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Output Consensus for Interconnected Heterogeneous Systems via a Combined Model Predictive Control and Integral Sliding Mode Control with Application to CSTRs

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Abstract

Interconnected structures are commonly found in process networks. In this paper, an output consensus framework is proposed for a class of continuous interconnected linear heterogeneous systems subject to disturbances and constraints. The distributed output consensus control strategy is developed by combining integral sliding mode control with model predictive control. The integral sliding mode control is designed to eliminate a class of matched disturbances. The model predictive control plays two main roles: On the one hand, it drives the system states to track the steady state values so as to achieve output consensus; on the other hand, it helps to deal with interconnections and constraints existing in systems. In the meantime, a distributed iterative algorithm is designed to acquire the system steady states. A simulation example and an experiment relating to control of systems of interconnected CSTRs are presented to validate the effectiveness and superiority of the proposed method.

Keywords: Output consensus, Integral sliding mode control, Model predictive control, Heterogeneous system, Interconnected system.

1. Introduction

In industrial processes, plants frequently manufacture product using multiple pieces of equipment simultaneously to improve production efficiency and expand production scale. This is common in mass production. Consider for example the saponification reaction as a representative industrial process. This is widely used in detergent manufacturing and food processing. A typical saponification reaction is the reaction of sodium hydroxide with ethyl acetate to produce sodium acetate and ethanol. The process is usually carried out in a continuous stirred-tank reactor (CSTR) to increase the conversion rate (Danish et al., 2015). In addition, part of the product is recycled to the original reactor in order to speed up the reaction rate and increase the yield (Chupa et al., 2012). Moreover, there are typically several tanks performing the same chemical reaction so that production is increased and it is necessary to maintain the same temperature and concentration for all reactors. This makes the control of the saponification reaction system more complicated. Such complex process systems can be summed up as coupled systems, or called interconnected systems, and the interconnection refers to the recycling process of the material.

Distributed control has become a popular candidate for the control of complex process systems in recent years. It can reduce communication costs and improve production efficiency (Antonelli, 2013; Li et al., 2017; Zhou et al.,

2020). Using distributed control and with an objective of coordination, cooperative control across controlled objects characterized by multiple subsystems has been widely used in various fields such as chemical process networks (Tippett et al., 2015), unmanned aerial vehicle formation (Murray, 2007) and collaborative surgery using medical robots (Haidegger, 2019). Consensus control is a class of cooperative control which requires the outputs of all subsystems to converge to the same value. The approach has broad application in mass production systems and it is significant to solve the consensus problem for interconnected systems from the perspective of both theoretical research and practical application.

From the point of view of practical application in process networks, there are several open problems of interest in the consensus control of interconnected systems. First, external disturbances widely exist in reality. For a distributed system, perturbations in one subsystem may affect the other subsystems either through information networks or the presence of interconnection. Secondly, constraints on a distributed system are important elements that must be considered when control algorithms are applied in practice. Unlike in theoretical calculations, the inputs and outputs of an actual system are usually limited. For instance, there is a threshold value for a reactor valve in a chemical process, and the temperature and pressure of a product in the process must be kept within a certain range. Thirdly, interconnections are non-negligible and

difficult to decouple in a chemical process that is dominated by material and energy exchange (Zhang et al., 2019). Once one subsystem fails and the corresponding state is unstable, it may have an adverse impact on the other subsystems due to the coupling. Moreover, the existence of coupling makes the study of stability for autonomous systems challenging. Fourthly, heterogeneity also brings challenges to the control of distributed systems. In a physical production process, even if the same product is produced, different equipment may have different intrinsic characteristics with different structure, which would be described by different mathematical models. Different dimensions and values within the system structure bring different challenges to the control design of the whole system when compared with homogeneous systems. Overall, a consensus problem should be solved for an interconnected heterogeneous system in the presence of disturbances and constraints. Unfortunately, the existing results for such a problem are not yet available.

Recent research work has considered consensus control for individual sub-problems. When considering the disturbance rejection problem, robust control methods are an effective approach. In John et al. (2023), a distributed H_∞ consensus control is applied to a physically interconnected linear system to suppress disturbances. In Wang et al. (2020), a disturbance observer based synchronization controller is proposed to estimate and eliminate exogenous disturbances for networked interconnected motion systems. In Wang et al. (2016), a distributed output consensus problem is considered based on an adaptive technique for a class of non-identical dynamic systems with uncertain interconnections and external disturbances. Compared with these methods, an integral sliding mode control (ISMC) is simple and mature in design and has strong robustness. From the initial instant, systems have insensitivity to matched disturbances under the action of ISMC with a well-designed sliding surface (Utkin et al., 1996; Zhang et al., 2022; Castaños et al., 2006; Rubagotti et al., 2011). In the case where there are constraints on the control signal, in some research, anti-windup compensators are utilized to deal with the potential constraint, i.e., the input saturation problem (Rehan et al., 2020; Lu et al., 2022). However, the nonlinear theory related to the saturation problem is still maturing, and frequent input saturation may not be suitable for the long-term operation of actuators. Model predictive control (MPC) seeks to minimize operating costs while accommodating constraints, and thus input and output violations can be avoided wherever possible through optimal control. In addition, MPC can not only deal with constraints, but also achieve a cooperative control that is able to stabilize strongly interconnected systems by means of synthesizing global performance indices (Conte et al., 2013).

Motivated by the above discussions, in this paper an output consensus control framework is proposed for distributed interconnected heterogeneous linear systems that are subject to disturbances and constraints. Firstly, out-

put consensus is achieved by virtue of a consensus algorithm and MPC tracking method. By using a finite-time consensus algorithm, a target output is acquired. The corresponding steady states are optimized by a distributed optimization algorithm. Proving the convergence of the optimization algorithm to the steady-state solution is a particularly challenging problem. Then state tracking is implemented using MPC. When the tracking errors go to zero, the system outputs correspondingly come to consensus. Because output consensus is achieved by tracking, the exact consensus value can be obtained and adjusted by setting different initial values. In Zhang et al. (2022) and Li et al. (2014), consensus is achieved by distributed ISM-C and distributed adaptive control, respectively. However, the obtained results provide input-to-state stability or ultimate boundedness, and hence the final consensus values are only available following convergence of the algorithm. In Wang et al. (2016), though the system matrices can have different parameters, the dimensions are required to be the same for all the interconnected systems, which limits the scope of application of the output consensus algorithm. In Liu et al. (2023) and Liu et al. (2024), distributed partial state and output consensus is studied, respectively, but the interconnections between the systems is limited and the systems are assumed to be in series. The coupling characteristics have no such requirements in this paper. Secondly, a class of disturbances are dealt with by ISM-C where matched disturbances are eliminated. Compared with only using MPC in Xiao et al. (2022), the proposed method is more robust and less affected by disturbances. Thirdly, state couplings and input/output constraints are dealt with by the use of MPC, and an optimal MPC gain can be achieved. In Hirche et al. (2020), output consensus for heterogeneous systems is realized using an MPC tracking method by augmenting systems. Nevertheless, the method is only suitable for discrete systems and cannot be applied in the continuous case. The main contributions of this paper are twofold. On the one hand, a distributed iterative optimization algorithm is presented to determine the steady state solution of continuous coupled systems and the corresponding convergence proof is given. On the other hand, a novel output consensus control framework is proposed using an MPC tracking strategy for a continuous heterogeneous system.

The remainder of the paper is organized as follows. In Section II, some basic concepts are presented and the problem considered is formulated. In Section III, an ISMC is designed to suppress disturbances. In Section IV, a distributed MPC tracking control is designed and analyzed. In Section V, a consensus algorithm is given and output consensus is proved to be achieved asymptotically. In Section VI, simulation and experiment results are analysed and finally Section VII concludes the paper.

Notation: 0_n and $0_{n \times m}$ denote an n -row vector and an $n \times m$ matrix respectively where the elements are 0. I_m denotes an $m \times m$ identity matrix. $\dim(\cdot)$ denotes the dimension of a matrix. $\|\cdot\|$ denotes the Euclidean

norm unless specifically stated. For a vector $w \in R^n$, $\|w\|_P^2 \triangleq w^T P w$, where $P \in R^{n \times n}$. For a symmetric matrix P , $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ denote the minimal and maximal eigenvalues of P respectively. $Q > 0 (\geq 0)$ or $Q < 0 (\leq 0)$ indicate Q is positive (semi-)definite or negative (semi-)definite. For a matrix $D \in R^{n \times m}$, the columns of $D^\perp \in R^{n \times (n-m)}$ span the null space of D^T , i.e., $D^T D^\perp = 0_{m \times (n-m)}$. For a vector $z = [z_1, \dots, z_n]^T \in R^n$, $\text{sig}(z)^\alpha = [|z_1|^\alpha \text{sgn}(z_1), \dots, |z_n|^\alpha \text{sgn}(z_n)]^T$, where $\text{sgn}(\cdot)$ is the sign function defined by

$$\text{sgn}(z_i) = \begin{cases} 1, & z_i > 0 \\ 0, & z_i = 0 \\ -1, & z_i < 0 \end{cases} \quad (1)$$

for $i = 1, 2, \dots, n$.

Definition 1: (Undirected Graph) Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ denote an undirected graph which consists of N nodes $\mathcal{V} = \{v_1, \dots, v_N\}$, a set of undirected edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, and a weighted adjacency matrix $\mathcal{A} = (a_{ij})_{N \times N}$. An undirected edge $\mathcal{E}_{ij}$ in the undirected graph $\mathcal{G}$ is denoted by a pair of unordered nodes (v_i, v_j) , which indicates v_i can exchange information with v_j , v_j and v_i are neighbours. The weights $a_{ij} = a_{ji} > 0$ in the weighted adjacency matrix $\mathcal{A}$ if and only if the edge (v_i, v_j) exists, and $a_{ij} = 0$ otherwise. Assume there is no self loop in the undirected graph, hence, $a_{ii} = 0$.

Definition 2: (Connectivity) A path is a sequence of connected edges in an undirected graph, and an undirected graph is connected if there is a path between every pair of vertices.

2. Preliminaries and problem formulation

Consider a collection of N subsystems with continuous linear dynamics

$$\begin{aligned} \dot{x}_i(t) &= A_{ii}x_i(t) + \sum_{j \in \mathcal{N}_i} A_{ij}x_j(t) + B_i(u_i(t) + \phi_i(x, t)) \\ y_i(t) &= C_i x_i(t), i \in \mathbf{N} \end{aligned} \quad (2)$$

where $x_i \in \mathcal{R}^{n_i}$, $u_i \in \mathcal{R}^{m_i}$ and $y_i \in \mathcal{R}^p$ denote the state, input, and output. $\phi_i \in \mathcal{R}^{m_i}$ denotes a class of disturbance contributions which are matched. This is a function of the full system state and time. A_{ii} , B_i , and C_i are the state, input and output matrices of a subsystem and have compatible dimensions. The term $\sum_{j \in \mathcal{N}_i} A_{ij}x_j$ is a state coupling term. $\mathbf{N} \triangleq \{1, \dots, N\}$, which represents the set of all subsystems. $\mathcal{N}_i \triangleq \{j | A_{ij} \neq 0_{n_i \times n_j}, j \in \mathbf{N}, j \neq i\}$, which includes subsystems that are coupled to subsystem i . Besides, $\bar{\mathcal{N}}_i \triangleq \{j | a_{ij} > 0, j \in \mathbf{N}\}$, which is the set of subsystems that can exchange information with subsystem i , and j is named as the neighbour of i .

Remark 1: In process system control, interconnection is a common form indicating the exchange of matter and energy among subsystems. Some cascade and parallel systems are typical coupled systems abstracted from the practical process control (Huang et al., 2010; Zhang et al., 2021). Therefore, the dynamics (2) is a class of systems with practical significance.

The control input for system (2) is shown as

$$u_i(t) = u_i^{ism}(t) + u_i^{mpc}(t) \quad (3)$$

where u_i^{ism} is a discontinuous control and u_i^{mpc} is an MPC that will be designed later.

The inputs and outputs are subject to constraints as follows

$$\|u_i^{ism}(t)\| + \|u_i^{mpc}(t)\| \leq \mathcal{U}_i \quad (4)$$

$$\|y_i(t)\| \leq \mathcal{Y} \quad (5)$$

where $\mathcal{U}_i$ and $\mathcal{Y}$ are known constants.

Remark 2: It is well known that sliding mode control is robust to matched disturbances on the sliding phase. Because the initial states are known, the ISMC eliminates the reaching phase on which the system is only sensitive to matched disturbances. Therefore, it has better robustness than the conventional sliding mode control. Traditional ISMC utilizes state feedback in nominal control, which may not guarantee the performance of the system. The application of MPC into the nominal control not only can ensure this by optimizing the performance index, but also can solve input and output constraints of the system.

To facilitate the exposition, system (2) can be equivalently described as

$$\begin{aligned} \dot{x}_i(t) &= A_{ii}x_i(t) + A_{\mathcal{N}_i}x_{\mathcal{N}_i}(t) + B_i(u_i(t) + \phi_i(x, t)) \\ y_i(t) &= C_i x_i(t), i \in \mathbf{N} \end{aligned} \quad (6)$$

where $x_{\mathcal{N}_i} \in \mathcal{R}^{\sum_{j \in \mathcal{N}_i} n_j}$ denotes a stack of subsystem states in $\mathcal{N}_i$, and $A_{\mathcal{N}_i} \in \mathcal{R}^{n_i \times \sum_{j \in \mathcal{N}_i} n_j}$.

In addition, system (2) can be expressed in a compact form as follows

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B(u(t) + \phi(x, t)) \\ y(t) &= Cx(t) \end{aligned} \quad (7)$$

where $x \in \mathcal{R}^{\sum_{i=1}^N n_i}$, $u \in \mathcal{R}^{\sum_{i=1}^N m_i}$ and $y \in \mathcal{R}^{Np}$ denote lumped state, input and output vectors. $\phi \in \mathcal{R}^{\sum_{i=1}^N m_i}$ denotes the lumped matched disturbance effects. A, B, C are the system state matrix, input matrix and output matrix which have appropriate dimensions.

The following basic assumptions are initially imposed on system (2).

Assumption 1: The matrix pair (A, B) in system (7) is stabilizable.

Assumption 2: The matrix B_i in system (2) has full column rank, i.e., $\text{rank}(B_i) = m_i$.

Assumption 3: The disturbance term $\phi_i(x, t)$ is unknown but bounded, i.e., $\|\phi_i(x, t)\| \leq \beta_i < \mathcal{U}_i$, where $\beta_i \in R$ is a known constant, and $\mathcal{U}_i$ satisfies (4).

Assumption 4: The undirected communication graph $\mathcal{G}$ is connected, and $\mathcal{N}_i \subseteq \tilde{\mathcal{N}}_i$.

Remark 3: The assumptions above are essential for many control systems. Assumption 1 is conventionally used in the distributed control of interconnected heterogeneous systems (Mark et al., 2021; Bai et al., 2019), and the assumption can be easily extended from discrete-time systems to continuous-time systems. Assumption 2 is widely used in sliding mode control to facilitate controller design (Castaños et al., 2006; Rubagotti et al., 2011). Assumption 3 specifies a class of disturbance effects, and is a common condition used to bound the external disturbances (Galván-Guerra et al., 2022; Incremona et al., 2017; Liao et al., 2015). Assumption 4 indicates that any two subsystems with state coupling have information transfer.

Definition 3: (Output Consensus) Output consensus is said to be achieved for the distributed heterogeneous system (2) if

$$\lim_{t \rightarrow \infty} \|y_i(t) - y_j(t)\| = 0, \forall i, j \in \mathbf{N} \quad (8)$$

This paper concentrates on utilizing local information to develop a distributed control for system (2) such that output consensus can be achieved even if each subsystem (2) is affected by a class of disturbances and subjected to constraints.

Before presenting the main results, the following lemmas are given.

Lemma 1: (Kothare et al., 1996) (Schur Complement) Let $S_{11} = S_{11}^T, S_{22} = S_{22}^T$. Then the linear matrix inequality

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix} > 0 \quad (9)$$

is equivalent to the matrix inequalities

$$S_{22} > 0, S_{11} - S_{12}S_{22}^{-1}S_{12}^T > 0 \quad (10)$$

and

$$S_{11} > 0, S_{22} - S_{12}^TS_{11}^{-1}S_{12} > 0 \quad (11)$$

Lemma 2: Let $V(t)$ be decreasing and piecewise first-order continuous differentiable. For any $v \in R^n$, if there exists $\alpha, \beta, \gamma > 0$ such that

$$\alpha \|v(t)\|^2 \leq V(t) \leq \beta \|v(t)\|^2 \quad (12)$$

$$\dot{V}(t) \leq -\gamma \|v(t)\|^2, a.e. \quad (13)$$

then

$$\|v(t)\|^2 \leq \beta \alpha^{-1} e^{-\gamma t / \beta} \|v(0)\|^2 \quad (14)$$

Thus, $v(t) \rightarrow 0$ when $t \rightarrow \infty$.

The proof of Lemma 2 is provided in the Appendix.

Lemma 3: (Xu et al., 2006) (Admissibility of Singular System) For the unforced discrete-time singular system

$$Ez(k+1) = \bar{A}z(k) \quad (15)$$

where $E, \bar{A} \in \mathbb{R}^{n \times n}$ and $\text{rank}(E) \leq n$. The pair $(E, \bar{A})$ is admissible if and only if there exists a symmetric matrix P such that

$$\begin{aligned} E^T P E &\geq 0 \\ \bar{A}^T P \bar{A} - E^T P E &< 0 \end{aligned} \quad (16)$$

where the matrix P is nonsingular.

Remark 4: Admissibility includes regularity, causality and stability, which are crucial characteristics of a singular system. Regularity guarantees the existence and uniqueness of a solution to a given singular system. Causality implies the state of a singular system at a particular instant only depends on its state in the past rather than in the future (Xu et al., 2006). Stability means the system is asymptotically stable in the Lyapunov sense (Zhang et al., 2003).

3. Distributed ISMC design

This section aims to design an ISMC for system (2). To simplify the notation, some of the function arguments will be omitted for brevity.

The sliding function is presented as follows

$$\begin{aligned} s_i(t) &= B_i^+ (x_i(t) - x_i(0)) \\ &\quad - B_i^+ \left(\int_0^t A_{ii}x_i(\tau) + A_{\mathcal{N}_i}x_{\mathcal{N}_i}(\tau) + B_i u_i^{mpc}(\tau) d\tau \right), \\ i &\in \mathbf{N} \end{aligned} \quad (17)$$

where $B_i^+ \triangleq (B_i^T B_i)^{-1} B_i^T \in R^{m_i \times n_i}$ is the left inverse of B_i , $x_i(0)$ is the initial value.

The corresponding sliding surface is

$$\{x_i | s_i(t) = 0_{m_i}, \forall i = 1, 2, \dots, N\} \quad (18)$$

where $s_i(t)$ is defined in (17).

The discontinuous control u_i^{ism} is given by

$$u_i^{ism}(t) = -\rho_i \frac{s_i(t)}{\|s_i(t)\|} \quad (19)$$

where the control gain ρ_i satisfies $\beta_i < \rho_i < \mathcal{U}_i$ with β_i and $\mathcal{U}_i$ satisfying Assumption 3. $s_i(t)$ is defined in (17).

Theorem 1: Suppose Assumptions 2-4 hold. The discontinuous control (19) can keep the system (2) on the sliding surface (18) from the initial time.

The proof of Theorem 1 is provided in the Appendix.

It follows from Theorem 1 that the system (2) will be driven to the sliding surface and maintain a sliding motion on the surface (18) in spite of the presence of the matched disturbances (Utkin et al., 1996).

4. Distributed MPC design

In this section, the steady state for each subsystem is acquired by solving a steady-state target tracking optimization problem, and state tracking for the distributed disturbed system is then considered utilizing the MPC method.

When the system (2) is restricted to the sliding surface (18), combining (17) and let $\dot{s}_i(t) = 0_{m_i}$, then the equivalent discontinuous control u_{ieq}^{ism} is given by

$$u_{ieq}^{ism}(t) = -\phi_i(x, t) \quad (20)$$

Substituting (20) into (6), the system dynamics can be described as

$$\begin{aligned} \dot{x}_i(t) &= A_{ii}x_i(t) + A_{\mathcal{N}_i}x_{s\mathcal{N}_i}(t) + B_i u_i^{mpc}(t) \\ y_i(t) &= C_i x_i(t), i \in \mathbf{N} \end{aligned} \quad (21)$$

In order to design an MPC that satisfies the input constraint (4), the scope for u_i^{mpc} must be acquired. According to (19), the norm of the local discontinuous control input is $\|u_i^{ism}(t)\| = \rho_i$, and then u_i^{mpc} fulfills the following condition

$$\|u_i^{mpc}(t)\| \leq \mathcal{U}_i - \rho_i \quad (22)$$

4.1. Steady-state target problem

In this subsection, a local steady state subject to constraints is defined and determined to achieve distributed system tracking.

Define a feasible initial output target $y_{si}(0)$ for each subsystem. Exchange $y_{si}(0)$ with its neighbours, and acquire a new y_{si} through the following finite time consensus law (Wang et al., 2010)

$$\dot{y}_{si} = \delta_i \sum_{j \in \mathcal{N}_i} a_{ij} \text{sig}(y_{sj} - y_{si})^{\alpha_i} \quad (23)$$

where $0 < \alpha_i < 1$, $\delta_i > 0$.

According to system (21), the dynamics for a steady state system are given by

$$\dot{x}_{si} = A_{ii}x_{si} + A_{\mathcal{N}_i}x_{s\mathcal{N}_i} + B_i u_{si}^{mpc} \quad (24)$$

$$y_{si} = C_i x_{si}, i \in \mathbf{N} \quad (25)$$

where $x_{si} \in \mathcal{R}^{n_i}$ and $u_{si}^{mpc} \in \mathcal{R}^{m_i}$ denote the steady state and steady MPC input respectively, while $x_{s\mathcal{N}_i} \in \mathcal{R}^{\sum_{j \in \mathcal{N}_i} n_j}$ denotes a stack of steady states in $\mathcal{N}_i$. For every given target output y_{si} , when the subsystem achieves output tracking and reaches the equilibrium point (not necessarily the origin), it can be obtained that $\dot{x}_{si} = 0_{n_i}$. Therefore, (24) is equivalent to

$$A_{ii}x_{si} + A_{\mathcal{N}_i}x_{s\mathcal{N}_i} + B_i u_{si}^{mpc} = 0_{n_i} \quad (26)$$

The candidate target function is therefore given by

$$\begin{aligned} \min_{x_{si}, u_{si}^{mpc}} & \frac{1}{2} \left(\|x_{si}\|_{Q_{si}}^2 + \|u_{si}^{mpc}\|_{R_{si}}^2 \right) \\ \text{s.t.} & \quad A_{ii}x_{si} + A_{\mathcal{N}_i}x_{s\mathcal{N}_i} + B_i u_{si}^{mpc} = 0_{n_i} \\ & \quad C_i x_{si} = y_{si} \\ & \quad \|C_i x_{si}\| \leq \omega \mathcal{Y} \\ & \quad \|u_{si}^{mpc}\| \leq \omega (\mathcal{U}_i - \rho_i) \\ & \quad i \in \mathbf{N} \end{aligned} \quad (27)$$

where $Q_{si}, R_{si} > 0$ are weighting matrices, $\omega \in (0, 1)$, and $\mathcal{U}_i$ and $\mathcal{Y}$ satisfy (4) and (5) respectively. The above optimization problem can be solved using the CVX toolbox (Grant et al., 2005).

Remark 5: In (27), the parameter ω is limited to the range $(0, 1)$ in order to ensure that there are feasible solutions not only for the steady state x_{si} and steady input u_{si}^{mpc} , but also for the state x_i and input u_i^{mpc} . Since ω can be arbitrarily close to 1, the constraints on the steady state and steady input in (27) can be used as a reliable approximation to the maximal constraints (Hirche et al., 2020; Limon et al., 2008; Gilbert et al., 1991).

The solvability of equations (25) and (26) is vital for solving optimization problem (27), so the condition for solvability of the equations is given subsequently.

Write (25) and (26) in centralised form

$$\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}_{Np \times \sum_{i=1}^N m_i} \begin{bmatrix} x_s \\ u_s^{mpc} \end{bmatrix} = \begin{bmatrix} 0 \\ \sum_{i=1}^N n_i \\ y_s \end{bmatrix} \quad (28)$$

where $x_s \in \mathcal{R}^{\sum_{i=1}^N n_i}$, $u_s^{mpc} \in \mathcal{R}^{\sum_{i=1}^N m_i}$ and $y_s \in \mathcal{R}^{Np}$ denote lumped steady state, steady input and target output vectors.

The above equation (28) is a typical nonhomogeneous linear matrix equation whose solution exists if and only if

$$\begin{aligned} \text{rank} \left(\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}_{Np \times \sum_{i=1}^N m_i} \right) \\ = \text{rank} \left(\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}_{Np \times \sum_{i=1}^N m_i}, \begin{bmatrix} 0 \\ \sum_{i=1}^N n_i \\ y_s \end{bmatrix} \right) \end{aligned} \quad (29)$$

In what follows, a distributed iterative algorithm is designed to solve the optimization problem (27). Before that, to clarify, in the following algorithm, the candidate function (27) is rewritten as

$$\begin{aligned} \min_{x_{si}(\kappa+1), u_{si}^{mpc}(\kappa+1)} & \frac{1}{2} \left(\|x_{si}(\kappa+1)\|_{Q_{si}}^2 + \|u_{si}^{mpc}(\kappa+1)\|_{R_{si}}^2 \right) \\ \text{s.t.} & \quad A_{ii}x_{si}(\kappa+1) + A_{\mathcal{N}_i}x_{s\mathcal{N}_i}(\kappa) + B_i u_{si}^{mpc}(\kappa+1) = 0_{n_i} \\ & \quad C_i x_{si}(\kappa+1) = y_{si} \\ & \quad \|C_i x_{si}(\kappa+1)\| \leq \omega \mathcal{Y} \\ & \quad \|u_{si}^{mpc}(\kappa+1)\| \leq \omega (\mathcal{U}_i - \rho_i), i \in \mathbf{N} \end{aligned} \quad (30)$$

where κ denotes the iteration count.

Algorithm 1. (Steady state target algorithm)

Step 1 (Initialization). For each subsystem, after giving y_{si} and setting an arbitrary initial value $x_{si}(0)$, let $\kappa = 1$ and $x_{si}(\kappa) = x_{si}(0)$.

Step 2 (Exchange and Solving). Exchange $x_{si}(\kappa)$ with its neighbours, solve the steady state target problem (30), acquire $x_{si}(\kappa + 1)$ and $u_{si}^{mpc}(\kappa + 1)$.

Step 3 (Judgement). If $\|x_{si}(\kappa + 1) - x_{si}(\kappa)\| \leq \epsilon$ with ϵ being a pre-decided small enough constant, stop iteration, and $x_{si} = x_{si}(\kappa + 1)$, $u_{si}^{mpc} = u_{si}^{mpc}(\kappa + 1)$; otherwise, let $\kappa = \kappa + 1$ and go back to Step 2.

Theorem 2: Suppose condition (29) holds, after applying distributed iterative steady-state target Algorithm 1, the solution of the optimization problem (30) will eventually converge to some optimal value if there exists a symmetric nonsingular matrix $\bar{P}$ such that

$$\begin{aligned} \bar{Z}_1^T \bar{P} \bar{Z}_1 &\geq 0_{\dim(\bar{Z}_1) \times \dim(\bar{Z}_1)} \\ \bar{Z}_2^T \bar{P} \bar{Z}_2 - \bar{Z}_1^T \bar{P} \bar{Z}_1 &< 0_{\dim(\bar{Z}_1) \times \dim(\bar{Z}_1)} \end{aligned} \quad (31)$$

where $\bar{Z}_1$ and $\bar{Z}_2$ are square matrices augmented by Z_1 and Z_2 . The diagonal blocks of Z_1 are $\begin{bmatrix} A_{ii} & B_i \\ C_i & 0_{p \times m_i} \end{bmatrix}$ for $i \in \mathbf{N}$. The diagonal blocks of Z_2 are zeros, and the rest are $\begin{bmatrix} A_{ij} & 0_{n_i \times m_j} \\ 0_{p \times n_j} & 0_{p \times m_j} \end{bmatrix}$ for $i, j \in \mathbf{N}$. If $Np > m_i$, supplement the columns of Z_1 and Z_2 with zeros such that $\bar{Z}_1, \bar{Z}_2 \in \mathcal{R}^{\left(\sum_{i=1}^N n_i + Np\right) \times \left(\sum_{i=1}^N n_i + Np\right)}$; if $Np < m_i$, supplement the rows of Z_1 and Z_2 with zeros such that $\bar{Z}_1, \bar{Z}_2 \in \mathcal{R}^{\sum_{i=1}^N (n_i + m_i) \times \sum_{i=1}^N (n_i + m_i)}$; else, $\bar{Z}_1 = Z_1$ and $\bar{Z}_2 = Z_2$.

Proof: As condition (29) holds, (28) is solvable. Assume there exists a solution pair $(x_s^*, u_s^{mpc,*})$ making (25) and (26) true, that is,

$$\begin{aligned} A_{ii}x_{si}^* + A_{N_i}x_{sN_i}^* + B_i u_{si}^{mpc,*} &= 0_{n_i} \\ C_i x_{si}^* &= y_{si} \end{aligned} \quad (32)$$

Define the iteration errors as

$$\begin{aligned} e_{x_{si}}(\kappa + 1) &= x_{si}(\kappa + 1) - x_s^* \\ e_{u_{si}^{mpc}}(\kappa + 1) &= u_{si}^{mpc}(\kappa + 1) - u_s^{mpc,*} \end{aligned} \quad (33)$$

Subtract (33) from (30), and describe the result in the following centralized fashion

$$Z_1 e(\kappa + 1) = -Z_2 e(\kappa) \quad (34)$$

where $e = [e_{x_{s1}}^T, e_{u_{s1}^{mpc}}^T, \dots, e_{x_{sN}}^T, e_{u_{sN}^{mpc}}^T]^T$.

When $Np > m_i$, $\left(\sum_{i=1}^N n_i + Np\right) > \sum_{i=1}^N (n_i + m_i)$, i.e., the row dimensions are larger than the column dimensions of Z_1 and Z_2 . After making the matrices square, the rows

of iteration error e should be augmented by supplementing variables so that the following new equation (35) still holds. Consider the case when $Np < m_i$, $\left(\sum_{i=1}^N n_i + Np\right) < \sum_{i=1}^N (n_i + m_i)$, i.e., the row dimensions are smaller than the column dimensions of Z_1 and Z_2 . After making the matrices square, the original iteration error e does not need to be augmented. Note that, in both cases, extending matrices and variables will not affect the convergence of the iteration error e . As $\bar{Z}_1 = Z_1$ and $\bar{Z}_2 = Z_2$ when $Np = m_i$, the extended equation in all three cases can be expressed as

$$\bar{Z}_1 \bar{e}(\kappa + 1) = -\bar{Z}_2 \bar{e}(\kappa) \quad (35)$$

According to Lemma 3, if the symmetric nonsingular matrix $\bar{P}$ fulfills (31), then (35) is stable, $\bar{e}$ goes to zero asymptotically, so $x_{si}(\kappa + 1) \rightarrow x_s^*$ and $u_{si}^{mpc}(\kappa + 1) \rightarrow u_s^{mpc,*}$ when $\kappa \rightarrow \infty$.

4.2. Distributed MPC tracking problem

Given the steady-state solutions, the following tracking objective function will be minimized

$$\begin{aligned} J_i(t) &= \int_0^\infty \|x_i(t + \tau|t) - x_{si}\|_{Q_{ii}}^2 \\ &\quad + \sum_{j \in \mathcal{N}_i} \|x_j(t + \tau|t) - x_{sj}\|_{Q_{ij}}^2 \\ &\quad + \|u_i^{mpc}(t + \tau|t) - u_{si}^{mpc}\|_{R_i}^2 d\tau \end{aligned} \quad (36)$$

where $\tau \geq 0$, $Q_{ii}, Q_{ij}, R_i > 0$, $x_j(t + \tau|t)$ and $u_i^{mpc}(t + \tau|t)$ denote the predicted state and MPC respectively at the instant $t + \tau$ computed by the optimization problem at the instant t .

For the sliding dynamics (21), the MPC input is designed as

$$u_i^{mpc}(t + \tau|t) = K_i(x_i(t + \tau|t) - x_{si}) + u_{si}^{mpc}, \quad (37)$$

where K_i is a control gain that is updated at each sampling instant kT ($k = 0, 1, 2, \dots$), and T denotes the sampling interval; $x_i(t + \tau|t)$ is a continuous function that follows dynamics (21) in the interval $\tau \in [0, T]$. Thus, $u_i^{mpc}(t + \tau|t)$ is actually a piecewise continuous function.

Theorem 3: Consider system (21), and suppose Assumptions 1-4 hold. The MPC gain matrix $K_i = Y_i X_i^{-1}$ can minimize $J_i(t)$ subject to the MPC constraint (22) and output constraint (5) with Y_i, X_i being determined from the solution of the following optimization problem

$$\min_{\eta_i, X_i, Y_i} \eta_i \quad (38)$$

$$s.t. \quad \begin{bmatrix} -X_i & Y_i^T \\ * & -(\mathcal{U}_i - \rho_i - \|u_{si}^{mpc}\|)^2 I_{m_i} \end{bmatrix} \leq 0, \quad (39)$$

$$\begin{bmatrix} -X_i & X_i C_i^T \\ * & -(\mathcal{Y} - \|C_i x_{si}\|)^2 I_p \end{bmatrix} \leq 0, \quad (40)$$

$$\begin{bmatrix} -\eta_i I_{n_i} & 0_{n_i \times m_i} & Q_{ii}^{1/2} X_i & 0_{n_i \times \sum_{j \in \mathcal{N}_i} n_j} \\ * & -\eta_i I_{m_i} & R_i^{1/2} Y_i & 0_{m_i \times \sum_{j \in \mathcal{N}_i} n_j} \\ * & * & A_{ii} X_i + B_i Y_i + X_i A_{ii}^T + Y_i^T B_i^T & \eta_i A_{\mathcal{N}_i} \\ * & * & * & \eta_i Q_{\mathcal{N}_i} \end{bmatrix} \leq 0, i \in \mathbf{N} \quad (42)$$

$$\begin{bmatrix} -X_i & x_i(t) - x_{si} \\ * & -1 \end{bmatrix} \leq 0, \quad (41)$$

where $*$ denotes the symmetric parts of matrices, $\eta_i > 0$, $X_i > 0$, $X_i = P_i^{-1} \eta_i$, $Q_{\mathcal{N}_i} \in \mathcal{R}^{\sum_{j \in \mathcal{N}_i} n_j \times \sum_{j \in \mathcal{N}_i} n_j}$ is a diagonal block matrix composed of Q_{ij} ($j \in \mathcal{N}_i$), and $A_{\mathcal{N}_i} \in \mathcal{R}^{n_i \times \sum_{j \in \mathcal{N}_i} n_j}$ is a row block matrix composed of A_{ij} ($j \in \mathcal{N}_i$).

Proof: Combining (21), (24) and (37), it can be obtained that

$$\begin{aligned} \dot{x}_i(t + \tau|t) - \dot{x}_{si} \\ = (A_{ii} + B_i K_i) (x_i(t + \tau|t) - x_{si}) \\ + A_{\mathcal{N}_i} (x_{\mathcal{N}_i}(t + \tau|t) - x_{s\mathcal{N}_i}) \end{aligned} \quad (43)$$

where $x_{\mathcal{N}_i}, x_{s\mathcal{N}_i} \in \mathcal{R}^{\sum_{j \in \mathcal{N}_i} n_j}$ denotes column vectors composed by x_j and x_{sj} ($j \in \mathcal{N}_i$) respectively.

Term (36) is similarly dealt with. It can be rewritten as

$$\begin{aligned} J_i(t) = \int_0^\infty \|x_i(t + \tau|t) - x_{si}\|_{K_i^T R_i K_i + Q_{ii}}^2 \\ + \|x_{\mathcal{N}_i}(t + \tau|t) - x_{s\mathcal{N}_i}\|_{Q_{\mathcal{N}_i}}^2 d\tau, i \in \mathbf{N} \end{aligned} \quad (44)$$

For system (43), consider a candidate Lyapunov function as

$$V_i(x_i(t), x_{si}) = \|x_i(t) - x_{si}\|_{P_i}^2, i \in \mathbf{N} \quad (45)$$

where $P_i > 0$.

From (43) and (45), the derivative of $V_i(x_i(t), x_{si})$ is shown as

$$\begin{aligned} \dot{V}_i(x_i(t + \tau|t), x_{si}) \\ = (x_i(t + \tau|t) - x_{si})^T P_i (A_{ii} + B_i K_i) (x_i(t + \tau|t) - x_{si}) \\ + (x_i(t + \tau|t) - x_{si})^T (A_{ii} + B_i K_i)^T P_i (x_i(t + \tau|t) - x_{si}) \\ + (x_{\mathcal{N}_i}(t + \tau|t) - x_{s\mathcal{N}_i})^T A_{\mathcal{N}_i}^T P_i (x_i(t + \tau|t) - x_{si}) \\ + (x_i(t + \tau|t) - x_{si})^T P_i A_{\mathcal{N}_i} (x_{\mathcal{N}_i}(t + \tau|t) - x_{s\mathcal{N}_i}) \end{aligned} \quad (46)$$

It is first necessary to prove that $J_i(t)$ is minimized by the specified K_i .

From Lemma 1 and $K_i = Y_i X_i^{-1}$, (42) is equivalent to

$$\begin{bmatrix} K_i^T R_i K_i + Q_{ii} + P_i (A_{ii} + B_i K_i) + (A_{ii} + B_i K_i)^T P_i & P_i A_{\mathcal{N}_i} \\ * & Q_{\mathcal{N}_i} \end{bmatrix} \leq 0 \quad (47)$$

Pre-multiply the above inequality by $\begin{bmatrix} (x_i(t + \tau|t) - x_{si})^T \\ (x_{\mathcal{N}_i}(t + \tau|t) - x_{s\mathcal{N}_i})^T \end{bmatrix}$ and post-multiply by the transpose. It is obtained that

$$\begin{aligned} (x_i(t + \tau|t) - x_{si})^T (K_i^T R_i K_i + Q_{ii}) (x_i(t + \tau|t) - x_{si}) \\ + (x_i(t + \tau|t) - x_{si})^T P_i (A_{ii} + B_i K_i) (x_i(t + \tau|t) - x_{si}) \\ + (x_i(t + \tau|t) - x_{si})^T (A_{ii} + B_i K_i)^T P_i (x_i(t + \tau|t) - x_{si}) \\ + (x_{\mathcal{N}_i}(t + \tau|t) - x_{s\mathcal{N}_i})^T A_{\mathcal{N}_i}^T P_i (x_i(t + \tau|t) - x_{si}) \\ + (x_i(t + \tau|t) - x_{si})^T P_i A_{\mathcal{N}_i} (x_{\mathcal{N}_i}(t + \tau|t) - x_{s\mathcal{N}_i}) \\ + (x_{\mathcal{N}_i}(t + \tau|t) - x_{s\mathcal{N}_i})^T Q_{\mathcal{N}_i} (x_{\mathcal{N}_i}(t + \tau|t) - x_{s\mathcal{N}_i}) \leq 0 \end{aligned} \quad (48)$$

In conjunction with (46), the above inequality is equivalent to

$$\begin{aligned} \dot{V}_i(x_i(t + \tau|t), x_{si}) \\ \leq -\|x_i(t + \tau|t) - x_{si}\|_{K_i^T R_i K_i + Q_{ii}}^2 \\ - \|x_{\mathcal{N}_i}(t + \tau|t) - x_{s\mathcal{N}_i}\|_{Q_{\mathcal{N}_i}}^2 \end{aligned} \quad (49)$$

Integrate the inequality (49) on both sides with respect to the predicting time τ from 0 to ∞ , and consider $V_i(x_i(t + \infty|t), x_{si}) \geq 0$. Then, the upper bound of the performance function can be written as

$$J_i(t) \leq V_i(x_i(t), x_{si}) \quad (50)$$

On the other hand, from Lemma 1, (41) is equivalent to

$$\|x_i(t) - x_{si}\|_{X_i^{-1}}^2 \leq 1 \quad (51)$$

Further, from $X_i = P_i^{-1} \eta_i$, the above inequality can be transformed to

$$V_i(x_i(t), x_{si}) \leq \eta_i \quad (52)$$

Using (50) and (52), it is concluded that η_i is the upper bound of $J_i(t)$, i.e., $J_i(t) \leq \eta_i$. Therefore, the minimization of $J_i(t)$ can be transformed into the minimization of η_i . This yields the expression (38).

Next, it will be shown that the MPC constraint (22) and output constraint (5) are satisfied.

From Lemma 1, (39) is equivalent to

$$Y_i X_i^{-1} Y_i^T \leq (\mathcal{U}_i - \rho_i - \|u_{si}^{mpc}\|)^2 I_{m_i} \quad (53)$$

which is also equivalent to

$$\lambda_{\max} \left(X_i^{-1/2} Y_i^T Y_i X_i^{-1/2} \right) \leq (\mathcal{U}_i - \rho_i - \|u_{si}^{mpc}\|)^2 \quad (54)$$

From (49), (51) can be written as

$$\left\| X_i^{-1/2} (x_i(t + \tau|t) - x_{si}) \right\|^2 \leq 1 \quad (55)$$

From (55) and $K_i = Y_i X_i^{-1}$, the left side of the inequality (54) is

$$\begin{aligned} & \lambda_{\max} \left(X_i^{-1/2} Y_i^T Y_i X_i^{-1/2} \right) \\ & \geq \lambda_{\max} \left(X_i^{-1/2} Y_i^T Y_i X_i^{-1/2} \right) \left\| X_i^{-1/2} (x_i(t + \tau|t) - x_{si}) \right\|^2 \\ & = \max_{\tau \geq 0} \left\| Y_i X_i^{-1} (x_i(t + \tau|t) - x_{si}) \right\|^2 \\ & = \max_{\tau \geq 0} \|K_i (x_i(t + \tau|t) - x_{si})\|^2 \end{aligned} \quad (56)$$

It follows that

$$\max_{\tau \geq 0} \|K_i (x_i(t + \tau|t) - x_{si})\| \leq \mathcal{U}_i - \rho_i - \|u_{si}^{mpc}\| \quad (57)$$

Further,

$$\begin{aligned} & \max_{\tau \geq 0} (\|K_i (x_i(t + \tau|t) - x_{si})\| + \|u_{si}^{mpc}\|) \\ & \geq \max_{\tau \geq 0} \|K_i (x_i(t + \tau|t) - x_{si}) + u_{si}^{mpc}\| \\ & = \max_{\tau \geq 0} \|u_i^{mpc}(t + \tau|t)\| \end{aligned} \quad (58)$$

As a result, $\max_{\tau \geq 0} \|u_i^{mpc}(t + \tau|t)\| \leq \mathcal{U}_i - \rho_i$, i.e., MPC constraint (22) is fulfilled.

From Lemma 1, (40) is equivalent to

$$C_i X_i C_i^T \leq (\mathcal{Y} - \|C_i x_{si}\|)^2 I_p \quad (59)$$

The inequality (59) holds if and only if the following condition holds

$$\lambda_{\max} \left(X_i^{1/2} C_i^T C_i X_i^{1/2} \right) \leq (\mathcal{Y} - \|C_i x_{si}\|)^2 \quad (60)$$

From (55), the left side of the above inequality is

$$\begin{aligned} & \lambda_{\max} \left(X_i^{1/2} C_i^T C_i X_i^{1/2} \right) \\ & \geq \lambda_{\max} \left(X_i^{1/2} C_i^T C_i X_i^{1/2} \right) \left\| X_i^{-1/2} (x_i(t + \tau|t) - x_{si}) \right\|^2 \\ & = \max_{\tau \geq 0} \|C_i (x_i(t + \tau|t) - x_{si})\|^2 \end{aligned} \quad (61)$$

Combining (60) and (61), it follows that

$$\max_{\tau \geq 0} \|C_i (x_i(t + \tau|t) - x_{si})\| \leq \mathcal{Y} - \|C_i x_{si}\| \quad (62)$$

In addition,

$$\begin{aligned} & \max_{\tau \geq 0} (\|C_i (x_i(t + \tau|t) - x_{si})\| + \|C_i x_{si}\|) \\ & \geq \max_{\tau \geq 0} \|C_i (x_i(t + \tau|t) - x_{si}) + C_i x_{si}\| \\ & = \max_{\tau \geq 0} \|C_i x_i(t + \tau|t)\| \\ & = \max_{\tau \geq 0} \|y_i(t + \tau|t)\| \end{aligned} \quad (63)$$

Hence, $\max_{\tau \geq 0} \|y_i(t + \tau|t)\| \leq \mathcal{Y}$, i.e., output constraint (5) is fulfilled.

Remark 6: There are three points that need to be illustrated regarding the calculation burden resulting from the linear matrix inequalities required to solve the optimization problem (38)-(42). Firstly, a distributed approach is adopted so that the linear matrix inequalities can be calculated in parallel and no global information is required, thus reducing the computational load. Secondly, process systems, such as petrochemical production processes, unlike multi-agent systems with rigid bodies as nodes, generally have only several devices as nodes, and the dimensions of the node state matrices are not high, so the linear matrix inequalities will not be large. Thirdly, process systems are mostly slowly time-varying systems and tend to possess long operating periods, which gives sufficient time for the calculation.

The following algorithm is provided to design the MPC.

Algorithm 2. (MPC algorithm)

Step 1 (Initialization). For each subsystem i , set $k = 0$, give an initial value $x_i(0)$ and a sampling interval T .

Step 2 (Exchange and Optimization). At each instant t , exchange $x_i(t)$ with its neighbours. At the instant $t = kT$, solve the optimization problem (38)-(42) to obtain $K_{i,k}$, where $K_{i,k}$ denotes the acquired control gain at the time instant kT .

Step 3 (MPC). For $t \in [kT, (k+1)T)$, apply $u_i^{mpc}(t) = K_{i,k}(x_i(t) - x_{si}) + u_{si}^{mpc}$ to each system. Go back to Step 2.

Theorem 4: Suppose Assumptions 1-4 hold. Given initial state $x_i(0)$, the MPC input (37) can make (21) achieve steady state tracking asymptotically with the state feedback matrix K_i satisfying Theorem 3.

Proof: The proof is divided into two parts. In Part I, the feasibility of the optimization problem (38)-(42) is proved. In Part II, the robust stability of the closed-loop system formed by applying the control (37) in the system (21) is analysed. In fact, X_i, Y_i, K_i, P_i should be written as $X_{i,k}, Y_{i,k}, K_{i,k}, P_{i,k}$ to distinguish the different optimal values acquired at different sampling instants. In Part I, the subscript k is omitted for brevity, while in Part II, they are displayed for the robust stability proof.

Part I. In the optimization problem (38)-(42), only (41) depends on time t explicitly, thus, only the feasibility of (41) needs to be proved. Since (49) holds, and $R_i, Q_{ii}, Q_{ij} > 0$ ($j \in \bar{N}_i$), then $\dot{V}_i(x_i(t + \tau|t), x_{si}) < 0$. As a result, if (52) holds, $V_i(x_i(t + \tau|t), x_{si}) < \eta_i$ for any $\tau > 0$.

According to (43) and $\dot{x}_{si} = 0_{n_i}$, for any $\tau \geq 0$,

$$\begin{aligned} x_i(t + \tau|t) &= x_i(t) + \int_0^\tau (A_{ii} + B_i K_i) (x_i(t + \bar{\tau}|t) - x_{si}) \\ &\quad + A_{\mathcal{N}_i} (x_{\mathcal{N}_i}(t + \bar{\tau}|t) - x_{s\mathcal{N}_i}) d\bar{\tau} \\ &= x_i(t + \tau|t + \tau) \\ &= x_i(t + \tau), \end{aligned} \quad (64)$$

where $\bar{\tau}$ denotes the variable of integration, $x_i(t + \tau) \triangleq x_i(t + \tau|t + \tau)$ denotes the measured state at sampling instant $t + \tau$. Thus, $V_i(x_i(t + \tau), x_{si}) < \eta_i$ for any $\tau \geq 0$. Hence, the optimization problem (38)-(42) is feasible.

Part II. It is clear that the candidate Lyapunov function (45) meets condition (12). $x_i(t)$ is continuously differentiable and $P_{i,k}$ is a constant, so $V_i(x_i(t), x_{si})$ is continuously differentiable in each time interval $t \in [kT, (k+1)T)$. Thus, for all $t \geq 0$, $V_i(x_i(t), x_{si})$ is piecewise continuously differentiable.

From (49) and (64), in each interval $t \in [kT, (k+1)T)$, it follows that

$$\begin{aligned} \dot{V}_i(x_i(t), x_{si}) &< -\|x_i(t) - x_{si}\|_{K_{i,k}^T R_i K_{i,k} + Q_{ii}}^2 \\ &\leq -\lambda_{\min}(K_{i,k}^T R_i K_{i,k} + Q_{ii}) \|x_i(t) - x_{si}\|^2 \end{aligned} \quad (65)$$

which satisfies condition (13).

According to Part I, $\dot{V}_i(x_i(t + \tau|t), x_{si}) < 0$, therefore,

$$\|x_i((k+1)T|kT) - x_{si}\|_{P_{i,k}}^2 < \|x_i(kT + \tau|kT) - x_{si}\|_{P_{i,k}}^2 \quad (66)$$

for $\tau \in [0, T)$. Note that $P_{i,k+1}$ is an optimal value for $V_i(x_i((k+1)T), x_{si})$, while $P_{i,k}$ is only a feasible value for $V_i(x_i((k+1)T), x_{si})$, thus,

$$\|x_i((k+1)T) - x_{si}\|_{P_{i,k+1}}^2 \leq \|x_i((k+1)T) - x_{si}\|_{P_{i,k}}^2 \quad (67)$$

Recall (64). It can be obtained that

$$\|x_i((k+1)T) - x_{si}\|_{P_{i,k+1}}^2 < \|x_i(kT + \tau) - x_{si}\|_{P_{i,k}}^2 \quad (68)$$

for $\tau \in [0, T)$. Therefore, $V_i(x_i(t), x_{si})$ is decreasing at the boundary point $(k+1)T$. From (65), in each time interval $t \in [kT, (k+1)T)$, $V_i(x_i(t), x_{si})$ is decreasing. As a result, $V_i(x_i(t), x_{si})$ is decreasing in $t \in [0, \infty)$.

From Lemma 2, it can be concluded that when $t \rightarrow \infty$, $x_i \rightarrow x_{si}$ for all $i \in \mathbf{N}$, i.e., all subsystems achieve steady state tracking.

Remark 7: Note that the ISMC (19) can reject matched disturbances completely. As a result, the MPC only needs to stabilize the nominal system (21) in this paper. Both the analysis and execution of MPC are done in a piecewise continuous manner. According to the recursive feasibility and stability in Theorem 4, when the system fulfills Assumptions 1-4, as long as the control gain is initially feasible, i.e., $K_{i,0}$ is feasible, $K_{i,0}$ is also feasible in the following time $t \geq 0$. That is to say, the corresponding control $u_i^{mpc}(t) = K_{i,0}(x_i(t) - x_{si}) + u_{si}^{mpc}(t \geq 0)$, can stabilize the system regardless of the sampling interval T in theory.

5. Output consensus for distributed heterogeneous system

In this section, an output consensus algorithm is first proposed and then the consensus for the distributed system is analysed and proved.

Algorithm 3. (Output consensus algorithm)

Step 1 (Initialization). For each subsystem, give feasible $y_{si}(0)$ and arbitrary $x_{si}(0)$.

Step 2 (Output Consensus). Exchange $y_{si}(0)$ with its neighbours, and solve (23) to acquire a new y_{si} .

Step 3 (Steady State Target Algorithm). Apply Algorithm 1 and acquire x_{si} and u_{si}^{mpc} .

Step 4 (ISMC and MPC). When $t = 0$, give $x_i(0)$, exchange x_i , apply ISMC (19) to make subsystems reach the corresponding sliding surfaces; Solve K_i through optimization problem (38)-(41) such that a new $x_i(t)$ can be obtained using Algorithm 2.

Step 5 (Output Calculation). Obtain $y_i(t)$ through system dynamics (2), and go back to Step 4.

Theorem 5: Suppose Assumptions 1-4 hold. Applying the control input (3) to the subsystem dynamics (2), the corresponding closed-loop system can achieve output consensus and all the subsystems can achieve steady state tracking.

Proof: In Algorithm 3, after initializing, take step 2, and average consensus can be achieved in finite time Wang et al. (2010), i.e., $y_{sj} = y_{sj} = \sum_{k=1}^N y_{sk}(0)/N, \forall i, j \in \mathbf{N}$. As a result, the new y_{si} is feasible.

Take step 3. Feasible x_{si} and u_{si}^{mpc} can be acquired.

Take step 4. According to Theorem 1, system (2) is on the sliding surface (18) from the initial instant. The system dynamics (2) are transformed to (21). According to Theorem 2, by applying MPC (37), when $t \rightarrow \infty$, $x_i(t) \rightarrow x_{si}$ for all $i \in \mathbf{N}$.

Take step 5. It can be obtained that

$$\begin{aligned} &\sum_{j=1}^N \|y_i(t) - y_j(t)\| \\ &= \sum_{j=1}^N \|y_i(t) - y_{si} - (y_j(t) - y_{sj})\| \\ &= \sum_{j=1}^N \|C_i(x_i(t) - x_{si}) - C_j(x_j(t) - x_{sj})\| \\ &\leq N \|C_i(x_i(t) - x_{si})\| + \sum_{j=1}^N \|C_j(x_j(t) - x_{sj})\| \end{aligned} \quad (69)$$

From the above inequality and $x_i(t) \rightarrow x_{si}$ when $t \rightarrow \infty$, it can be obtained that $\lim_{t \rightarrow \infty} \sum_{j=1, j \neq i}^N \|y_i(t) - y_j(t)\| = 0$, i.e., output consensus is achieved asymptotically.

6. Simulation and experimental results using interconnected CSTRs

In this section, a simulation example and an experiment are presented to demonstrate the validity of the

proposed method. First, a practical example is simulated. Then, temperature consensus control for the saponification reaction is carried out for interconnected CSTRs. Both the simulation and the experimental trial are performed with and without ISMC to show the effectiveness and superiority of the proposed method in the real world.

6.1. Practical example

The process control of four interconnected CSTR systems is used for simulation by referring Sun et al. (2008) to demonstrate the effectiveness of the proposed method. Fig. 1 shows the process flow diagram of the CSTRs. The four CSTRs are assumed to be well-mixed and non-isothermal. The interconnection of the two groups of systems is broadly the same. The only difference is the recycle rate F_r and F_{r1} . Note that the neighbour set of each reactor is $\mathcal{N}_1 = \{2\}$, $\mathcal{N}_2 = \{1, 3\}$, $\mathcal{N}_3 = \{2, 4\}$, $\mathcal{N}_4 = \{3\}$. For an introduction to process flow, please refer to Sun et al. (2008); Kazempour et al. (2013); Zhang et al. (2015). According to Sun et al. (2008), the plant model is given as

$$\begin{aligned} \dot{T}_1 &= \frac{F_0}{V_1} (T_0 - T_1) + \frac{F_r}{V_1} (T_2 - T_1) + \sum_{i=1}^3 G_i (T_1) C_{A1} + \frac{Q_1}{\rho c_p V_1} \\ \dot{C}_{A1} &= \frac{F_0}{V_1} (C_{A0} - C_{A1}) + \frac{F_r}{V_1} (C_{A2} - C_{A1}) + \sum_{i=1}^3 R_i (T_1) C_{A1} \\ \dot{T}_2 &= \frac{F_1}{V_2} (T_1 - T_2) + \frac{F_3}{V_2} (T_{03} - T_2) + \sum_{i=1}^3 G_i (T_2) C_{A2} + \frac{Q_2}{\rho c_p V_2} \\ \dot{C}_{A2} &= \frac{F_1}{V_2} (C_{A1} - C_{A2}) + \frac{F_3}{V_2} (C_{A03} - C_{A2}) - \sum_{i=1}^3 R_i (T_2) C_{A2} \\ \dot{T}_5 &= \frac{F_4}{V_5} (T_4 - T_5) + \frac{F_{r6}}{V_5} (T_6 - T_5) + \sum_{i=1}^3 G_i (T_5) C_{A5} + \frac{Q_5}{\rho c_p V_5} \\ \dot{C}_{A5} &= \frac{F_4}{V_5} (C_{A4} - C_{A5}) + \frac{F_{r6}}{V_5} (C_{A6} - C_{A5}) + \sum_{i=1}^3 R_i (T_5) C_{A5} \\ \dot{T}_6 &= \frac{F_5}{V_6} (T_5 - T_6) + \frac{F_7}{V_6} (T_{07} - T_6) + \sum_{i=1}^3 G_i (T_6) C_{A6} + \frac{Q_6}{\rho c_p V_6} \\ \dot{C}_{A6} &= \frac{F_5}{V_6} (C_{A5} - C_{A6}) + \frac{F_7}{V_6} (C_{A07} - C_{A6}) - \sum_{i=1}^3 R_i (T_6) C_{A6} \end{aligned} \quad (70)$$

where T_1, T_2, T_5 and T_6 denote fluid temperatures in CSTR 1, 2, 3 and 4 respectively, C_{A1}, C_{A2}, C_{A5} and C_{A6} denote fluid molar concentration of the four CSTRs, while Q_1, Q_2, Q_5 and Q_6 denote heat input rates of the corresponding CSTRs. $R_i(T_j) = k_{i0} \exp(-E_i/RT_j)$, $G_i(T_j) = (-\Delta H_i R_i(T_j))/\rho c_p$ for $j = 1, 2, 5, 6$. Other parameters are shown in Table 1 with values and physical meanings from Sun et al. (2008).

Note that different from Sun et al. (2008), in this paper, it is assumed that C_{A0} , C_{A4} , C_{A03} and C_{A07} are known constant parameters, which is more realistic in reality because constantly changing feed concentration is difficult to deal with in practice.

Consider the operating point

$$\begin{aligned} &(T_1, C_{A1}, T_2, C_{A2}, T_5, C_{A5}, T_6, C_{A6}) \\ &= (324K, 0.718kmol/m^3, 324K, 0.432kmol/m^3, 324K, \\ &0.719kmol/m^3, 324K, 0.883kmol/m^3) \end{aligned} \quad (71)$$

at which the open-loop system is not stable. The control objective is to achieve temperature consensus across four reactors by exchanging information in the presence of input/output constraints and disturbances using the heat input rates Q_1, Q_2, Q_5, Q_6 as control inputs.

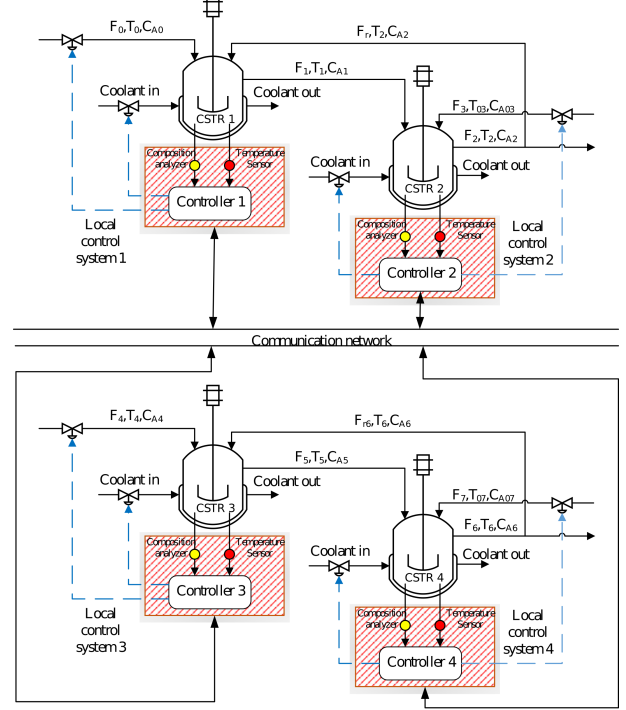


Figure 1: Process flow diagram of four interconnected CSTRs

By linearizing the plant model in (70) via evaluating the Jacobian matrices around the given operating points in (71), the following linear systems are obtained

$$\begin{aligned} \dot{x}_1 &= A_{11}x_1 + B_1u_1 + A_{12}x_2 + d_1 \\ y_1 &= C_1x_1 \\ \dot{x}_2 &= A_{22}x_2 + B_2u_2 + A_{21}x_1 + d_2 \\ y_2 &= C_2x_2 \\ \dot{x}_3 &= A_{33}x_3 + B_3u_3 + A_{34}x_4 + d_3 \\ y_3 &= C_3x_3 \\ \dot{x}_4 &= A_{44}x_4 + B_4u_4 + A_{43}x_3 + d_4 \\ y_4 &= C_4x_4 \end{aligned} \quad (72)$$

Table 1. Values and physical meanings of plant model parameters

Parameters & values	Physical meanings
$F_0 = F_4 = 4.998m^3/h$	Fresh feed flow rate of CSTR 1, 3
$F_1 = F_5 = 39.996m^3/h$	Discharge flow rate of CSTR 1, 3
$F_3 = F_7 = 30.0m^3/h$	Fresh feed flow rate of CSTR 2, 4
$F_r = 34.998m^3/h, F_{r6} = 20.9988m^3/h$	Recycle feed flow rate of CSTR 1, 3
$V_1 = V_5 = 1.0m^3$	Volume of CSTR 1, 3
$V_2 = V_6 = 3.0m^3$	Volume of CSTR 2, 4
$T_0 = T_4 = 300.0K$	Fresh feed temperature of CSTR 1, 3
$T_{03} = T_{07} = 300.0K$	Fresh feed temperature of CSTR 2, 4
$C_{A0} = 2.72kmol/m^3, C_{A4} = 2.36kmol/m^3$	Molar concentration of fresh feed of CSTR 1, 3
$C_{A03} = 0.0515kmol/m^3, C_{A07} = 0.504kmol/m^3$	Molar concentration of fresh feed of CSTR 2, 4
$\Delta H_1 = -5.0 \times 10^4 kJ/kmol$	Enthalpies of reactions
$\Delta H_2 = -5.2 \times 10^4 kJ/kmol$	Enthalpies of reactions
$\Delta H_3 = -5.4 \times 10^4 kJ/kmol$	Enthalpies of reactions
$k_{10} = 3.0 \times 10^5 h^{-1}$	Pre-exponential constants of reactions
$k_{20} = 3.0 \times 10^5 h^{-1}$	Pre-exponential constants of reactions
$k_{30} = 3.0 \times 10^5 h^{-1}$	Pre-exponential constants of reactions
$E_1 = 5.0 \times 10^4 kJ/kmol$	Activation energies of reactions
$E_2 = 7.53 \times 10^4 kJ/kmol$	Activation energies of reactions
$E_3 = 7.53 \times 10^4 kJ/kmol$	Activation energies of reactions
$\rho = 1000.0kg/m^3$	Density of fluid
$c_p = 0.231kJ/(kg \cdot K)$	Heat capacity of fluid
$R = 8.314kJ/(kmol \cdot K)$	Gas constant

where

$$\begin{aligned}
x_1 &= [CA_1, T_1]^T, u_1 = Q_1, y_1 = T_1, \\
x_2 &= [CA_2, T_2]^T, u_2 = Q_2, y_2 = T_2, \\
x_3 &= [CA_5, T_5]^T, u_3 = Q_5, y_3 = T_5, \\
x_4 &= [CA_6, T_6]^T, u_4 = Q_6, y_4 = T_6
\end{aligned} \tag{73}$$

The system matrices are given as follows

$$\begin{aligned}
A_{11} &= \begin{bmatrix} -39.9986 & -1.0721 \times 10^{-4} \\ 0.5641 & -39.9728 \end{bmatrix}, \\
A_{12} &= \begin{bmatrix} 34.9980 & 0 \\ 0 & 34.9980 \end{bmatrix}, \\
B_1 &= \begin{bmatrix} 0 \\ 0.0043 \end{bmatrix}, d_1 = \begin{bmatrix} 13.63456 \\ 1481.5582 \end{bmatrix}, C_1 = [0 \quad 1], \\
A_{22} &= \begin{bmatrix} -23.3346 & -6.4505 \times 10^{-5} \\ 0.5641 & -23.3180 \end{bmatrix}, \\
A_{21} &= \begin{bmatrix} 13.3320 & 0 \\ 0 & 13.3320 \end{bmatrix}, \\
B_2 &= \begin{bmatrix} 0 \\ 0.0014 \end{bmatrix}, d_2 = \begin{bmatrix} 0.5291 \\ 3004.3523 \end{bmatrix}, C_2 = [0 \quad 1], \\
A_{33} &= \begin{bmatrix} -25.9994 & -1.0721 \times 10^{-4} \\ 0.5641 & -25.9736 \end{bmatrix}, \\
A_{34} &= \begin{bmatrix} 20.9988 & 0 \\ 0 & 20.9988 \end{bmatrix}, \\
B_3 &= \begin{bmatrix} 0 \\ 0.0043 \end{bmatrix}, d_3 = \begin{bmatrix} 7.8315 \\ 1481.5582 \end{bmatrix}, C_3 = [0 \quad 1],
\end{aligned}$$

$$\begin{aligned}
A_{44} &= \begin{bmatrix} -23.3346 & -6.4505 \times 10^{-5} \\ 0.5641 & -23.3180 \end{bmatrix}, \\
A_{43} &= \begin{bmatrix} 13.3320 & 0 \\ 0 & 13.3320 \end{bmatrix}, \\
B_4 &= \begin{bmatrix} 0 \\ 0.0014 \end{bmatrix}, d_4 = \begin{bmatrix} 5.0541 \\ 3004.3523 \end{bmatrix}, C_4 = [0 \quad 1]
\end{aligned} \tag{74}$$

It should be noted that there is a constant d_i in the i -th subsystem in (72) for $i = 1, 2, 3, 4$. This is because the system is not linearized at the origin. Since these constants are known, it does not affect the controller design.

To make this example better align with reality, the following input and output constraints are assumed

$$\begin{aligned}
0 &\leq u_i \leq 10^6 \\
273.15 &\leq y_i \leq 353.15
\end{aligned} \tag{75}$$

Take a coordinate transformation of the above inequalities so that the constraints satisfy (4) and (5) as follows

$$\begin{aligned}
\|u_i - 5 \times 10^5\| &\leq 5 \times 10^5 \\
\|y_i - 313.15\| &\leq 40
\end{aligned} \tag{76}$$

For the simulation purpose, it is assumed that the persistent disturbances are given by

$$\begin{aligned}
\phi_1 &= 10^4 \cos(x_{12}), \phi_2 = -8 \times 10^3 \sin(x_{12} + x_{22}), \\
\phi_3 &= 2 \times 10^4 \sin(x_{32}), \phi_4 = 2 \times 10^4 \cos(t)
\end{aligned} \tag{77}$$

where x_{12} , x_{22} and x_{32} denote the second components of x_1 , x_2 and x_3 respectively. In addition, when $t = 4s$,

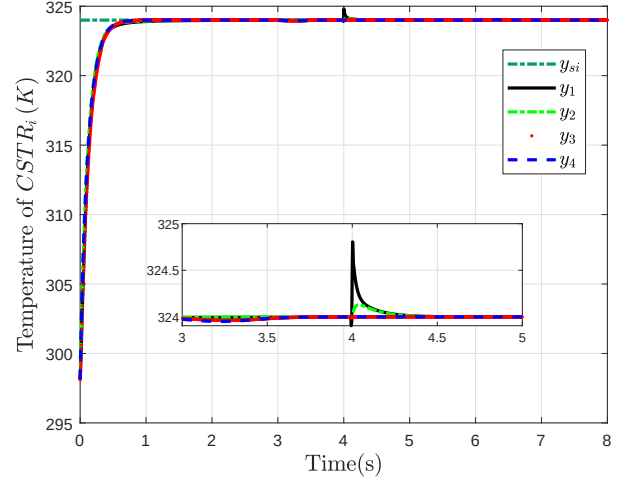
a transient pulse disturbance with amplitude of 10^5 is applied to $CSTR_1$, so as to observe the response characteristics to the disturbance after the system reaches the steady state. The other parameters are selected as

$$\begin{aligned}
y_{s1}(0) &= 320, x_{s1}(0) = [4.0 \quad 320]^T, \\
x_1(0) &= [0 \quad 298.15]^T, \beta_1 = 10^4, \\
y_{s2}(0) &= 336, x_{s2}(0) = [2.0 \quad 336]^T, \\
x_2(0) &= [0 \quad 298.15]^T, \beta_2 = 8 \times 10^3, \\
y_{s3}(0) &= 328, x_{s3}(0) = [4.0 \quad 328]^T, \\
x_3(0) &= [0 \quad 298.15]^T, \beta_3 = 2 \times 10^4, \\
y_{s4}(0) &= 312, x_{s4}(0) = [2.0 \quad 312]^T, \\
x_4(0) &= [0 \quad 298.15]^T, \beta_4 = 2 \times 10^4, \\
T &= 1, R_{si} = I_{m_i}, Q_{si} = 10^{10}, R_i = 10^{-25} I_{m_i}, \\
Q_{ii} &= 10^{-25} I_{n_i}, Q_{ij} = 5 \times 10^{-24} I_{n_j} \quad (j \in \bar{N}_i), \\
\rho_i &= \beta_i \|B_i^+\|_F + 0.1, \alpha_i = 0.5, \delta_i = 0.01, \\
\epsilon &= 10^{-4}, \omega = 0.99, i \in \{1, 2, 3, 4\}
\end{aligned}$$

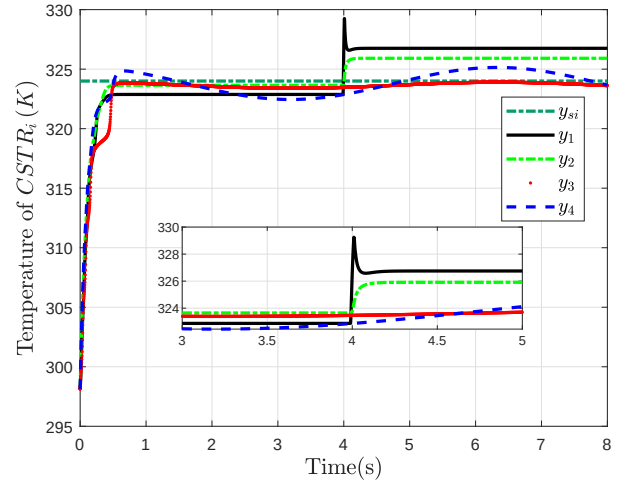
The simulation results are shown in Figs. 2-3, where ISMPC denotes the method proposed in this paper, while MPC denotes an approach that only adopts MPC alone. It follows from Fig. 2 that disturbances have a reduced effect on system states when ISMPC is used in contrast to using MPC alone, and the states can quickly return to the desired temperature after the transient pulse disturbance applied. This attributes to the strong robustness of the sliding mode control. Hence, ISMPC is much better than MPC regarding tracking accuracy of temperature. This is very important in practical applications such as fine chemical process where there are strict requirements on temperature. From Fig. 3, disturbances have no obvious impact on the molar concentration in both ISMPC and MPC, but the anti-disturbance capability is slightly better with ISMPC than MPC from the zooms. For ISMPC, the tracking accuracy depends on the size of ρ_i . On the one hand, from the perspective of suppressing disturbances, a bigger ρ_i results in improved accuracy. On the other hand, as can be seen from the range $\beta_i < \rho_i < \mathcal{U}_i$, the control gain ρ_i should be as small as possible. A small ρ_i not only reduces the chattering that is produced by sliding mode control, but also increases the scope for u_i^{mpc} to be feasible. Therefore, a suitable parameter value must be sought to achieve a balance between different performance requirements.

6.2. Experimental testing

In this part, the temperature consensus control experiment of a saponification reaction process is carried out in two serial CSTRs to further validate the proposed method using the chemical process control network platform as



(a) Time response of temperature with ISMPC

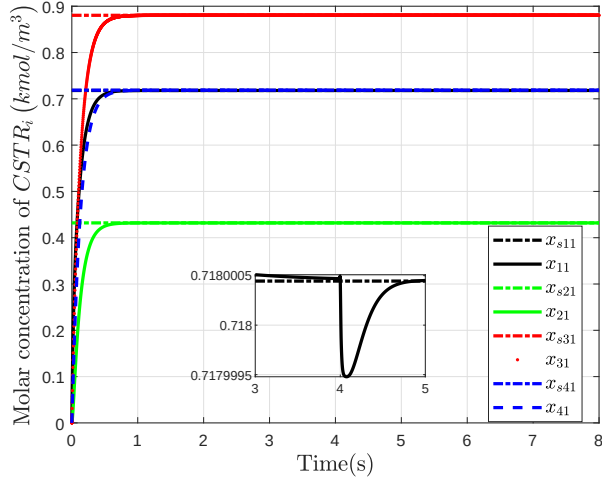


(b) Time response of temperature with MPC

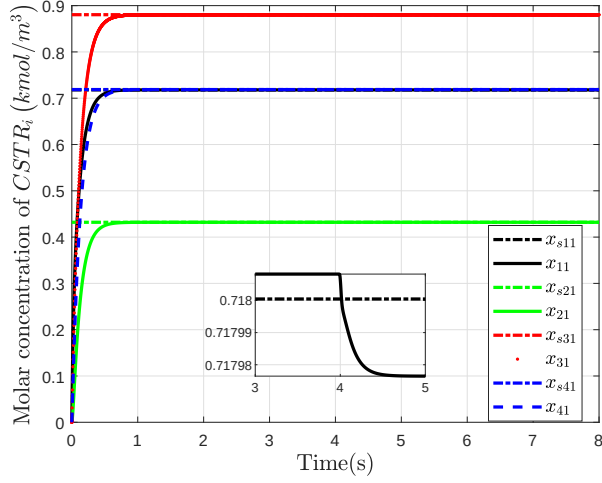
Figure 2: Temperature consensus response of four interconnected CSTRs with respect to time

shown in Fig. 4. The CSTRs can be connected in series, parallel and series-parallel. The chemical reaction is carried out after feeding. The control process can be continuously operated, and flow rate, liquid level and temperature can be measured and controlled.

Fig. 5 shows the platform operation interface. In this experiment, CSTRs R-102 and R-104 are used in series, and feedstock tanks V-111 and V-112 contain 0.02 kmol/m^3 ethyl acetate and sodium hydroxide solutions, respectively. The tank V-113 contains warm water to store the heat needed for the experiment. The raw material first flows through R-102 for reaction, and then part of the solution in R-102 enters R-104 for further reaction to improve the conversion rate. In this process, the control input is the flow of warm water into the jacket of the reactors, and this experiment seeks to control the temperature



(a) Time response of molar concentration with ISMPC



(b) Time response of molar concentration with MPC

Figure 3: Molar concentration response of four interconnected CSTRs with respect to time

in both reactors to be in consensus and maintain a desired temperature. The initial temperature of the feedstock in V-111 and V-112 is 26°C . While in R-102 and R-104, the initial temperatures are 26°C and 25°C respectively, and the target temperature is 32°C . The ambient temperature varies between 22°C and 24°C during the experiment. Despite the insulation layer, both the tank V-113 and the water pipeline still exchange heat with the environment, resulting in heat loss, which can be regarded as external disturbances. However, because the ambient temperature varies very slowly, and the impact on the experiment is indirect, the influence of the ambient temperature on the experimental results is quite limited.

Both ISMPC and MPC methods are adopted in this experiment, and the final temperature response over time is shown in Fig. 6. In Fig. 7, a temperature tracking

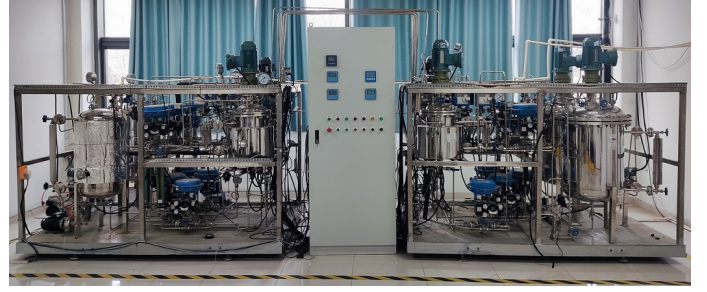


Figure 4: Experiment equipment

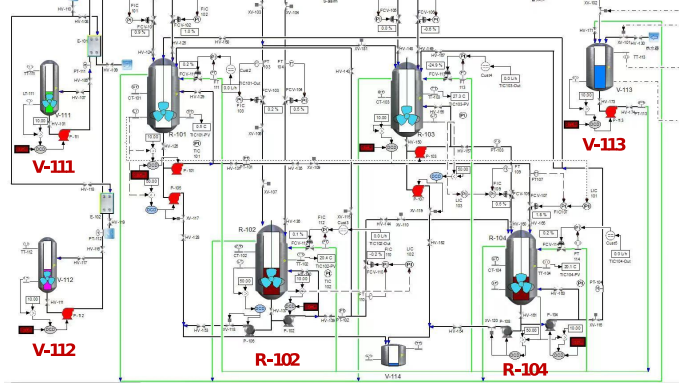


Figure 5: Experiment operation interface

cost function is used to further demonstrate the system performance

$$\mathcal{J}_{Ti}(t) = \|T_i(t) - T_{si}(t)\|^2, \quad (78)$$

$$i \in \{R-102, R-104\}$$

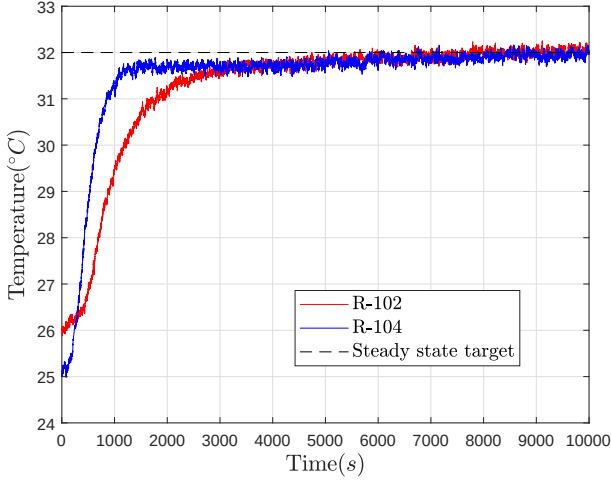
where T_i and T_{si} represent the real-time temperature ($^{\circ}\text{C}$) and steady temperature of the corresponding CSTR. Table 2 shows the overall performance indexes of the two methods within 10000s, i.e.

$$\mathcal{J}_i = \sum_t \left(\|T_i(t) - T_{si}(t)\|^2 + \|M_i(t) - M_{si}(t)\|^2 \right. \\ \left. + \|T_j(t) - T_{sj}(t)\|^2 + \|M_j(t) - M_{sj}(t)\|^2 \right. \\ \left. + \|F_i(t) - F_{si}(t)\|^2 \right) \quad (79)$$

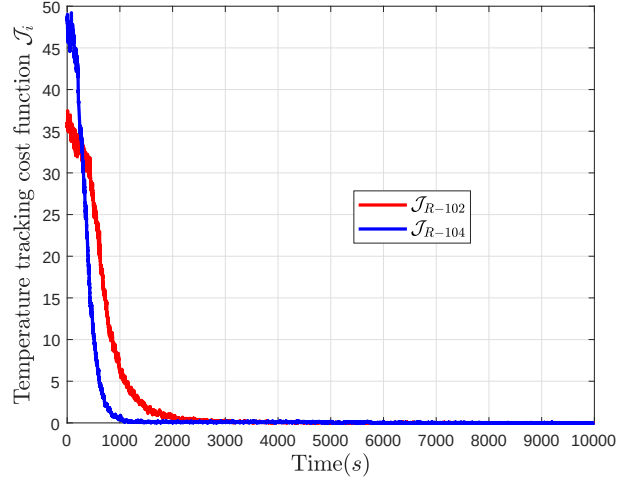
$$i, j \in \{R-102, R-104\}$$

where M_i and F_i represent the real-time molar concentration (kmol/m^3) and flow rate of water (L/h), M_{si} and F_{si} denote the corresponding steady values.

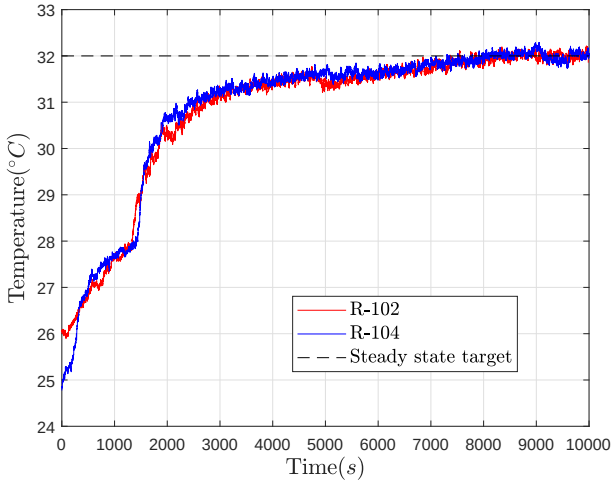
From Fig. 6, both methods achieve temperature consensus and maintain the desired value eventually. However, systems achieve consensus faster at the desired temperature with ISMPC, and the transient process in (a) of Fig. 6 is better due to the stronger robustness of ISMPC. From Fig. 7, the curves are steeper before they arrive



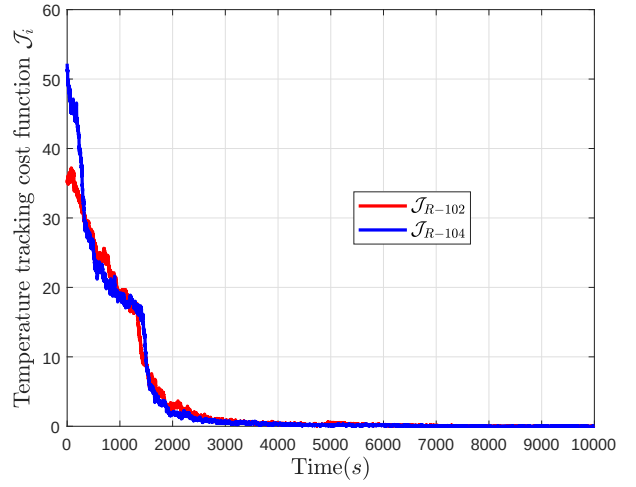
(a) Time response of temperature with ISMPC



(a) Temperature tracking cost function with ISMPC



(b) Time response of temperature with MPC



(b) Temperature tracking cost function with MPC

Figure 6: Temperature consensus response of two interconnected CSTRs with respect to time

at 0 in ISMPC. Thus systems approach the desired temperature much faster with ISMPC than that with MPC. From Table 2, the overall performance index also reflects the better control performance of the proposed method. From the experimental results, the ISMPC method shows good effect, and it also performs well when compared with MPC.

Table 2. The overall performance index within 10000s

Method	$\mathcal{J}_{R-102}$	$\mathcal{J}_{R-104}$
ISMPC	1.0130×10^8	1.0445×10^8
MPC	1.4267×10^8	1.4975×10^8

Figure 7: Temperature tracking cost function with respect to time

7. Conclusions

A combined ISMC and MPC method is proposed to solve the output consensus problem for a class of heterogeneous interconnected systems which are subject to constraints and affected by disturbances. The ISMC element reduces the systems' sensitivity to disturbances whilst the MPC is utilised to achieve output consensus by state tracking control. The simulation example and the experimental testing using the CSTRs shows the validity and practicality of the proposed method. Future work will focus on the case of output feedback control.

Declaration of conflicting interests

The author(s) declare no potential conflicts of interest with respect to the research, authorship, and/or publica-

tion of this article.

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Appendix A. The proof of Lemma 2

The inequalities (12) and (13) imply

$$\dot{V}(t) \leq -\gamma \|v(t)\|^2 \leq -\gamma/\beta V(t) \quad (\text{A.1})$$

Because $V(t)$ is positive,

$$\dot{V}(t)/V(t) \leq -\gamma/\beta, a.e. \quad (\text{A.2})$$

Integrate the inequality (A.2) on both sides with respect to t . Combining (12), (14) can be obtained, then it can be concluded that when $t \rightarrow \infty$, $v(t) \rightarrow 0$.

Appendix B. The proof of Theorem 1

Combining (6) and (19), the time derivative of $s_i(t)$ is given by

$$\begin{aligned} \dot{s}_i(t) &= B_i^+ (\dot{x}_i(t) - A_{ii}x_i(t) - A_{N_i}x_{N_i}(t) - B_i u_i^{mpc}(t)) \\ &= -\rho_i s_i(t)/\|s_i(t)\| + \phi_i(x, t) \end{aligned} \quad (\text{B.1})$$

A Lyapunov candidate function is selected as

$$V^{ism}(t) = \frac{1}{2} \sum_{i=1}^N s_i^T(t) s_i(t) \quad (\text{B.2})$$

Based on (B.1), (B.2) and Assumption 3, the time derivative of $V_i^{ism}(t)$ is obtained as

$$\begin{aligned} \dot{V}_i^{ism}(t) &= \sum_{i=1}^N s_i^T(t) \dot{s}_i(t) \\ &= \sum_{i=1}^N s_i^T(t) (-\rho_i s_i(t)/\|s_i(t)\| + \phi_i(x, t)) \\ &\leq \sum_{i=1}^N (-\rho_i + \beta_i) \|s_i(t)\| \end{aligned} \quad (\text{B.3})$$

Further,

$$\begin{aligned} \dot{V}_i^{ism}(t) &= \sum_{i=1}^N -\sigma_i \|s_i(t)\| \\ &\leq -\sigma_{min} \sum_{i=1}^N s_i(t) \\ &\leq -\sigma_{min} \sqrt{2V(t)} \end{aligned} \quad (\text{B.4})$$

where $\sigma_{min} = \min_i \{\sigma_i\}$ and $\sigma_i = \rho_i - \beta_i$.

As a result, the system (2) can be kept on the sliding surface (18) from the initial time.

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