DECOMPOSITION MATRICES FOR CATEGORY $\mathcal{O}$ OF RATIONAL CHEREDNIK ALGEBRAS

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Contents

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Introduction

Rational Cherednik algebras $H_c(W)$ for any complex reflection group W (and depending on a parameter c) were defined by Etingof and Ginzburg [11]. We are interested in fundamental questions about their representation theory such as:

- Which irreducible representations are finite-dimensional and what are their dimensions?
- What are the graded characters of the irreducible representations?
- What are the multiplicities of irreducible representations in standard (i.e. Verma) modules?

These questions have been answered when W is a dihedral group by Chmutova [6], when W is the Coxeter group H_3 by Balagovic-Puranik [1], and when W is the wreath product of a cyclic group with a symmetric group by several groups of authors [30, 35, 24]. Calculating the decomposition matrix of the category $\mathcal{O}_c(W)$ associated to $H_c(W)$ yields the answers to all three questions. The category $\mathcal{O}_c(W)$, studied in [15], is a lowest weight category defined analogously to the category $\mathcal{O}$ of a complex semisimple Lie algebra. It contains Verma modules $\mathrm{M}(\tau)$ and it contains all finite-dimensional representations of $H_c(W)$. The decomposition matrix of $\mathcal{O}_c(W)$ encodes the multiplicities $[\mathrm{M}(\tau) : \mathrm{L}(\sigma)]$ of irreducible representations $\mathrm{L}(\sigma)$ appearing in the composition series of Verma modules $\mathrm{M}(\tau)$. The decomposition matrix is an important invariant of the category $\mathcal{O}_c(W)$.

This paper seeks to answer the three questions above for W one of the following exceptional Coxeter groups: H_4, F_4 (in the case of equal parameters), E_6, E_7, and E_8.

The main result of this paper is the computation of the decomposition matrices of the categories $\mathcal{O}_c(W)$ for all parameters $c = 1/d$ with d a divisor of one of the fundamental degrees of W and $d > 2$ (and in the case of E_8, if $d \in \{3, 4, 6\}$ then we also exclude the decomposition matrix of the principal block).

The character of $\mathrm{L}(\tau)$ may be read off the row labeled by τ in the inverse of the decomposition matrix, from which the graded character of $\mathrm{L}(\tau)$ follows from a simple formula. The graded character is a rational function in t, and the order of its pole at $t = 1$ coincides with the dimension of the support of $\mathrm{L}(\tau)$. The irreducible representations with support of dimension less than the rank of W are of particular interest; these are the representations killed by the Knizhnik-Zalomodchikov functor. Among them, the representations whose support has dimension 0 consist of the finite-dimensional representations. Evaluating the character of a finite-dimensional representation at $t = 1$ gives its dimension. Both the characters of finite-dimensional representations, which are polynomials in $\mathbb{Z}[t^{-1}, t]$ symmetric under interchanging t and t^{-1}, and the dimensions of finite-dimensional representations have meanings in other contexts. The characters should be Markov traces of braids of type W, as has been studied in [17] and [33]. In some cases, the dimensions of finite-dimensional representations are dimensions of the homology of affine Springer fibers [27], [34]. And according to the philosophy of Miyachi, the finite-dimensional representations of the rational Cherednik algebra should correspond to cuspidal modular representations of a finite group of Lie type with Weyl group W.

The decomposition matrices we compute in this paper for $\mathcal{O}_c(W)$ turn out to be closely related to the unipotent decomposition matrices of exceptional finite groups of Lie type in cross-characteristic. If $G(q)$ is a finite group of Lie type with Weyl group W, then its representation theory in characteristic ℓ when ℓ divides the order of $G(q)$ and $\ell \nmid q$ depends on which cyclotomic polynomial $\Phi_d(q)$ the prime ℓ divides, provided that ℓ is large enough. The relevant integers d are the divisors of the fundamental degrees of W. In many of the examples in this paper, the decomposition matrix of $\mathcal{O}_c(W)$ embeds as a square submatrix in the unipotent decomposition matrix of $G(q)$, for $\ell >> 0$ dividing $\Phi_d(q)$ with $c = 1/d$. In particular, this is true for all the blocks of $\mathcal{O}_c(W)$ for $W = E_6$, E_7, and E_8 at $c = 1/4$ corresponding to the unipotent decomposition matrices at $d = 4$ computed by Dudas and Malle [9].

Let d divide one of the fundamental degrees of W and let $\ell | \Phi_d(q)$ be a prime. Let $\mathcal{B}$ be a block of $\mathcal{O}_c(W)$ and take any $\tau \in \operatorname{Irr} W$ such that $\mathrm{L}(\tau) \in \mathcal{B}$. Let $G(q)$ be a finite group of Lie type with Weyl group W and let $\widetilde{\mathcal{B}}$ be the unipotent block of $G(q)$ over a field of characteristic $\ell >> 0$ containing the unipotent representation labeled by $\tau \otimes \mathrm{sgn}$. Let $\sigma \in \operatorname{Irr} W$. The following conjecture is naively based on the existing low-rank examples.

Conjecture 1. *The rational Cherednik algebra decomposition number* $[\mathrm{M}(\tau) : \mathrm{L}(\sigma)]$ *is a lower bound for the* $G(q)$ *decomposition number in row* $\tau \otimes \mathrm{sgn}$ *and column* $\sigma \otimes \mathrm{sgn}$ *of the decomposition matrix of the unipotent block* $\widetilde{\mathcal{B}}$.

Note that the relevant square submatrix of the decomposition matrix of $G(q)$ is the one labeled by unipotent characters lying in the principal series. Also, note that the conjecture holds for those columns corresponding to simple representations of the Hecke algebra at a d'th root of unity, since this rectangular matrix embeds in both the rational Cherednik algebra decomposition matrix by [5], [4] and the $G(q)$ decomposition matrix for $\ell >> 0$ by James' Conjecture. One might imagine that what happens exactly is that the columns of the $G(q)$ decomposition matrix are nonnegative linear combinations of the columns of the $\mathcal{O}_c(W)$ decomposition matrix.

We have only stated the conjecture for the principal series, but a version of the conjecture should also hold for the other Harish-Chandra series of $G(q)$, which are parametrized by cuspidal representations in characteristic 0.

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1. Rational Cherednik algebras and their character theory

Rational Cherednik algebras were defined by Etingof and Ginzburg in [11]. They can be constructed starting from any complex reflection group. In this paper, we will only treat rational Cherednik algebras associated to finite Coxeter groups. In particular, many of our techniques are valid only in this setting. The representation theory of rational Cherednik algebras for exceptional Coxeter groups was done by Chmutova for dihedral groups [6] and Balagovic-Puranik for H_3 [1]. In this section, we recall the background material on rational Cherednik algebras $H_c(W)$ for finite Coxeter groups W and their representations in the category $\mathcal{O}_c(W)$ that we need for computing the decomposition matrix of $\mathcal{O}_c(W)$. Only Section 1.8 contains original material.

1.1. **Rational Cherednik algebras.** Let W be a finite Coxeter group. Let $\mathfrak{h}$ denote the reflection representation of W over $\mathbb{C}$ and $\mathfrak{h}^* \cong \mathfrak{h}$ its dual representation. Let $\operatorname{rk} W$ denote the dimension of $\mathfrak{h}$ as a $\mathbb{C}$-vector space, which is equal to the number of simple reflections generating W. Denote by $S \subset W$ the set of *all* reflections in W. We will denote by $\operatorname{Irr} W$ a complete set of isomorphism classes of irreducible representations of W over $\mathbb{C}$. Given $\tau \in \operatorname{Irr} W$, we will write $\tau' := \tau \otimes \operatorname{sign} \cong \tau \otimes \bigwedge^{\operatorname{rk} W} \mathfrak{h}^*$.

To W we will associate an associative $\mathbb{C}$-algebra $H_c(W)$ called the rational Cherednik algebra, depending on a parameter $c \in \mathbb{C}$. More generally, we may define $H_c(W)$ depending on a collection c of parameters, one for each W-conjugacy class of reflections. Thus $H_c(W)$ depends on a single parameter if W is of types A, D, E, H, or is a dihedral group $I_2(m)$ with m odd, and on two parameters if W is of types B, F, G, or is a dihedral group $I_2(m)$ with m even. When W has two conjugacy classes of reflections, the case that the two parameters are equal is a basic and important one as it arises naturally from geometry [27],[34]. We will exclusively study the case of equal parameters in this paper.

Fix a parameter $c \in \mathbb{C}$. Let $T(\mathfrak{h} \oplus \mathfrak{h}^*)$ be the tensor algebra on $\mathfrak{h} \oplus \mathfrak{h}^*$ and let $\mathbb{C}[W]$ be the group algebra of W over $\mathbb{C}$. Since W acts on $\mathfrak{h}$ and $\mathfrak{h}^*$, we may form the semidirect product algebra $T(\mathfrak{h} \oplus \mathfrak{h}^*) \rtimes \mathbb{C}[W]$.

Definition 2. [11] *The rational Cherednik algebra $H_c(W)$ is the quotient of $T(\mathfrak{h} \oplus \mathfrak{h}^*) \rtimes \mathbb{C}[W]$ by the relations*

$$[x, x'] = 0, \quad [y, y'] = 0, \quad [y, x] = (y, x) - c \sum_{s \in S} (\alpha_s^\vee, x)(y, \alpha) s$$

for all $x, x' \in \mathfrak{h}^*$ and all $y, y' \in \mathfrak{h}$. Here, $(-,-)$ is the natural pairing between $\mathfrak{h}$ and $\mathfrak{h}^*$, and $\alpha_s \in \mathfrak{h}^*$, $\alpha_s^\vee \in \mathfrak{h}$ are dual eigenvectors for the reflection s, normalized so that $(\alpha_s, \alpha_s^\vee) = 2$.

An important first result about $H_c(W)$, known as the PBW theorem by analogy with the triangular decomposition of the universal enveloping algebra of a complex semisimple Lie algebra, states that elements of $H_c(W)$ may be put in normal form with respect to a preferred order of x's, y's, and elements of $\mathbb{C}[W]$.

Theorem 3. *(Etingof-Ginzburg,* [11, Theorem 1.3]*) There is an isomorphism of $\mathbb{C}$-vector spaces $H_c(W) \cong \mathbb{C}[\mathfrak{h}^*] \otimes \mathbb{C}[W] \otimes \mathbb{C}[\mathfrak{h}]$.*

There are subalgebras $\mathbb{C}[\mathfrak{h}] \rtimes \mathbb{C}[W] \subset H_c(W)$ and $\mathbb{C}[\mathfrak{h}^*] \rtimes \mathbb{C}[W] \subset H_c(W)$, analogous to positive and negative Borel subalgebras. We remark that fixing dual bases $y_1, \ldots, y_n$ and $x_1, \ldots, x_n$ for $\mathfrak{h}$ and $\mathfrak{h}^*$, $\mathbb{C}[\mathfrak{h}^*] = \mathrm{Sym}(\mathfrak{h}) \cong \mathbb{C}[y_1, \ldots, y_n]$ denotes polynomials in the y-variables, and $\mathbb{C}[\mathfrak{h}] = \mathrm{Sym}(\mathfrak{h}^*) \cong \mathbb{C}[x_1, \ldots, x_n]$ denotes polynomials in the x-variables.

1.2. **Category $\mathcal{O}$ for rational Cherednik algebras.** We now discuss some of the representation theory of $H_c(W)$. Throughout the text, we will use the terms "simple module" and "irreducible representation" to mean the same thing. The main types of modules or representations that we will be interested in are simple modules, standard (or Verma) modules, and projective modules. The standard modules have a concrete construction and description. We will study a certain nice category of $H_c(W)$-modules containing the standard modules and their simple composition factors. All of these modules are infinite-dimensional except for some simple modules. The finite-dimensional simple modules are rare and play an important role in the theory.

The category $\mathcal{O}_c(W)$ is defined to be the full subcategory of left $H_c(W)$-modules whose objects consist of those representations of $H_c(W)$ that are (i) finitely generated over $\mathbb{C}[\mathfrak{h}]$, and (ii) locally nilpotent for $\mathbb{C}[\mathfrak{h}^*]$ [15, Section 2.2, Section 3.2.1]. This definition follows the definition of category $\mathcal{O}$ of a complex semisimple Lie algebra using the triangular decomposition given by the PBW theorem. By [15, Theorem 2.19], the category $\mathcal{O}_c(W)$ is a highest weight category in the sense of [7]. However, we will use lowest weights instead of highest weights. The structure of the category $\mathcal{O}_c(W)$ depends on the parameter c. For generic $c \in \mathbb{C}$, $\mathcal{O}_c(W)$ is semisimple and equivalent to $\mathbb{C}[W]$-mod. We will assume from now on that $c \in \mathbb{R}$, as this is a precondition for $\mathcal{O}_c(W)$ to not be semisimple. The relevant parameters are discussed in Section 1.6.

Given $\tau \in \operatorname{Irr} W$, we define the *standard module with lowest weight* τ as follows:

$$\mathrm{M}(\tau) := H_c(W) \otimes_{\mathbb{C}[\mathfrak{h}^*]\rtimes\mathbb{C}[W]} \tau$$

In this construction, τ is considered as a representation of $\mathbb{C}[\mathfrak{h}^*] \rtimes W$ by letting W act as it acts on τ, and extending this to an action of $\mathbb{C}[\mathfrak{h}^*] \rtimes \mathbb{C}[W]$ by having $\mathfrak{h}$ act by 0. We then induce the resulting representation from $\mathbb{C}[\mathfrak{h}^*] \rtimes \mathbb{C}[W]$ to $H_c(W)$ to obtain $\mathrm{M}(\tau)$. The standard modules $\mathrm{M}(\tau)$, $\tau \in \operatorname{Irr} W$, are the most basic examples of objects in $\mathcal{O}_c(W)$. The standard module $\mathrm{M}(\tau)$ has a unique simple quotient $\mathrm{L}(\tau)$ and all simple objects in $\mathcal{O}_c(W)$ are obtained in this way exactly once [15, Proposition 2.11]. Thus the irreducible $H_c(W)$-representations in $\mathcal{O}_c(W)$ are naturally labeled by $\operatorname{Irr} W$.

The poset structure on $\operatorname{Irr} W$ used to give $\mathcal{O}_c(W)$ the structure of a highest weight category is then defined as follows [15]. Let $n = \operatorname{rk} W$ and let $\{x_i\}$ and $\{y_i\}$, $i = 1, ..., n$, be dual orthonormal bases of $\mathfrak{h}^*$ and $\mathfrak{h}$ respectively. $H_c(W)$ contains a canonical $\mathfrak{sl}_2$-triple (E, H, F) with $E = \frac{1}{2}\sum_{i=1}^{n} x_i^2$, $H = \sum_{i=1}^{n} x_iy_i + y_ix_i$, and $F = -\frac{1}{2}\sum_{i=1}^{n} y_i^2$. This $\mathfrak{sl}_2$-triple commutes with the action of W on any representation in $\mathcal{O}_c(W)$, and H acts on the lowest weight τ of any indecomposable object in $\mathcal{O}_c(W)$ by a scalar $h_c(\tau) \in \mathbb{R}$. Let χ_τ be the character of $\tau \in \operatorname{Irr} W$. In our setting of equal parameters, the formula for $h_c(\tau)$ is given by:

$$h_c(\tau) = \frac{\dim \mathfrak{h}^*}{2} - c\frac{\sum_{s\in S} \chi_\tau(s)}{\dim \tau}.$$

In the case that W has only one root length, this formula becomes

$$h_c(\tau) = \frac{\dim \mathfrak{h}^*}{2} - c|S|\frac{\chi_\tau(s)}{\dim \tau}$$

where $s \in S$ is any reflection [1, Section 2.2]. The partial order on $\operatorname{Irr} W$ is given by:

$$\sigma \geq_c \tau \text{ if and only if } h_c(\sigma) - h_c(\tau) \in \mathbb{Z}_{\geq 0}.$$

Following the strategy of [1], we will visualize the values $h_c(\tau)$, $\tau \in \operatorname{Irr} W$, using "$h_c$-weight lines" marking τ at the point $h_c(\tau)$ on the real number line. Moreover, representations $\sigma, \tau \in \operatorname{Irr} W$ will be graphed on the same line only if $h_c(\tau) - h_c(\sigma) \in \mathbb{Z}$. This will provide us with a helpful visual tool for identifying which standard modules can appear in the character of a given simple module (defined in Section 1.4 below), or equivalently, which simple modules can appear in the composition series of a given standard module.

The $\mathfrak{sl}_2$-triple (E, H, F) induces a natural $\mathbb{Z}$-grading on $H_c(W)$ and on the modules in $\mathcal{O}_c(W)$ by assigning W to degree 0, $\mathfrak{h}^*$ to degree 1, and $\mathfrak{h}$ to degree -1. The standard module $\mathrm{M}(\tau)$ is $\mathbb{N}$-graded with a shift by $h_c(\tau)$, and is generated in degree $h_c(\tau)$. The graded pieces are obtained by taking the tensor product of τ with the symmetric powers of $\mathfrak{h}^*$ and are W-invariant: there is an isomorphism of graded $\mathbb{C}[W]$-modules

$$\mathrm{M}(\tau) \cong \bigoplus_{k=0}^{\infty} \mathrm{Sym}^k \mathfrak{h}^* \otimes \tau.$$

The simple module $\mathrm{L}(\tau)$ inherits a grading from $\mathrm{M}(\tau)$ and its graded piece of lowest degree, namely $\mathrm{Sym}^0 \mathfrak{h}^* \otimes \tau = \tau$, lies in degree $h_c(\tau)$. We will denote by $M[z]$ the graded piece in degree z of a graded $H_c(W)$-module M. For a simple module $\mathrm{L} \in \mathcal{O}_c(W)$, we have $\mathrm{L} \cong \oplus_{k\geq 0}\mathrm{L}[h_c(\tau)+k]$ as a $\mathbb{C}[W]$-module. Observe that for all $k \geq 0$, the graded piece $\mathrm{L}[h_c(\tau)+k)]$ is finite-dimensional of dimension at most $\dim \mathrm{M}(\tau)[h_c(\tau)+k]$. Since a finite-dimensional $\mathfrak{sl}_2$-representation must have integer weights with lowest weight in $\mathbb{Z}_{\leq 0}$, the same is true for a finite-dimensional $H_c(W)$-representation. It follows that

Lemma 4. *If* $\mathrm{L}(\tau)$ *is finite-dimensional then* $h_c(\tau) \in \mathbb{Z}_{\leq 0}$.

Because $\mathcal{O}_c(W)$ is a highest-weight category [15, Theorem 2.19], it holds for any simple module $\mathrm{L}(\tau) \in \mathcal{O}_c(W)$ that $\mathrm{Ext}(\mathrm{L}(\tau), \mathrm{L}(\tau)) = 0$ [7, Lemma 3.2(b)]. Every simple module $\mathrm{L}(\tau) \in \mathcal{O}_c(W)$ has a projective cover $\mathrm{P}(\tau) \in \mathcal{O}_c(W)$ with a finite-length filtration by standard modules $\mathrm{M}(\rho)$ satisfying $h_c(\rho) \leq h_c(\tau)$ [15, Corollary 2.10]. Any module $N \in \mathcal{O}_c(W)$ has a filtration of finite length whose successive quotients are simple modules $\mathrm{L}(\sigma)$, and the resulting list of simple modules with multiplicities is unique up to reordering [15, Corollary 2.16]. If the simple module $\mathrm{L}(\sigma)$ occurs as a subquotient of N then we say that $\mathrm{L}(\sigma)$ is a composition factor of N.

Following [21, Section 1.13], and using the duality from [15, §4] that fixes simples, we say that two non-isomorphic simple modules $\mathrm{L}(\tau)$ and $\mathrm{L}(\sigma)$ in $\mathcal{O}_c(W)$ are *in the same block* $\mathcal{B}$ if there are simple modules

$$\mathrm{L}(\tau) = \mathrm{L}(\rho_0), \mathrm{L}(\rho_1), \ldots, \mathrm{L}(\rho_{r-1}), \mathrm{L}(\rho_r) = \mathrm{L}(\sigma) \in \mathcal{O}_c(W)$$

such that for each $i = 0, \ldots, r-1$ we have

$$\mathrm{Ext}^1(\mathrm{L}(\rho_i), \mathrm{L}(\rho_{i+1})) \neq 0.$$

We say that an object M is in $\mathcal{B}$ if all of its composition factors are in $\mathcal{B}$. We refer to the block containing $\mathrm{L}(\mathrm{Triv})$, where $\mathrm{Triv} \in \mathrm{Irr}\, W$ is the trivial representation, as the *principal block* of $\mathcal{O}_c(W)$. As a highest-weight category, $\mathcal{O}_c(W)$ is equivalent to A-mod for some quasi-hereditary algebra A [7, Theorem 3.6]). Thus blocks may be defined more formally using the theory of finite-dimensional algebras, for instance one may study a block as representations of a connected quiver with relations [21, Section 13.9]. For our character-theoretic purposes, we think of the blocks informally and roughly as a partitioning of $\mathrm{Irr}\, W$. Our favorite lowest-weight objects $\mathrm{L}(\tau), \mathrm{M}(\tau), \mathrm{P}(\tau)$ can only be related homologically to other lowest-weight objects $\mathrm{L}(\sigma), \mathrm{M}(\sigma), \mathrm{P}(\sigma)$ that are in the same block.

1.3. **The decomposition matrix of $\mathcal{O}_c(W)$.**

Definition 5. *(1) Let $\tau, \sigma \in \operatorname{Irr} W$. The decomposition number $[\mathrm{M}(\tau) : \mathrm{L}(\sigma)]$ is the multiplicity of $\mathrm{L}(\sigma)$ as a composition factor of $\mathrm{M}(\tau)$.*

(2) The decomposition matrix of $\mathcal{O}_c(W)$ is the matrix with rows and columns indexed by $\operatorname{Irr} W$, and entry $[\mathrm{M}(\tau) : \mathrm{L}(\sigma)]$ in row τ and column σ for $\tau,\ \sigma \in \operatorname{Irr} W$.

Recall that $\mathrm{P}(\sigma)$ denotes the projective cover of $\mathrm{L}(\sigma)$. BGG reciprocity states that

$$[\mathrm{P}(\sigma) : \mathrm{M}(\tau)] = [\mathrm{M}(\tau) : \mathrm{L}(\sigma)]$$

for all $\sigma, \tau \in \operatorname{Irr} W$ [7, Theorem 3.11],[15, Proposition 3.3]. Here, $[\mathrm{P}(\sigma) : \mathrm{M}(\tau)]$ denotes the multiplicity of $\mathrm{M}(\tau)$ in a filtration of $\mathrm{P}(\sigma)$ whose subquotients are standard modules. Thus the decomposition matrix can be defined equivalently as having entry $[\mathrm{P}(\sigma) : \mathrm{M}(\tau)]$ in row τ and column σ for $\tau, \sigma \in \operatorname{Irr} W$.

Let $\sigma, \tau \in \operatorname{Irr} W$, $\sigma \ncong \tau$. It holds that $[\mathrm{M}(\tau) : \mathrm{L}(\tau)] = 1$, and $[\mathrm{M}(\tau) : \mathrm{L}(\sigma)] > 0$ implies that $h_c(\sigma) - h_c(\tau) \in \mathbb{Z}_{>0}$. Therefore, when we order the rows and columns so that h_c is non-decreasing from the top to the bottom row and from the leftmost to the rightmost column, the decomposition matrix of $\mathcal{O}_c(W)$ is upper triangular with 1's on the diagonal. Additionally, if we order the rows and the columns of the decomposition matrix block by block then the decomposition matrix is block diagonal. So we can study the decomposition matrix one block at a time, and this is what we will do in this paper.

1.4. **Graded characters of simple modules.**

The decomposition matrix D of $\mathcal{O}_c(W)$ can be interpreted as the change of basis matrix of the Grothendieck group of $\mathcal{O}_c(W)$ from the basis of simple modules to the basis of standard modules. The inverse $D^{-1} =: \left(d^{-1}_{\tau,\sigma}\right)$ of the decomposition matrix D expresses the simple modules in the basis of standard modules. The expression for $\mathrm{L}(\tau)$ in the basis of standard modules $\mathrm{M}(\sigma)$ is called the *character* of $\mathrm{L}(\tau)$. It is encoded by the row labeled by τ in D^{-1}. Since we focus on character theory in this paper and work on the level of Grothendieck groups, we will abuse notation slightly and write the character of $\mathrm{L}(\tau)$ as

$$\mathrm{L}(\tau) = \sum_{\sigma \in \operatorname{Irr} W} d^{-1}_{\tau,\sigma} \mathrm{M}(\sigma)$$

where what we mean precisely by such an expression is that in the Grothendieck group of $\mathcal{O}_c(W)$, the class of the simple module $\mathrm{L}(\tau)$ is given by the linear combination of the classes of standard modules $\mathrm{M}(\sigma)$ with the specified coefficients $d^{-1}_{\tau,\sigma} \in \mathbb{Z}$.

We now introduce characters of standard modules and their quotients. Let $c \in \mathbb{Q}$. The *graded character* of a module $M \in \mathcal{O}_c(W)$ of lowest weight τ is a Laurent series in t times some possibly fractional power of t (corresponding to the shift in the grading given by $h_c(\tau)$):

$$\chi_M(t) = t^{h_c(\tau)} \sum_{k=0}^{\infty} \dim M[k + h_c(\tau)] t^k$$

For $\mathrm{M}(\tau)$ a standard module, the graded character formula follows from the isomorphism $\mathrm{M}(\tau) \cong \mathrm{Sym}\,\mathfrak{h}^* \otimes \tau$:

$$\chi_{\mathrm{M}(\tau)}(t) = t^{h_c(\tau)} \dim \tau \sum_{k=0}^{\infty} \binom{n-1+k}{n-1} t^k$$

where $n = \mathrm{rk}\, W$. The graded character of $\mathrm{L}(\tau)$ is thus given by the formula:

$$\chi_{\mathrm{L}(\tau)}(t) = \frac{\sum_{\sigma \in \mathrm{Irr}\, W} \left(d^{-1}_{\tau,\sigma} \dim \sigma\right) t^{h_c(\sigma)}}{(1-t)^n}.$$

That is, we obtain the character of a simple module $\mathrm{L}(\tau)$ by looking at the row labeled τ in the inverse of the decomposition matrix, writing $\mathrm{L}(\tau)$ accordingly as a linear combination of standard modules $\mathrm{M}(\sigma)$, and then applying the formula above for the graded character of a standard module. Canceling all occurrences of $(1-t)$ in the numerator of $\chi_{\mathrm{L}(\tau)}(t)$, we have that $\mathrm{L}(\tau)$ is finite-dimensional if and only if $\chi_{\mathrm{L}(\tau)}(t)$ is a Laurent polynomial. For a Laurent series or polynomial $q(t)$ let $q[t^m]$ denote the coefficient of t^m in $q(t)$. Recalling the structure of $\mathfrak{sl}_2$-representation on $\mathrm{L}(\tau)$, we obtain:

Lemma 6. *If $\mathrm{L}(\tau)$ is finite-dimensional then $\chi_{\mathrm{L}(\tau)}[t^m] = \chi_{\mathrm{L}(\tau)}[t^{-m}]$ for all $m \in \mathbb{Z}$.*

1.5. **Supports of simple modules in $\mathcal{O}_c(W)$ and parabolic induction and restriction.** An important geometric object associated to an irreducible representation in $\mathcal{O}_c(W)$ is its support, a W-invariant algebraic subvariety of $\mathfrak{h}$. The *support* $\mathrm{Supp}\, M$ of a module $M \in \mathcal{O}_c(W)$ is defined to be its support as a $\mathbb{C}[\mathfrak{h}]$-module, i.e. the zero locus of the annihilator in $\mathbb{C}[\mathfrak{h}]$ of M. The support of a simple module $\mathrm{L}(\tau)$ is an important invariant and provides the starting point for a Harish-Chandra theory for rational Cherednik algebras. On the one extreme, if $\mathrm{Supp}\,\mathrm{L}(\tau) = \mathfrak{h}$ then we say that $\mathrm{L}(\tau)$ has *full support*. On the other extreme, $\mathrm{Supp}\,\mathrm{L}(\tau) = \{0\}$ if and only if $\mathrm{L}(\tau)$ is finite-dimensional. If $\mathrm{Supp}\,\mathrm{L}(\tau)$ is a proper subvariety of $\mathfrak{h}$ then we will say that $\mathrm{L}(\tau)$ has *less than full support*. The graded character of $\mathrm{L}(\tau)$ detects the dimension of the support of $\mathrm{L}(\tau)$: $\dim \mathrm{Supp}\,\mathrm{L}(\tau)$ coincides with the power of $(1-t)$ appearing in the denominator of $\chi_{\mathrm{L}(\tau)}(t)$ when written in lowest terms.

The algebra $H_c(W)$ need not have any finite-dimensional simple modules. For example, if $W = S_n$ then $H_c(W)$ has a finite-dimensional module if and only if $c = r/n$ for some $r \in \mathbb{Z}$ such that $\gcd(r,n) = 1$ [2]. It may still be interesting in such cases to distinguish simple modules by their supports. The modules of minimal support were defined and studied by Etingof-Gorsky-Losev in [10]. A module $M \in \mathcal{O}_c(W)$ has *minimal support* if no subset of $\mathrm{Supp}\, M$ of smaller dimension is the support of a nonzero object of $\mathcal{O}_c(W)$ [10, Definition 1.1]. The modules of minimal support have an important property which does not hold for arbitrary modules in $\mathcal{O}_c(W)$: if $M \in \mathcal{O}_c(W)$ has minimal support then M is a Cohen-Macaulay module over $\mathbb{C}[\mathfrak{h}]$ of dimension $\dim \mathrm{Supp}\, M$ [10, Theorem 1.2].

The supports of irreducible representations have a meaning in terms of parabolic induction and restriction functors. Fix a parameter c and let $W' \subseteq W$ be a parabolic subgroup of W. Parabolic induction and restriction functors

$$^{\mathcal{O}}\mathrm{Ind}_{W'}^{W} : \mathcal{O}_c(W') \to \mathcal{O}_c(W), \qquad {}^{\mathcal{O}}\mathrm{Res}_{W'}^{W} : \mathcal{O}_c(W) \to \mathcal{O}_c(W')$$

were defined by Bezrukavnikov-Etingof [3]. These functors are exact, biadjoint, transitive with respect to inclusions $W'' \subseteq W' \subseteq W$, and satisfy a Mackey formula [3], [31], [23], [26], [22]. Induction of $M \in \mathcal{O}_c(W')$ for a maximal parabolic $W' \subset W$ raises the dimension of support by 1 while restriction of M to $\mathcal{O}_c(W'')$ for $W'' \subset W'$ a maximal parabolic lowers its dimension of support by 1 or sends it to 0. In particular, it may happen that $\mathrm{Res}_{W'}^{W} M = 0$ for some parabolic subgroup $W' \subset W$ even though $\dim \mathrm{Supp}\, M > 0$. The finite-dimensional modules are exactly those modules $M \in \mathcal{O}_c(W)$ such that $\mathrm{Res}_{W'}^{W} M = 0$ for any proper parabolic subgroup $W' \subset W$. Thus the finite-dimensional modules play the role of cuspidal modules in the theory. Every simple module $\mathrm{L} \in \mathcal{O}_c(W)$ arises as a quotient of $\mathrm{Ind}_{W'}^{W} \mathrm{L}'$ for some parabolic subgroup $W' \subseteq W$ and some finite-dimensional representation $\mathrm{L}' \in \mathcal{O}_c(W')$, and the pair (W', L') is unique up to W-conjugation with the support of L being given by $\mathrm{Supp}\, \mathrm{L} = W\mathfrak{h}^{W'}$ [3, Proposition 3.23].

On the level of Grothendieck groups, the rule for induction and restriction of standard modules labeled by $\mathrm{Irr}\, W'$ and $\mathrm{Irr}\, W$ is the same as the induction and restriction rule for those characters (Proposition 3.14, [3]). That is, on the level of characters, for $W' \subseteq W$ and $\tau \in \mathrm{Irr}\, W'$ it holds that $\mathrm{Ind}_{W'}^{W} \mathrm{M}(\tau) = \sum\limits_{\sigma \subset \mathrm{Ind}_{W'}^{W} \tau} \mathrm{M}(\sigma)$. This will help us to compute characters of irreducible modules in $\mathcal{O}_c(W)$ by induction. If $\mathrm{L}(\tau) \in \mathcal{O}_c(W')$, $W' \subset W$, is an irreducible module whose character is known as a linear combination of standard modules, then the character of $\mathrm{Ind}_{W'}^{W} \mathrm{L}(\tau)$ is given by the corresponding linear combination of induced standard modules, which we have just seen how to compute. Conversely, restriction gives important information about characters of simple modules in $\mathcal{O}_c(W)$: if a candidate expression for the character of $\mathrm{L}(\tau) \in \mathcal{O}_c(W)$ produces the character of a virtual module upon restriction to $\mathcal{O}_c(W')$ then it cannot be correct.

Using BGG reciprocity, we may study the columns of the decomposition matrix as characters of projective modules in the basis of standard modules. Combining this perspective with parabolic induction and restriction, one may induce projective characters $\mathrm{P}(\lambda) \in \mathcal{O}_c(W')$, $W' \subset W$ a parabolic subgroup, to $\mathcal{O}_c(W')$ and then try to decompose the resulting projective character as a sum of characters of indecomposable projective modules in $\mathcal{O}_c(W)$ to obtain columns of the decomposition matrix of $\mathcal{O}_c(W)$. The following lemma is standard, see [8] for a proof:

Lemma 7. *Let $W' \subset W$ be a parabolic subgroup and let $\mathrm{P}(\lambda) \in \mathcal{O}_c(W')$ be projective indecomposable. If $\mathrm{P}(\tau)$ is a summand of ${}^{\mathcal{O}}\mathrm{Ind}_{W'}^{W} \mathrm{P}(\lambda)$ then $\dim \mathrm{Supp}\, \mathrm{L}(\tau) \geq \dim \mathrm{Supp}\, \mathrm{L}(\lambda) + \mathrm{rk}\, W - \mathrm{rk}\, W'$. Moreover, if ${}^{\mathcal{O}}\mathrm{Ind}_{W'}^{W} \mathrm{L}(\lambda) \twoheadrightarrow \mathrm{L}(\sigma)$ then $\mathrm{P}(\sigma)$ is a summand of ${}^{\mathcal{O}}\mathrm{Ind}_{W'}^{W} \mathrm{P}(\lambda)$.*

1.6. **The Knizhnik-Zalomodchikov functor.** There is an important functor named the *Knizhnik-Zalomodchikov* functor, but our mathematical fathers just called it KZ. The functor $\mathsf{KZ} : \mathcal{O}_c(W) \to \mathcal{H}_q(W)$-mod has its target in the category of finitely generated modules over the Hecke algebra $\mathcal{H}_q(W)$ at parameter $q = \exp(-2\pi i c)$. The functor KZ is exact and induces an equivalence $\mathcal{H}_q(W)\text{-mod} \simeq \mathcal{O}_c(W)/\mathcal{O}_c(W)^{\mathrm{tor}}$ where $\mathcal{O}_c(W)^{\mathrm{tor}}$ is the Serre subcategory of $\mathcal{O}_c(W)$ generated by all simple modules of less than full support [15, Theorem 5.14]. By the Double Centralizer Theorem [15], the blocks of $\mathcal{O}_c(W)$ are in bijection with the blocks of $\mathcal{H}_q(W)$-mod, i.e. $\mathrm{M}(\tau)$ and $\mathrm{M}(\sigma)$ are in the same block of $\mathcal{O}_c(W)$ if and only if $\mathsf{KZ}\mathrm{M}(\tau)$ and $\mathsf{KZ}\mathrm{M}(\sigma)$ are in the same block of $\mathcal{H}_q(W)$-mod. Moreover, $\mathsf{KZ}\mathrm{M}(\tau) = \mathrm{S}(\tau')$ where $\mathrm{S}(\tau')$ denotes the standard or cell module of $\mathcal{H}_q(W)$ labeled by τ'; and $\mathsf{KZ}\mathrm{L}(\tau) = \mathrm{D}(\tau')$ if the quotient of $\mathrm{S}(\tau')$ by its radical is a simple $\mathcal{H}_q(W)$-module $\mathrm{D}(\tau')$, and otherwise $\mathsf{KZ}\mathrm{L}(\tau) = 0$ [15, Theorem 6.8]. By [31, Theorem 2.1], KZ intertwines the induction and restriction functors on the respective categories.

Remark 8. *If $c < 0$ then $\mathsf{KZ}\mathrm{M}(\tau) = \mathrm{S}(\tau)$ and $\mathsf{KZ}\mathrm{L}(\tau) = \mathrm{D}(\tau)$ or* 0 [15, Remark 6.9]. *Although in* [15, Theorem 6.8] *it is written that $\mathsf{KZ}\mathrm{M}(\tau) = \mathrm{S}(\tau)$ if the parameter $k_{H,1} > 0$, the discrepancy comes from a change of parameters in which there is an implicit sign difference in the defining relations of the Cherednik algebra used by* [15]. *The twist by the sign character as we have written it is the correct one in order for the conventions used for labeling representations of Hecke algebras to match those used for labeling representations of Cherednik algebras when $c > 0$. For example, we see in the tables in* [14] *that the trivial representation always labels a column of the decomposition matrix of the Hecke algebra at a root of unity while the sign representation never does, whereas when $c > 0$ and $\mathcal{O}_c(W)$ is not semisimple, it is the sign representation that always labels a simple representation of full support in $\mathcal{O}_c(W)$ while the trivial representation never does.*

The blocks and decomposition matrices of $\mathcal{H}_q(W)$-mod have been calculated for all exceptional Coxeter groups W for q a root of unity and are collected into a series of tables at the end of [13]. The effect of KZ on the decomposition matrix of $\mathcal{O}_c(W)$ is to delete the columns labeled by irreducible representations of less than full support, and the result is the decomposition matrix of $\mathcal{H}_q(W)$ with respect to the canonical basic sets ([4] Prop. 5.7 and Prop. 5.12; [5]):

Lemma 9. *The decomposition matrix of $\mathcal{H}_q(W)$ for $q = \exp(-2\pi i c)$ is a submatrix of the decomposition matrix of $\mathcal{O}_c(W)$, with row and column labels τ, σ replaced by τ', σ'.*

Proof. The Knizhnik-Zalomodchikov functor is exact, takes $\mathrm{M}(\tau)$ to $\mathrm{S}(\tau')$, and takes $\mathrm{L}(\tau)$ to $\mathrm{D}(\tau')$ or 0 [15]. $\square$

The *defect* of a block is a nonnegative integer measuring, approximately, the complexity of the block. This notion is well-defined for Hecke algebras and is part of the data labeling blocks in [14]. Using the Double Centralizer Theorem of [15] which matches up the blocks of $\mathcal{H}_q(W)$-mod with the blocks of $\mathcal{O}_c(W)$, we say a block $\mathcal{B}$ of the rational Cherednik algebra has defect j if $\mathsf{KZ}(\mathcal{B})$ has defect j as a block of $\mathcal{H}_q(W)$-mod. A block of defect 0 contains a unique simple module which is also standard and projective. In particular, $\mathcal{O}_c(W)$ is semisimple if and only if every block is a defect 0 block. Blocks of $\mathcal{O}_c(W)$ of defect 1 were studied in [29]. These blocks are sent by KZ to Brauer tree blocks of $\mathcal{H}_q(W)$-mod. In a defect 1 block of $\mathcal{O}_c(W)$, there is a unique irreducible $\mathrm{L}(\tau)$ of less than full support, and its lowest weight τ is positioned at the leftmost end of the h_c-weight line for the block. The decomposition matrix of a defect 1 block has 1's on the diagonal and the super-diagonal, and 0's everywhere else [29, Theorem 5.15]. Therefore we will not print the defect 1 blocks in this paper, and we will investigate the decomposition matrices of blocks of defect 2 and greater.

Let us discuss the generality of the results from the parameters c treated in this paper. We follow the strategy of [1] to reduce the parameters c we need to consider to the list $\{\frac{1}{d} \mid d \text{ divides a regular number of } W\}$. Their rationale is as follows. As a consequence of the bijection on blocks, the category $\mathcal{O}_c(W)$ is semisimple if and only if $\mathcal{H}_q(W)$-mod is semisimple. It thus suffices to compute the decomposition matrix of $\mathcal{O}_c(W)$ for those parameters $c = r/d$ where d divides one of the fundamental degrees of W, that is, one of the degrees of the generators of $\mathbb{C}[\mathfrak{h}^*]^W$. Second, an isomorphism of algebras $H_c(W) \cong H_{-c}(W)$ yields an equivalence $\mathcal{O}_c(W) \simeq \mathcal{O}_{-c}(W)$, so it is sufficient to consider the case that $c > 0$. Finally, we can ignore the numerator r, reducing to the case $r = 1$ by equivalences of categories by the work of [28] and [2], provided that $d \neq 2$.

1.7. **A duality functor.** Following [15, Section 4.1], there is a derived equivalence $\mathsf{R} : D^b(\mathcal{O}_c(W)) \xrightarrow{\sim} D^b(\mathcal{O}_c(W))$ which is very close to being the Ringel duality. Let us recall the construction of R from [15, Section 4.1], and following some of the explanation in [25, Section 2].

First, by [15, Section 4.1, Lemma 4.1] there is a contravariant derived equivalence

$$\begin{aligned} R\mathrm{Hom}_{H_c(W)}(-, H_c(W))[\mathrm{rk}\, W] : D^b(\mathcal{O}_c(W)) &\xrightarrow{\sim} D^b(\mathcal{O}_c(W)^{\mathrm{r,opp}}) \\ \mathrm{M}(\tau) &\longmapsto \mathrm{M}(\tau')^{\mathrm{r,opp}} \end{aligned}$$

where $\mathcal{O}_c(W)^{\mathrm{r}}$ is the "right-handed analog" of $\mathcal{O}_c(W)$ arising from an algebra isomorphism in which the roles of $\mathfrak{h}$ and $\mathfrak{h}^*$ are swapped, and $\mathcal{C}^{\mathrm{opp}}$ denotes the opposite category of a category $\mathcal{C}$. Let us now explain how the target category of this functor gets identified with $\mathcal{O}_c(W)$, and what this identification does to the standard objects.

First, as explained in [25, Section 2], if $\mathcal{C}$ is a lowest weight category with respect to a poset Λ then so is its opposite category $\mathcal{C}^{\mathrm{opp}}$ with respect to the same poset Λ, and the standard objects $\mathrm{M}(\tau)^{\mathrm{opp}} \in \mathcal{C}^{\mathrm{opp}}$ are given by the costandard objects $\mathrm{N}(\tau) \in \mathcal{C}$ for each $\tau \in \Lambda$. There is a contravariant equivalence of categories:

$$\begin{aligned} (-)^{\mathrm{opp}} : \quad \mathcal{C}^{\mathrm{opp}} &\xrightarrow{\sim} \mathcal{C} \\ \mathrm{M}(\tau)^{\mathrm{opp}} &\longmapsto \mathrm{N}(\tau) \end{aligned}$$

exchanging standard and costandard objects with the same labels. Next, there is a *naive duality* $\mathcal{O}_c(W)^{\mathrm{r}} \xrightarrow{\sim} \mathcal{O}_c(W)$ defined in [15, Section 4.2.1]. Under our assumption that W is a real reflection group, there is a contravariant equivalence of categories

$$\begin{aligned} (-)^\vee : \mathcal{O}_c(W)^{\mathrm{r}} &\xrightarrow{\sim} \mathcal{O}_c(W) \\ \mathrm{N}(\tau)^{\mathrm{r}} &\longmapsto \mathrm{M}(\tau) \end{aligned}$$

sending costandard to standard objects and preserving labels [15, Proposition 4.7, Remark 4.9]. When W is a complex reflection group in which some of the simple reflections have order greater than 2, then this functor maps $\mathrm{N}(\tau)^{\mathrm{r}}$ to $\mathrm{M}(\tau^\vee)$ where $\tau^\vee := \mathrm{Hom}(\tau, \mathbb{C})$, and so then there can be a nontrivial involution on the labels of costandard and standard modules [15, Proposition 4.7]. However, when W is a real reflection group then $\tau^\vee \cong \tau$ and so the label τ does not change [15, Remark 4.9].

Now, as in [15, Proposition 4.10, Corollary 4.11, and Remark 4.12], we define

$$\mathsf{R} := (-)^\vee \circ (-)^{\mathrm{opp}} \circ R\mathrm{Hom}_{H_c(W)}(-, H_c(W))[\mathrm{rk}\, W],$$

obtaining a contravariant duality functor $\mathsf{R} : D^b(\mathcal{O}_c(W)) \xrightarrow{\sim} D^b(\mathcal{O}_c(W))$ such that $\mathsf{R}\mathrm{M}(\tau) = \mathrm{M}(\tau')$. This functor realizes the Ringel duality, up to an abelian equivalence [15, Remark 4.12]. Moreover, by general properties of Ringel duality, R restricts to an equivalence on the exact subcategory $\mathcal{O}_c(W)^\Delta$ of objects having a filtration by standard objects, as observed in [23, Section 2.5].

Given a block $\mathcal{B}$ of $\mathcal{O}_c(W)$ containing standard modules $\{\mathrm{M}(\tau)\}$, we will denote by $\mathsf{R}\mathcal{B}$ the block $\mathcal{O}_c(W)$ containing the standard modules $\{\mathrm{M}(\tau')\} = \{\mathsf{R}\mathrm{M}(\tau)\}$ and refer to $\mathsf{R}\mathcal{B}$ as the *Ringel dual block* of $\mathcal{B}$.

1.8. **Some lemmas on characters and decomposition numbers.** In this section we add some lemmas to our tool-box for computing the decomposition matrix of $\mathcal{O}_c(W)$. Our strategy for computing the decomposition matrix of $\mathcal{O}_c(W)$ is to start from the decomposition matrix of the Hecke algebra $\mathcal{H}_q(W)$ at the root of unity $q = \exp(-2\pi i c)$ using Lemma 9. We then try to "complete the decomposition matrix of $\mathcal{H}_q(W)$ to a square matrix" to recover the columns of the decomposition matrix of $\mathcal{O}_c(W)$ deleted by the KZ functor. The characters of the finite-dimensional representations, which play the role of cuspidal representations for the rational Cherednik algebra, are key to the success of this endeavor. For this reason, our lemmas mostly focus on properties of the characters of these finite-dimensional representations. We then use the Bezrukavnikov-Etingof functors of parabolic induction and restriction to build up the characters of the non-cuspidal irreducible representations. The matrix recording the characters of irreducible representations is the inverse of the decomposition matrix, so when this approach works, we obtain the decomposition matrix of $\mathcal{O}_c(W)$.

The following lemma of Etingof-Stoica was used by Balagovic-Puranik in their computation of the characters of irreducible $H_c(H_3)$-representations [1], and we will also find it useful.

Lemma 10. *(Etingof-Stoica [12]) Let $\sigma, \tau \in \operatorname{Irr} W$. If $h_c(\sigma) - h_c(\tau) = 1$ and $\sigma \subset \mathfrak{h}^* \otimes \tau$, then σ generates an $H_c(W)$-subrepresentation of $\mathrm{M}(\tau)$.*

It is known by work of Varagnolo and Vasserot that the "spherical representation" $\mathrm{L}(\mathrm{Triv})$ with lowest weight the trivial rep is finite-dimensional if and only if the denominator d of the parameter $c = r/d$ is an elliptic number of W [34]. Elliptic numbers were defined by Springer [32] and are also important in the study of epipelagic representations of p-adic groups [37]; a list of elliptic numbers for exceptional Weyl groups can be found in [37, Section 4.8].

Recall the canonical $\mathfrak{sl}_2$-triple $(E, H, F) \subset H_c(W)$ from Section 1.2.

Lemma 11. *Let $\mathrm{L} \in \mathcal{O}_c(W)$ be an irreducible $H_c(W)$-representation. Suppose there exists a nonzero element $v \in \mathrm{L}$ such that $Ev = 0$. Then L is finite-dimensional.*

Proof. We will show that L is finite-dimensional by showing this is true for L as an $\mathfrak{sl}_2$-module. Take $v \in \mathrm{L}$, $v \neq 0$, such that $Ev = 0$. Since L is irreducible, v generates L as an $H_c(W)$-module. Since $\mathrm{L} \in \mathcal{O}_c(W)$, there exists $N \in \mathbb{N}$ such that $y_{i_N} \cdots y_{i_2} y_{i_1} v = 0$ for all $y_{i_N}, \ldots, y_{i_2}, y_{i_1} \in \mathfrak{h}$. Let $m = hv$ be an arbitrary element of L, $h \in H_c(W)$. By the PBW theorem, we can write

$$h = \sum_{a_1,\ldots,a_n,b_1,\ldots,b_n} C_{b_1,\ldots,b_n,a_1,\ldots,a_n} x_1^{b_1} \cdots x_n^{b_n} y_1^{a_1} \cdots y_n^{a_n}$$

for $a_i, b_j \in \mathbb{Z}_{\geq 0}$ and $C_{b_1,\ldots,b_n,a_1,\ldots,a_n} \in \mathbb{C}[W]$.

We claim that E^N annihilates $z y_1^{a_1} \cdots y_n^{a_n} v$ for any $z \in \mathbb{C}[x_1, \ldots, x_n] \rtimes W$. If $a_1 + \ldots + a_n \geq N$ then $y_1^{a_1} \cdots y_n^{a_n} v = 0$ and there is nothing to show. So suppose $M := a_1 + \ldots + a_n < N$. For any $j = 1, \ldots, n$, the defining relations of Definition 2 imply that $E y_j = y_j E + \zeta_j$ for some element $\zeta_j \in \mathbb{C}[x_1, \ldots, x_n] \rtimes W$. Thus $E y_j v = \zeta_j v$. It follows by induction that $E^M y_1^{a_1} \cdots y_n^{a_n} v = \zeta v$ for some element $\zeta \in \mathbb{C}[x_1, \ldots, x_n] \rtimes W$. Since E commutes with W and $E \in \mathbb{C}[x_1, \ldots, x_n]$, E commutes with ζ. We then have $E^N(z y_1^{a_1} \cdots y_n^{a_n} v) = z E^N y_1^{a_1} \cdots y_n^{a_n} v = z E^{N-M} \zeta v = z \zeta E^{N-M} v = 0$. Therefore E^N annihilates L.

If $u \in \mathrm{L}$ is any homogeneous element, say $u \in \mathrm{L}[m]$, then the cyclic $\mathfrak{sl}_2$-submodule $\langle u \rangle \subset \mathrm{L}$ generated by u must be finite-dimensional because the degrees of the nonzero weight spaces of $\langle u \rangle$ are bounded below by $h_c(\tau)$ and bounded above by $2N + m - 1$. Then $h_c(\tau) \leq 0$ and each cyclic $\mathfrak{sl}_2$-submodule of L has dimension at most $-2h_c(\tau) + 1$, and $\mathrm{L} = \sum_{k=0}^{2h_c(\tau)} \mathrm{L}[h_c(\tau) + k]$. Since each weight space $\mathrm{L}[h_c(\tau) + k]$ is finite-dimensional, we conclude that L is a finite-dimensional $\mathfrak{sl}_2$-module. $\square$

Corollary 12. *Let L be an irreducible $H_c(W)$-representation. If $\dim \mathrm{L}[2] < \dim \mathrm{L}[0]$ then L is finite-dimensional.*

The next lemma concerns symmetry in the characters of minimal support representations. Thanks to the fact that the Ringel duality R is a perverse equivalence, R induces an involution r on the labeling set of simple modules. If we tensor the lowest weights σ of the standard modules $\mathrm{M}(\sigma)$ with the sign representation, then the character of a minimal support irreducible representation $\mathrm{L}(\tau)$ is sent to the character of its Ringel-dual $\mathsf{R}\mathrm{L}(\tau)$, which is again a minimal support irreducible representation $\mathrm{L}(\mathsf{r}(\tau))$. Since the dimension of support of a finite-dimensional representation is 0, finite-dimensional representations are always minimal support representations. In practice, the finite-dimensional representations $\mathrm{L}(\tau)$ turn out to be sent to themselves by the Ringel duality and thus possess a symmetry that is very useful for completing their characters when part of the character is known. A conceptual explanation of this phenomenon coming from the comparison of different dualities on $D^b(\mathcal{O}_c(W))$ lies outside the scope of this paper. For our computational purposes, we give a low-tech criterion for confirming that $\mathrm{L}(\tau) = \mathsf{R}\mathrm{L}(\tau)$ for all finite-dimensional representations $\mathrm{L}(\tau)$ that we encounter.

Lemma 13. *(1) Suppose the dimension d of support of the irreducible representation $\mathrm{L}(\tau) \in \mathcal{O}_c(W)$ is minimal over all dimensions of support of irreducible representations in the block $\mathcal{B} \subset \mathcal{O}_c(W)$ containing $\mathrm{L}(\tau)$. Let $B = \{\sigma \in \mathrm{Irr}\, W \mid \mathrm{L}(\sigma) \in \mathcal{B}\}$ and let $B' = \{\sigma' \in \mathrm{Irr}\, W \mid \sigma \in B\}$. Let $\mathrm{L}(\tau) = \sum_{\sigma \in \mathrm{Irr}\, W} d^{-1}_{\tau,\sigma}\mathrm{M}(\sigma)$ be the character of $\mathrm{L}(\tau)$. Then*

$$(-1)^{\mathrm{rk}\, W - d} \sum_{\sigma \in \mathrm{Irr}\, W} d^{-1}_{\tau,\sigma}\mathrm{M}(\sigma') = \mathsf{R}\mathrm{L}(\tau) = \mathrm{L}(\mathsf{r}(\tau)).$$

(2) Suppose $\mathrm{L}(\tau) \in \mathcal{O}_c(W)$ is finite-dimensional, and there is no other finite-dimensional $\mathrm{L}(\sigma)$ in the same block as $\mathrm{L}(\tau)$, $\sigma \neq \tau$, satisfying $h_c(\sigma) = h_c(\tau)$ and $\dim \tau = \dim \sigma$. Then $\mathrm{L}(\tau) = \mathsf{R}\mathrm{L}(\tau)$.

Proof. (1) The functor R is a self-duality of $D^b(\mathcal{O}_c(W))$ which sends the standard module $\mathrm{M}(\tau)$ to the standard module $\mathrm{M}(\tau')$ [15]. It is proved in [25, Lemma 2.5] that the functor R is perverse with respect to the dimension of support of simple modules. In particular, R induces an equivalence of abelian categories between the Serre subcategories of minimal support modules in $\mathcal{B}$ and the Ringel dual block $\mathsf{R}\mathcal{B}$. If $\mathrm{L}(\tau)$ is a minimal support irreducible module in the block $\mathcal{B}$ then $\mathsf{R}\mathrm{L}(\tau)$ is a minimal support irreducible module in $\mathsf{R}\mathcal{B}$. Considering characters we obtain the stated formula, where the sign comes from the homological shift by dimension of support.

(2) Suppose $\mathrm{L}(\tau)$ is finite-dimensional and let $\sum_\sigma d^{-1}_{\tau,\sigma}\mathrm{M}(\sigma)$ be its character. Since $\mathsf{R}\mathrm{L}(\tau) = \pm \sum_\sigma d^{-1}_{\tau,\sigma}\mathrm{M}(\sigma')$ is the character of a finite-dimensional irreducible module $\mathrm{L}(\mathsf{r}(\tau))$, there is a unique element $\mathsf{r}(\tau) \in \mathrm{Irr}\, W$ such that $d_{\tau,\mathsf{r}(\tau)} = \pm 1$ and $h_c(\mathsf{r}(\tau))$ is maximal over all $h_c(\sigma)$ such that $d_{\tau,\sigma} \neq 0$. By Lemma 6, $\dim \mathrm{L}(\tau)[m] = \dim \mathrm{L}(\tau)[-m]$ for all $m \in \mathbb{Z}$, so $\dim \mathsf{r}(\tau)' = \dim \mathsf{r}(\tau) = \dim \tau$ and $h_c(\mathsf{r}(\tau)') = (\mathrm{rk}\, W)/2 - h_c(\mathsf{r}(\tau)) = h_c(\tau)$. Now by the assumptions, $\mathrm{L}(\tau) = \mathrm{L}(\mathsf{r}(\tau))$. ∎

Example 14. *Let $W = S_2 = A_1$ and let $c = \frac{1}{2}$. The category $\mathcal{O}_{\frac{1}{2}}(S_2)$ has two simple objects up to isomorphism,* $\mathrm{L}(2)$ *and* $\mathrm{L}(1^2)$*. The module* $\mathrm{L}(2)$ *is finite-dimensional and has character* $\mathrm{L}(2) = \mathrm{M}(2) - \mathrm{M}(1^2)$ [2]*. We have*

$$\mathsf{R}\mathrm{L}(2) = (-1)^1 \left(\mathsf{R}\mathrm{M}(2) - \mathsf{R}\mathrm{M}(1^2)\right) = -\left(\mathrm{M}(1^2) - \mathrm{M}(2)\right) = \mathrm{L}(2)$$

as predicted by Lemma 13.

Lemma 15. *For any $\sigma, \tau \in \operatorname{Irr} W$, it holds that*

$$\dim \operatorname{Hom}(\mathrm{M}(\sigma), \mathrm{M}(\tau)) = \dim \operatorname{Hom}(\mathrm{M}(\tau'), \mathrm{M}(\sigma')).$$

Proof. As discussed in Section 1.7, the duality functor $\mathsf{R} : D^b(\mathcal{O}_c(W)) \xrightarrow{\sim} D^b(\mathcal{O}_c(W))$ is contravariant, sends $\mathrm{M}(\lambda)$ to $\mathrm{M}(\lambda')$ for any $\lambda \in \operatorname{Irr} W$, and restricts to an equivalence on the exact subcategory of standardly filtered objects. Therefore $\operatorname{Hom}(\mathrm{M}(\sigma), \mathrm{M}(\tau)) \cong \operatorname{Hom}(\mathsf{R}\mathrm{M}(\tau), \mathsf{R}\mathrm{M}(\sigma)) = (\mathrm{M}(\tau'), \mathrm{M}(\sigma'))$ for any $\sigma, \tau \in \operatorname{Irr} W$. $\square$

In general, $[\mathrm{M}(\tau) : \mathrm{L}(\sigma)] \geq \dim \operatorname{Hom}(\mathrm{M}(\sigma), \mathrm{M}(\tau))$. When equality holds, we say the multiplicity is *tight*. The following lemma gives a condition for tightness of a decomposition number and will be used in conjunction with Lemma 15 to recover decomposition numbers near the diagonal of the decomposition matrix.

Lemma 16. *Suppose that for any $\tau, \nu, \sigma \in \operatorname{Irr} W$ satisfying $\tau <_c \nu <_c \sigma$, it holds that $[\mathrm{M}(\tau) : \mathrm{L}(\nu)][\mathrm{M}(\nu) : \mathrm{L}(\sigma)] = 0$. Then $\dim \operatorname{Hom}(\mathrm{M}(\sigma), \mathrm{M}(\tau)) = [\mathrm{M}(\tau) : \mathrm{L}(\sigma)]$.*

Proof. Recall that in a lowest weight category $\mathcal{C}$, for $M \in \mathcal{C}$ and $\mathrm{L}(\lambda) \in \mathcal{C}$ a simple module with projective cover $\mathrm{P}(\lambda)$, we have $[M : \mathrm{L}(\lambda)] = \dim \operatorname{Hom}(\mathrm{P}(\lambda), M)$. So what we need to show is that $\operatorname{Hom}(\mathrm{M}(\sigma), \mathrm{M}(\tau)) \cong \operatorname{Hom}(\mathrm{P}(\sigma), \mathrm{M}(\tau))$. Since $\operatorname{Hom}(\mathrm{M}(\sigma), \mathrm{M}(\tau)) = \operatorname{Ext}^0(\mathrm{M}(\sigma), \mathrm{M}(\tau))$, we use a projective resolution of $\mathrm{M}(\sigma)$ to show this.

Let

$$\mathrm{P}_\bullet := 0 \to \mathrm{P}_r \overset{\alpha_{r-1}}{\to} \ldots \overset{\alpha_1}{\to} \mathrm{P}_1 \overset{\alpha_0}{\to} \mathrm{P}_0 \to 0$$

be a minimal projective resolution of $\mathrm{M}(\sigma)$, with P_i in degree $-i$. Thus the complex $\mathrm{P}_\bullet$ is exact except for in degree 0, where $H^0(\mathrm{P}_\bullet) = \mathrm{M}(\sigma)$. Let $\pi : \mathrm{P}(\sigma) \to \mathrm{M}(\sigma)$ be the natural surjection. Then $\mathrm{P}_0 = \mathrm{P}(\sigma)$. By the definition of a lowest weight category, $\ker \pi = \operatorname{Im} \alpha_0$ has a finite filtration whose successive quotients are standard modules $\mathrm{M}(\nu)$, $\nu <_c \sigma$. So P_1 is a direct sum of projective indecomposable modules $\mathrm{P}(\nu)$ with $\nu <_c \sigma$ and such that $[\mathrm{P}(\sigma) : \mathrm{M}(\nu)] > 0$.

Now we apply the contravariant functor $\operatorname{Hom}(-, \mathrm{M}(\tau))$ obtaining the complex

$$\operatorname{Hom}(\mathrm{P}_\bullet, \mathrm{M}(\tau)) := 0 \to \operatorname{Hom}(\mathrm{P}(\sigma), \mathrm{M}(\tau)) \overset{\beta_0}{\to} \operatorname{Hom}(\mathrm{P}_1, \mathrm{M}(\tau)) \overset{\beta_1}{\to} \ldots$$

Then $\operatorname{Hom}(\mathrm{M}(\sigma), \mathrm{M}(\tau)) = \operatorname{Ext}^0(\operatorname{Hom}(\mathrm{P}_\bullet, \mathrm{M}(\tau)) = \ker \beta_0$. If $\operatorname{Hom}(\mathrm{P}(\sigma), \mathrm{M}(\tau)) = 0$ then there is nothing to show. Otherwise, consider a nonzero homomorphism $\xi : \mathrm{P}(\sigma) \to \mathrm{M}(\tau)$. We will show that $\beta_0(\xi)$ is 0 on any indecomposable direct summand $\mathrm{P}(\nu)$ of P_1. By definition of the connecting homomorphism β_0, $\beta_0(\xi) = \xi \circ \alpha_0$.

Case 1. Suppose that $\tau <_c \nu <_c \sigma$. We have $0 < [\mathrm{P}(\sigma) : \mathrm{M}(\nu)] = [\mathrm{M}(\nu) : \mathrm{L}(\sigma)]$. By the assumption of the theorem, then $\dim \mathrm{Hom}(\mathrm{P}(\nu), \mathrm{M}(\tau)) = [\mathrm{M}(\tau) : \mathrm{L}(\nu)] = 0$.

Case 2. Suppose that $\nu <_c \tau$. Then $\dim \mathrm{Hom}(\mathrm{P}(\nu), \mathrm{M}(\tau)) = [\mathrm{M}(\tau) : \mathrm{L}(\nu)] = 0$.

Case 3. Suppose that $\nu = \tau$. Since $\sigma \neq \tau$, the map $\xi : \mathrm{P}(\sigma) \to \mathrm{M}(\tau)$ cannot be surjective, and therefore the map $\beta_0(\xi)(\mathrm{P}(\tau)) = (\xi \circ \alpha_0)(\mathrm{P}(\tau))$ cannot be surjective onto $\mathrm{M}(\tau)$ either. We know that $\dim \mathrm{Hom}(\mathrm{P}(\tau), \mathrm{M}(\tau)) = [\mathrm{M}(\tau) : \mathrm{L}(\tau)] = 1$, so any nonzero map $\mathrm{P}(\tau) \to \mathrm{M}(\tau)$ is a nonzero scalar multiple of the natural surjection of $\mathrm{P}(\tau)$ onto $\mathrm{M}(\tau)$. It follows that $\beta_0(\xi)(\mathrm{P}(\tau)) = 0$.

Since $\beta_0(\xi)$ is the zero map on any indecomposable summand $P(\nu)$ of P_1, it follows that $\beta_0(\xi) = 0$, that is, $\xi \in \ker(\beta_0)$. Since ξ was an arbitrary element of $\mathrm{Hom}(\mathrm{P}(\sigma), \mathrm{M}(\tau))$, we conclude that $\mathrm{Hom}(\mathrm{M}(\sigma, \mathrm{M}(\tau)) = \mathrm{Ext}^0(\mathrm{M}(\sigma), \mathrm{M}(\tau)) = \ker(\beta_0) = \mathrm{Hom}(\mathrm{P}(\sigma), \mathrm{M}(\tau))$. This proves the lemma. □

Remark 17. *It is proven in* [18, Lemma 4.5] *that $\mathcal{O}_c(W)$ has a minimal partial order $\leq_{\mathrm{Hom}}$ with respect to which it possesses a highest weight (or in our conventions here, lowest weight) structure. This minimal partial order $\leq_{\mathrm{Hom}}$ (in our lowest weight convention) is generated by the relation $\tau \leq_{\mathrm{Hom}} \sigma$ if there exists a nonzero homomorphism from* $\mathrm{M}(\sigma)$ *to* $\mathrm{M}(\tau)$.

1.9. **Overview of the results.** In this paper we obtain the following results on the decomposition matrices of $\mathcal{O}_c(W)$. For $c = \frac{1}{d}$, we compute all decomposition matrices of blocks of $\mathcal{O}_c(W)$ of defect at least 2 for $W = H_4$. For $c = \frac{1}{d}$ and $d \neq 2$, we compute all decomposition matrices of blocks of defect at least 2 for $W = F_4, E_6$, and E_7. For $W = E_8$ we compute all decomposition matrices of blocks of defect 2 for $c = \frac{1}{d}$ and $d \neq 2$. This determines the characters of simple modules in $\mathcal{O}_c(W)$ for the afore-mentioned parameters.

The decomposition matrices of $\mathcal{O}_c(W)$ for W a dihedral group were computed by Chmutova [6]. The decomposition matrices of $\mathcal{O}_c(H_3)$ were computed by Balagovic-Puranik [1]. The decomposition matrices of defect 1 blocks of $\mathcal{O}_c(W)$ for an arbitrary finite Coxeter group W were computed by Rouquier [29, Theorem 5.15]. Our work nearly completes the project of computing the decomposition matrices of $\mathcal{O}_c(W)$ for W an exceptional Coxeter group, but leaves open the following cases: $W = F_4$, E_6, E_7, or E_8 and $c = 1/2$; and the principal block of $\mathcal{O}_c(E_8)$ for $c \in \{\frac{1}{3}, \frac{1}{4}, \frac{1}{6}\}$.

In the interest of space, we omit printing the inverses of the decomposition matrices (which record the characters of simple modules) and the decomposition matrices of the defect 1 blocks (since these are known by [29, Theorem 5.15]).

1.10. **Notation for decomposition matrices.** We will present the decomposition matrices of blocks of $\mathcal{O}_c(W)$ using the following notation. The decomposition matrix of a block of $\mathcal{O}_c(W)$ is denoted by D and its inverse is denoted D^{-1}. The entry in row τ and column σ of D^{-1} is denoted $d^{-1}_{\tau,\sigma}$. The symbol $\star$ next to row τ of D indicates that $\mathrm{L}(\tau)$ is finite-dimensional. The symbol `(k)` next to row τ indicates that $\dim \mathrm{Supp}\, \mathrm{L}(\tau) = k$.

In the course of working out the decomposition matrices, some versions of the matrix in different stages of completeness may appear with the symbol • next to row τ. This records that $\mathrm{L}(\tau)$ has less than full support, i.e. that the Knizhnik-Zalomodchikov functor sends $\mathrm{L}(\tau)$ to 0.

2. The decomposition matrices of $\mathcal{O}_c(H_4)$

In this section we will compute the decomposition matrices of blocks of $\mathcal{O}_{\frac{1}{d}}(H_4)$ of defect at least 2 for d a divisor of a regular number of H_4. The labels for the thirty-four characters of H_4 used throughout this section are those used by Grove in his paper where the character table of H_4 first appeared [20]. In his notation, **3** is the reflection representation and **1** is the trivial representation.

2.1. **Parameters** $c = 1/10$ **and** $c = 1/6$**.** The h_c-weight lines of values $h_{\frac{1}{10}}(\tau)$, $h_{\frac{1}{6}}(\tau)$ for $\tau \in \operatorname{Irr} H_4$ such that $\mathrm{L}(\tau)$ belongs to the principal block of $\mathcal{O}_{1/10}(H_4)$, $\mathcal{O}_{\frac{1}{6}}(H_4)$ respectively, are given in Table 1.

Table 1. Weight lines for the principal blocks of $\mathcal{O}_{\frac{1}{10}}(H_4)$ and $\mathcal{O}_{\frac{1}{6}}(H_4)$

−4	−1	0	1	2	3	4	5	8
1	**5**	**13**	**31**	**26**	**32**	**14**	**6**	**2**
				30				
				33				

−8	−3	0	2	4	7	12
1	**3**	**27**	**22**	**28**	**4**	**2**
	5		**25**		**6**	
			26			

Theorem 18. *The decomposition matrices of the principal blocks of* $\mathcal{O}_{\frac{1}{10}}(H_4)$ *and* $\mathcal{O}_{\frac{1}{6}}(H_4)$ *are given by Table 2.*

Proof. All columns except the leftmost four columns are given by the decomposition matrices of the Hecke algebras at the appropriate parameters [14]. The unknown entries in the remaining four columns are obtained by applying Lemmas 15 and 16. □

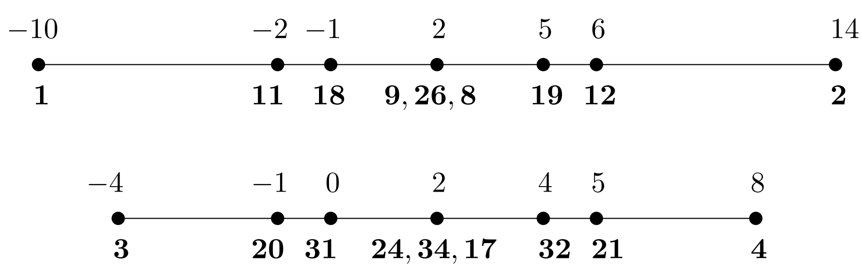
TABLE 2. Decomposition matrix of the principal blocks of $\mathcal{O}_{\frac{1}{10}}(H_4)$ and $\mathcal{O}_{\frac{1}{6}}(H_4)$

$$
\begin{array}{ccc}
c=\frac{1}{10} & c=\frac{1}{6} & \\
\mathbf{1} & \mathbf{1} & \star \\
\mathbf{5} & \mathbf{5} & \star \\
\mathbf{13} & \mathbf{3} & \star \\
\mathbf{31} & \mathbf{27} & (1) \\
\mathbf{33} & \mathbf{26} & \\
\mathbf{30} & \mathbf{25} & \\
\mathbf{26} & \mathbf{22} & \\
\mathbf{32} & \mathbf{28} & \\
\mathbf{14} & \mathbf{4} & \\
\mathbf{6} & \mathbf{6} & \\
\mathbf{2} & \mathbf{2} &
\end{array}
\begin{pmatrix}
1 & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{pmatrix}
$$

2.2. **Parameters $c = 1/5$ and $c = 1/3$.** The rational Cherednik algebra at each of these parameters has two blocks of defect 2. It will turn out that all four blocks in question have the same decomposition matrix with simple modules of matching dimensions of support.

The labelings of rows and columns for these four blocks (from top to bottom and left to right) and their h_c-weight lines are as follows:

$$c = \frac{1}{5}: \qquad \{\mathbf{1},\ \mathbf{11},\ \mathbf{18},\ \mathbf{9},\ \mathbf{26},\ \mathbf{8},\ \mathbf{19},\ \mathbf{14},\ \mathbf{2}\}, \qquad \{\mathbf{3},\ \mathbf{20},\ \mathbf{31},\ \mathbf{24},\ \mathbf{34},\ \mathbf{17},\ \mathbf{32},\ \mathbf{21},\ \mathbf{4}\}$$



$$c = \frac{1}{3}: \qquad \{\mathbf{1},\ \mathbf{18},\ \mathbf{27},\ \mathbf{15},\ \mathbf{33},\ \mathbf{9},\ \mathbf{28},\ \mathbf{19},\ \mathbf{2}\}, \qquad \{\mathbf{3},\ \mathbf{5},\ \mathbf{20},\ \mathbf{16},\ \mathbf{17},\ \mathbf{10},\ \mathbf{21},\ \mathbf{6},\ \mathbf{4}\}$$

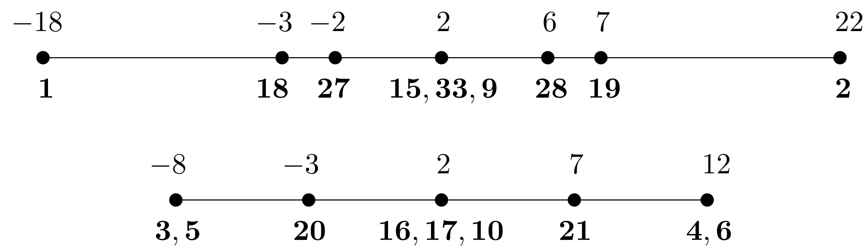


Table 3. Decomposition matrices of defect 2 blocks in $\mathcal{O}_c(H_4)$, $c \in \{\frac{1}{3}, \frac{1}{5}\}$

$$\begin{array}{c} \star \\ \star \\ (2) \\ \\ \\ (2) \\ \\ \\ \\ \end{array}
\begin{pmatrix}
1 & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & 1 & 1 & 1 & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{pmatrix}$$

Theorem 19. *The decomposition matrices of the blocks of defect* 2 *of* $H_c(H_4)$ *for* $c \in \{\frac{1}{3}, \frac{1}{5}\}$ *are all given by the matrix in Table 3.*

Proof. The argument for how to complete the matrix $([\mathrm{M}(\tau) : \mathrm{L}(\sigma)])$ above from the corresponding decomposition matrix of the Hecke algebra, which contains all but the first, second, third, and sixth columns [14], is the same in all four cases up to relabeling the rows and columns. We give the argument in the abstract in a way that applies to all four of the cases at hand. Label the rows and columns from top to bottom and left to right by $\tau_1, ..., \tau_9 \in \operatorname{Irr} H_4$. Then $\tau_k = \tau'_{9+1-k}$ for $k = 1, 2, 3$, while $\tau_4,\ \tau_5,$ and τ_6 are sign-invariant. We have $\tau_n \geq_c \tau_{n-1}$ for each $n = 2, .., 9$, and so it follows from $[\mathrm{M}(\tau_8) : \mathrm{L}(\tau_9)] = 0$ that $\dim \operatorname{Hom}(\mathrm{M}(\tau_9), \mathrm{M}(\tau_8)) = 0$. Since $\mathrm{L}(\tau_7) = \mathrm{M}(\tau_7) - \mathrm{M}(\tau_8) - \mathrm{M}(\tau_9)$ but τ_8 and τ_9 are not linked, it must be that $\dim \operatorname{Hom}(\mathrm{M}(\tau_8), \mathrm{M}(\tau_7)) = \dim \operatorname{Hom}(\mathrm{M}(\tau_9), \mathrm{M}(\tau_7)) = 1$. Then by Lemma 15, $\dim \operatorname{Hom}(\mathrm{M}(\tau_2), \mathrm{M}(\tau_1)) = 0$ and $\dim \operatorname{Hom}(\mathrm{M}(\tau_3), \mathrm{M}(\tau_1)) = \dim \operatorname{Hom}(\mathrm{M}(\tau_3), \mathrm{M}(\tau_2)) = 1$. This gives the first three columns of the decomposition matrix by Lemma 16.

As for the column labeled τ_6: $[\mathrm{M}(\tau_4) : \mathrm{L}(\tau_6)] = [\mathrm{M}(\tau_5) : \mathrm{L}(\tau_6)] = 0$ since $h_c(\tau_k) = 2$ for $k = 4, 5, 6$ and reps with the same h_c-weight can't be linked. From the decomposition $\mathrm{L}(\tau_6) = \mathrm{M}(\tau_6) - \mathrm{M}(\tau_7) + \mathrm{M}(\tau_8) + \mathrm{M}(\tau_9)$, we see that $1 = \dim \operatorname{Hom}(\mathrm{M}(\tau_7), \mathrm{M}(\tau_6)) =$

$\dim \operatorname{Hom}(\mathrm{M}(\tau_6), \mathrm{M}(\tau_3))$. Since there are no reps with h_c-weight intermediate between τ_3 and τ_6, it must be that $[\mathrm{M}(\tau_3) : \mathrm{L}(\tau_6)] = 1$. For the top two entries of column τ_6, we look instead to the characters of $\mathrm{L}(\tau_1)$ and $\mathrm{L}(\tau_2)$. We know so far that $\mathrm{L}(\tau_1) = \mathrm{M}(\tau_1) - \mathrm{M}(\tau_3) + \mathrm{M}(\tau_5)$... and $\mathrm{L}(\tau_2) = \mathrm{M}(\tau_2) - \mathrm{M}(\tau_3) + \mathrm{M}(\tau_4)$. In all four blocks under consideration, it must hold that the coefficient c_6 of $\mathrm{M}(\tau_6)$ in the character of $\mathrm{L}(\tau_i)$ is at least 1 for $i = 1, 2$, because $\dim \mathrm{L}(\tau_i)[2] \geq \dim \mathrm{L}(\tau_i)[-2]$. On the other hand, $\dim \operatorname{Hom}(\mathrm{M}(\tau_6), \mathrm{M}(\tau_3)) = 1$ implies that $c_6 \leq 1$, since $c_6\mathrm{M}(\tau_6)$ occurring next in the decomposition of $\mathrm{L}(\tau_i) = \mathrm{M}(\tau_i) - \mathrm{M}(\tau_3)$.... with $c_6 > 0$ would have to correspond to c_6 copies of τ_6 generating H_c-subreps inside $\mathrm{M}(\tau_3)$. Thus $c_6 = 1$ for $i = 1, 2$, and it follows that $[\mathrm{M}(\tau_i) : \mathrm{L}(\tau_6)] = 0$. This completes the decomposition matrix $([\mathrm{M}(\tau) : \mathrm{L}(\sigma)])$. □

2.3. **Parameter** $c = 1/4$**.** The h_c-weight line for the principal block of $\mathcal{O}_{\frac{1}{4}}(H_4)$ is given in Table 4.

Table 4. Weight line for the principal block of $\mathcal{O}_{\frac{1}{4}}(H_4)$

-13	-3	-1	2	5	7	17
1	**11, 13**	**27**	**23, 24, 10**	**28**	**14, 12**	**2**

Table 5. Decomposition matrix of the principal block of $\mathcal{O}_{\frac{1}{4}}(H_4)$

$$\begin{array}{cc}
 & \begin{array}{ccccccccccc} \mathbf{1} & \mathbf{11} & \mathbf{13} & \mathbf{27} & \mathbf{23} & \mathbf{24} & \mathbf{10} & \mathbf{28} & \mathbf{14} & \mathbf{12} & \mathbf{2} \end{array} \\
\begin{array}{cc}
\star & \mathbf{1} \\
\star & \mathbf{11} \\
\star & \mathbf{13} \\
 & \mathbf{27} \\
 & \mathbf{23} \\
 & \mathbf{24} \\
(1) & \mathbf{10} \\
 & \mathbf{28} \\
 & \mathbf{14} \\
 & \mathbf{12} \\
 & \mathbf{2}
\end{array} &
\begin{pmatrix}
1 & 1 & 1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{pmatrix}
\end{array}$$

Theorem 20. *The decomposition matrix of the principal block of $\mathcal{O}_{1/4}(H_4)$ is as given in Table 5.*

Proof. All but the columns labeled $\mathbf{1}, \mathbf{11}, \mathbf{13}$, and $\mathbf{10}$ of the decomposition matrix are given by the decomposition matrix of the Hecke algebra at $q = \exp(-2\pi i/4)$. The column labeled $\mathbf{1}$ is immediate as it's the leftmost column of the matrix, and the columns labeled $\mathbf{11}$ and $\mathbf{13}$ are deduced from Lemmas 15 and 16.

It remains to find the column labeled $\mathbf{10}$. We have $[\mathrm{M}(\mathbf{23}) : \mathrm{L}(\mathbf{10})] = [\mathrm{M}(\mathbf{24}) : \mathrm{L}(\mathbf{10})] = 0$ since $h_c(\mathbf{23}) = h_c(\mathbf{10}) = h_c(\mathbf{24})$. By Lemmas 16 and 15, then $[\mathrm{M}(\mathbf{27}) : \mathrm{L}(\mathbf{10})] = 1$. To determine the entries of this column in the first three rows, first observe that $\mathrm{L}(\mathbf{1})$ must be finite-dimensional since 4 is an elliptic number of H_4. By our partial knowledge of the decomposition matrix at this point, we know the expression for $\mathrm{L}(\mathbf{1})$ as a linear combination of standard modules up through those standard modules $\mathrm{M}(\tau)$ with $h_c(\tau) \leq 0$. We can thus compute the graded character of $\mathrm{L}(\mathbf{1})$, which must be a polynomial because $\mathrm{L}(\mathbf{1})$ is finite-dimensional. We compute it by calculating the dimensions of its graded pieces, which is determined by the part of its character that we already know. Multiplying the resulting polynomial by $\frac{(1-t)^4}{(1-t)^4}$ we have:

$$\chi_{\mathrm{L}(\mathbf{1})}(t) = \frac{t^{-13} - 18t^{-3} + 25t^{-1} - 16t^2 + 25t^5 - 18t^7 + t^{17}}{(1-t)^4}.$$

It follows that -2 must be the coefficient of $\mathrm{M}(\mathbf{10})$ in the decomposition of $\mathrm{L}(\mathbf{1})$, as neither $\mathrm{M}(\mathbf{23})$ nor $\mathrm{M}(\mathbf{24})$ occurs in the character.

Similarly, we look at the characters of $\mathrm{L}(\mathbf{11})$ and $\mathrm{L}(\mathbf{13})$ up to the standard modules of h_c-weight 2 which appear in those characters. We then apply Lemma 11 to conclude that $\mathrm{L}(\mathbf{11})$ and $\mathrm{L}(\mathbf{13})$ are finite-dimensional. This determines the graded character of these modules, since $\dim \mathrm{L}[m] = \dim \mathrm{L}[-m]$ for any finite-dimensional module $\mathrm{L} \in \mathcal{O}_c(W)$ and any $m \in \mathbb{Z}$. Next, a look at the graded character of these modules then shows that the coefficient of $\mathrm{M}(\mathbf{10})$ must be 1 in the characters of both $\mathrm{L}(\mathbf{11})$ and $\mathrm{L}(\mathbf{13})$. This implies that $[\mathrm{M}(\mathbf{13}) : \mathrm{L}(\mathbf{10})] = [\mathrm{M}(\mathbf{11}) : \mathrm{L}(\mathbf{10})] = 0$ and $[\mathrm{M}(\mathbf{1}) : \mathrm{L}(\mathbf{10})] = 1$. This completes the decomposition matrix. $\square$

2.4. **Parameter** $c = 1/2$**.** The h_c-weight line for the principal block of $\mathcal{O}_{\frac{1}{2}}(H_4)$ is given in Table 6.

TABLE 6. Weight line for the principal block of $\mathcal{O}_{\frac{1}{2}}(H_4)$

-28	-13	-8	-4	-3	2	7	8	12	17	32
$\mathbf{1}$	$\mathbf{5}$	$\mathbf{13}$	$\mathbf{27}$	$\mathbf{31}$	$\mathbf{30}, \mathbf{29}$	$\mathbf{32}$	$\mathbf{28}$	$\mathbf{14}$	$\mathbf{6}$	$\mathbf{2}$
	$\mathbf{3}$	$\mathbf{11}$			$\mathbf{22}, \mathbf{15}$			$\mathbf{12}$	$\mathbf{4}$	
					$\mathbf{8}, \mathbf{7}$					

By [26, §4.8.2] the finite-dimensional irreducible representations are $\mathrm{L}(\mathbf{27})$, $\mathrm{L}(\mathbf{3})$, $\mathrm{L}(\mathbf{5})$, $\mathrm{L}(\mathbf{13})$, $\mathrm{L}(\mathbf{11})$, and $\mathrm{L}(\mathbf{1})$. Applying Lemma 10, we have $\mathrm{L}(\mathbf{27}) = \mathrm{M}(\mathbf{27}) - \mathrm{M}(\mathbf{31}) + \ldots$, and referring to the weight line in Table 6, we see that all remaining $\mathrm{M}(\tau)$ appearing with a non-zero coefficient in the character of $\mathrm{L}(\mathbf{27})$ satisfy $h_{\frac{1}{2}}(\tau) \geq 2$. This determines the graded character of $\mathrm{L}(\mathbf{27})$ up through degree 1. Applying Lemma 6, we obtain the complete graded character of $\mathrm{L}(\mathbf{27})$. The standards $M(\tau)$ in the character with $h_c(\tau) > 2$ come from Lemma 13. Dimension considerations together with the requirement that ${}^{\mathcal{O}}\mathrm{Res}_{H_3}^{H_4}\mathrm{L}(\mathbf{27}) = 0$, as $\mathrm{L}(\mathbf{27})$ is finite-dimensional, then determine the standard modules $\mathrm{M}(\tau)$ with $h_c(\tau) = 2$ appearing in the character (these are the τ fixed under tensoring with sign). We obtain the following character and graded character:

$$\mathrm{L}(\mathbf{27}) = \mathrm{M}(\mathbf{27}) - \mathrm{M}(\mathbf{31}) + \mathrm{M}(\mathbf{7}) + \mathrm{M}(\mathbf{8}) + \mathrm{M}(\mathbf{15}) - \mathrm{M}(\mathbf{32}) + \mathrm{M}(\mathbf{28})$$
$$\chi_{\mathrm{L}(\mathbf{27})}(t) = 25(t^{-4} + t^4) + 64(t^{-3} + t^3) + 106(t^{-2} + t^2) + 140(t^{-1} + t) + 155$$
$$\dim \mathrm{L}(\mathbf{27}) = 825$$

The characters of $\mathrm{L}(\mathbf{3})$ and $\mathrm{L}(\mathbf{5})$ require some work. For $\mathrm{L}(\mathbf{3})$, induce $\mathrm{L}(\mathbf{3}_-)$ from $\mathcal{O}_{\frac{1}{2}}(H_3)$:

$${}^{\mathcal{O}}\mathrm{Ind}\mathrm{L}(\mathbf{3}_-) = \mathrm{M}(\mathbf{3}) - \mathrm{M}(\mathbf{13}) - \mathrm{M}(\mathbf{27}) + \mathrm{M}(\mathbf{31}) - \mathrm{M}(\mathbf{32}) + \mathrm{M}(\mathbf{28}) + \mathrm{M}(\mathbf{14}) - \mathrm{M}(\mathbf{4})$$

Thus $\dim \mathrm{Hom}(\mathrm{M}(\mathbf{13}), \mathrm{M}(\mathbf{3})) \geq 1$. The decomposition of $S^5\mathfrak{h}^* \otimes \mathbf{3}$ into irreducible H_4-representations implies that $\dim \mathrm{Hom}(\mathrm{M}(\mathbf{13}), \mathrm{M}(\mathbf{3})) \leq 1$. Since $\mathbf{11}^{\oplus 2} \subset S^5\mathfrak{h}^* \otimes \mathbf{3}$ it is not immediately obvious that $\dim \mathrm{Hom}(\mathrm{M}(\mathbf{11}), \mathrm{M}(\mathbf{3})) = 0$. However, considering the decomposition of $S^7\mathfrak{h}^* \otimes \mathbf{3}$ into irreducible H_4-representations, we see that it does not contain $\mathbf{15}$ as a subrep, while $S^2\mathfrak{h}^* \otimes \mathbf{11}$ does – and as $\mathrm{L}(\mathbf{3})[-6] = S^7\mathfrak{h}^* \otimes \mathbf{3} - S^2\mathfrak{h}^* \otimes \mathbf{13} - c_{11}S^2\mathfrak{h}^* \otimes \mathbf{11}$ can't contain $\mathbf{15}$ with negative multiplicity, $-\mathrm{M}(\mathbf{11})$ must not appear in the character. This implies that $\dim \mathrm{Hom}(\mathrm{M}(\mathbf{27}), \mathrm{M}(\mathbf{3})) \geq 1$. We must argue that it equals 1; this can be done by considering the decomposition of $\mathrm{L}(\mathbf{3})[-2] = S^{11}\mathfrak{h}^* \otimes \mathbf{3} - S^6\mathfrak{h}^* \otimes \mathbf{13} - c_{27}S^2\mathfrak{h}^* \otimes \mathbf{27}$ into irreducible H_4-representations. The irreducible H_4-representation $\mathbf{22}$ appears with multiplicity 3 in $S^{11}\mathfrak{h}^* \otimes \mathbf{3}$ and with multiplicity 2 in $S^6\mathfrak{h}^* \otimes \mathbf{13}$, implying $c_{27} \leq 1$. Thus $\mathrm{L}(\mathbf{3}) = \mathrm{M}(\mathbf{3}) - \mathrm{M}(\mathbf{13}) - \mathrm{M}(\mathbf{27}) + \mathrm{M}(\mathbf{31})$... where the rest of the character consists of Vermas $\mathrm{M}(\tau)$ with $h_c(\tau) \geq 2$.

By Lemma 13, we may write $\mathrm{L}(\mathbf{3}) = \mathrm{M}(\mathbf{3}) - \mathrm{M}(\mathbf{13}) - \mathrm{M}(\mathbf{27}) + \mathrm{M}(\mathbf{31}) + M + \mathrm{M}(\mathbf{32}) - \mathrm{M}(\mathbf{28}) - \mathrm{M}(\mathbf{14}) + \mathrm{M}(\mathbf{4})$ where M is some linear combination of Vermas $M(\tau)$ such that $h_c(\tau) = 2$. Now restrict this expression to $\mathcal{O}_{\frac{1}{2}}(H_3)$ and recall that the parabolic restriction of a finite-dimensional representation of a rational Cherednik algebra is always 0. For ease on the eyes we drop the notation M of Vermas and simply write the restriction of their lowest weight H_4-reps (it is the same rule on the level of the Grothendieck group anyways):

$$\begin{aligned}
0 &= {}^{\mathcal{O}}\mathrm{Res}^{H_4}_{H_3}\mathrm{L}(\mathbf{3})\\
&= \mathrm{Res}^{H_4}_{H_3}\mathbf{3} - \mathrm{Res}^{H_4}_{H_3}\mathbf{13} - \mathrm{Res}^{H_4}_{H_3}\mathbf{27} + \mathrm{Res}^{H_4}_{H_3}\mathbf{31} + \mathrm{Res}^{H_4}_{H_3}M\\
&\qquad + \mathrm{Res}^{H_4}_{H_3}\mathbf{32} - \mathrm{Res}^{H_4}_{H_3}\mathbf{28} - \mathrm{Res}^{H_4}_{H_3}\mathbf{14} + \mathrm{Res}^{H_4}_{H_3}\mathbf{4}\\
&= (\mathbf{1}_+ \oplus \mathbf{3}_-) - (\mathbf{1}_+ \oplus \tilde{\mathbf{3}}_- \oplus \mathbf{5}_+) - (\mathbf{1}_+ \oplus \mathbf{3}_- \oplus \tilde{\mathbf{3}}_- \oplus \mathbf{4}_+ \oplus \mathbf{4}_- \oplus {\mathbf{5}_+}^{\oplus 2})\\
&\qquad + (\mathbf{1}_+ \oplus {\mathbf{3}_-}^{\oplus 2} \oplus \tilde{\mathbf{3}}_-^{\oplus 2} \oplus \mathbf{4}_+ \oplus \mathbf{4}_- \oplus {\mathbf{5}_+}^{\oplus 2} \oplus \mathbf{5}_-) + \mathrm{Res}^{H_4}_{H_3}M\\
&\qquad + (\mathbf{1}_- \oplus {\mathbf{3}_+}^{\oplus 2} \oplus \tilde{\mathbf{3}}_+^{\oplus 2} \oplus \mathbf{4}_+ \oplus \mathbf{4}_- \oplus \mathbf{5}_+ \oplus {\mathbf{5}_-}^{\oplus 2})\\
&\qquad - (\mathbf{1}_- \oplus \mathbf{3}_+ \oplus \tilde{\mathbf{3}}_+ \oplus \mathbf{4}_+ \oplus \mathbf{4}_- \oplus {\mathbf{5}_-}^{\oplus 2}) - (\mathbf{1}_- \oplus \tilde{\mathbf{3}}_+ \oplus \mathbf{5}_-) + (\mathbf{1}_- \oplus \mathbf{3}_+)\\
&= {\mathbf{3}_+}^{\oplus 2} \oplus {\mathbf{3}_-}^{\oplus 2} + \mathrm{Res}^{H_4}_{H_3}M
\end{aligned}$$

Therefore (after examining the restrictions of the possible constituents of M):

$$\mathrm{L}(\mathbf{3}) = \mathrm{M}(\mathbf{3}) - \mathrm{M}(\mathbf{13}) - \mathrm{M}(\mathbf{27}) + \mathrm{M}(\mathbf{31}) - 2\mathrm{M}(\mathbf{7}) + \mathrm{M}(\mathbf{32}) - \mathrm{M}(\mathbf{28}) - \mathrm{M}(\mathbf{14}) + \mathrm{M}(\mathbf{4})$$

Using similar arguments starting with $\mathrm{Ind}\mathrm{L}(\tilde{\mathbf{3}}_-)$ in place of $\mathrm{Ind}\mathrm{L}(\mathbf{3}_-)$, we obtain the character of $\mathrm{L}(\mathbf{5})$:

$$\mathrm{L}(\mathbf{5}) = \mathrm{M}(\mathbf{5}) - \mathrm{M}(\mathbf{11}) - \mathrm{M}(\mathbf{27}) + \mathrm{M}(\mathbf{31}) - 2\mathrm{M}(\mathbf{8}) + \mathrm{M}(\mathbf{32}) - \mathrm{M}(\mathbf{28}) - \mathrm{M}(\mathbf{12}) + \mathrm{M}(\mathbf{6})$$

Using Lemma 15 and comparing the characters of the finite-dimensional representations with the rows of the inverse decomposition matrix, we deduce the full decomposition matrix given in Table 7, where the characters of $\mathrm{L}(\mathbf{1})$, $\mathrm{L}(\mathbf{11})$, and $\mathrm{L}(\mathbf{13})$ are obtained by similar arguments. We have $\dim \mathrm{L}(\mathbf{1}) = 65625$ and $\dim \mathrm{L}(\mathbf{11}) = \dim \mathrm{L}(\mathbf{13}) = 5625$.

3. The decomposition matrices of $\mathcal{O}_c(F_4)$ with equal parameters

Throughout this section, we consider the "equal parameters" case

$$(c_1, c_2) = (c, c) = \left(\frac{1}{d}, \frac{1}{d}\right)$$

for some fixed $d \in \mathbb{Z}_{\geq 2}$. We write $\mathcal{O}_c(F_4)$ for $\mathcal{O}_{(c,c)}(F_4)$. The values of the denominator d of c such that $\mathcal{O}_c(F_4)$ is not semisimple are $d = 2, 3, 4, 6, 8, 12$. We will not treat the case $d = 2$. For $d \in \{8, 12\}$ all blocks of $\mathcal{O}_c(F_4)$ are of defect 0 or 1 ([14], Table F.3), so the decomposition matrices are known by [29]. For $d \in \{3, 4, 6\}$, all blocks are of defect 0 or 2 ([14], Table F.3). We will find the decomposition matrices of the defect 2 blocks.

3.1. **Parameter $c = 1/6$.** Let $c = \frac{1}{6}$. The principal block of $\mathcal{O}_c(F_4)$ is of defect 2, while all other blocks are of defect 0 [13, Table 7.7]. The h_c-weight line for the principal block is given in Table 8.

Table 7. Decomposition matrix of the principal block of $\mathcal{O}_{\frac{1}{2}}(H_4)$

$$
\begin{array}{cc}
 & \begin{array}{cccccccccccccccccccc} \mathbf{1} & \mathbf{5} & \mathbf{3} & \mathbf{13} & \mathbf{11} & \mathbf{27} & \mathbf{31} & \mathbf{30} & \mathbf{29} & \mathbf{22} & \mathbf{15} & \mathbf{8} & \mathbf{7} & \mathbf{32} & \mathbf{28} & \mathbf{14} & \mathbf{12} & \mathbf{6} & \mathbf{4} & \mathbf{2} \end{array} \\
\begin{array}{rr}
\star & \mathbf{1} \\
\star & \mathbf{5} \\
\star & \mathbf{3} \\
\star & \mathbf{13} \\
\star & \mathbf{11} \\
\star & \mathbf{27} \\
(1) & \mathbf{31} \\
(3) & \mathbf{30} \\
(3) & \mathbf{29} \\
(2) & \mathbf{22} \\
(2) & \mathbf{15} \\
(1) & \mathbf{8} \\
(1) & \mathbf{7} \\
(3) & \mathbf{32} \\
 & \mathbf{28} \\
 & \mathbf{14} \\
 & \mathbf{12} \\
 & \mathbf{6} \\
 & \mathbf{4} \\
 & \mathbf{2}
\end{array}
&
\begin{pmatrix}
1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & 1 & \cdot & \cdot & 1 & 1 & 1 & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & 1 & 1 & \cdot & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & 1 & \cdot & \cdot & 1 & 1 & \cdot & 1 & \cdot & 1 & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & 1 & 1 & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & 1 & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 3 & 1 & 1 & 1 & 1 & 1 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & \cdot & \cdot & 2 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & 1 & 2 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & 1 & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{pmatrix}
\end{array}
$$

Table 8. Weight line for the principal block of $H_{\frac{1}{6}}(F_4)$

-2	0	1	2	3	4	6
1_1	2_3	8_3	12	8_4	2_4	1_4
	2_1	8_1	9_2	8_2	2_2	
			9_3			

Theorem 21. *The decomposition matrix of the principal block of $\mathcal{O}_{\frac{1}{6}}(F_4)$ is given by Table 9.*

Proof. All but the leftmost five columns of the decomposition matrix are given by the decomposition matrix of the Hecke algebra. All unknown entries in the leftmost five columns follow from Lemmas 15 and 16 except for $[\mathrm{M}(1_1) : \mathrm{L}(8_3)] =: a$ and $[\mathrm{M}(1_1) : \mathrm{L}(8_1)] =: b$. Since $\mathrm{L}(1_1)$ is finite-dimensional [34], there is an equality of graded dimensions

TABLE 9. Decomposition matrix of the principal block of $\mathcal{O}_{\frac{1}{6}}(F_4)$

$$
\begin{array}{cc}
 & \begin{array}{ccccccccccccc} 1_1 & 2_3 & 2_1 & 8_3 & 8_1 & 12 & 9_2 & 9_3 & 8_4 & 8_2 & 2_4 & 2_2 & 1_4 \end{array} \\
\begin{array}{rr}
\star & 1_1 \\
\star & 2_3 \\
\star & 2_1 \\
(1) & 8_3 \\
(1) & 8_1 \\
 & 12 \\
 & 9_2 \\
 & 9_3 \\
 & 8_4 \\
 & 8_2 \\
 & 2_4 \\
 & 2_2 \\
 & 1_4
\end{array} &
\begin{pmatrix}
1 & \cdot & \cdot & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & 1 & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{pmatrix}
\end{array}
$$

$\dim \mathrm{L}(1_1)[1] = \dim \mathrm{L}(1_1)[-1]$ in degrees 1 and -1 of $\mathrm{L}(1_1)$. This forces $a+b=2$. Lemma 13 implies that $a=b$. We conclude that $a=1=b$. $\square$

3.2. **Parameter** $c = 1/4$**.** The principal block of $\mathcal{O}_{\frac{1}{4}}(F_4)$ has defect 2 [13] (Table 7.7). Its h_c-weight line is given in Table 10.

TABLE 10. Weight line for the principal block of $\mathcal{O}_{\frac{1}{4}}(F_4)$

-4	-1	0	2	4	5	8
1_1	4_2	9_1	12	9_4	4_5	1_4
			6_1			
			4_1			

Theorem 22. *The decomposition matrix of the principal block of* $\mathcal{O}_{\frac{1}{4}}(F_4)$ *is given by Table 11.*

Proof. All but four columns are given by the decomposition matrix of the Hecke algebra. Of the four remaining columns, all unknown entries follow from Lemmas 15 and 16 except for $[\mathrm{M}(1_1) : \mathrm{L}(4_1)] =: a$ and $[\mathrm{M}(4_2) : \mathrm{L}(4_1)] =: b$. The inequality $\dim \mathrm{L}(1_1)[2] \geq \dim \mathrm{L}(1_1)[-2]$ implies $a = 0$. The inequality $\dim \mathrm{L}(4_2)[2] \geq 0$ implies $b = 0$. $\square$

TABLE 11. Decomposition matrix of the principal block of $\mathcal{O}_{\frac{1}{4}}(F_4)$

$$
\begin{array}{ll}
 & \begin{array}{ccccccccc} 1_1 & 4_2 & 9_1 & 12 & 6_1 & 4_1 & 9_4 & 4_5 & 1_4 \end{array} \\
\begin{array}{rr}
\star & 1_1 \\
\star & 4_2 \\
(2) & 9_1 \\
 & 12 \\
 & 6_1 \\
(2) & 4_1 \\
 & 9_4 \\
 & 4_5 \\
 & 1_4
\end{array}
&
\begin{pmatrix}
1 & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & 1 & 1 & 1 & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{pmatrix}
\end{array}
$$

3.3. **Parameter** $c = 1/3$**.** There are two blocks of defect 2 in $\mathcal{O}_{\frac{1}{3}}(F_4)$, one containing $\mathrm{L}(1_1)$ and the other containing $\mathrm{L}(4_2)$, the irreducible representation with lowest weight the reflection representation of F_4. The decomposition matrices of these two blocks are identical, and moreover, if $\sigma_i, \tau_i \in \operatorname{Irr} F_4$ label the i'th rows and columns in the shared decomposition matrix of these two blocks then $\dim \tau_i = 4 \dim \sigma_i$, and $\mathrm{L}(\tau_i)$ and $\mathrm{L}(\sigma_i)$ have the same dimension of support. The h_c-weight lines for these two blocks are given in Table 12.

TABLE 12. Weight lines for the defect 2 blocks of $\mathcal{O}_{\frac{1}{3}}(F_4)$

-6	-2	2	6	10
1_1	2_3	4_1	2_4	1_4
	2_1	1_2	2_2	
		1_3		

-2	0	2	4	6
4_2	8_3	16	8_4	4_5
	8_1	4_3	8_2	
		4_4		

Theorem 23. *The decomposition matrices of the blocks of* $\mathcal{O}_{\frac{1}{3}}(F_4)$ *containing* $\mathrm{L}(1_1)$ *and* $\mathrm{L}(4_2)$ *are the same after relabeling and are given by Table 13.*

Proof. All but five columns are given by the decomposition matrix of the Hecke algebra. Lemmas 15 and 16 recover all the unknown entries in the remaining five columns except for $[\mathrm{M}(1_1) : \mathrm{L}(1_2)] =: a$ and $[\mathrm{M}(1_1) : \mathrm{L}(1_3)] =: b$ in the principal block, and $[\mathrm{M}(4_2) : \mathrm{L}(4_3)] =: a'$ and $[\mathrm{M}(4_2) : \mathrm{L}(4_4)] =: b'$ in the other block. Calculating the graded dimensions of $\mathrm{L}(1_1)$

TABLE 13. Decomposition matrix of the defect 2 blocks of $\mathcal{O}_{\frac{1}{3}}(F_4)$

$$\begin{array}{cccc}
 & & & \begin{matrix} 4_2 & 8_3 & 8_1 & 16 & 4_3 & 4_4 & 8_4 & 8_2 & 4_5 \\ 1_1 & 2_3 & 2_1 & 4_1 & 1_2 & 1_3 & 2_4 & 2_2 & 1_4 \end{matrix} \\
\begin{matrix} \star \\ (2) \\ (2) \\ \\ (2) \\ (2) \\ \\ \\ \\ \end{matrix} & \begin{matrix} 4_2 \\ 8_3 \\ 8_1 \\ 16 \\ 4_3 \\ 4_4 \\ 8_4 \\ 8_2 \\ 4_5 \end{matrix} & \begin{matrix} 1_1 \\ 2_3 \\ 2_1 \\ 4_1 \\ 1_2 \\ 1_3 \\ 2_4 \\ 2_2 \\ 1_4 \end{matrix} & \begin{pmatrix}
1 & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot \\
\cdot & \cdot & 1 & 1 & 1 & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & 1 & 1 \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{pmatrix}
\end{array}$$

and $\mathrm{L}(4_2)$ using the characters obtained from the inverses of the decompositions matrices forces $a = a' = b = b' = 0$:

$$31 = \dim \mathrm{L}(1_1)[-2] \leq \dim \mathrm{L}(1_1)[2] = 29 + 1 - a + 1 - b$$
$$4 = \dim \mathrm{L}(4_2)[-2] \leq \dim \mathrm{L}(4_2)[2] = -4 + 4(1 - a') + 4(1 - b').$$

This concludes the proof. □

4. THE DECOMPOSITION MATRICES OF $\mathcal{O}_c(E_6)$

The category $\mathcal{O}_c(E_6)$ for $c = \frac{1}{d}$ is semisimple unless $d \in \{2, 3, 4, 5, 6, 8, 9, 12\}$. We will calculate the decomposition matrices for all blocks of $\mathcal{O}_c(E_6)$ of defect greater than 1 for $c = \frac{1}{d}$ as above, $d \neq 2$. It turns out that the entries of the decomposition matrix of $\mathcal{O}_{\frac{1}{d}}(E_6)$ for $d \neq 2$ are all 0's and 1's. For $d \in \{5, 8, 9, 12\}$ all blocks of $\mathcal{O}_{\frac{1}{d}}(E_6)$ are of defect 0 or 1. We therefore study the cases $d \in \{3, 4, 6\}$.

4.1. **Parameter $c = 1/6$.** The principal block of $\mathcal{O}_{\frac{1}{6}}(E_6)$ is the unique block of non-zero defect. The h_c-weight line of the principal block is given in Table 14.

TABLE 14. Weight line for the principal block of $\mathcal{O}_{\frac{1}{6}}(E_6)$

-3	-1	0	1	2	3	4	5	6	7	9
1_p	6_p	20_p	30_p	24_p	80_s	$24'_p$	$30'_p$	$20'_p$	$6'_p$	$1'_p$
			15_q	60_p	60_s	$60'_p$	$15'_q$			

Theorem 24. *The decomposition matrix of the principal block of $\mathcal{O}_{\frac{1}{6}}(E_6)$ is given by Table 15.*

TABLE 15. Decomposition matrix of the principal block of $\mathcal{O}_{\frac{1}{6}}(E_6)$

$$\begin{array}{cc}
 & \begin{array}{cccccccccccccccc} 1_p & 6_p & 20_p & 30_p & 15_q & 24_p & 60_p & 80_s & 60_s & 24'_p & 60'_p & 30'_p & 15'_q & 20'_p & 6'_p & 1'_p \end{array} \\
\begin{array}{rr}
\star & 1_p \\
\star & 6_p \\
(2) & 20_p \\
 & 30_p \\
(1) & 15_q \\
(2) & 24_p \\
 & 60_p \\
 & 80_s \\
 & 60_s \\
 & 24'_p \\
 & 60'_p \\
 & 30'_p \\
 & 15'_q \\
 & 20'_p \\
 & 6'_p \\
 & 1'_p
\end{array} &
\begin{pmatrix}
1 & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & 1 & \cdot & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & 1 & 1 & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & 1 & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{pmatrix}
\end{array}$$

Proof. Let D be the decomposition matrix of the principal block of $\mathcal{O}_{\frac{1}{6}}(E_6)$ and let $D^{-1} =: \left(d^{-1}_{\tau,\sigma}\right)$ be the inverse matrix. All but five columns of D are given by the decomposition matrix of the Hecke algebra [14]. The five remaining columns are labeled by $1_p, 6_p, 20_p, 15_q, 24_p$.

First, since $h_c(30_p) = 1 = h_c(15_q)$, we have $[\mathrm{M}(30_p) : \mathrm{L}(15_q)] = 0$. Next, we observe from Lemma 16 that for all $\tau', \sigma' \in \{15'_q, 20'_p, 6'_p, 1'_p\}$ it holds that $\dim \mathrm{Hom}(\mathrm{M}(\tau'), \mathrm{M}(\sigma')) = [\mathrm{M}(\sigma') : \mathrm{L}(\tau')]$. Applying Lemma 15 we find that

$$\dim \mathrm{Hom}(\mathrm{M}(\sigma), \mathrm{M}(\tau)) = \dim \mathrm{Hom}(\mathrm{M}(\tau'), \mathrm{M}(\sigma')) = [\mathrm{M}(\sigma') : \mathrm{L}(\tau')]$$

for all $\sigma, \tau \in \{1_p, 6_p, 20_p, 15_q\}$. But each pair (τ, σ) with the aforementioned choices of σ, τ is either not related in the minimal partial order $\leq_{\mathrm{Hom}}$ or describes a minimal edge in this order. Lemma 16 then gives $[\mathrm{M}(\tau) : \mathrm{L}(\sigma)] = \dim \mathrm{Hom}(\mathrm{M}(\sigma), \mathrm{M}(\tau))$ for $\sigma, \tau \in \{1_p, 6_p, 20_p, 15_q\}$. This completes these four columns of the decomposition matrix.

It remains to find the upper-diagonal entries of column 24_p. Applying Lemmas 15 and 16 we obtain $[\mathrm{M}(\tau) : \mathrm{L}(24_p)] = [\mathrm{M}(24'_p) : \mathrm{L}(\tau')]$ for $\tau \in \{15_q, 20_p, 30_p\}$. This leaves two entries to determine: $[\mathrm{M}(1_p) : \mathrm{L}(24_p)] =: a$ and $[\mathrm{M}(6_p) : \mathrm{L}(24_p)] =: b$. Since we know the character of $\mathrm{L}(1_p)$ for all standards up to those of h_c-weight 0, and since $\mathrm{L}(1_p)$ is finite-dimensional [34], we may calculate the graded character of $\mathrm{L}(1_p)$ by Lemma 6:

$$\begin{aligned}\chi_{\mathrm{L}(1_p)}(t) &= t^{-3} + t^3 + 6(t^{-2} + t^2) + 21(t^{-1} + t) + 36 \\ &= \frac{t^{-3} - 20 + 15t + 84t^2 - 160t^3 + 84t^4 + 15t^5 - 20t^6 + t^9}{(1-t)^6}\end{aligned}$$

Since the coefficient of t^2 is 84 and $d^{-1}_{1_p,60_p} = 1$, we have $d^{-1}_{1_p,24_p} = 1$. The dot product of row 1_p of D^{-1} with column 24_p of D is 0, giving $a = 0$.

Next, we consider the character of $\mathrm{L}(6_p)$ up through $\mathrm{M}(\tau)$'s of h_c-weight 2. We have

$$\mathrm{L}(6_p) = \mathrm{M}(6_p) - \mathrm{M}(20_p) + (1-b)\mathrm{M}(24_p) + \mathrm{M}(60_p) + \dots$$

with $h_c(6_p) = -1$, $h_c(20_p) = 0$, and $h_c(24_p) = h_c(60_p) = 2$. We then calculate $\dim \mathrm{L}(6_p)[2]$, the graded dimension of $\mathrm{L}(6_p)$ in grading degree 2, a nonnegative integer. We calculate

$$\dim \mathrm{L}(6_p)[2] = 6\binom{8}{3} - 20\binom{7}{2} + (1-b)24 + 60 = -24 + (1-b)24 \geq 0$$

and conclude that $b = 0$. $\square$

4.2. **Parameter** $c = 1/4$**.** The principal block is the unique block of $\mathcal{O}_{\frac{1}{4}}(E_6)$ of defect greater than 1 [14]. The h_c-weight line of this block is given in Table 16.

TABLE 16. Weight line for the principal block of $\mathcal{O}_{\frac{1}{4}}(E_6)$

-6	-3	0	2	3	4	6	9	12
1_p	6_p	15_q	81_p	90_s	$81'_p$	$15'_q$	$6'_p$	$1'_p$
		15_p		80_s		$15'_p$		
				10_s				

Theorem 25. *The decomposition matrix of the principal block of $\mathcal{O}_{\frac{1}{4}}(E_6)$ is given by Table 17.*

Proof. All but the five leftmost columns come from the decomposition matrix of the Hecke algebra of E_6 with $e = 4$. Observing that $\mathrm{M}(15'_p) = \mathrm{L}(15'_p)$ and $\mathrm{M}(6'_p) = \mathrm{L}(6'_p)$, we have $\dim \mathrm{Hom}(\mathrm{M}(\sigma), \mathrm{M}(\tau)) = 0$ for $\sigma \in \{15_p, 6_p\}$ and for all τ by Lemma 15. Since $\mathrm{L}(15'_q) = \mathrm{M}(15'_q) - \mathrm{M}(6'_p) - \mathrm{M}(1'_p)$ and all standard modules to the right of 15_q are simple, $\dim \mathrm{Hom}(\mathrm{M}(15_q), \mathrm{M}(6_p)) = \dim \mathrm{Hom}(\mathrm{M}(6'_p), \mathrm{M}(15'_q)) = 1$ and likewise $\dim \mathrm{Hom}(\mathrm{M}(15_q), \mathrm{M}(1_p)) = 1$. This gives column 15_q. Since $\dim \mathrm{Hom}(\mathrm{M}(15'_q), \mathrm{M}(81'_p)) =$

Table 17. Decomposition matrix of the principal block of $\mathcal{O}_{\frac{1}{4}}(E_6)$

$$
\begin{array}{cc}
\begin{array}{rr}
(2) & 1_p \\
(2) & 6_p \\
(3) & 15_q \\
(2) & 15_p \\
(3) & 81_p \\
 & 90_s \\
 & 80_s \\
 & 10_s \\
 & 81'_p \\
 & 15'_q \\
 & 15'_p \\
 & 6'_p \\
 & 1'_p
\end{array}
&
\begin{pmatrix}
1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & 1 & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & 1 & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & 1 & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{pmatrix}
\end{array}
$$

0, and the remaining Vermas to the right are simple, column 81_p follows by the same reasoning. □

4.3. **Parameter $c = 1/3$.** The principal block of $\mathcal{O}_{\frac{1}{3}}(E_6)$ is the unique block of nonzero defect. The h_c-weight line for the principal block is given by Table 18. This block is considerably more difficult to deal with than the blocks for $c = \frac{1}{6}$ and $c = \frac{1}{4}$.

Table 18. Weight line for the principal block of $\mathcal{O}_{\frac{1}{3}}(E_6)$

-9	-5	-3	-1	0	1	3	5	6	7	9	11	15
1_p	6_p	20_p	30_p, 15_q, 15_p	64_p	24_p, 60_p	80_s, 60_s, 20_s, 10_s	$24'_p$, $60'_p$	$64'_p$	$30'_p$, $15'_q$, $15'_p$	$20'_p$	$6'_p$	$1'_p$

Theorem 26. *The decomposition matrix of the principal block of $\mathcal{O}_{\frac{1}{3}}(E_6)$ is given by Table 19.*

Proof. Let D denote the decomposition matrix of the principal block of $\mathcal{O}_{\frac{1}{3}}(E_6)$ and let $D^{-1} =: \left(d^{-1}_{\tau,\sigma}\right)$ be the inverse matrix.

TABLE 19. Decomposition matrix of the principal block of $\mathcal{O}_{\frac{1}{3}}(E_6)$

$$
\begin{array}{rr}
\star & 1_p \\
\star & 6_p \\
(4) & 20_p \\
(2) & 30_p \\
(2) & 15_q \\
\star & 15_p \\
(4) & 64_p \\
(4) & 24_p \\
(4) & 60_p \\
 & 80_s \\
 & 60_s \\
(4) & 20_s \\
(2) & 10_s \\
(2) & 24_p' \\
 & 60_p' \\
 & 64_p' \\
 & 30_p' \\
 & 15_q' \\
 & 15_p' \\
 & 20_p' \\
 & 6_p' \\
 & 1_p'
\end{array}
\begin{pmatrix}
1 & 1 & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & 1 & 1 & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & 1 & \cdot & 1 & 1 & 1 & \cdot & 1 & \cdot & 1 & 1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & 1 & 1 & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & 1 & 1 & 1 & \cdot & 1 & 1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & 1 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & \cdot & 1 & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & 1 & 1 & \cdot & 1 & 1 & 1 & 1 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & 1 & \cdot & 1 & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & 1 & 1 & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{pmatrix}
$$

Importing the columns from the Hecke algebra decomposition matrix of E_6 for $e = 3$ in [13] and then applying Lemmas 15 and 16 provides the first approximation to the decomposition matrix of $\mathcal{O}_{\frac{1}{3}}(E_6)$ shown in Table 20. The rows τ marked with $\bullet$ correspond to those $\mathrm{L}(\tau)$ with less than full support.

There are many unknown entries of D to determine, marked with red question marks in Table 20. Before we start working our way through the unknown entries systematically, we first establish that $\mathrm{L}(15_p)$ is finite-dimensional and simultaneously determine two of the unknown decomposition numbers. Setting $a = [\mathrm{M}(15_p) : \mathrm{L}(24_p)]$ and $b = [\mathrm{M}(15_p) : \mathrm{L}(60_p)]$, the beginning of the character of $\mathrm{L}(15_p)$ is $\mathrm{L}(15_p) = \mathrm{M}(15_p) - \mathrm{M}(64_p) + (1 -$

Table 20. Preliminary approximation of the decomposition matrix of the principal block of $\mathcal{O}_{\frac{1}{3}}(E_6)$

$$\begin{array}{r}
\bullet\quad 1_p \\
\bullet\quad 6_p \\
\bullet\quad 20_p \\
\bullet\quad 30_p \\
\bullet\quad 15_q \\
\bullet\quad 15_p \\
\bullet\quad 64_p \\
\bullet\quad 24_p \\
\bullet\quad 60_p \\
80_s \\
60_s \\
\bullet\quad 20_s \\
\bullet\quad 10_s \\
\bullet\quad 24'_p \\
60'_p \\
64'_p \\
30'_p \\
15'_q \\
15'_p \\
20'_p \\
6'_p \\
1'_p
\end{array}
\begin{pmatrix}
1 & 1 & ? & ? & ? & ? & ? & ? & ? & \cdot & \cdot & ? & ? & ? & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & 1 & 1 & ? & 1 & ? & ? & ? & \cdot & \cdot & ? & ? & ? & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & 1 & \cdot & 1 & ? & ? & \cdot & 1 & ? & ? & ? & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & ? & ? & 1 & \cdot & ? & ? & ? & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & ? & ? & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & ? & ? & \cdot & \cdot & ? & ? & ? & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & 1 & 1 & ? & ? & ? & 1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & ? & ? & ? & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & \cdot & 1 & ? & 1 & \cdot & \cdot & \cdot & 1 & \cdot & 1 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & \cdot & 1 & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & 1 & 1 & \cdot & 1 & 1 & 1 & 1 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & ? & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & ? & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & 1 & \cdot & 1 & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & 1 & 1 & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{pmatrix}$$

$a)\mathrm{M}(24_p) + (1-b)\mathrm{M}(60_p) + \ldots$ The dimension of $\mathrm{L}(15_p)$ in grading degree 1 is:

$$\begin{aligned}\dim \mathrm{L}(15_p)[1] &= \binom{7}{5}15 - \binom{6}{5}64 + (1-a)24 + (1-b)60 \\ &= -69 + (1-a)24 + (1-b)60 \geq 0.\end{aligned}$$

This forces $a = b = 0$. Then $\dim \mathrm{L}(15_p)[2] = 15\binom{8}{5} - 64\binom{7}{5} + 84\binom{6}{5} = 0$ and therefore $\mathrm{L}(15_p)$ is finite-dimensional, with $\dim \mathrm{L}(15_p) = 15 + 26 + 15 = 56$. The full character of $\mathrm{L}(15_p)$ as a linear combination of standard modules follows by applying Lemma 13, and observing that the dot product of row 15_p of D^{-1} with columns $6'_p$, $30'_p$, and $64'_p$ of D is equal to 0. The character of $\mathrm{L}(15_p)$ is:

$$\begin{aligned}\mathrm{L}(15_p) = \mathrm{M}(15_p) - \mathrm{M}(64_p) + \mathrm{M}(24_p) + \mathrm{M}(60_p) - \mathrm{M}(60_s) - \mathrm{M}(10_s) \\ + \mathrm{M}(60'_p) + \mathrm{M}(24'_p) - \mathrm{M}(64'_p) + \mathrm{M}(15'_p).\end{aligned}$$

Now we begin working our way through the unknown entries of the decomposition matrix, starting from the bottom right and working up and left. We begin by computing the character of $\mathrm{L}(24'_p)$ from the columns given in Table 20:

$$\mathrm{L}(24'_p) = \mathrm{M}(24'_p) - \mathrm{M}(64'_p) + \mathrm{M}(30'_p) + \mathrm{M}(15'_p) - \mathrm{M}(6'_p) + \mathrm{M}(1'_p).$$

The graded character of $\mathrm{L}(24'_p)$ has a pole of order 2 at $t = 1$, so $\dim \mathrm{Supp}\, \mathrm{L}(24'_p) = 2$. Restricting $\mathrm{L}(24'_p)$ to $\mathcal{O}_{\frac{1}{3}}(A_5)$ we find that ${}^{\mathcal{O}}\mathrm{Res}^{E_6}_{A_5}\mathrm{L}(24'_p) = \mathrm{L}(3^2)$, one of the two minimal support irreducible representations (with dimension of support equal to 1) in $\mathcal{O}_{\frac{1}{3}}(A_5)$ [36].

Set $\alpha = [\mathrm{M}(10_s) : \mathrm{L}(24'_p)]$. Then

$$\begin{aligned}\mathrm{L}(10_s) &= \big(\mathrm{M}(10_s) - \mathrm{M}(60'_p) + \mathrm{M}(64'_p) + \mathrm{M}(15'_q) - \mathrm{M}(15'_p) - \mathrm{M}(20'_p) + \mathrm{M}(6'_p)\big) - \alpha \mathrm{L}(24'_p)\\ &=: (*) - \alpha \mathrm{L}(24'_p).\end{aligned}$$

Since ${}^{\mathcal{O}}\mathrm{Res}^{E_6}_{A_5}(*) = \mathrm{L}(3^2)$, we have $\alpha \in \{0, 1\}$. Suppose $\alpha = 1$, i.e. $\mathrm{L}(10_s) = (*) - \mathrm{L}(24'_p)$. Then

$${}^{\mathcal{O}}\mathrm{Ind}^{E_6}_{A_2\times A_2\times A_1}{}^{\mathcal{O}}\mathrm{Res}^{E_6}_{A_2\times A_2\times A_1}\mathrm{L}(10_s) = \mathrm{M}(1_p) - 2\mathrm{M}(6_p) + \ldots$$

where $\ldots$ ranges over $\mathrm{M}(\tau)$ with $\tau >_c 6_p$. Subtracting $\mathrm{L}(1_p) = \mathrm{M}(1_p) - \mathrm{M}(6_p)$... leaves an expression with negative leading term $-\mathrm{M}(6_p) + ...$, a contradiction. Therefore $\alpha = 0$ and $(*) = \mathrm{L}(10_s)$. The graded character yields $\dim \mathrm{Supp}\, \mathrm{L}(10_s) = 2$.

We apply the same argument to $\mathrm{L}(20_s)$. Now set $\alpha = [\mathrm{M}(20_s) : \mathrm{L}(24'_p)]$ and write $\mathrm{L}(20_s) = \mathrm{M}(20_s) - \mathrm{M}(64'_p) + \mathrm{M}(30'_p) + \mathrm{M}(20'_p) - \mathrm{M}(6'_p) - \alpha\mathrm{L}(24'_p)$. Then compute that ${}^{\mathcal{O}}\mathrm{Ind}^{E_6}_{A_5}{}^{\mathcal{O}}\mathrm{Res}^{E_6}_{A_5}\mathrm{L}(20_s) = -\alpha\mathrm{M}(15_q) + \ldots$ where $\ldots$ denotes terms $\mathrm{M}(\tau)$ with $\tau \geq_c 15_q$. Therefore $\alpha = 0$. Computing the graded character of $\mathrm{L}(20_p)$ yields $\dim \mathrm{Supp}\, \mathrm{L}(20_s) = 4$.

Next, we determine the character of $\mathrm{L}(60_p)$. Set $\alpha = [\mathrm{M}(60_p) : \mathrm{L}(24'_p)]$ and write

$$\begin{aligned}\mathrm{L}(60_p) = \mathrm{M}(60_p) &- \mathrm{M}(80_s) - \mathrm{M}(60_s) - \mathrm{M}(10_s) + \mathrm{M}(24'_p) + 2\mathrm{M}(60'_p) - \mathrm{M}(64'_p) - \mathrm{M}(15'_q)\\ &+ 2\mathrm{M}(15'_p) - \mathrm{M}(6'_p) + \mathrm{M}(1'_p) - \alpha\mathrm{L}(24'_p).\end{aligned}$$

The coefficient of $\mathrm{M}(\tau)$ for τ of minimal h_c-value in ${}^{\mathcal{O}}\mathrm{Ind}^{E_6}_{A_2\times A_2\times A_1}{}^{\mathcal{O}}\mathrm{Res}^{E_6}_{A_2\times A_2\times A_1}\mathrm{L}(60_p) =:$ M is negative unless $\alpha < 3$. If $\alpha = 2$, we obtain $M = 4\mathrm{M}(20_p) - 4\mathrm{M}(30_p) - 4\mathrm{M}(15_q) + \ldots$, but since the coefficient of $\mathrm{M}(30_p)$ in the character of $\mathrm{L}(20_p)$ is 0, this is not the character of a module, only a virtual module. If $\alpha = 1$, we obtain $M = \mathrm{M}(6_p) + 3\mathrm{M}(20_p) - 3\mathrm{M}(30_p) + \ldots$. Using that $d^{-1}_{6_p,30_p} = -1$ and subtracting off the factor of $\mathrm{L}(6_p)$ from M leaves an expression that cannot be the character of a module. Therefore $\alpha = 0$. The character of $\mathrm{L}(60_p)$ has a pole of order 4 at $t = 1$, so $\dim \mathrm{Supp}\, \mathrm{L}(60_p) = 4$.

By Lemmas 15 and 16 we have $[\mathrm{M}(24_p) : \mathrm{L}(10_s)] = [\mathrm{M}(24_p) : \mathrm{L}(20_s)] = 0$. The value of $\alpha := [\mathrm{M}(24_p) : \mathrm{L}(24'_p)]$ will finish the computation of the character of $\mathrm{L}(24_p)$:

$$\mathrm{L}(24_p) = \mathrm{M}(24_p) - \mathrm{M}(60_s) + \mathrm{M}(24'_p) + \mathrm{M}(60'_p) - \mathrm{M}(64'_p) + \mathrm{M}(15'_p) + \mathrm{M}(1'_p) - \alpha\mathrm{L}(24'_p).$$

Computing ${}^{\mathcal{O}}\mathrm{Ind}_{A_2\times A_2\times A_1}^{E_6}\,{}^{\mathcal{O}}\mathrm{Res}_{A_2\times A_2\times A_1}^{E_6}\mathrm{L}(24_p)$ shows that α must be 0, for otherwise we get an expression with negative leading term (with respect to minimality in the c-order). Computing the character of $\mathrm{L}(24_p)$, we find that $\dim\mathrm{Supp}\,\mathrm{L}(24_p)=4$.

There are three missing entries in the decomposition matrix for row 64_p. The first comes for free from Lemmas 15 and 16 now that the decomposition numbers in row 20_s are known: $[\mathrm{M}(64_p):\mathrm{L}(20_s)]=1$. Next, applying parabolic restriction-induction, we consider the following module character:

$$\begin{aligned}M:=&\frac{1}{2}\left({}^{\mathcal{O}}\mathrm{Ind}_{A_2\times A_2\times A_1}^{E_6}\,{}^{\mathcal{O}}\mathrm{Res}_{A_2\times A_2\times A_1}^{E_6}\mathrm{L}(24p)\right)-2\mathrm{L}(24_p)-2\mathrm{L}(20_s)\\ =&\,\mathrm{M}(64_p)-\mathrm{M}(24_p)-\mathrm{M}(60_p)+\mathrm{M}(60_s)-\mathrm{M}(20_s)+\mathrm{M}(10_s)-2\mathrm{M}(24'_p)-2\mathrm{M}(60'_p)\\ &+3\mathrm{M}(64'_p)-\mathrm{M}(30'_p)+\mathrm{M}(15'_q)-2\mathrm{M}(15'_p)-\mathrm{M}(20'_p)+2\mathrm{M}(6'_p)-\mathrm{M}(1'_p).\end{aligned}$$

This character coincides with the character of $\mathrm{L}(64_p)$ up through $\mathrm{M}(20_s)$. So for some $a,b\geq 0$, $\mathrm{L}(64_p)=M-a\mathrm{L}(10_s)-b\mathrm{L}(24'_p)$. Then ${}^{\mathcal{O}}\mathrm{Res}_{A_2\times A_2\times A_1}^{E_6}\left(M-a\mathrm{L}(10_s)\right)=-a\mathrm{L}(1_p)+$ ... implies $a=0$, and ${}^{\mathcal{O}}\mathrm{Ind}_{A_2\times A_2\times A_1}^{E_6}\,{}^{\mathcal{O}}\mathrm{Res}_{A_2\times A_2\times A_1}^{E_6}\left(M-b\mathrm{L}(24'_p)\right)=-b\mathrm{L}(6_p)+\ldots$ implies $b=0$. Therefore $M=\mathrm{L}(64_p)$, and computing its character, $\dim\mathrm{Supp}\,\mathrm{L}(64_p)=4$. It follows that $[\mathrm{M}(64_p):\mathrm{L}(10_s)]=0$ and $[\mathrm{M}(64_p):\mathrm{L}(24'_p)]=1$.

Since $\mathrm{L}(15_p)$ has already been determined and all rows of the decomposition matrix below row 15_p are now known, $[\mathrm{M}(15_p):\mathrm{L}(20_s)]=1$ and $[\mathrm{M}(15_p):\mathrm{L}(10_s)]=[\mathrm{M}(15_p):\mathrm{L}(24'_p)]=0$.

Row 30_p of the decomposition matrix has five missing entries. The first three, $[\mathrm{M}(30_p):\mathrm{L}(24_p)]$, $[\mathrm{M}(30_p):\mathrm{L}(60_p)]$, and $[\mathrm{M}(30_p):\mathrm{L}(20_s)]$ are easily recovered by parabolically inducing $\mathrm{L}(3^2)$ from $\mathcal{O}_{\frac{1}{3}}(A_5)$ to $\mathcal{O}_{\frac{1}{3}}(E_6)$.

$$\begin{aligned}{}^{\mathcal{O}}\mathrm{Ind}_{A_5}^{E_6}\mathrm{L}(3^2)=&\,\mathrm{M}(30_p)+\mathrm{M}(15_q)-\mathrm{M}(64_p)+\mathrm{M}(24_p)-\mathrm{M}(60_p)+\mathrm{M}(80_s)+\mathrm{M}(20_s)+2\mathrm{M}(10_s)\\ &+2\mathrm{M}(24'_p)-2\mathrm{M}(60'_p)-\mathrm{M}(64'_p)+2\mathrm{M}(30'_p)+2\mathrm{M}(15'_q)-\mathrm{M}(20'_p)+\mathrm{M}(1'_p).\end{aligned}$$

This module contains $\mathrm{L}(30_p)$ and $\mathrm{L}(15_q)$ in its composition series. The first few terms of $\mathrm{L}(15_q)$ (up through the coefficient of $\mathrm{M}(20_s)$) are already known: $\mathrm{L}(15_q)=\mathrm{M}(15_q)-\mathrm{M}(60_p)+\mathrm{M}(80_s)\ldots$, and subtracting off this expression for $\mathrm{L}(15_q)$ reveals that the character of $\mathrm{L}(30_p)$ through the coefficient of $\mathrm{M}(20_s)$ is $\mathrm{L}(30_p)=\mathrm{M}(30_p)-\mathrm{M}(64_p)+\mathrm{M}(24_p)+\mathrm{M}(20_s)\ldots$ There cannot be any other composition factors of ${}^{\mathcal{O}}\mathrm{Ind}_{A_5}^{E_6}\mathrm{L}(3^2)$ until 10_s since there are no more simple modules of (at most) 2-dimensional support until then, and $\dim\mathrm{Supp}\,{}^{\mathcal{O}}\mathrm{Ind}_{A_5}^{E_6}\mathrm{L}(3^2)=2$. So $[\mathrm{M}(30_p):\mathrm{L}(24_p)]=0$, $[\mathrm{M}(30_p):\mathrm{L}(60_p)]=1$, and $[\mathrm{M}(30_p):\mathrm{L}(20_s)]=0$.

Row 15_q has two missing entries: $\alpha=[\mathrm{M}(15_q):\mathrm{L}(10_s)]$ and $\beta=[\mathrm{M}(15_q):\mathrm{L}(24'_p)]$. We set $\alpha=[\mathrm{M}(15_q):\mathrm{L}(10_s)]$ and $\beta=[\mathrm{M}(15_q):\mathrm{L}(24'_p)]$, and write:

$$\mathrm{L}(15_q)=\mathrm{M}(15_q)-\mathrm{M}(60_p)+\mathrm{M}(80_s)+\mathrm{M}(10_s)-\mathrm{M}(60'_p)+\mathrm{M}(15'_q)-\alpha\mathrm{L}(10_s)-\beta\mathrm{L}(24'_p).$$

Likewise, we set $\gamma = [\mathrm{M}(30_p) : \mathrm{L}(10_s)]$ and $\delta = [\mathrm{M}(30_p) : \mathrm{L}(24'_p)]$ and write:

$$\mathrm{L}(30_p) = \mathrm{M}(30_p) - \mathrm{M}(64_p) + \mathrm{M}(24_p) + \mathrm{M}(20_s) - \mathrm{M}(64'_p) + \mathrm{M}(30'_p) - \gamma\mathrm{L}(10_s) - \delta\mathrm{L}(24'_p).$$

Then

$$\begin{aligned}{}^{\mathcal{O}}\mathrm{Res}^{E_6}_{A_5}\mathrm{L}(15_q) &= \mathrm{L}(6) + \mathrm{L}(3^2) - (\alpha+\beta)\mathrm{L}(3^2),\\ {}^{\mathcal{O}}\mathrm{Res}^{E_6}_{A_5}\mathrm{L}(30_p) &= \mathrm{L}(6) + \mathrm{L}(3^2) - (\gamma+\delta)\mathrm{L}(3^2).\end{aligned}$$

It follows that $\alpha+\beta \leq 1$ and $\gamma+\delta \leq 1$.

The parabolic subgroup $A_2 \times A_2 \times A_1 \subset E_6$ contains some information about $\mathrm{L}(15_q)$ and $\mathrm{L}(30_p)$. The parabolic restrictions of the characters of $\mathrm{L}(10_s)$ and $\mathrm{L}(24'_p)$ do not contain any standard modules in common. The characters are identical in the first two components $A_2 \times A_2$ but whereas the character of ${}^{\mathcal{O}}\mathrm{Res}^{E_6}_{A_2\times A_2\times A_1}\mathrm{L}(10_s)$ only contains terms labeled with $(\lambda,\mu,(2))$, the character of ${}^{\mathcal{O}}\mathrm{Res}^{E_6}_{A_2\times A_2\times A_1}\mathrm{L}(24'_p)$ only contains terms labeled with $(\lambda,\mu,(1^2))$. For the rest of this section, let

$$\mathbf{1} = {}^{\mathcal{O}}\mathrm{Res}^{E_6}_{A_2\times A_2\times A_1}\mathrm{L}(10_s), \qquad \mathbf{2} = {}^{\mathcal{O}}\mathrm{Res}^{E_6}_{A_2\times A_2\times A_1}\mathrm{L}(24'_p)$$

Then ${}^{\mathcal{O}}\mathrm{Res}^{E_6}_{A_2\times A_2\times A_1}\mathrm{L}(30_p) = (1-\gamma)\mathbf{1} + (1-\delta)\mathbf{2}$ and ${}^{\mathcal{O}}\mathrm{Res}^{E_6}_{A_2\times A_2\times A_1}\mathrm{L}(15_q) = (1-\alpha)\mathbf{1} + (1-\beta)\mathbf{2}$.

Parabolically inducing $\mathbf{2}$ up to $\mathcal{O}_{\frac{1}{3}}(E_6)$ and then parabolically restricting it back down to $\mathcal{O}_{\frac{1}{3}}(A_2\times A_2\times A_1)$ will force one of ${}^{\mathcal{O}}\mathrm{Res}^{E_6}_{A_2\times A_2\times A_1}\mathrm{L}(15_q)$, ${}^{\mathcal{O}}\mathrm{Res}^{E_6}_{A_2\times A_2\times A_1}\mathrm{L}(30_p)$ to be equal to $\mathbf{2}$. We compute:

$$\begin{aligned}{}^{\mathcal{O}}\mathrm{Ind}^{E_6}_{A_2\times A_2\times A_1}\mathbf{2} = {} & \mathrm{M}(6_p) - \mathrm{M}(20_p) + \mathrm{M}(30_p) + 2\mathrm{M}(15_q) - \mathrm{M}(15_p) + \mathrm{M}(24_p) - 2\mathrm{M}(60_p)\\ & + \mathrm{M}(80_s) + \mathrm{M}(20_s) + 2\mathrm{M}(10_s) + 2\mathrm{M}(24'_p) - \mathrm{M}(60'_p) - 2\mathrm{M}(64'_p)\\ & + 2\mathrm{M}(30'_p) + \mathrm{M}(15'_q) + \mathrm{M}(15'_p) - \mathrm{M}(6'_p) + \mathrm{M}(1'_p).\end{aligned}$$

This module contains $\mathrm{L}(6_p)$ in its composition series, but the character of $\mathrm{L}(6_p)$ begins $\mathrm{L}(6_p) = \mathrm{M}(6_p) - \mathrm{M}(20_p) - \mathrm{M}(30_p)...$, so $2\mathrm{L}(30_p)$ also belongs to the composition series of ${}^{\mathcal{O}}\mathrm{Ind}^{E_6}_{A_2\times A_2\times A_1}\mathbf{2}$. If we write $\epsilon = [\mathrm{M}(6_p) : \mathrm{L}(15_q)]$ then $1-\epsilon$ is the coefficient of $\mathrm{M}(15_q)$ in the character of $\mathrm{L}(6_p)$. But $2\mathrm{M}(15_q)$ appears in ${}^{\mathcal{O}}\mathrm{Ind}^{E_6}_{A_2\times A_2\times A_1}\mathbf{2}$, and the coefficient of $\mathrm{M}(15_q)$ in the character of $\mathrm{L}(30_p)$ is equal to 0. Therefore $\mathrm{L}(15_q)$ is also a composition factor (possibly with multiplicity) of ${}^{\mathcal{O}}\mathrm{Ind}^{E_6}_{A_2\times A_2\times A_1}\mathbf{2}$. Parabolically restricting the latter back down yields

$$\mathrm{Res}^{E_6}_{A_2\times A_2\times A_1}\mathrm{Ind}^{E_6}_{A_2\times A_2\times A_1}\mathbf{2} = 2\cdot\mathbf{1} + 4\cdot\mathbf{2}.$$

Consequently, at least one of $\mathrm{Res}\mathrm{L}(15_q), \mathrm{Res}\mathrm{L}(30_p)$ is $\mathbf{2}$ rather than $\mathbf{1}$ or $\mathbf{1}+\mathbf{2}$. So at least one of α,γ equals 1. We note that since $\mathrm{L}(15_q)$ and $\mathrm{L}(30_p)$ are composition factors of $\mathrm{Ind}\mathbf{2}$, a module of support 2-dimensional support, and moreover, they both restrict to modules of 1-dimensional support in a maximal parabolic, we have established that $\dim \operatorname{Supp} \mathrm{L}(15_q) = \dim \operatorname{Supp} \mathrm{L}(30_p) = 2$.

It is now possible to prove that $\mathrm{L}(6_p)$ is finite-dimensional, and find its graded dimension. To do this, subtract off the expression for $2\mathrm{L}(30_p)+\mathrm{L}(15_q)$ up through the coefficient of $\mathrm{M}(20_s)$ from ${}^{\mathcal{O}}\mathrm{Ind}_{A_2\times A_2\times A_1}^{E_6}\mathbf{2}$, and call the resulting character M:

$$\begin{aligned} M = {} & \mathrm{M}(6_p) - \mathrm{M}(20_p) - \mathrm{M}(30_p) + \mathrm{M}(15_q) - \mathrm{M}(15_p) + 2\mathrm{M}(64_p) - \mathrm{M}(24_p) - \mathrm{M}(60_p) \\ & - \mathrm{M}(20_s)... \end{aligned}$$

Up through $\mathrm{M}(20_s)$, M coincides with $\mathrm{L}(6_p) + a\mathrm{L}(15_q)$, where $a := [\mathrm{M}(6_p) : \mathrm{L}(15_q)]$, by the same dimension of support and restriction arguments given in the previous paragraph. However, if $a > 0$ then the dimensions of the graded pieces of $\mathrm{L}(6_p)$ fail to satisfy the inequality $\dim \mathrm{L}(6_p)[-1] \leq \dim \mathrm{L}(6_p)[1]$. Therefore $a = 0$, and M coincides with the character of $\mathrm{L}(6_p)$ up through the term $\mathrm{M}(20_s)$, which occurs in degree 3. The dimensions of graded pieces of $\mathrm{L}(6_p)$ in degrees -2 through 2 are:

$$\begin{aligned} \dim \mathrm{L}(6_p)[-2] = \dim \mathrm{L}(6_p)[2] &= 216 \\ \dim \mathrm{L}(6_p)[-1] = \dim \mathrm{L}(6_p)[1] &= 306 \\ \dim \mathrm{L}(6_p)[0] &= 340 \end{aligned}$$

By Lemma 11 we deduce that $\mathrm{L}(6_p)$ is finite-dimensional. Computing the dimensions of graded pieces in degrees $[-5]$, $[-4]$, and $[-3]$ as well, we obtain:

$$\dim \mathrm{L}(6_p) = 2\,(6 + 36 + 106 + 216 + 306) + 340 = 1680$$

and completing the graded dimension formula for $\mathrm{L}(6_p)$:

$$\chi_{\mathrm{L}(6_p)}(t) = 6(t^{-5} + t^5) + 36(t^{-4} + t^4) + 106(t^{-3} + t^3) + 216(t^{-2} + t^2) + 306(t^{-1} + t) + 340$$

When $\chi_{\mathrm{L}(6_p)}(t)$ is expressed as a rational function with denominator $(1-t)^6$, the coefficient of t^3 in the numerator is 0. This forces the coefficient of $\mathrm{M}(10_s)$ in the character of $\mathrm{L}(6_p)$ to be 2. Then by Lemma 13,

$$\begin{aligned} \mathrm{L}(6_p) = {} & \mathrm{M}(6_p) - \mathrm{M}(20_p) - \mathrm{M}(30_p) + \mathrm{M}(15_q) - \mathrm{M}(15_p) + 2\mathrm{M}(64_p) - \mathrm{M}(24_p) - \mathrm{M}(60_p) \\ & - \mathrm{M}(20_s) + 2\mathrm{M}(10_s) - \mathrm{M}(24_p') - \mathrm{M}(60_p') + 2\mathrm{M}(64_p') - \mathrm{M}(30_p') + \mathrm{M}(15_q') \\ & - \mathrm{M}(15_p') - \mathrm{M}(20_p') + \mathrm{M}(6_p'). \end{aligned}$$

This character for $\mathrm{L}(6_p)$ then implies the decomposition numbers $[\mathrm{M}(6_p) : \mathrm{L}(64_p)] = 1$ (and $[\mathrm{M}(6_p) : \mathrm{L}(15_q)] = 0$ was recovered above). The rest of the missing decomposition numbers in this row depend on those in row 20_p, which haven't been found yet.

To understand $\mathrm{L}(20_p)$ we study parabolic induction and restriction with respect to categories $\mathcal{O}_{\frac{1}{3}}(W)$ for parabolic subgroups $W \subset E_6$, starting with $W = D_5$. Irreducible representations of D_5 are labeled by bipartitions $(\lambda^1.\lambda^2)$ of 5, up to swapping λ^1 and λ^2. First of all, we observe that ${}^{\mathcal{O}}\mathrm{Res}_{D_5}^{E_6}\mathrm{L}(24_p) = \mathrm{L}(41.\emptyset)$, and that ${}^{\mathcal{O}}\mathrm{Ind}_{D_5}^{E_6}\,{}^{\mathcal{O}}\mathrm{Res}_{D_5}^{E_6}\mathrm{L}(24_p) =$

${}^{\mathcal{O}}\mathrm{Ind}_{D_5}^{E_6}\mathrm{L}(41.\emptyset) = \mathrm{M}(20_p) - \mathrm{M}(15_q) + \mathrm{M}(64_p) + \mathrm{M}(24_p) - \mathrm{M}(60_p) - \mathrm{M}(60_s) + \mathrm{M}(20'_p) + \mathrm{M}(6'_p) + \mathrm{M}(1'_p)$ is a module containing $\mathrm{L}(20_p)$ in its composition series and whose support is 4-dimensional. This implies that $\dim \mathrm{Supp}\,\mathrm{L}(20_p) \leq 4$. So now the dimensions of supports of all irreducible modules in $\mathcal{O}_{\frac{1}{3}}(E_6)$ modules are known. We have found that there are four dimensions of support, namely 0, 2, 4, and 6. This implies that $\mathcal{O}_{\frac{1}{3}}(D_5)$ has no modules of even-dimensional support, as otherwise, inducing such a module would produce a module of odd-dimensional support in $\mathcal{O}_{\frac{1}{3}}(E_6)$. On the other hand, it is easy to check that if $W \subseteq D_5$ is a parabolic subgroup of rank greater than 2, then $\mathcal{O}_{\frac{1}{3}}(W)$ contains no finite-dimensional representation. This can be verified for most such W by checking that $h_{\frac{1}{3}}(\mathrm{Triv}_W) > 0$ where Triv_W is the trivial representation of W. Since $h_c(\mathrm{Triv}_W)$ is minimal over $h_c(\tau)$ for all $\tau \in \mathrm{Irr}\,W$, this implies the claim for such W by Lemma 4. In the case $W = A_3 \times A_1$, $\mathcal{O}_{\frac{1}{3}}(W) \simeq \mathcal{O}_{\frac{1}{3}}(A_3) \otimes \mathcal{O}_{\frac{1}{3}}(A_1)$ has no finite-dimensional representations by [2]. On the other hand, $\mathrm{L}(\mathrm{Triv}_{A_2}) \in \mathcal{O}_{\frac{1}{3}}(A_2)$ is finite-dimensional because 3 is the Coxeter number of A_2 [2]. It follows from [3, Corollary 3.24] that the dimensions of support occuring for modules in $\mathcal{O}_{\frac{1}{3}}(D_5)$ are 3 and 5.

It turns out that in $\mathcal{O}_{\frac{1}{3}}(D_5)$, there are five irreducible representations of minimal support, and they all have characters containing just three standard modules with alternating signs. Each such expression comprises all the standard modules in its block, which is of defect 1 and only contains three isomorphism classes of simple modules. Each such minimal support simple module is given by restricting a unique one of the five irreducible representations with 4-dimensional support in $\mathcal{O}_{\frac{1}{3}}(E_6)$:

$$\begin{aligned}
{}^{\mathcal{O}}\mathrm{Res}_{D_5}^{E_6}\mathrm{L}(20_s) &= \mathrm{L}(3.1^2) = \mathrm{M}(3.1^2) - \mathrm{M}(21.1^2) + \mathrm{M}(1^3.1^2)\\
{}^{\mathcal{O}}\mathrm{Res}_{D_5}^{E_6}\mathrm{L}(60_p) &= \mathrm{L}(3.2) = \mathrm{M}(3.2) - \mathrm{M}(21.2) + \mathrm{M}(1^3.2)\\
{}^{\mathcal{O}}\mathrm{Res}_{D_5}^{E_6}\mathrm{L}(24_p) &= \mathrm{L}(41.\emptyset) = \mathrm{M}(41.\emptyset) - \mathrm{M}(32.\emptyset) + \mathrm{M}(1^5.\emptyset)\\
{}^{\mathcal{O}}\mathrm{Res}_{D_5}^{E_6}\mathrm{L}(64_p) &= \mathrm{L}(4.1) = \mathrm{M}(4.1) - \mathrm{M}(2^2.1) + \mathrm{M}(1^4.1)\\
{}^{\mathcal{O}}\mathrm{Res}_{D_5}^{E_6}\mathrm{L}(20_p) &= \mathrm{L}(5.\emptyset) = \mathrm{M}(5.\emptyset) - \mathrm{M}(2^21.\emptyset) + \mathrm{M}(21^3.\emptyset)
\end{aligned}$$

The first four equations follow by direct calculation using the results found earlier in the proof. The fifth equation follows by process of elimination. The formula for $\mathrm{L}(5.\emptyset)$ we accept as something known, e.g. by the classification of defect 1 blocks by Rouquier [29].

Restricting ${}^{\mathcal{O}}\mathrm{Ind}_{D_5}^{E_6}\mathrm{L}(41.\emptyset)$ back down to $\mathcal{O}_{\frac{1}{3}}(D_5)$, we find:

$$\begin{aligned}
{}^{\mathcal{O}}\mathrm{Res}_{D_5}^{E_6}\,{}^{\mathcal{O}}\mathrm{Ind}_{D_5}^{E_6}\mathrm{L}(41.\emptyset) &= \mathrm{L}(5.\emptyset) + 3\mathrm{L}(41.\emptyset) + 2\mathrm{L}(4.1) + \mathrm{L}(3.1^2)\\
&= {}^{\mathcal{O}}\mathrm{Res}_{D_5}^{E_6}\mathrm{L}(20_p) + 3\,{}^{\mathcal{O}}\mathrm{Res}_{D_5}^{E_6}\mathrm{L}(24_p) + 2\,{}^{\mathcal{O}}\mathrm{Res}_{D_5}^{E_6}\mathrm{L}(64_p) + {}^{\mathcal{O}}\mathrm{Res}_{D_5}^{E_6}\mathrm{L}(20_s).
\end{aligned}$$

It follows that

$$\mathrm{L}(20_p) = {}^{\mathcal{O}}\mathrm{Ind}_{D_5}^{E_6}\mathrm{L}(41.\emptyset) - 3\mathrm{L}(24_p) - 2\mathrm{L}(64_p) - \mathrm{L}(20_s) - \text{modules of smaller support.}$$

Since $\mathrm{L}(10_s)$, $\mathrm{L}(24'_p)$, $\mathrm{L}(15_q)$, $\mathrm{L}(15_p)$, $\mathrm{L}(30_p)$ have 0- or 2-dimensional support, their restriction to $\mathcal{O}_{\frac{1}{3}}(D_5)$ is 0 and they could be composition factors of ${}^{\mathcal{O}}\mathrm{Ind}_{D_5}^{E_6}\mathrm{L}(41.\emptyset)$. (The modules $\mathrm{L}(6_p)$ and $\mathrm{L}(1_p)$ cannot be composition factors because $h_c(6_p), h_c(1_p) < h_c(20_p)$ and $\mathrm{M}(20_p)$ is the standard module of smallest h_c-weight appearing in ${}^{\mathcal{O}}\mathrm{Ind}_{D_5}^{E_6}\mathrm{L}(41.\emptyset)$). However, from the entries of row 20_p in the decomposition matrix that we have found so far, we see that none of $\mathrm{L}(15_q)$, $\mathrm{L}(15_p)$, $\mathrm{L}(30_p)$ is a composition factor of ${}^{\mathcal{O}}\mathrm{Ind}_{D_5}^{E_6}\mathrm{L}(41.\emptyset)$). Therefore we may write

$$\mathrm{L}(20_p) = {}^{\mathcal{O}}\mathrm{Ind}_{D_5}^{E_6}\mathrm{L}(41.\emptyset) - 3\mathrm{L}(24_p) - 2\mathrm{L}(64_p) - \mathrm{L}(20_s) - a\mathrm{L}(10_s) - b\mathrm{L}(24'_p)$$

for some $a, b \geq 0$. To determine a and b, take the right-hand-side of the equation above, restrict it to A_5, then induce the result back up to E_6 to get $\mathrm{M}(6_p) + 2\mathrm{M}(20_p) - (a + b + 1)\mathrm{M}(30_p)$... Since the coefficient of $\mathrm{M}(30_p)$ is 1 in the characters of both $\mathrm{L}(20_p)$ and $\mathrm{L}(6_p)$, this is not the character of a module unless $a = b = 0$. Therefore $\mathrm{L}(20_p) = {}^{\mathcal{O}}\mathrm{Ind}_{D_5}^{E_6}\mathrm{L}(41.\emptyset) - 3\mathrm{L}(24_p) - 2\mathrm{L}(64_p) - \mathrm{L}(20_s)$, and consequently:

$$\begin{aligned}\mathrm{L}(20_p) = \mathrm{M}(20_p) &- \mathrm{M}(15_q) - \mathrm{M}(64_p) + \mathrm{M}(60_p) + \mathrm{M}(20_s) - 2\mathrm{L}(10_s) + \mathrm{M}(24'_p) + \mathrm{M}(60'_p)\\ &- 2\mathrm{M}(64'_p) + \mathrm{M}(30'_p) - 2\mathrm{M}(15'_q) + \mathrm{M}(15'_p) + 2\mathrm{M}(20'_p) - 2\mathrm{M}(6'_p).\end{aligned}$$

There is enough information now to determine the character of $\mathrm{L}(1_p)$. As above, let $\mathbf{1}$ denote the module $\mathrm{Res}^{E_6}_{A_2\times A_2\times A_1}\mathrm{L}(10_s)$ and $\mathbf{2}$ the module $\mathrm{Res}^{E_6}_{A_2\times A_2\times A_1}\mathrm{L}(24'_p)$. Consider

$$\begin{aligned}{}^{\mathcal{O}}\mathrm{Ind}^{E_6}_{A_2\times A_2\times A_1}\mathbf{1} = \mathrm{M}(1_p) &- \mathrm{M}(6_p) + 2\mathrm{M}(30_p) + \mathrm{M}(15_q) + \mathrm{M}(15_p) - 2\mathrm{M}(64_p) + 2\mathrm{M}(24_p)\\ &- \mathrm{M}(60_p) + \mathrm{M}(80_s) + \mathrm{M}(20_s) + 2\mathrm{M}(10_s) + \mathrm{M}(24'_p) - 2\mathrm{M}(60'_p) + \mathrm{M}(30'_p)\\ &+ 2\mathrm{M}(15'_q) - \mathrm{M}(15'_p) - \mathrm{M}(20'_p) + \mathrm{M}(6'_p).\end{aligned}$$

The dimension of support of ${}^{\mathcal{O}}\mathrm{Ind}^{E_6}_{A_2\times A_2\times A_1}\mathbf{1}$ is 2. Its possible composition factors are the finite-dimensional representations $\mathrm{L}(1_p)$, $\mathrm{L}(6_p)$, and $\mathrm{L}(15_p)$, and the representations with 2-dimensional support $\mathrm{L}(30_p)$, $\mathrm{L}(15_q)$, $\mathrm{L}(10_s)$, and $\mathrm{L}(24'_p)$. Since $[\mathrm{M}(1_p) : \mathrm{L}(6_p)] = 1$, $\mathrm{L}(6_p)$ is not a composition factor. Furthermore, we see that $d^{-1}_{1_p,20_p} \leq 0$. But if $d^{-1}_{1_p,20_p} < 0$ then $\mathrm{L}(20_p)$ is a composition factor, contradicting the bound on dimension of supports. So $d^{-1}_{1_p,20_p} = 0$ and hence $[\mathrm{M}(1_p) : \mathrm{L}(20_p)] = 1$. We also observe that $d^{-1}_{1_p,15_p} \leq 1$.

The dimension of $\mathrm{L}(1_p)$ was calculated to be 4152 by Oblomkov and Yun in [27, Section 9.10], using the geometry of affine Springer fibers. On the other hand, by counting up dimensions of graded pieces, all of which except the ones in degrees -1 and 0 are known now, the dimension of $\mathrm{L}(1_p)$ is

$$\begin{aligned}\dim \mathrm{L}(1_p) &= 2\left(\sum_{n=h_{\frac{1}{3}}(1_p)}^{-1} \dim \mathrm{L}(1_p)[n]\right) + \dim \mathrm{L}(1_p)[0] \\ &= 2\left(\sum_{n=0}^{8} \binom{n+5}{5} - 6\binom{n+1}{5}\right) + 2\left(d^{-1}_{1_p,30_p}30 + d^{-1}_{1_p,15_q}15 + d^{-1}_{1_p,15_p}15\right) \\ &\qquad + \binom{14}{5} - 6\binom{10}{5} + \left(d^{-1}_{1_p,30_p}30 + d^{-1}_{1_p,15_q}15 + d^{-1}_{1_p,15_p}15\right)\binom{6}{5} + d^{-1}_{1_p,64_p}64.\end{aligned}$$

The character of ${}^{\mathcal{O}}\mathrm{Ind}_{A_2\times A_2\times A_1}^{E_6}\mathbf{1}$ imposes the bounds $d^{-1}_{1_p,30_p} \leq 2$, $d^{-1}_{1_p,15_q} \leq 1$, and $d^{-1}_{1_p,15_p} \leq 1$. Setting $\dim \mathrm{L}(1_p) = 4152$ then forces $d^{-1}_{1_p,30_p} = 1$, $d^{-1}_{1_p,15_q} = -1$, $d^{-1}_{1_p,15_p} = 1$, $d^{-1}_{1_p,64_p} = -1$.

As composition factors of ${}^{\mathcal{O}}\mathrm{Ind}_{A_2\times A_2\times A_1}^{E_6}\mathbf{1}$, we observe that $\mathrm{L}(30_p)$ occurs exactly once, $\mathrm{L}(15_q)$ occurs twice, and $\mathrm{L}(15_p)$ does not occur. Truncating characters at the term 20_s and subtracting $2\mathrm{L}(15_q) + \mathrm{L}(30_q)$ from ${}^{\mathcal{O}}\mathrm{Ind}_{A_2\times A_2\times A_1}^{E_6}\mathbf{1}$ yields the character of $\mathrm{L}(1_p)$ through $\mathrm{M}(20_s)$:

$$\mathrm{L}(1_p) = \mathrm{M}(1_p) - \mathrm{M}(6_p) + \mathrm{M}(30_p) - \mathrm{M}(15_q) + \mathrm{M}(15_p) - \mathrm{M}(64_p) + \mathrm{M}(24_p) + \mathrm{M}(60_p) - \mathrm{M}(80_s)...$$

Now we multiply the graded character of $\mathrm{L}(1_p)$ by $(1-t)^6/(1-t)^6$ and observe that the coefficient of t^3 in the numerator of the resulting expression is equal to -90, that is, $\sum\limits_{h_c(\tau)=3} d_{1_p,\tau} = -90$. We deduce that $d^{-1}_{1_p,10_s} = -1$, since $d^{-1}_{1_p,80_s} = -1$ while $d^{-1}_{1_p,60_s} = d^{-1}_{1_p,20_s} = 0$. Using Lemma 13 now yields the complete character of $\mathrm{L}(1_p)$:

$$\begin{aligned}\mathrm{L}(1_p) &= \mathrm{M}(1_p) - \mathrm{M}(6_p) + \mathrm{M}(30_p) - \mathrm{M}(15_q) + \mathrm{M}(15_p) - \mathrm{M}(64_p) + \mathrm{M}(24_p) + \mathrm{M}(60_p) - \mathrm{M}(80_s) \\ &\quad - \mathrm{M}(10_s) + \mathrm{M}(24'_p) + \mathrm{M}(60'_p) - \mathrm{M}(64'_p) + \mathrm{M}(30'_p) - \mathrm{M}(15'_q) + \mathrm{M}(15'_p) - \mathrm{M}(6'_p) + \mathrm{M}(1'_p).\end{aligned}$$

Using this character, the missing decomposition numbers $[\mathrm{M}(1_p) : \mathrm{L}(\tau)]$ may all be filled in now except for when $\tau = 10_s$ or $24'_p$. They are all 0 except for when $\tau = 20_p$ or 15_q, in which case they are 1.

There remain ten unknown decomposition numbers, lying in the first five rows of columns 10_s and $24'_p$. Some $\tau \in \{30_p, 15_q\}$ satisfies ${}^{\mathcal{O}}\mathrm{Res}_{A_2\times A_2\times A_1}^{E_6}\mathrm{L}(\tau) = \mathbf{2}$. Additionally, ${}^{\mathcal{O}}\mathrm{Res}_{A_2\times A_2\times A_1}^{E_6}\,{}^{\mathcal{O}}\mathrm{Ind}_{A_2\times A_2\times A_1}^{E_6}\mathbf{1} = 4\cdot\mathbf{1} + 2\cdot\mathbf{2}$. It follows that the other element σ of $\{30_p, 15_q\}$ satisfies ${}^{\mathcal{O}}\mathrm{Res}_{A_2\times A_2\times A_1}^{E_6}\mathrm{L}(\sigma) = \mathbf{1}$, as otherwise the coefficient of $\mathbf{2}$ would be at least 3. This means that the two-by-two square where rows 30_p, 15_q intersect columns 10_s, $24'_p$ in the decomposition matrix is either $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ or $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$. By considering the beginning of the character $\mathrm{L}(20_p) = \mathrm{M}(20_p) - \mathrm{M}(15_q) - \mathrm{M}(64_p) + \mathrm{M}(60_p) + \mathrm{M}(20_s) - 2\mathrm{M}(10_s) + \mathrm{M}(24'_p)...$, and likewise the characters of $\mathrm{L}(1_p)$, $\mathrm{L}(6_p)$ up through $\mathrm{M}(24'_p)$, we may determine what the remaining decomposition numbers would be in each case. That computation shows that if the two-by-two square is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ then $[\mathrm{M}(1_p) : \mathrm{L}(24'_p)] < 0$, a contradiction. So the correct option is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and the top 5 rows of columns 10_s and $24'_p$ are:

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

This completes the calculation of the decomposition matrix. $\square$

5. The decomposition matrices of $\mathcal{O}_c(E_7)$

5.1. **Parameter $c = 1/6$.** In this section we study the principal block of $\mathcal{O}_{\frac{1}{6}}(E_7)$. The h_c-weight line for the principal block of $\mathcal{O}_{\frac{1}{6}}(E_7)$ is given in Table 5.1.

Table 21. Weight line for the principal block of $\mathcal{O}_{\frac{1}{6}}(E_7)$

-7	-4	-2	-1	0	1	2	3	4	5	6	7	8	9	11	14
1_a	$7'_a$	$21'_b$	35_b	$105'_a$	210_a	$315'_a$	105_c	$105'_c$	315_a	$210'_a$	105_a	$35'_b$	21_b	7_a	$1'_a$
			21_a	$15'_a$	168_a	$280'_a$	420_a	$420'_a$	280_a	$168'_a$	15_a	$21'_a$			
					105_b	$70'_a$	210_b	$210'_b$	70_a	$105'_b$					
							84_a	$84'_a$							

Theorem 27. *The decomposition matrix of the principal block of $\mathcal{O}_{\frac{1}{6}}(E_7)$ is given by Table 23.*

Proof. First, we import the columns of the Hecke algebra decomposition matrix from [14], and then apply Lemmas 15 and 16. This brings the upper left hand corner of the decomposition matrix containing all unknown entries to the state shown in Table 22. From Table 22 we can identify some supports of simple modules. First, $\mathrm{L}(1_a)$ must be finite-dimensional since 6 is an elliptic number of E_7 [34]. Second, 105_b generates a subrep of $\mathrm{M}(15'_a)$. Then $\dim \mathrm{L}(15'_a)[1] = 15 \cdot 7 - 105 = 0$, so $\mathrm{L}(15'_a)$ is finite-dimensional.

There are two parabolic subgroups $W \subset E_7$ such that $\mathcal{O}_{\frac{1}{6}}(W)$ contains finite-dimensional representations: in $\mathcal{O}_{\frac{1}{6}}(E_6)$ we have $\mathrm{L}(1_p)$ and $\mathrm{L}(6_p)$ by Theorem 24, and in $\mathcal{O}_{\frac{1}{6}}(D_6)$ there is a unique finite-dimensional irreducible representation with lowest weight the trivial representation, denoted $\mathrm{L}(6.\emptyset)$ (a character formula for this representation was conjectured in [27] and proved in [19]). We have $\mathrm{P}(1_p) = \mathrm{M}(1_p)$ and $\mathrm{P}(6_p) = \mathrm{M}(6_p)$ in $\mathcal{O}_{\frac{1}{6}}(E_6)$ and $\mathrm{P}(6.\emptyset) = \mathrm{M}(6.\emptyset)$ in $\mathcal{O}_{\frac{1}{6}}(D_6)$. For $W \subset E_7$ let $\overline{{}^{\mathcal{O}}\mathrm{Ind}_W^{E_7}}$ denote ${}^{\mathcal{O}}\mathrm{Ind}_W^{E_7}$ followed by cutting to the principal block. We compute:

$$\overline{{}^{\mathcal{O}}\mathrm{Ind}_{E_6}^{E_7}}\mathrm{P}(1_p) = \mathrm{P}(21_b'), \quad \overline{{}^{\mathcal{O}}\mathrm{Ind}_{E_6}^{E_7}}\mathrm{P}(6_p) = \mathrm{P}(105_a'), \quad \overline{{}^{\mathcal{O}}\mathrm{Ind}_{D_6}^{E_7}}\mathrm{P}(6.\emptyset) = \mathrm{P}(35_b).$$

It follows that $\dim \mathrm{Supp}\, \mathrm{L}(\tau) = 1$ if and only if $\tau \in \{21_b', 105_a', 35_b\}$.

Now it is easily checked that ${}^{\mathcal{O}}\mathrm{Ind}_{E_6}^{E_7}\mathrm{L}(1_p)$ contains $\mathrm{L}(7_a')$ in its composition series, because $\mathrm{M}(7_a')$ appears in its character and there does not exist $\tau \in \mathrm{Irr}\, E_7$ with $h_c(\tau) < h_c(7_a')$ and $[\mathrm{M}(\tau) : \mathrm{L}(7_a')] > 0$. It follows that $\dim \mathrm{Supp}\, \mathrm{L}(7_a') \leq 1$. Since $\dim \mathrm{Supp}\, \mathrm{L}(7_a') \neq 1$, we deduce that $\mathrm{L}(7_a')$ is finite-dimensional. A similar argument inducing $\mathrm{L}(6_p) \in \mathcal{O}_{\frac{1}{6}}(E_6)$ shows that $\mathrm{L}(21_a)$ is finite-dimensional.

TABLE 22. Preliminary approximation of the decomposition matrix of the principal block of $\mathcal{O}_{\frac{1}{6}}(E_7)$

$$\begin{array}{cc}
\star & 1_a \\
\star? & 7_a' \\
\bullet & 21_b' \\
\bullet & 35_b \\
\star? & 21_a \\
\bullet & 105_a' \\
\star & 15_a' \\
\bullet & 210_a \\
\bullet & 168_a \\
\bullet & 105_b \\
 & 315_a' \\
 & 280_a' \\
\bullet & 70_a' \\
\bullet & 105_c \\
 & 420_a \\
 & 210_b \\
\bullet & 84_a \\
\bullet & 105_c'
\end{array}
\left(\begin{matrix}
1 & \cdot & 1 & 1 & \cdot & \cdot & 1 & ? & ? & ? & \cdot & \cdot & ? & ? & \cdot & 1 & ? & ? \\
\cdot & 1 & 1 & 1 & \cdot & 1 & \cdot & ? & ? & 1 & 1 & \cdot & ? & ? & \cdot & \cdot & ? & ? \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & 1 & \cdot & \cdot & ? & \cdot & 1 & ? & ? \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & \cdot & 1 & ? & \cdot & 1 & ? & ? \\
\cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & ? & ? & \cdot & \cdot & 1 & \cdot & ? & \cdot & \cdot & ? & ? \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & 1 & \cdot & 1 & 1 & \cdot & ? & 1 & \cdot & ? & ? \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & ? & \cdot & 1 & \cdot & ? \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & 1 & \cdot & ? & 1 & \cdot & \cdot & ? \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & ? & 1 & 1 & 1 & ? \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & ? & \cdot & 1 & \cdot & ? \\
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\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{matrix}\right)$$

TABLE 23. Decomposition matrix of the principal block of $\mathcal{O}_{\frac{1}{6}}(E_7)$

$$
\begin{array}{cc}
\star & 1_a \\
\star & 7'_a \\
\texttt{(1)} & 21'_b \\
\texttt{(1)} & 35_b \\
\star & 21_a \\
\texttt{(1)} & 105'_a \\
\star & 15'_a \\
\texttt{(3)} & 210_a \\
\texttt{(3)} & 168_a \\
\texttt{(2)} & 105_b \\
 & 315'_a \\
 & 280'_a \\
\texttt{(2)} & 70'_a \\
\texttt{(2)} & 105_c \\
 & 420_a \\
 & 210_b \\
\texttt{(3)} & 84_a \\
\texttt{(3)} & 105'_c \\
 & 420'_a \\
 & 210'_b \\
 & 84'_a \\
 & 315_a \\
 & 280_a \\
 & 70_a \\
 & 210'_a \\
 & 168'_a \\
 & 105'_b \\
 & 105_a \\
 & 15_a \\
 & 35'_b \\
 & 21'_a \\
 & 21_b \\
 & 7_a \\
 & 1'_a
\end{array}
\left(\begin{array}{cccccccccccccccccccccccccccccccccc}
1 & \cdot & 1 & 1 & \cdot & \cdot & 1 & \cdot & 1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & 1 & 1 & \cdot & 1 & \cdot & 1 & 1 & 1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
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\end{array}\right)
$$

Next, we induce the projective cover $\mathrm{P}(15_q) = \mathrm{M}(15_q) + \mathrm{M}(1_p)$ of the dimension 1 support simple $\mathrm{L}(15_q) \in \mathcal{O}_{\frac{1}{6}}(E_6)$, and then cut to the principal block (by throwing out all $\mathrm{M}(\tau)$ belonging to other blocks). This yields:

$$\overline{{}^{\mathcal{O}}\mathrm{Ind}_{E_6}^{E_7}}\mathrm{P}(15_q) = \mathrm{M}(105_b) + \mathrm{M}(35_b) + \mathrm{M}(15'_a) + \mathrm{M}(21'_b) + \mathrm{M}(7'_a) + \mathrm{M}(1_a).$$

Since $\dim \mathrm{Supp}\,\mathrm{L}(\tau) < 2$ for all τ such that $[\overline{{}^{\mathcal{O}}\mathrm{Ind}_{E_6}^{E_7}}\mathrm{P}(15_q) : \mathrm{M}(\tau)] \neq 0$, we deduce that $\overline{{}^{\mathcal{O}}\mathrm{Ind}_{E_6}^{E_7}}\mathrm{P}(15_q) = \mathrm{P}(105_b)$, $\dim \mathrm{Supp}\,\mathrm{L}(105_b) = 2$, and $[\mathrm{M}(1_a) : \mathrm{L}(105_b)] = 1$. Similarly, $\overline{{}^{\mathcal{O}}\mathrm{Ind}_{D_6}^{E_7}}\mathrm{P}(3.3-) = \mathrm{P}(105_b)$.

Irreducible representations of D_6 are labeled by bipartitions $(\lambda^1.\lambda^2)$ of 6 up to swapping λ^1 and λ^2 when $\lambda^1 \neq \lambda^2$, and when $\lambda^1 = \lambda^2 = \lambda$, a partition of 3, there are two irreducible representations of D_6 labeled $(\lambda.\lambda+)$ and $(\lambda.\lambda-)$. In $\mathcal{O}_{\frac{1}{6}}(D_6)$ we have the formula $\mathrm{P}(3.3-) = \mathrm{M}(3.3-) + \mathrm{M}(6.\emptyset)$. Inducing and cutting to the principal block yields

$$\begin{aligned}\overline{{}^{\mathcal{O}}\mathrm{Ind}_{D_6}^{E_7}}\mathrm{P}(3.3-) = {} & \mathrm{M}(105_c) + \mathrm{M}(70'_a) + \mathrm{M}(315'_a) + \mathrm{M}(210_a) \\ & + 2\mathrm{M}(35_b) + \mathrm{M}(7'_a) + \mathrm{M}(1_a).\end{aligned}$$

Since $\mathrm{L}(3.3-)$ has 1-dimensional support, the projective summands of $Q := \overline{{}^{\mathcal{O}}\mathrm{Ind}_{D_6}^{E_7}}\mathrm{P}(3.3-)$ are projective covers of simples of at least 2-dimensional support. By maximality of 105_c in the c-order among $\mathrm{M}(\tau)$ appearing in Q, $\mathrm{P}(105_c)$ is a summand of Q. Because $[\mathrm{P}(105_c) : \mathrm{M}(70'_a)] = 0$ by Table 22 and $70'_a$ is next-largest after 105_c in the c-order, $\mathrm{P}(70'_a)$ is also a summand of Q. For $\tau \notin \{105_c, 70_a\}$, $\mathrm{P}(\tau)$ cannot be a summand of Q by considering already known decomposition numbers or the dimension of support of $\mathrm{L}(\tau)$, so $Q = \mathrm{P}(105_c) \oplus \mathrm{P}(70'_a)$. Using Table 22 we obtain:

$$\begin{aligned}\mathrm{P}(70'_a) &= \mathrm{M}(70'_a) + \mathrm{M}(35_b) + \alpha\mathrm{M}(7'_a) + \beta\mathrm{M}(1_a)\\ \mathrm{P}(105_c) &= \mathrm{M}(105_c) + \mathrm{M}(315'_a) + \mathrm{M}(210_a) + \mathrm{M}(35_b) + (1-\alpha)\mathrm{M}(7'_a) + (1-\beta)\mathrm{M}(1_a)\end{aligned}$$

for some $\alpha, \beta \in \{0, 1\}$. Computing the restriction of these projective characters to $\mathcal{O}_{\frac{1}{6}}(E_6)$ and cutting to the principal block, we only get projective characters if $\alpha + \beta = 1$.

Next, we look at columns 168_a and 210_a. We compute that

$$\begin{aligned}\overline{{}^{\mathcal{O}}\mathrm{Ind}_{E_6}^{E_7}}\mathrm{P}(20_p) = {} & \mathrm{M}(168_a) + \mathrm{M}(210_a) + \mathrm{M}(35_b) + \mathrm{M}(105'_a) + \mathrm{M}(21'_b) + \mathrm{M}(21_a) \\ & + \mathrm{M}(7'_a) + \mathrm{M}(105'_a) + \mathrm{M}(1_a) + \mathrm{M}(7'_a) + \mathrm{M}(21'_b).\end{aligned}$$

Set $Q' = \overline{{}^{\mathcal{O}}\mathrm{Ind}_{E_6}^{E_7}}\mathrm{P}(20_p)$. Since $\dim \mathrm{Supp}\,\mathrm{L}(20_p) = 2$, the simple quotient of any indecomposable projective summand of Q' has dimension of support at least 3. By maximality in the c-order, $\mathrm{P}(168_a)$ is a summand of Q'. Since 210_a is next-biggest in the c-order and $[\mathrm{P}(168_a) : \mathrm{M}(210_a)] = 0$ by Table 22, $\mathrm{P}(210_a)$ is also a summand of Q'. The supports of all

$\mathrm{L}(\tau)$ such that $[Q' : \mathrm{M}(\tau)] > 0$ and $\tau \neq 210_a, 168_a$ satisfy $\dim \operatorname{Supp} \mathrm{L}(\tau) < 3$. Therefore $Q' = \mathrm{P}(168_a) \oplus \mathrm{P}(210_a)$. We obtain:

$$\mathrm{P}(168_a) = \mathrm{M}(168_a) + \mathrm{M}(105'_a) + \alpha\mathrm{M}(21_a) + \mathrm{M}(35_b) + \mathrm{M}(21'_b) + \beta\mathrm{M}(7'_a) + \gamma\mathrm{M}(1_a),$$
$$\mathrm{P}(210_a) = \mathrm{M}(210_a) + \mathrm{M}(105'_a) + (1-\alpha)\mathrm{M}(21_a) + \mathrm{M}(21'_b) + (2-\beta)\mathrm{M}(7'_a) + (1-\gamma)\mathrm{M}(1_a).$$

Since $\mathrm{L}(21_a)$ is finite-dimensional, we compute that $\dim \mathrm{L}(21_a)[1] = \dim \mathrm{L}(21_a)[-1] = 21$ forcing $\alpha = 0$. To determine β we use that $\mathrm{L}(7'_a)$ is finite-dimensional. We compute from the part of its character known so far that $\dim \mathrm{L}(7_a)[-1] = 406$. From what is known of its character, then

$$406 = \dim \mathrm{L}(7_a)[1] = 7\binom{11}{5} - 21\binom{9}{3} - 35\binom{8}{2} - 105\binom{7}{1} + 210\beta + 168(3-\beta) + 105.$$

Therefore $\beta = 1$. Now, by setting $\dim \mathrm{L}(7'_a)[2] = \dim \mathrm{L}(7'_a)[-2] = 175$, we find that $[\mathrm{M}(7'_a) : \mathrm{L}(70'_a)] = 0$. This completes the calculation of $\mathrm{P}(70'_a)$ and $\mathrm{P}(105_c)$ above. Finally, since $\mathrm{L}(1_a)$ is finite-dimensional, the same argument yields $\gamma = 1$. This completes all columns of the decomposition matrix except for columns 84_a and $105'_c$.

To determine the last two missing columns, we induce the projective cover $\mathrm{P}(24_p)$ of the other simple $\mathcal{O}_{\frac{1}{6}}(E_6)$-module with 2-dimensional support and cut to the principal block. A similar argument shows that $\overline{{}^{\mathcal{O}}\mathrm{Ind}_{E_6}^{E_7}}\mathrm{P}(24_p) = \mathrm{P}(84_a) \oplus \mathrm{P}(105'_c)$. We obtain the projective characters:

$$\mathrm{P}(105'_c) = \mathrm{M}(105'_c) + \mathrm{M}(420_a) + \mathrm{M}(168_a) + \mathrm{M}(210_a) + \alpha\mathrm{M}(105'_a) + \beta\mathrm{M}(35_b) + \gamma\mathrm{M}(21'_b),$$
$$\mathrm{P}(84_a) = \mathrm{M}(84_a) + \mathrm{M}(168_a) + (1-\alpha)\mathrm{M}(105'_a) + (1-\beta)\mathrm{M}(35_b) + (1-\gamma)\mathrm{M}(21'_b).$$

But since the characters of all finite-dimensional modules are known, the characters of $\mathrm{L}(21'_b)$, $\mathrm{L}(105'_a)$, and $\mathrm{L}(35'_b)$ are now also known implicitly by induction of the cuspidal characters $\mathrm{L}(1_p)$, $\mathrm{L}(6_p)$ (from $\mathcal{O}_{\frac{1}{6}}(E_6)$) and $\mathrm{L}(6.\emptyset)$ (from $\mathcal{O}_{\frac{1}{6}}(D_6)$). Computing them and solving, we obtain $\alpha = \gamma = 1$ and $\beta = 0$. This completes the decomposition matrix. □

5.2. **Parameter** $c = 1/4$**.** There are four nontrivial blocks in $\mathcal{O}_{\frac{1}{4}}(E_7)$: one pair of Ringel-dual blocks containing $\mathrm{L}(1_a)$ and $\mathrm{L}(1'_a)$ respectively, and a second pair of Ringel-dual blocks containing $\mathrm{L}(7'_a)$ and $\mathrm{L}(7_a)$ respectively. The h_c-weight lines for these blocks are given in Table 24.

Theorem 28. *The decomposition matrices of the four blocks of* $\mathcal{O}_{\frac{1}{4}}(E_7)$ *containing* $\mathrm{L}(7'_a)$, $\mathrm{L}(7_a)$, $\mathrm{L}(1_a)$, *and* $\mathrm{L}(1'_a)$ *are given in Tables 25, 26, 27, and 28 respectively.*

Proof. Tables 25 and 26. These two decomposition matrices are given immediately by the decomposition matrices of the corresponding blocks of the Hecke algebra together with Lemmas 15 and 16.

Table 27. All decomposition numbers are obtained from the columns of the Hecke algebra decomposition matrix and Lemmas 15 and 16 except for $[\mathrm{M}(56'_a) : \mathrm{L}(189_a)]$.

Table 24. Weight lines for blocks of $\mathcal{O}_{\frac{1}{4}}(E_7)$

-7.75	-1.75	$.25$	1.25	2.25	4.25	5.25	10.25	12.25
$7'_a$	$105'_a$	$189'_c$	280_b	$378'_a$	$105'_a$	216_a	$35'_b$	$27'_a$
	$15'_a$				$210'_b$		$21'_a$	

-5.25	-3.25	1.75	2.75	4.75	5.75	6.75	8.75	14.75
27_a	35_b	$216'_a$	105_c	378_a	$280'_b$	189_c	105_a	7_a
	21_a		210_b				15_a	

-12.25	-3.25	$-.25$	1.75	2.75	5.75	7.75	8.75	11.75
1_a	$56'_a$	210_a	405_a	$336'_a$	315_a	189_b	$120'_a$	21_b
		105_b	189_a		70_a			
					35_a			

-4.75	-1.75	$-.75$	1.25	4.25	5.25	7.25	10.25	19.25
$21'_b$	120_a	$189'_b$	$35'_a$	336_a	$189'_a$	$105'_b$	56_a	$1'_a$
			$70'_a$		$405'_a$	$210'_a$		
			$315'_a$					

This last entry is obtained from applying Lemma 13 to the characters of the Ringel-dual minimal support representations $\mathrm{L}(120_a)$ and $\mathrm{L}(56'_a)$, using the results of Table 28 below.

Table 28. All the entries of the decomposition matrix of the block containing $\mathrm{L}(1'_a)$ are given by the decomposition matrix of the Hecke algebra and Lemmas 15 and 16 except for three entries in column 336_a: $[\mathrm{M}(21'_b) : \mathrm{L}(336_a)]$, $[\mathrm{M}(120_a) : \mathrm{L}(336_a)]$, and $[\mathrm{M}(189'_b) : \mathrm{L}(336_a)]$.

Inducing $\mathrm{P}(81_p)$ from $\mathcal{O}_{\frac{1}{4}}(E_6)$ then cutting to the block containing $\mathrm{L}(1'_a)$ yields the projective character

$$Q := \overline{{}^{\mathcal{O}}\mathrm{Ind}_{E_6}^{E_7}\mathrm{P}(81_p)} = \mathrm{M}(336_a) + \mathrm{M}(315'_a) + \mathrm{M}(35'_a) + \mathrm{M}(120_a).$$

The projective indecomposable summands of Q must have simple quotients whose dimension of support is at least 4, because the dimension of support of $\mathrm{L}(81_p)$ is equal

TABLE 25. Decomposition matrix of the block of $\mathcal{O}_{\frac{1}{4}}(E_7)$ containing $\mathrm{L}(7'_a)$

		$7'_a$	$105'_a$	$15'_a$	$189'_c$	280_b	$378'_a$	$105'_c$	$210'_b$	216_a	$35'_b$	$21'_a$	$27'_a$
(3)	$7'_a$	1	·	1	·	1	·	·	1	·	·	·	·
(3)	$105'_a$	·	1	·	1	1	1	·	·	·	·	·	·
(4)	$15'_a$	·	·	1	·	·	·	·	1	·	1	·	·
(4)	$189'_c$	·	·	·	1	·	1	1	·	·	·	·	·
(4)	280_b	·	·	·	·	1	1	·	1	1	·	·	·
	$378'_a$	·	·	·	·	·	1	1	·	1	·	1	·
	$105'_c$	·	·	·	·	·	·	1	·	·	·	1	·
	$210'_b$	·	·	·	·	·	·	·	1	1	1	·	1
	216_a	·	·	·	·	·	·	·	·	1	·	1	1
	$35'_b$	·	·	·	·	·	·	·	·	·	1	·	1
	$21'_a$	·	·	·	·	·	·	·	·	·	·	1	·
	$27'_a$	·	·	·	·	·	·	·	·	·	·	·	1

TABLE 26. Decomposition matrix of the block of $\mathcal{O}_{\frac{1}{4}}(E_7)$ containing $\mathrm{L}(7_a)$

		27_a	35_b	21_a	$216'_a$	105_c	210_b	378_a	$280'_b$	189_c	105_a	15_a	7_a
(3)	27_a	1	1	·	1	·	1	·	·	·	·	·	·
(4)	35_b	·	1	·	·	·	1	·	·	·	·	1	·
(3)	21_a	·	·	1	1	1	·	1	·	·	·	·	·
(4)	$216'_a$	·	·	·	1	·	1	1	1	·	·	·	·
(4)	105_c	·	·	·	·	1	·	1	·	1	·	·	·
	210_b	·	·	·	·	·	1	·	1	·	·	1	1
	378_a	·	·	·	·	·	·	1	1	1	1	·	·
	$280'_b$	·	·	·	·	·	·	·	1	·	1	·	1
	189_c	·	·	·	·	·	·	·	·	1	1	·	·
	105_a	·	·	·	·	·	·	·	·	·	1	·	·
	15_a	·	·	·	·	·	·	·	·	·	·	1	1
	7_a	·	·	·	·	·	·	·	·	·	·	·	1

to 3. By maximality of 336_a in the c-order, $\mathrm{P}(336_a)$ is a summand of Q. From the information we have already, either $Q = \mathrm{P}(336_a)$ or $Q = \mathrm{P}(336_a) \oplus \mathrm{P}(120_a)$. However, $\mathrm{P}(120_a) = \mathrm{M}(120_a) = \overline{{}^{\mathcal{O}}\mathrm{Ind}_{E_6}^{E_7}}\mathrm{M}(6_p) = \overline{{}^{\mathcal{O}}\mathrm{Ind}_{E_6}^{E_7}}\mathrm{P}(6_p)$. Since the dimension of support of $\mathrm{L}(6_p)$ is equal to 2, we deduce that the dimension of support of $\mathrm{L}(120_a)$ is equal to 3. It follows that $Q = \mathrm{P}(336_a)$. $\square$

Table 27. Decomposition matrix of the block of $\mathcal{O}_{\frac{1}{4}}(E_7)$ containing $\mathrm{L}(1_a)$

		1_a	$56'_a$	210_a	105_b	405_a	189_a	$336'_a$	315_a	70_a	35_a	189_b	$120'_a$	21_b
(3)	1_a	1	·	·	1	·	·	·	·	1	·	·	·	·
(3)	$56'_a$	·	1	1	1	1	·	·	·	·	·	·	·	·
(4)	210_a	·	·	1	·	1	1	1	·	·	·	·	·	·
(4)	105_b	·	·	·	1	1	·	·	·	1	·	1	·	·
	405_a	·	·	·	·	1	·	1	1	·	·	1	·	·
(3)	189_a	·	·	·	·	·	1	1	·	·	1	·	·	·
	$336'_a$	·	·	·	·	·	·	1	1	·	1	·	·	·
	315_a	·	·	·	·	·	·	·	1	·	·	1	1	·
	70_a	·	·	·	·	·	·	·	·	1	·	1	·	1
	35_a	·	·	·	·	·	·	·	·	·	1	·	·	·
	189_b	·	·	·	·	·	·	·	·	·	·	1	1	1
	$120'_a$	·	·	·	·	·	·	·	·	·	·	·	1	·
	21_b	·	·	·	·	·	·	·	·	·	·	·	·	1

Table 28. Decomposition matrix of the block of $\mathcal{O}_{\frac{1}{4}}(E_7)$ containing $\mathrm{L}(1_a)'$

		$21'_b$	120_a	$189'_b$	$315'_a$	$70'_a$	$35'_a$	336_a	$405'_a$	$189'_a$	$210'_a$	$105'_b$	56_a	$1'_a$
(3)	$21'_b$	1	·	1	·	1	·	·	·	·	·	·	·	·
(3)	120_a	·	1	1	1	·	·	1	·	·	·	·	·	·
(4)	$189'_b$	·	·	1	1	1	·	·	·	·	·	1	·	·
	$315'_a$	·	·	·	1	·	·	1	1	·	·	1	·	·
	$70'_a$	·	·	·	·	1	·	·	·	·	·	1	·	1
(3)	$35'_a$	·	·	·	·	·	1	1	·	1	·	·	·	·
(4)	336_a	·	·	·	·	·	·	1	1	1	1	·	·	·
	$405'_a$	·	·	·	·	·	·	·	1	·	1	1	1	·
	$189'_a$	·	·	·	·	·	·	·	·	1	1	·	·	·
	$210'_a$	·	·	·	·	·	·	·	·	·	1	·	1	·
	$105'_b$	·	·	·	·	·	·	·	·	·	·	1	1	1
	56_a	·	·	·	·	·	·	·	·	·	·	·	1	·
	$1'_a$	·	·	·	·	·	·	·	·	·	·	·	·	1

5.3. **Parameter** $c = 1/3$**.** There are two blocks of $\mathcal{O}_{\frac{1}{3}}(E_7)$ of defect greater than 1 for us to study: the principal block and its Ringel-dual block. The h_c-weight lines for these two blocks are given in Table 29.

TABLE 29. Weight lines of the blocks of $\mathcal{O}_{\frac{1}{3}}(E_7)$ containing $\mathrm{L}(1_a)$ and $\mathrm{L}(1'_a)$

-17.5	-5.5	-3.5	-1.5	$.5$	2.5	3.5	4.5	6.5	10.5	12.5	14.5	18.5
1_a	35_b	120_a	210_a	280_b	105_c	$512'_a$	336_a	280_a	105_a	56_a	21_b	7_a
	21_a		168_a		420_a			70_a	15_a			
			105_b		210_b			35_a				
					84_a							

-11.5	-7.5	-5.5	-3.5	$.5$	2.5	3.5	4.5	6.5	8.5	10.5	12.5	24.5
$7'_a$	$21'_b$	$56'_a$	$105'_a$	$280'_a$	$336'_a$	512_a	$105'_c$	$280'_b$	$210'_a$	$120'_a$	$35'_b$	$1'_a$
			$15'_a$	$70'_a$			$420'_a$		$168'_a$		$21'_a$	
				$35'_a$			$210'_b$		$105'_b$			
							$84'_a$					

Theorem 29. *The decomposition matrices of the principal block of* $\mathcal{O}_{\frac{1}{3}}(E_7)$ *and its Ringel-dual block are given in Tables 30 and 31, respectively.*

Proof. The principal block contains 22 irreducible representations of which 12 have less than full support, and likewise for its Ringel-dual block. After the ten columns from each of the corresponding blocks of the Hecke algebra have been copied and Lemmas 15 and 16 have been applied, the decomposition matrices of these two blocks look as in Tables 32 and 33.

The character of $\mathrm{L}(35_a) = \mathrm{M}(35_a) - \mathrm{M}(105_a) + \mathrm{M}(56_a) + \mathrm{M}(21_b) - \mathrm{M}(7_a)$ follows from Table 32. Calculation reveals that $\dim \mathrm{Supp}\, \mathrm{L}(35_a) = 5$, and it restricts simply to $\mathrm{L}(20_s)$ in the principal block of $\mathcal{O}_{\frac{1}{3}}(E_6)$. Therefore $\mathrm{P}(35_a)$ is a direct summand of ${}^{\mathcal{O}}\mathrm{Ind}_{E_6}^{E_7}\mathrm{P}(20_s)$. Let Q_1 be the projective character obtained by cutting ${}^{\mathcal{O}}\mathrm{Ind}_{E_6}^{E_7}\mathrm{P}(20_s)$ to the block containing $\mathrm{L}(1_a)$, and similarly define Q'_1 by cutting ${}^{\mathcal{O}}\mathrm{Ind}_{E_6}^{E_7}\mathrm{P}(20_s)$ to the block containing $\mathrm{L}(1'_a)$. We have:

$$Q_1 = \mathrm{M}(35_a) + \mathrm{M}(336_a) + \mathrm{M}(420_a) + \mathrm{M}(280_b) + \mathrm{M}(168_a) + 2\mathrm{M}(210_a) + \mathrm{M}(120_a) + \mathrm{M}(21_a),$$
$$Q'_1 = 2\mathrm{M}(336'_a) + 2\mathrm{M}(35'_a) + 2\mathrm{M}(280'_a) + 2\mathrm{M}(105'_a).$$

Table 30. Decomposition matrix of the block of $\mathcal{O}_{\frac{1}{3}}(E_7)$ containing $\mathrm{L}(1_a)$

$$
\begin{array}{rr}
(1) & 1_a \\
(3) & 35_b \\
(1) & 21_a \\
(1) & 120_a \\
(5) & 210_a \\
(5) & 168_a \\
(3) & 105_b \\
(5) & 280_b \\
(3) & 105_c \\
(5) & 420_a \\
(3) & 210_b \\
 & 84_a \\
 & 512'_a \\
 & 336_a \\
 & 280_a \\
 & 70_a \\
(5) & 35_a \\
 & 105_a \\
 & 15_a \\
 & 56_a \\
 & 21_b \\
 & 7_a
\end{array}
\begin{pmatrix}
1 & 1 & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & 1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & 1 & 1 & 1 & 1 & \cdot & 1 & 1 & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & 1 & 1 & \cdot & \cdot & 1 & 1 & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot & 1 & 1 & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & 1 & 1 & 1 & \cdot & \cdot & 1 & 1 & 1 & \cdot & 1 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & 1 & 1 & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & 1 & \cdot & 1 & 1 & 1 & 1 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & 1 & 1 & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & 1 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{pmatrix}
$$

Comparing with Table 33 we immediately obtain $Q'_1 = 2\mathrm{P}(336'_a)$, while Q_1 gives upper bounds on all unknown entries of column 35_a in Table 32. In particular it determines many entries to be 0. We also deduce from the entries of column 336_a that $\mathrm{P}(336_a)$ is not a summand of Q_1, and therefore that $[\mathrm{M}(336_a) : \mathrm{L}(35_a)] = 1$. The character ${}^{\mathcal{O}}\mathrm{Ind}_{E_6}^{E_7}\mathrm{L}(80_s) - \mathrm{L}(210_b) - 2\mathrm{M}(512'_a) - \mathrm{L}(336_a)$ coincides with $\mathrm{L}(420_a)$ up through the place just to the left of $\mathrm{M}(35_a)$ and then has the term $\mathrm{M}(35_a)$ with coefficient -1, showing that $d^{-1}_{420_a,35_a} \leq -1$. This implies $[\mathrm{M}(420_a) : \mathrm{L}(35_a)] \geq 1$. Therefore $[\mathrm{M}(420_a) : \mathrm{L}(35_a)] = 1$ and $\mathrm{P}(420_a)$ is not a summand of Q_1. From the columns in Table 32 we see that $\mathrm{P}(280_b)$, $\mathrm{P}(168_a)$, and $\mathrm{P}(120_a)$ cannot be summands of Q_1. From the known entries in column 210_a it also follows that $[\mathrm{M}(210_a) : \mathrm{L}(35_a)] \geq 1$.

Let Q_2 be ${}^{\mathcal{O}}\mathrm{Ind}_{E_6}^{E_7}\mathrm{P}(24'_p)$ cut to the principal block. We compute that $Q_2 = \mathrm{P}(512_a) \oplus \mathrm{P}(84_a) \oplus P_2$, where

$$P_2 = \mathrm{M}(105_c) + \mathrm{M}(280_b) + 2\mathrm{M}(210_a) + \mathrm{M}(120_a) + \mathrm{M}(21_a) + \mathrm{M}(35_b).$$

Table 31. Decomposition matrix of the block of $\mathcal{O}_{\frac{1}{3}}(E_7)$ containing $\mathrm{L}(1_a')$

$$
\begin{array}{rr}
(1) & 7_a' \\
(1) & 21_b' \\
(3) & 56_a' \\
(5) & 105_a' \\
(3) & 15_a' \\
(5) & 280_a' \\
(3) & 70_a' \\
(1) & 35_a' \\
(5) & 336_a' \\
(5) & 512_a \\
(5) & 105_c' \\
 & 420_a' \\
 & 210_b' \\
(3) & 84_a' \\
 & 280_b' \\
 & 210_a' \\
 & 168_a' \\
 & 105_b' \\
 & 120_a' \\
 & 35_b' \\
 & 21_a' \\
 & 1_a'
\end{array}
\begin{pmatrix}
1 & 1 & 1 & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & 1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & 1 & \cdot & 1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & 1 & 1 & 1 & 1 & \cdot & 1 & 1 & 1 & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & 1 & \cdot & 1 & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & 1 & \cdot & \cdot & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & 1 & \cdot & 1 & 1 & 1 & 1 & 1 & \cdot & 1 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & 1 & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & 1 & 1 & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & 1 & \cdot & \cdot & 1 & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & 1 & 1 & 1 & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & 1 & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{pmatrix}
$$

By maximality of 105_c in the c-order, $\mathrm{P}(105_c)$ is a summand of P_2. We obtain $[\mathrm{M}(105_b) : \mathrm{L}(105_c)] = [\mathrm{M}(168_a) : \mathrm{L}(105_c)] = 0$. Moreover, by the known entries in column 210_a, $\mathrm{P}(210_a)$ occurs at most once as a summand of P_2. This yields $[\mathrm{M}(210_a) : \mathrm{L}(105_c)] \geq 1$.

We now know all entries of the decomposition matrix lying below row 210_a, and in row 210_a we have lower bounds of 1 on both unknown entries. We compute that ${}^{\mathcal{O}}\mathrm{Res}^{E_7}_{A_6}\mathrm{L}(210_a) = \mathrm{L}(6,1) \oplus \mathrm{L}(5,1^2)$ if $[\mathrm{M}(210_a) : \mathrm{L}(105_c)] = [\mathrm{M}(210_a) : \mathrm{L}(35_a)] = 1$, the direct sum of the two simple modules of 4-dimensional support in the dual defect 1 blocks of $H_{\frac{1}{3}}(A_6)$. However, ${}^{\mathcal{O}}\mathrm{Res}^{E_7}_{A_6}\mathrm{L}(105_c)$ and ${}^{\mathcal{O}}\mathrm{Res}^{E_7}_{A_6}\mathrm{L}(35_a)$ belong to a completely different block, the principal block of $H_{\frac{1}{3}}(A_6)$. If either $[\mathrm{M}(210_a) : \mathrm{L}(105_c)] > 1$ or $[\mathrm{M}(210_a) : \mathrm{L}(35_a)] > 1$ then the restriction of the character of $\mathrm{L}(210_a)$ cut to the principal block would not be the character of a module. Therefore $[\mathrm{M}(210_a) : \mathrm{L}(105_c)] = [\mathrm{M}(210_a) : \mathrm{L}(35_a)] = 1$.

It remains to determine the top four rows of the decomposition matrix of the principal block. Let $\overline{{}^{\mathcal{O}}\mathrm{Ind}^{E_7}_{E_6}}$ be the functor of induction followed by cutting to the principal block.

Table 32. Approximation to the decomposition matrix of the principal block of $\mathcal{O}_{\frac{1}{3}}(E_7)$

$$
\begin{array}{cr}
\bullet & 1_a \\
\bullet & 35_b \\
\bullet & 21_a \\
\bullet & 120_a \\
\bullet & 210_a \\
\bullet & 168_a \\
\bullet & 105_b \\
\bullet & 280_b \\
\bullet & 105_c \\
\bullet & 420_a \\
\bullet & 210_b \\
 & 84_a \\
 & 512'_a \\
 & 336_a \\
 & 280_a \\
 & 70_a \\
\bullet & 35_a \\
 & 105_a \\
 & 15_a \\
 & 56_a \\
 & 21_b \\
 & 7_a
\end{array}
\begin{pmatrix}
1 & 1 & \cdot & 1 & ? & ? & ? & ? & ? & ? & ? & \cdot & \cdot & \cdot & \cdot & \cdot & ? & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & ? & ? & ? & ? & 1 & \cdot & \cdot & \cdot & \cdot & ? & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & 1 & ? & ? & ? & ? & ? & ? & ? & \cdot & \cdot & \cdot & \cdot & 1 & ? & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & 1 & 1 & 1 & ? & ? & ? & ? & \cdot & 1 & \cdot & \cdot & 1 & ? & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & ? & 1 & \cdot & \cdot & 1 & 1 & \cdot & 1 & ? & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & ? & 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot & ? & 1 & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & ? & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & ? & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & 1 & 1 & 1 & \cdot & \cdot & ? & 1 & 1 & \cdot & 1 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & ? & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & 1 & 1 & \cdot & ? & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & 1 & ? & 1 & \cdot & 1 & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & ? & 1 & 1 & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & 1 & ? & 1 & 1 & 1 & 1 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & ? & 1 & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & 1 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{pmatrix}
$$

We compute that

$$
\overline{{}^{\mathcal{O}}\mathrm{Ind}_{E_6}^{E_7}}\mathrm{L}(15_p) = \mathrm{L}(21_a), \qquad \overline{{}^{\mathcal{O}}\mathrm{Ind}_{E_6}^{E_7}}\mathrm{L}(15_q) = \mathrm{L}(35_b)
$$
$$
\overline{{}^{\mathcal{O}}\mathrm{Ind}_{E_6}^{E_7}}\mathrm{L}(6_p) = \mathrm{L}(120_a), \qquad \overline{{}^{\mathcal{O}}\mathrm{Ind}_{E_6}^{E_7}}\mathrm{L}(1_p) = \mathrm{L}(1_a).
$$

This determines the top four rows of the inverse matrix, completing the decomposition matrix of the principal block.

Now we turn our attention to the block containing $\mathrm{L}(1'_a)$. Let $\overline{{}^{\mathcal{O}}\mathrm{Ind}_{E_6}^{E_7}}$ denote the functor of parabolic induction from $\mathcal{O}_{\frac{1}{3}}(E_6)$ to $\mathcal{O}_{\frac{1}{3}}(E_7)$ followed by cutting to the block containing $\mathrm{L}(1'_a)$. We compute that

$$
\overline{{}^{\mathcal{O}}\mathrm{Ind}_{E_6}^{E_7}}\mathrm{L}(1_p) = \mathrm{L}(21'_b), \qquad \overline{{}^{\mathcal{O}}\mathrm{Ind}_{E_6}^{E_7}}\mathrm{L}(6_p) = \mathrm{L}(7'_a)
$$
$$
\overline{{}^{\mathcal{O}}\mathrm{Ind}_{E_6}^{E_7}}\mathrm{L}(15_p) = \mathrm{L}(35'_a), \qquad \overline{{}^{\mathcal{O}}\mathrm{Ind}_{E_6}^{E_7}}\mathrm{L}(30_p) = \mathrm{L}(56'_a).
$$

Table 33. Approximation to the decomposition matrix of the block of $\mathcal{O}_{\frac{1}{3}}(E_7)$ containing $\mathrm{L}(1'_a)$

$$
\begin{array}{cc}
\bullet & 7'_a \\
\bullet & 21'_b \\
\bullet & 56'_a \\
\bullet & 105'_a \\
\bullet & 15'_a \\
\bullet & 280'_a \\
\bullet & 70'_a \\
\bullet & 35'_a \\
\bullet & 336'_a \\
\bullet & 512_a \\
\bullet & 105'_c \\
 & 420'_a \\
 & 210'_b \\
\bullet & 84'_a \\
 & 280'_b \\
 & 210'_a \\
 & 168'_a \\
 & 105'_b \\
 & 120'_a \\
 & 35'_b \\
 & 21'_a \\
 & 1'_a
\end{array}
\begin{pmatrix}
1 & 1 & 1 & ? & ? & ? & ? & ? & ? & ? & ? & \cdot & \cdot & ? & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & 1 & 1 & ? & \cdot & ? & ? & ? & ? & \cdot & \cdot & ? & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & 1 & \cdot & ? & 1 & ? & ? & ? & ? & \cdot & 1 & ? & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & 1 & ? & ? & ? & \cdot & 1 & 1 & 1 & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & ? & ? & 1 & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & ? & ? & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & 1 & \cdot & \cdot & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & 1 & \cdot & 1 & 1 & 1 & 1 & 1 & \cdot & 1 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & 1 & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & 1 & 1 & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & 1 & \cdot & \cdot & 1 & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & 1 & 1 & 1 & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & 1 & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{pmatrix}
$$

Since column $105'_c$ was found above, the decomposition matrix will be complete once we find the following three decomposition numbers: $[\mathrm{M}(280'_a) : \mathrm{L}(512_a)] =: \alpha$, $[\mathrm{M}(105'_a) : \mathrm{L}(512_a)] =: \beta$, and $[\mathrm{M}(105'_a) : \mathrm{L}(336'_a)] =: \gamma$. The character of ${}^{\mathcal{O}}\mathrm{Ind}_{E_6}^{E_7}\mathrm{L}(64_p)$ shows that $\alpha \geq 1$, while the restriction of $\mathrm{L}(280'_a)$ to $\mathcal{O}_{\frac{1}{3}}(A_6)$ does not yield a module unless $\alpha \leq 1$. Therefore $\alpha = 1$. Next, the dot product of row $56'_a$ of the inverse of the decomposition matrix with column $336'_a$ of the decomposition matrix is equal to 0. This yields the equation $1 + [\mathrm{M}(56'_a) : \mathrm{L}(336'_a)] - \gamma = 0$, and thus $\gamma \geq 1$. Similarly, we find that $\beta = [\mathrm{M}(56'_a) : \mathrm{L}(512_a)] = [\mathrm{M}(21'_b) : \mathrm{L}(512_a)]$. Using the equality involving β, the character of $\mathrm{L}(7'_a)$ then gives $\beta = [\mathrm{M}(7'_a) : \mathrm{L}(512_a)] + 1$, so $\beta \geq 1$. Restricting $\mathrm{Ind}_{E_6}^{E_7}\mathrm{L}(20_p) - \mathrm{L}(21'_b) - \mathrm{L}(56'_a) - \mathrm{L}(35'_a)$, which contains $\mathrm{L}(105'_a)$ with multiplicity 2, back down to E_6, one finds it contains $\mathrm{L}(60_p) = \mathrm{ResL}(512_a)$ just once, $\mathrm{L}(64_p) = \mathrm{ResL}(336'_a)$ twice, and $\mathrm{L}(24_p) = \mathrm{ResL}(105'_c)$. This immediately implies that $\beta \leq 1$. Therefore $\beta = 1$. Restricting $\mathrm{L}(105'_a)$ to A_6 then shows that $\mathrm{L}(4, 1^3) + \mathrm{L}(5, 1^2) = \mathrm{ResL}(336'_a)$ will appear with negative multiplicity unless $\gamma \leq 1$. So $\gamma = 1$ also. This completes the decomposition matrix of the block containing $\mathrm{L}(1'_a)$. $\square$

6. The decomposition matrices of defect 2 blocks of $\mathcal{O}_c(E_8)$

We will only consider blocks of $\mathcal{O}_c(E_8)$ of defect 2. The values of c we will need to consider are $c \in \{\frac{1}{12}, \frac{1}{10}, \frac{1}{8}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}\}$. Since all decomposition matrices of defect 1 blocks are known, this provides complete knowledge of the decomposition matrices of $\mathcal{O}_c(E_8)$ except for: the principal blocks of $\mathcal{O}_{\frac{1}{6}}(E_8)$, $\mathcal{O}_{\frac{1}{4}}(E_8)$, and $\mathcal{O}_{\frac{1}{3}}(E_8)$, and the blocks of $\mathcal{O}_{\frac{1}{2}}(E_8)$.

The weight lines for the blocks we study are printed in Tables 34, 36, 38, 44, 40, 42, and 46.

6.1. **Parameters** $c = 1/12,\ 1/10,\ 1/8,\ 1/5$**.**

Theorem 30. *For $c \in \{\frac{1}{12}, \frac{1}{10}, \frac{1}{8}, \frac{1}{5}\}$, the decomposition matrices of the defect 2 blocks of $\mathcal{O}_c(E_8)$ are as given in Tables 35, 37, 39, 41, 43.*

Proof. All but six columns are given by the decomposition matrix of the corresponding block of the Hecke algebra. The remaining columns follow by applying Lemmas 15 and 16. $\square$

Table 34. Weight line for the principal block of $\mathcal{O}_{\frac{1}{12}}(E_8)$

-6	-2	-1	0	1	2	3	4	5	6	7	8	9	10	14
1_x	35_x	112_z	50_x	400_z	1050_x	3360_x	1400_y	$3360'_x$	$1050'_x$	$400'_z$	$50'_x$	$112'_z$	$35'_x$	$1'_x$
			210_x		1400_x	2800_z	2688_y	$2800'_z$	$1400'_x$		$210'_x$			
					525_x		4536_y		$525'_x$					
							2100_y							

6.2. **Parameter** $c = 1/6$**.** At parameter $c = \frac{1}{6}$, $\mathcal{O}_c(E_8)$ has two blocks of defect larger than 1. The large principal block is not studied in this paper. We concentrate on the smaller defect 2 block. The h_c-weight line of the defect 2 block is given in table 44.

Theorem 31. *For $c = \frac{1}{6}$, the decomposition matrix of the defect 2 block of $\mathcal{O}_c(E_8)$ is as given in Table 45.*

Proof. All nonzero entries of the decomposition matrix of the block of $H_{1/6}(E_8)$ containing 112_z can be recovered from the Hecke algebra decomposition matrix using Lemmas 15 and 16 except for the entries $[\mathrm{M}(112_z) : \mathrm{L}(3200_x)]$ and $[\mathrm{M}(160_z) : \mathrm{L}(3200_x)]$. Set

$$a := [\mathrm{M}(160_z) : \mathrm{L}(3200_x)], \qquad b := [\mathrm{M}(112_z) : \mathrm{L}(3200_x)].$$

TABLE 35. Decomposition matrix of the principal block of $\mathcal{O}_{\frac{1}{12}}(E_8)$

$$
\begin{array}{cc}
\star & 1_x \\
\star & 35_x \\
\texttt{(2)} & 112_z \\
 & 210_x \\
\star & 50_x \\
\texttt{(1)} & 400_z \\
\texttt{(2)} & 525_x \\
 & 1400_x \\
 & 1050_x \\
 & 2800_z \\
 & 3360_z \\
 & 2100_y \\
 & 4536_y \\
 & 2688_y \\
 & 1400_y \\
 & 2800'_z \\
 & 3360'_z \\
 & 525'_x \\
 & 1400'_x \\
 & 1050'_x \\
 & 400'_z \\
 & 210'_x \\
 & 50'_x \\
 & 112'_z \\
 & 35'_x \\
 & 1'_x
\end{array}
\left(\begin{array}{cccccccccccccccccccccccccc}
1&\cdot&1&\cdot&\cdot&1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&1&1&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&1&1&\cdot&\cdot&1&1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&1&1&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&1&1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&1&1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&1&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&1&1&\cdot&1&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&1&1&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&1&1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&1&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&1&1&\cdot&\cdot&1&\cdot&1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&1&1&1&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&1&\cdot&\cdot&1&\cdot&1\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&1&\cdot&1&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&1&\cdot&\cdot&1\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&1&1&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&1&1\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1
\end{array}\right)
$$

TABLE 36. Weight line for the principal block of $\mathcal{O}_{\frac{1}{10}}(E_8)$

-8	-5	-2	0	1	2	3	4	5	6	7	8	10	13	16
1_x	8_z	28_x	567_x	448_z	972_x	4536_z	1400_y	$4536'_z$	$972'_x$	$448'_z$	$567'_x$	$28'_x$	$8'_z$	$1'_x$
		84_x		1008_z	2268_x		3150_y		$2268'_x$	$1008'_z$		$84'_x$		
				1400_z			4200_y			$1400'_z$				
							4480_y							

Table 37. Decomposition matrix of the principal block of $\mathcal{O}_{\frac{1}{10}}(E_8)$

$$
\begin{array}{cc}
\star & 1_x \\
\star & 8_z \\
\texttt{(2)} & 84_x \\
\star & 28_x \\
\texttt{(1)} & 567_x \\
 & 1400_z \\
 & 1008_z \\
 & 448_z \\
 & 2268_x \\
\texttt{(2)} & 972_x \\
 & 4536_z \\
 & 4480_y \\
 & 4200_y \\
 & 3150_y \\
 & 1400_y \\
 & 4536'_z \\
 & 2268'_x \\
 & 972'_x \\
 & 1400'_z \\
 & 1008'_z \\
 & 448'_z \\
 & 567'_x \\
 & 84'_x \\
 & 28'_x \\
 & 8'_z \\
 & 1'_x
\end{array}
\begin{pmatrix}
1&\cdot&1&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&1&1&\cdot&1&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&1&\cdot&\cdot&1&\cdot&1&\cdot&1&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&1&1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&1&1&1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&1&\cdot&1&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&1&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&1&\cdot&\cdot&1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&1&1&1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&1&1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&1&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&1&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&1&1&\cdot&1&\cdot&1&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&1&1&\cdot&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&1&1&\cdot&1&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&1&\cdot&1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&1&\cdot&\cdot&1\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&1&1&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&1&1\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1
\end{pmatrix}
$$

Table 38. Weight line of the principal block of $\mathcal{O}_{\frac{1}{8}}(E_8)$

-11	-5	-2	-1	1	3	4	5	7	9	10	13	19
1_x	35_x	160_z	567_x	525_x	6075_x	7168_w	$6075'_x$	$525'_x$	$567'_x$	$160'_z$	$35'_x$	$1'_x$
				1575_x	2835_x	5600_w	$2835'_x$	$1575'_x$				
				1400_x		2016_w		$1400'_x$				
				175_x				$175'_x$				

TABLE 39. Decomposition matrix of the principal block of $\mathcal{O}_{\frac{1}{8}}(E_8)$

$$
\begin{array}{ll}
\star & 1_x \\
\texttt{(3)} & 35_x \\
\star & 160_z \\
\texttt{(3)} & 567_x \\
\texttt{(3)} & 525_x \\
 & 1575_x \\
 & 1400_x \\
\texttt{(1)} & 175_x \\
 & 6075_x \\
 & 2835_x \\
 & 7168_w \\
 & 5600_w \\
 & 2016_w \\
 & 6075'_x \\
 & 2835'_x \\
 & 525'_x \\
 & 1575'_x \\
 & 1400'_x \\
 & 175'_x \\
 & 567'_x \\
 & 160'_z \\
 & 35'_x \\
 & 1'_x
\end{array}
\left(\begin{array}{ccccccccccccccccccccccc}
1&1&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&1&\cdot&1&\cdot&\cdot&1&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&1&1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&1&1&1&1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&1&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&1&1&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&1&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&1&1&\cdot&1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&1&\cdot&1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&1&1&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&1&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&1&1&1&\cdot&1&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&1&1&\cdot&\cdot&\cdot&1\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&1&1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&1&\cdot&1&1\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&1\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&1&1&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&1\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1
\end{array}\right)
$$

TABLE 40. Weight line of the principal block of $\mathcal{O}_{\frac{1}{5}}(E_8)$

-20	-8	-4	-2	0	1	4	7	8	10	12	16	28
1_x	84_x	567_x	1344_x	972_x	4096_z	168_y	$4096'_z$	$972'_x$	$1344'_x$	$567'_x$	$84'_x$	$1'_x$
	28_x			2268_x		1134_y		$2268'_x$			$28'_x$	
						2688_y						
						4536_y						

Write the characters of $\mathrm{L}(160_z)$ and $\mathrm{L}(112_z)$ in terms of a and b:

$$
\begin{aligned}
\mathrm{L}(160_z) = {} & \mathrm{M}(160_z) - \mathrm{M}(1344_x) + \mathrm{M}(2240_x) + (1-a)\mathrm{M}(3200_x) + (a-1)\mathrm{M}(7168_w) - \mathrm{M}(1344_w) \\
& + \mathrm{M}(3200'_x) + (1-a)\mathrm{M}(2240'_x) - \mathrm{M}(1344'_x) + a\mathrm{M}(400'_z) + \mathrm{M}(160'_z), \\
\mathrm{L}(112_z) = {} & \mathrm{M}(112_z) - \mathrm{M}(400_z) - \mathrm{M}(1344_x) + \mathrm{M}(2240_x) + \mathrm{M}(3360_z) + (1-b)\mathrm{M}(3200_x) \\
& + (b-2)\mathrm{M}(7168_w) + (1-b)\mathrm{M}(3200'_x) + \mathrm{M}(3360'_z) + (1-b)\mathrm{M}(2240'_x) \\
& + (b-1)\mathrm{M}(1344'_x) + (b-1)\mathrm{M}(400'_z) - b\mathrm{M}(160'_z) + (1-b)\mathrm{M}(112'_z).
\end{aligned}
$$

TABLE 41. Decomposition matrix of the principal block of $\mathcal{O}_{\frac{1}{5}}(E_8)$

$$
\begin{array}{ll}
\star & 1_x \\
(4) & 84_x \\
\star & 28_x \\
(4) & 567_x \\
(4) & 1344_x \\
(4) & 2268_x \\
 & 972_x \\
 & 4096_z \\
 & 4536_y \\
 & 2688_y \\
 & 1134_y \\
 & 168_y \\
 & 4096'_z \\
 & 2268'_x \\
 & 972'_x \\
 & 1344'_x \\
 & 567'_x \\
 & 84'_x \\
 & 28'_x \\
 & 1'_x
\end{array}
\begin{pmatrix}
1&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&1&\cdot&\cdot&1&\cdot&1&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&1&1&1&\cdot&1&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&1&\cdot&1&1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&1&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&1&\cdot&1&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&1&1&1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&1&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&1&\cdot&1&1&\cdot&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&1\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&1&\cdot&1&1&\cdot&1&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&1&\cdot&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&1&\cdot&1&\cdot&1\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&1&1&\cdot&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&1&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&1\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot\\
\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1
\end{pmatrix}
$$

TABLE 42. Weight line of the block of $\mathcal{O}_{\frac{1}{5}}(E_8)$ containing $\mathrm{L}(8_z)$

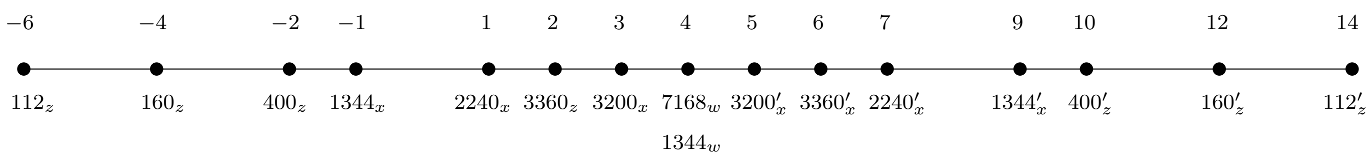
-14	-8	-2	0	1	2	4	6	7	8	10	16	22
8_z	112_z	1008_z	1296_z	4096_x	4536_z	1344_w	$4536'_z$	$4096'_x$	$1296'_z$	$1008'_z$	$112'_z$	$8'_z$
		448_z				2016_w				$448'_z$		
		56_z				7168_w				$56'_z$		
						448_w						

We see that $[\mathrm{M}(3200'_x) : \mathrm{L}(\sigma)] = 0$ unless $\sigma = 1344'_x$. Therefore $\dim \mathrm{Hom}(\mathrm{M}(\sigma), \mathrm{M}(3200'_x)) = 0$ unless possibly (i) $\sigma = 1344'_x$, or (ii) σ is the lowest weight of a simple appearing in the decomposition of $\mathrm{M}(1344'_x)$. However, we have the characters $\mathrm{L}(3200'_x) = \mathrm{M}(3200'_x) - \mathrm{M}(1344'_x) + \mathrm{M}(160'_z) + \mathrm{M}(112'_z)$ and $\mathrm{L}(1344'_x) = \mathrm{M}(1344'_x) - \mathrm{M}(160'_z) - \mathrm{M}(112'_z)$. It follows that $\dim \mathrm{Hom}(\mathrm{M}(160'_z), \mathrm{M}(1344'_x)) = \dim \mathrm{Hom}(\mathrm{M}(112'_z), \mathrm{M}(1344'_x)) = 0$ as well.

TABLE 43. Decomposition matrix of the block of $\mathcal{O}_{\frac{1}{5}}(E_8)$ containing $\mathrm{L}(8_z)$

$$
\begin{array}{rr}
\star & 8_z \\
(4) & 112_z \\
(4) & 1008_z \\
(4) & 448_z \\
\star & 56_z \\
(4) & 1296_z \\
 & 4096_x \\
 & 4536_z \\
 & 1344_w \\
 & 2016_w \\
 & 7168_w \\
 & 448_w \\
 & 4536'_z \\
 & 4096'_x \\
 & 1296'_z \\
 & 1008'_z \\
 & 448'_z \\
 & 56'_z \\
 & 112'_z \\
 & 8'_z
\end{array}
\begin{pmatrix}
1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & 1 & 1 & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & 1 & \cdot & 1 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{pmatrix}
$$

TABLE 44. Weight line of the block of $\mathcal{O}_{\frac{1}{6}}(E_8)$ containing $\mathrm{L}(112_z)$



So $\dim \mathrm{Hom}(\mathrm{M}(\sigma), \mathrm{M}(3200'_x)) = 0$ unless $\sigma = 1344'_x$. Then by Lemma 15, we have $a, b \in \{0, 1\}$.

We will argue that $a = b = 0$ by considering the dimensions of supports of $\mathrm{L}(160_z)$ and $\mathrm{L}(112)_z$. Suppose first that $a = 1$. Then the character of $\mathrm{L}(160_z)$ has a pole of order 8, i.e. $\mathrm{L}(160_z)$ has full support. But this is impossible, because the column labeled by $\mathrm{L}(160_z)$ does not appear as a column in the decomposition matrix of the Hecke algebra, which means that $\mathrm{L}(160_z)$ does not have full support. Therefore $a = 0$.

TABLE 45. Decomposition matrix of the block of $\mathcal{O}_{\frac{1}{6}}(E_8)$ containing $\mathrm{L}(112_z)$

$$
\begin{array}{rr}
(2) & 112_z \\
(2) & 160_z \\
(3) & 400_z \\
(4) & 1344_x \\
 & 2240_x \\
 & 3360_z \\
(4) & 3200_x \\
 & 7168_w \\
 & 1344_w \\
 & 3200'_x \\
 & 3360'_z \\
 & 2240'_x \\
 & 1344'_x \\
 & 400'_z \\
 & 160'_z \\
 & 112'_z
\end{array}
\begin{pmatrix}
1 & \cdot & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{pmatrix}
$$

Next, suppose that $b = 1$. Then the character of $\mathrm{L}(112_z)$ has a pole of order 4, i.e. the dimension of support of $\mathrm{L}(112_z)$ is 4-dimensional. Consider the restriction of $112_z \in \operatorname{Irr} E_8$ from E_8 to its subgroup E_7:

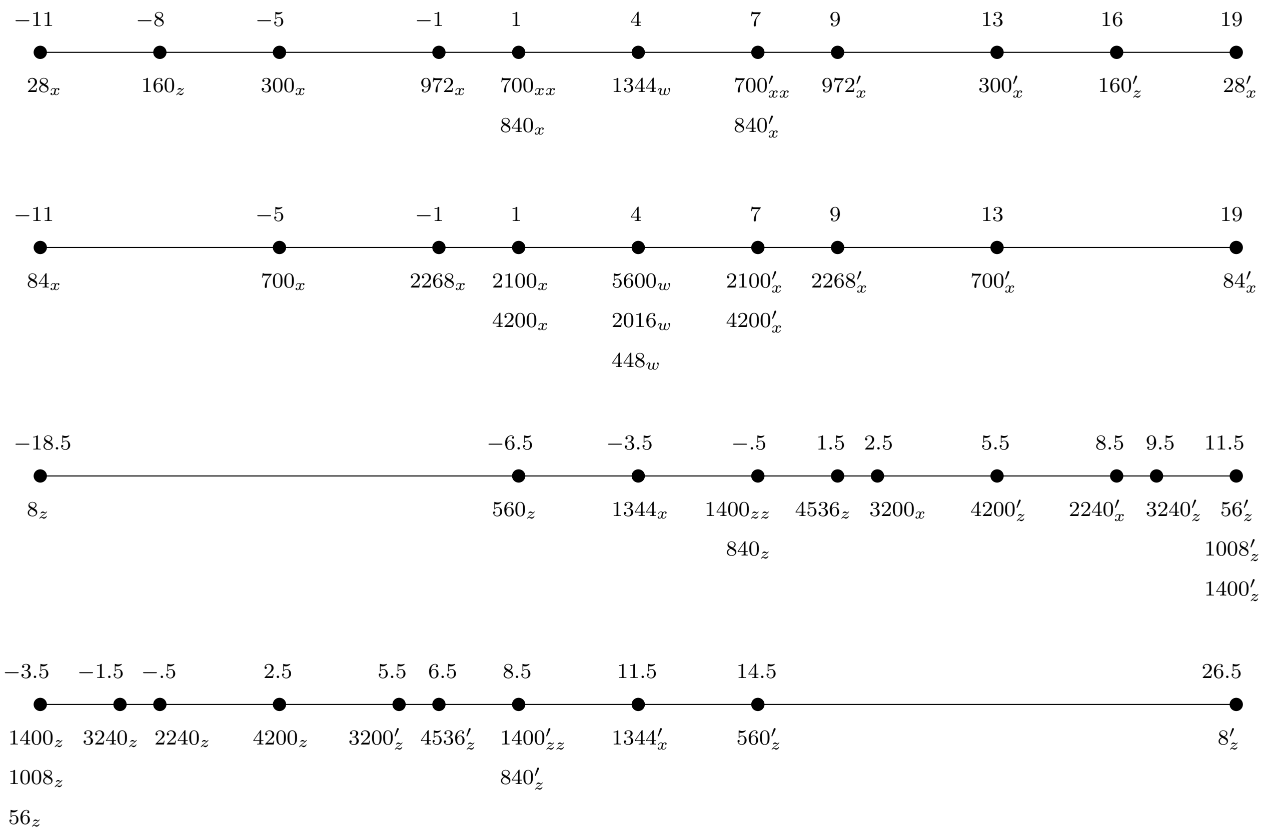
$$\mathrm{Res}^{E_8}_{E_7} 112_z = 1_a \oplus 27_a \oplus 7'_a \oplus 21'_b \oplus 56a'.$$

The dimension of support of the simple $H_{\frac{1}{6}}(E_7)$-module $\mathrm{L}(56'_a)$ is 2, and therefore the dimension of support of ${}^{\mathcal{O}}\mathrm{Ind}^{E_8}_{E_7}\mathrm{L}(56'_a)$ is 3. Moreover, the character of $\mathrm{L}(56'_a) \in \mathcal{O}_{1/6}(E_7)$ is given by $\mathrm{L}(56'_a) = \mathrm{M}(56'_a) - \mathrm{M}(120_a) + \mathrm{M}(336'_a) - \mathrm{M}(336_a) + \mathrm{M}(120'_a) - \mathrm{M}(56_a)$. Referring to the formula for $\mathrm{Res}^{E_8}_{E_7} 112_z$, we deduce that $\mathrm{M}(112_z)$ appears exactly once in the character of ${}^{\mathcal{O}}\mathrm{Ind}^{E_8}_{E_7}\mathrm{L}(56'_a)$. Since 112_z is smallest in its block with respect to the c-order, it then follows that $\mathrm{L}(112_z)$ occurs as a composition factor of ${}^{\mathcal{O}}\mathrm{Ind}^{E_8}_{E_7}\mathrm{L}(56'_a)$. Since $\dim \operatorname{Supp} \mathrm{Ind}^{E_8}_{E_7}\mathrm{L}(56'_a) = 3$, it must hold that $\dim \operatorname{Supp} \mathrm{L}(112_z) \leq 3$, and so $b = 0$. $\square$

6.3. **Parameter** $c = 1/4$**.** The principal block is not studied in this paper. There are also four smaller blocks, of which two are self-dual and two are dual to each other under tensoring $\tau \in \operatorname{Irr} E_8$ with the sign representation. The h_c-weight lines of these blocks are given in Table 46.

Theorem 32. *The decomposition matrices of the blocks of* $\mathcal{O}_{\frac{1}{4}}(E_8)$ *containing* $\mathrm{L}(28_x)$, $\mathrm{L}(84_x)$, $\mathrm{L}(8_z)$, *and* $\mathrm{L}(8'_z)$ *are given in Tables 47, 48, 49, and 50 respectively.*

TABLE 46. Weight lines of the defect 2 blocks of $\mathcal{O}_{\frac{1}{4}}(E_8)$



Proof. Table 47. All but the first five columns are given by the decomposition matrix of the Hecke algebra, and the entries for the first five columns follow immediately by applying Lemmas 15 and 16.

Table 48. The decomposition matrix has five columns that are not given by the decomposition matrix of the Hecke algebra, of which all but six upper diagonal entries are given by Lemmas 16 and 15. The entries left to find are $[\mathrm{M}(84_x) : \mathrm{L}(2268_x)]$, $[\mathrm{M}(84_x) : \mathrm{L}(2100_x)]$,$[\mathrm{M}(84_x) : \mathrm{L}(448_w)]$, $[\mathrm{M}(700_x) : \mathrm{L}(2100_x)]$, $[\mathrm{M}(700_x) : \mathrm{L}(448_w)]$, and $[\mathrm{M}(2268_x) : \mathrm{L}(448_w)]$. We will use induction and restriction to $\mathcal{O}_{\frac{1}{4}}(E_7)$ to find the missing entries.

Set $\alpha = [\mathrm{M}(2268_x) : \mathrm{L}(448_w)]$ and write

$$\begin{aligned}\mathrm{L}(2268_x) = {} & \mathrm{M}(2268_x) - \mathrm{M}(2100_x) - \mathrm{M}(4200_x) + \mathrm{M}(5600_w) + \mathrm{M}(2016_w) + (1-\alpha)\mathrm{M}(448_w) \\ & + (\alpha-1)\mathrm{M}(2100'_x) - \mathrm{M}(4200'_x) + (1-\alpha)\mathrm{M}(2268'_x) + \alpha\mathrm{M}(700'_x) - \alpha\mathrm{M}(84'_x).\end{aligned}$$

Table 47. Decomposition matrix of the block of $\mathcal{O}_{\frac{1}{4}}(E_8)$ containing $\mathrm{L}(28_x)$

$$
\begin{array}{cc}
\begin{array}{rr}
(4) & 28_x \\
(4) & 160_z \\
(4) & 300_x \\
(5) & 972_x \\
(5) & 700_{xx} \\
 & 840_x \\
 & 1344_w \\
 & 700'_{xx} \\
 & 840'_x \\
 & 972'_x \\
 & 300'_x \\
 & 160'_z \\
 & 28'_x
\end{array}
&
\begin{pmatrix}
1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & 1 & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & 1 & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & 1 & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{pmatrix}
\end{array}
$$

Table 48. Decomposition matrix of the block of $\mathcal{O}_{\frac{1}{4}}(E_8)$ containing $\mathrm{L}(84_x)$

$$
\begin{array}{cc}
\begin{array}{rr}
(4) & 84_x \\
(5) & 700_x \\
(4) & 2268_x \\
(5) & 2100_x \\
 & 4200_x \\
 & 5600_w \\
 & 2016_w \\
(4) & 448_w \\
 & 2100'_x \\
 & 4200'_x \\
 & 2268'_x \\
 & 700'_x \\
 & 84'_x
\end{array}
&
\begin{pmatrix}
1 & 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & 1 & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & 1 & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{pmatrix}
\end{array}
$$

We have $\dim \mathrm{Hom}(\mathrm{M}(448_w), \mathrm{M}(2268_x)) = \dim \mathrm{Hom}(\mathrm{M}(2268'_x), \mathrm{M}(448_w)) \leq [\mathrm{M}(448_w) : \mathrm{L}(2268'_x)] = 0$ by Lemma 15. This implies that $\alpha \in \{0, 1\}$. Let M be the virtual character given by setting $\alpha = 1$ in the character of $\mathrm{L}(2269_x)$ above. We have

$$\begin{aligned}{}^{\mathcal{O}}\mathrm{Res}_{E_7}^{E8} M = {} & \mathrm{M}(120_a) + \mathrm{M}(56'_a) - \mathrm{M}(35'_a) - \mathrm{M}(105_b) - \mathrm{M}(210_a) - \mathrm{M}(189'_b) + \mathrm{M}(405_a) \\ & + \mathrm{M}(189'_a) + \mathrm{M}(70_a) + \mathrm{M}(70'_a) - \mathrm{M}(210'_a) - \mathrm{M}(315_a) + \mathrm{M}(120'_a) + \mathrm{M}(56_a) \\ & - \mathrm{M}(21_b) - \mathrm{M}(1'_a).\end{aligned}$$

The standard modules of minimal h_c-weight in this expression are $\mathrm{M}(120_a)$ and $\mathrm{M}(56'_a)$ and these belong to different blocks, so $\mathrm{L}(120_a)$ and $\mathrm{L}(56'_a)$ must both appear in the composition series of ${}^{\mathcal{O}}\mathrm{Res}_{E_7}^{E_8} M$. Subtracting their characters, we obtain

$$\begin{aligned}{}^{\mathcal{O}}\mathrm{Res}_{E_7}^{E_8} M - \mathrm{L}(120_a) - \mathrm{L}(56'_a) = {} & -\mathrm{M}(35'_a) - \mathrm{M}(189_a) + \mathrm{M}(336'_a) + \mathrm{M}(336_a) - \mathrm{M}(405'_a) \\ & - \mathrm{M}(315_a) + \mathrm{M}(105'_b) + \mathrm{M}(189_b) - \mathrm{M}(21_b) - \mathrm{M}(1'_a).\end{aligned}$$

The standard module of minimal h_c-weight is $\mathrm{M}(35'_a)$ and it appears with a minus sign, so this cannot be the character of a module. Therefore M cannot be the character of $\mathrm{L}(2268_x)$, and $\alpha = 0$.

By a similar argument, considering the restriction of $\mathrm{L}(700_x)$ to the principal block of $\mathcal{O}_{\frac{1}{4}}(E_7)$ we find that $[\mathrm{M}(700_x) : \mathrm{L}(2100_x)] = [\mathrm{M}(700_x) : \mathrm{L}(448_w)] = 0$.

It remains to find three decomposition numbers in row 84_x. Let $M = \mathrm{M}(84_x) - \mathrm{M}(700_x) + \mathrm{M}(2268_x) - \mathrm{M}(2100_x) + \mathrm{M}(448_w)$. The character of $\mathrm{L}(84_x)$ is given by $M - d_{84_x,2268_x}\mathrm{L}(2268_x) - d_{84_x,2100_x}\mathrm{L}(2100_x) - d_{84_x,448_w}\mathrm{L}(448_w)$, where $d_{84_x,\tau}$ denotes $[\mathrm{M}(84_x) : \mathrm{L}(\tau)]$. Let F be the functor of parabolic restriction to $\mathcal{O}_{\frac{1}{4}}(E_7)$ followed by projecting to the principal block. We find that $F(M) = \mathrm{L}(1_a)$ and $F(\mathrm{L}(\tau)) \neq 0$ for $\tau \in \{2268_x, 2100_x, 448_w\}$. The character of $F(\mathrm{L}(84_x))$ is thus the character of a module if and only if $[\mathrm{M}(84_x) : \mathrm{L}(\tau)] = 0$ for $\tau \in \{2268_x, 2100_x, 448_w\}$.

Tables 49 and 50. These two blocks are dual to each other under tensoring $\tau \in \mathrm{Irr}\, E_8$ by the sign character, and under Ringel duality at the categorical level. We exploit this to complete their decomposition matrices from the columns given by the Hecke algebra.

In table 49, all decomposition numbers are given by the columns of the Hecke algebra decomposition matrix and then applying Lemmas 15 and 16 with a single exception, $[\mathrm{M}(560_z) : \mathrm{L}(840_z)] =: \alpha$. The character of $\mathrm{L}(560_z)$ depending on α is given by:

$$\begin{aligned}\mathrm{L}(560_z) &= \mathrm{M}(560_z) - \mathrm{M}(1344_x) - \mathrm{M}(1400_{zz}) + \mathrm{M}(840_z) + \mathrm{M}(4536_z) - \mathrm{M}(3200_x) \\ &\quad + \mathrm{M}(2240'_x) - \mathrm{M}(3240'_z) + \mathrm{M}(1008'_z) - \alpha\mathrm{L}(840_z) \\ &=: M - \alpha\mathrm{L}(840_z).\end{aligned}$$

Calculating the graded character shows that $\dim \mathrm{Supp}\, \mathrm{L}(560_z) = 4 = \dim \mathrm{Supp}\, \mathrm{L}(840_z)$, and this is the minimal dimension of support for simple modules in this block. Ringel

Table 49. Decomposition matrix of the block of $\mathcal{O}_{\frac{1}{4}}(E_8)$ containing $\mathrm{L}(8_z)$

$$
\begin{array}{rr}
(4) & 8_z \\
(4) & 560_z \\
(5) & 1344_x \\
(5) & 1400_{zz} \\
(4) & 840_z \\
 & 4536_z \\
 & 3200_x \\
 & 4200'_z \\
 & 2240'_x \\
 & 3240'_z \\
 & 1400'_z \\
 & 1008'_z \\
 & 56'_z
\end{array}
\begin{pmatrix}
1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & 1 & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & 1 & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{pmatrix}
$$

duality sends $\mathrm{L}(840_z)$ to $\mathrm{L}(1400_z)$ and sends $\mathrm{L}(560_z)$ to $\mathrm{L}(1008_z)$. Set

$$
\begin{aligned}
M' := \mathsf{R}(M) = \mathrm{M}(1008_z) - \mathrm{M}(3240_z) + \mathrm{M}(2240_x) - \mathrm{M}(3200'_x) + \mathrm{M}(4536'_z) \\
+ \mathrm{M}(840'_z) - \mathrm{M}(1400'_{zz}) - \mathrm{M}(1344'_x) + \mathrm{M}(560'_z).
\end{aligned}
$$

Applying Lemma 13 yields $\mathrm{L}(1008_z) = M' - \alpha\mathrm{L}(1400_z)$.

Table 50. Decomposition matrix of the block of $\mathcal{O}_{\frac{1}{4}}(E_8)$ containing $\mathrm{L}(8z')$

$$
\begin{array}{rr}
(4) & 1400_z \\
(4) & 1008_z \\
(4) & 56_z \\
(5) & 3240_z \\
 & 2240_x \\
 & 4200_z \\
(5) & 3200'_x \\
 & 4536'_z \\
 & 1400'_{zz} \\
 & 840'_z \\
 & 1344'_x \\
 & 560'_z \\
 & 8'_z
\end{array}
\begin{pmatrix}
1 & \cdot & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & 1 & \cdot & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & 1 & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & 1 & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & 1 & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1
\end{pmatrix}
$$

On the other hand, in Table 50 all decomposition numbers are given by the columns of the Hecke algebra decomposition matrix and then applying Lemmas 15 and 16 except for $[\mathrm{M}(1400_z) : \mathrm{L}(3200'_x)] =: \beta$, $[\mathrm{M}(1008_z) : \mathrm{L}(3200'_x)] =: \gamma$, and $[\mathrm{M}(3240_z) : \mathrm{L}(3200'_x)] =: \delta$. The three unknown characters of simple modules in this block are determined as:

$$\begin{aligned}
\mathrm{L}(3240_z) &= \mathrm{M}(3240_z) - \mathrm{M}(2240_x) - \mathrm{M}(4200_z) + (1-\beta)\mathrm{M}(3200'_x) + \beta\mathrm{M}(4536'_z) \\
&\quad + (1-\beta)\mathrm{M}(1400'_{zz}) + (\beta-1)\mathrm{M}(840'_z) - \beta\mathrm{M}(1344'_x) + (\beta-1)\mathrm{M}(560'_z) + \beta\mathrm{M}(8'_z), \\
\mathrm{L}(1008_z) &= \mathrm{M}(1008_z) - \mathrm{M}(3240_z) + \mathrm{M}(2240_x) - \gamma\mathrm{M}(3200'_x) + \gamma\mathrm{M}(4536'_z) - \gamma\mathrm{M}(1400'_{zz}), \\
&\quad + \gamma\mathrm{M}(840'_z) - \gamma\mathrm{M}(1344'_x) + \gamma\mathrm{M}(560'_z) + (\gamma-1)\mathrm{M}(8'_z) \\
\mathrm{L}(1400_z) &= \mathrm{M}(1400_z) - \mathrm{M}(3240_z) + \mathrm{M}(4200_z) - (\delta+1)\mathrm{M}(3200'_x) + \delta\mathrm{M}(4536'_z) \\
&\quad - \delta\mathrm{M}(1400'_{zz}) + (\delta+1)\mathrm{M}(840'_z) - \delta\mathrm{M}(1344'_x) + \delta\mathrm{M}(560'_z) + \delta\mathrm{M}(8'_z).
\end{aligned}$$

As computed above using Ringel duality, $\mathrm{L}(1008_z) = M' - \alpha\mathrm{L}(1400'_z)$, forcing $\alpha = 0$ and $\gamma = 1$. Likewise, $\mathrm{L}(1400_z) = \mathsf{R}(\mathrm{L}(840_z))$ and so by Lemma 13 and so its character is given from the character of $\mathrm{L}(840_z)$ by replacing $\mathrm{M}(\tau)$ with $\mathrm{M}(\tau')$. This yields $\delta = 0$.

We use a different argument to determine β because $\mathrm{L}(3240_z)$ does not have minimal support in its block. Let F be the functor of parabolic restriction to $\mathcal{O}_{\frac{1}{4}}(E_7)$ followed by projection to the block containing $\mathrm{L}(35_b)$. First, we establish that $\beta \in \{0,1\}$. From the characters of $\mathrm{L}(3200_x)$, $\mathrm{L}(2240'_x)$, and $\mathrm{L}(4200'_z)$ and from applying Lemma 15, it follows that $\dim\mathrm{Hom}(\mathrm{M}(3200'_x), \mathrm{M}(3240_z)) = \dim\mathrm{Hom}(\mathrm{M}(3200'_x), \mathrm{M}(2240_x)) = 0$, while $\dim\mathrm{Hom}(\mathrm{M}(3200'_x), \mathrm{M}(4200_z)) = 1$. This implies that β is either 0 or 1. Suppose $\beta = 1$, so that $\mathrm{M}(3240_z) - \mathrm{M}(2240_x) - \mathrm{M}(4200_z) + \mathrm{M}(4536'_z) - \mathrm{M}(1344'_x) + \mathrm{M}(8'_z) =: X$ would be the character of $\mathrm{L}(3240_z)$. We find that $F(X) = \mathrm{M}(35_b) - \mathrm{M}(210_b) - \mathrm{M}(105_c) + \mathrm{M}(378_a) - \mathrm{M}(105_a) + \mathrm{M}(7_a)$. Since 35_b has the smallest h_c-weight among τ such that $\mathrm{M}(\tau)$ appears with nonzero coefficient in $F(X)$, it holds that $\mathrm{L}(35_b)$ is a composition factor of $F(X)$. Subtracting off the character of $\mathrm{L}(35_b) = \mathrm{M}(35_b) - \mathrm{M}(210_b) + \mathrm{M}(280'_b) - \mathrm{M}(105_a)$ from $F(X)$, we obtain the virtual character $-\mathrm{M}(105_c) + \mathrm{M}(378_a) - \mathrm{M}(280'_b) + \mathrm{M}(7_a)$. But $F(X) - \mathrm{L}(35_b)$ cannot be the character of a module since the smallest h_c-weight term, $-\mathrm{M}(105_c)$, appears with a minus sign. Since $F(X)$ is not a module, neither is X. We conclude that $\beta = 0$. □

7. The finite-dimensional irreducible representations

Our method in this paper for finding finite-dimensional irreducible representations and their dimensions has been to calculate the inverse of the decomposition matrix and then test from there which irreducible representations have a polynomial for their graded character. This method has the advantage of not only giving a definitive answer to the question of which irreducible representations are finite-dimensional, but also producing the graded characters and dimensions of those representations at the same time. The disadvantage is that it's a lot of work to calculate the decomposition matrices if all that is wanted is a list of the finite-dimensional representations. In the case of E_8 and parameters c with denominators $2,3,4,6$, the principal blocks are very large and computing their decomposition matrices would be a horrible task.

In the time since the first of the preprints on which the current monograph is based appeared, more work has appeared by two different sets of authors on the problem of classifying the finite-dimensional representations of rational Cherednik algebras. Both approaches use the fact that the finite-dimensional representations of $H_c(W)$ coincide with the cuspidal representations: they are exactly the representations whose parabolic restriction is 0 for every proper parabolic of W [3]. The first approach, by Griffeth, Gusenbauer, Juteau, and Lanini, studies a modification of the parabolic restriction functor to obtain necessary conditions for an irreducible representation to be finite-dimensional [18, Theorem 1.3]. The authors then produce short-lists of the potential finite-dimensional representations at various parameters [18, §5]. The second approach, by Losev and Shelley-Abrahamson, adapts Howlett-Lehrer's methods to define Harish-Chandra series for rational Cherednik algebras [26]. The number of irreducible representations belonging to each series equals the number of irreducible representations of a Hecke algebra at certain parameters, so the number of finite-dimensional irreducible representations is the number of irreducible representations left over after all these are tallied [26].

Losev and Shelley-Abrahamson used the existing results in conjunction with their methods to provide a classification of the finite-dimensional irreducible representations of $H_c(W)$ for all Coxeter groups W, with the exception of the parameter $c = 1/3$ for $W = E_8$ where there was still ambiguity [26, Table 4]. In Section 7.1, we resolve that last remaining case. In Section 7.2, we provide the graded characters and dimensions of the finite-dimensional irreducible representations obtained from our decomposition matrices.

7.1. **The last step of the classification of finite-dimensional irreducible representations: the case of $H_{1/3}(E_8)$.** In this section, we finish the classification of finite-dimensional irreducible representations of rational Cherednik algebras $H_c(W)$ with equal parameters for exceptional Coxeter groups by resolving the last open case: when $W = E_8$ and the denominator of c is 3.

Let $V = \mathbb{C}^8$ be the reflection representation of E_8. According to [18, Section 5.10], if $\mathrm{L}(\tau)$ is finite-dimensional then:

$$\tau \in \{\mathrm{triv}, V, \phi_{28,8}, \phi_{35,2}, \phi_{50,8}, \phi_{160,7}, \phi_{175,12}, \phi_{300,8}, \phi_{840,13}\} \subset \mathrm{Irr}\, E_8.$$

Switching to the notation of [14] and [13] for $\mathrm{Irr}\, E_8$, if $\mathrm{L}(\tau)$ is finite-dimensional then:

$$\tau \in \{1_x, 8_z, 28_x, 35_x, 50_x, 160_z, 175_x, 300_x, 840_z\} \subset \mathrm{Irr}\, E_8.$$

The work of Losev and Shelley-Abrahamson on Harish-Chandra series for rational Cherednik algebras provides a count of the number of finite-dimensional representations [26]. They find that there are exactly eight finite-dimensional irreducibles [26, Table 4]. To finish the classification it therefore suffices to identify for which τ in the list above, $\mathrm{L}(\tau)$ is infinite-dimensional.

Theorem 33. *Let $c = 1/3$. Then $\mathrm{L}(50_x)$ is infinite-dimensional.*

Proof. If L is a finite-dimensional $H_c(W)$-module then by Lemma 6 the equality $\dim_{\mathbb{C}} \mathrm{L}[-i] = \dim_{\mathbb{C}} \mathrm{L}[i]$ must hold for any $i \in \mathbb{Z}$, where $\mathrm{L}[i]$ denotes the graded piece of L in degree i. We will show that $\dim_{\mathbb{C}} \mathrm{L}(50_x)[-1] < \dim_{\mathbb{C}} \mathrm{L}(50_x)[1]$, which means that $\mathrm{L}(50_x)$ cannot be finite-dimensional.

When $c = 1/3$, $h_c(\tau) = (\dim V)/2 - c\sum_{\text{reflections } s\in E_8} s|_\tau = 4 - (1/3)120\tau(s)/\dim\tau$. Table C.6 of [14] gives the values of $\tau(s)$, from which $h_c(50_x) = -12$. Next, the decomposition matrix of the Hecke algebra at a root of unity for $e = 3$ and $W = E_8$ given in Table 7.15 of [13] is a submatrix of the decomposition matrix for $\mathcal{O}_{1/3}(E_8)$, but one must tensor the labels τ for rows and columns by the sign representation. Write $\tau' := \tau \otimes \mathrm{sign}$. Look at the column for 50_x in Table 7.15 of [13], which corresponds to the column for $50'_x$ in $\mathcal{O}_{1/3}(E_8)$, and observe that every entry in the column 50_x of Table 7.15 is zero (excepting row 50_x itself of course) until the row labeled 700_{xx}, where there is a 1. Applying Lemma 16, this means that $\dim\mathrm{Hom}(\mathrm{M}(50'_x), \mathrm{M}(\tau')) = 0$ for all $\tau \in \mathrm{Irr}\, E_8$, $h_c(50_x) < h_c(\tau) < h_c(700_{xx})$, but $\dim\mathrm{Hom}(\mathrm{M}(50'_x), \mathrm{M}(700'_{xx})) = 1$. On the other hand, $h_c(700_{xx}) = 0$ and for all other σ such that $h_c(\sigma) = 0$, it follows from Table 7.15 of [13] and Lemma 16 that $\dim\mathrm{Hom}(\mathrm{M}(50'_x), \mathrm{M}(\sigma')) = 0$. Moreover, there is no $\tau \in \mathrm{Irr}\, E_8$ with $h_c(\tau) = 1$. By Lemma 15, it follows that for all $700_{xx} \neq \tau \in \mathrm{Irr}\, E_8$ satisfying $h_c(50_x) < h_c(\tau) < 2$, $\dim\mathrm{Hom}(\mathrm{M}(\tau), \mathrm{M}(50_x)) = 0$, while $\dim\mathrm{Hom}(\mathrm{M}(700_{xx}), \mathrm{M}(50_x)) = 1$.

The expression in the Grothendieck group of $\mathcal{O}_{1/3}(E_8)$ for $\mathrm{L}(50_x)$ in the basis of Vermas $\mathrm{M}(\tau)$ is therefore: $\mathrm{L}(50_x) = \mathrm{M}(50_x) - \mathrm{M}(700_{xx}) + \sum_{h_c(\tau)\geq 2} a_{50_x,\tau}\mathrm{M}(\tau)$. The graded character of $\mathrm{L}(50_x)$ truncated after degree 1 is then $\sum_{k=0}^{13}\binom{7+k}{7}50t^{-12+k} - \sum_{j=0}^{1}\binom{7+j}{7}700t^j$, and so the graded dimensions of $\mathrm{L}(50_x)$ in degrees -1 and 1 (the coefficients of t^{-1} and t in the graded character) are:

$$\dim\mathrm{L}(50_x)[-1] = \binom{18}{7}50 = 1,591,200$$

$$\dim\mathrm{L}(50_x)[1] = \binom{20}{7}50 - \binom{8}{7}700 = 3,870,400$$

Since $\dim\mathrm{L}(50_x)[-1] < \dim\mathrm{L}(50_x)[1]$, $\mathrm{L}(50_x)$ must be infinite-dimensional. $\square$

It follows that the finite-dimensional irreducible representations of $H_{1/3}(E_8)$ are $\mathrm{L}(\tau)$ where $\tau \in \{1_x, 8_z, 28_x, 35_x, 160_z, 175_x, 300_x, 840_z\} \subset \mathrm{Irr}\, E_8$. By category equivalences, the cases of $c = r/3$ for $r \neq 1$ reduce to the case $c = 1/3$ [15]. This completes the classification.

7.2. **Graded characters and dimensions of finite-dimensional representations.** In this section, we provide the graded characters and dimensions of the finite-dimensional irreducible representations belonging to the blocks whose decomposition matrices were found in the previous sections. Additionally, in the case that at the given parameter there are other finite-dimensional irreducible representations belonging to different blocks than those appearing in this paper, we print those as well. This situation occurs only in the following two cases: when $W = H_4$ and $c = 1/10$, or when $W = E_8$ and $c = 1/12$. Then, in addition to the finite-dimensional representations in the principal block, there is one irreducible representation belonging to a defect 1 block, whose character and dimension were found by Rouquier [29, §5.2.4].

We do not list the finite-dimensional representations at those parameters for which the category $\mathcal{O}$ only has defect 1 or defect 0 blocks as these were given in [29, §5.2.4]. When $W = E_8$ and $c = 1/6, 1/4, 1/3$, the defect 2 blocks do not contain finite-dimensional representations, and we do not know the characters and dimensions of the finite-dimensional representations belonging to the principal block. Finally, when $W = F_4, E_7$, or E_8, we omit consideration of the characters of the finite-dimensional irreducible representations when $c = 1/2$.

7.3. **Finite-dimensional representations of $H_c(H_4)$.**

7.3.1. *H_4, $c = 1/10$.* The irreducible representations $\mathrm{L}(\mathbf{1})$, $\mathrm{L}(\mathbf{5})$, and $\mathrm{L}(\mathbf{13})$ in the principal block are finite-dimensional and have the following graded characters and dimensions:

$$\chi_{\mathrm{L}(\mathbf{1})}(t) = t^{-4} + t^4 + 4(t^{-3} + t^3) + 10(t^{-2} + t^2) + 20(t^{-1} + t) + 35$$
$$\dim \mathrm{L}(\mathbf{1}) = 105$$

$$\chi_{\mathrm{L}(\mathbf{5})}(t) = 4(t^{-1} + t) + 16$$
$$\dim \mathrm{L}(\mathbf{5}) = 24$$

$$\chi_{\mathrm{L}(\mathbf{13})}(t) = 9$$
$$\dim \mathrm{L}(\mathbf{13}) = 9$$

Additionally, $\mathrm{L}(\mathbf{31})$ (belonging to a defect 1 block) is finite-dimensional and has the following graded character and dimension [29, §5.2.4]:

$$\chi_{\mathrm{L}(\mathbf{3})}(t) = 4(t^{-1} + t) + 7$$
$$\dim \mathrm{L}(\mathbf{3}) = 15$$

7.3.2. H_4, $c = 1/6$. The irreducible representations $\mathrm{L}(\mathbf{1})$, $\mathrm{L}(\mathbf{5})$, and $\mathrm{L}(\mathbf{3})$ are finite-dimensional and have the following graded characters and dimensions:

$$\begin{aligned}\chi_{\mathrm{L}(\mathbf{1})}(t) &= t^{-8} + t^{8} + 4(t^{-7} + t^{7}) + 10(t^{-6} + t^{6}) + 20(t^{-5} + t^{5}) + 35(t^{-4} + t^{4})\\ &\quad + 56(t^{-3} + t^{3}) + 84(t^{-2} + t^{2}) + 120(t^{-1} + t) + 140\\ \dim \mathrm{L}(\mathbf{1}) &= 800\end{aligned}$$

$$\begin{aligned}\chi_{\mathrm{L}(\mathbf{3})}(t) &= \chi_{\mathrm{L}(\mathbf{5})}(t) = 4(t^{-3} + t^{3}) + 16(t^{-2} + t^{2}) + 40(t^{-1} + t) + 55\\ \dim \mathrm{L}(\mathbf{3}) &= \dim \mathrm{L}(\mathbf{5}) = 175\end{aligned}$$

7.3.3. H_4, $c = 1/5$. The irreducible representations $\mathrm{L}(\mathbf{1})$, $\mathrm{L}(\mathbf{11})$, $\mathrm{L}(\mathbf{3})$, and $\mathrm{L}(\mathbf{20})$ are finite-dimensional and have the following graded characters and dimensions:

$$\begin{aligned}\chi_{\mathrm{L}(\mathbf{1})}(t) &= t^{-10} + t^{10} + 4(t^{-9} + t^{9}) + 10(t^{-8} + t^{8}) + 20(t^{-7} + t^{7}) + 35(t^{-6} + t^{6})\\ &\quad + 56(t^{-5} + t^{5}) + 84(t^{-4} + t^{4}) + 120(t^{-3} + t^{3}) + 165(t^{-2} + t^{2})\\ &\quad + 204(t^{-1} + t) + 222\\ \dim \mathrm{L}(\mathbf{1}) &= 1620\end{aligned}$$

$$\begin{aligned}\chi_{\mathrm{L}(\mathbf{11})}(t) &= 9(t^{-2} + t^{2}) + 20(t^{-1} + t) + 26\\ \dim \mathrm{L}(\mathbf{11}) &= 84\end{aligned}$$

$$\begin{aligned}\chi_{\mathrm{L}(\mathbf{3})}(t) &= 4(t^{-4} + t^{4}) + 16(t^{-3} + t^{3}) + 40(t^{-2} + t^{2}) + 80(t^{-1} + t) + 104\\ \dim \mathrm{L}(\mathbf{3}) &= 384\end{aligned}$$

$$\begin{aligned}\chi_{\mathrm{L}(\mathbf{20})}(t) &= 16(t^{-1} + t) + 28\\ \dim \mathrm{L}(\mathbf{20}) &= 60\end{aligned}$$

7.3.4. H_4, $c = 1/4$. The irreducible representations $\mathrm{L}(\mathbf{1})$, $\mathrm{L}(\mathbf{11})$, and $\mathrm{L}(\mathbf{13})$ are finite-dimensional and have the following graded characters and dimensions:

$$\begin{aligned}\chi_{L(\mathbf{1})}(t) &= t^{-13} + t^{13} + 4(t^{-12} + t^{12}) + 10(t^{-11} + t^{11}) + 20(t^{-10} + t^{10}) + 35(t^{-9} + t^{9}) \\ &+ 56(t^{-8} + t^{8}) + 84(t^{-7} + t^{7}) + 120(t^{-6} + t^{6}) + 165(t^{-5} + t^{5}) \\ &+ 220(t^{-4} + t^{4}) + 268(t^{-3} + t^{3}) + 292(t^{-2} + t^{2}) + 300(t^{-1} + t) + 300\end{aligned}$$
$$\dim L(\mathbf{1}) = 3450$$

$$\chi_{L(\mathbf{11})}(t) = \chi_{L(\mathbf{13})}(t) = 9(t^{-3} + t^{3}) + 36(t^{-2} + t^{2}) + 65(t^{-1} + t) + 80$$
$$\dim \chi_{L(\mathbf{11})} = \dim \chi_{L(\mathbf{13})} = 300$$

7.3.5. H_4, $c = 1/3$. The irreducible representations $\mathrm{L}(\mathbf{1})$, $\mathrm{L}(\mathbf{18})$, $\mathrm{L}(\mathbf{3})$, and $\mathrm{L}(\mathbf{5})$ are finite-dimensional and have the following graded characters and dimensions:

$$\begin{aligned}\chi_{\mathrm{L}(\mathbf{1})}(t) &= t^{-18} + t^{18} + 4(t^{-17} + t^{17}) + 10(t^{-16} + t^{16}) + 20(t^{-15} + t^{15}) + 35(t^{-14} + t^{14}) \\ &+ 56(t^{-13} + t^{13}) + 84(t^{-12} + t^{12}) + 120(t^{-11} + t^{11}) + 165(t^{-10} + t^{10}) \\ &+ 220(t^{-9} + t^{9}) + 286(t^{-8} + t^{8}) + 364(t^{-7} + t^{7}) + 455(t^{-6} + t^{6}) \\ &+ 560(t^{-5} + t^{5}) + 680(t^{-4} + t^{4}) + 816(t^{-3} + t^{3}) + 944(t^{-2} + t^{2}) \\ &+ 1040(t^{-1} + t) + 1080\end{aligned}$$
$$\dim \mathrm{L}(\mathbf{1}) = 12,800$$

$$\chi_{\mathrm{L}(\mathbf{18}}(t) = 16(t^{-3} + t^{3}) + 39(t^{-2} + t^{2}) + 60(t^{-1} + t) + 70$$
$$\dim \mathrm{L}(\mathbf{18}) = 300$$

$$\begin{aligned}\chi_{\mathrm{L}(\mathbf{3})}(t) &= 4(t^{-8} + t^{8}) + 16(t^{-7} + t^{7}) + 40(t^{-6} + t^{6}) + 80(t^{-5} + t^{5}) + 140(t^{-4} + t^{4}) \\ &+ 208(t^{-3} + t^{3}) + 272(t^{-2} + t^{2}) + 320(t^{-1} + t) + 340\end{aligned}$$
$$\dim \mathrm{L}(\mathbf{3}) = 2500$$

$$\begin{aligned}\chi_{\mathrm{L}(\mathbf{5})}(t) &= 4(t^{-8} + t^{8}) + 16(t^{-7} + t^{7}) + 40(t^{-6} + t^{6}) + 80(t^{-5} + t^{5}) + 140(t^{-4} + t^{4}) \\ &+ 208(t^{-3} + t^{3}) + 272(t^{-2} + t^{2}) + 320(t^{-1} + t) + 340\end{aligned}$$
$$\dim \mathrm{L}(\mathbf{5}) = 2500$$

7.3.6. H_4, $c = 1/2$. The irreducible representations $\mathrm{L}(\mathbf{1})$, $\mathrm{L}(\mathbf{5})$, $\mathrm{L}(\mathbf{3})$, $\mathrm{L}(\mathbf{13}$, $\mathrm{L}(\mathbf{11})$, and $\mathrm{L}(\mathbf{27})$ are finite-dimensional and have the following graded characters and dimensions:

$$\begin{aligned}\chi_{\mathrm{L}(\mathbf{1})}(t) = {} & 3775 + 3700(t^{-1}+t) + 3510(t^{-2}+t^2) + 3240(t^{-3}+t^3) + 2925(t^{-4}+t^4)\\ & + 2600(t^{-5}+t^5) + 2300(t^{-6}+t^6) + 2024(t^{-7}+t^7) + 1771(t^{-8}+t^8)\\ & + 1540(t^{-9}+t^9) + 1330(t^{-10}+t^{10}) + 1140(t^{-11}+t^{11}) + 969(t^{-12}+t^{12})\\ & + 816(t^{-13}+t^{13}) + 680(t^{-14}+t^{14}) + 560(t^{-15}+t^{15}) + 455(t^{-16}+t^{16})\\ & + 364(t^{-17}+t^{17}) + 286(t^{-18}+t^{18}) + 220(t^{-19}+t^{19}) + 165(t^{-20}+t^{20})\\ & + 120(t^{-21}+t^{21}) + 84(t^{-22}+t^{22}) + 56(t^{-23}+t^{23}) + 35(t^{-24}+t^{24})\\ & + 20(t^{-25}+t^{25}) + 10(t^{-26}+t^{26}) + 4(t^{-27}+t^{27}) + t^{-28} + t^{28}\end{aligned}$$

$\dim \mathrm{L}(\mathbf{1}) = 65625$

$$\chi_{\mathrm{L}(\mathbf{27})}(t) = 25(t^{-4}+t^4) + 64(t^{-3}+t^3) + 106(t^{-2}+t^2) + 140(t^{-1}+t) + 155$$

$\dim \mathrm{L}(\mathbf{27}) = 825$

$$\begin{aligned}\chi_{\mathrm{L}(\mathbf{3})}(t) = \chi_{\mathrm{L}(\mathbf{5})}(t) = {} & 4(t^{-13}+t^{13}) + 16(t^{-12}+t^{12}) + 40(t^{-11}+t^{11}) + 80(t^{-10}+t^{10})\\ & + 140(t^{-9}+t^9) + 215(t^{-8}+t^8) + 300(t^{-7}+t^7) + 390(t^{-6}+t^6)\\ & + 480(t^{-5}+t^5) + 540(t^{-4}+t^4) + 576(t^{-3}+t^3) + 594(t^{-2}+t^2)\\ & + 600(t^{-1}+t^1) + 600\end{aligned}$$

$\dim \mathrm{L}(\mathbf{3}) = \dim \mathrm{L}(\mathbf{5}) = 8550$

$$\begin{aligned}\chi_{\mathrm{L}(\mathbf{11})}(t) = \chi_{\mathrm{L}(\mathbf{13})}(t) = {} & 765 + 720(t^{-1}+t) + 612(t^{-2}+t^2) + 468(t^{-3}+t^3) + 315(t^{-4}+t^4)\\ & + 180(t^{-5}+t^5) + 90(t^{-6}+t^6) + 36(t^{-7}+t^7) + 9(t^{-8}+t^8)\end{aligned}$$

$\dim \mathrm{L}(\mathbf{11}) = \dim \mathrm{L}(\mathbf{13}) = 5625$

7.4. **Finite-dimensional representations of $H_c(F_4)$ with equal parameters.**

7.4.1. *F_4, $c = 1/6$.* The irreducible representations $\mathrm{L}(1_1)$, $\mathrm{L}(2_3)$, and $\mathrm{L}(2_1)$ are finite-dimensional and have the following graded characters and dimensions:

$$\chi_{\mathrm{L}(1_1)}(t) = t^{-2} + t^2 + 4(t^{-1} + t) + 10$$
$$\dim \mathrm{L}(1_1) = 20$$

$$\chi_{\mathrm{L}(2_1)}(t) = \chi_{\mathrm{L}}(2_3)(t) = 2$$
$$\dim \mathrm{L}(2_1) = \dim \mathrm{L}(2_3) = 2$$

7.4.2. *F_4, $c = 1/4$.* The irreducible representations $\mathrm{L}(1_1)$ and $\mathrm{L}(4_2)$ are finite-dimensional and have the following graded characters and dimensions:

$$\chi_{\mathrm{L}(1_1)}(t) = t^{-4} + t^4 + 4(t^{-3} + t^3) + 10(t^{-2} + t^2) + 20(t^{-1} + t) + 26$$
$$\dim \mathrm{L}(1_1) = 96$$

$$\chi_{\mathrm{L}(4_2)}(t) = 4(t^{-1} + t) + 7$$
$$\dim \mathrm{L}(4_2) = 15$$

7.4.3. *F_4, $c = 1/3$.* The irreducible representations $\mathrm{L}(1_1)$ and $\mathrm{L}(4_2)$ are finite-dimensional and have the following graded characters and dimensions:

$$\begin{aligned}\chi_{\mathrm{L}(1_1)}(t) = t^{-6} + t^6 + 4(t^{-5} + t^5) + 10(t^{-4} + t^4) + 20(t^{-3} + t^3) + 31(t^{-2} + t^2)\\ + 40(t^{-1} + t) + 44\end{aligned}$$
$$\dim \mathrm{L}(1_1) = 256$$

$$\chi_{\mathrm{L}(4_2)}(t) = 4(t^{-1} + t^2) + 16(t^{-1} + t) + 24$$
$$\dim \mathrm{L}(4_2) = 64$$

7.5. **Finite-dimensional representations of $H_c(E_6)$.**

7.5.1. *E_6, $c = 1/6$.* The irreducible representations $\mathrm{L}(1_p)$ and $\mathrm{L}(6_p)$ are finite-dimensional and have the following graded characters and dimensions:

$$\chi_{\mathrm{L}(1_p)}(t) = t^{-3} + t^3 + 6(t^{-2} + t^2) + 21(t^{-1} + t) + 36$$
$$\dim \mathrm{L}(1_p) = 92$$

$$\chi_{\mathrm{L}(6_p)}(t) = 6(t^{-1} + t) + 16$$
$$\dim \mathrm{L}(6_p) = 28$$

7.5.2. E_6, $c = 1/3$. The irreducible representations $\mathrm{L}(1_p)$, $\mathrm{L}(6_p)$, and $\mathrm{L}(15_p)$ are finite-dimensional and have the following graded characters and dimensions:

$$\begin{aligned}\chi_{\mathrm{L}(1_p)}(t) &= t^{-9} + t^9 + 6(t^{-8} + t^8) + 21(t^{-7} + t^7) + 56(t^{-6} + t^6) + 120(t^{-5} + t^5) \\ &\quad + 216(t^{-4} + t^4) + 336(t^{-3} + t^3) + 456(t^{-2} + t^2) + 561(t^{-1} + t) + 606\end{aligned}$$
$$\dim \mathrm{L}(1_p) = 4152$$

$$\chi_{\mathrm{L}(6_p)}(t) = 6(t^{-5} + t^5) + 36(t^{-4} + t^4) + 106(t^{-3} + t^3) + 216(t^{-2} + t^2) + 306(t^{-1} + t) + 340$$
$$\dim \mathrm{L}(6_p) = 1680$$

$$\chi_{\mathrm{L}(15_p)}(t) = 15(t^{-1} + t) + 26$$
$$\dim \mathrm{L}(15_p) = 56$$

7.6. Finite-dimensional representations of $H_c(E_7)$.

7.6.1. E_7, $c = 1/6$. The irreducible representations $\mathrm{L}(1_a)$, $\mathrm{L}(7'_a)$, $\mathrm{L}(21_a)$, and $\mathrm{L}(15'_a)$ are finite-dimensional and have the following graded characters and dimensions:

$$\begin{aligned}\chi_{\mathrm{L}(1_a)}(t) &= t^{-7} + t^7 + 7(t^{-6} + t^6) + 28(t^{-5} + t^5) + 84(t^{-4} + t^4) + 210(t^{-3} + t^3) \\ &\quad + 441(t^{-2} + t^2) + 742(t^{-1} + t) + 868\end{aligned}$$
$$\dim \mathrm{L}(1_a) = \chi_{\mathrm{L}(1_a)}(1) = 3894$$

$$\chi_{\mathrm{L}(7'_a)}(t) = 7(t^{-4} + t^4) + 49(t^{-3} + t^3) + 175(t^{-2} + t^2) + 406(t^{-1} + t) + 532$$
$$\dim \mathrm{L}(7'_a) = \chi_{\mathrm{L}(7'_a)}(1) = 1806$$

$$\chi_{\mathrm{L}(21_a)}(t) = 21(t + t^{-1}) + 42$$
$$\dim \mathrm{L}(21_a) = 84$$

$$\chi_{\mathrm{L}(15'_a)}(t) = 15$$
$$\dim \mathrm{L}(15'_a) = 15$$

7.7. Finite-dimensional representations of $H_c(E_8)$.

7.7.1. E_8, $c = 1/12$. The irreducible representations $\mathrm{L}(1_x)$, $\mathrm{L}(35_x)$, and $\mathrm{L}(50_x)$ are finite-dimensional and have the following graded characters and dimensions:

$$\begin{aligned}\chi_{\mathrm{L}(1_x)}(t) &= t^{-6}+t^{6}+8(t^{-5}+t^{5})+36(t^{-4}+t^{4})+120(t^{-3}+t^{3})\\ &\qquad +330(t^{-2}+t^{2})+680(t^{-1}+t)+1030\\ \dim \mathrm{L}(1_x) &= 3380\end{aligned}$$

$$\begin{aligned}\chi_{\mathrm{L}(35_x)}(t) &= 35(t^{-2}+t^{2})+168(t^{-1}+t)+364\\ \dim \mathrm{L}(35_x) &= 770\end{aligned}$$

$$\begin{aligned}\chi_{\mathrm{L}(50_x)}(t) &= 50\\ \dim \mathrm{L}(50_x) &= 50\end{aligned}$$

Additionally, $\mathrm{L}(28_x)$ (belonging to a defect 1 block) is finite-dimensional and has character and dimension [29, §5.2.4]:

$$\begin{aligned}\chi_{\mathrm{L}(28_x)}(t) &= 28(t+t^{-1})+64\\ \dim \mathrm{L}(28_x) &= 120\end{aligned}$$

7.7.2. E_8, $c = 1/10$. The irreducible representations $\mathrm{L}(1_x)$, $\mathrm{L}(8_z)$, and $\mathrm{L}(28_x)$ are finite-dimensional and have the following graded characters and dimensions:

$$\begin{aligned}\chi_{\mathrm{L}(1_x)}(t) &= t^{-8}+t^{8}+8(t^{-7}+t^{7})+36(t^{-6}+t^{6})+120(t^{-5}+t^{5})+330(t^{-4}+t^{4})\\ &\qquad +792(t^{-3}+t^{3})+1632(t^{-2}+t^{2})+2760(t^{-1}+t)+3411\\ \dim \mathrm{L}(1_x) &= 14,769\end{aligned}$$

$$\begin{aligned}\chi_{\mathrm{L}(8_z)}(t) &= 8(t^{-5}+t^{5})+64(t^{-4}+t^{4})+288(t^{-3}+t^{3})+876(t^{-2}+t^{2})+1968(t^{-1}+t)+2745\\ \dim \mathrm{L}(8_z) &= 9153\end{aligned}$$

$$\begin{aligned}\chi_{\mathrm{L}(28_x)}(t) &= 28(t^{-2}+t^{2})+224(t^{-1}+t)+441\\ \dim \mathrm{L}(28_x) &= 945\end{aligned}$$

7.7.3. E_8, $c = 1/8$. The irreducible representations $\mathrm{L}(1_x)$ and $\mathrm{L}(160_z)$ are finite-dimensional and have the following graded characters and dimensions:

$$\begin{aligned}\chi_{\mathrm{L}(1_x)}(t) &= t^{-11} + t^{11} + 8(t^{-10} + t^{10}) + 36(t^{-9} + t^{9}) + 120(t^{-8} + t^{8}) + 330(t^{-7} + t^{7}) \\ &\quad + 792(t^{-6} + t^{6}) + 1681(t^{-5} + t^{5}) + 3152(t^{-4} + t^{4}) + 5175(t^{-3} + t^{3}) \\ &\quad + 7240(t^{-2} + t^{2}) + 8465(t^{-1} + t) + 8640\end{aligned}$$

$\dim \mathrm{L}(1_x) = 62,640$

$$\chi_{\mathrm{L}(160_z)}(t) = 160(t^{-2} + t^{2}) + 713(t^{-1} + t) + 1224$$

$\dim \mathrm{L}(160_z) = 2970$

7.7.4. E_8, $c = 1/5$. The irreducible representations $\mathrm{L}(1_x)$, $\mathrm{L}(28_x)$, $\mathrm{L}(8_z)$, and $\mathrm{L}(56_z)$ are finite-dimensional and have the following graded characters and dimensions:

$$\begin{aligned}\chi_{\mathrm{L}(1_x)}(t) &= t^{-20} + t^{20} + 8(t^{-19} + t^{19}) + 36(t^{-18} + t^{18}) + 120(t^{-17} + t^{17}) + 330(t^{-16} + t^{16}) \\ &\quad + 792(t^{-15} + t^{15}) + 1716(t^{-14} + t^{14}) + 3432(t^{-13} + t^{13}) + 6435(t^{-12} + t^{12}) \\ &\quad + 11440(t^{-11} + t^{11}) + 19448(t^{-10} + t^{10}) + 31824(t^{-9} + t^{9}) + 50304(t^{-8} + t^{8}) \\ &\quad + 76848(t^{-7} + t^{7}) + 113256(t^{-6} + t^{6}) + 160464(t^{-5} + t^{5}) + 217437(t^{-4} + t^{4}) \\ &\quad + 279576(t^{-3} + t^{3}) + 337900(t^{-2} + t^{2}) + 380264(t^{-1} + t) + 395874\end{aligned}$$

$\dim \mathrm{L}(1_x) = 3,779,136$

$$\begin{aligned}\chi_{\mathrm{L}(28_x)}(t) &= 28(t^{-8} + t^{8}) + 224(t^{-7} + t^{7}) + 1008(t^{-6} + t^{6}) + 3360(t^{-5} + t^{5}) + 8673(t^{-4} + t^{4}) \\ &\quad + 17640(t^{-3} + t^{3}) + 28980(t^{-2} + t^{2}) + 38808(t^{-1} + t) + 42750\end{aligned}$$

$\dim \mathrm{L}(28_x) = 240,192$

$$\begin{aligned}\chi_{\mathrm{L}(8_z)}(t) &= 8(t^{-14} + t^{14}) + 64(t^{-13} + t^{13}) + 288(t^{-12} + t^{12}) + 960(t^{-11} + t^{11}) \\ &\quad + 2640(t^{-10} + t^{10}) + 6336(t^{-9} + t^{9}) + 13616(t^{-8} + t^{8}) + 26560(t^{-7} + t^{7}) \\ &\quad + 47448(t^{-6} + t^{6}) + 78080(t^{-5} + t^{5}) + 118624(t^{-4} + t^{4}) + 165888(t^{-3} + t^{3}) \\ &\quad + 212368(t^{-2} + t^{2}) + 247424(t^{-1} + t) + 260640\end{aligned}$$

$\dim \mathrm{L}(8_z) = 2,101,248$

$$\chi_{\mathrm{L}(56_z)}(t) = 56(t^{-2} + t^{2}) + 448(t^{-1} + t) + 720$$

$\dim \mathrm{L}(56_z) = 1728$

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