk}, u_r^{DWk}, u_r^{UWk}, u_r^{DS}, u_r^{US}, u_r^{DWS}, u_r^{UWS}, u_0^k} \sum_{k\in H}\sum_{i\in I^k} v_i^k x_{io}^{k*} + \sum_{i\in I^S} v_i^S x_{io}^{S*}$$

$$s.t. \begin{cases} \sum_{k\in H}\sum_{i\in I^k} v_i^k x_{io}^{k*} + \sum_{i\in I^S} v_i^S x_{io}^{S*} - \sum_{k\in H}\sum_{r\in O^{Dk}} u_r^{Dk} y_{ro}^{Dk*} \\ + \sum_{k\in H}\sum_{r\in O^{Uk}} u_r^{Uk} y_{ro}^{Uk*} - \sum_{k\in H}\sum_{r\in O^{DWk}} u_r^{DWk} y_{ro}^{DWk*} \\ + \sum_{k\in H}\sum_{r\in O^{UWk}} u_r^{UWk} y_{ro}^{UWk*} - \sum_{r\in O^{DS}} u_r^{DS} y_{ro}^{DS*} + \sum_{r\in O^{US}} u_r^{US} y_{ro}^{US*} \\ - \sum_{r\in O^{DWS}} u_r^{DWS} y_{ro}^{DWS*} + \sum_{r\in O^{UWS}} u_r^{UWS} y_{ro}^{UWS*} + \sum_{k\in H} u_0^k = 0, \\ - \sum_{k\in H}\sum_{r\in O^{Dk}} u_r^{Dk} y_{ro}^{Dk*} + \sum_{k\in H}\sum_{r\in O^{Uk}} u_r^{Uk} y_{ro}^{Uk*} \\ - \sum_{k\in H}\sum_{r\in O^{DWk}} u_r^{DWk} y_{ro}^{DWk*} + \sum_{k\in H}\sum_{r\in O^{UWk}} u_r^{UWk} y_{ro}^{UWk*} \\ - \sum_{r\in O^{DS}} u_r^{DS} y_{ro}^{DS*} + \sum_{r\in O^{US}} u_r^{US} y_{ro}^{US*} \\ - \sum_{r\in O^{DWS}} u_r^{DWS} y_{ro}^{DWS*} + \sum_{r\in O^{UWS}} u_r^{UWS} y_{ro}^{UWS*} = 1, \\ \gamma_j^k - \sum_{i\in I^k} v_i^k x_{ij}^{k*} - \sum_{i\in I^S} v_i^S x_{ij}^{Sk*} + \sum_{r\in O^{Dk}} u_r^{Dk} y_{rj}^{Dk*} \\ - \sum_{r\in O^{Uk}} u_r^{Uk} y_{rj}^{Uk*} + \sum_{r\in O^{DWk}} u_r^{DWk} y_{rj}^{DWk*} \\ - \sum_{r\in O^{UWk}} u_r^{UWk} y_{rj}^{UWk*} + \sum_{r\in O^{DS}} u_r^{DS} y_{rj}^{DSk*} - \sum_{r\in O^{US}} u_r^{US} y_{rj}^{USk*} \\ + \sum_{r\in O^{DWS}} u_r^{DWS} y_{rj}^{DWSk*} - \sum_{r\in O^{UWS}} u_r^{UWS} y_{rj}^{UWSk*} - u_0^k \le 0\ (\forall j,k), \\ -\gamma_j^k - \sum_{r\in O^{DWk}} u_r^{DWk} y_{rj}^{DWk*} + \sum_{r\in O^{UWk}} u_r^{UWk} y_{rj}^{UWk*} \\ - \sum_{r\in O^{DWS}} u_r^{DWS} y_{rj}^{DWSk*} + \sum_{r\in O^{UWS}} u_r^{UWS} y_{rj}^{UWSk*} \le 0\ (\forall j,k), \\ \gamma_j^k \ge 0\ (\forall k,j), v_i^k, v_i^S \ge 0\ (\forall k,i), u_r^{Dk}, u_r^{Uk}, u_r^{DWk}, \\ u_r^{DS}, u_r^{US}, u_r^{DWS}, u_0^k \ge 0\ (\forall k, r \in O \setminus (\bigcup_{k\in H} O^{DWk} \cup O^{UWS})), \\ u_r^{UWk}, u_r^{UWS}, u_0^k \ \textit{signfree}\ (\forall k, r \in \bigcup_{k\in H} O^{DWk} \cup O^{UWS}), \end{cases} \tag{5.11}$$

$$\varepsilon^{-} = \max_{\gamma_j^k, v_i^k, v_i^S, u_r^{Dk}, u_r^{Uk}, u_r^{DWk}, u_r^{UWk}, u_r^{DS}, u_r^{US}, u_r^{DWS}, u_r^{UWS}, u_0^k} \sum_{k\in H}\sum_{i\in I^k} v_i^k x_{io}^{k*} + \sum_{i\in I^S} v_i^S x_{io}^{S*}$$

*s.t. All constrains in model* (5.11). (5.12)

It is worth noting that the overall system cannot be assumed to be additive because of the existence of undesirable outputs in $\bigcup_{k\in H} O^{Uk} \cup O^{US}$. Therefore, the overall SE cannot be decomposed in line with the method of decomposing SE in Section 3.2.3.

### 5.2.3 Case study

Considering the fact that the retraction of scholarly publications is an shared undesirable output in the case study of 43 UK universities (data sourced from

Restraction Watch Database (https://retractiondatabase.org/)), this subsection carries out two possible forms of production technologies, in which one of them assumes weakly disposability of undesirable outputs while another assumes strongly disposability of undesirable outputs. The original data of retraction is shown in Table 5.1.

Table 5.1: Retraction data of 43 UK universities

| DMU | $y^{R1}$ Retraction NS | $y^{R2}$ Retraction SS | DMU | $y^{R1}$ Retraction NS | $y^{R2}$ Retraction SS | DMU | $y^{R1}$ Retraction NS | $y^{R2}$ Retraction SS |
|---|---|---|---|---|---|---|---|---|
| DMU1 | 6 | 0 | DMU16 | 9 | 2 | DMU31 | 1 | 0 |
| DMU2 | 4 | 0 | DMU17 | 0 | 0 | DMU32 | 0 | 0 |
| DMU3 | 6 | 0 | DMU18 | 0 | 0 | DMU33 | 0 | 0 |
| DMU4 | 5 | 4 | DMU19 | 0 | 1 | DMU34 | 2 | 0 |
| DMU5 | 3 | 4 | DMU20 | 1 | 0 | DMU35 | 1 | 0 |
| DMU6 | 8 | 0 | DMU21 | 5 | 0 | DMU36 | 0 | 0 |
| DMU7 | 1 | 0 | DMU22 | 4 | 0 | DMU37 | 1 | 0 |
| DMU8 | 0 | 0 | DMU23 | 0 | 0 | DMU38 | 0 | 0 |
| DMU9 | 6 | 1 | DMU24 | 3 | 1 | DMU39 | 0 | 0 |
| DMU10 | 4 | 0 | DMU25 | 1 | 0 | DMU40 | 0 | 0 |
| DMU11 | 3 | 0 | DMU26 | 3 | 0 | DMU41 | 1 | 0 |
| DMU12 | 4 | 0 | DMU27 | 3 | 0 | DMU42 | 0 | 0 |
| DMU13 | 1 | 0 | DMU28 | 0 | 0 | DMU43 | 0 | 0 |
| DMU14 | 0 | 0 | DMU29 | 1 | 0 | | | |
| DMU15 | 0 | 0 | DMU30 | 8 | 0 | | | |

Let $y^R = y^{R1} + y^{R2}$ represent the number of retractions, where $y^{R1}$ and $y^{R2}$ represent the parts of retraction allocated to NS and SS, respectively.

**Scenario 1**

Assuming $y^R \in O^{UWS}$, $y^S \in O^{DWS}$ and $y_1^1,\ y_1^2,\ y_2^1,\ y_2^2 \in \bigcup_{k=1,2} O^{Dk}$, based on the production technology (5.6), the according production technology of

universities is stated as:

$$T_{\text{Uni1}} = \left\{ (X,Y) \in \mathbb{R}_+^9 \;\middle|\; \begin{array}{l} \sum_{j=1}^n \lambda_j^1 x_j^1 \le x_o^1, \sum_{j=1}^n \lambda_j^2 x_j^2 \le x_o^2, \\ \sum_{j=1}^n \lambda_j^1 x_j^{S1} + \sum_{j=1}^n \lambda_j^2 x_j^{S2} \le x_o^S, \\ \sum_{j=1}^n \lambda_j^1 y_{1j}^1 \ge y_{1o}^1, \sum_{j=1}^n \lambda_j^1 y_{2j}^1 \ge y_{2o}^1, \\ \sum_{j=1}^n \lambda_j^2 y_{1j}^2 \ge y_{1o}^2, \sum_{j=1}^n \lambda_j^2 y_{2j}^2 \ge y_{2o}^2, \\ \sum_{j=1}^n (\lambda_j^1 - \mu_j^1) y_j^{S1} + \sum_{j=1}^n (\lambda_j^2 - \mu_j^2) y_j^{S2} \ge y_o^S, \\ \sum_{j=1}^n (\lambda_j^1 - \mu_j^1) y_j^{R1} + \sum_{j=1}^n (\lambda_j^2 - \mu_j^2) y_j^{R2} = y_o^R, \\ \sum_{j=1}^n \lambda_j^1 = \sum_{j=1}^n \lambda_j^2 = 1, \lambda_j^1, \lambda_j^2, \mu_j^1, \mu_j^2 \ge 0 \, (\forall j), \\ \lambda_j^1 \ge \mu_j^1 \, (\forall j), \lambda_j^2 \ge \mu_j^2 \, (\forall j). \end{array} \right\} \tag{5.13}$$

Then, we use the output-oriented model to project all inefficient DMUs to the efficient frontier of the production technology $T_{\text{Uni1}}$:

$$\begin{aligned} \frac{1}{E_0^{UP-O}} = \max_{\theta, \lambda_j^1, \lambda_j^2, \mu_j^1, \mu_j^2} \; & 1+\theta \\ \text{s.t.} \; & \left\{ \begin{array}{l} \sum_{j=1}^n \lambda_j^1 x_j^1 \le x_o^1, \sum_{j=1}^n \lambda_j^2 x_j^2 \le x_o^2, \\ \sum_{j=1}^n \lambda_j^1 x_j^{S1} + \sum_{j=1}^n \lambda_j^2 x_j^{S2} \le x_o^S, \\ \sum_{j=1}^n \lambda_j^1 y_{1j}^1 \ge (1+\theta) y_{1o}^1, \sum_{j=1}^n \lambda_j^1 y_{2j}^1 \ge (1+\theta) y_{2o}^1, \\ \sum_{j=1}^n \lambda_j^2 y_{1j}^2 \ge (1+\theta) y_{1o}^2, \sum_{j=1}^n \lambda_j^2 y_{2j}^2 \ge (1+\theta) y_{2o}^2, \\ \sum_{j=1}^n (\lambda_j^1 - \mu_j^1) y_j^{S1} + \sum_{j=1}^n (\lambda_j^2 - \mu_j^2) y_j^{S2} \ge (1+\theta) y_o^S, \\ \sum_{j=1}^n (\lambda_j^1 - \mu_j^1) y_j^{R1} + \sum_{j=1}^n (\lambda_j^2 - \mu_j^2) y_j^{R2} = (1+\theta) y_o^R, \\ \sum_{j=1}^n \lambda_j^1 = \sum_{j=1}^n \lambda_j^2 = 1, \lambda_j^1, \lambda_j^2, \mu_j^1, \mu_j^2 \ge 0 \, (\forall j), \\ \lambda_j^1 \ge \mu_j^1 \, (\forall j), \lambda_j^2 \ge \mu_j^2 \, (\forall j), \theta \text{ signfree}. \end{array} \right. \end{aligned} \tag{5.14}$$

Let the optimal solution vector of model (5.14) be $(\theta^*, \lambda_j^{1*}, \lambda_j^{2*}, \mu_j^*)$, each inefficient $\text{DMU}_o(X_o, Y_o)$ is projected as $\text{DMU}_o^*(X_o^*, Y_o^*)$ by applying Eqs. (5.8). The results of projections and efficient scores are shown in Table B.4. Based on Theorem 7, we can calculate the right-hand and left-hand SE

by applying the following models:

$$\varepsilon^{+}_{\text{Uni1}} = \min_{\gamma^1_j,\gamma^2_j,v^1,v^2,v^S,u^1_1,u^2_1,u^1_2,u^2_2,u^S,u^R,u^1_0,u^2_0} v^1x^{1*}_o + v^2x^{2*}_o + v^Sx^{S*}_o$$

$$\text{s.t.} \begin{cases} v^1x^{1*}_o + v^2x^{2*}_o + v^Sx^{S*}_o - u^1_1y^{1*}_{1o} - u^1_2y^{1*}_{2o} - u^2_1y^{2*}_{1o} \\ -u^2_2y^{2*}_{2o} - u^Sy^{S*}_o + u^Ry^{R*}_o + u_{10} + u_{20} = 0, \\ \gamma^1_j - v^1x^{1*}_o - v^Sx^{S1*}_o + u^1_1y^{1*}_{1o} + u^1_2y^{1*}_{2o} \\ +u^Sy^{S1*}_o - u^Ry^{R1*}_o - u_{10} \le 0\ (\forall j), \\ \gamma^2_j - v^2x^{2*}_o - v^Sx^{S2*}_o + u^2_1y^{2*}_{1o} + u^2_2y^{2*}_{2o} \\ +u^Sy^{S2*}_o - u^Ry^{R2*}_o - u_{20} \le 0\ (\forall j), \\ -\gamma^1_j - u^Sy^{S1*}_o + u^Ry^{R1*}_o \le 0\ (\forall j), \\ -\gamma^2_j - u^Sy^{S2*}_o + u^Ry^{R2*}_o \le 0\ (\forall j), \\ u^1_1y^{1*}_{1o} + u^1_2y^{1*}_{2o} + u^2_1y^{2*}_{1o} + u^2_2y^{2*}_{2o} + u^Sy^{S*}_o - u^Ry^{R*}_o = 1, \\ \gamma^1_j, \gamma^2_j \ge 0(\forall j), v^1, v^2, v^S, u^1_1, u^2_1, u^1_2, u^2_2, u^S \ge 0, \\ u_{10}, u_{20}, u^R \text{ signfree}, \end{cases} \tag{5.15}$$

$$\varepsilon^{-}_{\text{Uni1}} = \max_{\gamma^1_j,\gamma^2_j,v^1,v^2,v^S,u^1_1,u^2_1,u^1_2,u^2_2,u^S,u^R,u^1_0,u^2_0} v^1x^{1*}_o + v^2x^{2*}_o + v^Sx^{S*}_o \tag{5.16}$$
$$\text{s.t. All constrains in model (5.15).}$$

By solving model (5.15) and (5.16), the results of the right-hand and left-hand SE are shown in Table 5.2.

Table 5.2: Right-hand and left-hand SE for 43 UK universities with undesirable outputs (selective weakly disposability)

| DMU | $\varepsilon^{+}_{\text{Uni1}}$ | $\varepsilon^{-}_{\text{Uni1}}$ | DMU | $\varepsilon^{+}_{\text{Uni1}}$ | $\varepsilon^{-}_{\text{Uni1}}$ | DMU | $\varepsilon^{+}_{\text{Uni1}}$ | $\varepsilon^{-}_{\text{Uni1}}$ |
|---|---|---|---|---|---|---|---|---|
| $DMU_1$ | 0.46 | 1.01 | $DMU_{16}$ | 0.15 | 0.76 | $DMU_{31}$ | 0.74 | 1.10 |
| $DMU_2$ | 0.00 | 0.64 | $DMU_{17}$ | 0.15 | 0.93 | $DMU_{32}$ | 0.31 | 0.75 |
| $DMU_3$ | 0.69 | 0.79 | $DMU_{18}$ | 0.43 | 0.54 | $DMU_{33}$ | 0.56 | 0.64 |
| $DMU_4$ | 0.00 | 0.98 | $DMU_{19}$ | 0.73 | 0.88 | $DMU_{34}$ | 0.00 | 2.70 |
| $DMU_5$ | 0.00 | 0.82 | $DMU_{20}$ | 0.28 | 0.94 | $DMU_{35}$ | 0.66 | 1.23 |
| $DMU_6$ | 0.00 | 0.69 | $DMU_{21}$ | 0.55 | 0.60 | $DMU_{36}$ | 0.96 | $+\infty$ |
| $DMU_7$ | 0.03 | 1.01 | $DMU_{22}$ | 0.81 | 0.86 | $DMU_{37}$ | 0.50 | 0.63 |
| $DMU_8$ | 0.30 | 0.66 | $DMU_{23}$ | 0.78 | 0.85 | $DMU_{38}$ | 0.66 | 0.80 |
| $DMU_9$ | 0.75 | 0.77 | $DMU_{24}$ | 0.48 | 0.51 | $DMU_{39}$ | 0.64 | 0.90 |
| $DMU_{10}$ | 0.72 | 0.74 | $DMU_{25}$ | 0.44 | 13.03 | $DMU_{40}$ | 1.32 | 1.53 |

| DMU | $\varepsilon^{+}_{\text{Uni1}}$ | $\varepsilon^{-}_{\text{Uni1}}$ | DMU | $\varepsilon^{+}_{\text{Uni1}}$ | $\varepsilon^{-}_{\text{Uni1}}$ | DMU | $\varepsilon^{+}_{\text{Uni1}}$ | $\varepsilon^{-}_{\text{Uni1}}$ |
|---|---|---|---|---|---|---|---|---|
| $\text{DMU}_{11}$ | 0.34 | 0.86 | $\text{DMU}_{26}$ | 0.30 | 0.85 | $\text{DMU}_{41}$ | 0.62 | 1.02 |
| $\text{DMU}_{12}$ | 0.74 | 0.87 | $\text{DMU}_{27}$ | 0.71 | 1.13 | $\text{DMU}_{42}$ | 0.78 | $+\infty$ |
| $\text{DMU}_{13}$ | 0.37 | 0.75 | $\text{DMU}_{28}$ | 0.77 | 0.94 | $\text{DMU}_{43}$ | 0.81 | 1.12 |
| $\text{DMU}_{14}$ | 0.51 | 0.96 | $\text{DMU}_{29}$ | 0.47 | 0.85 | | | |
| $\text{DMU}_{15}$ | 0.84 | 1.14 | $\text{DMU}_{30}$ | 0.45 | 1.17 | | | |

Comparing with the results in Tables 3.4 and 5.2, projected $\text{DMU}_{36}$ and $\text{DMU}_{43}$ cannot further increase the scale within $T_{\text{Uni1}}$. Therefore, we can say that considering retraction as an undesirable output under the assumption of selective weakly disposability has a negative impact on the scale expansion of $\text{DMU}_{36}$ and $\text{DMU}_{43}$. In this case, only $\text{DMU}_{40}$ can further increase the scale of inputs.

**Scenario 2**

Assuming $y^R \in O^{US}$, $y^S \in O^{DS}$ and $y^1_1,\ y^2_1,\ y^1_2,\ y^2_2 \in \bigcup_{k=1,2} O^{Dk}$, based on the production technology (5.6), the according production technology of universities is stated as:

$$T_{\text{Uni2}} = \left\{ (X,Y) \in \mathbb{R}^9_+ \;\middle|\; \begin{array}{l} \sum_{j=1}^n \lambda^1_j x^1_j \le x^1_o, \sum_{j=1}^n \lambda^2_j x^2_j \le x^2_o, \\ \sum_{j=1}^n \lambda^1_j x^{S1}_j + \sum_{j=1}^n \lambda^2_j x^{S2}_j \le x^S_o, \\ \sum_{j=1}^n \lambda^1_j y^1_{1j} \ge y^1_{1o}, \sum_{j=1}^n \lambda^1_j y^1_{2j} \ge y^1_{2o}, \\ \sum_{j=1}^n \lambda^2_j y^2_{1j} \ge y^2_{1o}, \sum_{j=1}^n \lambda^2_j y^2_{2j} \ge y^2_{2o}, \\ \sum_{j=1}^n \lambda^1_j y^{S1}_j + \sum_{j=1}^n \lambda^2_j y^{S2}_j \ge y^S_o, \\ \sum_{j=1}^n \lambda^1_j y^{R1}_j + \sum_{j=1}^n \lambda^2_j y^{R2}_j \le y^R_o, \\ \sum_{j=1}^n \lambda^1_j = \sum_{j=1}^n \lambda^2_j = 1, \lambda^1_j, \lambda^2_j \ge 0. \end{array} \right\} \tag{5.17}$$

Then, we use output-oriented model to project all inefficient DMUs to

the efficient frontier of the production technology $T_{\text{Uni2}}$:

$$\frac{1}{E_0^{UP-O}} = \max_{\theta,\lambda_j^1,\lambda_j^2} 1+\theta$$
$$\text{s.t.} \begin{cases} \sum_{j=1}^n \lambda_j^1 x_j^1 \le x_o^1, \sum_{j=1}^n \lambda_j^2 x_j^2 \le x_o^2, \\ \sum_{j=1}^n \lambda_j^1 x_j^{S1} + \sum_{j=1}^n \lambda_j^2 x_j^{S2} \le x_o^S, \\ \sum_{j=1}^n \lambda_j^1 y_{1j}^1 \ge (1+\theta) y_{1o}^1, \sum_{j=1}^n \lambda_j^1 y_{2j}^1 \ge (1+\theta) y_{2o}^1, \\ \sum_{j=1}^n \lambda_j^2 y_{1j}^2 \ge (1+\theta) y_{1o}^2, \sum_{j=1}^n \lambda_j^2 y_{2j}^2 \ge (1+\theta) y_{2o}^2, \\ \sum_{j=1}^n \lambda_j^1 y_j^{S1} + \sum_{j=1}^n \lambda_j^2 y_j^{S2} \ge (1+\theta) y_o^S, \\ \sum_{j=1}^n \lambda_j^1 y_j^{R1} + \sum_{j=1}^n \lambda_j^2 y_j^{R2} \le (1-\theta) y_o^R, \\ \sum_{j=1}^n \lambda_j^1 = \sum_{j=1}^n \lambda_j^2 = 1, \lambda_j^1, \lambda_j^2 \ge 0, \theta \text{ signfree.} \end{cases} \tag{5.18}$$

Let the optimal solution vector of model (5.18) be $(\theta^*, \lambda_j^{1*}, \lambda_j^{2*})$, each inefficient $\text{DMU}_o(X_o, Y_o)$ is projected as $\text{DMU}_o^*(X_o^*, Y_o^*)$ by applying Eqs. (5.8). The results of projections and efficient scores are shown in Table B.5. Based on Theorem 7, we can calculate the right-hand and left-hand SE by applying the following models:

$$\varepsilon_{\text{Uni1}}^+ = \min_{v^1,v^2,v^S,u_1^1,u_1^2,u_2^1,u_2^2,u^S,u^R,u_0^1,u_0^2} v^1 x_o^{1*} + v^2 x_o^{2*} + v^S x_o^{S*}$$
$$\text{s.t.} \begin{cases} v^1 x_o^{1*} + v^2 x_o^{2*} + v^S x_o^{S*} - u_1^1 y_{1o}^{1*} - u_2^1 y_{2o}^{1*} - u_1^2 y_{1o}^{2*} \\ -u_2^2 y_{2o}^{2*} - u^S y_o^{S*} + u^R y_o^{R*} + u_{10} + u_{20} = 0, \\ -v^1 x_o^{1*} - v^S x_o^{S1*} + u_1^1 y_{1o}^{1*} + u_2^1 y_{2o}^{1*} \\ +u^S y_o^{S1*} - u^R y_o^{R1*} - u_{10} \le 0 \, (\forall j), \\ -v^2 x_o^{2*} - v^S x_o^{S2*} + u_1^2 y_{1o}^{2*} + u_2^2 y_{2o}^{2*} \\ +u^S y_o^{S2*} - u^R y_o^{R2*} - u_{20} \le 0 \, (\forall j), \\ u_1^1 y_{1o}^{1*} + u_2^1 y_{2o}^{1*} + u_1^2 y_{1o}^{2*} + u_2^2 y_{2o}^{2*} + u^S y_o^{S*} + u^R y_o^{R*} = 1, \\ \gamma_j^1, \gamma_j^2 \ge 0 (\forall j), v^1, v^2, v^S, u_1^1, u_1^2, u_2^1, u_2^2, u^S \ge 0, \\ u_{10}, u_{20}, u^R \text{ signfree,} \end{cases} \tag{5.19}$$

$$\varepsilon_{\text{Uni1}}^- = \max_{v^1,v^2,v^S,u_1^1,u_1^2,u_2^1,u_2^2,u^S,u^R,u_0^1,u_0^2} v^1 x_o^{1*} + v^2 x_o^{2*} + v^S x_o^{S*}$$
$$\text{s.t. All constrains in model (5.19).} \tag{5.20}$$

By solving models (5.19) and (5.20), the results of the right-hand and left-hand SE in are shown in Table 5.3.

Table 5.3: Right-hand and left-hand SE for 43 UK universities with undesirable outputs (strongly disposability)

| DMU | $\varepsilon^{+}_{\text{Uni2}}$ | $\varepsilon^{-}_{\text{Uni2}}$ | DMU | $\varepsilon^{+}_{\text{Uni2}}$ | $\varepsilon^{-}_{\text{Uni2}}$ | DMU | $\varepsilon^{+}_{\text{Uni2}}$ | $\varepsilon^{-}_{\text{Uni2}}$ |
|---|---|---|---|---|---|---|---|---|
| $DMU_1$ | 0.56 | 0.97 | $DMU_{16}$ | 0.61 | 0.84 | $DMU_{31}$ | 0.70 | 0.93 |
| $DMU_2$ | 0.00 | 0.27 | $DMU_{17}$ | 0.00 | 0.63 | $DMU_{32}$ | 0.24 | 0.74 |
| $DMU_3$ | 0.69 | 0.83 | $DMU_{18}$ | 0.00 | 0.63 | $DMU_{33}$ | 0.59 | 0.59 |
| $DMU_4$ | 0.00 | 0.98 | $DMU_{19}$ | 0.69 | 0.89 | $DMU_{34}$ | 0.00 | 2.70 |
| $DMU_5$ | 0.22 | 0.55 | $DMU_{20}$ | 0.24 | 0.85 | $DMU_{35}$ | 0.49 | 1.15 |
| $DMU_6$ | 0.00 | 0.69 | $DMU_{21}$ | 0.54 | 0.56 | $DMU_{36}$ | 0.73 | $+\infty$ |
| $DMU_7$ | 0.00 | 0.99 | $DMU_{22}$ | 0.74 | 0.85 | $DMU_{37}$ | 0.62 | 0.74 |
| $DMU_8$ | 0.30 | 0.64 | $DMU_{23}$ | 0.80 | 0.85 | $DMU_{38}$ | 0.60 | 0.75 |
| $DMU_9$ | 0.54 | 0.65 | $DMU_{24}$ | 0.40 | 0.52 | $DMU_{39}$ | 0.32 | 1.00 |
| $DMU_{10}$ | 0.49 | 0.49 | $DMU_{25}$ | 0.36 | 13.09 | $DMU_{40}$ | 0.78 | 1.53 |
| $DMU_{11}$ | 0.23 | 0.85 | $DMU_{26}$ | 0.30 | 0.65 | $DMU_{41}$ | 0.23 | 0.59 |
| $DMU_{12}$ | 0.67 | 0.88 | $DMU_{27}$ | 0.67 | 1.12 | $DMU_{42}$ | 0.82 | $+\infty$ |
| $DMU_{13}$ | 0.00 | 0.19 | $DMU_{28}$ | 0.73 | 0.91 | $DMU_{43}$ | 0.93 | 1.12 |
| $DMU_{14}$ | 0.53 | 0.96 | $DMU_{29}$ | 0.45 | 0.84 | | | |
| $DMU_{15}$ | 0.82 | 1.14 | $DMU_{30}$ | 0.72 | 0.76 | | | |

Comparing with the results in Tables 3.4 and 5.3, projected $DMU_{36}$, $DMU_{40}$ and $DMU_{43}$ cannot be further increase the scale within $T_{\text{Uni2}}$. Therefore, considering retraction as an undesirable output under the assumption of strongly disposability, all of the universities cannot increase their scales.

## 5.3 Estimating DSE in multi-stage technologies with undesirable outputs

### 5.3.1 Expressions of multi-stage technologies with undesirable outputs

On the basis of a closed two-stage production technology $T_S$ as shown in (3.41), we further consider an open production system represented by $T_S^U$, which includes multiple undesirable outputs. The set of all intermediate products and outputs need to be decomposed into the following disjoint subsets:

$$\begin{aligned} C = C^D \cup C^{DW}, O^{(1)} = O^{U(1)} \cup O^{D(1)} \cup O^{DW(1)} \cup O^{UW(1)}, \\ O^{(2)} = O^{U(2)} \cup O^{D(2)} \cup O^{DW(2)} \cup O^{UW(2)}. \end{aligned} \tag{5.21}$$

We assume that the undesirable outputs produced in stage 1 cannot be used as inputs for stage 2. Specifically, $O^{(1)}$ represents the union of exogenous outputs produced in stage 1, in which $O^{D(1)} \cup O^{DW(1)}$ and $O^{U(1)} \cup O^{UW(1)}$ represent the set of desirable and undesirable outputs, respectively. Similarly, $O^{(2)}$ represents the union of final outputs produced in stage 2, in which $O^{D(2)} \cup O^{DW(2)}$ and $O^{U(2)} \cup O^{UW(2)}$ represent the set of desirable and undesirable outputs, respectively. Moreover, in stage 1, the undesirable outputs in $O^{UW(1)}$ and desirable outputs in $C^{DW} \cup O^{DW(1)}$ are jointly weakly disposable and can be reduced by the same abatement factor $\pi^{(1)} \in [0,1]$. In stage 2, the undesirable outputs in $O^{UW(2)}$ and desirable outputs in $O^{DW(2)}$ are jointly weakly disposable and can be reduced by the same abatement factor $\pi^{(2)} \in [0,1]$. The desirable outputs in $O^{D(1)}$ and $O^{D(1)}$, and the undesirable outputs in $O^{U(1)}$ and $O^{U(1)}$ are assumed as strongly disposability. Note that $T_S^U$ is equivalent to $T_S$ when $O^{(1)} \cup O^{U(2)} \cup O^{DW(2)} \cup O^{UW(2)} = \varnothing$ holds.

Based on the decomposed set of intermediate products and outputs shown in (5.21), the vector of intermediate products, exogenous outputs and final outputs are stated as follows:

$$\begin{aligned} Z = (Z^D; Z^{DW}), Y^{(1)} = (Y^{U(1)}; Y^{D(1)}; Y^{DW(1)}; Y^{UW(1)}), \\ Y^{(2)} = (Y^{U(2)}; Y^{D(2)}; Y^{DW(2)}; Y^{UW(2)}), \end{aligned} \tag{5.22}$$

where $Z^D = (z_1^D, \ldots, z_{|C^D|}^D)^\top$, $Z^{DW} = (z_1^{DW}, \ldots, z_{|C^{DW}|}^{DW})^\top$, $Y^{D(1)} = (y_1^{D(1)}, \ldots, y_{|O^{D(1)}|}^{D(1)})^\top$, $Y^{U(1)} = (y_1^{U(1)}, \ldots, y_{|O^{U(1)}|}^{U(1)})^\top$, $Y^{DW(1)} = (y_1^{DW(1)}, \ldots, y_{|O^{DW(1)}|}^{DW(1)})^\top$, $Y^{UW(1)} = (y_1^{UW(1)}, \ldots, y_{|O^{UW(1)}|}^{UW(1)})^\top$, $Y^{D(2)} = (y_1^{D(2)}, \ldots, y_{|O^{D(2)}|}^{D(2)})^\top$, $Y^{U(2)} = (y_1^{U(2)}, \ldots, y_{|O^{U(2)}|}^{U(2)})^\top$, $Y^{DW(2)} = (y_1^{DW(2)}, \ldots, y_{|O^{DW(2)}|}^{DW(2)})^\top$ and $Y^{UW(2)} = (y_1^{UW(2)}, \ldots, y_{|O^{UW(2)}|}^{UW(2)})^\top$.

Then, the axioms of $T_S$ presented in Section 3.2.4 need to be modified. We assume that $T_S^U$ satisfies several standard axioms as follows:

- **A1*** Selective strongly disposability: If $(X, Z, Y) \in T_S^U$, we have $(X^*, Z^D, Z^{DW}, Y^{U(1)*}, Y^{D(1)*}, Y^{DW(1)*}, Y^{UW(1)*}, Y^{U(2)*}, Y^{D(2)*}, Y^{DW(2)*}, Y^{UW(2)*}) \in T_S^U$ (for all $X^* \geq X, 0, \leq Y^{D(1)*} \leq Y^{D(1)}, Y^{U(1)*} \geq Y^{U(1)}, 0 \leq Y^{DW(1)*} \leq Y^{DW(1)}, 0, \leq Y^{D(2)*} \leq Y^{D(2)}, Y^{U(2)*} \geq Y^{U(2)}$ and $0 \leq Y^{DW(2)*} \leq Y^{DW(2)}$).

- **A2** Convexity: Technology $T_S^U$ is a convex set.

- **A3** Feasibility of all observed DMUs: For any $j \in N$, $\left(X_j, Z_j, Y_j\right) \in T_S^U$.

- **A4** Selective weakly disposability: If $(X, Z, Y) \in T_S^U$, we have $(X, Z^D, \pi^{(1)} Z^{DW}, Y^{U(1)}, Y^{D(1)}, \pi^{(1)} Y^{DW(1)}, \pi^{(1)} Y^{UW(1)}, Y^{U(2)}, Y^{D(2)}, \pi^{(2)} Y^{DW(2)},$

$\pi^{(2)}Y^{UW(2)}) \in T_S^U$ (for all $\pi^{(1)}, \pi^{(2)} \in [0,1]$).

By employing the above axioms, the production technology $T_S^U$ is stated as:

$$T_S^U = \left\{ (X,Z,Y) \in \mathbb{R}_+^{|I|+|C|+|O|} \;\middle|\; \begin{array}{l} \sum_{j=1}^n \lambda_j^{(1)} x_{ij} \le x_i (\forall i), \\ \sum_{j=1}^n \lambda_j^{(1)} z_{fj}^D \ge z_f^D (f \in C^D), \\ \sum_{j=1}^n \lambda_j^{(2)} z_{fj}^D \le z_f (f \in C^D), \\ \sum_{j=1}^n \lambda_j^{(1)} \pi_j^{(1)} z_{fj}^{DW} \ge z_f^{DW} (f \in C^{DW}), \\ \sum_{j=1}^n \lambda_j^{(1)} z_{fj}^{DW} \le z_f^{DW} (f \in C^{DW}), \\ \sum_{j=1}^n \lambda_j^{(1)} y_{rj}^{U(1)} \le y_r^{U(1)} (r \in O^{U(1)}), \\ \sum_{j=1}^n \lambda_j^{(1)} y_{rj}^{D(1)} \ge y_r^{D(1)} (r \in O^{D(1)}), \\ \sum_{j=1}^n \lambda_j^{(1)} \pi_j^{(1)} y_{rj}^{DW(1)} \ge y_r^{DW(1)} (r \in O^{DW(1)}), \\ \sum_{j=1}^n \lambda_j^{(1)} \pi_j^{(1)} y_{rj}^{UW(1)} = y_r^{UW(1)} (r \in O^{UW(1)}), \\ \sum_{j=1}^n \lambda_j^{(2)} y_{rj}^{U(2)} \le y_r^{U(2)} (r \in O^{U(2)}), \\ \sum_{j=1}^n \lambda_j^{(2)} y_{rj}^{D(2)} \ge y_r^{D(2)} (r \in O^{D(2)}), \\ \sum_{j=1}^n \lambda_j^{(2)} \pi_j^{(2)} y_{rj}^{DW(2)} \ge y_r^{DW(2)} (r \in O^{DW(2)}), \\ \sum_{j=1}^n \lambda_j^{(2)} \pi_j^{(2)} y_{rj}^{UW(2)} = y_r^{UW(2)} (r \in O^{UW(2)}), \\ \sum_{j=1}^n \lambda_j^{(1)} = 1, \sum_{j=1}^n \lambda_j^{(2)} = 1, \\ \lambda_j^{(1)}, \lambda_j^{(2)} \ge 0 (\forall j), \pi_j^{(1)}, \pi_j^{(2)} \in [0,1]. \end{array} \right\} \tag{5.23}$$

To linearise the nonlinear terms $\lambda_j^{(1)}\pi_j^{(1)}$ and $\lambda_j^{(2)}\pi_j^{(2)}$, let $\lambda_j^{(1)} = \lambda_j^{(1)} - \mu_j^{(1)}$ and $\lambda_j^{(2)} = \lambda_j^{(2)} - \mu_j^{(2)}$ hold, technology (5.23) can be restated as:

$$T_S^U = \left\{ (X,Z,Y) \in \mathbb{R}_+^{|I|+|C|+|O|} \;\middle|\; \begin{array}{l} \sum_{j=1}^n \lambda_j^{(1)} x_{ij} \le x_i (\forall i), \\ \sum_{j=1}^n \lambda_j^{(1)} z_{fj}^D \ge z_f^D (f \in C^D), \\ \sum_{j=1}^n \lambda_j^{(2)} z_{fj}^D \le z_f^D (f \in C^D), \\ \sum_{j=1}^n (\lambda_j^{(1)} - \mu_j^{(1)}) z_{fj}^{DW} \ge z_f^{DW} (f \in C^{DW}), \\ \sum_{j=1}^n \lambda_j^{(1)} z_{fj}^{DW} \le z_f^{DW} (f \in C^{DW}), \\ \sum_{j=1}^n \lambda_j^{(1)} y_{rj}^{U(1)} \le y_r^{U(1)} (r \in O^{U(1)}), \\ \sum_{j=1}^n \lambda_j^{(1)} y_{rj}^{D(1)} \ge y_r^{D(1)} (r \in O^{D(1)}), \\ \sum_{j=1}^n (\lambda_j^{(1)} - \mu_j^{(1)}) y_{rj}^{DW(1)} \ge y_r^{DW(1)} (r \in O^{DW(1)}), \\ \sum_{j=1}^n (\lambda_j^{(1)} - \mu_j^{(1)}) y_{rj}^{UW(1)} = y_r^{UW(1)} (r \in O^{UW(1)}), \\ \sum_{j=1}^n \lambda_j^{(2)} y_{rj}^{U(2)} \le y_r^{U(2)} (r \in O^{U(2)}), \\ \sum_{j=1}^n \lambda_j^{(2)} y_{rj}^{D(2)} \ge y_r^{D(2)} (r \in O^{D(2)}), \\ \sum_{j=1}^n (\lambda_j^{(2)} - \mu_j^{(2)}) y_{rj}^{DW(2)} \ge y_r^{DW(2)} (r \in O^{DW(2)}), \\ \sum_{j=1}^n (\lambda_j^{(2)} - \mu_j^{(2)}) y_{rj}^{UW(2)} = y_r^{UW(2)} (r \in O^{UW(2)}), \\ \sum_{j=1}^n \lambda_j^{(1)} = 1, \sum_{j=1}^n \lambda_j^{(2)} = 1, \lambda_j^{(1)} - \mu_j^{(1)} \ge 0, \\ \lambda_j^{(2)} - \mu_j^{(2)} \ge 0, \lambda_j^{(1)}, \lambda_j^{(2)}, \mu_j^{(1)}, \mu_j^{(2)} \ge 0 (\forall j). \end{array} \right\} \tag{5.24}$$

The following subsection will develop the model of estimating and decomposing SE on the frontier of technology (5.24).

### 5.3.2 DSE estimation

Regarding the production technology (5.24), the output oriented DEA model is stated as follows:

$$\frac{1}{E_0^{US-O}} = \max_{\theta,\lambda_j^{(1)},\lambda_j^{(2)},\mu_j^{(1)},\mu_j^{(2)},z_f^D,z_f^{DW}} 1+\theta$$

$$\text{s.t.} \begin{cases} \sum_{j=1}^n \lambda_j^{(1)} x_{ij} \le x_{io} (\forall i), \\ \sum_{j=1}^n \lambda_j^{(1)} z_{fj}^D \ge z_f^D (f \in C^D), \\ \sum_{j=1}^n \lambda_j^{(2)} z_{fj}^D \le z_f^D (f \in C^D), \\ \sum_{j=1}^n (\lambda_j^{(1)} - \mu_j^{(1)}) z_{fj}^{DW} \ge z_f^{DW} (f \in C^{DW}), \\ \sum_{j=1}^n \lambda_j^{(1)} z_{fj}^{DW} \le z_f^{DW} (f \in C^{DW}), \\ \sum_{j=1}^n \lambda_j^{(1)} y_{rj}^{U(1)} \le y_{ro}^{U(1)} (r \in O^{U(1)}), \\ \sum_{j=1}^n \lambda_j^{(1)} y_{rj}^{D(1)} \ge y_{ro}^{D(1)} (r \in O^{D(1)}), \\ \sum_{j=1}^n (\lambda_j^{(1)} - \mu_j^{(1)}) y_{rj}^{DW(1)} \ge y_{ro}^{DW(1)} (r \in O^{DW(1)}), \\ \sum_{j=1}^n (\lambda_j^{(1)} - \mu_j^{(1)}) y_{rj}^{UW(1)} = y_{ro}^{UW(1)} (r \in O^{UW(1)}), \\ \sum_{j=1}^n \lambda_j^{(2)} y_{rj}^{U(2)} \le (1-\theta) y_{ro}^{U(2)} (r \in O^{U(2)}), \\ \sum_{j=1}^n \lambda_j^{(2)} y_{rj}^{D(2)} \ge (1+\theta) y_{ro}^{D(2)} (r \in O^{D(2)}), \\ \sum_{j=1}^n (\lambda_j^{(2)} - \mu_j^{(2)}) y_{rj}^{DW(2)} \ge (1+\theta) y_{ro}^{DW(2)} (r \in O^{DW(2)}), \\ \sum_{j=1}^n (\lambda_j^{(2)} - \mu_j^{(2)}) y_{rj}^{UW(2)} = (1+\theta) y_{ro}^{UW(2)} (r \in O^{UW(2)}), \\ \sum_{j=1}^n \lambda_j^{(1)} = 1, \sum_{j=1}^n \lambda_j^{(2)} = 1, \\ \lambda_j^{(1)} - \mu_j^{(1)} \ge 0, \lambda_j^{(2)} - \mu_j^{(2)} \ge 0, \\ \lambda_j^{(1)}, \lambda_j^{(2)}, \mu_j^{(1)}, \mu_j^{(2)} \ge 0 (\forall j), \\ \theta, z_f^D, z_f^{DW} \text{ signfree } (f \in C^D \cup C^{DW}). \end{cases} \tag{5.25}$$

Let $(\theta^*, z_f^{D*}, z_f^{DW*}, \lambda_j^{(1)*}, \mu_j^{(1)*}, \lambda_j^{(2)*}, \mu_j^{(2)*})$ represent the optimal value of model (5.25). $\text{DMU}_o$ is defined as the output radial efficient DMU if and only if the optimal solution of the acquired from model (5.25) satisfies $\theta^* = 1$. If not, the inefficient DMUs can be projected onto the efficient frontier by applying the following equations:

$$x_{io}^* = \sum_{j=1}^n \lambda_j^{(1)*} x_{ij} (\forall i), z_{fo}^{D*} = z_f^{D*} (f \in C^D), z_{fo}^{DW*} = z_f^{DW*} (f \in C^{DW})$$

$$y_{ro}^{D(1)*} = \sum_{j=1}^n \lambda_j^{(1)*} y_{rj}^D (r \in O^{D(1)}), y_{ro}^{U(1)*} = \sum_{j=1}^n \lambda_j^{(1)*} y_{rj}^U (r \in O^{U(1)}),$$

$$y_{ro}^{DW(1)*} = \sum_{j=1}^n (\lambda_j^{(1)*} - \mu_j^{(1)*}) y_{rj}^{DW} (r \in O^{DW(1)}),$$

$$
\begin{aligned}
y_{ro}^{UW(1)*} &= \sum_{j=1}^{n}(\lambda_j^{(2)*} - \mu_j^{(1)*})y_{rj}^{UW}(r \in O^{UW(1)}),\\
y_{ro}^{D(2)*} &= \sum_{j=1}^{n}\lambda_j^{(2)*}y_{rj}^{D}(r \in O^{D(2)}), y_{ro}^{U(2)*} = \sum_{j=1}^{n}\lambda_j^{(2)*}y_{rj}^{U}(r \in O^{U(2)}),\\
y_{ro}^{DW(2)*} &= \sum_{j=1}^{n}(\lambda_j^{(2)*} - \mu_j^{(2)*})y_{rj}^{DW}(r \in O^{DW(2)}),\\
y_{ro}^{UW(2)*} &= \sum_{j=1}^{n}(\lambda_j^{(2)*} - \mu_j^{(2)*})y_{rj}^{UW}(r \in O^{UW(2)}).
\end{aligned}
\tag{5.26}
$$

All efficient $\mathrm{DMU}_o(X_o^*, Z_o^*, Y_o^*)$ by applying model (5.25) and Eqs. (5.26) can be obtained. For all output-radial efficient $\mathrm{DMU}_o^*$ in the technology $T_S^U$, the response function of outputs is defined as follows:

$$
\begin{aligned}
\beta(\varphi(t)) = \max\Big\{\beta \,|\Big((1+t)X_o^*, (1+\varphi)Z_o^*, (1-\varphi)Y_o^{U(1)*}, (1+\varphi)Y_o^{D(1)*},\\
(1+\varphi)Y_o^{DW(1)*}, (1+\varphi)Y_o^{UW(1)*}, (1-\beta)Y_o^{U(2)*}, (1+\beta)Y_o^{D(2)*},\\
(1+\beta)Y_o^{DW(2)*}, (1+\beta)Y_o^{UW(2)*}\Big) \in T_P^U, \quad \beta \in \mathbb{R}\Big\}.
\end{aligned}
\tag{5.27}
$$

where we assume $\beta = \varphi = 0$ when $t = 0$ to indicate that $Y_o^*$ and $Z_o^*$ cannot be changed within $T_P^U$ on condition that $X_o^*$ keeps unchanged.

To estimate SE for efficient $\mathrm{DMU}_o^*(X_o^*, Z_o^*, Y_o^*) \in T_S^U$ in the overall system, the first stage and the second stage, $\beta(t)$, $\varphi(t)$ and $\beta(\varphi)$ are expressed as

following linear programs, respectively:

$$\beta(t) = \max_{\beta,\varphi,\lambda_j^{(1)},\lambda_j^{(2)},\mu_j^{(1)},\mu_j^{(2)}} \beta$$

$$\text{s.t.} \begin{cases} \sum_{j=1}^{n} \lambda_j^{(1)} x_{ij}^{*} \le (1+t) x_{io}^{*} (\forall i), \\ \sum_{j=1}^{n} \lambda_j^{(1)} z_{fj}^{D*} \ge (1+\varphi) z_{fo}^{D*} (f \in C^{D}), \\ \sum_{j=1}^{n} \lambda_j^{(2)} z_{fj}^{D*} \le (1+\varphi) z_{fo}^{D*} (f \in C^{D}), \\ \sum_{j=1}^{n} (\lambda_j^{(1)} - \mu_j^{(1)}) z_{fj}^{DW*} \ge (1+\varphi) z_{fo}^{DW*} (f \in C^{DW}), \\ \sum_{j=1}^{n} \lambda_j^{(1)} z_{fj}^{DW*} \le (1+\varphi) z_{fo}^{DW*} (f \in C^{DW}), \\ \sum_{j=1}^{n} \lambda_j^{(1)} y_{rj}^{U(1)*} \le (1-\varphi) y_{ro}^{U(1)*} (r \in O^{U(1)}), \\ \sum_{j=1}^{n} \lambda_j^{(1)} y_{rj}^{D(1)*} \ge (1+\varphi) y_{ro}^{D(1)*} (r \in O^{D(1)}), \\ \sum_{j=1}^{n} (\lambda_j^{(1)} - \mu_j^{(1)}) y_{rj}^{DW(1)*} \ge (1+\varphi) y_{ro}^{DW(1)*} (r \in O^{DW(1)}), \\ \sum_{j=1}^{n} (\lambda_j^{(1)} - \mu_j^{(1)}) y_{rj}^{UW(1)*} = (1+\varphi) y_{ro}^{UW(1)*} (r \in O^{UW(1)}), \\ \sum_{j=1}^{n} \lambda_j^{(2)} y_{rj}^{U(2)*} \le (1-\beta) y_{ro}^{U(2)*} (r \in O^{U(2)}), \\ \sum_{j=1}^{n} \lambda_j^{(2)} y_{rj}^{D(2)*} \ge (1+\beta) y_{ro}^{D(2)*} (r \in O^{D(2)}), \\ \sum_{j=1}^{n} (\lambda_j^{(2)} - \mu_j^{(2)}) y_{rj}^{DW(2)*} \ge (1+\beta) y_{ro}^{DW(2)*} (r \in O^{DW(2)}), \\ \sum_{j=1}^{n} (\lambda_j^{(2)} - \mu_j^{(2)}) y_{rj}^{UW(2)*} = (1+\beta) y_{ro}^{UW(2)*} (r \in O^{UW(2)}), \\ \sum_{j=1}^{n} \lambda_j^{(1)} = 1, \sum_{j=1}^{n} \lambda_j^{(2)} = 1, \\ \lambda_j^{(1)} - \mu_j^{(1)} \ge 0, \lambda_j^{(2)} - \mu_j^{(2)} \ge 0, \\ \lambda_j^{(1)}, \lambda_j^{(2)}, \mu_j^{(1)}, \mu_j^{(2)} \ge 0 (\forall j), \varphi, \beta \text{ signfree}, \end{cases} \tag{5.28}$$

$$\varphi(t) = \max_{\beta,\varphi,\lambda_j^{(1)},\lambda_j^{(2)},\mu_j^{(1)},\mu_j^{(2)}} \varphi \quad \text{s.t. all constrains in model (5.28)}, \tag{5.29}$$

$$\beta(\varphi) = \max_{\beta,t,\lambda_j^{(1)},\lambda_j^{(2)},\mu_j^{(1)},\mu_j^{(2)}} \beta$$

s.t. all constraints in model (5.28), where $\varphi$ is converted into a scalar, $t$ is converted into an unknown variable satisfying $t \ge 0$. (5.30)

Based on the DDO method, the methods of estimating SE in the production technology (5.24) is proposed as following theorem:

**Theorem 8.** *Let* $(X_o^*, Z_o^*, Y_o^*) \in T_S^U$ *be defined by* (5.24) *and* $(X_o^*, Z_o^*, Y_o^*)$ *be an output-radial efficient point, then the right-hand SE in the overall system,*

*the first stage and the second stage are estimated by the following problems, respectively:*

$$
\begin{aligned}
\varepsilon_U^{(S)+} = \min_{\substack{v_i,\tau_f^{D(1)},\tau_f^{D(2)},\tau_f^{DW(1)},\tau_f^{DW(2)},u_r^{U(1)},u_r^{D(1)},u_r^{DW(1)},\\ u_r^{UW(1)},u_r^{U(2)},u_r^{D(2)},u_r^{DW(2)},u_r^{UW(2)},u_{10},u_{20}}} & \sum_{i=1}^{m} v_i x_{io}^* \\
s.t. \quad & \left\{ \begin{array}{l}
\sum_{i=1}^{m} v_i x_{io}^* - \sum_{f\in C^D} \tau_f^{D(1)} z_{fo}^{D*} + \sum_{f\in C^D} \tau_f^{D(2)} z_{fo}^{D*} \\
- \sum_{f\in C^{DW}} \tau_f^{DW(1)} z_{fo}^{DW*} + \sum_{f\in C^{DW}} \tau_f^{DW(2)} z_{fo}^{DW*} + \sum_{r\in O^{U(1)}} u_r y_{ro}^{U(1)*} \\
- \sum_{r\in O^{D(1)}} u_r y_{ro}^{D(1)*} - \sum_{r\in O^{DW(1)}} u_r y_{ro}^{DW(1)*} + \sum_{r\in O^{UW(1)}} u_r y_{ro}^{UW(1)*} \\
+ \sum_{r\in O^{U(2)}} u_r y_{ro}^{U(2)*} - \sum_{r\in O^{D(2)}} u_r y_{ro}^{D(2)*} - \sum_{r\in O^{DW(2)}} u_r y_{ro}^{DW(2)*} \\
+ \sum_{r\in O^{UW(2)}} u_r y_{ro}^{UW(2)*} + u_{10} + u_{20} = 0, \\
\gamma_j^1 - \sum_{i=1}^{m} v_i x_{ij}^* + \sum_{f\in C^D} \tau_f^{D(1)} z_{fj}^{D*} + \sum_{f\in C^{DW}} \tau_f^{DW(1)} z_{fj}^{DW*} \\
- \sum_{r\in O^{U(1)}} u_r y_{rj}^{U(1)*} + \sum_{r\in O^{D(1)}} u_r y_{rj}^{D(1)*} + \sum_{r\in O^{DW(1)}} u_r y_{rj}^{DW(1)*} \\
- \sum_{r\in O^{UW(1)}} u_r y_{rj}^{UW(1)*} - u_{10} \le 0 \ (\forall j), \\
\gamma_j^2 - \sum_{f\in C^D} \tau_f^{D(2)} z_{fj}^{D*} - \sum_{f\in C^{DW}} \tau_f^{DW(2)} z_{fj}^{DW*} - \sum_{r\in O^{U(2)}} u_r y_{rj}^{U(2)*} \\
+ \sum_{r\in O^{D(2)}} u_r y_{rj}^{D(2)*} + \sum_{r\in O^{DW(2)}} u_r y_{rj}^{DW(2)*} - \sum_{r\in O^{UW(2)}} u_r y_{rj}^{UW(2)*} \\
- u_{20} \le 0 \ (\forall j), \\
-\gamma_j^1 - \sum_{f\in C^{DW}} \tau_f^{DW(1)} z_{fj}^{DW*} - \sum_{r\in O^{DW(1)}} u_r y_{rj}^{DW(1)*} \\
+ \sum_{r\in O^{UW(1)}} u_r y_{rj}^{UW(1)*} \le 0 \ (\forall j), \\
-\gamma_j^2 - \sum_{r\in O^{DW(2)}} u_r y_{rj}^{DW(2)*} + \sum_{r\in O^{UW(2)}} u_r y_{rj}^{UW(2)*} \le 0 \ (\forall j), \\
\sum_{f\in C^D} \tau_f^{D(1)} z_{fo}^{D*} - \sum_{f\in C^D} \tau_f^{D(2)} z_{fo}^{D*} + \sum_{f\in C^{DW}} \tau_f^{DW(1)} z_{fo}^{DW*} \\
- \sum_{f\in C^{DW}} \tau_f^{DW(2)} z_{fo}^{DW*} + \sum_{r\in O^{U(1)}} u_r y_{ro}^{U(1)*} + \sum_{r\in O^{D(1)}} u_r y_{ro}^{D(1)*} \\
+ \sum_{r\in O^{DW(1)}} u_r y_{ro}^{DW(1)*} - \sum_{r\in O^{UW(1)}} u_r y_{ro}^{UW(1)*} = 0, \\
\sum_{r\in O^{U(2)}} u_r y_{ro}^{U(2)*} + \sum_{r\in O^{D(2)}} u_r y_{ro}^{D(2)*} + \sum_{r\in O^{DW(2)}} u_r y_{ro}^{DW(2)*} \\
- \sum_{r\in O^{UW(2)}} u_r y_{ro}^{UW(2)*} = 1, \\
v_i, \tau_f^{D(1)}, \tau_f^{D(2)}, \tau_f^{DW(1)}, \tau_f^{DW(2)}, u_r^{U(1)}, u_r^{D(1)}, u_r^{DW(1)}, u_r^{U(2)}, \\
u_r^{D(2)}, u_r^{DW(2)} \ge 0 \ (\forall i, f \in C^D \cup C^{DW}, r \in (O^{(1)} \cup O^{(2)}) \\
\backslash (O^{UW1} \cup O^{UW2})), u_r^{UW(1)}, u_r^{UW(2)}, \\
u_{10}, u_{20} \ \ signfree(r \in O^{UW1} \cup O^{UW2}).
\end{array} \right.
\end{aligned}
\tag{5.31}
$$

$$\varepsilon^{(SU1)+} = \min_{\substack{v_i,\tau_f^{D(1)},\tau_f^{D(2)},\tau_f^{DW(1)},\tau_f^{DW(2)},u_r^{U(1)},u_r^{D(1)},u_r^{DW(1)},\\ u_r^{UW(1)},u_r^{U(2)},u_r^{D(2)},u_r^{DW(2)},u_r^{UW(2)},u_{10},u_{20}}} \sum_{i=1}^{m} v_i x_{io}^*$$

$$s.t. \begin{cases} \text{The first } 1+4*n \text{ constrains in model (5.31)}, \\ \sum_{f\in C^D} \tau_f^{D(1)} z_{fo}^{D*} - \sum_{f\in C^D} \tau_f^{D(2)} z_{fo}^{D*} + \sum_{f\in C^{DW}} \tau_f^{DW(1)} z_{fo}^{DW*} \\ - \sum_{f\in C^{DW}} \tau_f^{DW(2)} z_{fo}^{DW*} + \sum_{r\in O^{U(1)}} u_r y_{ro}^{U(1)*} + \sum_{r\in O^{D(1)}} u_r y_{ro}^{D(1)*} \\ + \sum_{r\in O^{DW(1)}} u_r y_{ro}^{DW(1)*} - \sum_{r\in O^{UW(1)}} u_r y_{ro}^{UW(1)*} = 1, \\ \sum_{r\in O^{U(2)}} u_r y_{ro}^{U(2)*} + \sum_{r\in O^{D(2)}} u_r y_{ro}^{D(2)*} + \sum_{r\in O^{DW(2)}} u_r y_{ro}^{DW(2)*} \\ - \sum_{r\in O^{UW(2)}} u_r y_{ro}^{UW(2)*} = 0, \\ v_i, \tau_f^{D(1)}, \tau_f^{D(2)}, \tau_f^{DW(1)}, \tau_f^{DW(2)}, u_r^{U(1)}, u_r^{D(1)}, u_r^{DW(1)}, u_r^{U(2)}, \\ u_r^{D(2)}, u_r^{DW(2)} \geq 0 \, (\forall i, f \in C^D \cup C^{DW}, r \in (O^{(1)} \cup O^{(2)}) \\ \backslash (O^{UW1} \cup O^{UW2})), u_r^{UW(1)}, u_r^{UW(2)}, \\ u_{10}, u_{20} \ \textit{signfree}(r \in O^{UW1} \cup O^{UW2}), \end{cases} \tag{5.32}$$

$$\begin{aligned} \varepsilon^{(SU2)+} = \min_{\substack{v_i,\tau_f^{D(1)},\tau_f^{D(2)},\tau_f^{DW(1)},\tau_f^{DW(2)},u_r^{U(1)},u_r^{D(1)},u_r^{DW(1)},\\ u_r^{UW(1)},u_r^{U(2)},u_r^{D(2)},u_r^{DW(2)},u_r^{UW(2)},u_{10},u_{20}}} & - \sum_{f\in C^D} \tau_f^{D(1)} z_{fo}^{D*} + \sum_{f\in C^D} \tau_f^{D(2)} z_{fo}^{D*} \\ - \sum_{f\in C^{DW}} \tau_f^{DW(1)} z_{fo}^{DW*} + \sum_{f\in C^{DW}} \tau_f^{DW(2)} z_{fo}^{DW*} & - \sum_{r\in O^{U(1)}} u_r y_{ro}^{U(1)*} - \sum_{r\in O^{D(1)}} u_r y_{ro}^{D(1)*} \\ - \sum_{r\in O^{DW(1)}} u_r y_{ro}^{DW(1)*} + \sum_{r\in O^{UW(1)}} u_r y_{ro}^{UW(1)*} & \end{aligned}$$

$$s.t. \begin{cases} \text{The first } 1+4*n \text{ constrains in model (5.31)}, \\ -\sum_{i=1}^{m} v_i x_{io}^* = 0, \\ \sum_{r\in O^{U(2)}} u_r y_{ro}^{U(2)*} + \sum_{r\in O^{D(2)}} u_r y_{ro}^{D(2)*} + \sum_{r\in O^{DW(2)}} u_r y_{ro}^{DW(2)*} \\ - \sum_{r\in O^{UW(2)}} u_r y_{ro}^{UW(2)*} = 1, \\ v_i, \tau_f^{D(1)}, \tau_f^{D(2)}, \tau_f^{DW(1)}, \tau_f^{DW(2)}, u_r^{U(1)}, u_r^{D(1)}, u_r^{DW(1)}, u_r^{U(2)}, \\ u_r^{D(2)}, u_r^{DW(2)} \geq 0 \, (\forall i, f \in C^D \cup C^{DW}, r \in (O^{(1)} \cup O^{(2)}) \\ \backslash (O^{UW1} \cup O^{UW2})), u_r^{UW(1)}, u_r^{UW(2)}, \\ u_{10}, u_{20} \ \textit{signfree}(r \in O^{UW1} \cup O^{UW2}), \end{cases} \tag{5.33}$$

*The left-hand SE in the overall system, the first stage and the second stage*

*are estimated by modifying the objective functions of linear programs* (5.31), (5.32) *and* (5.33) *by changing minimization to maximization.*

The proof of Theorem 8 is similar to that of Theorem 2.

### 5.3.3 Case study

This subsection incorporates nitrogen oxides and carbon dioxide as exogenous undesirable outputs in the case study of cement companies (data sourced from the annual ESG reports of each company), the original data of undesirable outputs is shown in Table 5.4.

Table 5.4: Undesirable outputs of 15 cement companies

| DMU | $y_1^U$ | $y_2^U$ | DMU | $y_1^U$ | $y_2^U$ |
|---|---|---|---|---|---|
| | $NO_x$ | $CO_2$ | | $NO_x$ | $CO_2$ |
| $DMU_1$ | 52908 | 16657.97 | $DMU_9$ | 6026.638 | 1346.686 |
| $DMU_2$ | 52817 | 18377.86 | $DMU_{10}$ | 4497 | 1517.349 |
| $DMU_3$ | 22660 | 4175.51 | $DMU_{11}$ | 4810.97 | 1271.44 |
| $DMU_4$ | 16715.73 | 3453.119 | $DMU_{12}$ | 1878 | 749.619 |
| $DMU_5$ | 11623.03 | 5821.92 | $DMU_{13}$ | 2763.44 | 754.35 |
| $DMU_6$ | 11514.35 | 5761.582 | $DMU_{14}$ | 1240 | 352.458 |
| $DMU_7$ | 4431.23 | 1087.895 | $DMU_{15}$ | 11136.01 | 3529.308 |
| $DMU_8$ | 4582.841 | 1851.027 | | | |

Note: $NO_x$ is measured in tons, while $CO_2$ is measured in million tons.

We assume that $NO_x$ is strongly disposable while $CO_2$ is selective weakly disposable and related to clinker & cement, i.e., $y_1^U \in \boldsymbol{O}^{U(1)}$, $y_2^U \in \boldsymbol{O}^{UW(1)}$, $z_1 \in \boldsymbol{C}^{DW}$ and $z_2 \in \boldsymbol{C}^{D}$. Note that $\boldsymbol{Y}^{(2)} = \varnothing$ holds in this case. Then we have

the following production technology for 15 Chinese cement companies:

$$T_{\text{Cem}} = \left\{ (X,Z,Y) \in \mathbb{R}_+^8 \left| \begin{array}{l} \sum_{j=1}^n \lambda_j^{(1)} x_{1j} \le x_{1o}, \sum_{j=1}^n \lambda_j^{(1)} x_{2j} \le x_{2o}, \\ \sum_{j=1}^n (\lambda_j^{(1)} - \mu_j) z_{1j} \ge z_1, \sum_{j=1}^n \lambda_j^{(2)} z_{1j} \le z_1, \\ \sum_{j=1}^n \lambda_j^{(1)} z_{2j} \ge z_2, \sum_{j=1}^n \lambda_j^{(2)} z_{2j} \le z_2, \\ \sum_{j=1}^n \lambda_j^{(1)} y_{1j}^U \le y_{1o}^U, \\ \sum_{j=1}^n (\lambda_j^{(1)} - \mu_j) y_{2j}^U = y_{2o}^U, \\ \sum_{j=1}^n \lambda_j^{(2)} y_{1j} \ge y_{1o}, \sum_{j=1}^n \lambda_j^{(2)} y_{2j} \ge y_{2o}, \\ \sum_{j=1}^n \lambda_j^{(1)} = 1, \sum_{j=1}^n \lambda_j^{(2)} = 1, \lambda_j^{(1)} \ge \mu_j (\forall j), \\ \lambda_j^{(1)}, \lambda_j^{(2)}, \mu_j \ge 0 (\forall j). \end{array} \right. \right\} \tag{5.34}$$

According to production technology (5.34), the output-oriented model is stated as:

$$\frac{1}{E_0^{\text{Cem}}} = \max_{\theta, \lambda_j^{(1)}, \lambda_j^{(2)}, \mu_j, z_1, z_2} 1 + \theta$$

$$\text{s.t.} \left\{ \begin{array}{l} \sum_{j=1}^n \lambda_j^{(1)} x_{1j} \le x_{1o}, \sum_{j=1}^n \lambda_j^{(1)} x_{2j} \le x_{2o}, \\ \sum_{j=1}^n (\lambda_j^{(1)} - \mu_j) z_{1j} \ge z_1, \sum_{j=1}^n \lambda_j^{(2)} z_{1j} \le z_1, \\ \sum_{j=1}^n \lambda_j^{(1)} z_{2j} \ge z_2, \sum_{j=1}^n \lambda_j^{(2)} z_{2j} \le z_2, \\ \sum_{j=1}^n \lambda_j^{(1)} y_{1j}^U \le y_{1o}^U, \\ \sum_{j=1}^n (\lambda_j^{(1)} - \mu_j) y_{2j}^U = y_{2o}^U, \\ \sum_{j=1}^n \lambda_j^{(2)} y_{1j} \ge (1+\theta) y_{1o}, \sum_{j=1}^n \lambda_j^{(2)} y_{2j} \ge (1+\theta) y_{2o}, \\ \sum_{j=1}^n \lambda_j^{(1)} = 1, \sum_{j=1}^n \lambda_j^{(2)} = 1, \lambda_j^{(1)} \ge \mu_j (\forall j), \\ \lambda_j^{(1)}, \lambda_j^{(2)}, \mu_j \ge 0 (\forall j), \theta, z_1, z_2 \text{ signfree}. \end{array} \right. \tag{5.35}$$

Let the optimal solution vector of model (5.35) be $(\theta^*, \lambda_j^{1*}, \lambda_j^{2*}, \mu_j^{1*}, \mu_j^{2*})$, each inefficient $\text{DMU}_o(X_o, Z_o, Y_o)$ is projected as $\text{DMU}_o^*(X_o^*, Z_o^*, Y_o^*)$ by applying Eqs. (5.26). The results of projections and efficient scores are shown in Table B.6. Based on Theorem 8, we can calculate the right-hand and

left-hand SE by applying the following models:

$$\varepsilon_{\text{Cem}}^{(S)+} = \min_{\gamma_j,v_1,v_2,\tau_1,\tau_2,\tau_3,\tau_4,u_1^U,u_2^U,u_1,u_2,u_{10},u_{20}} v_1x_{1o}^* + v_2x_{2o}^*$$
$$\text{s.t.} \begin{cases} v_1x_{1o}^* + v_2x_{2o}^* - \tau_1z_{1o}^* + \tau_2z_{1o}^* - \tau_3z_{2o}^* + \tau_4z_{2o}^* \\ +u_1^Uy_{1o}^{U*} + u_2^Uy_{2o}^{U*} - u_1y_{1o}^* - u_2y_{2o}^* + u_{10} + u_{20} = 0, \\ \gamma_j - v_1x_{1j}^* - v_2x_{2j}^* + \tau_1z_{1j}^* + \tau_3z_{2j}^* - u_1^Uy_{1j}^{U*} \\ -u_2^Uy_{2j}^{U*} - u_{10} \le 0\,(\forall j), \\ -\tau_2z_{1j}^* - \tau_4z_{2j}^* + u_1y_{1j}^* + u_2y_{2j}^* - u_{20} \le 0\,(\forall j), \\ -\gamma_j - \tau_1z_{1j}^* + u_2^Uy_{2j}^{U*} \le 0\,(\forall j), \\ \tau_1z_{1o}^* - \tau_2z_{1o}^* + \tau_3z_{2o}^* - \tau_4z_{2o}^* + u_1^Uy_{1o}^{U*} - u_2^Uy_{2o}^{U*} = 0, \\ u_1y_{1o}^* + u_2y_{2o}^* = 1, \\ \gamma_j \ge 0(\forall j), v_1, v_2, \tau_1, \tau_2, \tau_3, \tau_4, u_1^U \ge 0, \\ u_2^U, u_{10}, u_{20} \text{ signfree}, \end{cases} \tag{5.36}$$

$$\varepsilon_{\text{Cem}}^{(S1)+} = \min_{\gamma_j,v_1,v_2,\tau_1,\tau_2,\tau_3,\tau_4,u_1^U,u_2^U,u_1,u_2,u_{10},u_{20}} v_1x_{1o}^* + v_2x_{2o}^*$$
$$\text{s.t.} \begin{cases} \text{The first 46 constrains in model (5.36)}, \\ \tau_1z_{1o}^* - \tau_2z_{1o}^* + \tau_3z_{2o}^* - \tau_4z_{2o}^* + u_1^Uy_{1o}^{U*} - u_2^Uy_{2o}^{U*} = 1, \\ u_1y_{1o}^* + u_2y_{2o}^* = 0, \\ \gamma_j \ge 0(\forall j), v_1, v_2, \tau_1, \tau_2, \tau_3, \tau_4, u_1^U \ge 0, \\ u_2^U, u_{10}, u_{20} \text{ signfree}, \end{cases} \tag{5.37}$$

$$\varepsilon_{\text{Cem}}^{(S2)+} = \min_{\gamma_j,v_1,v_2,\tau_1,\tau_2,\tau_3,\tau_4,u_1^U,u_2^U,u_1,u_2,u_{10},u_{20}} -\tau_1z_{1o}^* + \tau_2z_{1o}^* - \tau_3z_{2o}^* + \tau_4z_{2o}^* - u_1^Uy_{1o}^{U*} + u_2^Uy_{2o}^{U*}$$
$$\text{s.t.} \begin{cases} \text{The first 46 constrains in model (5.36)}, \\ -v_1x_{1o}^* - v_2x_{2o}^* = 0, \\ u_1y_{1o}^* + u_2y_{2o}^* = 1, \\ \gamma_j \ge 0(\forall j), v_1, v_2, \tau_1, \tau_2, \tau_3, \tau_4, u_1^U \ge 0, \\ u_2^U, u_{10}, u_{20} \text{ signfree}. \end{cases} \tag{5.38}$$

The left-hand SE of the overall system, stage 1, and stage 2 can be formulated by changing the objective functions of model (5.31), (5.32) and (5.33)

from minimization to maximization, respectively. Therefore, the results of SEs can be calculated and shown in Table 5.5.

Table 5.5: SE results and decomposition of cement companies with undesirable outputs

| DMU | $\varepsilon_{\text{Cem}}^{(N)+}$ | $\varepsilon_{\text{Cem}}^{(1)+}$ | $\varepsilon_{\text{Cem}}^{(2)+}$ | $\varepsilon_{\text{Cem}}^{(N)-}$ | $\varepsilon_{\text{Cem}}^{(1)-}$ | $\varepsilon_{\text{Cem}}^{(2)-}$ |
|---|---|---|---|---|---|---|
| $\text{DMU}_1$ | 0.00 | 0.00 | inf. | 0.62 | 0.71 | 0.88 |
| $\text{DMU}_2$ | 0.00 | 0.00 | inf. | 0.95 | 1.00 | 0.95 |
| $\text{DMU}_3$ | 0.46 | 0.72 | 0.63 | 0.76 | 0.99 | 0.77 |
| $\text{DMU}_4$ | 0.00 | 0.00 | inf. | 0.83 | 1.06 | 0.78 |
| $\text{DMU}_5$ | 0.00 | 0.00 | inf. | 0.93 | 0.97 | 0.95 |
| $\text{DMU}_6$ | 0.00 | 0.00 | inf. | 0.57 | 0.66 | 0.86 |
| $\text{DMU}_7$ | 0.00 | 0.00 | inf. | 0.68 | 1.18 | 0.58 |
| $\text{DMU}_8$ | 0.00 | 0.06 | 0.00 | 0.00 | 0.91 | 0.00 |
| $\text{DMU}_9$ | 0.00 | 0.28 | 0.00 | 1.33 | 2.54 | 0.53 |
| $\text{DMU}_{10}$ | 0.00 | 0.00 | inf. | 0.96 | 0.91 | 1.04 |
| $\text{DMU}_{11}$ | 0.08 | 0.27 | 0.29 | 0.78 | 1.80 | 0.43 |
| $\text{DMU}_{12}$ | 0.00 | 0.00 | inf. | 0.80 | 2.33 | 0.34 |
| $\text{DMU}_{13}$ | 0.00 | 0.02 | 0.23 | 0.43 | 1.88 | 0.23 |
| $\text{DMU}_{14}$ | 0.00 | 0.00 | inf. | $+\infty$ | $+\infty$ | $+\infty$ |
| $\text{DMU}_{15}$ | 0.49 | 0.87 | 0.56 | 0.67 | 0.96 | 0.70 |

Comparing with the results in table 3.11 and 5.5, the left-hand RTS of $\text{DMU}_{10}$ have changed, shifting from increasing to decreasing. This implies that, under a technology $\boldsymbol{T}_{\text{Cem}}$ that incorporates undesirable outputs, $\text{DMU}_{10}$ is no longer recommended to maintain its current scale but rather to reduce it. This also indicates that, when considering pollutant emissions, it becomes more difficult for cement companies to expand their scale without reducing productivity.

## 5.4 Chapter summary

By assuming that undesirable measures in a sub-system cannot be related to desirable outputs in the others since each sub-system was independent of the other in network production technologies, this chapter first proposed multi-activity and multi-stage production technologies with multiple undesirable outputs, respectively. In particular, strong and selective weakly disposability assumptions were considered for different undesirable outputs. Then, the

models of SE estimation were developed in the proposed production technologies. Especially, the overall SE in two-stage production technologies can be decomposed into its sub-stage specific SEs. For illustrating the developed models of estimating SE, retractions and pollutions are considered undesirable outputs in the two case studies, respectively.

The contributions of this chapter are summarized below:

- This chapter developed the method of SE estimation in proposed novel multi-activity production technologies with multiple undesirable outputs under different disposability assumptions.
- This chapter developed the method of estimating and decomposing SE in proposed novel two-stage production technologies with multiple undesirable outputs and intermediate products.

# Chapter 6

# SE Estimation and Decomposition for Production Technologies with Closed Network Structures

## 6.1 Introduction

In the previous chapters, we proposed methods of decomposing SE in the two basic network structures (i.e., parallel and series structures). Existing research focused on efficiency evaluating and decomposing in network structures, for example, Kao (2020) proposed a slacks-based method to decompose efficiency in network technologies without considering shared measures. Considering a two-stage parallel production system, in which the first stage includes two activities while the second stage is a joint production process, Amirteimoori et al. (2024) separately calculated the sub-technology SE and aggregated them into the overall SE. There is a lack of methods to decompose SE in closed network production technologies including shared measures. Therefore, this chapter focuses on decomposing SE in proposed network production technologies.

The remainder of this chapter is divided into the following sections: Section 6.2 provides the notations and structures of network production technologies with a mixture of parallel and two-stage structures. Section 6.3 proposes a method of decomposing the overall SE into each smallest sub-technology SE. In Section 6.4, a case study is carried out to illustrate the proposed method. Section 6.5 provides conclusions and a chapter summary.

## 6.2 Two-stage parallel production technologies

### 6.2.1 Notations of the two-stage parallel system

To embody both two-stage and parallel structure in a production system, this section will first propose a two-stage parallel production system as shown in Fig. 6.1.

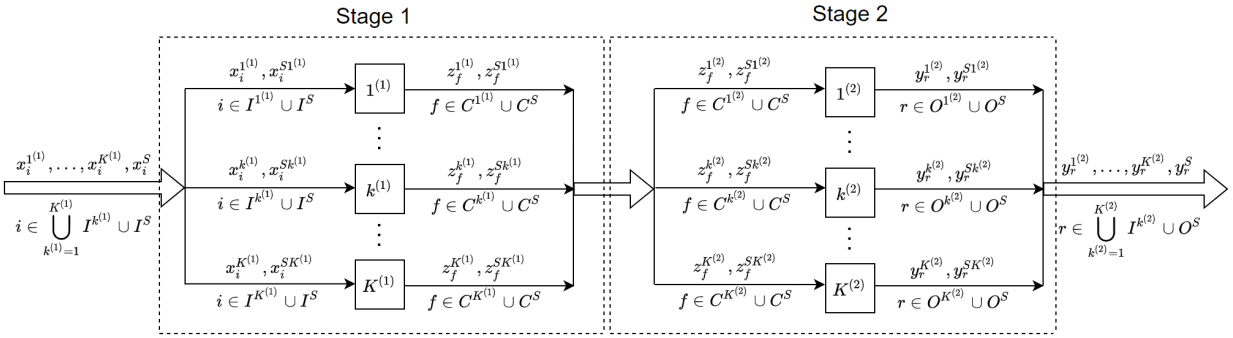



Figure 6.1: Network production system.

In Fig. 6.1, we assume that the parallel structures can be different in stage 1 and 2, while the properties of intermediate products remain unchanged. In other words, the number of activities in the two stages can be different, while the intermediate products regarded as shared (or activity-specific) outputs in stage 1 are also regarded as shared (or activity-specific) inputs in stage 2. Therefore, we have $K^{(1)}$ and $K^{(2)}$ activities in stage 1 and 2, respectively. For each activity $k^{(1)} \in H^{(1)} = \{1^{(1)}, \ldots, K^{(1)}\}(K^{(1)} \geq 1)$ in stage 1, $I^{k^{(1)}}$ and $Z^{k^{(1)}}$ represent the activity-specific input and intermediate product set, $I^S$ and $Z^S$ represent the shared input and intermediate product set. For each activity $k^{(2)} \in H^{(2)} = \{1^{(2)}, \ldots, K^{(2)}\}(K^{(2)} \geq 1)$ in stage 2, $Z^{k^{(2)}}$ and $Y^{k^{(2)}}$ represent the activity-specific intermediate product and output set, $O^S$ represents the shared output set. We still assume that the inputs and outputs are allocated into each activities with known proportions. Thus, the input, intermediate product and output set are decomposed into the two disjoint subsets as follows:

$$\begin{aligned} I = I^S \cup \left( \bigcup_{k^{(1)} \in H^{(1)}} I^{k^{(1)}} \right), C^{(1)} = C^S \cup \left( \bigcup_{k^{(1)} \in H^{(1)}} C^{k^{(1)}} \right), \\ C^{(2)} = C^S \cup \left( \bigcup_{k^{(2)} \in H^{(2)}} C^{k^{(2)}} \right), O = O^S \cup \left( \bigcup_{k^{(2)} \in H^{(2)}} O^{k^{(2)}} \right). \end{aligned} \tag{6.1}$$

Note that $C^{(1)} = C^{(2)}$ holds since we assume that the properties of intermediate products remain unchanged. Corresponding to the decomposition of input and output sets as shown in (6.1), we have:

$$\begin{aligned} X = (X^{1^{(1)}}; \ldots; X^{K^{(1)}}; X^S), Z^{(1)} = (Z^{1^{(1)}}; \ldots; Z^{K^{(1)}}; Z^S), \\ Z^{(2)} = (Z^{1^{(2)}}; \ldots; Z^{K^{(2)}}; Z^S), Y^{(2)} = (Y^{1^{(2)}}; \ldots; Y^{K^{(2)}}; Y^S), \end{aligned} \tag{6.2}$$

where $X^{k^{(1)}} = (x_1^{k^{(1)}}, \ldots, x_{|I^{k^{(1)}}|}^{k^{(1)}})^\top (\forall k^{(1)})$, $X^S = (x_1^S, \ldots, x_{|I^S|}^S)^\top$, $Z^{k^{(1)}} = (z_1^{k^{(1)}}, \ldots, z_{|C^{k^{(1)}}|}^{k^{(1)}})^\top (\forall k^{(1)})$, $Z^{k^{(2)}} = (z_1^{k^{(2)}}, \ldots, z_{|C^{k^{(2)}}|}^{k^{(2)}})^\top (\forall k^{(2)})$, $Z^S = (z_1^S, \ldots, z_{|C^S|}^S)^\top$, $Y^{k^{(2)}} = (y_1^{k^{(2)}}, \ldots, y_{|O^{k^{(2)}}|}^{k^{(2)}})^\top (\forall k^{(2)})$ and $Y^S = (y_1^S, \ldots, y_{|O^S|}^S)^\top$.

For the smallest sub-systems $k^{(1)}$ or $k^{(2)}$ (i.e., the activities in stage 1 or 2), the input and output sets are decomposed as follows:

$$\begin{aligned} I^{Ak^{(1)}} = I^{k^{(1)}} \cup I^S, Z^{Ak^{(1)}} = Z^{k^{(1)}} \cup Z^S, \\ Z^{Ak^{(2)}} = Z^{k^{(2)}} \cup Z^S, O^{Ak^{(2)}} = O^{k^{(2)}} \cup O^S. \end{aligned} \tag{6.3}$$

In stage 1, we assume that each shared input $x_i^S (i \in I^S)$ and each shared intermediate product $z_f^S (f \in C^S)$ are allocated into each activity $k^{(1)}$ by known proportion vectors $Q_i = (q_i^{1^{(1)}}, \ldots, q_i^{K^{(1)}})^\top (i \in I^S)$ and $D_f^{(1)} = (d_f^{1^{(1)}}, \ldots, d_f^{K^{(1)}})^\top (f \in C^S)$, respectively, such that $\sum_{k \in H} q_i^{k^{(1)}} = 1$ and $\sum_{k \in H} d_f^{k^{(1)}} = 1$. In stage 2, we assume that each shared intermediate product $z_f^S (f \in C^S)$ and each shared output $y_r^S (r \in O^S)$ are allocated into each activity $k^{(2)}$ by known proportion vectors $D_f^{(2)} = (d_f^{1^{(2)}}, \ldots, d_f^{K^{(d)}})^\top (f \in C^S)$ and $P_r = (p_r^{1^{(2)}}, \ldots, p_r^{K^{(2)}})^\top (r \in O^S)$, respectively, such that $\sum_{k \in H} d_f^{k^{(2)}} = 1$ and $\sum_{k \in H} p_r^{k^{(2)}} = 1$. Let $x_i^{Sk^{(1)}} = q_i^{k^{(1)}} x_i^S$, $z_f^{Sk^{(1)}} = d_f^{k^{(1)}} z_f^S$, $z_f^{Sk^{(2)}} = d_f^{k^{(2)}} z_f^S$ and $y_r^{Sk^{(2)}} = p_r^{k^{(2)}} y_r^S$ represent the allocations of shared inputs, intermediate products of stage 1, intermediate products of stage 2 and outputs. Corresponding to the decomposition of input and output sets as shown in (6.3), we have:

$$\begin{aligned} X^{Ak^{(1)}} = (X^{k^{(1)}}; X^{Sk^{(1)}}), Z^{Ak^{(1)}} = (Z^{k^{(1)}}; Z^{Sk^{(1)}}), \\ Z^{Ak^{(2)}} = (Z^{k^{(2)}}; Z^{Sk^{(2)}}), Y^{Ak^{(2)}} = (Y^{k^{(2)}}; Y^{Sk^{(2)}}), \end{aligned} \tag{6.4}$$

where $X^{Sk^{(1)}} = (x_1^{Sk^{(1)}}, \ldots, x_{|I^S|}^{Sk^{(1)}})^\top (\forall k^{(1)})$, $Z^{Sk^{(1)}} = (z_1^{Sk^{(1)}}, \ldots, z_{|C^S|}^{Sk^{(1)}})^\top (\forall k^{(1)})$, $Z^{Sk^{(2)}} = (z_1^{Sk^{(2)}}, \ldots, z_{|C^S|}^{Sk^{(2)}})^\top (\forall k^{(2)})$ and $Y^{Sk^{(2)}} = (y_1^{Sk^{(2)}}, \ldots, y_{|O^S|}^{Sk^{(2)}})^\top (\forall k^{(2)})$.

### 6.2.2 Two-stage parallel DEA models

Let $T_N$ represent the production technologies describing the above production system, we assume that $T_N$ and all its sub-technologies satisfy all axioms and assumptions as shown in Section 3.2.3 and 3.3.3, such as convexity, no waste in intermediate products and known allocations of shared measures. Then, the technology $T_N$ is expressed as the following form:

$$T_N = \left\{ (X, Z, Y) \in \mathbb{R}_+^{|I|+|C|+|O|} \;\middle|\; \begin{array}{l} \sum_{j=1}^{n} \lambda_j^{k^{(1)}} x_{ij}^{k^{(1)}} \le x_i^{k^{(1)}} \; (\forall k^{(1)}, i \in I^{k^{(1)}}), \\ \sum_{k^{(1)}=1}^{K^{(1)}} \sum_{j=1}^{n} \lambda_j^{k^{(1)}} x_{ij}^{Sk^{(1)}} \le x_i^S \; (i \in I^S), \\ \sum_{j=1}^{n} \lambda_j^{k^{(1)}} z_{fj}^{k^{(1)}} \ge z_f^{k^{(1)}} \; (\forall k^{(1)}, f \in C^{k^{(1)}}), \\ \sum_{k^{(1)}=1}^{K^{(1)}} \sum_{j=1}^{n} \lambda_j^{k^{(1)}} z_{fj}^{Sk^{(1)}} \ge z_f^S \; (f \in C^S), \\ \sum_{j=1}^{n} \lambda_j^{k^{(2)}} z_{fj}^{k^{(2)}} \le z_f^{k^{(2)}} \; (\forall k^{(2)}, f \in C^{k^{(2)}}), \\ \sum_{k^{(2)}=1}^{K^{(2)}} \sum_{j=1}^{n} \lambda_j^{k^{(2)}} z_{fj}^{Sk^{(2)}} \le z_f^S \; (f \in C^S), \\ \sum_{j=1}^{n} \lambda_j^{k^{(2)}} y_{rj}^{k^{(2)}} \ge y_r^{k^{(2)}} \; (\forall k^{(2)}, r \in O^{k^{(2)}}), \\ \sum_{k^{(2)}=1}^{K^{(2)}} \sum_{j=1}^{n} \lambda_j^{k^{(2)}} y_{rj}^{Sk^{(2)}} \ge y_r^S \; (r \in O^S), \\ \sum_{j=1}^{n} \lambda_j^{k^{(1)}} = 1 \; (\forall k^{(1)}), \sum_{j=1}^{n} \lambda_j^{k^{(2)}} = 1 \; (\forall k^{(2)}), \\ z_f^{k^{(1)}} = z_f^{k^{(2)}} \; (\forall f), \lambda_j^{k^{(1)}}, \lambda_j^{k^{(2)}} \ge 0 \; (\forall j, k), \\ z_f^{k^{(1)}}, z_f^{k^{(2)}}, z_f^S \text{ signfree.} \end{array} \right\} \tag{6.5}$$

Note that when $K^{(1)} = K^{(2)} = 1$, $T_N$ is equal to the technology (3.41).

Accordingly, the output-oriented DEA model is formulated as follows:

$$\frac{1}{E_0^{N-O}} = \max_{\theta,\lambda_j^{k(1)},\lambda_j^{k(2)},z_f^{k(1)},z_f^{k(2)},z_f^S} \theta$$

$$\text{s.t.} \begin{cases} \sum_{j=1}^{n} \lambda_j^{k(1)} x_{ij}^{k(1)} \le x_i^{k(1)} \ (\forall k^{(1)}, i \in I^{k^{(1)}}), \\ \sum_{k^{(1)}=1}^{K^{(1)}} \sum_{j=1}^{n} \lambda_j^{k(1)} x_{ij}^{Sk^{(1)}} \le x_i^S \ (i \in I^S), \\ \sum_{j=1}^{n} \lambda_j^{k(1)} z_{fj}^{k(1)} \ge z_f^{k(1)} \ (\forall k^{(1)}, f \in C^{k^{(1)}}), \\ \sum_{k^{(1)}=1}^{K^{(1)}} \sum_{j=1}^{n} \lambda_j^{k(1)} z_{fj}^{Sk^{(1)}} \ge z_f^S \ (f \in C^S), \\ \sum_{j=1}^{n} \lambda_j^{k(2)} z_{fj}^{k(2)} \le z_f^{k(2)} \ (\forall k^{(2)}, f \in C^{k^{(2)}}), \\ \sum_{k^{(2)}=1}^{K^{(2)}} \sum_{j=1}^{n} \lambda_j^{k(2)} z_{fj}^{Sk^{(2)}} \le z_f^S \ (f \in C^S), \\ \sum_{j=1}^{n} \lambda_j^{k(2)} y_{rj}^{k(2)} \ge \theta y_r^{k(2)} \ (\forall k^{(2)}, r \in O^{k^{(2)}}), \\ \sum_{k^{(2)}=1}^{K^{(2)}} \sum_{j=1}^{n} \lambda_j^{k(2)} y_{rj}^{Sk^{(2)}} \ge \theta y_r^S \ (r \in O^S), \\ \sum_{j=1}^{n} \lambda_j^{k(1)} = 1 \ (\forall k^{(1)}), \sum_{j=1}^{n} \lambda_j^{k(2)} = 1 \ (\forall k^{(2)}), \\ z_f^{k(1)} = z_f^{k(2)} \ (\forall f), \lambda_j^{k(1)}, \lambda_j^{k(2)} \ge 0 \ (\forall j, k), \\ \theta, z_f^{k(1)}, z_f^{k(2)}, z_f^S \text{ signfree.} \end{cases} \tag{6.6}$$

The output-oriented DEA model (6.6) is used to project inefficient DMUs, let $(\theta^*, \lambda_j^{k(1)*}, \lambda_j^{k(2)*}, z_f^{k(1)*}, z_f^{k(2)*}, z_f^{S*})(\forall j, k^{(1)}, k^{(2)}, f)$ represent the optimal value of model (6.6). $\text{DMU}_o$ is defined as the output radial efficient DMU if and only if the optimal solution of the acquired from model (6.6) satisfies $\theta^* = 1$. If not, the inefficient DMUs can be projected onto the efficient frontier by applying the following equations:

$$\begin{aligned} x_{io}^{k(1)*} &= \sum_{j=1}^{n} \lambda_j^{k(1)*} x_{ij}^{k(1)} (\forall k, i \in I^{k^{(1)}}), x_{io}^{Sk(1)*} = \sum_{j=1}^{n} \lambda_j^{k(1)*} x_{ij}^{Sk^{(1)}} (\forall k, i \in I^S), \\ z_{fo}^{k(1)*} &= z_f^{k(1)*} (\forall k, f \in C^{k^{(1)}}), z_{fo}^{k(2)*} = z_f^{k(2)*} (\forall k, f \in C^{k^{(2)}}), \\ z_{fo}^{Sk(1)*} &= \sum_{j=1}^{n} \lambda_j^{k(1)*} z_{fj}^{Sk^{(1)}} (\forall k, f \in C^S), z_{fo}^{Sk(2)*} = \sum_{j=1}^{n} \lambda_j^{k(2)*} z_{fj}^{Sk^{(2)}} (\forall k, f \in C^S), \\ y_{ro}^{k(2)*} &= \sum_{j=1}^{n} \lambda_j^{k(2)*} y_{rj}^{k(2)} (\forall k, r \in O^{k^{(2)}}), y_{ro}^{Sk(2)*} = \sum_{j=1}^{n} \lambda_j^{k(2)*} y_{rj}^{Sk^{(2)}} (\forall k, r \in O^S). \end{aligned} \tag{6.7}$$

Consequently, we obtain all output-radial efficient $\text{DMUs}_o(X_o^*, Z_o^*, Y_o^*)$ by applying model (6.6) and Eqs. (6.7). In the next section, the methods of esti-

mating and decomposing SE in the two-stage parallel production technology (6.5) will be proposed.

## 6.3 SE estimation and decomposition in network production technologies

For all output-radial efficient $\mathrm{DMU}_o^*$ in the technology $T_N$, the response function of final outputs is defined as follows:

$$\beta(t) = \max\left\{\beta \mid \left((1+t)X_o^*, (1+\varphi)Z_o^*, (1+\beta)Y_o^*\right) \in T_N, \varphi, \beta \in \mathbb{R}\right\}, \tag{6.8}$$

where we assume $\beta = \varphi = 0$ when $t = 0$ to indicate that $Y_o^*$ and $Z_o^*$ cannot be changed within $T_N$ on condition that $X_o^*$ keeps unchanged.

Assume the derivatives of $\beta(t)$ exist when $t = 0$, the overall SE system, the first stage and the second stage, the right-hand and left-hand SE are also defined as Eqs. (3.45). Moreover, it is obvious that the relationship as shown in (3.46) still holds (i.e., $\varepsilon^{(N)+}$ and $\varepsilon^{(N)-}$ are the product of $\varepsilon^{(N1)+}$ and $\varepsilon^{(N2)+}$, $\varepsilon^{(N1)-}$ and $\varepsilon^{(N2)-}$, respectively).

To estimate SE for efficient $\mathrm{DMU}_o^*(X_o^*, Z_o^*, Y_o^*) \in T_N$ in the overall system, the first stage and the second stage, $\beta(t)$, $\varphi(t)$ and $\beta(\varphi)$ are expressed as following linear programs, respectively:

$$\begin{aligned}
\beta(t) = \max_{\varphi,\beta,\lambda_j^{k(1)},\lambda_j^{k(2)}} \ & \beta \\
\text{s.t.} \ & \begin{cases}
\sum_{j=1}^{n} \lambda_j^{k(1)} x_{ij}^{k(1)*} \le (1+t) x_i^{k(1)*} \ (\forall k^{(1)}, i \in I^{k(1)}), \\
\sum_{k^{(1)}=1}^{K^{(1)}} \sum_{j=1}^{n} \lambda_j^{k(1)} x_{ij}^{Sk(1)*} \le (1+t) x_i^{S*} \ (i \in I^S), \\
\sum_{j=1}^{n} \lambda_j^{k(1)} z_{fj}^{k(1)*} \ge (1+\varphi) z_f^{k(1)*} \ (\forall k^{(1)}, f \in C^{k(1)}), \\
\sum_{k^{(1)}=1}^{K^{(1)}} \sum_{j=1}^{n} \lambda_j^{k(1)} z_{fj}^{Sk(1)*} \ge (1+\varphi) z_f^{S*} \ (f \in C^S), \\
\sum_{j=1}^{n} \lambda_j^{k(2)} z_{fj}^{k(2)*} \le (1+\varphi) z_f^{k(2)*} \ (\forall k^{(2)}, f \in C^{k(2)}), \\
\sum_{k^{(2)}=1}^{K^{(2)}} \sum_{j=1}^{n} \lambda_j^{k(2)} z_{fj}^{Sk(2)*} \le (1+\varphi) z_f^{S*} \ (f \in C^S), \\
\sum_{j=1}^{n} \lambda_j^{k(2)} y_{rj}^{k(2)*} \ge (1+\beta) y_r^{k(2)*} \ (\forall k^{(2)}, r \in O^{k(2)}), \\
\sum_{k^{(2)}=1}^{K^{(2)}} \sum_{j=1}^{n} \lambda_j^{k(2)} y_{rj}^{Sk(2)*} \ge (1+\beta) y_r^{S*} \ (r \in O^S), \\
\sum_{j=1}^{n} \lambda_j^{k(1)} = 1 \ (\forall k^{(1)}), \sum_{j=1}^{n} \lambda_j^{k(2)} = 1 \ (\forall k^{(2)}), \\
\lambda_j^{k(1)}, \lambda_j^{k(2)} \ge 0 \ (\forall j, k), \varphi, \beta \text{ signfree}.
\end{cases}
\end{aligned} \tag{6.9}$$

$$\varphi(t) = \max_{\varphi,\beta,\lambda_j^{k(1)},\lambda_j^{k(2)}} \varphi$$
$$\text{s.t. all constrains in model (6.9).} \tag{6.10}$$

$$\beta(\varphi) = \max_{t,\beta,\lambda_j^{k(1)},\lambda_j^{k(2)}} \beta$$
$$\text{s.t. all constraints in model (6.9), where } \varphi \text{ is converted into a scalar, } t \text{ is converted into an unknown variable satisfying } t \geq 0. \tag{6.11}$$

Then, based on the DDO method, the following Theorem is proposed to estimate SE for each sub-stage and the overall system:

**Theorem 9.** *Let $(X_o^*, Z_o^*, Y_o^*) \in T_N$ be defined by (6.5) and $(X_o^*, Z_o^*, Y_o^*)$ be an output-radial efficient point, then the right-hand SE in the overall system, the first stage and the second stage are estimated by the following problems, respectively:*

$$\varepsilon^{(N)+} = \min_{v_i^{k(1)},v_i^S,\tau_f^{k(1)},\tau_f^{k(2)},\tau_f^{S(1)},\tau_f^{S(2)},u_r^{k(2)},u_r^S,u_{10}^{k(1)},u_{20}^{k(2)}} \sum_{k\in H}\sum_{i\in I^k} v_i^k x_{io}^{k(1)*} + \sum_{i\in I^S} v_i^S x_{io}^{S*}$$

$$s.t. \begin{cases} \sum_{k^{(1)}\in H^{(1)}} \sum_{i\in I^{k(1)}} v_i^{k(1)} x_{io}^{k(1)*} + \sum_{i\in I^S} v_i^S x_{io}^{S*} \\ - \sum_{k^{(1)}\in H^{(1)}} \sum_{f\in C^{k(1)}} \tau_f^{k(1)} z_{fo}^{k(1)*} - \sum_{f\in C^S} \tau_f^{S(1)} z_{fo}^{S*} \\ + \sum_{k^{(1)}\in H^{(1)}} u_{10}^{k(1)} + \sum_{k^{(2)}\in H^{(2)}} \sum_{f\in C^{k(2)}} \tau_f^{k(2)} z_{fo}^{k(2)*} + \sum_{f\in C^S} \tau_f^{S(2)} z_{fo}^{S*} \\ - \sum_{k^{(2)}\in H^{(2)}} \sum_{r\in O^{k(2)}} u_r^{k(2)} y_{ro}^{k(2)*} - \sum_{r\in O^S} u_r^S y_{ro}^{S*} + \sum_{k^{(2)}\in H^{(2)}} u_{20}^{k(2)} = 0, \\ \sum_{k^{(1)}\in H^{(1)}} \sum_{i\in I^{k(1)}} v_i^{k(1)} x_{ij}^{k(1)*} + \sum_{i\in I^S} v_i^S x_{ij}^{Sk(1)*} \\ - \sum_{k^{(1)}\in H^{(1)}} \sum_{f\in C^{k(1)}} \tau_f^{k(1)} z_{fj}^{k(1)*} - \sum_{f\in C^S} \tau_f^{S(1)} z_{fj}^{Sk(1)*} + u_{10}^{k(1)} \geq 0\ (\forall j, k^{(1)}), \\ \sum_{k^{(2)}\in H^{(2)}} \sum_{f\in C^{k(2)}} \tau_f^{k(2)} z_{fj}^{k(2)*} + \sum_{f\in C^S} \tau_f^{S(2)} z_{fj}^{Sk(2)*} \\ - \sum_{k^{(2)}\in H^{(2)}} \sum_{r\in O^{k(2)}} u_r^{k(2)} y_{rj}^{k(2)*} - \sum_{r\in O^S} u_r^S y_{rj}^{Sk(2)*} + u_{20}^{k(1)} \geq 0\ (\forall j, k^{(2)}), \\ \sum_{k^{(1)}\in H^{(1)}} \sum_{f\in C^{k(1)}} \tau_f^{k(1)} z_{fo}^{k(1)*} + \sum_{f\in C^S} \tau_f^{S(1)} z_{fo}^{S*} \\ - \sum_{k^{(2)}\in H^{(2)}} \sum_{f\in C^{k(2)}} \tau_f^{k(2)} z_{fo}^{k(2)*} - \sum_{f\in C^S} \tau_f^{S(2)} z_{fo}^{S*} = 0, \\ \sum_{k^{(2)}\in H^{(2)}} \sum_{r\in O^{k(2)}} u_r^{k(2)} y_{ro}^{k(2)*} + \sum_{r\in O^S} u_r^S y_{ro}^{S*} = 1, \\ v_i^{k(1)}, v_i^S, \tau_f^{k(1)}, \tau_f^{k(2)}, \tau_f^{S(1)}, \tau_f^{S(2)}, u_r^{k(2)}, u_r^S, \geq 0\ (\forall i, f, r), \\ u_{10}^{k(1)}, u_{20}^{k(2)}\ \ signfree\ (\forall k^{(1)}, k^{(2)}), \end{cases} \tag{6.12}$$

$$\varepsilon^{(N1)+} = \min_{v_i^{k(1)}, v_i^S, \tau_f^{k(1)}, \tau_f^{k(2)}, \tau_f^{S(1)}, \tau_f^{S(2)}, u_r^{k(2)}, u_r^S, u_{10}^{k(1)}, u_{20}^{k(2)}} \sum_{k \in H} \sum_{i \in I^k} v_i^k x_{io}^{k(1)*} + \sum_{i \in I^S} v_i^S x_{io}^{S*}$$

$$s.t. \begin{cases} \textit{The first } 1 + n * k^{(1)} + n * k^{(2)} \textit{ constrains in model (6.12)}, \\ \sum_{k^{(1)} \in H^{(1)}} \sum_{f \in C^{k^{(1)}}} \tau_f^{k(1)} z_{fo}^{k(1)*} + \sum_{f \in C^S} \tau_f^{S(1)} z_{fo}^{S*} \\ - \sum_{k^{(2)} \in H^{(2)}} \sum_{f \in C^{k^{(2)}}} \tau_f^{k(2)} z_{fo}^{k(2)*} - \sum_{f \in C^S} \tau_f^{S(2)} z_{fo}^{S*} = 1, \\ \sum_{k^{(2)} \in H^{(2)}} \sum_{r \in O^{k^{(2)}}} u_r^{k(2)} y_{ro}^{k(2)*} + \sum_{r \in O^S} u_r^S y_{ro}^{S*} = 0, \\ v_i^{k(1)}, v_i^S, \tau_f^{k(1)}, \tau_f^{k(2)}, \tau_f^{S(1)}, \tau_f^{S(2)}, u_r^{k(2)}, u_r^S, \geq 0 \, (\forall i, f, r), \\ u_{10}^{k(1)}, u_{20}^{k(2)} \textit{ signfree } (\forall k^{(1)}, k^{(2)}), \end{cases} \tag{6.13}$$

$$\varepsilon^{(N2)+} = \min_{v_i^{k(1)}, v_i^S, \tau_f^{k(1)}, \tau_f^{k(2)}, \tau_f^{S(1)}, \tau_f^{S(2)}, u_r^{k(2)}, u_r^S, u_{10}^{k(1)}, u_{20}^{k(2)}} \sum_{k^{(1)} \in H^{(1)}} \sum_{f \in C^{k^{(1)}}} \tau_f^{k(1)} z_{fo}^{k(1)*} + \sum_{f \in C^S} \tau_f^{S(1)} z_{fo}^{S*} - \sum_{k^{(2)} \in H^{(2)}} \sum_{f \in C^{k^{(2)}}} \tau_f^{k(2)} z_{fo}^{k(2)*} - \sum_{f \in C^S} \tau_f^{S(2)} z_{fo}^{S*}$$

$$s.t. \begin{cases} \textit{The first } 1 + n * k^{(1)} + n * k^{(2)} \textit{ constrains in model (6.12)}, \\ \sum_{k \in H} \sum_{i \in I^k} v_i^k x_{io}^{k(1)*} + \sum_{i \in I^S} v_i^S x_{io}^{S*} = 0, \\ \sum_{k^{(2)} \in H^{(2)}} \sum_{r \in O^{k^{(2)}}} u_r^{k(2)} y_{ro}^{k(2)*} + \sum_{r \in O^S} u_r^S y_{ro}^{S*} = 1, \\ v_i^{k(1)}, v_i^S, \tau_f^{k(1)}, \tau_f^{k(2)}, \tau_f^{S(1)}, \tau_f^{S(2)}, u_r^{k(2)}, u_r^S, \geq 0 \, (\forall i, f, r), \\ u_{10}^{k(1)}, u_{20}^{k(2)} \textit{ signfree } (\forall k^{(1)}, k^{(2)}). \end{cases} \tag{6.14}$$

*The left-hand SE in the overall system, the first stage and the second stage are estimated by changing min linear programs* (6.12)*,* (6.13) *and* (6.14) *to max linear programs, respectively.*

The proof of Theorem 9 is similar to that of the proof of Theorem 2. Then, to decompose $\varepsilon^{(N1)+}$ and $\varepsilon^{(N2)+}$ into each sub-activity $\varepsilon^{k(1)+}$ and $\varepsilon^{k(2)+}$, respectively, we assume that the following transformation function holds for

output-radial efficient DMUs in stage 1 and 2:

$$
\begin{aligned}
F(X_o^*, Z_o^*, Y_o^*) = & \sum_{k^{(1)} \in H^{(1)}} \sum_{i \in I^{k^{(1)}}} v_i^{k^{(1)}*} x_{io}^{k^{(1)}*} + \sum_{i \in I^S} v_i^{S*} x_{io}^{S*} \\
& - \sum_{k^{(1)} \in H^{(1)}} \sum_{f \in C^{k^{(1)}}} \tau_f^{k^{(1)}*} z_{fo}^{k^{(1)}*} - \sum_{f \in C^S} \tau_f^{S(1)*} z_{fo}^{S*} + \sum_{k^{(1)} \in H^{(1)}} u_{10}^{k^{(1)}*} \\
& + \sum_{k^{(2)} \in H^{(2)}} \sum_{f \in C^{k^{(2)}}} \tau_f^{k^{(2)}*} z_{fo}^{k^{(2)}*} + \sum_{f \in C^S} \tau_f^{S(2)*} z_{fo}^{S*} \\
& - \sum_{k^{(2)} \in H^{(2)}} \sum_{r \in O^{k^{(2)}}} u_r^{k^{(2)}*} y_{ro}^{k^{(2)}*} - \sum_{r \in O^S} u_r^{S*} y_{ro}^{S*} + \sum_{k^{(2)} \in H^{(2)}} u_{20}^{k^{(2)}*} = 0,
\end{aligned} \tag{6.15}
$$

where $v_i^{k^{(1)}*}, v_i^{S*}, \tau_f^{k^{(1)}*}, \tau_f^{S(1)*}, u_{10}^{k^{(1)}*}, u_{20}^{k^{(2)}*}$ are assumed as the unique optimal values obtained by solving model (6.13) or (6.14).

Let

$$
\begin{aligned}
& \left\langle X_o^*, \nabla_X F\left(X_o^*, Z_o^*, Y_o^*\right)\right\rangle \\
& = \sum_{k^{(1)} \in H^{(1)}} \sum_{i \in I^{k^{(1)}}} \frac{\partial F\left(X_o^*, Z_o^*, Y_o^*\right)}{\partial x_i^{k^{(1)}*}} x_{io}^{k^{(1)}*} + \sum_{i \in I^S} \frac{\partial F\left(X_o^*, Z_o^*, Y_o^*\right)}{\partial x_i^{S*}} x_{io}^{S*}, \\
& \left\langle Z_o^*, \nabla_Z F\left(X_o^*, Z_o^*, Y_o^*\right)\right\rangle \\
& = \sum_{k^{(1)} \in H^{(1)}} \sum_{f \in C^{k^{(1)}}} \frac{\partial F\left(X_o^*, Z_o^*, Y_o^*\right)}{\partial z_f^*} z_{fo}^{k^{(1)}*} + \sum_{k^{(1)} \in H^{(2)}} \sum_{f \in C^{k^{(2)}}} \frac{\partial F\left(X_o^*, Z_o^*, Y_o^*\right)}{\partial z_f^*} z_{fo}^{k^{(2)}*} \\
& + \sum_{f \in C^S} \frac{\partial F\left(X_o^*, Z_o^*, Y_o^*\right)}{\partial z_f^{S*}} z_{fo}^{S*}, \\
& \left\langle Y_o^*, \nabla_Y F\left(X_o^*, Z_o^*, Y_o^*\right)\right\rangle \\
& = \sum_{k^{(2)} \in H^{(2)}} \sum_{r \in O^{k^{(2)}}} \frac{\partial F\left(X_o^*, Z_o^*, Y_o^*\right)}{\partial y_r^{k^{(2)}*}} y_{ro}^{k^{(2)}*} + \sum_{r \in O^S} \frac{\partial F\left(X_o^*, Z_o^*, Y_o^*\right)}{\partial y_r^{S*}} y_{ro}^{S*}
\end{aligned}
$$

hold, where $\langle \cdot, \cdot \rangle$ represents the inner product of the two vector with the same dimensions, $\nabla_X F\left(X_o^*, Z_o^*, Y_o^*\right)$, $\nabla_Z F\left(X_o^*, Z_o^*, Y_o^*\right)$ and $\nabla_Y F\left(X_o^*, Z_o^*, Y_o^*\right)$ are the partial gradient of the function $F(X, Z, Y)$ evaluated at $(X_o^*, Z_o^*, Y_o^*)$. Furthermore, let

$$
\left\langle X_o^{Ak^{(1)}*}, \nabla_{X^{Ak^{(1)}}} F\left(X_o^*, Z_o^*, Y_o^*\right)\right\rangle
$$

$$
\begin{aligned}
&= \sum_{i \in I^{k^{(1)}}} \frac{\partial F\left(X_o^*, Z_o^*, Y_o^*\right)}{\partial x_i^{k^{(1)}*}} x_{io}^{k^{(1)}*} + \sum_{i \in I^S} \frac{\partial F\left(X_o^*, Z_o^*, Y_o^*\right)}{\partial x_i^{Sk^{(1)}*}} x_{io}^{Sk^{(1)}*}, \\
&\left\langle Z_o^{Ak^{(1)}*}, \nabla_{Z^{Ak^{(1)}}} F\left(X_o^*, Z_o^*, Y_o^*\right) \right\rangle \\
&= \sum_{f \in C^{k^{(1)}}} \frac{\partial F\left(X_o^*, Z_o^*, Y_o^*\right)}{\partial z_f^*} z_{fo}^{k^{(1)}*} + \sum_{f \in C^S} \frac{\partial F\left(X_o^*, Z_o^*, Y_o^*\right)}{\partial z_f^{Sk^{(1)}*}} z_{fo}^{Sk^{(1)}*}, \\
&\left\langle Z_o^{Ak^{(2)}*}, \nabla_{Z^{Ak^{(2)}}} F\left(X_o^*, Z_o^*, Y_o^*\right) \right\rangle \\
&= \sum_{f \in C^{k^{(2)}}} \frac{\partial F\left(X_o^*, Z_o^*, Y_o^*\right)}{\partial z_f^*} z_{fo}^{k^{(2)}*} + \sum_{f \in C^S} \frac{\partial F\left(X_o^*, Z_o^*, Y_o^*\right)}{\partial z_f^{Sk^{(2)}*}} z_{fo}^{Sk^{(2)}*}, \\
&\left\langle Y_o^{Ak^{(2)}*}, \nabla_{Y^{Ak^{(2)}}} F\left(X_o^*, Z_o^*, Y_o^*\right) \right\rangle \\
&= \sum_{r \in O^{k^{(2)}}} \frac{\partial F\left(X_o^*, Z_o^*, Y_o^*\right)}{\partial y_r^{k^{(2)}*}} y_{ro}^{k^{(2)}*} + \sum_{r \in O^S} \frac{\partial F\left(X_o^*, Z_o^*, Y_o^*\right)}{\partial y_r^{Sk^{(2)}*}} y_{ro}^{Sk^{(2)}*}.
\end{aligned}
$$

hold. Then, the right-hand SE in stage 1 and each activity $k^{(1)}$ are calculated as:

$$
\varepsilon^{(N1)+}(X_o^*, Z_o^*) = -\frac{\left\langle X_o^*, \nabla_X F\left(X_o^*, Z_o^*, Y_o^*\right)\right\rangle}{\left\langle Z_o^*, \nabla_Z F\left(X_o^*, Z_o^*, Y_o^*\right)\right\rangle} = \sum_{k^{(1)} \in H^{(1)}} \sum_{i \in I^{k^{(1)}}} v_i^{k^{(1)}*} x_{io}^{k^{(1)}*} + \sum_{i \in I^S} v_i^{S*} x_{io}^{S*},
$$

$$
\begin{aligned}
&\varepsilon^{k^{(1)}+}(X_o^{Ak^{(1)}*}, Z_o^{Ak^{(1)}*}) \\
&= -\frac{\left\langle X_o^{Ak^{(1)}*}, \nabla_{X^{Ak^{(1)}}} F\left(X_o^*, Z_o^*, Y_o^*\right)\right\rangle}{\left\langle Z_o^{Ak^{(1)}*}, \nabla_{Z^{Ak^{(1)}}} F\left(X_o^*, Z_o^*, Y_o^*\right)\right\rangle + \left\langle Z_o^{Ak^{(2)}*}, \nabla_{Z^{Ak^{(2)}}} F\left(X_o^*, Z_o^*, Y_o^*\right)\right\rangle} \\
&= \frac{\sum_{i \in I^{k^{(1)}}} v_i^{k^{(1)}*} x_{io}^{k^{(1)}*} + \sum_{i \in I^S} v_i^{S*} x_{io}^{Sk^{(1)}*}}{\sum_{f \in C^{k^{(1)}}} \tau_f^{k^{(1)}*} z_{fo}^{k^{(1)}*} + \sum_{f \in C^S} \tau_f^{S(1)*} z_{fo}^{Sk^{(1)}*} - \sum_{f \in C^{k^{(2)}}} \tau_f^{k^{(2)}*} z_{fo}^{k^{(2)}*} - \sum_{f \in C^S} \tau_f^{S(2)*} z_{fo}^{Sk^{(2)}*}},
\end{aligned} \tag{6.16}
$$

where $v_i^{k^{(1)}*}, v_i^{S*}, \tau_f^{k^{(1)}*}, \tau_f^{S(1)*}, u_{10}^{k^{(1)}*}, u_{20}^{k^{(2)}*}$ are assumed as the unique optimal values obtained by solving model (6.13).

Let $\sum_{f \in C^{k^{(1)}}} \tau_f^{k^{(1)}*} z_{fo}^{k^{(1)}*} + \sum_{f \in C^S} \tau_f^{S(1)*} z_{fo}^{Sk^{(1)}*} - \sum_{f \in C^{k^{(2)}}} \tau_f^{k^{(2)}*} z_{fo}^{k^{(2)}*} - \sum_{f \in C^S} \tau_f^{S(2)*} z_{fo}^{Sk^{(2)}*} = R^{k^{(1)}} (\forall k^{(1)})$ hold, we have:

$$
\varepsilon^{(N1)+}(X_o^*, Z_o^*) = \sum_{k^{(1)} \in H^{(1)}} R^{k^{(1)}} \varepsilon^{k^{(1)}+}(X_o^{Ak^{(1)}*}, Z_o^{Ak^{(1)}*}). \tag{6.17}
$$

Note that $\sum_{k^{(1)}\in H^{(1)}} R^{k^{(1)}} = 1$ holds due to the second constrain in model (6.13). Similarly, for the right-hand SE in stage 2 and each activity $k^{(2)}$, we have

$$\begin{aligned}
&\varepsilon^{(N2)+}(Z_o^*, Y_o^*) = -\frac{\left\langle Z_o^*, \nabla_Z F\left(X_o^*, Z_o^*, Y_o^*\right)\right\rangle}{\left\langle Y_o^*, \nabla_Y F\left(X_o^*, Z_o^*, Y_o^*\right)\right\rangle} \\
&= \sum_{k^{(1)}\in H^{(1)}} \sum_{f\in C^{k^{(1)}}} \tau_f^{k^{(1)}} z_{fo}^{k^{(1)}*} + \sum_{f\in C^S} \tau_f^{S(1)} z_{fo}^{S*} - \sum_{k^{(2)}\in H^{(2)}} \sum_{f\in C^{k^{(2)}}} \tau_f^{k^{(2)}} z_{fo}^{k^{(2)}*} - \sum_{f\in C^S} \tau_f^{S(2)} z_{fo}^{S*}, \\
&\varepsilon^{k^{(2)}+}(Z_o^{Ak^{(2)}*}, Y_o^{Ak^{(2)}*}) \\
&= -\frac{\left\langle Z_o^{Ak^{(1)}*}, \nabla_{Z^{Ak^{(1)}}} F\left(X_o^*, Z_o^*, Y_o^*\right)\right\rangle + \left\langle Z_o^{Ak^{(2)}*}, \nabla_{Z^{Ak^{(2)}}} F\left(X_o^*, Z_o^*, Y_o^*\right)\right\rangle}{\left\langle Y_o^{Ak^{(2)}*}, \nabla_{Y^{Ak^{(2)}}} F\left(X_o^*, Z_o^*, Y_o^*\right)\right\rangle} \\
&= \frac{\sum_{f\in C^{k^{(1)}}} \tau_f^{k^{(1)}*} z_{fo}^{k^{(1)}*} + \sum_{f\in C^S} \tau_f^{S(1)*} z_{fo}^{Sk^{(1)}*} - \sum_{f\in C^{k^{(2)}}} \tau_f^{k^{(2)}*} z_{fo}^{k^{(2)}*} - \sum_{f\in C^S} \tau_f^{S(2)*} z_{fo}^{Sk^{(2)}*}}{\sum_{r\in O^{k^{(2)}}} u_r^{k^{(2)}*} y_{ro}^{k^{(2)}*} + \sum_{r\in O^S} u_r^{S*} y_{ro}^{Sk^{(2)}*}},
\end{aligned} \tag{6.18}$$

where $v_i^{k^{(1)}*}, v_i^{S*}, \tau_f^{k^{(1)}*}, \tau_f^{S(1)*}, u_{10}^{k^{(1)}*}, u_{20}^{k^{(2)}*}$ are assumed as the unique optimal values obtained by solving model (6.14).

Let $\sum_{r\in O^{k^{(2)}}} u_r^{k^{(2)}*} y_{ro}^{k^{(2)}*} + \sum_{r\in O^S} u_r^{S*} y_{ro}^{Sk^{(2)}*} = R^{k^{(2)}} (\forall k^{(2)})$ hold, we have:

$$\varepsilon^{(N2)+}(X_o^*, Z_o^*) = \sum_{k^{(2)}\in H^{(2)}} R^{k^{(2)}} \varepsilon^{k^{(2)}+}(Z_o^{Ak^{(2)}*}, Y_o^{Ak^{(2)}*}). \tag{6.19}$$

Note that $\sum_{k^{(2)}\in H^{(2)}} R^{k^{(2)}} = 1$ holds due to the third constrain in model (6.14).

Consequently, the right-hand SE in stages 1 and 2 are expressed as the weighted sum of all SE in activities of stages 1 and 2, respectively, in which the weights are allocated by models (6.13) and (6.14), respectively. Consider that $\varepsilon^{(N)+}$ and is the product of $\varepsilon^{(N1)+}$ and $\varepsilon^{(N2)+}$, we have:

$$\begin{aligned}
\varepsilon^{(N)+} &= \varepsilon^{(N1)+} * \varepsilon^{(N2)+} \\
&= \sum_{k^{(1)}\in H^{(1)}} R^{k^{(1)}} \varepsilon^{k^{(1)}+} * \sum_{k^{(2)}\in H^{(2)}} R^{k^{(2)}} \varepsilon^{k^{(2)}+}.
\end{aligned} \tag{6.20}$$

The left-hand SE can also be decomposed in the same way.

Accordingly, the right-hand RTS of each smallest sub-system is defined as follows:

**Definition 6.1.** *In the $k^{(l)}(\forall k, l = 1, 2)$ sub-system, $DMU_o(X_o^*, Z_o^*, Y_o^*)$ exhibits:*

- *Increasing RTS on the right-hand side if* $\varepsilon^{k^{(l)}+} > 0$.
- *Decreasing RTS on the right-hand side if* $\varepsilon^{k^{(l)}+} < 0$.
- *Constant RTS on the right-hand side if* $\varepsilon^{k^{(l)}+} = 0$.

## 6.4 Numerical study

As the continuous rapid growth of green loan scale, the Chinese bank industry is playing an increasingly important role in achieving sustainable development goals. To ensure the integrity of data, the selection of DMUs is based on the completeness of disclosed information. Due to limitations in data disclosure, particularly regarding green loans and their associated $CO_2$ emission reductions, only a portion of banks provide such information. Specifically, the dataset includes 18 DMUs, consisting of 5 large state-owned commercial banks, 6 national joint-stock commercial banks and 7 local joint-stock commercial banks. The original data is shown in the Table A.3, and their descriptive statistics are listed as Table 6.1.

Table 6.1: Data description of 18 commercial banks

| Variables | Units | Observations | Mean | SD | Maximum | Minimum |
|---|---|---|---|---|---|---|
| | | | Inputs | | | |
| Employees | Thousand | 18 | 113.42 | 144.538 | 442.288 | 1.826 |
| IC | Billion CNY | 18 | 156.174 | 189.978 | 589.688 | 3.07288 |
| | | | Intermediates | | | |
| BGL | Billion CNY | 18 | 1093.15 | 1671.88 | 5400 | 4.498 |
| BNL | Billion CNY | 18 | 5869.33 | 6851.18 | 19606.6 | 112.328 |
| | | | Outputs | | | |
| CER | Thousand ton | 18 | 72034.8 | 158523 | 687120 | 1.186 |
| IIG | Billion CNY | 18 | 30.5633 | 44.2243 | 145.26 | 0.19134 |
| IING | Billion CNY | 18 | 806.585 | 4.77822 | 247.321 | 276.191 |

Suppose that the number of employees and interest costs (represented as $x_1$ and $x_2$, respectively) are initial inputs to jointly produce green and non-green loans (represented as $z_1$ and $z_2$, respectively) in the first stage. In the second stage, two activities are identified: The first activity uses green loans ($z_1$) as the activity-specific input to produce CO2 emission reduction (represented as $y_1$) as activity-specific output, and interest incomes as a part of the shared output (represented as $y^{S1}$); The second activity uses non-green

loans ($z_2$) as the activity-specific input to produce interest incomes as another part of the shared output (represented as $y^{S2}$). Note that the total interest income (represented as $y^S$) is the sum of $y^{S1}$ and $y^{S2}$. Fig. 6.2 describes the sustainability production of the 18 commercial banks.

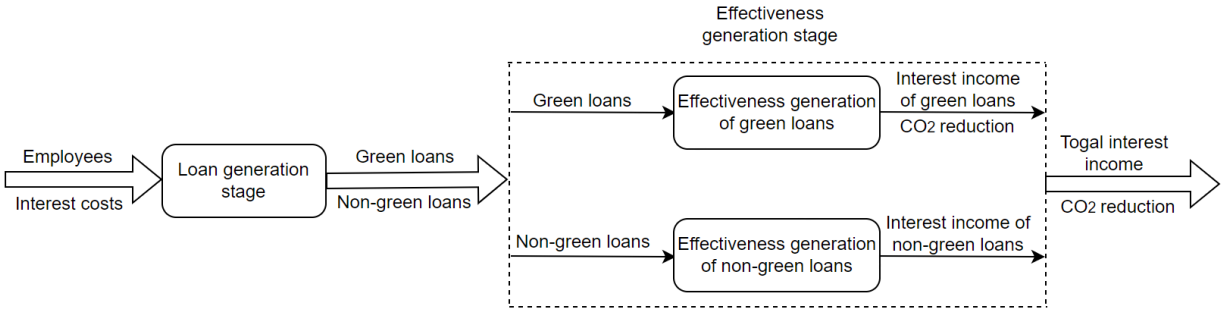



Figure 6.2: Two-stage network structure of 18 commercial banks.

As shown in Fig. 6.2, the numbers of activities in the first and second stage are equal to 1 and 2, respectively (i.e., $K^{(1)} = 1$ and $K^{(2)} = 2$). Therefore, based on the production technology (6.5), the according production technology of banks is formulated as:

$$T_{\text{Bank}} = \left\{ (x_1, x_2, z_1, z_2, y_1, y^S) \;\middle|\; \begin{array}{l} \sum_{j=1}^n \lambda_j^1 x_{1j} \le x_1, \sum_{j=1}^n \lambda_j^1 x_{2j} \le x_2, \\ \sum_{j=1}^n \lambda_j^1 z_{1j} \ge z_1, \sum_{j=1}^n \lambda_j^1 z_{2j} \ge z_2, \\ \sum_{j=1}^n \lambda_j^2 z_{1j} \le z_1, \sum_{j=1}^n \lambda_j^3 z_{2j} \le z_2, \\ \sum_{j=1}^n \lambda_j^2 y_{1j} \ge y_1, \\ \sum_{j=1}^n \lambda_j^2 y^{S1} + \sum_{j=1}^n \lambda_j^3 y^{S2} \ge y^S, \\ \sum_{j=1}^n \lambda_j^1 = \sum_{j=1}^n \lambda_j^2 = \sum_{j=1}^n \lambda_j^3 = 1, \\ \lambda_j^1, \lambda_j^2, \lambda_j^3 \ge 0. \end{array} \right\} \quad (6.21)$$

Then, we use output-oriented model (i.e., model (6.22)) to project all

inefficient DMUs to the efficient frontier of the production technology $T_{\text{Bank}}$:

$$\frac{1}{E_0^{N-O}} = \max_{\theta,\lambda_j^1,\lambda_j^2,\lambda_j^3,z_1,z_2} \theta$$
$$\text{s.t.} \begin{cases} \sum_{j=1}^n \lambda_j^1 x_{1j} \le x_1, \sum_{j=1}^n \lambda_j^1 x_{2j} \le x_2, \\ \sum_{j=1}^n \lambda_j^1 z_{1j} \ge z_1, \sum_{j=1}^n \lambda_j^1 z_{2j} \ge z_2, \\ \sum_{j=1}^n \lambda_j^2 z_{1j} \le z_1, \sum_{j=1}^n \lambda_j^3 z_{2j} \le z_2, \\ \sum_{j=1}^n \lambda_j^2 y_{1j} \ge \theta y_1, \sum_{j=1}^n \lambda_j^2 y^{S1} + \sum_{j=1}^n \lambda_j^3 y^{S2} \ge \theta y^S, \\ \sum_{j=1}^n \lambda_j^1 = \sum_{j=1}^n \lambda_j^2 = \sum_{j=1}^n \lambda_j^3 = 1, \lambda_j^1, \lambda_j^2, \lambda_j^3 \ge 0, \\ \theta, z_1, z_2 \text{ signfree.} \end{cases} \tag{6.22}$$

Let the optimal solution vector of model (6.22) be $(\theta^*, \lambda_j^{1*}, \lambda_j^{2*}, \lambda_j^{3*}, z_1^*, z_2^*)$, each inefficient $\text{DMU}_o(X_o, Z_o, Y_o)$ can be projected as output radial $\text{DMU}_o^*(X_o^*, Z_o^*, Y_o^*)$ by applying $x_{1o}^* = \sum_{j=1}^n \lambda_j^{1*} x_{1j}$, $x_{2o}^* = \sum_{j=1}^n \lambda_j^{1*} x_{2j}$, $z_{1o}^* = z_1^*$, $z_{2o}^* = z_2^*$, $y_{1o}^* = \sum_{j=1}^n \lambda_j^{2*} y_{1j}$, $y_o^{S1*} = \sum_{j=1}^n \lambda_j^{2*} y^{S1}$ and $y_o^{S2*} = \sum_{j=1}^n \lambda_j^{3*} y^{S2}$ based on Eqs. (6.7). The results of projections and efficient scores are shown in Table B.3. Then we calculate the right-hand SE for overall production system and each sub-system based on the methods proposed in Section 6.3. Specifically, based on the Theorem 9, we have the following models to calculate $\varepsilon_{\text{Bank}}^{N+}$, $\varepsilon_{\text{Bank}}^{N1+}$ and $\varepsilon_{\text{Bank}}^{N2+}$:

$$\varepsilon_{\text{Bank}}^{N+} = \min_{v_1,v_2,\tau_1,\tau_2,\tau_3,\tau_4,u_1,u_2,u_{10},u_{20},u_{30}} v_1 x_{1o}^* + v_2 x_{2o}^*$$
$$\text{s.t.} \begin{cases} v_1 x_{1o}^* + v_2 x_{2o}^* - \tau_1 z_{1o}^* - \tau_2 z_{2o}^* + \tau_3 z_{1o}^* + \tau_4 z_{2o}^* \\ -u_1 y_{1o}^* - u_2 y_o^{S*} + u_{10} + u_{20} + u_{30} = 0, \\ -v_1 x_{1j}^* - v_2 x_{2j}^* + \tau_1 z_{1j}^* + \tau_2 z_{2j}^* - u_{10} \le 0 \, (\forall j), \\ -\tau_3 z_{1j}^* + u_1 y_{1j}^* + u_2 y_j^{S1*} - u_{20} \le 0 \, (\forall j), \\ -\tau_4 z_{1j}^* + u_2 y_j^{S2*} - u_{30} \le 0 \, (\forall j), \\ \tau_1 z_{1o}^* + \tau_2 z_{2o}^* - \tau_3 z_{1o}^* - \tau_4 z_{2o}^* = 0, \\ u_1 y_{1o}^* + u_2 y_o^{S*} = 1, \\ v_1, v_2, \tau_1, \tau_2, \tau_3, \tau_4, u_1, u_2 \ge 0, \\ u_{10}, u_{20}, u_{30} \text{ signfree,} \end{cases} \tag{6.23}$$

$$\varepsilon_{\text{Bank}}^{N1+} = \min_{v_1,v_2,\tau_1,\tau_2,\tau_3,\tau_4,u_1,u_2,u_{10},u_{20},u_{30}} v_1 x_{1o}^* + v_2 x_{2o}^*$$
$$\text{s.t.} \begin{cases} \text{The first 55 constrains in model (6.23)}, \\ \tau_1 z_{1o}^* + \tau_2 z_{2o}^* - \tau_3 z_{1o}^* - \tau_4 z_{2o}^* = 1, \\ u_1 y_{1o}^* + u_2 y_o^{S*} = 0, \\ v_1, v_2, \tau_1, \tau_2, \tau_3, \tau_4, u_1, u_2 \geq 0, u_{10}, u_{20}, u_{30} \text{ signfree}, \end{cases} \tag{6.24}$$

$$\varepsilon_{\text{Bank}}^{N2+} = \min_{v_1,v_2,\tau_1,\tau_2,\tau_3,\tau_4,u_1,u_2,u_{10},u_{20},u_{30}} -\tau_1 z_{1o}^* - \tau_2 z_{2o}^* + \tau_3 z_{1o}^* + \tau_4 z_{2o}^*$$
$$\text{s.t.} \begin{cases} \text{The first 55 constrains in model (6.23)}, \\ -v_1 x_{1o}^* - v_2 x_{2o}^* = 0, \\ u_1 y_{1o}^* + u_2 y_o^{S*} = 1, \\ v_1, v_2, \tau_1, \tau_2, \tau_3, \tau_4, u_1, u_2 \geq 0, u_{10}, u_{20}, u_{30} \text{ signfree}. \end{cases} \tag{6.25}$$

Let $(v_1^*, v_2^*, \tau_1^*, \tau_2^*, \tau_3^*, \tau_4^*, u_1^*, u_2^*, u_{10}^*, u_{20}^*, u_{30}^*)$ be the optimal solution vector of model (6.25). Then, for the two activities in the stage 2, $\varepsilon_{\text{Bank}}^{N2+}$ can be further decomposed by $\varepsilon_{\text{Bank}}^{N2+} = R^1 * \varepsilon_{\text{Bank}}^{N21+} + R^2 * \varepsilon_{\text{Bank}}^{N22+}$, where $R^1 = u_1^* y_{1o}^* + u_2^* y_o^{S1*}$, $R^2 = u_2^* y_o^{S2*}$, $\varepsilon_{\text{Bank}}^{N21+} = \frac{\tau_1^* z_{1o}^* + \tau_2^* z_{2o}^* - \tau_3^* z_{1o}^* - \tau_4^* z_{2o}^*}{u_1^* y_{1o}^* + u_2^* y_o^{S1*}}$ and $\varepsilon_{\text{Bank}}^{N22+} = \frac{\tau_1^* z_{1o}^* + \tau_2^* z_{2o}^* - \tau_3^* z_{1o}^* - \tau_4^* z_{2o}^*}{u_2^* y_o^{S2*}}$.

By solving model (6.23), (6.24) and (6.25), the results of the right-hand SE in the overall system and each sub-system are shown in Table 6.2.

Table 6.2: Right-hand SE results and decomposition

| DMU | $\varepsilon_{\text{Bank}}^{N+}$ | RTS | $\varepsilon_{\text{Bank}}^{N1+}$ | RTS | $\varepsilon_{\text{Bank}}^{N2+}$ | RTS | $\varepsilon_{\text{Bank}}^{N21+}$ | RTS | $\varepsilon_{\text{Bank}}^{N22+}$ | RTS |
|---|---|---|---|---|---|---|---|---|---|---|
| $DMU_1$ | 0.00 | D | 0.00 | D | inf. | undef. | inf. | undef. | inf. | undef. |
| $DMU_2$ | 0.09 | D | 0.10 | D | 0.93 | D | 7.75 | I | 1.05 | I |
| $DMU_3$ | 0.09 | D | 0.10 | D | 0.93 | D | 7.61 | I | 1.06 | I |
| $DMU_4$ | 0.00 | D | 0.62 | D | 0.00 | D | 0.00 | D | undef. | undef. |
| $DMU_5$ | 0.75 | D | 0.88 | D | 0.85 | D | 9.70 | I | 0.93 | D |
| $DMU_6$ | 0.66 | D | 0.82 | D | 0.80 | D | 14.23 | I | 0.85 | D |
| $DMU_7$ | 0.56 | D | 0.74 | D | 0.75 | D | 6.87 | I | 0.84 | D |
| $DMU_8$ | 0.56 | D | 0.81 | D | 0.70 | D | 9.19 | I | 0.75 | D |
| $DMU_9$ | 0.64 | D | 0.85 | D | 0.75 | D | 9.53 | I | 0.81 | D |
| $DMU_{10}$ | 0.89 | D | 0.93 | D | 0.96 | D | 10.43 | I | 1.05 | I |
| $DMU_{11}$ | 1.04 | I | 1.03 | I | 1.01 | I | 14.56 | I | 1.09 | I |
| $DMU_{12}$ | 0.51 | D | 0.77 | D | 0.66 | D | 0.66 | D | undef. | undef. |
| $DMU_{13}$ | 0.53 | D | 0.53 | D | 1.01 | I | 16.27 | I | 1.07 | I |
| $DMU_{14}$ | 1.05 | I | 1.03 | I | 1.01 | I | 14.10 | I | 1.09 | I |
| $DMU_{15}$ | 1.06 | I | 1.04 | I | 1.02 | I | 11.71 | I | 1.12 | I |
| $DMU_{16}$ | 1.78 | I | 1.29 | I | 1.38 | I | 35.83 | I | 1.43 | I |
| $DMU_{17}$ | 1.11 | I | 1.08 | I | 1.04 | I | 15.19 | I | 1.11 | I |
| $DMU_{18}$ | 1.15 | I | 1.10 | I | 1.05 | I | 14.68 | I | 1.13 | I |

From the SE results shown in Table 6.2, we find that there are total 6 banks can further increase the scale of the whole system, including 5 local joint-stock commercial banks and 1 national joint-stock commercial bank. It is worth noting that increasing RTS prevails on the right-hand side for 15 DMUs in the effectiveness generation stage of green loans.

To further explore the influence of adjusting loan structures on increasing scales of projected banks, $\delta_1$ and $\delta_2$ are introduced to represent the directions of green loans and non-green loans, respectively, where $\alpha_1 + \alpha_2 = 2$ holds for standardization. By varying $\alpha_1$ and $\alpha_2$ by a fixed step-length of 0.01, the largest DSE and the corresponding directions are shown in Table 6.3.

Table 6.3: Optimal loan structures and corresponding DSE

| DMU | $\varepsilon_{\max}^{N+}$ | $(\alpha_1, \alpha_2)$ | DMU | $\varepsilon_{\max}^{N+}$ | $(\alpha_1, \alpha_2)$ |
|---|---|---|---|---|---|
| $DMU_1$ | 0.00 | inf. | $DMU_{10}$ | 0.93 | (0.49,1.51) |
| $DMU_2$ | 0.25 | (1.88,0.12) | $DMU_{11}$ | 1.05 | (1.03,0.97) |
| $DMU_3$ | 0.25 | (1.88,0.12) | $DMU_{12}$ | 0.66 | (1.22,0.78) |
| $DMU_4$ | 0.00 | inf. | $DMU_{13}$ | 0.59 | (1.47,0.53) |
| $DMU_5$ | 0.80 | (1.33,0.67) | $DMU_{14}$ | 1.05 | (1.03,0.97) |
| $DMU_6$ | 0.75 | (1.56,0.44) | $DMU_{15}$ | 1.07 | (1.03,0.97) |
| $DMU_7$ | 0.56 | (1.06,0.94) | $DMU_{16}$ | 1.86 | (1.46,0.54) |

| DMU | $\varepsilon^{N+}_{\max}$ | $(\alpha_1, \alpha_2)$ | DMU | $\varepsilon^{N+}_{\max}$ | $(\alpha_1, \alpha_2)$ |
|---|---|---|---|---|---|
| $DMU_8$ | 0.63 | (1.41,0.59) | $DMU_{17}$ | 1.13 | (1.09,0.91) |
| $DMU_9$ | 0.69 | (1.33,0.67) | $DMU_{18}$ | 1.16 | (1.1,0.9) |

According to Tables 6.3 and 6.2, no bank changes its overall RTS status by adjusting its loan structure, although SE can generally be higher. Especially, without considering $DMU_1$ and $DMU_4$, $\varepsilon^{N+}$ always maximizes when allocating more proportions to green loans. To carry out in-depth analysis for the right-hand DSE of each DMU, we exhibit the relationship between $\varepsilon^{N+}$ and $\alpha_1$ for all banks as shown in Fig. 6.3.

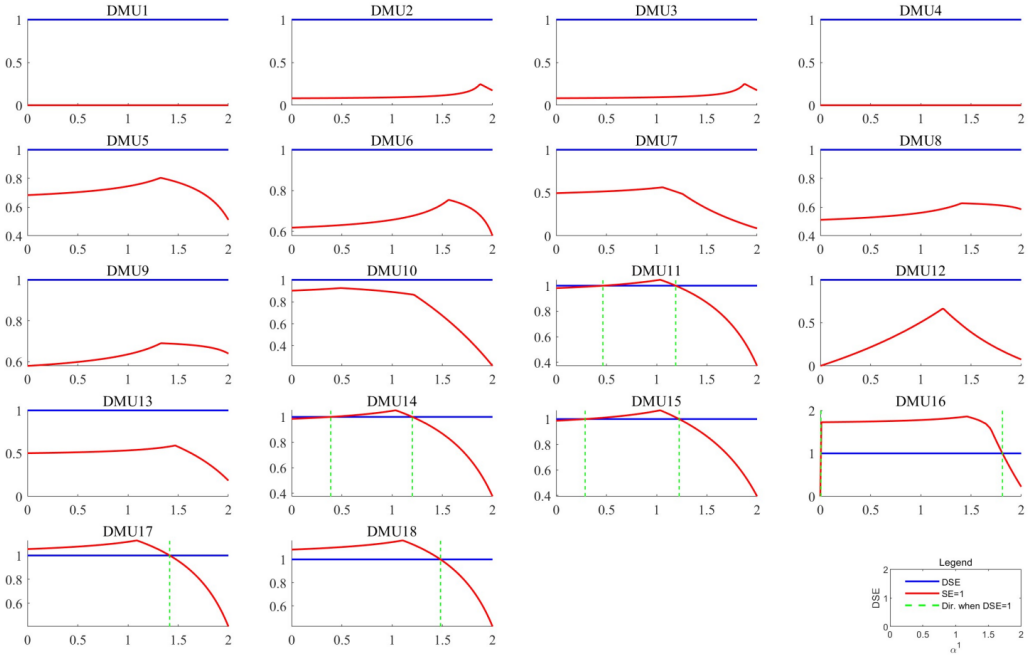



Figure 6.3: Relationship between DSE and directions of green and non-green loans.

The results shown in Fig. 6.3 indicate that allocating too many proportions to green loans is not conducive to increasing final output. For example, increasing RTS prevails in $DMU_{11}$ when $\alpha_1$ is between two dotted green lines ($\alpha_1 \in [0.47, 1.19]$), while decreasing RTS prevails when $\alpha_1 > 1.19$.

## 6.5 Chapter summary

This chapter proposed a closed production technology with a mixture of parallel and two-stage structures, which considers shared inputs and outputs in each stage. This was followed by proposing the method to decompose the SE of the proposed network production technology into each smallest sub-technology SE. Considering the effectiveness of green loans in CO2 emission reduction, a case study was carried out to illustrate the SE decomposition method in proposed closed network technologies. The results showed that for most banks in the dataset, implementing appropriate increases in the proportion of green loans can help increase final outputs more effectively.

The contributions of this chapter are summarized below:

- This chapter proposed a production technology with a mixture of parallel and two-stage structures, which considered shared inputs and outputs in each stage.
- This chapter proposed the method to decompose the SE of the proposed network production technology into each smallest sub-technology SE.

# Chapter 7

# Conclusions and Further Research

## 7.1 Conclusions

This thesis has achieved the four objectives set in Chapter 1. Specifically, in Chapter 3, the first objective was achieved by the DDO method and the duality theory of linear programming. In Chapter 4, the second objective was achieved by introducing different standardization conditions and applying heuristic search. In Chapter 5, the third objective was achieved by incorporating undesirable outputs with different disposability assumptions into the basic network production technologies. In Chapter 6, the fourth objective was achieved by integrating the SE decomposition methods in the two basic network production technologies proposed in Chapter 3. These achievements contribute to a deeper understanding of the notion of SE in network production technologies.

With the achievements of the four research objectives, this thesis has generated several key contributions as follows:

- Chapter 3 proposed a novel decomposition method of overall SE in multi-activity production technologies. The method is demonstrated to be a weighted sum of SE in each sub-activity, in which the weight allocated to each activity is automatically generated by the model. For two-stage production technologies, this chapter proposed a novel method of decomposing overall SE into each stage, in which the decomposition results obtained by the proposed method are unique.

- Given the existence of activity-specific and shared inputs and outputs, Chapter 4 considered that the standardization conditions of change directions need to be modified from different management perspectives,

which reflect the mutual compensation differences in the specified direction of changes in inputs. In addition, this chapter developed DSE into two-stage production technologies by introducing a direction vector for intermediate products, in which the change directions of intermediate products represent different production structures. Furthermore, bi-level models were proposed to find the optimal input (and intermediate product) directions that maximize output changes. By using heuristic searching, DSE corresponding to different input directions can be presented. Therefore, the definition of absolute decreasing RTS is proposed to describe the DMUs with no possible direction to achieve IRS. Additionally, the definition of relative decreasing RTS is proposed to describe the DMUs with at least one direction to let them be in the status of decreasing RTS. Based on the data set in Chapter 3, by applying the bi-level model, we found that most universities should increase the proportions of inputs in NS activity to expand the scale, and adjusting production structures cannot make increasing RTS prevail in any cement companies.

- In Chapter 5, the methods of SE estimation were developed in the proposed production technologies, in which strong and selective weak disposability assumptions were considered for different undesirable outputs. For illustrating the developed models of estimating SE, retraction of papers and pollution are considered undesirable outputs in the two case studies, respectively.

- Chapter 6 proposed a novel method to decompose the SE of the proposed network production technology into each smallest sub-technology SE, in which the production technology holds a mixture of parallel and two-stage structures. Considering the effectiveness of green loans in CO2 emission reduction, a case study was carried out to illustrate the SE decomposition method in proposed closed network technologies. The results showed that for most banks in the dataset, implementing an appropriate increase in the proportion of green loans can help increase final outputs more effectively.

## 7.2 Limitations and Directions for Future Research

This thesis develops new methodologies for the estimation and decomposition of SE in network production technologies. Because SE is a marginal

characteristic of points on the production frontier, it captures how marginal changes in inputs at a given point on the production frontier lead to proportional changes in output. In this sense, the results of SE already represent a form of sensitivity analysis. Specifically, the results of SE are estimated by conducting sensitivity analysis of output response functions, which are expressed as linear programs (e.g., (3.18), (5.10) and (6.9)). The results presented in this thesis, therefore, provide sensitivity-based insights into DMUs on the frontier of network production technologies.

Nevertheless, this thesis inevitably faces several limitations. These limitations highlight the assumptions that can be relaxed in network production technologies, other types of scale properties and the influence of different projection points to SE. Addressing them involves significant challenges, but also provides important directions for future research that can broaden the applicability of SE in network production technologies and enhance its practical relevance for decision-making. Specific limitations, possible challenges and future research directions are carried out as follows:

- Chapter 3 assumed that the allocation proportions of shared measures in multi-activity production technologies are known and fixed. In reality, such proportions may be unknown, only partially observed, or subject to measurement error, which may limit applicability. Further research can employ the approach proposed by Podinovski (2022) and Papaioannou & Podinovski (2023) to linearise the expression of production technologies with parallel structures. Therefore, the method can be further developed to estimate and decompose SE in multi-activity production technologies with unknown allocation proportions of shared measures. Such advances would enhance the applicability of the method to empirical settings. Moreover, with respect to multi-stage production technologies, Chapter 3 assumed that all intermediate products are fully used in the subsequent stage. In a more general setting, the intermediate products are allowed to be wasted. Future research could investigate the decomposition of SE in production technologies with more general series structures, for example, by considering two-stage models using the concept of a non-cooperative approach, which is characterised by Stackelberg game theory. Overall, the challenge here is that employing network production technologies with relaxing these assumptions in Chapter 3 makes the decomposition of SE considerably more difficult. For example, if we assume that the quantity of an intermediate product as the output of the first stage may differ from its quantity as the input of the second stage, Eq. (3.46) does not necessarily hold, which makes reconsideration of SE decomposition in

multi-stage production technologies.

- In Chapter 4, heuristic searching is used to solve the developed bi-level models (i.e., model (4.25), (4.29), (4.30) and (4.31)) that find the optimal direction of inputs or intermediate products. However, in the case of the requirement for higher accuracy and the existence of a large number of indicators (e.g., three or more inputs) in production technologies, high computational costs can be generated due to the multitude of repeated iterations. The challenge here is to find an appropriate method to solve such non-linear models, which incur lower computational costs and yield higher computational accuracy. The possible directions include global optimisation techniques and local nonlinear programming solvers.

- In terms of empirical studies, several improvements can be made in Chapter 5 and Chapter 6. First, regarding the production process of universities, each department can be further divided into two distinct activities (i.e., teaching and researching). Evidently, undesirable outputs (i.e., retractions of scholarly publications) are only associated with the research activity. Second, in the case study of cement companies, only environmental pollution indicators were considered as undesirable outputs. However, financial risks (e.g., debt-to-equity ratio, leverage ratio, and current ratio) are also worth considering. Third, in Chapter 6, the bank production process did not consider non-performing loans as undesirable outputs because of the difficulty of obtaining reliable data, particularly on non-performing green loans. Furthermore, loans can be classified into more refined categories according to additional attributes, such as term structures, interest rate types, and borrower types. Nevertheless, these directions of improvements in Chapter 5 and Chapter 6 face a key practical challenge: the availability and accessibility of high-quality data. With the improvement of the information disclosure policies, future research will further refine the construction of these production processes.

- Overall, this thesis concentrates on SE alone, leaving other marginal characteristics of network production technologies unexplored. Examples include marginal rates of substitution and transformation between inputs and outputs. These rates can reflect the trade-off between two inputs (or between two outputs) when the other factors are held constant. Specifically, they measure the amount of one input (or output) that must be increased to compensate for a marginal reduction in another input (or output) in order to maintain the same level of pro-

duction (or resource). The challenge of extending the methodology to these measures is that it is not always clear how such measures can be consistently decomposed in network production technologies. Nonetheless, studying various marginal characteristics in network production technologies is necessary in the future since it would not only yield a more comprehensive understanding of network production technologies but also generate practical insights for resource allocation and policy design.

In summary, the methodologies developed in this thesis represent an important step forward in the analysis of SE in network production technologies. The limitations identified above highlight both the complexity of further development and the opportunities for meaningful extensions. Future research can propose more flexible methods related to the analysis of scale properties in network production technologies.

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# Appendix A

# Original Data

Table A.1: Original data of 15 Chinese cement companies

| DMU | Employees | Energy | Clinker & Cement | Aggregate | Profit | Tax | Company |
|---|---|---|---|---|---|---|---|
| $DMU_1$ | 61662 | 32.49 | 248.52 | 188.14 | 1965.14 | 1546.66 | TSGF |
| $DMU_2$ | 50769 | 25.11 | 272.26 | 110.41 | 10430.14 | 2297.51 | HLSN |
| $DMU_3$ | 17939 | 6.76 | 64.76 | 45.58 | 643.80 | 295.28 | HYJC |
| $DMU_4$ | 20804 | 5.38 | 65.26 | 177.79 | 2762.12 | 705.99 | HXSN |
| $DMU_5$ | 45144 | 7.34 | 86.33 | 34.53 | 25.26 | 973.37 | JYJT |
| $DMU_6$ | 22344 | 8.66 | 86.53 | 41.46 | -1498.37 | 6.80 | JDSN |
| $DMU_7$ | 8297 | 1.36 | 19.80 | 4.05 | 421.30 | 195.44 | XBSN |
| $DMU_8$ | 2957 | 11.30 | 24.05 | 0.00 | 106.09 | 96.36 | YZSN |
| $DMU_9$ | 2047 | 1.90 | 17.62 | 0.00 | 741.51 | 125.35 | TPJT |
| $DMU_{10}$ | 6892 | 2.22 | 25.20 | 43.60 | -633.88 | 629.82 | ZGTR |
| $DMU_{11}$ | 2793 | 1.63 | 16.87 | 13.27 | 744.00 | 186.40 | SFSN |
| $DMU_{12}$ | 1579 | 0.90 | 8.33 | 0.00 | -322.59 | 13.49 | FJSN |
| $DMU_{13}$ | 2644 | 0.88 | 12.21 | 5.33 | 297.40 | 38.39 | NXJC |
| $DMU_{14}$ | 1241 | 0.20 | 3.92 | 0.00 | 230.15 | 74.16 | NBFD |
| $DMU_{15}$ | 16032 | 4.59 | 52.61 | 0.00 | -884.00 | 51.00 | SSSN |

Note: The unit of energy is million tons of standard coals. The units of clinker & cement and aggregate are million tons. The units of profits and tax are million Chinese yuan.

Table A.2: QS top 500 universities in UK

| DMU | Staff | | Expen. | | Undergrad. | | Postgrad. | | Pub. | | Uni. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | NS | SS | NS | SS | NS | SS | NS | SS | NS | SS | |
| $DMU_1$ | 5520 | 480 | 342.631 | 72.459 | 11840 | 70 | 7000 | 1930 | 8860 | 457 | ICL |
| $DMU_2$ | 8170 | 2795 | 421.141 | 280.131 | 5840 | 6485 | 4150 | 4485 | 10196 | 1971 | Oxf. |
| $DMU_3$ | 6940 | 1960 | 282.145 | 186.723 | 7665 | 5570 | 3945 | 3530 | 8288 | 1482 | Cam. |
| $DMU_4$ | 8270 | 2165 | 465.074 | 176.14 | 14015 | 10330 | 12680 | 9295 | 11735 | 1848 | UCL |
| $DMU_5$ | 4655 | 2310 | 365.133 | 160.127 | 11475 | 13040 | 5205 | 5205 | 5782 | 1086 | Edin. |
| $DMU_6$ | 5850 | 1980 | 339.183 | 157.712 | 18330 | 13110 | 6175 | 5385 | 5925 | 989 | UOM |
| $DMU_7$ | 3190 | 1335 | 287.653 | 159.41 | 12130 | 10035 | 5120 | 6220 | 5556 | 1038 | KCL |
| $DMU_8$ | 370 | 1435 | 28.718 | 158.927 | 845 | 5060 | 660 | 6305 | 523 | 859 | LSE |
| $DMU_9$ | 3490 | 1455 | 198.018 | 126.342 | 12115 | 10625 | 3105 | 4050 | 3800 | 633 | UoB |
| $DMU_{10}$ | 2610 | 1590 | 158.25 | 153.206 | 8325 | 10065 | 2605 | 4030 | 2951 | 620 | Warw. |
| $DMU_{11}$ | 3565 | 1450 | 175.852 | 110.06 | 11420 | 9620 | 4855 | 7730 | 3875 | 754 | UofG |
| $DMU_{12}$ | 3015 | 930 | 156.078 | 69.047 | 10100 | 5530 | 3380 | 5475 | 3539 | 543 | Soton. |
| $DMU_{13}$ | 3520 | 1535 | 184.759 | 123.048 | 12470 | 13975 | 3520 | 5520 | 3661 | 756 | Leeds |
| $DMU_{14}$ | 1225 | 1235 | 75.425 | 87.605 | 6840 | 10900 | 1105 | 2715 | 1824 | 653 | Durham |
| $DMU_{15}$ | 940 | 625 | 52.089 | 40.709 | 4305 | 4690 | 950 | 1120 | 1143 | 311 | St Andrews |
| $DMU_{16}$ | 3710 | 1130 | 186.968 | 77.168 | 12235 | 8055 | 4865 | 3910 | 3366 | 613 | TUOS |
| $DMU_{17}$ | 2560 | 915 | 158.726 | 77.149 | 9850 | 8040 | 3110 | 3730 | 3240 | 506 | QM |
| $DMU_{18}$ | 3000 | 970 | 174.405 | 63.977 | 11505 | 9410 | 3055 | 2240 | 1775 | 334 | NEWC |
| $DMU_{19}$ | 1130 | 790 | 70.741 | 56.72 | 6015 | 7560 | 1115 | 2245 | 1441 | 404 | LUSU |
| $DMU_{20}$ | 1270 | 500 | 85.474 | 47.421 | 9120 | 5345 | 1540 | 1375 | 1382 | 430 | Bath |
| $DMU_{21}$ | 3335 | 925 | 235.399 | 77.379 | 12565 | 8845 | 2735 | 2380 | 3675 | 570 | UOL |
| $DMU_{22}$ | 1825 | 1055 | 124.126 | 94.224 | 8660 | 13530 | 2570 | 3230 | 2695 | 663 | Exon |
| $DMU_{23}$ | 865 | 900 | 61.382 | 77.642 | 4985 | 7270 | 1150 | 1515 | 1000 | 414 | UoR |
| $DMU_{24}$ | 3075 | 1335 | 204.854 | 84.486 | 12550 | 8970 | 3145 | 2580 | 2593 | 629 | CU |
| $DMU_{25}$ | 2080 | 745 | 118.512 | 52.683 | 10300 | 6380 | 2935 | 2145 | 2025 | 428 | QUB |
| $DMU_{26}$ | 1275 | 515 | 78.138 | 35.884 | 9075 | 5825 | 1825 | 1475 | 1180 | 351 | Lboro |
| $DMU_{27}$ | 1170 | 525 | 62.291 | 43.158 | 4625 | 4510 | 1420 | 1910 | 1465 | 303 | Aberd. |
| $DMU_{28}$ | 855 | 880 | 71.924 | 75.672 | 5205 | 9280 | 1220 | 2905 | 1274 | 554 | Sussex |
| $DMU_{29}$ | 1955 | 890 | 102.49 | 55.426 | 8195 | 6005 | 2875 | 3785 | 1586 | 309 | Strath. |
| $DMU_{30}$ | 1510 | 735 | 82.706 | 43.531 | 6845 | 5470 | 2110 | 2370 | 1950 | 290 | Leicester |
| $DMU_{31}$ | 1155 | 500 | 75.974 | 42.581 | 6355 | 4235 | 2055 | 1505 | 1801 | 384 | Surrey |
| $DMU_{32}$ | 1730 | 725 | 93.244 | 46.497 | 9785 | 5855 | 1335 | 2530 | 1245 | 301 | Swansea |
| $DMU_{33}$ | 1215 | 675 | 75.018 | 45.838 | 6830 | 5625 | 1470 | 1845 | 1142 | 316 | UEA |
| $DMU_{34}$ | 660 | 440 | 56.694 | 36.297 | 5795 | 45560 | 3110 | 3085 | 932 | 264 | Brunel |
| $DMU_{35}$ | 600 | 805 | 46.107 | 81.953 | 4690 | 7545 | 1905 | 4270 | 1205 | 245 | CUL |
| $DMU_{36}$ | 225 | 330 | 13.563 | 25.132 | 790 | 2285 | 600 | 1185 | 279 | 193 | BBK |
| $DMU_{37}$ | 695 | 615 | 53.102 | 45.145 | 4970 | 7630 | 1310 | 1470 | 279 | 197 | OBU |
| $DMU_{38}$ | 1710 | 485 | 77.489 | 30.81 | 6215 | 3980 | 1350 | 2635 | 958 | 167 | Dundee |
| $DMU_{39}$ | 840 | 425 | 64.141 | 37.389 | 6085 | 5370 | 1795 | 2075 | 725 | 196 | Aston |
| $DMU_{40}$ | 465 | 450 | 26.988 | 34.991 | 3730 | 4705 | 900 | 1855 | 485 | 189 | Stirling |
| $DMU_{41}$ | 655 | 880 | 37.544 | 51.662 | 4645 | 6555 | 2355 | 2940 | 583 | 474 | Essex |
| $DMU_{42}$ | 660 | 285 | 33.186 | 19.376 | 3920 | 2160 | 980 | 2130 | 534 | 80 | Bangor |
| $DMU_{43}$ | 460 | 530 | 36.265 | 45.28 | 3155 | 6740 | 1035 | 1255 | 645 | 265 | RHUL |

Note: The unit of expenditures is million pounds.

Table A.3: 18 Chinese commercial banks

| DMU | Employees | IC | BGL | BNGL | CER | IIG | IING | Company |
|---|---|---|---|---|---|---|---|---|
| $DMU_1$ | 396.28 | 589.69 | 5400.00 | 19606.61 | 144132.80 | 145.26 | 806.59 | ICBC |
| $DMU_2$ | 352.16 | 468.00 | 3882.94 | 19113.29 | 172540.00 | 104.06 | 773.85 | CCB |
| $DMU_3$ | 442.29 | 475.53 | 4048.70 | 17298.82 | 128950.00 | 106.08 | 702.60 | ABC |
| $DMU_4$ | 267.02 | 450.85 | 3106.70 | 16370.17 | 687120.00 | 84.19 | 666.41 | BOC |
| $DMU_5$ | 195.43 | 202.67 | 637.88 | 7277.37 | 48217.50 | 19.77 | 299.74 | PSBC |
| $DMU_6$ | 97.16 | 128.81 | 447.77 | 6061.10 | 17786.70 | 13.88 | 254.36 | CMB |
| $DMU_7$ | 60.74 | 112.91 | 809.02 | 4524.46 | 22920.00 | 28.23 | 208.05 | CIB |
| $DMU_8$ | 42.76 | 75.54 | 139.65 | 3180.46 | 1191.32 | 4.90 | 178.95 | PAIC |
| $DMU_9$ | 60.84 | 98.30 | 264.24 | 4059.67 | 8075.30 | 7.93 | 178.64 | CMBC |
| $DMU_{10}$ | 40.29 | 42.75 | 269.27 | 1987.32 | 2608.40 | 9.48 | 98.17 | HB |
| $DMU_{11}$ | 13.78 | 24.45 | 31.26 | 901.39 | 474.20 | 1.22 | 42.60 | CBHB |
| $DMU_{12}$ | 15.85 | 39.35 | 156.05 | 1808.74 | 56217.30 | 5.51 | 75.42 | BB |
| $DMU_{13}$ | 15.90 | 42.65 | 287.02 | 1460.45 | 4994.80 | 12.00 | 76.59 | BJ |
| $DMU_{14}$ | 13.48 | 22.18 | 68.16 | 738.93 | 174.50 | 2.11 | 33.01 | BH |
| $DMU_{15}$ | 14.59 | 16.64 | 61.98 | 614.73 | 595.90 | 2.67 | 26.47 | CRCB |
| $DMU_{16}$ | 1.83 | 3.07 | 4.50 | 112.33 | 1.19 | 0.19 | 4.78 | SZRCB |
| $DMU_{17}$ | 5.91 | 9.92 | 31.34 | 281.14 | 502.34 | 1.38 | 14.93 | BG |
| $DMU_{18}$ | 5.28 | 7.80 | 30.18 | 250.96 | 124.00 | 1.28 | 10.64 | BS |

Note: The number of employees and the amount of CO2 emission reduction is in thousands and thousand tons, respectively, while all other indicators are measured in billions of Chinese yuan. IC, BGL, BNGL, CER, IIG AND IING represent interest costs of deposits, balance of green loans, balance of non-green loans, CO2 emission reduction, Interest income of green loans and Interest income of non-green loans, respectively.

# Appendix B

# Projection Data

Table B.1: Projections and efficiencies of 15 Chinese cement companies

| DMU | Employees | Energy | Clinker & Cement | Aggregate | Profit | Tax | Efficiency |
|---|---|---|---|---|---|---|---|
| $DMU_1$ | 50769 | 25.11 | 272.26 | 110.41 | 11930.14 | 2297.51 | 0.67 |
| $DMU_2$ | 50769 | 25.11 | 272.26 | 110.41 | 11930.14 | 2297.51 | 1.00 |
| $DMU_3$ | 17939 | 6.76 | 77.16 | 90.87 | 4598.88 | 729.49 | 0.52 |
| $DMU_4$ | 20804 | 5.38 | 65.26 | 177.79 | 4262.12 | 705.99 | 1.00 |
| $DMU_5$ | 41144 | 7.34 | 86.23 | 60.10 | 3599.39 | 1041.81 | 0.93 |
| $DMU_6$ | 22344 | 8.66 | 97.94 | 63.27 | 4123.54 | 1120.81 | 0.09 |
| $DMU_7$ | 5640 | 1.36 | 17.61 | 39.06 | 2233.64 | 227.21 | 0.88 |
| $DMU_8$ | 2957 | 9.78 | 22.99 | 2.33 | 2445.88 | 171.17 | 0.71 |
| $DMU_9$ | 2047 | 1.90 | 17.62 | 0.00 | 2241.51 | 125.35 | 1.00 |
| $DMU_{10}$ | 6892 | 2.22 | 26.73 | 44.01 | 934.76 | 640.17 | 0.98 |
| $DMU_{11}$ | 2793 | 1.63 | 16.87 | 13.27 | 2244.00 | 186.40 | 1.00 |
| $DMU_{12}$ | 1579 | 0.90 | 9.59 | 0.03 | 1941.90 | 95.48 | 0.69 |
| $DMU_{13}$ | 2644 | 0.88 | 12.21 | 5.33 | 2051.79 | 130.79 | 0.90 |
| $DMU_{14}$ | 1241 | 0.20 | 3.92 | 0.00 | 1730.15 | 74.16 | 1.00 |
| $DMU_{15}$ | 16032 | 4.59 | 55.11 | 121.31 | 3811.21 | 579.00 | 0.26 |

Table B.2: Projections and efficiencies of 43 UK universities

| DMU | Staff | | Expen. | | Undergrad. | | Postgrad. | | Pub. | | Efficiency |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | NS | SS | NS | SS | NS | SS | NS | SS | NS | SS | |
| $DMU_1$ | 5520 | 480 | 342.63 | 72.46 | 11840 | 70 | 7000 | 1930 | 8860 | 457 | 1.00 |
| $DMU_2$ | 8170 | 2556 | 460.62 | 240.65 | 13936 | 7945 | 12473 | 6311 | 11630 | 1924 | 0.90 |
| $DMU_3$ | 6940 | 786 | 405.86 | 63.01 | 12963 | 30762 | 9933 | 3949 | 10345 | 585 | 0.89 |
| $DMU_4$ | 8270 | 2165 | 465.07 | 176.14 | 14015 | 10330 | 12680 | 9295 | 11735 | 1848 | 1.00 |
| $DMU_5$ | 4655 | 1900 | 329.44 | 154.65 | 13855 | 15745 | 6642 | 8341 | 6688 | 1605 | 0.83 |
| $DMU_6$ | 5850 | 1337 | 339.18 | 110.20 | 18330 | 13110 | 6175 | 5773 | 5925 | 989 | 1.00 |
| $DMU_7$ | 3190 | 1335 | 287.65 | 107.03 | 12130 | 10035 | 5120 | 6220 | 5556 | 1038 | 1.00 |
| $DMU_8$ | 370 | 1435 | 26.15 | 111.38 | 2298 | 8230 | 1105 | 7268 | 637 | 956 | 0.87 |
| $DMU_9$ | 3490 | 1311 | 220.14 | 104.22 | 13480 | 11822 | 3986 | 5437 | 3861 | 1072 | 0.90 |
| $DMU_{10}$ | 2610 | 1320 | 206.20 | 105.25 | 11402 | 13785 | 3690 | 5572 | 3813 | 1078 | 0.73 |
| $DMU_{11}$ | 3565 | 1450 | 175.85 | 110.06 | 11420 | 9620 | 4855 | 7730 | 3875 | 754 | 1.00 |
| $DMU_{12}$ | 3010 | 930 | 156.08 | 69.05 | 10100 | 5530 | 3982 | 5475 | 3539 | 543 | 1.00 |
| $DMU_{13}$ | 3520 | 1131 | 219.33 | 88.48 | 13364 | 14977 | 4002 | 5916 | 3929 | 805 | 0.93 |
| $DMU_{14}$ | 1225 | 856 | 97.37 | 65.66 | 7247 | 11549 | 2317 | 3326 | 1967 | 658 | 0.94 |
| $DMU_{15}$ | 940 | 387 | 64.23 | 28.57 | 5536 | 6031 | 1847 | 1440 | 1634 | 236 | 0.78 |
| $DMU_{16}$ | 3710 | 654 | 215.86 | 48.27 | 12696 | 8359 | 5048 | 4057 | 3873 | 352 | 0.96 |
| $DMU_{17}$ | 2560 | 655 | 187.46 | 48.42 | 10736 | 8763 | 3466 | 4066 | 3729 | 354 | 0.92 |
| $DMU_{18}$ | 3000 | 785 | 179.89 | 58.49 | 12589 | 10296 | 3343 | 2513 | 3060 | 410 | 0.91 |
| $DMU_{19}$ | 1130 | 729 | 72.90 | 54.57 | 7029 | 9100 | 1346 | 2623 | 1612 | 544 | 0.86 |
| $DMU_{20}$ | 1270 | 500 | 85.47 | 47.42 | 9120 | 5345 | 1540 | 1375 | 1382 | 430 | 1.00 |
| $DMU_{21}$ | 3335 | 925 | 209.31 | 95.05 | 13189 | 9284 | 3791 | 4064 | 3654 | 802 | 0.95 |
| $DMU_{22}$ | 1825 | 866 | 151.38 | 66.97 | 9181 | 14345 | 2961 | 3509 | 2895 | 666 | 0.94 |
| $DMU_{23}$ | 865 | 900 | 64.37 | 74.66 | 6712 | 9788 | 2232 | 3440 | 1165 | 739 | 0.74 |
| $DMU_{24}$ | 3075 | 901 | 185.46 | 75.98 | 12750 | 9113 | 3367 | 3712 | 3172 | 499 | 0.98 |
| $DMU_{25}$ | 2080 | 300 | 150.13 | 21.06 | 10478 | 6490 | 2986 | 2225 | 2640 | 98 | 0.98 |
| $DMU_{26}$ | 1275 | 515 | 78.14 | 35.88 | 9075 | 5825 | 1825 | 1475 | 1180 | 351 | 1.00 |
| $DMU_{27}$ | 1170 | 358 | 78.96 | 26.49 | 5742 | 16499 | 2235 | 2371 | 2028 | 167 | 0.81 |
| $DMU_{28}$ | 855 | 880 | 63.83 | 83.76 | 6079 | 10838 | 1908 | 3668 | 1385 | 750 | 0.86 |
| $DMU_{29}$ | 1830 | 687 | 107.33 | 50.59 | 9140 | 6697 | 3207 | 4221 | 1842 | 372 | 0.90 |
| $DMU_{30}$ | 1510 | 368 | 100.09 | 26.15 | 7428 | 5936 | 2290 | 2572 | 2286 | 145 | 0.92 |
| $DMU_{31}$ | 1155 | 500 | 83.65 | 34.91 | 6510 | 6289 | 2105 | 1542 | 1905 | 333 | 0.98 |
| $DMU_{32}$ | 1730 | 433 | 110.96 | 28.79 | 10045 | 6011 | 2006 | 2597 | 1838 | 158 | 0.97 |
| $DMU_{33}$ | 1215 | 462 | 86.13 | 34.73 | 8395 | 23775 | 1807 | 2268 | 1503 | 289 | 0.81 |
| $DMU_{34}$ | 660 | 440 | 56.69 | 36.30 | 5795 | 45560 | 3110 | 3085 | 932 | 264 | 1.00 |
| $DMU_{35}$ | 600 | 805 | 46.94 | 64.57 | 4962 | 32419 | 2209 | 4518 | 967 | 567 | 0.95 |
| $DMU_{36}$ | 225 | 330 | 13.56 | 25.13 | 790 | 2285 | 600 | 1185 | 279 | 193 | 1.00 |
| $DMU_{37}$ | 695 | 611 | 58.35 | 39.90 | 5986 | 9189 | 3020 | 2623 | 958 | 250 | 0.83 |
| $DMU_{38}$ | 1135 | 485 | 68.99 | 39.30 | 7794 | 41884 | 1693 | 3304 | 1121 | 290 | 0.80 |
| $DMU_{39}$ | 840 | 425 | 65.19 | 30.31 | 6776 | 5980 | 2647 | 2311 | 1065 | 188 | 0.90 |
| $DMU_{40}$ | 465 | 450 | 26.99 | 34.99 | 3730 | 4705 | 900 | 1855 | 485 | 211 | 1.00 |
| $DMU_{41}$ | 644 | 520 | 51.27 | 37.94 | 5348 | 7496 | 2693 | 3362 | 953 | 256 | 0.87 |
| $DMU_{42}$ | 542 | 285 | 33.19 | 19.38 | 3920 | 2160 | 994 | 2130 | 534 | 80 | 1.00 |
| $DMU_{43}$ | 460 | 530 | 29.74 | 51.81 | 3560 | 7606 | 1168 | 1710 | 575 | 452 | 0.89 |

Table B.3: Projections and efficiencies of 18 Chinese commercial banks

| DMU | Employees | IC | BGL | BNGL | CER | IIG | IING | Efficiency |
|---|---|---|---|---|---|---|---|---|
| $DMU_1$ | 396.28 | 589.69 | 5400.00 | 19606.61 | 144132.80 | 145.26 | 806.59 | 1.00 |
| $DMU_2$ | 352.16 | 468.00 | 3882.94 | 19113.29 | 175703.85 | 106.28 | 787.74 | 0.98 |
| $DMU_3$ | 354.89 | 475.53 | 3976.83 | 19143.82 | 143150.58 | 108.82 | 788.90 | 0.90 |
| $DMU_4$ | 267.02 | 431.05 | 3106.70 | 15938.02 | 687120.00 | 84.19 | 666.41 | 1.00 |
| $DMU_5$ | 152.68 | 202.67 | 1195.75 | 8903.12 | 65730.04 | 37.95 | 397.61 | 0.73 |
| $DMU_6$ | 97.16 | 128.81 | 447.77 | 6061.10 | 20274.25 | 16.74 | 289.02 | 0.88 |
| $DMU_7$ | 60.74 | 112.91 | 809.02 | 4524.46 | 25073.27 | 28.18 | 230.30 | 0.91 |
| $DMU_8$ | 42.76 | 75.54 | 386.59 | 3316.38 | 8413.80 | 15.09 | 184.14 | 0.92 |
| $DMU_9$ | 60.84 | 98.30 | 509.71 | 4305.79 | 12641.71 | 18.92 | 221.95 | 0.77 |
| $DMU_{10}$ | 40.29 | 42.75 | 269.27 | 1987.32 | 4647.82 | 11.26 | 111.45 | 0.88 |
| $DMU_{11}$ | 13.78 | 24.45 | 109.64 | 1115.32 | 1527.43 | 4.64 | 62.12 | 0.66 |
| $DMU_{12}$ | 15.85 | 39.35 | 207.74 | 1607.37 | 67269.71 | 6.89 | 89.95 | 0.84 |
| $DMU_{13}$ | 15.90 | 39.44 | 156.81 | 1811.92 | 6092.49 | 6.53 | 101.53 | 0.82 |
| $DMU_{14}$ | 13.48 | 22.18 | 102.59 | 1009.53 | 1389.64 | 4.35 | 56.13 | 0.58 |
| $DMU_{15}$ | 14.59 | 16.64 | 93.37 | 753.11 | 1209.38 | 3.97 | 41.63 | 0.64 |
| $DMU_{16}$ | 1.83 | 3.07 | 4.50 | 112.33 | 1.19 | 0.19 | 4.78 | 1.00 |
| $DMU_{17}$ | 5.91 | 9.92 | 39.27 | 433.86 | 778.65 | 1.71 | 23.57 | 0.65 |
| $DMU_{18}$ | 5.28 | 7.80 | 31.21 | 334.76 | 499.84 | 1.37 | 17.96 | 0.62 |

Table B.4: Projections and efficiencies of 43 UK universities with retraction (selective weak disposability)

| DMU | Staff | | Expen. | | Undergrad. | | Postgrad. | | Pub. | | Retraction | | Efficiency |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | NS | SS | NS | SS | NS | SS | NS | SS | NS | SS | NS | SS | |
| $DMU_1$ | 5520 | 480 | 342.63 | 72.46 | 11840 | 70 | 7000 | 1930 | 8860 | 457 | 6.00 | 0.00 | 1.00 |
| $DMU_2$ | 8170 | 2795 | 421.14 | 280.13 | 5840 | 6485 | 4150 | 4485 | 10196 | 1971 | 4.00 | 0.00 | 1.00 |
| $DMU_3$ | 6940 | 866 | 400.13 | 68.74 | 12718 | 6216 | 9656 | 3940 | 10249 | 655 | 5.51 | 1.19 | 0.90 |
| $DMU_4$ | 8270 | 2165 | 465.07 | 176.14 | 14015 | 10330 | 12680 | 9295 | 11735 | 1848 | 5.00 | 4.00 | 1.00 |
| $DMU_5$ | 4655 | 1916 | 310.17 | 155.97 | 13561 | 15411 | 6151 | 8399 | 6497 | 1620 | 4.85 | 3.42 | 0.85 |
| $DMU_6$ | 5850 | 1980 | 339.18 | 157.71 | 18330 | 13110 | 6175 | 5385 | 5925 | 989 | 8.00 | 0.00 | 1.00 |
| $DMU_7$ | 3190 | 1335 | 287.65 | 159.41 | 12130 | 10035 | 5120 | 6220 | 5556 | 1038 | 1.00 | 0.00 | 1.00 |
| $DMU_8$ | 370 | 1435 | 28.46 | 144.45 | 1160 | 9738 | 707 | 6751 | 519 | 961 | 0.00 | 0.00 | 0.93 |
| $DMU_9$ | 3490 | 1447 | 208.68 | 115.68 | 13299 | 11663 | 3993 | 6055 | 3671 | 1195 | 5.41 | 2.27 | 0.91 |
| $DMU_{10}$ | 2610 | 1474 | 193.19 | 118.27 | 11218 | 13562 | 3667 | 6272 | 3593 | 1219 | 3.04 | 2.35 | 0.74 |
| $DMU_{11}$ | 3565 | 1450 | 175.85 | 110.06 | 11420 | 9620 | 4855 | 7730 | 3875 | 754 | 3.00 | 0.00 | 1.00 |
| $DMU_{12}$ | 3015 | 930 | 156.08 | 69.05 | 10100 | 5530 | 3380 | 5475 | 3539 | 543 | 4.00 | 0.00 | 1.00 |
| $DMU_{13}$ | 3520 | 983 | 233.44 | 74.36 | 12775 | 14316 | 4253 | 5655 | 3976 | 549 | 1.02 | 0.00 | 0.98 |
| $DMU_{14}$ | 1225 | 1127 | 75.67 | 87.36 | 6901 | 10997 | 1119 | 3122 | 1770 | 729 | 0.00 | 0.00 | 0.99 |
| $DMU_{15}$ | 940 | 444 | 60.93 | 31.86 | 5445 | 5388 | 1091 | 1402 | 1381 | 290 | 0.00 | 0.00 | 0.87 |
| $DMU_{16}$ | 3710 | 1130 | 186.97 | 77.17 | 12235 | 8055 | 4865 | 3910 | 3366 | 613 | 9.00 | 2.00 | 1.00 |
| $DMU_{17}$ | 2560 | 915 | 159.61 | 76.26 | 9895 | 8077 | 3124 | 3747 | 3143 | 620 | 0.00 | 0.00 | 1.00 |
| $DMU_{18}$ | 3000 | 947 | 175.10 | 63.29 | 11837 | 9682 | 3143 | 3008 | 1617 | 553 | 0.00 | 0.00 | 0.97 |
| $DMU_{19}$ | 1130 | 717 | 73.87 | 53.59 | 7023 | 8827 | 1452 | 2621 | 1624 | 531 | 0.65 | 0.52 | 0.86 |
| $DMU_{20}$ | 1270 | 500 | 85.47 | 47.42 | 9120 | 5345 | 1540 | 1375 | 1382 | 430 | 1.00 | 0.00 | 1.00 |
| $DMU_{21}$ | 3335 | 925 | 209.22 | 95.06 | 13169 | 9270 | 3860 | 4064 | 3647 | 802 | 4.17 | 1.07 | 0.95 |
| $DMU_{22}$ | 1825 | 1055 | 134.24 | 84.11 | 9109 | 14232 | 2703 | 4519 | 2692 | 840 | 2.78 | 1.43 | 0.95 |
| $DMU_{23}$ | 865 | 900 | 59.35 | 79.67 | 5792 | 8447 | 1419 | 2125 | 972 | 671 | 0.00 | 0.00 | 0.86 |
| $DMU_{24}$ | 3075 | 1017 | 185.46 | 84.73 | 12750 | 9113 | 3367 | 2910 | 3172 | 666 | 3.76 | 0.30 | 0.98 |
| $DMU_{25}$ | 2080 | 300 | 150.14 | 21.05 | 10423 | 6456 | 2970 | 2225 | 2384 | 98 | 1.01 | 0.00 | 0.99 |
| $DMU_{26}$ | 1275 | 515 | 78.14 | 35.88 | 9075 | 5825 | 1825 | 1475 | 1180 | 351 | 3.00 | 0.00 | 1.00 |
| $DMU_{27}$ | 1170 | 425 | 74.90 | 30.55 | 5772 | 5395 | 2270 | 2285 | 1887 | 228 | 3.43 | 0.16 | 0.84 |
| $DMU_{28}$ | 855 | 880 | 68.81 | 78.79 | 5301 | 9451 | 1242 | 2958 | 1216 | 645 | 0.00 | 0.00 | 0.98 |
| $DMU_{29}$ | 1847 | 686 | 107.37 | 50.54 | 9133 | 6692 | 3204 | 4218 | 1740 | 372 | 1.11 | 0.00 | 0.90 |
| $DMU_{30}$ | 1510 | 553 | 83.29 | 42.95 | 6988 | 5584 | 2154 | 2420 | 1913 | 373 | 7.67 | 0.50 | 0.98 |
| $DMU_{31}$ | 1155 | 450 | 86.99 | 31.57 | 6498 | 4694 | 2101 | 1539 | 1951 | 283 | 1.02 | 0.00 | 0.98 |
| $DMU_{32}$ | 1730 | 714 | 94.74 | 45.00 | 9807 | 11663 | 1392 | 2536 | 1140 | 410 | 0.00 | 0.00 | 1.00 |
| $DMU_{33}$ | 1215 | 675 | 75.41 | 45.45 | 7436 | 6124 | 1600 | 2009 | 1163 | 424 | 0.00 | 0.00 | 0.92 |
| $DMU_{34}$ | 660 | 440 | 56.69 | 36.30 | 5795 | 45560 | 3110 | 3085 | 932 | 264 | 2.00 | 0.00 | 1.00 |
| $DMU_{35}$ | 600 | 805 | 46.76 | 81.30 | 4903 | 27969 | 2144 | 4464 | 958 | 558 | 1.05 | 0.00 | 0.96 |
| $DMU_{36}$ | 225 | 330 | 13.56 | 25.13 | 790 | 2285 | 600 | 1185 | 279 | 193 | 0.00 | 0.00 | 1.00 |
| $DMU_{37}$ | 695 | 581 | 58.35 | 39.90 | 5986 | 9189 | 3020 | 2550 | 614 | 121 | 1.20 | 0.00 | 0.83 |
| $DMU_{38}$ | 1251 | 480 | 73.69 | 34.61 | 7427 | 4756 | 1613 | 3149 | 1121 | 223 | 0.00 | 0.00 | 0.84 |
| $DMU_{39}$ | 840 | 425 | 64.26 | 37.27 | 6164 | 22092 | 1893 | 2102 | 642 | 291 | 0.00 | 0.00 | 0.99 |
| $DMU_{40}$ | 465 | 450 | 26.99 | 34.99 | 3730 | 4705 | 900 | 1855 | 485 | 211 | 0.00 | 0.00 | 1.00 |
| $DMU_{41}$ | 655 | 475 | 52.84 | 36.37 | 5283 | 23186 | 2553 | 3188 | 895 | 251 | 1.08 | 0.00 | 0.92 |
| $DMU_{42}$ | 581 | 285 | 33.19 | 19.38 | 3920 | 2160 | 1126 | 2130 | 534 | 80 | 0.00 | 0.00 | 1.00 |
| $DMU_{43}$ | 460 | 530 | 33.40 | 48.14 | 3353 | 7162 | 1100 | 1509 | 534 | 433 | 0.00 | 0.00 | 0.94 |

Table B.5: Projections and efficiencies of 43 UK universities with retraction (strong disposability)

| DMU | Staff | | Expen. | | Undergrad. | | Postgrad. | | Pub. | | Retraction | | Efficiency |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | NS | SS | NS | SS | NS | SS | NS | SS | NS | SS | NS | SS | |
| $DMU_1$ | 5520 | 480 | 342.63 | 72.46 | 11840 | 70 | 7000 | 1930 | 8860 | 457 | 6.00 | 0.00 | 1.00 |
| $DMU_2$ | 8170 | 2795 | 421.14 | 280.13 | 5840 | 6485 | 4150 | 4485 | 10196 | 1971 | 4.00 | 0.00 | 1.00 |
| $DMU_3$ | 6940 | 927 | 407.08 | 61.79 | 13016 | 6185 | 10006 | 3920 | 10323 | 526 | 5.34 | 0.00 | 0.90 |
| $DMU_4$ | 8270 | 2165 | 465.07 | 176.14 | 14015 | 10330 | 12680 | 9295 | 11735 | 1848 | 5.00 | 4.00 | 1.00 |
| $DMU_5$ | 4655 | 2009 | 329.52 | 176.46 | 13845 | 15733 | 6648 | 6996 | 6694 | 1593 | 3.26 | 2.30 | 0.83 |
| $DMU_6$ | 5850 | 1980 | 339.18 | 157.71 | 18330 | 13110 | 6175 | 5385 | 5925 | 989 | 8.00 | 0.00 | 1.00 |
| $DMU_7$ | 3190 | 1335 | 287.65 | 159.41 | 12130 | 10035 | 5120 | 6220 | 5556 | 1038 | 1.00 | 0.00 | 1.00 |
| $DMU_8$ | 370 | 1435 | 28.46 | 144.45 | 1160 | 9738 | 707 | 6751 | 519 | 961 | 0.00 | 0.00 | 0.93 |
| $DMU_9$ | 3490 | 1311 | 220.14 | 104.22 | 13480 | 11822 | 3986 | 5437 | 3861 | 1072 | 4.03 | 1.95 | 0.90 |
| $DMU_{10}$ | 2610 | 1263 | 211.97 | 99.49 | 11350 | 13723 | 3789 | 5495 | 3949 | 920 | 1.45 | 1.10 | 0.73 |
| $DMU_{11}$ | 3565 | 1450 | 175.85 | 110.06 | 11420 | 9620 | 4855 | 7730 | 3875 | 754 | 3.00 | 0.00 | 1.00 |
| $DMU_{12}$ | 3010 | 930 | 156.08 | 69.05 | 10100 | 5530 | 3982 | 5475 | 3539 | 543 | 2.13 | 0.00 | 1.00 |
| $DMU_{13}$ | 3520 | 1535 | 184.76 | 123.05 | 12470 | 13975 | 3520 | 5520 | 3661 | 756 | 1.00 | 0.00 | 1.00 |
| $DMU_{14}$ | 1225 | 1128 | 75.51 | 87.52 | 6895 | 10988 | 1137 | 3122 | 1767 | 730 | 0.00 | 0.00 | 0.99 |
| $DMU_{15}$ | 940 | 444 | 60.93 | 31.86 | 5445 | 5388 | 1091 | 1402 | 1381 | 290 | 0.00 | 0.00 | 0.87 |
| $DMU_{16}$ | 3710 | 654 | 215.86 | 48.27 | 12696 | 8359 | 5048 | 4057 | 3873 | 352 | 6.01 | 0.00 | 0.96 |
| $DMU_{17}$ | 2560 | 915 | 158.73 | 77.15 | 9850 | 8040 | 3110 | 3730 | 3240 | 571 | 0.00 | 0.00 | 1.00 |
| $DMU_{18}$ | 3000 | 857 | 174.41 | 63.98 | 11505 | 9410 | 3055 | 2240 | 1775 | 521 | 0.00 | 0.00 | 1.00 |
| $DMU_{19}$ | 1130 | 772 | 72.83 | 54.63 | 7012 | 8813 | 1342 | 2748 | 1616 | 535 | 0.40 | 0.43 | 0.86 |
| $DMU_{20}$ | 1270 | 500 | 85.47 | 47.42 | 9120 | 5345 | 1540 | 1375 | 1382 | 430 | 1.00 | 0.00 | 1.00 |
| $DMU_{21}$ | 3335 | 925 | 209.66 | 98.26 | 13186 | 9282 | 3797 | 4099 | 3662 | 792 | 3.86 | 0.90 | 0.95 |
| $DMU_{22}$ | 1825 | 866 | 151.38 | 66.97 | 9181 | 14345 | 2961 | 3509 | 2895 | 666 | 1.00 | 0.88 | 0.94 |
| $DMU_{23}$ | 865 | 900 | 59.35 | 79.67 | 5792 | 8447 | 1419 | 2125 | 972 | 671 | 0.00 | 0.00 | 0.86 |
| $DMU_{24}$ | 3075 | 918 | 185.46 | 76.88 | 12750 | 9113 | 3367 | 2621 | 3172 | 556 | 3.76 | 0.00 | 0.98 |
| $DMU_{25}$ | 2080 | 300 | 150.16 | 21.04 | 10366 | 6421 | 2954 | 2224 | 2690 | 98 | 0.99 | 0.00 | 0.99 |
| $DMU_{26}$ | 1275 | 515 | 78.14 | 35.88 | 9075 | 5825 | 1825 | 1475 | 1180 | 351 | 3.00 | 0.00 | 1.00 |
| $DMU_{27}$ | 1170 | 358 | 78.96 | 26.49 | 5742 | 16499 | 2235 | 2371 | 2028 | 167 | 1.29 | 0.00 | 0.81 |
| $DMU_{28}$ | 855 | 880 | 68.81 | 78.79 | 5301 | 9451 | 1242 | 2958 | 1216 | 645 | 0.00 | 0.00 | 0.98 |
| $DMU_{29}$ | 1904 | 646 | 110.47 | 47.44 | 8682 | 6362 | 3046 | 4010 | 2054 | 343 | 0.94 | 0.00 | 0.94 |
| $DMU_{30}$ | 1510 | 368 | 100.09 | 26.15 | 7428 | 5936 | 2290 | 2572 | 2286 | 145 | 2.24 | 0.00 | 0.92 |
| $DMU_{31}$ | 1155 | 446 | 87.24 | 31.31 | 6497 | 4620 | 2101 | 1539 | 1955 | 279 | 0.98 | 0.00 | 0.98 |
| $DMU_{32}$ | 1730 | 589 | 93.24 | 46.50 | 9785 | 5855 | 1335 | 2530 | 1245 | 324 | 0.00 | 0.00 | 1.00 |
| $DMU_{33}$ | 1215 | 675 | 77.50 | 43.36 | 7365 | 6125 | 1585 | 2094 | 1163 | 409 | 0.00 | 0.00 | 0.93 |
| $DMU_{34}$ | 660 | 440 | 56.69 | 36.30 | 5795 | 45560 | 3110 | 3085 | 932 | 264 | 2.00 | 0.00 | 1.00 |
| $DMU_{35}$ | 600 | 805 | 45.46 | 82.60 | 4849 | 28224 | 1970 | 4415 | 934 | 565 | 0.97 | 0.00 | 0.97 |
| $DMU_{36}$ | 225 | 330 | 13.56 | 25.13 | 790 | 2285 | 600 | 1185 | 279 | 193 | 0.00 | 0.00 | 1.00 |
| $DMU_{37}$ | 695 | 550 | 54.37 | 43.88 | 5546 | 8514 | 2220 | 1640 | 775 | 263 | 0.88 | 0.00 | 0.90 |
| $DMU_{38}$ | 1163 | 464 | 74.99 | 33.31 | 7222 | 4625 | 1569 | 3062 | 1096 | 211 | 0.00 | 0.00 | 0.86 |
| $DMU_{39}$ | 840 | 425 | 64.14 | 33.01 | 6085 | 5370 | 1795 | 2075 | 725 | 208 | 0.00 | 0.00 | 1.00 |
| $DMU_{40}$ | 465 | 450 | 26.99 | 34.99 | 3730 | 4705 | 900 | 2054 | 485 | 252 | 0.00 | 0.00 | 1.00 |
| $DMU_{41}$ | 655 | 880 | 37.54 | 51.66 | 4645 | 6555 | 2355 | 2940 | 583 | 474 | 1.00 | 0.00 | 1.00 |
| $DMU_{42}$ | 559 | 285 | 32.51 | 19.38 | 4020 | 2160 | 980 | 2130 | 534 | 80 | 0.00 | 0.00 | 1.00 |
| $DMU_{43}$ | 460 | 529 | 36.27 | 45.28 | 3155 | 6740 | 1035 | 1376 | 645 | 393 | 0.00 | 0.00 | 1.00 |

Table B.6: Projections and efficiencies of 15 Chinese cement companies with undesirable outputs

| DMU | Employees | Energy | Clinker & Cement | Aggregate | Profit | Tax | $NO_x$ | $CO_2$ | Efficiency |
|---|---|---|---|---|---|---|---|---|---|
| $DMU_1$ | 61662 | 32.49 | 248.52 | 188.14 | 10867.00 | 2137.26 | 52908.00 | 16657.97 | 0.72 |
| $DMU_2$ | 50769 | 25.11 | 272.26 | 110.41 | 11930.14 | 2297.51 | 52817.00 | 18377.86 | 1.00 |
| $DMU_3$ | 17939 | 6.13 | 69.81 | 118.89 | 4358.66 | 693.95 | 16801.50 | 4175.51 | 0.49 |
| $DMU_4$ | 20804 | 5.38 | 65.26 | 177.79 | 4262.12 | 705.99 | 16715.73 | 3453.12 | 1.00 |
| $DMU_5$ | 45144 | 7.34 | 86.33 | 34.53 | 1525.26 | 973.37 | 11623.03 | 5821.92 | 1.00 |
| $DMU_6$ | 22344 | 8.66 | 86.53 | 41.46 | 2092.73 | 993.28 | 11514.35 | 5761.58 | 0.01 |
| $DMU_7$ | 7966 | 1.36 | 19.43 | 8.49 | 2218.55 | 225.68 | 4406.48 | 1087.90 | 0.87 |
| $DMU_8$ | 2957 | 11.30 | 24.05 | 0.00 | 2241.51 | 125.35 | 4582.84 | 1851.03 | 0.77 |
| $DMU_9$ | 2047 | 1.90 | 17.62 | 0.00 | 2241.51 | 125.35 | 6026.64 | 1346.69 | 1.00 |
| $DMU_{10}$ | 6892 | 2.22 | 25.20 | 43.60 | 866.13 | 629.82 | 4497.00 | 1517.35 | 1.00 |
| $DMU_{11}$ | 2793 | 1.63 | 16.87 | 13.27 | 2244.00 | 186.40 | 4810.97 | 1271.44 | 1.00 |
| $DMU_{12}$ | 1579 | 0.90 | 8.33 | 0.00 | 1894.83 | 90.65 | 1878.00 | 749.62 | 0.62 |
| $DMU_{13}$ | 2644 | 0.88 | 12.21 | 5.33 | 2051.79 | 130.79 | 2763.44 | 754.35 | 0.88 |
| $DMU_{14}$ | 1241 | 0.20 | 3.92 | 0.00 | 1730.15 | 74.16 | 1240.00 | 352.46 | 1.00 |
| $DMU_{15}$ | 16032 | 4.59 | 52.86 | 26.96 | 3609.08 | 483.92 | 9944.11 | 3529.31 | 0.17 |