Reliability assessment of discrete-time k/n(G) retrial system based on different failure types and the δ-shock model

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Abstract

The k/n(G) system is a typical reliability system in engineering practice. However, research on this system under discrete-time and retrial conditions is still limited. This paper establishes a k/n(G) retrial system model under a discrete-time condition based on different failure types and the δ-shock model, whose objective is to investigate its various reliability measures and the relationship between measures and parameters. Two methodologies are utilized in this paper to evaluate the system's various measures: the analytical method for the discrete-time Markov process and the Monte Carlo simulation method. The two methodologies are compared, and optimization is performed on certain measures concerning system parameters. A discrete PH distribution representation method for component lifetime under the δ-shock is presented, the relationship between various reliability measures of the system and various parameters is determined, and a novel approach for system reliability research based on the discrete PH distribution and the δ-shock is provided in this paper.
Keywords: Discrete time; PH distribution; δ-shock; Different failure types; Retrial

1. Introduction

The k/n(G) and k/n(F) systems can use redundancy technology to effectively improve system reliability. Therefore, numerous studies have been conducted on the reliability of the k/n(G) and k/n(F) systems over the past decades. Following the introduction of the k/n(G) system by Birnbaum et al.[1], a k/n(G) non-repairable system with warm standby components was studied by She and Pecht[2]. Moreover, some researchers studied the k/n(G) system based on different repair strategies and unreliable repair equipment[3, 4]. Recently, some researchers have carried out innovative studies on k/n(G) systems from a variety of unique perspectives. Xiahou et al.[5] conducted a study that undertakes the reliability modeling of modular k/n(G) systems, which exhibit functional dependency. Bian et al.[6] conducted a reliability analysis for the k/n(G) systems, and investigated the impact of dependent competing failure processes on the system's performance. Xu et al.[7] utilized a new dynamic k/n(G) model to present a reliability analysis for a satellite power supply system. Zhang et al.[8] proposed a general reliability optimization model for k/n(G) subsystems based on a mixed redundancy strategy. Additionally, some researchers carried out investigations on the k/n(F) systems[9, 10]. In addition to the k/n(G) and k/n(F) system designs,

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retrial also plays a crucial role in reliability systems. Krishnamoorthy and Ushakumari[11] investigated the reliability of a k/n(G) retrial system, involving three scenarios: cold, warm, and hot. Ke et al.[12] conducted a study on the availability of a repairable retrial system equipped with warm standby components. Kuo et al.[13] presented a study on the reliability and availability of a retrial and repairable multi-component system with mixed warm and cold standby components. Yang and Tsao[14] presented a reliability and availability analysis of repairable systems with standby components, working vacations and retrial of failed components. Similarly, numerous researchers[15–17] explored retrial systems with various types of standby components. Subsequently, Gao[18] studied the reliability of a repairable retrial system with an optional two-stage repair mode. Hu et al.[19] investigated the reliability of a k/n(G) repairable retrial system with two failure modes.

In actual manufacturing processes, components may fail due to external interference (shock) in addition to the typical wear and tear associated with usage. Shock models have been classified into different types by researchers based on different failure criteria: run shock models[20, 21] (the component fails when a sequence of shocks above a threshold arrive in succession), cumulative shock models[22, 23] (the component fails when the cumulative damage due to shocks exceeds the handling capacity of the component), extreme shock models[24, 25] (the component fails due to a single shock that is above a critical level[26]), and δ-shock models[27, 28] (the component fails when the interval between two consecutive shocks is less than a given threshold δ[26]). Various shock models have garnered a lot of interest, and many researchers have studied system reliability based on different types of shocks. The reliability of a performance-balanced system in a random shock environment was explored by Wang et al.[29]. Shamstabar et al.[30] presented a system reliability monitoring method for analyzing the failure probability of multi-component systems affected by random shocks. Recently, Lorvand and Kelkinnama[31] presented a study on a k/n(G) system subject to random shocks, and discussed the reliability measures of the system. Lyu et al.[32] explored a reliability model based on the time-varying δ-shock model. Eryilmaz and Unlu[33] investigated a generalized δ-shock model and elucidated its practical application in enhancing the reliability of the 1-out-of-(m+1): G cold standby system. Lyu et al.[34] presented an approach to the reliability analysis of multi-state systems in complex environments, where a mixed shock model was introduced. Chadjiconstantinidis and Eryilmaz[35] presented a study on the reliability of a mixed shock model, where the distribution of shock magnitude changes at a random point. Levitin et al.[36] proposed and optimized a multi-threshold mission abort policy for systems exposed to random shocks. Additionally, they[37] proposed a model and an optimization method for a 1-out-of-n standby system with heterogeneous components exposed to different random shocks.

Previous research was conducted under the continuous-time condition. However, in actual production, system state information can only be obtained at specific times due to practical constraints such as measurement techniques or the cost of observation. Therefore, it is crucial to analyze system reliability under the discrete-time condition. Neuts[38] noted that discrete distributions, which only take on a finite number of positive values, can be expressed as a discrete PH distribution. Subsequently, Neuts[39] discussed the properties of the PH distribution and the method of solving the probability vector of Markov processes with repeated structures using matrix geometric methods. Based on the relevant research on discrete distributions, Ruiz-Castro et al.[40] studied the reliability of a discrete system with loss of components. Dembińska[41] presented a comprehensive reliability analysis of k-out-of-n systems with heterogeneous components, focusing on systems with discrete lifetime distributions. Davies and Dembińska[42] explored the distribution of failed components at system failure in k-out-of-n systems with discretely distributed component lifetimes. Kan and Eryilmaz[43] conducted a comprehensive reliability assessment of a discrete-time cold standby repairable system. A discrete-time Bayesian network-based approach for assessing the reliability of dynamic systems was presented by Guo et al.[44]. Temraz[45] presented an analysis of a discrete-time semi-Markov model for a two-unit warm standby system, focusing on system effectiveness measures like availability and reliability.

Just like systems under the continuous-time condition, systems under the discrete-time condition are also susceptible to external shocks that might cause systems to fail. Eryilmaz[46] studied two different discrete-time shock models in the Markovian environment. Chadjiconstantinidis and Eryilmaz[47] conducted an investigation into the distributions of random variables involved in discrete censored δ-shock models. Ruiz-Castro[48] investigated a preventive maintenance plan for a standby system subjected to both external shocks and internal breakdowns. Furthermore, Eryilmaz et al.[49, 50] represented the lifetime distribution of different shock models using probability-generating functions when the time intervals between two continuous shocks follow a geometric distribution or a discrete PH distribution. Lorvand et al.[51] studied a mixed δ-shock model under a discrete-time condition, which considered the impact of the interval time and amplitude of external shocks on system reliability. A discrete-time k/n(G) retrial system was investigated by Yu et al.[52] in a Bernoulli shock environment.

Given the research background and current situation of the k/n(G) system, it is evident that there are still some limitations in the study of the k/n(G) retrial system based on the δ-shock:

- There is less research under the discrete-time condition for the δ-shock model than under the continuous-time condition.
- When the time interval between two consecutive shocks follows a discrete PH distribution, the component lifetime distribution based on the δ-shock is given by the probability generating function, which poses challenges to apply it to the reliability system.
- In the k/n(G) repairable system, the failure type of components is often considered repairable. However, due to high repair costs or severe damage, repairs for certain components may be abandoned. The failure type of these components should be classified as non-repairable.

Considering some limitations in the research on k/n(G) repairable retrial systems, a k/n(G) retrial system is established based on different failure types and the δ-shock model under the discrete-time condition. Additionally, since the discrete PH distribution not only retains the characteristic of the geometric distribution that is easy to handle analytically but also avoids the unreasonableness brought about by the lack of aftereffects to a certain extent, it is assumed that the interval between two consecutive shocks in the k/n(G) retrial system follows a discrete PH distribution. A reliability analysis and optimization design for the aforementioned k/n(G) retrial system is conducted. The following are the specific work and contributions:

- A discrete-time k/n(G) retrial system, which is based on different failure types and the δ-shock model, is established. The failures of components are divided into two types: repairable and non-repairable.
- Given that the interval between two consecutive shocks follows a discrete PH distribution, a discrete PH distribution is constructed to represent the lifetime distribution of a component under the δ-shock.
- The temporal variations of different reliability measures of the system under different parameters are explored, the sensitivity of system reliability measures to different parameters are investigated, and some parameters of the system for different optimization goals of the system model are optimized.
- To overcome certain disadvantages of the analytical method for the discrete-time Markov process, the Monte Carlo simulation method is employed to simulate some reliability measures of the system. Furthermore, a comparison is made between the advantages and disadvantages of both the analytical method and the Monte Carlo simulation method.

The rest of this paper is arranged as follows: Section 2 describes the system model in detail, including the construction and analysis, and proposes the discrete PH distribution of the component lifetime under the δ-shock. Section 3 presents the calculation methods for different reliability measures of the system and constructs the T_C function and CER function of the system. Section 4 gives the numerical examples of the system model, shows the relationship between various system reliability measures and system parameters, and optimizes some parameters for different optimization objectives. Section 5 introduces the Monte Carlo simulation method for solving system reliability measures, and compares it with the analytical method. Section 6 summarizes the research content of this paper, and suggests several possible extended research directions.

2. Model construction and analysis

2.1. Model construction

2.1.1. Background of modeling

The hardware load balancer group is a type of k/n(G) retrial system based on different failure types and the δ-shock model, primarily used for implementing load balancing in computer networks. When a primary load balancer fails, an immediate switch can be made to the standby load balancer. During operation, hardware load balancers may fail due to internal factors such as wear and tear (internal failure). When an internal failure occurs, repair equipment may be utilized to repair it (failure is repairable), or repair may be abandoned due to exorbitant repair expenses or other reasons (failure is non-repairable). In addition to internal failures, hardware load balancers may also fail due to external factors (external failure). A Distributed denial-of-service (DDoS) attack is a common example of an external factor, which disrupts the operation of hardware load balancers by sending a large amount of traffic. DDoS attacks opt to strike at certain times, such as during the periods of high network traffic, to guarantee the attack's specificity. As a result, the timing of DDoS attacks is discrete. Moreover, as hardware load balancers are equipped with protection measures like firewalls, a DDoS attack is only considered effective (shock) when the DDoS attack traffic exceeds the maximum traffic that the hardware load balancer can handle at one time. Additionally, the DDoS attack traffic at the next time is related to the current DDoS attack traffic (increasing attack traffic to enhance interference or reducing attack traffic to evade detection). When the interval between two consecutive shocks is less than a given threshold, the hardware load balancer may malfunction properly due to heavy load (this external failure is repairable); it is also possible that the hardware may fail due to overheating caused by excessive load (this external failure is non-reparable). A hardware load balancer is non-reparable if it experiences a non-repairable failure (internal or external). The non-repairable hardware load balancer will be replaced with a brand-new one from inventory. If a hardware load balancer experiences a repairable failure and no non-reparable failures occur, then this hardware load balancer is repairable. The repair equipment can only repair one hardware load balancer at a time, and it will not stop operating when the system fails. When the repair equipment is repairing a repairable hardware load balancer, the failure information of the remaining repairable hardware load balancers is stored in the failure information storage and call center, and a repair request is sent to the repair equipment after a period of time. The schematic diagram of hardware load balancer groups is shown in Figure 1.

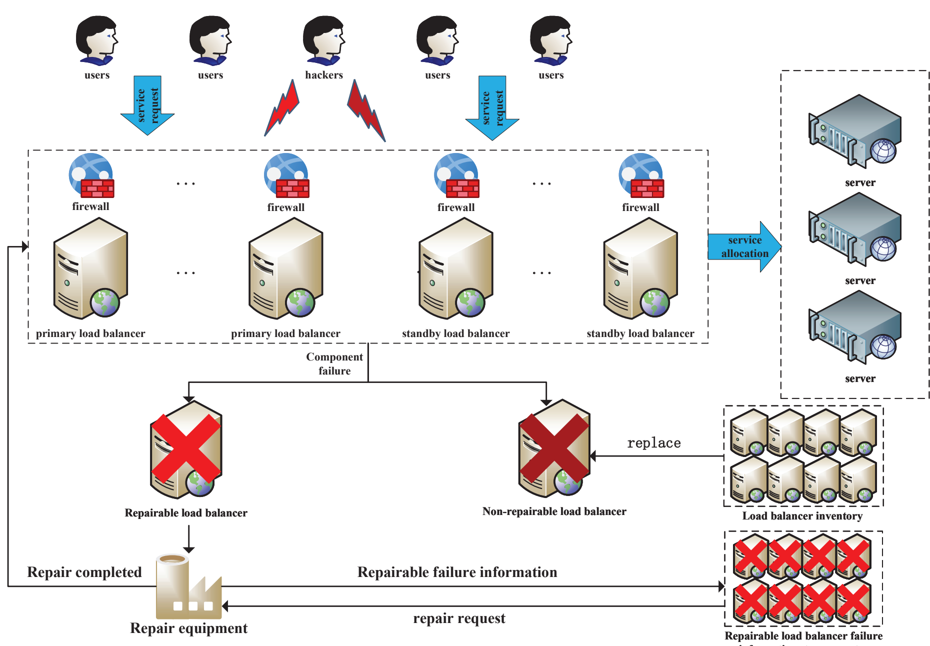


Fig. 1. Hardware load balancer group system diagram

2.1.2. Model description

Based on the modeling background, a k/n(G) retrial system model under the discrete-time condition based on different failure types and the δ-shock model is established. This model can be considered for the design and application of hardware load balancer groups. A k/n(G) system consists of n independent and identical components. When the number of main components (main components refer to the working components during system operation or the non-failed components when the system fails) in the system is less than k, the system will fail. When the repair equipment is repairing a failed repairable component, the remaining failed repairable components are stored in the retrial orbit. The retrial orbit is virtual rather than actual physical space. It is used to store failure signals or repair request signals of repairable failed components. Failed repairable components that have not yet been repaired are merely considered as being stored in the retrial orbit. The failed repairable components in the retrial orbit are independent of each other, and each component will send a repair request signal to the repair equipment after a period of time. If the repair equipment is idle at this time, it will start repairing a component that sends a repair request. Under the discrete-time condition, events are confined to occurring at discrete time points which means that component failure, retrial of the failed repairable component, and completion of the failed repairable component's repair may occur at the same time point. In addition, multiple main components may fail simultaneously. The types of component failures include both repairable and non-reparable. At time $t = 0$, the repair equipment is idle, and all of the components are brand new, a total of k components are operating, while the remaining $n-k$ are cold standby components (standby components won't fail before they start operating). The following model assumptions are further proposed in section 2.1.3.

2.1.3. Model assumption and operation diagram

(1) Assumption 1: The following priority order is established at the same time:

- **(i)** When a component fails and a repair completion of a failed component occurs simultaneously, the completion of the repair takes precedence.
- **(ii)** When a component fails and a retrial of a failed component occurs simultaneously, the failure of the component takes precedence.

(2) Assumption 2: While the system is operating, the main components may experience both internal and external failures. However, when the system fails, the main components only suffer from external failures.

(3) Assumption 3: When a component is non-repairable, it is instantly replaced by a brand new component from outside the system, ensuring that the number of components in the system remains at n. The replacement component is identical to the original component in the system, and there are enough components for replacing failed non-repairable components.

(4) Assumption 4: Cold standby components will immediately replace the failed repairable components. This replacement process continues until there are no standby components left or the system resumes operation.

(5) Assumption 5: The lifetime up to an internal failure of the main components, X, follows a geometric distribution with parameter θ, that is

$$P(X = t) = \theta(1-\theta)^{t-1}, \qquad t = 1, 2, \ldots.$$

An internal failure is a repairable failure with probability μ and a non-repairable failure with probability $1-\mu$.

(6) Assumption 6: The arrival interval of shock, Y, follows a discrete PH distribution PH$(\boldsymbol{\eta}, \boldsymbol{H})$ of order m, and $\boldsymbol{I} - \boldsymbol{H}$ is non-singular, where $\boldsymbol{I}$ represents an identity matrix of appropriate order. That is

$$P(Y = t) = \boldsymbol{\eta}\boldsymbol{H}^{t-1}\boldsymbol{H}^0, \qquad t = 1, 2, \ldots,$$

where $\boldsymbol{H}^0 = \boldsymbol{e} - \boldsymbol{H}\boldsymbol{e} = \begin{pmatrix} H_1^0 & \cdots & H_m^0 \end{pmatrix}_{m\times 1}^{\mathrm{T}}$, with $\boldsymbol{e}$ being a column vector of appropriate dimension with all elements equal to 1. $\boldsymbol{H} = (H_{v_1 v_2})(1 \leq v_1, v_2 \leq m)$ satisfies $H_{v_1 v_2} \geq 0$, and the one-step transition probability of transitioning from transient state v_1 to transient state v_2 is indicated by element $H_{v_1 v_2}$. $H_{v_1}^0(1 \leq v_1 \leq m)$ denotes the one-step transition probability of transitioning from transient state v_1 to the absorbing state (shock reaches fully).

When the interval between two consecutive shocks is less than δ, the component fails. If this interval is less than $\tilde{\delta}(\tilde{\delta} < \delta)$, the failure is non-repairable; otherwise, it is repairable.

(7) Assumption 7: There is just one repair equipment in the system, and the time, Z, for the repair equipment to repair a failed repairable component follows a geometric distribution with parameter α, that is

$$P(Z = t) = \alpha(1-\alpha)^{t-1}, \qquad t = 1, 2, \ldots.$$

Once a failed repairable component is successfully repaired, if the system is operating, the component will immediately enter the standby status; if the system is not operating, it will immediately become a main component waiting to work.

(8) Assumption 8: The retrial time, L, of the component in the retrial orbit follows a geometric distribution with parameter β, that is

$$P(L = t) = \beta(1-\beta)^{t-1}, \qquad t = 1, 2, \ldots.$$

(9) Assumption 9: All random variables in the system model are mutually independent.

Based on the model description in section 2.1.2 and model assumptions above, the system operation diagram is shown in Fig. 2.

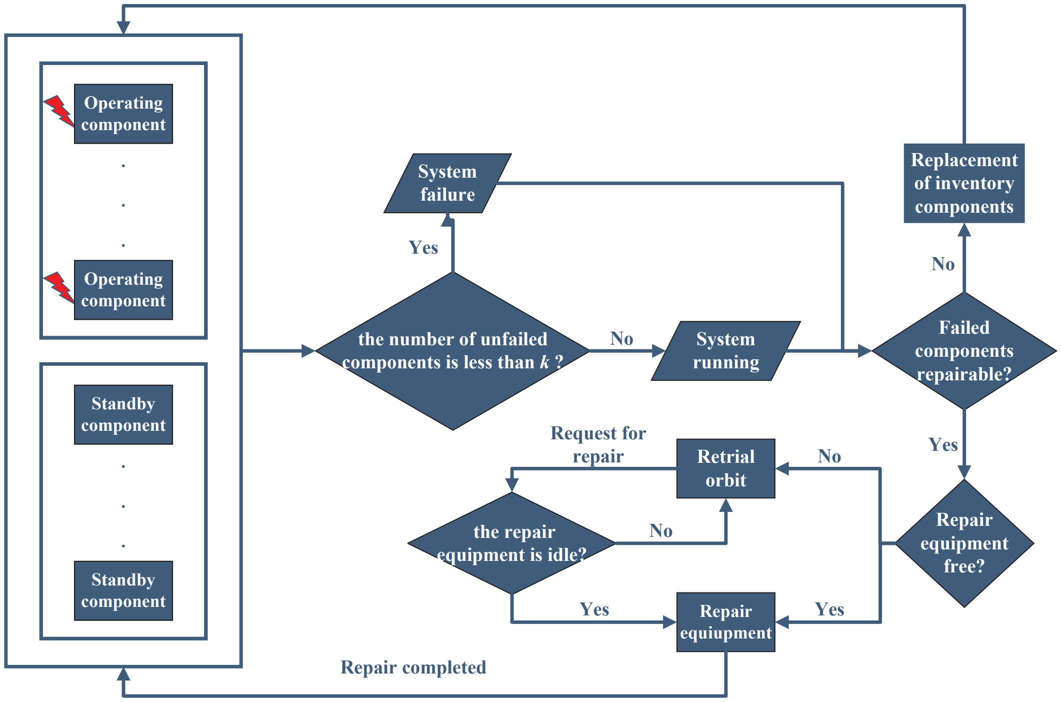


Fig. 2. Operation diagram of a k/n(G) retrial system based on different types of failures

2.2. Model analysis

2.2.1. δ-shock lifetime analysis

The lifetime distribution of a component affected only by the δ-shock is first shown (the lifetime is referred to as δ-shock lifetime). Since the shock can only occur at discrete time points, δ is replaced by $[\delta]+1$ when δ takes a non-integer value greater than 1 (in this case, the failure of the component is identical to the failure before). The value of δ is thus assumed to be an integer bigger than 1.

Instead of obtaining the δ-shock lifetime distribution with the method of probability generation function, a discrete PH distribution is used to express the δ-shock lifetime in order to make the construction of the one-step transition probability matrix of the system state easier. First, two new matrices are constructed for a fixed parameter δ, and they are referred to as the δ-shock lifetime generation matrices. Then the discrete PH distribution that the δ-shock lifetime follows is built using these two matrices. The δ-shock lifetime generation matrices are as follows:

$$\mathbf{\Delta}_\delta^{(1)} = \begin{pmatrix} 1\ 0\ 0\ \cdots\ 0 \\ \boldsymbol{I}_\delta \end{pmatrix}_{(\delta+1)\times\delta}, \qquad \mathbf{\Delta}_\delta^{(2)} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{(\delta+1)\times 1}, \tag{1}$$

where $\boldsymbol{I}_\delta$ denotes the identity matrix of order δ. Then the δ-shock lifetime of the component follows the discrete PH distribution $\mathrm{PH}(\boldsymbol{\xi}, \boldsymbol{\Delta}_\delta)$ of order $m(\delta+1)$, where

$$\boldsymbol{\xi} = \begin{pmatrix} \boldsymbol{\eta} & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \boldsymbol{\Delta}_\delta = \begin{pmatrix} \boldsymbol{\Delta}_\delta^{(1)} \otimes \boldsymbol{H} & \boldsymbol{\Delta}_\delta^{(2)} \otimes (\boldsymbol{H}^0 \boldsymbol{\eta}) \end{pmatrix}. \tag{2}$$

Next, the theorem about the δ-shock lifetime is presented as follows, and the proof is located in appendix A.

Theorem 1. In discrete time, for a component subjected only to external shocks, $X_v \sim \mathrm{PH}(\boldsymbol{\eta}, \boldsymbol{H})$, where $v = 0, 1, \ldots,$ and $\boldsymbol{I} - \boldsymbol{H}$ is non-singular. Here, $X_v (v \geq 1)$ denotes the inter-arrival time between the v-th and the $(v+1)$-th shocks, X_0 represents the duration from the initial time to the first shock of the component. The component fails when the interval between two consecutive shocks is less than δ. The δ-shock lifetime of the component is denoted by $T(\delta)$, then $T(\delta) \sim \mathrm{PH}(\boldsymbol{\xi}, \boldsymbol{\Delta}_\delta)$, where $\boldsymbol{\xi}$ and $\boldsymbol{\Delta}_\delta$ are given by formula (2).

For the discrete PH distribution $\mathrm{PH}(\boldsymbol{\xi}, \boldsymbol{\Delta}_\delta)$, $\boldsymbol{\Delta}_\delta = (\boldsymbol{\Delta}_{\delta, v_1 v_2})(1 \leq v_1, v_2 \leq m(\delta+1))$ satisfies $\boldsymbol{\Delta}_{\delta, v_1 v_2} \geq 0$, and one-step transition probability of transitioning from transient state v_1 to transient state v_2 is indicated by $\boldsymbol{\Delta}_{\delta, v_1 v_2}$. $\boldsymbol{\Delta}_\delta^0 = \begin{pmatrix} \Delta_{\delta,1}^0 & \cdots & \Delta_{\delta, m(\delta+1)}^0 \end{pmatrix}^{\mathrm{T}}$ is a column vector of dimension $m(\delta+1)$, $\Delta_{\delta,v}^0$ denotes the one-step transition probability of transitioning from transient state v $(1 \leq v \leq m(\delta+1))$ to the absorbing state (component's failure due to the δ-shock).

Remark 1. *As a transition probability matrix, $\boldsymbol{\Delta}_\delta$ can be understood as follows: Firstly, δ and $\tilde{\delta}$ are given, when the interval between two consecutive shocks is less than δ, the component fails. Moreover, if the interval between two consecutive shocks is less than $\tilde{\delta}$, the failure type of the component is non-repairable; otherwise, it is repairable. Next, an abstract concept called "virtual durability" that is related to both external shocks and the component is introduced. The "virtual durability" has the following characteristics: for a fixed parameter δ, the initial "virtual durability" of the component is $\delta+1$, and the maximum "virtual durability" is also $\delta+1$. Each time the component is subjected to a shock, the "virtual durability" of the component decreases by δ. Moreover, for every unit of time that passes, the "virtual durability" of the component increases by 1 (if a shock occurs at this time, the component's "virtual durability" recovery comes first, followed by the shock), until it increases to the maximum "virtual durability" of $\delta+1$. For a component subjected to δ-shock, when the "virtual durability" of the component decreases to 0 or less, the component fails. When the "virtual durability" of the component is lower than or equivalent to $\tilde{\delta} - \delta$, the failure type of the component is non-repairable; otherwise, it is repairable.*

For the matrix in (1), the matrix $\boldsymbol{\Delta}_\delta^{(1)}$ represents that the component does not experience any shock from the current time point to the next time point. During this period, the component's "virtual durability" is either in a recovery phase or maintained at the maximum "virtual durability" level. Conversely, the matrix $\boldsymbol{\Delta}_\delta^{(2)}$ indicates that the component will undergo a shock in the next moment, but this shock will not lead to component failure. Regarding (2), it considers both the different phases of the shock and the operational failure of the component simultaneously. The formula (2) encapsulates both the phase state of the shock and the state of the component. It employs the Kronecker product to integrate two elements (shock and component) of the δ-shock *model, which effectively represents the discrete PH distribution PH$(\boldsymbol{\xi}, \boldsymbol{\Delta}_\delta)$ that the component's lifetime follows under δ-shock.*

Let $\Delta^0_{\delta,v,\mathrm{nr}}$ denotes the one-step transition probability from transient state v of the discrete PH distribution, PH$(\boldsymbol{\xi}, \boldsymbol{\Delta}_\delta)$, to the absorbing state (non-repairable failure), $\Delta^0_{\delta,v,\mathrm{r}}$ denotes the one-step transition probability from transient state v of the discrete PH distribution, PH$(\boldsymbol{\xi}, \boldsymbol{\Delta}_\delta)$, to the absorbing state (repairable failure) for $1 \le v \le m(\delta+1)$. It is known from remark 1 that the following vectors for computing one-step transition probability can be constructed: $\boldsymbol{\Delta}^0_{\delta,\mathrm{nr}} = \begin{pmatrix} \Delta^0_{\delta,1,\mathrm{nr}} & \cdots & \Delta^0_{\delta,m(\delta+1),\mathrm{nr}} \end{pmatrix}^{\mathrm{T}}_{m(\delta+1)\times 1}$ is a column vector with dimension $m(\delta+1)$, and is composed of: the last $m(\tilde{\delta}-1)$ elements are identical to the last $m(\tilde{\delta}-1)$ elements of $\boldsymbol{\Delta}^0_\delta$, while the remaining elements are all 0. $\boldsymbol{\Delta}^0_{\delta,\mathrm{r}} = \boldsymbol{\Delta}^0_\delta - \boldsymbol{\Delta}^0_{\delta,\mathrm{nr}} = \begin{pmatrix} \Delta^0_{\delta,1,\mathrm{r}} & \cdots & \Delta^0_{\delta,m(\delta+1),\mathrm{r}} \end{pmatrix}^{\mathrm{T}}_{m(\delta+1)\times 1}$ is a column vector with dimension $m(\delta+1)$.

2.2.2. System state

Let $J(t)$ represent the state of the repair equipment at time t:

$$J(t) = \begin{cases} 1, & \text{the repair equipment is working at time } t, \\ 0, & \text{the repair equipment is idle at time } t. \end{cases}$$

$U(t) = u, u = 0, 1, 2, \ldots, n-1$, represents the number of components in the retrial orbit at time t, and $J(t) + U(t)$ represents the number of failed repairable components after all components that need to be replaced have been replaced at time t. The state space Ω of the system is as follows:

$$\Omega = \{(0,0), (0,1), \ldots, (0, n-1), (1,0), (1,1), \ldots, (1, n-1)\},$$

Ω can be denoted as $\Omega = E_0 \cup E_1 \cup \cdots \cup E_n$, where

$$E_0 = (0,0), \quad E_i = (0,i) \cup (1, i-1), 1 \le i \le n-1, \quad E_n = (1, n-1).$$

The working state space W and the fault state space F are as follows:

$$W = E_0 \cup E_1 \cup \cdots \cup E_{n-k},$$

$$F = E_{n-k+1} \cup \cdots \cup E_n.$$

Based on the δ-shock lifetime analysis, it can be deduced that when the main component does not fail, its δ-shock lifetime can be split into $m(\delta+1)$ states based on the discrete PH distribution PH$(\boldsymbol{\xi}, \boldsymbol{\Delta}_\delta)$. Therefore, each state (j, u) of the system is specifically constituted as follows:

$$(j,u) = \begin{cases} \{(j, u; l_1, l_2, \ldots, l_k), 1 \le l_1, l_2, \ldots, l_k \le m(\delta+1), j = 0, 1; u \ge 0\}, 0 \le j+u \le n-k, \\ \{(j, u; l_1, l_2, \ldots, l_{n-j-u}), 1 \le l_1, l_2, \ldots, l_{n-j-u} \le m(\delta+1), j = 0, 1; u \ge 0\}, \\ n-k < j+u < n, \end{cases}$$

the meaning of $(j, u; l_1, l_2, \ldots, l_k)$ and $(j, u; l_1, l_2, \ldots, l_{n-j-u})$ is as follows: j is the state of repair equipment, the number of failed repairable components in the retrial orbit is represented by u. The $\min\{k, n-j-u\}$ main components are numbered $1, \ldots, \min\{k, n-j-u\}$ respectively, $l_v (1 \le v \le \min\{k, n-j-u\})$ stands for the state of the vth main component's δ-shock lifetime. Each state in the state set (j, u) is sorted according to the lexicographic order, then $\{J(t), U(t), l_1, \cdots, l_{\min\{k,n-j-u\}}, t \ge 1\}$ constitutes the discrete-time Markov process of the system.

2.2.3. One-step transition probability analysis

Considering the composition of the system state set (j, u), the one-step transition probability matrix of the system can be represented by submatrices. In order to clearly express the matrices of transition probabilities between different state sets of the system, the matrix $\boldsymbol{B}(j; u; h_1; h_2)$ is defined in the appendix B. Similarly, $\tilde{\boldsymbol{B}}(n, k, u, h_1)$ is defined as follows:

$$\tilde{\boldsymbol{B}}(n, k, u, h_1) = \begin{cases} \boldsymbol{\xi}, & n-1-u-k < h_1, \\ 1, & n-1-u-k \ge h_1. \end{cases}$$

As mentioned in section 2.2.2, both E_i and (j, u) are the system state set. Therefore, the one-step transition probability matrix from E_i to $E_{\tilde{i}}$, which is denoted as $\boldsymbol{P}_{i,\tilde{i}}$, is composed of the transition probabilities between the system states it contains. Similarly, the one-step transition probability matrix from (j_1, u_1) to (j_2, u_2) is denoted as $\boldsymbol{P}_{(j_1,u_1),(j_2,u_2)}$. It is known from assumption 1 in section 2.1.3 that $\boldsymbol{P}_{(0,u_1),(0,u_2)} = \boldsymbol{0}$ $(u_1 \ne u_2)$, $\boldsymbol{P}_{(0,u_1),(1,u_2)} = \boldsymbol{0}$ $(1 + u_2 < u_1)$, and $\boldsymbol{P}_{(1,u_1),(0,u_2)} = \boldsymbol{0}$ $(1 + u_1 \le u_2)$. The remaining one-step transition probability analyses are as follows:

(1) $E_i \to E_{\tilde{i}}$ $(\tilde{i} < i - 1$ or $\tilde{i} > i + k)$

At the same time, at most one component is successfully repaired by the repair equipment, and at most k components fail at the same time point, then

$$\boldsymbol{P}_{i,\tilde{i}} = \boldsymbol{0}.$$

(2) $E_i \to E_{i-1}$

(i) $1 \le i \le n - k$

(a) $(1, i-1) \to (0, i-1)$

The repair equipment successfully restores a failed repairable component, no components in the retrial orbit retry, and no repairable failures occur in the main components, then

$$\boldsymbol{P}_{(1,i-1),(0,i-1)} = \alpha(1-\beta)^{i-1} \sum_{h_2=0}^{k} \boldsymbol{B}(1; i-1; 0; h_2).$$

(b) $(1, i-1) \to (1, i-2)$ $(2 \le i \le n - k)$

The repair equipment successfully restores a failed repairable component, at least one component in the retrial orbit retries and no repairable failures occur in the main components, then

$$\boldsymbol{P}_{(1,i-1),(1,i-2)} = \alpha(1-(1-\beta)^{i-1}) \sum_{h_2=0}^{k} \boldsymbol{B}(1; i-1; 0; h_2).$$

(ii) $i > n-k$

(a) $(1, i-1) \to (0, i-1)$

The repair equipment successfully restores a failed repairable component, no components in the retrial orbit retry, and there are no repairable failures in the main components, then

$$\boldsymbol{P}_{(1,i-1),(0,i-1)} = \alpha(1-\beta)^{i-1} \sum_{h_2=0}^{n-i} \boldsymbol{B}(1; i-1; 0; h_2) \otimes \boldsymbol{\xi}.$$

(b) $(1, i-1) \to (1, i-2)$

The repair equipment successfully restores a failed repairable component, at least one component in the retrial orbit retries and there are no repairable failures in the main components, then

$$\boldsymbol{P}_{(1,i-1),(1,i-2)} = \alpha(1-(1-\beta)^{i-1}) \sum_{h_2=0}^{n-i} \boldsymbol{B}(1; i-1; 0; h_2) \otimes \boldsymbol{\xi}.$$

(3) $E_i \to E_i$

(i) $i = 0$

(a) $(0, 0) \to (0, 0)$

There are no repairable failures in the main components, then

$$\boldsymbol{P}_{0,0} = \sum_{h_2=0}^{k} \boldsymbol{B}(0; 0; 0; h_2).$$

(ii) $1 \leq i \leq n-k$

(a) $(0, i) \to (0, i)$

No components in the retrial orbit retry, and there are no repairable failures in the main components, then

$$\boldsymbol{P}_{(0,i),(0,i)} = (1-\beta)^{i} \sum_{h_2=0}^{k} \boldsymbol{B}(0; i; 0; h_2).$$

(b) $(0, i) \to (1, i-1)$

At least one component in the retrial orbit retries and there are no repairable failures in the main components, then

$$\boldsymbol{P}_{(0,i),(1,i-1)} = (1-(1-\beta)^{i}) \sum_{h_2=0}^{k} \boldsymbol{B}(0; i; 0; h_2).$$

(c) $(1, i-1) \to (1, i-1)$

The repair equipment does not successfully restore the failed repairable components, and there are no repairable failures in the main components, retries may or may not occur; or the repair completes, only one of main components in the system experiences a repairable failure, retries may or may not occur, then

$$\boldsymbol{P}_{(1,i-1),(1,i-1)} = (1-\alpha)\sum_{h_2=0}^{k} \boldsymbol{B}(1;i-1;0;h_2) + \alpha \sum_{h_2=0}^{k-1} \boldsymbol{B}(1;i-1;1;h_2) \otimes \tilde{\boldsymbol{B}}(n,k,i-1,1).$$

(iii) $n-k < i < n$

(a) $(0,i) \to (0,i)$

No components in the retrial orbit retry, and there are no repairable failures in the main components, then

$$\boldsymbol{P}_{(0,i),(0,i)} = (1-\beta)^i \sum_{h_2=0}^{n-i} \boldsymbol{B}(0;i;0;h_2).$$

(b) $(0,i) \to (1,i-1)$

At least one component in the retrial orbit retries and there are no repairable failures in the main components, then

$$\boldsymbol{P}_{(0,i),(1,i-1)} = (1-(1-\beta)^i) \sum_{h_2=0}^{n-i} \boldsymbol{B}(0;i;0;h_2).$$

(c) $(1,i-1) \to (1,i-1)$

The repair equipment does not successfully restore the failed repairable components, there are no repairable failures in the main components, and retries may or may not occur; or the repair completes, only one of main components experiences a repairable failure and retries may or may not occur, then

$$\boldsymbol{P}_{(1,i-1),(1,i-1)} = (1-\alpha)\sum_{h_2=0}^{n-i} \boldsymbol{B}(1;i-1;0;h_2) + \alpha \sum_{h_2=0}^{n-i-1} \boldsymbol{B}(1;i-1;1;h_2) \otimes \boldsymbol{\xi}.$$

(iv) $i=n$

(a) $(1,i-1) \to (1,i-1)$

The repair equipment does not successfully restore the failed repairable components, and retries may or may not occur, then

$$P_{(1,i-1),(1,i-1)} = 1-\alpha.$$

(4) $E_i \to E_{i+q}$ $(1 \le q \le \min\{k, n-i\})$

(i) $i=0$

(a) $(0,0) \to (1,q-1)$

q main components experience repairable failures, then

$$\boldsymbol{P}_{(0,0),(1,q-1)} = \sum_{h_2=0}^{k-q} \boldsymbol{B}(0;0;q;h_2).$$

(ii) $1 \leq i \leq n-k$

(a) $(0,i) \rightarrow (1,i+q-1)$

q main components experience repairable failures, and retries may or may not occur, then

$$\boldsymbol{P}_{(0,i),(1,i+q-1)} = \sum_{h_2=0}^{k-q} \boldsymbol{B}(0;i;q;h_2).$$

(b) $(1,i-1) \rightarrow (1,i+q-1)$

The repair equipment does not successfully restore the failed repairable components, q main components experience repairable failures and retries may or may not occur; or the repair completes, $q+1$ main components experience repairable failures, and retries may or may not occur, then

$$\begin{aligned}\boldsymbol{P}_{(1,i-1),(1,i+q-1)} = (1-\alpha) &\sum_{h_2=0}^{k-q} \boldsymbol{B}(1;i-1;q;h_2) \\ &+ \alpha \sum_{h_2=0}^{k-q-1} \boldsymbol{B}(1;i-1;q+1;h_2) \otimes \tilde{\boldsymbol{B}}(n,k,i-1,q+1).\end{aligned}$$

(iii) $n-k < i \leq n-1$

(a) $(0,i) \rightarrow (1,i+q-1)$

q main components experience repairable failures, and retries may or may not occur, then

$$\boldsymbol{P}_{(0,i),(1,i+q-1)} = \sum_{h_2=0}^{n-i-q} \boldsymbol{B}(0;i;q;h_2).$$

(b) $(1,i-1) \rightarrow (1,i+q-1)$

The repair equipment does not successfully restore the failed repairable components, q main components experience repairable failures, and retries may or may not occur; or the repair completes, $q+1$ main components experience repairable failures, and retries may or may not occur, then

$$\begin{aligned}\boldsymbol{P}_{(1,i-1),(1,i+q-1)} = (1-\alpha) &\sum_{h_2=0}^{n-i-q} \boldsymbol{B}(1;i-1;q;h_2) \\ &+ \alpha \sum_{h_2=0}^{n-i-q-1} \boldsymbol{B}(1;i-1;q+1;h_2) \otimes \boldsymbol{\xi}.\end{aligned}$$

2.2.4. One-step transition probability matrix

To construct a one-step transition probability matrix of the system state, for each state E_i, they are arranged in the order of $(0,i) \rightarrow (1,i-1)$ (if both states exist), and the one-step transition probability matrix is expressed in the form of a block matrix as follows:

$$
\boldsymbol{P}_1 = \begin{array}{c} \\ E_0 \\ E_1 \\ E_2 \\ \vdots \\ E_n \end{array}
\begin{array}{c}
\begin{array}{ccccccccccc} E_0 & E_1 & \cdots & E_k & E_{k+1} & E_{k+2} & E_{k+3} & \cdots & E_{n-1} & E_n \end{array} \\
\begin{pmatrix}
\boldsymbol{P}_{0,0} & \boldsymbol{P}_{0,1} & \cdots & \boldsymbol{P}_{0,k} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0} \\
\boldsymbol{P}_{1,0} & \boldsymbol{P}_{1,1} & \cdots & \boldsymbol{P}_{1,k} & \boldsymbol{P}_{1,k+1} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0} \\
\boldsymbol{0} & \boldsymbol{P}_{2,1} & \cdots & \boldsymbol{P}_{2,k} & \boldsymbol{P}_{2,k+1} & \boldsymbol{P}_{2,k+2} & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{P}_{n,n-1} & P_{n,n}
\end{pmatrix}
\end{array},
$$

where

$$
\boldsymbol{P}_{0,0} = \begin{array}{cc} & (0,0) \\ (0,0) & \begin{pmatrix} \boldsymbol{P}_{(0,0),(0,0)} \end{pmatrix} \end{array},
$$

$$
\boldsymbol{P}_{0,\tilde{i}} = \begin{array}{cc} & \begin{array}{cc} (0,\tilde{i}) & (1,\tilde{i}-1) \end{array} \\ (0,0) & \begin{pmatrix} \boldsymbol{P}_{(0,0),(0,\tilde{i})} & \boldsymbol{P}_{(0,0),(1,\tilde{i}-1)} \end{pmatrix} \end{array}, \ \ 1 \le \tilde{i} \le n-1, \ \ \boldsymbol{P}_{0,n} = \boldsymbol{P}_{(0,0),(1,n-1)},
$$

$$
\boldsymbol{P}_{i,0} = \begin{array}{cc} & (0,0) \\ \begin{array}{c} (0,i) \\ (1,i-1) \end{array} & \begin{pmatrix} \boldsymbol{P}_{(0,i),(0,0)} \\ \boldsymbol{P}_{(1,i-1),(0,0)} \end{pmatrix} \end{array}, \ \ 1 \le i \le n-1, \ \ \boldsymbol{P}_{n,0} = \boldsymbol{P}_{(1,n-1),(0,0)},
$$

$$
\boldsymbol{P}_{i,\tilde{i}} = \begin{array}{cc} & \begin{array}{cc} (0,\tilde{i}) & (1,\tilde{i}-1) \end{array} \\ \begin{array}{c} (0,i) \\ (1,i-1) \end{array} & \begin{pmatrix} \boldsymbol{P}_{(0,i),(0,\tilde{i})} & \boldsymbol{P}_{(0,i),(1,\tilde{i}-1)} \\ \boldsymbol{P}_{(1,i-1),(0,\tilde{i})} & \boldsymbol{P}_{(1,i-1),(1,\tilde{i}-1)} \end{pmatrix} \end{array}, \ \ 1 \le i, \tilde{i} \le n-1,
$$

$$
\boldsymbol{P}_{i,n} = \begin{array}{cc} & (1,n-1) \\ \begin{array}{c} (0,i) \\ (1,i-1) \end{array} & \begin{pmatrix} \boldsymbol{P}_{(0,i),(1,n-1)} \\ \boldsymbol{P}_{(1,i-1),(1,n-1)} \end{pmatrix} \end{array}, \ \ 1 \le i \le n-1,
$$

$$
\boldsymbol{P}_{n,\tilde{i}} = \begin{array}{cc} & \begin{array}{cc} (0,\tilde{i}) & (1,\tilde{i}-1) \end{array} \\ (1,n-1) & \begin{pmatrix} \boldsymbol{P}_{(1,n-1),(0,\tilde{i})} & \boldsymbol{P}_{(1,n-1),(1,\tilde{i}-1)} \end{pmatrix} \end{array}, \ \ 1 \le \tilde{i} \le n-1,
$$

$$
P_{n,n} = \begin{array}{cc} & (1,n-1) \\ (1,n-1) & \begin{pmatrix} P_{(1,n-1),(1,n-1)} \end{pmatrix} \end{array}.
$$

In order to study the reliability of the system, the system's failure state space F is taken as the one-dimensional absorbing state $F^{'}$ and a new state space is defined as follows:

$$
\Omega^{'} = \{(0,0),(0,1),\ldots,(0,n-k),(1,0),(1,1),\ldots,(1,n-k-1),F^{'}\},
$$

$\Omega^{'}$ can be denoted as $\Omega^{'} = E_0 \cup E_1 \cup \cdots \cup E_{n-k} \cup F^{'}$, where

$$
E_0 = (0,0), \quad E_i = (0,i) \cup (1,i-1), \ 1 \le i \le n-k.
$$

In block matrix form, the system state's one-step transition probability matrix is stated as follows:

$$\boldsymbol{P}_2 = \begin{array}{c} \\ E_0 \\ \vdots \\ E_{n-k} \\ F^{'} \end{array} \begin{array}{c} \begin{array}{cccc} E_0 & \cdots & E_{n-k} & F^{'} \end{array} \\ \begin{pmatrix} \boldsymbol{P}_{0,0} & \cdots & \boldsymbol{P}_{0,n-k} & \sum\limits_{u=n-k+1}^{n} \boldsymbol{P}_{0,u}\boldsymbol{e} \\ \vdots & \vdots & \vdots & \vdots \\ \boldsymbol{P}_{n-k,0} & \cdots & \boldsymbol{P}_{n-k,n-k} & \sum\limits_{u=n-k+1}^{n} \boldsymbol{P}_{n-k,u}\boldsymbol{e} \\ \boldsymbol{0} & \cdots & \boldsymbol{0} & 1 \end{pmatrix} \end{array}.$$

3. Reliability measures

3.1. Reliability and mean time to first failure

3.1.1. Reliability $R(t)$

The reliability $R(t)$ of the system is considered first, the one-step transition probability matrix $\boldsymbol{P}_2$ is used for analysis, and the state probability vector of the system at time t is as follows:

$$\begin{aligned} \boldsymbol{Q}_{j,u}(t) &= P(J(t) = j, U(t) = u, (j,u) \in W) \\ &= (Q_{j,u}^{l_1,\cdots,l_k}(t))_{1\times(m(\delta+1))^k}, 1 \le l_1, \cdots, l_k \le m(\delta+1), \end{aligned}$$

where $Q_{j,u}^{l_1,\cdots,l_k}(t)$ denotes the probability of $J(t) = j,\ U(t) = u$ and the δ-shock lifetime states of k main components are $l_1, \ldots, l_k$ respectively. Let $\boldsymbol{Q}(t)$ denote the state probability vector of the system at time t $(t \ge 1)$ when the system's failure state is $F^{'}$, it is denoted as follows:

$$\boldsymbol{Q}(t) = (\boldsymbol{Q}_0(t), \cdots, \boldsymbol{Q}_{n-k}(t), Q_{F'}(t)),$$

where

$$\boldsymbol{Q}_0(t) = \boldsymbol{Q}_{0,0}(t),$$

$$\boldsymbol{Q}_i(t) = (\boldsymbol{Q}_{0,i}(t), \boldsymbol{Q}_{1,i-1}(t)), 1 \le i \le n-k,$$

$$Q_{F'}(t) = P\{\text{the system is in state } F^{'} \text{ at time } t\}.$$

The calculation method for $\boldsymbol{Q}(t)$ is as follows:

$$\boldsymbol{Q}(t) = \boldsymbol{Q}(0)\boldsymbol{P}_2^t,$$

for the system in which $F^{'}$ is the failure state, the initial state probability vector is represented by $\boldsymbol{Q}(0)$. Specifically as follows:

$$\boldsymbol{Q}(0) = (\underbrace{\boldsymbol{\xi} \otimes \cdots \otimes \boldsymbol{\xi}}_{\text{the number of } \boldsymbol{\xi} \text{ is } k}, \boldsymbol{0})_{1\times((2(n-k)+1)(m(\delta+1))^k+1)}.$$

The reliability of the system at time t can be calculated as follows:

$$R(t) = \boldsymbol{Q}(t)\boldsymbol{e} - Q_{F'}(t).$$

3.1.2. Mean time to first failure MTTFF

When the system is in operation, the one-step transition probability matrix appears just like:

$$\boldsymbol{P}_2^* = \begin{array}{c} \\ E_0 \\ \vdots \\ E_{n-k} \end{array} \begin{array}{c} \begin{array}{ccc} E_0 & \cdots & E_{n-k} \end{array} \\ \begin{pmatrix} \boldsymbol{P}_{0,0} & \cdots & \boldsymbol{P}_{0,n-k} \\ \vdots & \vdots & \vdots \\ \boldsymbol{P}_{n-k,0} & \cdots & \boldsymbol{P}_{n-k,n-k} \end{pmatrix} \end{array},$$

that is, $\boldsymbol{P}_2$ can be denoted as

$$\boldsymbol{P}_2 = \begin{pmatrix} \boldsymbol{P}_2^* & \boldsymbol{P}_2^{*,0} \\ \boldsymbol{0} & 1 \end{pmatrix}, \ \boldsymbol{P}_2^{*,0} = \begin{pmatrix} \sum\limits_{u=n-k+1}^{n} (\boldsymbol{P}_{0,u}\boldsymbol{e}) & \cdots & \sum\limits_{u=n-k+1}^{n} (\boldsymbol{P}_{n-k,u}\boldsymbol{e}) \end{pmatrix}^{\mathrm{T}}_{((2(n-k)+1)(m(\delta+1))^k)\times 1}.$$

It is known from the structure of $\boldsymbol{P}_2$ that $\boldsymbol{P}_2$ can be regarded as a transition probability matrix of a discrete PH distribution, $\boldsymbol{P}_2^* = (P_{2,v_1v_2}^*)$ $(1 \le v_1, v_2 \le (2(n-k)+1)(m(\delta+1))^k)$ satisfies $P_{2,v_1v_2}^* \ge 0$, the probability of switching from transient state v_1(one of the working state) to transient state v_2 (one of the working state) in a single step is indicated by element $P_{2,v_1v_2}^*$, $(1 \le v_1, v_2 \le (2(n-k)+1)(m(\delta+1))^k)$. $P_{2,v_1}^{*,0}$ denotes the probability of transitioning from transient state v_1 (one of the working state) to the absorbing state (state F') in one step $(1 \le v_1 \le (2(n-k)+1)(m(\delta+1))^k)$. MTTFF is the mean of the discrete PH distribution $\mathrm{PH}(\boldsymbol{Q}^*(0), \boldsymbol{P}_2^*)$, where

$$\boldsymbol{Q}^*(0) = (\underbrace{\boldsymbol{\xi} \otimes \cdots \otimes \boldsymbol{\xi}}_{\text{the number of } \boldsymbol{\xi} \text{ is } k}, \boldsymbol{0})_{1\times(2(n-k)+1)(m(\delta+1))^k}.$$

MTTFF can calculated as follows based on the relevant theory of discrete PH distribution:

$$\mathrm{MTTFF} = \boldsymbol{Q}^*(0)(\boldsymbol{I} - \boldsymbol{P}_2^*)^{-1}\boldsymbol{e}.$$

3.2. Transient availability and steady-state availability

This subsection studies the transient availability $A(t)$ and the steady-state availability $A(\infty)$ of the system, using the one-step transition probability matrix $\boldsymbol{P}_1$ to analyze these two measures.

3.2.1. Transient availability $A(t)$

The state probability vector of the system at time t is denoted as

$$\begin{aligned} \boldsymbol{\pi}_{j,u}(t) &= P(J(t) = j, U(t) = u, (j,u) \in \Omega) \\ &= (\pi_{j,u}^{l_1,\cdots,l_{\min\{k,n-j-u\}}}(t)), 1 \le l_1, \cdots, l_{\min\{k,n-j-u\}} \le m(\delta+1), \end{aligned}$$

where $\pi_{j,u}^{l_1,\cdots,l_{\min\{k,n-j-u\}}}(t)$ denotes the probability of $J(t) = j,\ U(t) = u$ and the δ-shock lifetime states of $\min\{k, n-j-u\}$ main components are $l_1, \ldots, l_{\min\{k,n-j-u\}}$ respectively. Let

$$\boldsymbol{\pi}(t) = (\boldsymbol{\pi}_0(t), \cdots, \boldsymbol{\pi}_{n-1}(t), \pi_n(t)),$$

where

$$\boldsymbol{\pi}_0(t) = \boldsymbol{\pi}_{0,0}(t),$$

$$\boldsymbol{\pi}_i(t) = (\boldsymbol{\pi}_{0,i}(t), \boldsymbol{\pi}_{1,i-1}(t))_{2\times(m(\delta+1))^{\min\{k,n-j-u\}}}, 1 \le i \le n-1,$$

$$\pi_n(t) = \pi_{1,n-1}(t),$$

$\boldsymbol{\pi}(t)$ is calculated as follows:

$$\boldsymbol{\pi}(t) = \boldsymbol{\pi}(0)\boldsymbol{P}_1^t,$$

where $\boldsymbol{\pi}(0)$ denotes the state probability vector of the system initially. Specifically as follows:

$$\boldsymbol{\pi}(0) = (\underbrace{\boldsymbol{\xi} \otimes \cdots \otimes \boldsymbol{\xi}}_{\text{the number of } \boldsymbol{\xi} \text{ is } k}, \mathbf{0}).$$

In order to calculate $A(t)$, matrix $\boldsymbol{R}_{(j,u)}$ is first given as follows:

$$\boldsymbol{R}_{(j,u)} = \left(\boldsymbol{R}_{(j,u)}^{(0,0)}, \boldsymbol{R}_{(j,u)}^{(0,1)}, \boldsymbol{R}_{(j,u)}^{(1,0)}, \cdots, \boldsymbol{R}_{(j,u)}^{(0,n-1)}, \boldsymbol{R}_{(j,u)}^{(1,n-2)}, \boldsymbol{R}_{(j,u)}^{(1,n-1)}\right), (j,u) \in \Omega,$$

where

$$\boldsymbol{R}_{(j,u)}^{(\tilde{j},\tilde{u})} = \begin{cases} \boldsymbol{I}_{(m(\delta+1))^{\min\{k,n-j-u\}}}, & \text{if}(\tilde{j},\tilde{u}) = (j,u), \\ \mathbf{0}_{(m(\delta+1))^{\min\{k,n-j-u\}} \times (m(\delta+1))^{\min\{k,n-\tilde{j}-\tilde{u}\}}}, & \text{if}(\tilde{j},\tilde{u}) \ne (j,u), \end{cases}$$

then $\boldsymbol{\pi}_{(j,u)}(t) = \boldsymbol{\pi}(t)\boldsymbol{R}_{(j,u)}^{\mathrm{T}}$, and $A(t)$ can be calculated as follows:

$$A(t) = \sum_{(j,u)\in W} \left(\boldsymbol{\pi}_{(j,u)}(t)\boldsymbol{e}\right) = \sum_{(j,u)\in W} \left(\boldsymbol{\pi}(t)\boldsymbol{R}_{(j,u)}^{\mathrm{T}}\boldsymbol{e}\right) = \sum_{(j,u)\in W} \left(\boldsymbol{\pi}(0)\boldsymbol{P}_1^t\boldsymbol{R}_{(j,u)}^{\mathrm{T}}\boldsymbol{e}\right).$$

3.2.2. Steady-state availability $A(\infty)$

The steady-state probability vector of the system is denoted as

$$\begin{aligned} \boldsymbol{\pi}_{j,u} &= \lim_{t\to\infty} \boldsymbol{\pi}_{j,u}(t) \\ &= \lim_{t\to\infty} \boldsymbol{P}(J(t) = j, U(t) = u, (j,u) \in \Omega) \\ &= (\pi_{j,u}^{l_1,\cdots,l_{\min\{k,n-j-u\}}}), 1 \le l_1, \cdots, l_{\min\{k,n-j-u\}} \le m(\delta+1), \end{aligned}$$

where $\pi_{j,u}^{l_1,\cdots,l_{min\{k,n-j-u\}}}$ denotes the probability that at time $t \to \infty$, $J(t) = j$, $U(t) = u$ and the δ-shock lifetime states of $\min\{k, n-j-u\}$ main components are $l_1, \ldots, l_{\min\{k,n-j-u\}}$ respectively. Let

$$\boldsymbol{\pi} = (\boldsymbol{\pi}_0, \cdots, \boldsymbol{\pi}_{n-1}, \pi_n),$$

where

$$\boldsymbol{\pi}_0 = \boldsymbol{\pi}_{0,0},$$

$$\boldsymbol{\pi}_i = (\boldsymbol{\pi}_{0,i}, \boldsymbol{\pi}_{1,i-1})_{2(m(\delta+1))^{\min\{k,n-j-u\}}\times 1}, 1 \le i \le n-1,$$

$$\pi_n = \pi_{1,n-1},$$

$\boldsymbol{\pi}$ satisfies the following equation set:

$$\begin{cases} \boldsymbol{\pi}\boldsymbol{P}_1 = \boldsymbol{\pi} \\ \boldsymbol{\pi}\boldsymbol{e} = 1 \end{cases}. \tag{3}$$

Based on the solution of equation set (3) and $\boldsymbol{\pi}_{(j,u)} = \boldsymbol{\pi}\boldsymbol{R}_{(j,u)}^{\mathrm{T}}$, $A(\infty)$ can be calculated as follows:

$$A(\infty) = \sum_{(j,u)\in W} \left(\boldsymbol{\pi}_{(j,u)}\boldsymbol{e}\right) = \sum_{(j,u)\in W} \left(\boldsymbol{\pi}\boldsymbol{R}_{(j,u)}^{\mathrm{T}}\boldsymbol{e}\right).$$

3.3. Conditional failure probability $M(t)$ and steady-state conditional failure probability M

The conditional failure probability $M(t)$ refers to the probability that the system does not fail at time $t-1$ but fails at time t, and the steady-state conditional failure probability M refers to the conditional failure probability that $t \to \infty$.

$M(t)$ can be calculated as follows:

$$\begin{aligned} M(t) &= \sum_{\substack{(j,u)\in \mathrm{W} \\ (\tilde{j},\tilde{u})\in \mathrm{F}}} \left(\boldsymbol{\pi}_{(j,u)}(t-1) \boldsymbol{P}_{(j,u),(\tilde{j},\tilde{u})} \boldsymbol{e} \right) \\ &= \sum_{\substack{(j,u)\in \mathrm{W} \\ (\tilde{j},\tilde{u})\in \mathrm{F}}} \left(\boldsymbol{\pi}(t-1) \boldsymbol{R}^{\mathrm{T}}_{(j,u)} \boldsymbol{P}_{(j,u),(\tilde{j},\tilde{u})} \boldsymbol{e} \right) \\ &= \sum_{\substack{(j,u)\in \mathrm{W} \\ (\tilde{j},\tilde{u})\in \mathrm{F}}} \left(\boldsymbol{\pi}(0) \boldsymbol{P}_1^{t-1} \boldsymbol{R}^{\mathrm{T}}_{(j,u)} \boldsymbol{P}_{(j,u).(\tilde{j},\tilde{u})} \boldsymbol{e} \right), \end{aligned}$$

Based on the solution of equation set (3), M can be calculated as follows:

$$\begin{aligned} M &= \sum_{\substack{(j,u)\in \mathrm{W} \\ (\tilde{j},\tilde{u})\in \mathrm{F}}} \left(\boldsymbol{\pi}_{(j,u)} \boldsymbol{P}_{(j,u),(\tilde{j},\tilde{u})} \boldsymbol{e} \right) \\ &= \sum_{\substack{(j,u)\in \mathrm{W} \\ (\tilde{j},\tilde{u})\in \mathrm{F}}} \left(\boldsymbol{\pi} \boldsymbol{R}^{\mathrm{T}}_{(j,u)} \boldsymbol{P}_{(j,u),(\tilde{j},\tilde{u})} \boldsymbol{e} \right). \end{aligned}$$

3.4. Transient working probability of repair equipment $P_{re}(t)$ and steady working probability of repair equipment P_{re}

The transient working probability of repair equipment $P_{\mathrm{re}}(t)$ refers to the probability that the repair equipment is repairing a component at time t, and the steady working probability of repair equipment P_{re} refers to the transient working probability of repair equipment that $t \to \infty$.

$P_{\mathrm{re}}(t)$ can be calculated as follows:

$$P_{\mathrm{re}}(t) = \sum_{u=0}^{n-1} (\boldsymbol{\pi}_{1,u}(t)\boldsymbol{e}) = \sum_{u=0}^{n-1} \left(\boldsymbol{\pi}(t) \boldsymbol{R}^{\mathrm{T}}_{(1,u)} \boldsymbol{e} \right) = \sum_{u=0}^{n-1} \left(\boldsymbol{\pi}(0) \boldsymbol{P}_1^{t} \boldsymbol{R}^{\mathrm{T}}_{(1,u)} \boldsymbol{e} \right).$$

Based on the solution of equation set (3), P_{re} can be calculated as follows:

$$P_{\mathrm{re}} = \sum_{u=0}^{n-1} (\boldsymbol{\pi}_{1,u}\boldsymbol{e}) = \sum_{u=0}^{n-1} \left(\boldsymbol{\pi} \boldsymbol{R}^{\mathrm{T}}_{(1,u)} \boldsymbol{e} \right).$$

3.5. Sensitivity analysis

Sensitivity analysis is done on some of the aforementioned reliability measures to find out how system parameters impact them. This section provides the calculation formulas for the sensitivity and relative sensitivity of the mean time to first failure MTTFF, the steady-state availability $A(\infty)$, the steady-state conditional failure probability M, and the steady working probability of repair equipment P_{re} to α, β, θ, μ.

The sensitivity and relative sensitivity of reliability measure y (MTTFF, $A(\infty)$, M, P_{re}) to parameter x (α, β, θ, μ) is caiculated as follows:

$$y_x = \frac{\partial y}{\partial x},\ y_x^* = \frac{\partial y}{\partial x} \times \frac{x}{y}.$$

3.6. Cost and cost-effectiveness ratio

The ratio of the total system cost per unit time, T_C, to the steady-state availability, $A(\infty)$, is known as the cost-effectiveness ratio (CER). In retrial system, an increase in $A(\infty)$ often implies an increase in T_C, and excessive costs can sometimes be unacceptable. Under the premise that $A(\infty)$ is not lower than a given threshold, the pursuit is generally for the lowest T_C or CER. This approach can reduce costs while achieving the expected functions of the system, thereby providing valuable reference suggestions for decision-making by facility managers. Choosing repair equipment with an appropriate repair rate and retrial orbit with a suitable retrial rate can help reduce T_C or CER. Therefore, the repair rate α and the retrial rate β are taken as decision variables of the T_C function or CER function, and the minimum T_C or CER is determined. Based on the following cost elements:

$$C_0 = \text{unit time cost for repair equipment when it is working,}$$

$$C_1 = \text{unit time cost for repair equipment when it is idle,}$$

$$C_2 = \text{unit time cost for retrial orbit,}$$

$$C_3 = \text{unit time loss when the system fails,}$$

$$C_4 = \text{unit time cost for maintenance when the system is operating,}$$

T_C function and CER function with respect to the parameter α and β are constructed as follows:

$$T_C(\alpha,\beta) = C_0 P_{\mathrm{re}} + C_1(1 - P_{\mathrm{re}}) + C_2 + C_3(1 - A(\infty)) + C_4 A(\infty),$$

$$CER(\alpha,\beta) = \frac{C_0 P_{\mathrm{re}} + C_1(1 - P_{\mathrm{re}}) + C_2 + C_3(1 - A(\infty)) + C_4 A(\infty)}{A(\infty)}.$$

4. Numerical examples and analysis

In this section, a hardware load balancer group that includes four hardware load balancers and one repair equipment is studied. As long as two or more hardware load balancers in the group can work normally, the group can achieve load balancing in the computer networks(that means the hardware load balancer group is a 2/4(G) system). Each hardware load balancer has a probability of $\theta = 0.15$ of failing due to internal wear at each time point when it is operating. When it fails due to internal original failure, the probability that the hard load balancer can be repaired by the repair equipment is $\mu = 0.4$. The time interval between the arrival of two consecutive external shocks follows a discrete PH distribution PH$(\boldsymbol{\eta}, \boldsymbol{H})$ of order 2, where $\boldsymbol{\eta} = (0.2, 0.8)$, $\boldsymbol{H} = \begin{pmatrix} 0.3 & 0.5 \\ 0.4 & 0.3 \end{pmatrix}$. When the time interval between the arrival of two consecutive external shocks is less than $\delta = 3$, the hardware load balancer will fail due to external shocks. If the time interval between the arrival of two consecutive shocks is less than $\tilde{\delta} = 2$, the hardware load balancer cannot be repaired by the repair equipment. When the repair equipment is idle, the repairable hardware load balancer will be repaired by the repair equipment immediately. Otherwise, the fault information of the failed repairable hardware load balancer will enter the failure information storage and call center, and at each discrete time point, the failure information storage and call center will submit a repair request to the repair equipment with a probability of $\beta = 0.6$. The time for the repair equipment to repair the repairable hardware load balancer follows a geometric distribution with parameter $\alpha = 0.4$. In the following numerical analysis, the "hardware load balancer group" in the aforementioned example is referred to as the "2/4(G) retrial system". Each "hardware load balancer" is termed a "component", the "failure information storage and call center" is denoted as the "retrial orbit", and the "failed repairable hardware load balancer's fault information" stored in the failure information storage and call center is referred to as "failed repairable components stored in the retrial orbit".

4.1. Reliability measures analysis

Partial reliability measures of the 2/4(G) retrial system are shown in table 1.

Table 1. partial reliability measures

MTTFF	$A(\infty)$	M	P_{re}
147.5851	0.9755	0.0089	0.3832

Next, relationships between reliability measures and parameters of the system are shown as follows:

(1) $R(t)$ over time under different parameters

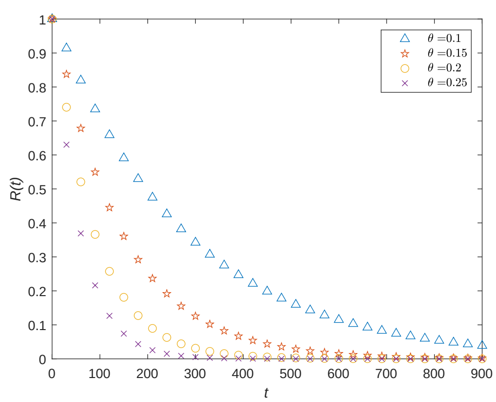


Fig. 3. $R(t)$ over time under different θ

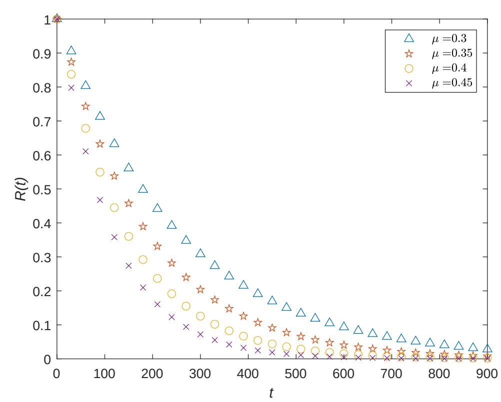


Fig. 4. $R(t)$ over time under different μ

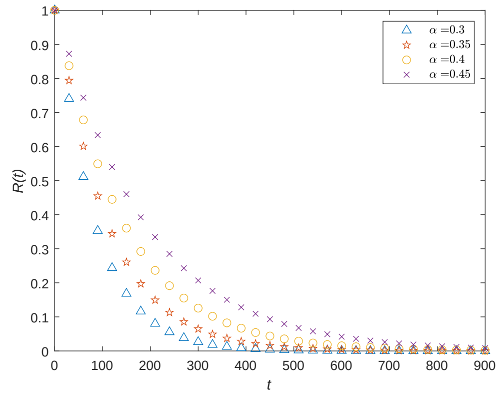


Fig. 5. $R(t)$ over time under different α

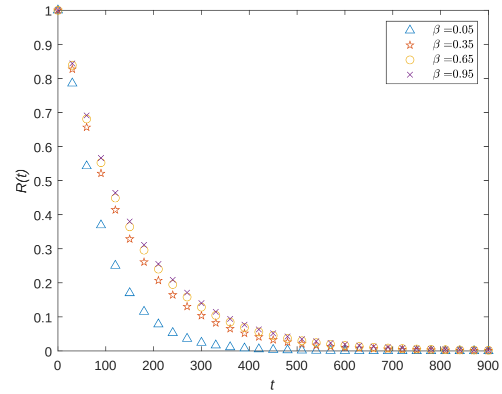


Fig. 6. $R(t)$ over time under different β

Figs 3 to 6 illustrate the effect of different parameters on $R(t)$, which are elaborated as follows: at the same time point, $R(t)$ decreases as θ and μ increase, and increases as α and β increase. $R(t)$ depends strongly on θ, moderately on μ and α, and weakly on β.

(2) $A(t)$ over time under different parameters

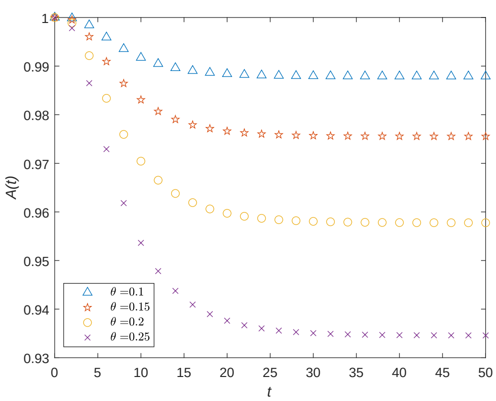


Fig. 7. $A(t)$ over time under different θ

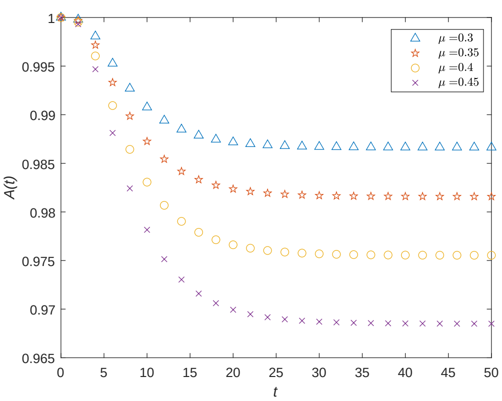


Fig. 8. $A(t)$ over time under different μ

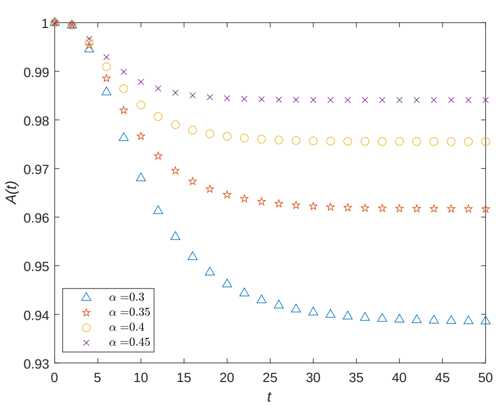


Fig. 9. $A(t)$ over time under different α

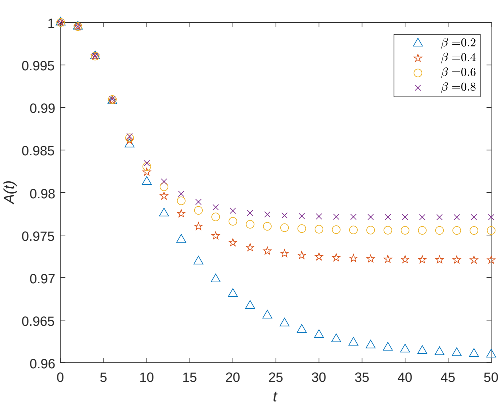


Fig. 10. $A(t)$ over time under different β

Figs 7 to 10 illustrate the effect of different parameters on $A(t)$, which are elaborated as follows: at the same time point, $A(t)$ decreases as θ and μ increase, and increases as α and β increase. $A(t)$ depends strongly on θ and α, moderately on μ, and weakly on β.

(3) $M(t)$ over time under different parameters

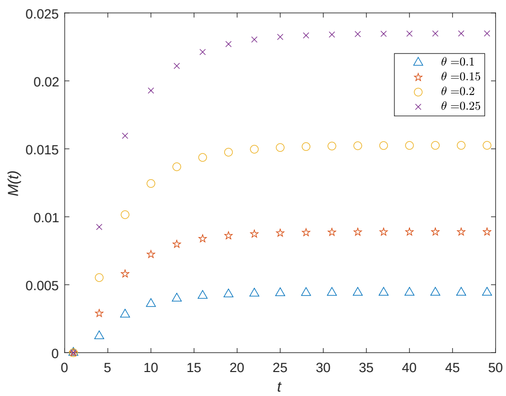


Fig. 11. $M(t)$ over time under different θ

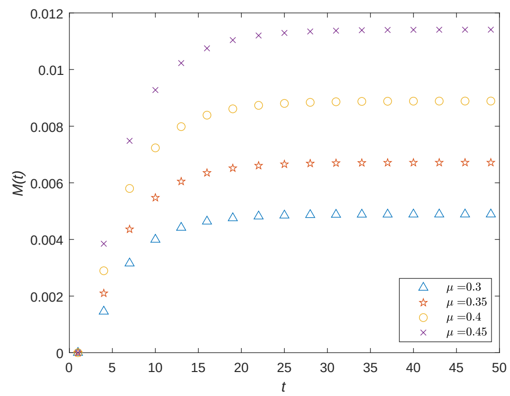


Fig. 12. $M(t)$ over time under different μ

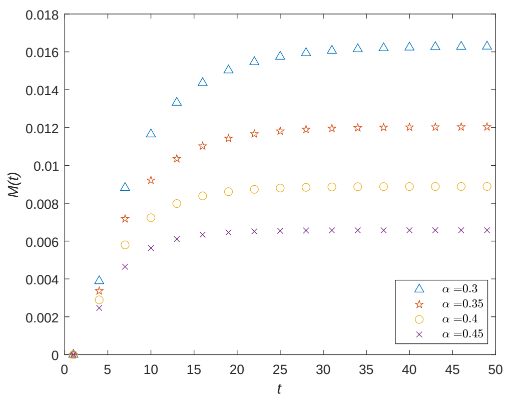


Fig. 13. $M(t)$ over time under different α

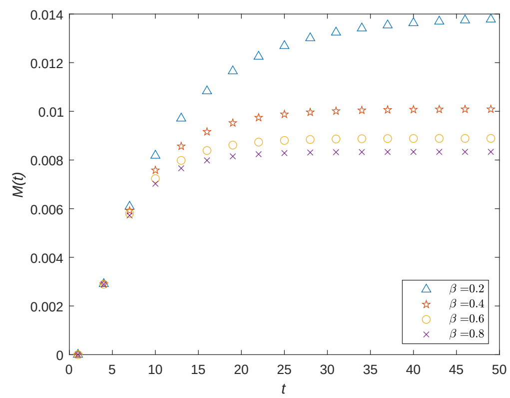


Fig. 14. $M(t)$ over time under different β

Figs 11 to 14 illustrate the effect of different parameters on $M(t)$, which are elaborated as follows: at the same time point, $M(t)$ increases as θ and μ increase, and decreases as α and β increase. $M(t)$ depends strongly on θ, moderately on μ and α, and weakly on β.

(4) $P_{\mathrm{re}}(t)$ over time under different parameters

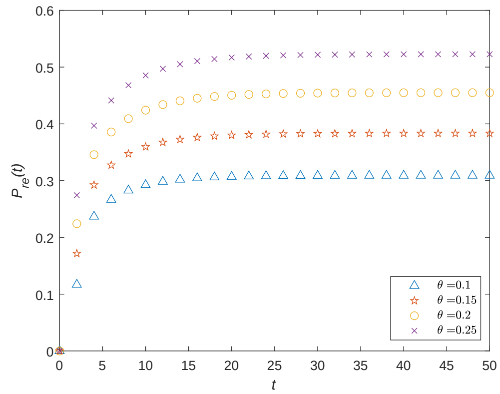


Fig. 15. $P_{re}(t)$ over time under different θ

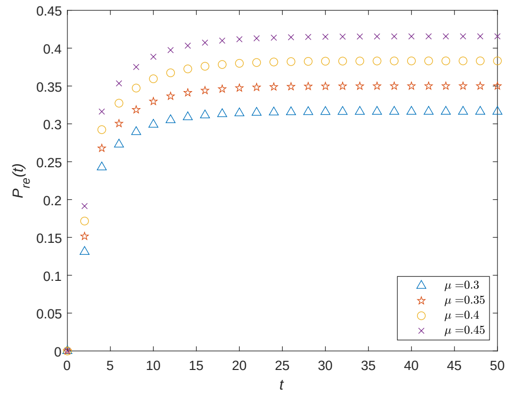


Fig. 16. $P_{re}(t)$ over time under different μ

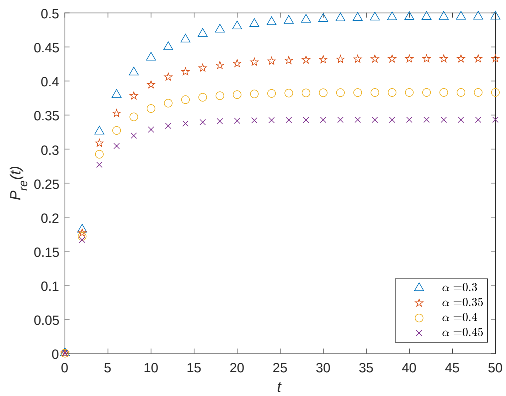


Fig. 17. $P_{re}(t)$ over time under different α

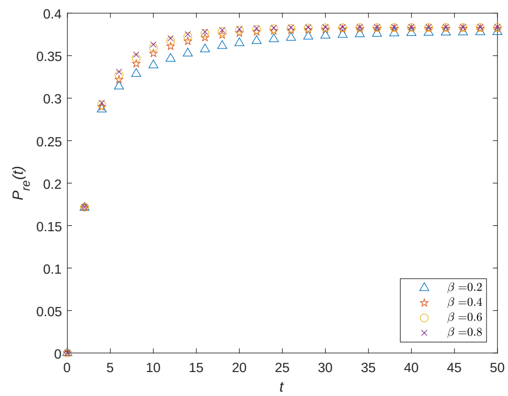


Fig. 18. $P_{re}(t)$ over time under different β

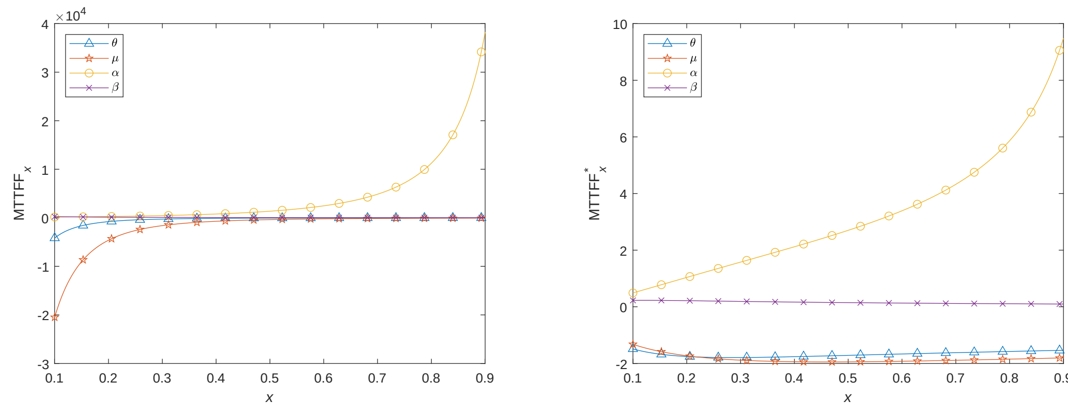
Figs 15 to 18 illustrate the effect of different parameters on $P_{\mathrm{re}}(t)$, which are elaborated as follows: at the same time point, $P_{\mathrm{re}}(t)$ increases as θ, μ and β increase, and decreases as α increases. That is to say, as the failure probability of the main component increases, or as the retrial probability of components in the retrial orbit increases, failed repairable components are more likely to be repaired by the repair equipment. Consequently, the transient working probability of the repair equipment increases. However, as the repair capability of the repair equipment strengthens, its working probability decreases. $P_{\mathrm{re}}(t)$ depends strongly on θ, moderately on μ and α, and weakly on β.

Based on the analysis of Figs 3 to 18, the following results can be obtained:

(1) For the aforementioned four reliability measures, the effect of θ is quite significant, while the effect of μ is average among the parameters studied. This implies that if facility managers want to increase $R(t)$, $A(t)$, or reduce $M(t)$, a more direct method is to replace components that are reliable in quality, wear-resistant, and easy to repair. However, this method may reduce the transient working probability of the repair equipment, resulting in a waste of repair resources. Therefore, while replacing components, it is advisable to consider optimizing the repair strategy for the repair equipment.

(2) For the aforementioned four reliability measures, the effect of β is relatively small. Therefore, pursuing a retrial orbit that retries more frequently to improve $R(t)$, $A(t)$, reduce $M(t)$, or increase $P_{\rm re}(t)$, that is, improve the repair resource utilization, are all inefficient ways. Therefore, when choosing the retrial orbit, the maintenance cost of the retrial orbit should be the main criterion.

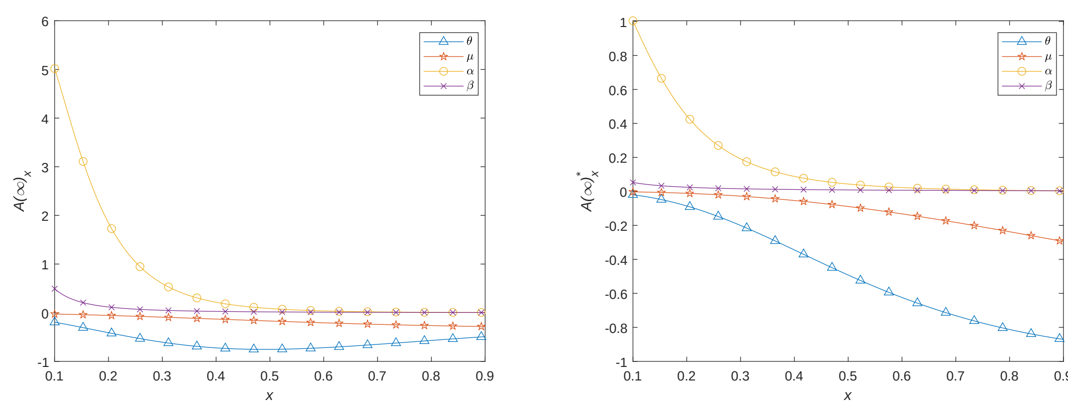
4.2. Sensitivity analysis



(a) (b)

Fig. 19. sensitivity and relative sensitivity of MTTFF to parameters θ,μ,α,β

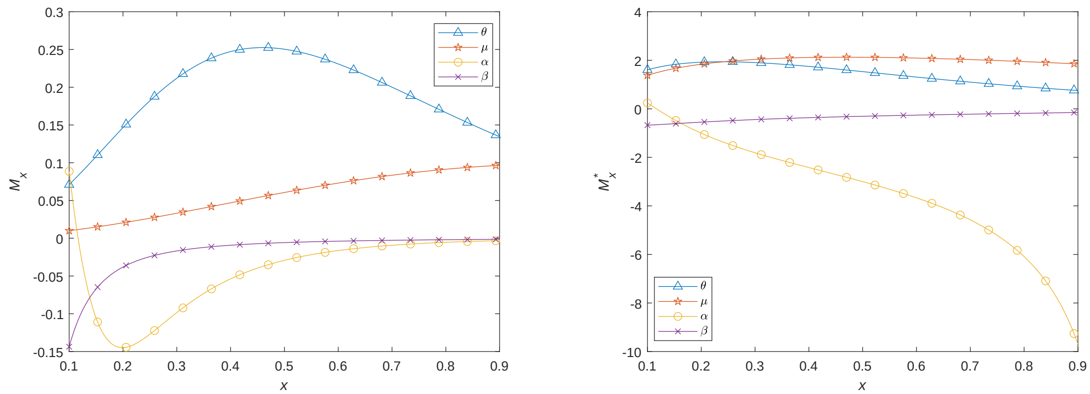
Fig 19 shows sensitivity and relative sensitivity of MTTFF to θ, μ, α and β, which are elaborated as follows: MTTFF is sensitive to μ and θ when their values are small. When the value of α exceeds a certain value, the sensitivity of MTTFF to it increases significantly. The relative sensitivity of MTTFF to θ, μ, and β remains basically unchanged, while the relative sensitivity of MTTFF to α increases with the increase of its value.



(a) (b)

Fig. 20. sensitivity and relative sensitivity of $A(\infty)$ to parameters θ,μ,α,β

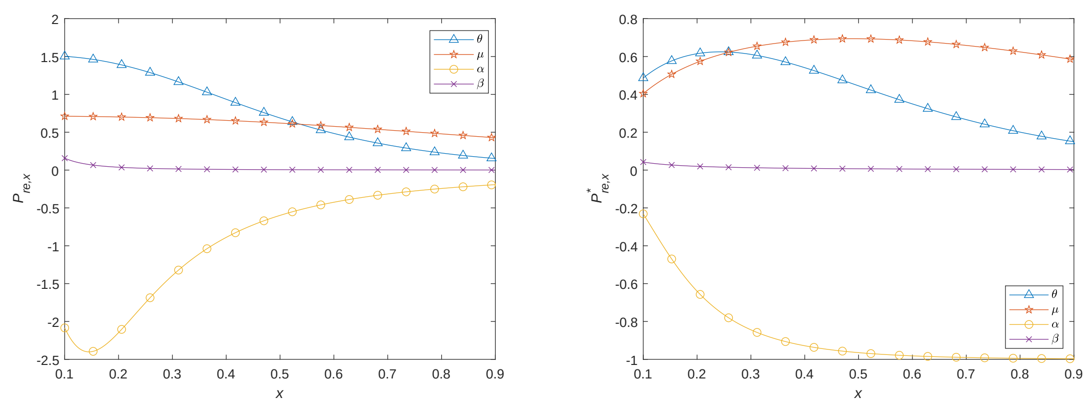
Fig 20 shows the sensitivity and relative sensitivity of $A(\infty)$ to θ, μ, α and β, which are elaborated as follows: when the value of α is small, $A(\infty)$ is sensitive to α. The relative sensitivity of $A(\infty)$ to β is essentially constant, and relative sensitivity to α decreases as α increases, to μ, θ increase as μ, θ increase.



(a) (b)

Fig. 21. sensitivity and relative sensitivity of M to parameters θ,μ,α,β

Fig 21 shows the sensitivity and relative sensitivity of M to θ, μ, α and β, which are elaborated as follows: as α increases, the sensitivity of M to α first decreases, then increases, and finally decreases again. As μ increases, the sensitivity of M to μ increases, while as β increases, the sensitivity of M to β decreases. As θ increases, the sensitivity of M to θ first increases, then decreases. The relative sensitivity of M to θ, μ, β is essentially constant, and to α first decreases, then increases as α increases.



(a) (b)

Fig. 22. sensitivity and relative sensitivity of P_{re} to parameters θ,μ,α,β

Fig 22 shows the sensitivity and relative sensitivity of P_{re} to θ, μ, α and β, which are elaborated as follows: the sensitivity of P_{re} to μ and β is essentially constant, to θ decreases as θ increases, and sensitivity to α first increases, then decreases as α increases. The relative sensitivity of P_{re} to α increases as α increases, the relative sensitivity of P_{re} to θ and μ first increase, then decrease as θ and μ increase, the relative sensitivity of P_{re} to β is essentially constant.

Based on the analyses of Figs 19 to 22, the following results can be obtained:

(1) When facility managers require increasing MTTFF of the system, selecting repair equipment with higher efficiency and level is more efficient than selecting better-quality components or a retrial orbit that retries more frequently.

(2) When facility managers require improving $A(\infty)$, replacing the repair equipment with lower efficiency and level with the ones with higher efficiency and level is an effective method; if the component quality is poor, replacing the components with better quality is also an effective method.

(3) When facility managers require reducing M, it should be noted that if the repair equipment's efficiency and level are low, a slight increase in the repair equipment's efficiency and level may result in an increase in M, which contradicts facility managers' requirement. In this case, a substantial increase in the repair equipment's efficiency and level is a more appropriate method.

(4) When facility managers require improving P_{re}, that is, to increase the utilization rate of the repair resources, the system availability, that is, the stable operation, should not be ignored. Therefore, selecting repair equipment with low efficiency and level is not a reasonable method, while selecting components with suitable quality (appropriately increasing θ, μ) can meet facility managers' demand and also reduce the procurement cost of the components.

4.3. Cost optimization and cost-effectiveness ratio optimization

Based on subsection 3.6, it is assumed that $C_0 = 800\alpha^3$, $C_1 = 200\alpha^4$, $C_2 = 100\beta^2$, $C_3 = 1500$, $C_4 = 85$, θ, μ, $\boldsymbol{\eta}$, and $\boldsymbol{H}$ are all constant (equal to the initial value given in section 4), and there exist repair equipments with varying repair rates ($0.2 \leq \alpha \leq 0.8$) and retrial orbits with varying retrial rates ($0.1 \leq \beta \leq 0.9$) to select. Figs 23 and 24 demonstrate T_C and CER as a function of α and β, and it is evident from these figures that optimal values α^* and β^* exist, which minimize the T_C function or CER function.

As can be seen from Figs 23 and 24, when α and β are relatively small (the repair efficiency and level are low and retries are sparse), both T_C and CER are quite sensitive to α and β. If facility managers want to effectively reduce the system's T_C or CER at this time, it is more effective to increase parameter α (choose a repair equipment with higher repair efficiency and level) than to increase parameter β (choose a retrial orbit with more frequent retries). This means that for facility managers, if they want to pursue a lower T_C or CER, they should abandon repair equipment with too low repair efficiency and level.

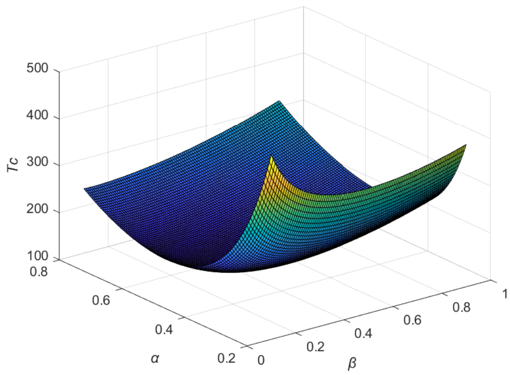


Fig. 23. Effect of different α, β on T_c

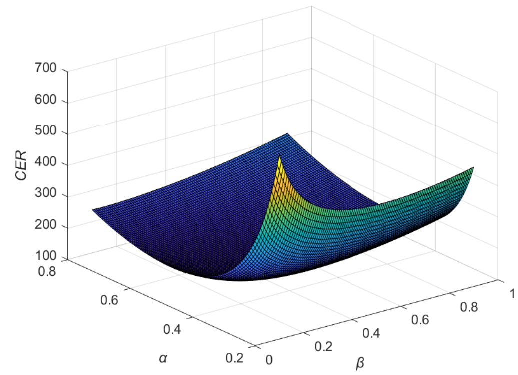


Fig. 24. Effect of different α, β on CER

Next, on the basis of requiring $A(\infty)$ to be not lower than 0.98, T_C and CER are optimized by varying α and β. PSO is applied to obtain the following results: under the premise of satisfying the given conditions, when $\alpha^* = 0.48408,\ \beta^* = 0.26175$, T_c reaches the minimum value 152.5038, in this case, $A(\infty) = 0.9829$; when $\alpha^* = 0.492,\ \beta^* = 0.26593$, CER reaches the minimum value 155.0678, in this case, $A(\infty) = 0.9840$. The results indicate that the optimal parameter values vary when the optimization objectives are different. Moreover, when the optimization objective reaches its optimal value, $A(\infty)$ may not necessarily be exactly its minimum required $A(\infty)$. This implies that a lower $A(\infty)$ does not necessarily result in a lower T_C or CER for the system.

5. Monte Carlo simulation

In this research, the order of $\mathbf{\Delta}_\delta$ and the one-step transition probability matrix of the system state are both large. Moreover, they increase significantly with the increase of n, k and δ. The excessive order of the matrix makes it difficult to solve the system state probability using the proposed analytical method. To compensate for the shortcomings of the analytical method, this section uses the Monte Carlo simulation method to solve the δ-shock lifetime of a component and some reliability measures of the system.

5.1. Simulation of δ-shock lifetime

In this subsection, the δ-shock lifetime is simulated by means of Monte Carlo simulation for different δ, and we will compare it with the δ-shock lifetime calculated by the PH distribution $\mathrm{PH}(\boldsymbol{\xi}, \mathbf{\Delta}_\delta)$. The simulation Algorithm 1 of the δ-shock lifetime is as follows:

Algorithm 1 Simulation of the δ-shock lifetime

Input: parameter: δ, $\boldsymbol{\eta}$, $\boldsymbol{H}$ and simulationnumbers.

1: Initialize unitlife as an empty array
2: **for** i = 1 to simulationnumbers **do**
3: Initialize time(t), time of shock arrival(shocktime1,shocktime2)
4: Generate a random number and determine the shock interval state(statelast) using $\boldsymbol{\eta}$
5: **while** shocktime2 - shocktime1 >= δ or shocktime1 == 0 **do**

t = t + 1
if the shock has not occurred **then**
Generate a random number and update statelast using $\boldsymbol{H}$
else
shocktime1 = shocktime2, shocktime2 = t, generate a random number and update statelast using $\boldsymbol{\eta}$
end if
end while
end for
Calculate the simulated value of the component's δ-shock lifetime

---

For different δ, mutiple simulations are conducted, and the average of the results of these simulations is taken as the δ-shock lifetime of the component simulated by Monte Carlo simulation method (denoted as simulated lifetime). Next, the lifetime is calculated according to the PH distribution $\mathrm{PH}(\boldsymbol{\xi}, \boldsymbol{\Delta}_\delta)$ (denoted as $\boldsymbol{\Delta}_\delta$-lifetime), and the $\boldsymbol{\Delta}_\delta$-lifetime is compared with the simulated lifetime. The absolute error = $\boldsymbol{\Delta}_\delta$-lifetime − simulated lifetime, and the relative error = absolute error / $\boldsymbol{\Delta}_\delta$-lifetime. The results are shown in Table 2 (the values of the remaining parameters refer to the numerical example in section 4). As can be seen from Table 2, with the increase of δ, the δ-shock lifetime shows a downward trend. This is because a larger δ implies that the component has a weaker shock resistance capability.

Table 2. Simulation of δ-shock lifetime

δ	$\boldsymbol{\Delta}_\delta$-lifetime	simulated lifetime	absolute error	relative error
3	12.29453	12.27667	0.01786	0.1453%
8	8.281387	8.26582	0.015567	0.1880%
13	7.841003	7.84999	-0.008987	-0.1146%
18	7.750742	7.75937	-0.008628	-0.1113%
23	7.730302	7.72264	0.007662	0.0991%
28	7.725572	7.71516	0.010412	0.1348%

5.2. Simulation of reliability measures

This section uses the Monte Carlo simulation method to simulate the reliability measures of the system model under different n, k, α, μ and compares them with the reliability measures calculated using the proposed analytical method. In addition, this section introduces a new measure that is difficult to calculate with the analytical method: the average number of components replaced due to non-repairable failures per unit time (denoted as N_{rpnf}). In the following simulation algorithm, the elements in the array "systemunitstate" can only take $1, \ldots, m+1$, where $1, \ldots, m$ represent the m transient states in assumption 6, and $m+1$ represents the absorbing state in assumption 6. In addition, −1 indicates that the component is being repaired by the repair equipment, −2 indicates that the component is in the retrial orbit, and 0 indicates that the component is in the standby state. The simulation Algorithm 2 for $A(\infty)$ and N_{rpnf} are as follows:

Algorithm 2 Simulation of $A(\infty)$ and N_{rpnf}

Input: parameters n, k, δ, $\tilde{\delta}$, $\boldsymbol{\eta}$, $\boldsymbol{H}$, θ, μ, α, β and number of simulations (simulationnumbers), simulation duration (simulationtime)

Initialize non-repairable replacement array (replacematrix), availability array (availmatrix)
for i = 1 to simulationnumbers **do**
  Initialize shock arrival time array (shocktime), unrepairable replacement number (replacenumber), system state (systemstate), determine the initial component state array (systemunitstate)
  for t= 1 to simulationtime **do**
    Determine whether the component under repair has been repaired. If it has been repaired, update its state to 0 and determine the main component array(workingunitposition)
    for i=1 to length(workingunitposition) **do**
      if the ith main component is non-repairable **then**
        update systemunitstate and shocktime, replacenumber=replacenumber+1
      else if the ith main component is repairable **then**
        Update the state of the component to −1 (if the repair equipment is idle) or −2, and update shocktime
      else
        if the ith main component suffers an external shock **then**
          Generate a random number and update systemunitstate and shocktime based on $\boldsymbol{\eta}$
        else
          Generate a random number and update systemunitstate based on $\boldsymbol{H}$
        end if
      end if
    end for
    if there are components in the retrial orbit and the repair equipment is idle **then**
      update the state of a retry component to −1
    end if
    update the state of the standby component
    if the number of main components is k **then**
      systemstate= systemstate+1
    end if
  end for
end for
Calculate the simulated value of $A(\infty)$ and N_{rpnf}

For different n, k, α, μ, $A(\infty)$ is simulated for multiple times, and the average of these simulation results is taken as the Monte Carlo simulated $A(\infty)$ (denoted as simulated $A(\infty)$). Then, $A(\infty)$ is calculated using the proposed analytical method (denoted as $\boldsymbol{\Delta}_{\delta}$-$A(\infty)$), and $\boldsymbol{\Delta}_{\delta}$-$A(\infty)$ is compared with the simulated $A(\infty)$. The absolute error = $\boldsymbol{\Delta}_{\delta}$-$A(\infty)$ − simulated $A(\infty)$, and the relative error = the absolute error / $\boldsymbol{\Delta}_{\delta}$-$A(\infty)$. In addition, for different n, k, N_{rpnf} is simulated for multiple times, and the average of these simulation results is denoted as simulated N_{rpnf}. The calculation and simulation results are shown in Tables 3 to 6 (the values of the remaining parameters refer to the numerical example in section 4).

Table 3. Simulation of $A(\infty)$ for different n, k

reliability measures	n	$k=2$			
		$\boldsymbol{\Delta}_{\delta}$-$A(\infty)$	simulated $A(\infty)$	absolute error	relative error
$A(\infty)$	4	0.9755373	0.973497	0.00204	0.2091%
	6	0.9979826	0.99774	0.00024	0.0243%
	8	0.999834	0.999784	5E-05	5E-03%
	10	0.9999864	0.999985	1.4E-06	1.4E-04%
	12	0.9999989	0.999999	-1.0E-06	-1.0-04%
reliability measures	n	$k=3$			
		$\boldsymbol{\Delta}_{\delta}$-$A(\infty)$	simulated $A(\infty)$	absolute error	relative error
$A(\infty)$	4	0.8168536	0.809077	0.0077766	0.952%
	6	0.9599354	0.955931	0.0040044	0.417%
	8	0.9906087	0.98896	0.0016487	0.166%
	10	0.9977582	0.997323	0.0004352	0.044%
	12	0.9994623	0.999306	0.0001563	0.016%

Table 4. Simulation of $A(\infty)$ for different α

reliability measures	α	$\boldsymbol{\Delta}_{\delta}$-$A(\infty)$	simulated $A(\infty)$	absolute error	relative error
$A(\infty)$	0.3	0.938438	0.933815	0.004623	0.4926%
	0.4	0.975537	0.97337	0.002168	0.2222%
	0.5	0.989487	0.988494	9.93E-04	0.1004%
	0.6	0.995282	0.994827	4.55E-04	0.0457%
	0.7	0.997872	0.997667	2.05E-04	0.0206%

Table 5. Simulation of $A(\infty)$ for different μ

reliability measures	μ	$\boldsymbol{\Delta}_\delta$-$A(\infty)$	simulated $A(\infty)$	absolute error	relative error
$A(\infty)$	0.2	0.993959	0.993517	0.000442	0.044%
	0.3	0.986636	0.985624	0.001012	0.103%
	0.4	0.975537	0.973778	0.001759	0.180%
	0.5	0.96046	0.956875	0.003585	0.373%
	0.6	0.941524	0.936509	0.005015	0.533%

Table 6. Simulated N_{rpnf} for different n, k

k	$n=7$	$n=14$	$n=21$	$n=28$	$n=35$	$n=42$
4	0.467342	0.510232	0.514438	0.514904	0.515025	0.515033
8		0.633312	0.639710	0.641792	0.644053	0.646161
12		0.616946	0.636698	0.638764	0.641189	0.643352
16			0.633230	0.635668	0.638047	0.640310
20			0.629422	0.634205	0.636620	0.638753
24				0.633922	0.636246	0.638429

As shown in Table 6, when k remains constant, as n increases, N_{rpnf} also shows a slow upward trend. However, when n remains constant, as k increases, N_{rpnf} does not completely show a gradual upward trend. This is because as k increases, although the number of components working at the same time is increasing, and thus the number of components that may fail irreparably is also increasing, but the failure probability of the system is also increasing. When the system fails, the probability of the main components failing irreparably is smaller than the situation when the system is operating, so it is difficult to determine the trend of N_{rpnf} based solely on the change in k. This finding provides an important reference for further understanding and optimizing the system. Since it is difficult to have an infinite number of components for replacement in production practice, the Monte Carlo simulation method is used to first estimate N_{rpnf}, and then make purchases according to the production procurement cycle combined with the simulation results, so it can be assumed that there are sufficient components available for replacing the failed non-reparable components.

5.3. Comparison between analytical methods and simulation methods

In comparison with the Monte Carlo simulation method proposed above for the constructed discrete-time retrial system shock model, the analytical method provided in this paper has the following advantages and disadvantages:

(1) The analytical method provides a discrete PH distribution representation of component lifetime under δ-shock. The PH distribution can not only be applied to the model analysis in this paper, but also brings new insights to the analysis of other systems involving δ-shock model.

(2) The analytical method provides a one-step transition probability matrix of the system state. Based on the one-step transition probability matrix, the exact solutions for most reliability measures of the system can be calculated.

(3) The order of the one-step transition probability matrix given by the analytical method is large, which brings difficulties to our calculation.

(4) The analytical method can only determine the current state of a component, and cannot determine whether the component has undergone replacement by brand new components from outside the system. Therefore, it encounters difficulties when calculating certain system measures (such as N_{rpnf}).

6. Conclusion

In actual production processes, due to constraints of objective conditions such as measurement methods and observation costs, facility managers can only obtain system state information at specific moments. Therefore, the evaluation of system reliability under the discrete-time condition has important practical significance. This study introduces two failure types of components, repairable and non-reparable, into the k/n(G) retrial system under the discrete-time condition, and also includes the δ-shock into this system to study the reliability measures of this system. It is assumed that the repair time, retrial time, and duration before the main component experiences internal failure all follow the geometric distribution, while the interval between two consecutive shocks follows a discrete PH distribution. Firstly, two matrices named δ-shock lifetime generation matrices were constructed in the paper, and based on these two matrices, a discrete PH distribution representation method for the δ-shock lifetime was derived. Secondly, based on the discrete PH distribution that the component's δ-shock lifetime follows, the one-step transition probability matrix of the system state was given, and the solutions of multiple reliability measures were given. Then, using numerical examples, relationships between different reliability measures and various parameters were demonstrated, and optimization results for different objectives were given. Finally, the Monte Carlo simulation method was used to simulate the δ-shock lifetime and different reliability measures, pointing out the advantages and disadvantages of the analytical method and the Monte Carlo simulation method.

The discrete-time retrial system model studied in this paper still has certain limitations. In order to further improve and perfect this model, it can be expanded and deepened from the following directions:

(1) Treating the number of inventory components used to replace non-repairable failed components as a fixed value can be considered.

(2) The reliability evaluation of system models under mixed shocks from various types of shocks can be considered.

(3) Multi-decision variable optimization problems based on reliability measure control can be considered.

(4) Reformulating the corresponding times in assumptions 5, 7 and 8 using discrete PH distributions can make this research more generalized.

CRediT authorship contribution statement

Zebin Hu: Methodology, Software, Writing - original draft , Visualization. **Linmin Hu**: Conceptualization, Writing – review & editing, Funding acquisition. **Shaomin Wu**: Writing – review & editing, Validation. **Xiaoyun Yu**: Writing – review & editing.

Declaration of competing interest

The authors state that none of the work described in this paper have been influenced by any known conflicting financial interests or personal ties.

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Data availability

No data was used for the research described in the article.

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Appendix A

Before proving that the δ-shock lifetime follows the discrete PH distribution PH$(\boldsymbol{\xi}, \boldsymbol{\Delta}_\delta)$, the following lemmas that will be used in the proof are first presented.

Lemma 1. Block matrix $\boldsymbol{V} = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{pmatrix}$, and matrix $\boldsymbol{D}$ and $\boldsymbol{A} - \boldsymbol{B}\boldsymbol{D}^{-1}\boldsymbol{C}$ are invertible,

$$\tilde{\boldsymbol{V}} = \begin{pmatrix} (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{D}^{-1}\boldsymbol{C})^{-1} & -(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{D}^{-1}\boldsymbol{C})^{-1}\boldsymbol{B}\boldsymbol{D}^{-1} \\ -\boldsymbol{D}^{-1}\boldsymbol{C}(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{D}^{-1}\boldsymbol{C})^{-1} & \boldsymbol{D}^{-1} + \boldsymbol{D}^{-1}\boldsymbol{C}(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{D}^{-1}\boldsymbol{C})^{-1}\boldsymbol{B}\boldsymbol{D}^{-1} \end{pmatrix},$$

then $\tilde{\boldsymbol{V}} = \boldsymbol{V}^{-1}$.

Lemma 2. Block matrix $\boldsymbol{U} = \begin{pmatrix} \boldsymbol{I} & & & & & \\ \boldsymbol{S}_1 & \boldsymbol{I} & & & & \\ & \boldsymbol{S}_2 & \boldsymbol{I} & & & \\ & & \ddots & \ddots & & \\ & & & \boldsymbol{S}_{d-2} & \boldsymbol{I} & \\ & & & & \boldsymbol{S}_{d-1} & \boldsymbol{I} \end{pmatrix}$, where $\boldsymbol{S}_{\boldsymbol{v}} = \boldsymbol{S}$, for $1 \leq v \leq d-1$, $\boldsymbol{S}$ is a square matrix,

$$\tilde{\boldsymbol{U}} = \begin{pmatrix} \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ -\boldsymbol{S} & \boldsymbol{I} & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ (-\boldsymbol{S})^2 & -\boldsymbol{S} & \boldsymbol{I} & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (-\boldsymbol{S})^{d-2} & (-\boldsymbol{S})^{d-3} & (-\boldsymbol{S})^{d-4} & \cdots & \boldsymbol{I} & \boldsymbol{0} \\ (-\boldsymbol{S})^{d-1} & (-\boldsymbol{S})^{d-2} & (-\boldsymbol{S})^{d-3} & \cdots & -\boldsymbol{S} & \boldsymbol{I} \end{pmatrix},$$

then $\tilde{\boldsymbol{U}} = \boldsymbol{U}^{-1}$.

Lemma 3. [50] In discrete time, for a component subjected only to external shocks, $X_v \sim$ PH$(\boldsymbol{\eta}, \boldsymbol{H})$, where $v = 0, 1, \ldots,$ and $\boldsymbol{I} - \boldsymbol{H}$ is non-singular. Here, $X_v (v \geq 1)$ denotes the inter-arrival time between the v-th and the $(v+1)$-th shocks, X_0 represents the duration from the initial time to the first shock of the component. The component fails when the interval between two consecutive shocks is less than δ. The δ-shock lifetime of the component is denoted by $T(\delta)$. Then, the probability-generating function of $T(\delta)$, denoted by $\Phi_{T(\delta)}(z)$, satisfies

$$\Phi_{T(\delta)}(z) = \frac{\Phi(z)[\Phi(z) - A_\delta(z)]}{1 - A_\delta(z)},$$

where $\Phi(z) = z\boldsymbol{\eta}(\boldsymbol{I} - z\boldsymbol{H})^{-1}(\boldsymbol{I} - \boldsymbol{H})\boldsymbol{e}$, $A_\delta(z) = z\boldsymbol{\eta}(\boldsymbol{I} - \boldsymbol{H})(\boldsymbol{I} - z\boldsymbol{H})^{-1}(z\boldsymbol{H})^{\delta-1}\boldsymbol{e}$.

The proof of theorem 1 is as follows:

Proof. It is known from formula (2) that

$$\boldsymbol{\Delta}_\delta = \begin{pmatrix} \boldsymbol{H} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{H}^0\boldsymbol{\eta} \\ \boldsymbol{H} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{H}^0\boldsymbol{\eta} \\ \boldsymbol{0} & \boldsymbol{H} & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{H} & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{H} & \boldsymbol{0} \end{pmatrix}_{m(\delta+1)\times m(\delta+1)}, \quad \boldsymbol{\Delta}_\delta^0 = \boldsymbol{e} - \boldsymbol{\Delta}_\delta \boldsymbol{e} = \begin{pmatrix} \boldsymbol{0} \\ \boldsymbol{0} \\ \boldsymbol{H}^0 \\ \boldsymbol{H}^0 \\ \vdots \\ \boldsymbol{H}^0 \end{pmatrix}_{m(\delta+1)\times 1},$$

then

$$\boldsymbol{I} - z\boldsymbol{\Delta}_\delta = \begin{pmatrix} \boldsymbol{I} - z\boldsymbol{H} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} & -z\boldsymbol{H}^0\boldsymbol{\eta} \\ -z\boldsymbol{H} & \boldsymbol{I} & \boldsymbol{0} & \cdots & \boldsymbol{0} & -z\boldsymbol{H}^0\boldsymbol{\eta} \\ \boldsymbol{0} & -z\boldsymbol{H} & \boldsymbol{I} & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & -z\boldsymbol{H} & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \cdots & -z\boldsymbol{H} & \boldsymbol{I} \end{pmatrix}_{m(\delta+1)\times m(\delta+1)}.$$

Let

$$\boldsymbol{D}_{m(\delta-1)} = \begin{pmatrix} \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ -z\boldsymbol{H} & \boldsymbol{I} & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & -z\boldsymbol{H} & \boldsymbol{I} & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \cdots & -z\boldsymbol{H} & \boldsymbol{I} \end{pmatrix}_{m(\delta-1)\times m(\delta-1)},$$

it is known from Lemma 2 that

$$\boldsymbol{D}_{m(\delta-1)}^{-1} = \begin{pmatrix} \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ z\boldsymbol{H} & \boldsymbol{I} & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ (z\boldsymbol{H})^2 & z\boldsymbol{H} & \boldsymbol{I} & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (z\boldsymbol{H})^{\delta-2} & (z\boldsymbol{H})^{\delta-3} & (z\boldsymbol{H})^{\delta-4} & \cdots & z\boldsymbol{H} & \boldsymbol{I} \end{pmatrix}_{m(\delta-1)\times m(\delta-1)}.$$

Let

$$\boldsymbol{D}_{m\delta} = \begin{pmatrix} \boldsymbol{A}_{m(\delta-1)} & \boldsymbol{B}_{m(\delta-1)} \\ \boldsymbol{C}_{m(\delta-1)} & \boldsymbol{D}_{m(\delta-1)} \end{pmatrix}, \ \boldsymbol{D}_{m\delta}^{-1} = \begin{pmatrix} \boldsymbol{D}_{m\delta,11}^{-1} & \boldsymbol{D}_{m\delta,12}^{-1} \\ \boldsymbol{D}_{m\delta,21}^{-1} & \boldsymbol{D}_{m\delta,22}^{-1} \end{pmatrix},$$

where

$$\boldsymbol{A}_{m(\delta-1)} = \boldsymbol{I},\ \boldsymbol{B}_{m(\delta-1)} = \begin{pmatrix} \boldsymbol{0} & \cdots & \boldsymbol{0} & -z\boldsymbol{H}^0\boldsymbol{\eta} \end{pmatrix}_{m\times m(\delta-1)},\ \boldsymbol{C}_{m(\delta-1)} = \begin{pmatrix} -z\boldsymbol{H} & \boldsymbol{0} & \cdots & \boldsymbol{0} \end{pmatrix}^{\mathrm{T}}_{m(\delta-1)\times m},$$

it is known from Lemma 1 that

$$\boldsymbol{D}_{m\delta,11}^{-1} = (\boldsymbol{A}_{m(\delta-1)} - \boldsymbol{B}_{m(\delta-1)}\boldsymbol{D}_{m(\delta-1)}^{-1}\boldsymbol{C}_{m(\delta-1)})^{-1} = (\boldsymbol{I} - z^\delta \boldsymbol{H}^0\boldsymbol{\eta}\boldsymbol{H}^{\delta-1})^{-1},$$

$\boldsymbol{D}_{m\delta,11}^{-1}$is denoted by$\boldsymbol{\Lambda}$, then

$$\boldsymbol{D}_{m\delta,12}^{-1} = -\boldsymbol{\Lambda}\boldsymbol{B}_{m(\delta-1)}\boldsymbol{D}_{m(\delta-1)}^{-1} = \boldsymbol{\Lambda}\begin{pmatrix} z^{\delta-1}\boldsymbol{H}^0\boldsymbol{\eta}\boldsymbol{H}^{\delta-2} & \cdots & z^2\boldsymbol{H}^0\boldsymbol{\eta}\boldsymbol{H} & z\boldsymbol{H}^0\boldsymbol{\eta}\end{pmatrix}_{m\times m(\delta-1)},$$

$$\boldsymbol{D}_{m\delta,21}^{-1} = -\boldsymbol{D}_{m(\delta-1)}^{-1}\boldsymbol{C}_{m(\delta-1)}\boldsymbol{\Lambda} = \begin{pmatrix} z\boldsymbol{H} \\ \vdots \\ (z\boldsymbol{H})^{\delta-1}\end{pmatrix}_{m(\delta-1)\times m}\boldsymbol{\Lambda},$$

$$\begin{aligned}\boldsymbol{D}_{m\delta,22}^{-1} &= \boldsymbol{D}_{m(\delta-1)}^{-1} + \boldsymbol{D}_{m(\delta-1)}^{-1}\boldsymbol{C}_{m(\delta-1)}\boldsymbol{\Lambda}\boldsymbol{B}_{m(\delta-1)}\boldsymbol{D}_{m(\delta-1)}^{-1} \\ &= \begin{pmatrix} \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ z\boldsymbol{H} & \boldsymbol{I} & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ (z\boldsymbol{H})^2 & z\boldsymbol{H} & \boldsymbol{I} & \cdots & \boldsymbol{0} & \boldsymbol{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (z\boldsymbol{H})^{\delta-2} & (z\boldsymbol{H})^{\delta-3} & (z\boldsymbol{H})^{\delta-4} & \cdots & z\boldsymbol{H} & \boldsymbol{I}\end{pmatrix}_{m(\delta-1)\times m(\delta-1)} \\ &\quad + \begin{pmatrix} z\boldsymbol{H} \\ \vdots \\ (z\boldsymbol{H})^{\delta-1}\end{pmatrix}_{m(\delta-1)\times m}\boldsymbol{\Lambda}\begin{pmatrix} z^{\delta-1}\boldsymbol{H}^0\boldsymbol{\eta}\boldsymbol{H}^{\delta-2} & \cdots & z^2\boldsymbol{H}^0\boldsymbol{\eta}\boldsymbol{H} & z\boldsymbol{H}^0\boldsymbol{\eta}\end{pmatrix}_{m\times m(\delta-1)}.\end{aligned}$$

Let

$$\boldsymbol{I}-z\boldsymbol{\Delta}_\delta = \begin{pmatrix} \boldsymbol{A}_{m\delta} & \boldsymbol{B}_{m\delta} \\ \boldsymbol{C}_{m\delta} & \boldsymbol{D}_{m\delta}\end{pmatrix}_{m(\delta+1)\times m(\delta+1)},\ (\boldsymbol{I}-z\boldsymbol{\Delta}_\delta)^{-1} = \begin{pmatrix} (\boldsymbol{I}-z\boldsymbol{\Delta}_\delta)_{11}^{-1} & (\boldsymbol{I}-z\boldsymbol{\Delta}_\delta)_{12}^{-1} \\ (\boldsymbol{I}-z\boldsymbol{\Delta}_\delta)_{21}^{-1} & (\boldsymbol{I}-z\boldsymbol{\Delta}_\delta)_{22}^{-1}\end{pmatrix}_{m(\delta+1)\times m(\delta+1)},$$

where

$$\boldsymbol{A}_{m\delta} = \boldsymbol{I}-z\boldsymbol{H}, \boldsymbol{B}_{m\delta} = \begin{pmatrix} \boldsymbol{0} & \cdots & \boldsymbol{0} & -z\boldsymbol{H}^0\boldsymbol{\eta}\end{pmatrix}_{m\times m\delta},\ \boldsymbol{C}_{m\delta} = \begin{pmatrix} -z\boldsymbol{H} & \boldsymbol{0} & \cdots & \boldsymbol{0}\end{pmatrix}_{m\delta\times m}^{\mathrm{T}}.$$

It is known from Lemma 1 that

$$(\boldsymbol{I}-z\boldsymbol{\Delta}_\delta)_{12}^{-1} = -(\boldsymbol{A}_{m\delta}-\boldsymbol{B}_{m\delta}\boldsymbol{D}_{m\delta}^{-1}\boldsymbol{C}_{m\delta})^{-1}\boldsymbol{B}_{m\delta}\boldsymbol{D}_{m\delta}^{-1},$$

where

$$\begin{aligned}\boldsymbol{B}_{m\delta}\boldsymbol{D}_{m\delta}^{-1} &= \begin{pmatrix} \boldsymbol{0} & \cdots & \boldsymbol{0} & -z\boldsymbol{H}^0\boldsymbol{\eta}\end{pmatrix}_{m\times m\delta}\begin{pmatrix} \boldsymbol{D}_{m\delta,11}^{-1} & \boldsymbol{D}_{m\delta,12}^{-1} \\ \boldsymbol{D}_{m\delta,21}^{-1} & \boldsymbol{D}_{m\delta,22}^{-1}\end{pmatrix}_{m\delta\times m\delta} \\ &= -z\boldsymbol{H}^0\boldsymbol{\eta}\begin{pmatrix} (z\boldsymbol{H})^{\delta-1}\boldsymbol{\Lambda} \\ (z\boldsymbol{H})^{\delta-2} + z^{2\delta-2}\boldsymbol{H}^{\delta-1}\boldsymbol{\Lambda}\boldsymbol{H}^0\boldsymbol{\eta}\boldsymbol{H}^{\delta-2} \\ \vdots \\ z\boldsymbol{H} + z^{\delta+1}\boldsymbol{H}^{\delta-1}\boldsymbol{\Lambda}\boldsymbol{H}^0\boldsymbol{\eta}\boldsymbol{H} \\ \boldsymbol{I} + z^{\delta}\boldsymbol{H}^{\delta-1}\boldsymbol{\Lambda}\boldsymbol{H}^0\boldsymbol{\eta}\end{pmatrix}_{m\times m\delta}^{\mathrm{T}},\end{aligned}$$

$$\boldsymbol{B}_{m\delta}\boldsymbol{D}_{m\delta}^{-1}\boldsymbol{C}_{m\delta} = z^{\delta+1}\boldsymbol{H}^0\boldsymbol{\eta}\boldsymbol{H}^{\delta-1}\boldsymbol{\Lambda}\boldsymbol{H},$$

hence,

$$(\boldsymbol{I}-z\boldsymbol{\Delta}_\delta)_{12}^{-1}=(\boldsymbol{I}-z\boldsymbol{H}-z^{\delta+1}\boldsymbol{H}^0\boldsymbol{\eta}\boldsymbol{H}^{\delta-1}\boldsymbol{\Lambda}\boldsymbol{H})^{-1}(z\boldsymbol{H}^0\boldsymbol{\eta})$$
$$\times\begin{pmatrix}(z\boldsymbol{H})^{\delta-1}\boldsymbol{\Lambda}\\ (z\boldsymbol{H})^{\delta-2}+z^{2\delta-2}\boldsymbol{H}^{\delta-1}\boldsymbol{\Lambda}\boldsymbol{H}^0\boldsymbol{\eta}\boldsymbol{H}^{\delta-2}\\ \vdots\\ z\boldsymbol{H}+z^{\delta+1}\boldsymbol{H}^{\delta-1}\boldsymbol{\Lambda}\boldsymbol{H}^0\boldsymbol{\eta}\boldsymbol{H}\\ \boldsymbol{I}+z^{\delta}\boldsymbol{H}^{\delta-1}\boldsymbol{\Lambda}\boldsymbol{H}^0\boldsymbol{\eta}\end{pmatrix}^{\mathrm{T}}_{m\times m\delta},$$

then the probability-generating function of $\mathrm{PH}(\boldsymbol{\xi},\boldsymbol{\Delta_\delta})$ is as follows:

$$\begin{aligned}\Phi_{\mathrm{PH}(\boldsymbol{\xi},\boldsymbol{\Delta}_\delta)}(z)&=z\boldsymbol{\xi}(\boldsymbol{I}-z\boldsymbol{\Delta}_\delta)^{-1}\boldsymbol{\Delta}_\delta^0\\
&=z\boldsymbol{\eta}(\boldsymbol{I}-z\boldsymbol{H}-z^{\delta+1}\boldsymbol{H}^0\boldsymbol{\eta}\boldsymbol{H}^{\delta-1}\boldsymbol{\Lambda}\boldsymbol{H})^{-1}(z\boldsymbol{H}^0\boldsymbol{\eta})\\
&\quad\times\begin{pmatrix}(z\boldsymbol{H})^{\delta-2}+z^{2\delta-2}\boldsymbol{H}^{\delta-1}\boldsymbol{\Lambda}\boldsymbol{H}^0\boldsymbol{\eta}\boldsymbol{H}^{\delta-2}\\ \vdots\\ z\boldsymbol{H}+z^{\delta+1}\boldsymbol{H}^{\delta-1}\boldsymbol{\Lambda}\boldsymbol{H}^0\boldsymbol{\eta}\boldsymbol{H}\\ \boldsymbol{I}+z^{\delta}\boldsymbol{H}^{\delta-1}\boldsymbol{\Lambda}\boldsymbol{H}^0\boldsymbol{\eta}\end{pmatrix}^{\mathrm{T}}_{m\times m(\delta-1)}\begin{pmatrix}\boldsymbol{H}^0\\ \vdots\\ \boldsymbol{H}^0\end{pmatrix}_{m(\delta-1)\times m}\\
&=\sum_{k=1}^{\delta-1}z^2\boldsymbol{\eta}(\boldsymbol{I}-z\boldsymbol{H}-z^{\delta+1}\boldsymbol{H}^0\boldsymbol{\eta}\boldsymbol{H}^{\delta-1}\boldsymbol{\Lambda}\boldsymbol{H})^{-1}\boldsymbol{H}^0\boldsymbol{\eta}(\boldsymbol{I}+z^{\delta}\boldsymbol{H}^{\delta-1}\boldsymbol{\Lambda}\boldsymbol{H}^0\boldsymbol{\eta})(z\boldsymbol{H})^{\delta-1-k}\boldsymbol{H}^0\\
&=z^2\boldsymbol{\eta}(\boldsymbol{I}-z\boldsymbol{H}-z^{\delta+1}\boldsymbol{H}^0\boldsymbol{\eta}\boldsymbol{H}^{\delta-1}\boldsymbol{\Lambda}\boldsymbol{H})^{-1}\boldsymbol{H}^0\boldsymbol{\eta}(\boldsymbol{I}+z^{\delta}\boldsymbol{H}^{\delta-1}\boldsymbol{\Lambda}\boldsymbol{H}^0\boldsymbol{\eta})(\boldsymbol{I}-(z\boldsymbol{H})^{\delta-1})\\
&\quad\times(\boldsymbol{I}-z\boldsymbol{H})^{-1}\boldsymbol{H}^0\\
&=z\boldsymbol{\eta}(\boldsymbol{I}-z\boldsymbol{H}-z^{\delta+1}\boldsymbol{H}^0\boldsymbol{\eta}\boldsymbol{H}^{\delta-1}\boldsymbol{\Lambda}\boldsymbol{H})^{-1}\boldsymbol{H}^0\boldsymbol{\eta}z(\boldsymbol{I}-(z\boldsymbol{H})^{\delta-1})(\boldsymbol{I}-z\boldsymbol{H})^{-1}\boldsymbol{H}^0+z\boldsymbol{\eta}(\boldsymbol{I}-z\boldsymbol{H}\\
&\quad-z^{\delta+1}\boldsymbol{H}^0\boldsymbol{\eta}\boldsymbol{H}^{\delta-1}\boldsymbol{\Lambda}\boldsymbol{H})^{-1}\boldsymbol{H}^0\boldsymbol{\eta}z^{\delta}\boldsymbol{H}^{\delta-1}\boldsymbol{\Lambda}\boldsymbol{H}^0\boldsymbol{\eta}z(\boldsymbol{I}-(z\boldsymbol{H})^{\delta-1})(\boldsymbol{I}-z\boldsymbol{H})^{-1}\boldsymbol{H}^0.\end{aligned}$$

It follows from Lemma 1 that

$$\Phi(z)-A_\delta(z)=\boldsymbol{\eta}z(\boldsymbol{I}-(z\boldsymbol{H})^{\delta-1})(\boldsymbol{I}-z\boldsymbol{H})^{-1}\boldsymbol{H}^0,$$

hence,

$$\Phi_{\mathrm{PH}(\boldsymbol{\xi},\boldsymbol{\Delta}_\delta)}(z)=z\boldsymbol{\eta}(\boldsymbol{I}-z\boldsymbol{H}-z^{\delta+1}\boldsymbol{H}^0\boldsymbol{\eta}\boldsymbol{H}^{\delta-1}\boldsymbol{\Lambda}\boldsymbol{H})^{-1}(\boldsymbol{I}+\boldsymbol{H}^0\boldsymbol{\eta}z^{\delta}\boldsymbol{H}^{\delta-1}\boldsymbol{\Lambda})\boldsymbol{H}^0(\Phi(z)-A_\delta(z)).$$

By the definition of $\boldsymbol{\Lambda}$, it is known that:

$$\boldsymbol{H}^0\boldsymbol{\eta}z^{\delta}\boldsymbol{H}^{\delta-1}=\boldsymbol{I}-\boldsymbol{\Lambda}^{-1},\tag{A.1}$$

then $\boldsymbol{I}+\boldsymbol{H}^0\boldsymbol{\eta}z^{\delta}\boldsymbol{H}^{\delta-1}\boldsymbol{\Lambda}=\boldsymbol{\Lambda}$, hence,

$$\Phi_{\mathrm{PH}(\boldsymbol{\xi},\boldsymbol{\Delta}_\delta)}(z)=z\boldsymbol{\eta}(\boldsymbol{I}-z\boldsymbol{H}-z^{\delta+1}\boldsymbol{H}^0\boldsymbol{\eta}\boldsymbol{H}^{\delta-1}\boldsymbol{\Lambda}\boldsymbol{H})^{-1}\boldsymbol{\Lambda}\boldsymbol{H}^0(\Phi(z)-A_\delta(z)),$$

it is known from formula (A.1) that

$$z^{\delta+1}\boldsymbol{H}^0\boldsymbol{\eta}\boldsymbol{H}^{\delta-1}\boldsymbol{\Lambda}\boldsymbol{H}=z(z^{\delta}\boldsymbol{H}^0\boldsymbol{\eta}\boldsymbol{H}^{\delta-1}\boldsymbol{\Lambda})\boldsymbol{H}=z(\boldsymbol{\Lambda}-\boldsymbol{I})\boldsymbol{H},$$

$$\Phi_{\mathrm{PH}(\boldsymbol{\xi},\boldsymbol{\Delta}_\delta)}(z)=z\boldsymbol{\eta}(\boldsymbol{I}-z\boldsymbol{\Lambda}\boldsymbol{H})^{-1}\boldsymbol{\Lambda}\boldsymbol{H}^0(\Phi(z)-A_\delta(z)),\tag{A.2}$$

hence,

$$(1-A_\delta(z))\Phi_{\mathrm{PH}(\boldsymbol{\xi},\boldsymbol{\Delta}_\delta)}(z) = z\boldsymbol{\eta}(\boldsymbol{I}-z\boldsymbol{\Lambda}\boldsymbol{H})^{-1}\boldsymbol{\Lambda}\boldsymbol{H}^0(\Phi(z)-A_\delta(z)) - z\boldsymbol{\eta}(\boldsymbol{I}-z\boldsymbol{\Lambda}\boldsymbol{H})^{-1}\boldsymbol{\Lambda}\boldsymbol{H}^0\boldsymbol{\eta}z(\boldsymbol{I}-z\boldsymbol{H})^{-1} \times (z\boldsymbol{H})^{\delta-1}\boldsymbol{H}^0(\Phi(z)-A_\delta(z)).$$

It is known from formula (A.1) that

$$\boldsymbol{I}-\boldsymbol{H}^0\boldsymbol{\eta}z(\boldsymbol{I}-z\boldsymbol{H})^{-1}(z\boldsymbol{H})^{\delta-1} = \boldsymbol{I}-(\boldsymbol{I}-\boldsymbol{\Lambda}^{-1})(\boldsymbol{I}-z\boldsymbol{H})^{-1}, \tag{A.3}$$

furthermore,

$$\begin{aligned}
&(\boldsymbol{I}-z\boldsymbol{\Lambda}\boldsymbol{H})^{-1}\boldsymbol{\Lambda}(\boldsymbol{I}-(\boldsymbol{I}-\boldsymbol{\Lambda}^{-1})(\boldsymbol{I}-z\boldsymbol{H})^{-1})(\boldsymbol{I}-z\boldsymbol{H})\\
&=(\boldsymbol{I}-z\boldsymbol{\Lambda}\boldsymbol{H})^{-1}\boldsymbol{\Lambda}(\boldsymbol{I}-z\boldsymbol{H})-(\boldsymbol{I}-z\boldsymbol{\Lambda}\boldsymbol{H})^{-1}\boldsymbol{\Lambda}(\boldsymbol{I}-\boldsymbol{\Lambda}^{-1})\\
&=(\boldsymbol{I}-z\boldsymbol{\Lambda}\boldsymbol{H})^{-1}\boldsymbol{\Lambda}(\boldsymbol{\Lambda}^{-1}-z\boldsymbol{H})\\
&=\boldsymbol{I},
\end{aligned}$$

therefore,

$$(\boldsymbol{I}-z\boldsymbol{\Lambda}\boldsymbol{H})^{-1}\boldsymbol{\Lambda}(\boldsymbol{I}-(\boldsymbol{I}-\boldsymbol{\Lambda}^{-1})(\boldsymbol{I}-z\boldsymbol{H})^{-1}) = (\boldsymbol{I}-z\boldsymbol{H})^{-1}. \tag{A.4}$$

It follows from formula(A.3)and(A.4)that

$$(1-A_\delta(z))\Phi_{\mathrm{PH}(\boldsymbol{\xi},\boldsymbol{\Delta}_\delta)}(z) = \Phi(z)(\Phi(z)-A_\delta(z)),$$

therefore,

$$\Phi_{\mathrm{PH}(\boldsymbol{\xi},\boldsymbol{\Delta}_\delta)}(z) = \Phi_{T(\delta)}(z),$$

that means $T(\delta)\sim \mathrm{PH}(\boldsymbol{\xi},\boldsymbol{\Delta}_\delta)$. □

Appendix B

Before defining the matrix $\boldsymbol{B}(j;u;h_1;h_2)$, the matrix $\boldsymbol{C}(j;u;w_1,\ldots,w_{h_1};s_1,\ldots,s_{h_2})$ is defined firstly. The definitions of these two matrices are as follows:

(1) Matrix $\boldsymbol{C}(j;u;w_1,\ldots,w_{h_1};s_1,\ldots,s_{h_2})$

The matrix $\boldsymbol{C}(j;u;w_1,\ldots,w_{h_1};s_1,\ldots,s_{h_2})$ is a probability matrix, with the specific meanings as follows: j represents the state of the repair equipment, u represents the number of failed repairable components in the retrial orbit, h_1 represents the number of components that may experience repairable failures at the next time, and h_2 represents the number of components that may experience non-repairable failures at the next time. The main components $w_1,\ldots,w_{h_1}$ are expected to experience repairable failures at the next time, and $s_1,\ldots,s_{h_2}$ are expected to experience non-repairable failures at the next time.

(i) $j+u\le n-k$

(a) $h_1 > 0, h_2 > 0$

$$\boldsymbol{C}(j; u; w_1, \ldots, w_{h_1}; s_1, \ldots, s_{h_2}) = \boldsymbol{\Delta}_{\delta,1} \otimes \cdots \otimes \boldsymbol{\Delta}_{\delta,k},$$

where

$$\boldsymbol{\Delta}_{\delta,v} = \begin{cases} ((1-\theta(1-\mu))\boldsymbol{\Delta}^0_{\delta,\mathrm{r}} + \theta\mu\boldsymbol{\Delta}_\delta \boldsymbol{e}) \otimes \boldsymbol{\xi}, & \text{if } v = w_1, \ldots, w_{\min(h_1, n-j-u-k)}, \\ (1-\theta(1-\mu))\boldsymbol{\Delta}^0_{\delta,\mathrm{r}} + \theta\mu\boldsymbol{\Delta}_\delta \boldsymbol{e}, & \text{if } v = w_{\min(h_1, n-j-u-k)+1}, \ldots, w_{h_1}, \\ ((1-\theta(1-\mu))\boldsymbol{\Delta}^0_{\delta,\mathrm{nr}} + \theta(1-\mu)\boldsymbol{e}) \otimes \boldsymbol{\xi}, & \text{if } v = s_1, \ldots, s_{h_2}, \\ (1-\theta)\boldsymbol{\Delta}_\delta, & \text{if } v = \text{otherwise}. \end{cases}$$

(b) $h_1 = 0, h_2 > 0$

$$\boldsymbol{C}(j; u; 0; s_1, \ldots, s_{h_2}) = \boldsymbol{\Delta}_{\delta,1} \otimes \cdots \otimes \boldsymbol{\Delta}_{\delta,k},$$

where

$$\boldsymbol{\Delta}_{\delta,v} = \begin{cases} ((1-\theta(1-\mu))\boldsymbol{\Delta}^0_{\delta,\mathrm{nr}} + \theta(1-\mu)\boldsymbol{e}) \otimes \boldsymbol{\xi}, & \text{if } v = s_1, \ldots, s_{h_2}, \\ (1-\theta)\boldsymbol{\Delta}_\delta, & \text{if } v = \text{otherwise}. \end{cases}$$

(c) $h_1 > 0, h_2 = 0$

$$\boldsymbol{C}(j; u; w_1, \ldots, w_{h_1}; 0) = \boldsymbol{\Delta}_{\delta,1} \otimes \cdots \otimes \boldsymbol{\Delta}_{\delta,k},$$

where

$$\boldsymbol{\Delta}_{\delta,v} = \begin{cases} ((1-\theta(1-\mu))\boldsymbol{\Delta}^0_{\delta,\mathrm{r}} + \theta\mu\boldsymbol{\Delta}_\delta \boldsymbol{e}) \otimes \boldsymbol{\xi}, & \text{if } v = w_1, \ldots, w_{\min(h_1, n-j-u-k)}, \\ (1-\theta(1-\mu))\boldsymbol{\Delta}^0_{\delta,\mathrm{r}} + \theta\mu\boldsymbol{\Delta}_\delta \boldsymbol{e}, & \text{if } v = w_{\min(h_1, n-j-u-k)+1}, \ldots, w_{h_1}, \\ (1-\theta)\boldsymbol{\Delta}_\delta, & \text{if } v = \text{otherwise}. \end{cases}$$

(d) $h_1 = 0, h_2 = 0$

$$\boldsymbol{C}(j; u; 0; 0) = \boldsymbol{\Delta}_{\delta,1} \otimes \cdots \otimes \boldsymbol{\Delta}_{\delta,k},$$

where

$$\boldsymbol{\Delta}_{\delta,v} = (1-\theta)\boldsymbol{\Delta}_\delta, 1 \le v \le k.$$

(ii) $j + u > n - k$

(a) $h_1 > 0, h_2 > 0$

$$\boldsymbol{C}(j; u; w_1, \ldots, w_{h_1}; s_1, \ldots, s_{h_2}) = \boldsymbol{\Delta}_{\delta,1} \otimes \cdots \otimes \boldsymbol{\Delta}_{\delta,n-j-u},$$

where

$$\boldsymbol{\Delta}_{\delta,v} = \begin{cases} \boldsymbol{\Delta}^0_{\delta,\mathrm{r}}, & \text{if } v = w_1, \ldots, w_{h_1}, \\ \boldsymbol{\Delta}^0_{\delta,\mathrm{nr}} \otimes \boldsymbol{\xi}, & \text{if } v = s_1, \ldots, s_{h_2}, \\ \boldsymbol{\Delta}_\delta, & \text{if } v = \text{otherwise}. \end{cases}$$

(b) $h_1 = 0, h_2 > 0$

$$\boldsymbol{C}(j;u;0;s_1,\ldots,s_{h_2}) = \boldsymbol{\Delta}_{\delta,1} \otimes \cdots \otimes \boldsymbol{\Delta}_{\delta,n-j-u},$$

where

$$\boldsymbol{\Delta}_{\delta,v} = \begin{cases} \boldsymbol{\Delta}^0_{\delta,\mathrm{nr}} \otimes \boldsymbol{\xi}, & \text{if } v = s_1,\ldots,s_{h_2}, \\ \boldsymbol{\Delta}_{\delta}, & \text{if } v = \text{otherwise}. \end{cases}$$

(c) $h_1 > 0, h_2 = 0$

$$\boldsymbol{C}(j;u;w_1,\ldots,w_{h_1};0) = \boldsymbol{\Delta}_{\delta,1} \otimes \cdots \otimes \boldsymbol{\Delta}_{\delta,n-j-u},$$

where

$$\boldsymbol{\Delta}_{\delta,v} = \begin{cases} \boldsymbol{\Delta}^0_{\delta,\mathrm{r}}, & \text{if } v = w_1,\ldots,w_{h_1}, \\ \boldsymbol{\Delta}_{\delta}, & \text{if } v = \text{otherwise}. \end{cases}$$

(d) $h_1 = 0, h_2 = 0$

$$\boldsymbol{C}(j;u;0;0) = \boldsymbol{\Delta}_{\delta,1} \otimes \cdots \otimes \boldsymbol{\Delta}_{\delta,n-j-u},$$

where

$$\boldsymbol{\Delta}_{\delta,v} = \boldsymbol{\Delta}_{\delta}, 1 \le v \le n-j-u.$$

(2) Matrix $\boldsymbol{B}(j;u;h_1;h_2)$

The matrix $\boldsymbol{B}(j;u;h_1;h_2)$ is a probability matrix, with the specific meanings as follows: j represents the state of the repair equipment, u represents the number of failed repairable components in the retrial orbit, h_1 represents the number of components that may experience repairable failures at the next time, and h_2 represents the number of components that may experience non-repairable failures at the next time.

(i) $j + u \le n - k$

(a) $h_1 > 0, h_2 > 0$

$$\boldsymbol{B}(j;u;h_1;h_2) = \sum_{w_1=1}^{k-h_1+1} \sum_{w_2=w_1+1}^{k-h_1+2} \cdots \sum_{w_{h_1}=w_{h_1-1}+1}^{k} \sum_{\substack{s_1=1,\\ s_1\neq w_v,\\ v=1\ldots h_1}}^{k-h_2+1} \sum_{\substack{s_2=s_1+1,\\ s_2\neq w_v,\\ v=1\ldots h_1}}^{k-h_2+2} \cdots \sum_{\substack{s_{h_2}=s_{h_2-1}+1,\\ s_{h_2}\neq w_v,\\ v=1\ldots h_1}}^{k} \boldsymbol{C}(j;u;w_1,\ldots,w_{h_1};s_1,\ldots,s_{h_2}).$$

(b) $h_1 = 0, h_2 > 0$

$$\boldsymbol{B}(j;u;0;h_2) = \sum_{s_1=1}^{k-h_2+1} \sum_{s_2=s_1+1}^{k-h_2+2} \cdots \sum_{s_{h_2}=s_{h_2-1}+1}^{k} \boldsymbol{C}(j;u;0;s_1,\ldots,s_{h_2}).$$

(c) $h_1 > 0, h_2 = 0$

$$\boldsymbol{B}(j;u;h_1;0) = \sum_{w_1=1}^{k-h_1+1} \sum_{w_2=w_1+1}^{k-h_1+2} \cdots \sum_{w_{h_1}=w_{h_1-1}+1}^{k} \boldsymbol{C}(j;u;w_1,\ldots,w_{h_1};0).$$

(d) $h_1 = 0, h_2 = 0$

$$\boldsymbol{B}(j;u;0;0) = \boldsymbol{C}(j;u;0;0).$$

(ii) $j + u > n - k$

(a) $h_1 > 0, h_2 > 0$

$$\boldsymbol{B}(j;u;h_1;h_2) = \sum_{w_1=1}^{n-j-u-h_1+1} \sum_{w_2=w_1+1}^{n-j-u-h_1+2} \cdots \sum_{w_{h_1}=w_{h_1-1}+1}^{n-j-u} \sum_{\substack{s_1=1,\\ s_1\neq w_v,\\ v=1\ldots h_1}}^{n-j-u-h_2+1} \sum_{\substack{s_2=s_1+1,\\ s_2\neq w_v,\\ v=1\ldots h_1}}^{n-j-u-h_2+2} \cdots \sum_{\substack{s_{h_2}=s_{h_2-1}+1,\\ s_{h_2}\neq w_v,\\ v=1\ldots h_1}}^{n-j-u} \boldsymbol{C}(j;u;w_1,\ldots,w_{h_1};s_1,\ldots,s_{h_2}).$$

(b) $h_1 = 0, h_2 > 0$

$$\boldsymbol{B}(j;u;0;h_2) = \sum_{s_1=1}^{n-j-u-h_2+1} \sum_{s_2=s_1+1}^{n-j-u-h_2+2} \cdots \sum_{s_{h_2}=s_{h_2-1}+1}^{n-j-u} \boldsymbol{C}(j;u;0;s_1,\ldots,s_{h_2}).$$

(c) $h_1 > 0, h_2 = 0$

$$\boldsymbol{B}(j;u;h_1;0) = \sum_{w_1=1}^{n-j-u-h_1+1} \sum_{w_2=w_1+1}^{n-j-u-h_1+2} \cdots \sum_{w_{h_1}=w_{h_1-1}+1}^{n-j-u} \boldsymbol{C}(j;u;w_1,\ldots,w_{h_1};0).$$

(d) $h_1 = 0, h_2 = 0$

$$\boldsymbol{B}(j;u;0;0) = \boldsymbol{C}(j;u;0;0).$$