THE HOROFUNCTION COMPACTIFICATION OF ℓ^1-PRODUCTS OF METRIC SPACES

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Abstract. We study the horofunction compactification of the ℓ^1-product of proper geodesic metric spaces. We provide a complete characterisation of the horofunction compactification of the product space in terms of the horofunctions of the constituent spaces, and provide a complete characterisation of the Busemann points in terms of the Busemann points of the constituent spaces. We also identify the parts of the horofunction boundary and the detour distance. The results are applied to show that the horofunction compactification of the ℓ^1-product of finite dimensional normed spaces with polyhedral or smooth unit balls is naturally homeomorphic to the closed dual unit ball.

1. Introduction

The horofunction compactification of metric spaces has been studied in various contexts for about 50 years, and is generally attributed to Gromov [11]. It associates a boundary at infinity to locally compact geodesic metric spaces, which captures certain geometric features of the metric space. Since its inception it has found uses in numerous areas, ranging from geometric group theory [6, 11], to Teichmüller theory [8, 10, 18, 28, 34], to dynamical systems [3, 9, 17, 24], to complex analysis [1, 2, 4, 5, 37]. Rieffel used the horofunction compactification in his study of noncommutative geometry [30], where he introduced a special class of horofunctions, the so-called Busemann points. The study of Busemann points is particularly useful, as they can be used to study the group of isometries of metric spaces and the isometric embeddings between metric spaces [21, 27, 35, 36].

Even though the horofunction compactification of various metric spaces has been studied for many decades, the explicit form of a particular horofunction compactification is often difficult to compute. The horoboundary of various normed spaces has been calculated to varying degrees of explicitness [12, 13, 15, 22, 25, 33, 36]. There has also been significant work done on calculating the horofunction boundary of cones under the Thompson metric and Hilbert geometries [24, 25, 26, 36], as well as symmetric spaces of non-compact type with invariant Finsler metrics [7, 14, 22], and Teichmüller spaces [10, 34].

Although there is no real unifying approach to identifying the horofunction boundary for general classes of metric spaces, there is some structure for products of metric spaces. Given a finite collection of metric spaces, say (M_i, d_i) for $i \in \{1, \ldots, m\}$, a natural way to equip the product

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$M = \prod_{i=1}^m M_i$ with a metric is by equipping it with an *ℓ^p-metric*,

$$d^p(x,y) = \left(\sum_{i=1}^m d_i(x_i,y_i)^p\right)^{1/p}, \text{ for } p \in [1,\infty), \text{ and } d^\infty(x,y) = \max_{1\leq i\leq m} d_i(x_i,y_i), \tag{1.1}$$

where $x = (x_1,\dots,x_m), y = (y_1,\dots,y_m) \in M$.

In this context it is interesting to investigate if the horofunction compactification of (M, d^p) can be expressed in terms of the horofunctions of the component metric spaces (M_i, d_i). In [21], Lemmens showed that, if each (M_i, d_i) is a proper geodesic metric space, every horofunction h of (M, d^∞) is of the form,

$$h(z) = \max_{j\in J} h_j(z_j) - \alpha_j,$$

for $z = (z_1,\dots,z_m) \in M$. Here $\emptyset \neq J \subseteq \{1,\dots,m\}$, $\min_{j\in J} \alpha_j = 0$, and each h_j is a horofunction of (M_i, d_i).

The main goal of this paper is to provide a detailed analysis for the ℓ^1-metric on M. Among other results we will prove the following:

Theorem 1.1. *If (M_i, d_i) is a proper geodesic metric space for $i \in \{1,\dots,m\}$, then the horofunction compactification of (M, d^1) consists precisely of the functions h of the form,*

$$h(z) = \sum_{i=1}^m h_i(z_i) \tag{1.2}$$

for $z = (z_1,\dots,z_m) \in M$, where each h_i lies in the horofunction compactification of (M_i, d_i). Moreover, h is a horofunction of (M, d^1) if and only if there exists a $j \in \{1,\dots,m\}$ such that h_j is a horofunction of (M_j, d_j).

Throughout we shall simply denote the functions of the form (1.2) by $h = \sum_{i=1}^m h_i$. In [21] the Busemann points of (M, d^∞) were characterised, under some additional assumptions. Due to the nicer interplay between the ℓ^1-metric and almost-geodesics, we are able to get a characterisation without these assumptions, which is another result of this paper.

Theorem 1.2. *The horofunction $h = \sum_{i=1}^m h_i$ of (M, d^1), as in (1.2), is a Busemann point if and only if for each i the function h_i is an internal point or Busemann point of (M_i, d_i), and there exists a j such that h_j is a Busemann point.*

This result allows us to characterise the parts of the horofunction boundary of (M, d^1) and determine the detour distance in Theorem 4.1. In Section 5 we will use the result to establish some new results regarding the geometry and global topology of the horofunction compactifications of l^1-products of finite dimensional normed spaces in terms of the unit ball of the dual space.

2. Preliminaries

A metric space (M, d) is said to be *proper* if its closed balls are compact. A prototypical example is a finite-dimensional normed space. Proper metric spaces are complete and separable. A path $\gamma\colon I \to M$ is called *geodesic* if $d(\gamma(t), \gamma(s)) = |t - s|$ for all $t, s \in I$. We call (M, d) a *geodesic space* if for every $x, y \in M$ there exists a closed interval $[a, b]$ and a geodesic path $\gamma\colon [a, b] \to M$ with $\gamma(a) = x$ and $\gamma(b) = y$.

In the sequel the following fact will be useful (see [29, Proposition 2.6.6] for a proof).

Lemma 2.1. *Suppose that (M_i, d_i) is a proper geodesic metric space for $i \in \{1,\dots,m\}$ and $M = \prod_{i=1}^m M_i$. Then (M, d^p), where d^p is the ℓ^p-metric given by (1.1), is a proper geodesic metric space for each $p \in [1,\infty]$.*

Let (M, d) be any metric space. We let $\mathbb{R}^M$ be the space of all real-valued functions on M, equipped with the topology of pointwise convergence. We fix some *basepoint* $b \in M$ and denote by $\mathrm{Lip}_b^1(M)$ the set of all $h \in \mathbb{R}^M$ with $h(b) = 0$ and h 1-Lipschitz, i.e., $|h(x) - h(y)| \le d(x, y)$ for all $x, y \in M$.

Lemma 2.2. $\mathrm{Lip}_b^1(M)$ *is a closed subset of* $\mathbb{R}^M$.

Proof. We show that the complement is open in $\mathbb{R}^M$. So, suppose that $h \notin \mathrm{Lip}_b^1(M)$. If $|h(b)| \ge \delta > 0$, then $U = \{g \in \mathbb{R}^M : |g(b) - h(b)| < \delta\}$ is an open set disjoint from $\mathrm{Lip}_b^1(M)$. On the other hand, if h is not 1-Lipschitz, there exist $x, y \in M$ and $\epsilon > 0$ such that $|h(x) - h(y)| \ge d(x, y) + \epsilon$. In that case

$$W = \{g \in \mathbb{R}^M : |g(x) - h(x)| < \epsilon/3 \text{ and } |g(y) - h(y)| < \epsilon/3\}$$

is open set disjoint from $\mathrm{Lip}_b^1(M)$. Indeed, for $g \in W$ we have

$$|g(x) - g(y)| \ge |h(x) - h(y)| - |g(x) - h(x)| - |g(y) - h(y)| \ge d(x, y) + \epsilon/3,$$

hence $g \notin \mathrm{Lip}_b^1(M)$. □

Note that for any $h \in \mathrm{Lip}_b^1(M)$, $|h(x)| = |h(x) - h(b)| \le d(x, b)$, so

$$\mathrm{Lip}_b^1(M) \subseteq [-d(x, b), d(x, b)]^M = \prod_{x \in M} [-d(x, b), d(x, b)].$$

By Tychonoff's Theorem this set is compact in the product topology, which is equivalent to the topology of pointwise convergence. As $\mathrm{Lip}_b^1(M)$ is closed, it follows that it too must be compact. For any $y \in M$ we define the *internal point* h_y by

$$h_y(x) = d(x, y) - d(b, y). \quad \text{for } x \in M. \tag{2.1}$$

It follows immediately from the triangle inequality that $h_y \in \mathrm{Lip}_b^1(M)$, hence, the map $\iota \colon y \mapsto h_y$ maps M into $\mathrm{Lip}_b^1(M)$. This map is injective, as $h_y = h_z$ implies $0 = h_y(z) - h_y(y) + h_z(y) - h_z(z) = 2d(y, z)$. The closure of $\iota(M)$ in $\mathrm{Lip}_b^1(M)$ is denote by $\overline{M}^h$, and is called the *horofunction compactification* of M. The boundary $\partial\overline{M}^h = \overline{M}^h \backslash \iota(M)$ is called the *horofunction boundary* of M, and its elements are called *horofunctions*. The choice of basepoint is not important, as the horofunction compactifications with a different basepoint are homeomorphic. Some readers may be more familiar with the definition of horofunctions in which $\mathrm{Lip}_b^1(M)$ is equipped with the topology of uniform convergence on compact sets. However, $\mathrm{Lip}_b^1(M)$ is an equicontinuous family and uniformly bounded, so these topologies are equal on $\mathrm{Lip}_b^1(M)$ as a consequence of the generalised Ascoli Theorem [19, Ch. 7, Thm. 15]. For any metric space, the map ι is a continuous injection, but it may fail to have a continuous inverse. If, however, (M, d) is a proper geodesic metric space, then ι is a homeomorphism onto its image (see e.g. [34, Proposition 2.2]). So, in this case $\overline{M}^h$ is a compactification in the usual topological sense of the word. Furthermore, if (M, d) is separable, then $\mathrm{Lip}_b^1(M)$ is second countable, meaning that sequences suffice to characterise the topology of $\overline{M}^h$. In particular, this leads to the following useful lemma, see [23, Lemma 2.1].

Proposition 2.3. *If* (M, d) *is a proper geodesic metric space, a function* $h \in \mathrm{Lip}_b^1(M)$ *is a horofunction if and only if there exists a sequence* (y^n) *in* M*, such that* $d(b, y^n) \to \infty$ *and* $h_{y^n} \to h$*.*

In a proper geodesic metric space (M, d), it is easy to verify that, if $\gamma \colon [a, \infty) \to M$ is a geodesic, $\lim_{t \to \infty} h_{\gamma(t)}(z)$ exists for every $z \in M$. Thus every unbounded geodesic gives rise to a horofunction. In [30], Rieffel introduced the notion of an *almost-geodesic*, which also always give rise to a horofunction. Recall that an *almost-geodesic sequence* is a sequence $(x^n) \subseteq M$, such that $d(b, x^n) \to \infty$, and for all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$, so that

$$d(x^m, x^k) + d(x^k, x^0) - d(x^m, x^0) < \varepsilon \quad \text{for all } m \geq k \geq N.$$

In the context of proper geodesic metric spaces we call a horofunction $h \in \partial \overline{M}^h$ a *Busemann point* if there exists an almost-geodesic sequence (x^n) such that, for all $z \in M$,

$$h(z) = \lim_{n\to\infty} h_{x^n}(z).$$

We shall use the notation $\partial_B \overline{M}^h$ to represent the set of Busemann points of M. As mentioned, in a proper geodesic metric space every almost-geodesic sequence gives rise to a horofunction [30]. The converse, however, need not be true, not even in finite dimensional normed spaces, see [33].

The set of Busemann points in a proper geodesic metric space can be equipped with a metric using the *detour cost*, $H\colon \partial_B \overline{M}^h(M) \times \partial_B \overline{M}^h(M) \to [0, \infty]$, which is given by

$$H(h, h') = \inf_{(z^n)\subseteq M} \{\liminf_n d(z^n, b) + h'(z^n)\colon h_{z^n} \to h\}.$$

Now one defines the *detour distance*, $\delta\colon \partial_B \overline{M}^h(M) \times \partial_B \overline{M}^h(M) \to \mathbb{R}^+ \cup \{\infty\}$, by

$$\delta(h, h') = H(h, h') + H(h', h).$$

It should be noted that the detour distance can be infinite. To determine the detour distance it is often useful to use the following fact [27, Lemma 3.1]:

Lemma 2.4. *If h, h' are Busemann points, and (x^n) is an almost geodesic such that (h_{x^n}) converges to h, then*

$$H(h, h') = \lim_{n\to\infty} d(b, x^n) + h'(x^n).$$

The detour distance induces a partition of $\partial_B \overline{M}^h$ into different equivalence classes, where $h, h' \in \partial \overline{M}^h$ are equivalent if and only if $\delta(h, h') < \infty$. The resulting equivalence classes are metric spaces, which are called the *parts* of $\partial_B \overline{M}^h$. There is another natural equivalence relation on the whole horofunction compactification $\overline{M}^h$, where the equivalence relation is given by $h \sim h'$ if

$$\sup_{x\in M} |h(x) - h'(x)| < \infty. \tag{2.2}$$

It is known that the restriction of the equivalence classes of $\sim$ to the set of Busemann points coincides with the parts, see [35, Proposition 4.5].

3. Metric compactification of ℓ^1-product spaces

For $i = 1, \dots, m$, let (M_i, d_i), be a proper geodesic metric space with basepoint $b_i \in M_i$. Now consider the ℓ^1-product metric space (M, d^1) with basepoint $b = (b_1, \dots, b_m)$. In what follows, whenever we refer to $\overline{M}^h$, or $\overline{M}_i^h$, it will be with respect to these basepoints.

Lemma 3.1. *For any $h \in \overline{M}^h$ and any $j \in \{1, \dots, m\}$, there exists a unique $h_j \in \overline{M}_j^h$, so that*

$$h(y) = \sum_{i=1}^{m} h_j(y_j) \quad \textit{for } y = (y_1, \dots, y_m) \in M. \tag{3.1}$$

Furthermore, if $h \in \partial \overline{M}^h$, then there exists a j_0, so that $h_{j_0} \in \partial \overline{M}_{j_0}^h$. If, instead, h is an internal point, then h_j is an internal point for M_j for all j.

Proof. There must exist a sequence $(x^n) = ((x_1^n, \dots, x_m^n)) \subseteq M$ such that $h_{x^n} \to h$. As $\overline{M}_i^h$ is compact, we can assume, after taking a subsequence, that $h_{x_i^n}$ converges to some $h_i \in \overline{M}_i$ for each $i \in \{1, \dots, m\}$. Thus, for every $y \in M$,

$$h(y) = \lim_{n\to\infty} d^1(y, x^n) - d^1(b, x^n) = \lim_{n\to\infty} \sum_{j=1}^m h_{x_j^n}(y_j) = \sum_{j=1}^m h_j(y_j).$$

To prove the second assertion assume by way of contradiction that $h \in \partial\overline{M}^h$, but $h_j \notin \partial\overline{M}_j^h$ for all $j \in \{1, \dots, m\}$. Then for each j there exists a $x_j \in M_j$ such that $h_j = h_{x_j}$. Set $x = (x_1, \dots, x_m) \in M$. Now for any $y \in M$, we get that $h(y) = \sum_{j=1}^m h_{x_j}(y_j) = \sum_{j=1}^m (d_j(y_j, x_j) - d_j(b_j, x_j)) = d^1(y, x) - d^1(b, x) = h_x(y)$, which contradicts the fact that h is a horofunction.

Now suppose that $h = h_x$ for some $x = (x_1, \dots, x_m) \in M$. Then for each $y = (y_1, \dots, y_m) \in M$ we have that $h(y) = d^1(y, x) - d^1(b, x) = \sum_{j=1}^m (d_j(y_j, x_j) - d_j(b_j, x_j)) = \sum_{j=1}^m h_{x_j}(y_j)$.

All that is left is to prove uniqueness. Suppose $h \in \overline{M}^h$ and $h = \sum_{i=1}^m h_i = \sum_{i=1}^m h'_i$ for some $h_i, h'_i \in \overline{M}_i^h$. Then for each i and $y_i \in M_i$ we consider $z \in M$ where $z_i = y_i$, and $z_j = b_j$ for all $j \neq i$. The above equation immediately gives that $h_i(y_i) = h'_i(y_i)$, which completes the proof. □

We thus see that every horofunction on M is of the form (3.1). Conversely:

Lemma 3.2. *If $h_j \in \overline{M}_j^h$ for $j \in \{1, \dots, m\}$ and $h\colon M \to \mathbb{R}$ is given by $h(y) = \sum_{i=1}^m h_j(y_j)$ for $y = (y_1, \dots, y_m) \in M$, then $h \in \overline{M}^h$. If, in addition, there exists a $j_0 \in \{1, \dots, m\}$ with $h_{j_0} \in \partial\overline{M}_{j_0}^h$, then $h \in \partial\overline{M}^h$.*

Proof. As each M_j is proper, there exists a sequence (x_j^n) in M_j such that $h_{x_j^n} \to h_j$. Let $x^n = (x_1^n, \dots, x_m^n) \in M$, then for each $y = (y_1, \dots, y_m) \in M$ and $n \in \mathbb{N}$ we have that

$$h_{x^n}(y) = \sum_{j=1}^m \big(d_j(y_j, x_j^n) - d_j(b_j, x_j^n)\big) = \sum_{j=1}^m h_{x_j^n}(y_j) \xrightarrow{n} h(y),$$

hence $h \in \overline{M}^h$.

Suppose there exists a $j_0 \in \{1, \dots, m\}$ such that $h_{j_0} \in \partial\overline{M}_{j_0}^h$. Using Proposition 2.3, we know that $d_{j_0}(b_{j_0}, x_{j_0}^n) \to \infty$. By the definition of the metric d^1, we get that $d^1(b, x^n) \to \infty$ as well, hence $h \in \partial\overline{M}^h$ by Proposition 2.3. □

Lemmas 3.1 and 3.2 immediately give Theorem 1.1. Theorem 1.1 has some immediate consequences.

Example 3.2. In [12], Guttiérez gives a complete description of the horoboundary compactification of $(\mathbb{R}^n, \|\cdot\|_1)$. Taking the basepoint $b = 0$ he showed that $h \in \partial\overline{(\mathbb{R}^n, \|\cdot\|_1)}^h$ if and only if $h = h_{\epsilon,z}^I$ where $\emptyset \neq I \subseteq \{1, \dots, n\}$, an $\epsilon \in \{1, -1\}^I$ and $z \in \mathbb{R}^I$, such that, for all $x \in \mathbb{R}^n$,

$$h_{\epsilon,z}^I(x) = \sum_{i\in I} \epsilon_i x_i + \sum_{j\notin I} |x_j - z_j| - |z_j|.$$

This, in fact, now follows immediately from Theorem 1.1 applied to $(M_i, d_i) = (\mathbb{R}, |\cdot|)$, as it can be easily checked that the horofunction boundary on $(\mathbb{R}, |\cdot|)$ consists only of the two maps: $x \mapsto x$ and $x \mapsto -x$.

We also get a topological characterisation of the horofunction compactification of (M,d) as a corollary of Theorem 1.1:

Corollary 3.3. *The map* $\varphi\colon \overline{M}^h \to \prod_{i=1}^m \overline{M}_i^h$ *defined by* $h \mapsto (h_1,\dots,h_m)$*, where* $h=\sum_{i=1}^m h_i \in \overline{M}^h$ *is given by (3.1), is a homeomorphism.*

Proof. Theorem 1.1 guarantees that φ is well defined and bijective. Both $\overline{M}^h$ and $\prod_{i=1}^m \overline{M}_i^h$ are metrisable, so it suffices to check the sequential continuity of φ. Let (h^n) be a sequence converging to some $h\in\overline{M}^h$ where $h^n=\sum_{i=1}^m h_i^n$. For any $n\in\mathbb{N}$, $\varphi(h^n)=(h_1^n,\dots,h_m^n)$, which provides, for each $j\in\{1,\dots,m\}$, a sequence $(h_j^n)\subseteq \overline{M}_j^h$. We also know that $\varphi(h)=(h_1,\dots,h_m)$, with each $h_j\in\overline{M}_j^h$. Furthermore, for any $i\in\{1,\dots,m\}$, and any $y_i\in M_i$, we can define a $z\in M$ by setting $z_i=y_i$, and $z_j=b_j$ for all $j\neq i$, which means

$$h_i^n(y_i)=h_i^n(y_i)+\sum_{j\neq i}h_j^n(b_j)=h^n(z)\to h(z)=h_i(y_i)+\sum_{j\neq i}h_j(b_j)=h_i(y_i).$$

Thus, each (h_i^n) converges to h_i in $\overline{M}_i^h$, so $\varphi(h^n)$ converges to $\varphi(h)$ in $\prod_{i=j}^m \overline{M}_j^h$. Therefore, φ is a continuous bijection between compact Hausdorff spaces, hence φ is a homeomorphism. □

4. The Busemann Points and Boundary Structure of (M,d^1)

We are now in a position to prove Theorem 1.2 and classify the Busemann points.

Proof of Theorem 1.2. Let $h=\sum_{i=1}^m h_i\in\overline{M}^h$. For each $i\in\{1,\dots,m\}$ with $h_i\in\partial_B\overline{M}^h$, there exists an almost-geodesic sequence (x_i^n) in M_i such that $h_{x_i^n}$ converges to h_i. For all other i there exists $x_i\in M_i$ such that $h_i=h_{x_i}$. Now let $x^n=(x_1^n,\dots,x_m^n)$ and note that $d^1(b,x^n)\to\infty$, as $d_j(b_j,x_j^n)\to\infty$. It follows from the definition of the ℓ^1-metric that (x^n) is an almost-geodesic sequence in M. As (h_{x^n}) converges to h, h is a Busemann point.

Conversely, suppose that h is a Busemann point and that $h_{x^n}\to h$, where (x^n) is an almost-geodesic sequence in M. Fix an $\varepsilon>0$ and write $x^n=(x_1^n,\dots,x_m^n)$. Note that, by Proposition 2.3, there exists j such that $d_j(b_j,x_j^n)\to\infty$. By definition there exists an L, so that

$$\sum_{i=1}^m d_i(b_i,x_i^r)\ge\sum_{i=1}^m d_i(b_i,x_i^s)+d_i(x_i^r,x_i^s)-\varepsilon\quad\text{for all } r\ge s\ge L.$$

For any $k\in\{1,\dots,m\}$, we can subtract $\sum_{i\neq k}d_i(b_i,x_i^r)$ from both sides in the above inequality to get

$$\begin{aligned}d_k(b_k,x_k^r)&\ge d_k(b_k,x_k^s)+d_k(x_k^r,x_k^s)+\sum_{i\neq k}^m\left(d_i(b_i,x_i^s)+d_i(x_i^r,x_i^s)-d_i(b_i,x_i^r)\right)-\varepsilon\\&\ge d_k(b_k,x_k^s)+d_k(x_k^r,x_k^s)-\varepsilon.\end{aligned}$$

This implies that the sequence (x_k^n) is an almost geodesic for each k with $d_k(b_k,x_k^n)\to\infty$, hence $h_{x_k^n}$ converges to a Busemann point h_k of $\partial\overline{M}_k^h$ for those k. If $d_k(b_k,x_k^n)$ remains bounded, then we claim that (x_k^n) is a Cauchy sequence in (M_k,d_k). To see this we first show that $d_k(b_k,x_k^n)$ converges to $\Delta=\limsup_n d_k(b_k,x_k^n)$. Indeed, if we let $\epsilon>0$ be given, then there exists an s such that $\Delta-d_k(b_k,x_k^s)<\epsilon$, and for all $r\ge s$ we have that $\Delta+\epsilon>d_k(b_k,x_k^r)$ and $d_k(b_k,x_k^r)\ge d_k(b_k,x_k^s)+d_k(x_k^r,x_k^s)-\epsilon$. Now note that for $r\ge s$ we have that

$$-\epsilon<\Delta-d_k(b_k,x_k^r)\le\Delta-(d_k(b_k,x_k^s)+d_k(x_k^r,x_k^s)-\epsilon)\le\Delta-d_k(b_k,x_k^s)+\epsilon<2\epsilon,$$

which gives the desired conclusion. Thus for all $r \geq s$ we have that

$$d_k(x_k^r, x_k^s) \leq d_k(b_k, x_k^r) - d_k(b_k, x_k^s) + \epsilon \leq 2\epsilon,$$

which implies that (x_k^n) is a Cauchy sequence. As (M_k, d_k) is complete, (x_k^n) converges to some x_k, hence $h_{x_k^n}$ converges to the internal point h_{x_k}. $\square$

The above lemma means that, for any $h \in \partial_B \overline{M}^h$, there exists a $\emptyset \neq I \subseteq \{1, \ldots, m\}$ and a decomposition

$$h = \sum_{i \in I} h_i + \sum_{j \notin I} h_{x_j},$$

where h_i is a Busemann point for each $i \in I$.

Theorem 4.1. *Two Busemann points*

$$h = \sum_{i \in I} h_i + \sum_{j \notin I} h_{x_j} \quad \textit{and} \quad h' = \sum_{i \in I'} h_i' + \sum_{j \notin I'} h_{y_j}$$

are in the same part of the Busemann boundary if and only if $I' = I$ and, for each $i \in I$, the Busemann points h_i and h_i' are in the same part. Furthermore, if h and h' are Busemann points in the same part, then the detour distance between them is given by

$$\delta(h, h') = \sum_{i \in I} \delta(h_i, h_i') + 2 \sum_{j \notin I} d_j(x_j, y_j).$$

Proof. First assume we are given two Busemann points

$$h = \sum_{i \in I} h_i' + \sum_{j \notin I} h_{x_j} \quad \text{and} \quad h' = \sum_{i \in I'} h_i' + \sum_{j \notin I'} h_{y_j}, \tag{4.1}$$

where $I' = I$ and for each $i \in I$ the Busemann point h_i and h_i' are in the same part. Let (x^n) be an almost-geodesic sequence such that $h_{x^n} \to h$, and (y^n) an almost geodesic sequence such that $h_{y^n} \to h'$. The proof of Theorem 1.2 shows that, for all $i \in I$, the sequences (x_i^n) and (y_i^n) are also almost-geodesic sequences in M_i, and for all $j \notin I$ we have that $x_j^n \to x_j$ and $y_j^n \to y_j$. By Lemma 2.4 we know that

$$\begin{aligned} H(h, h') &= \lim_{n \to \infty} d^1(b, x^n) + h'(x^n) \\ &= \lim_{n \to \infty} \sum_{i \in I} d_i(b_i, x_i^n) + h_i'(x_i^n) + \sum_{j \notin I} d_j(b_j, x_j^n) + h_{y_j}(x_j^n) \\ &= \sum_{i \in I} H(h_i, h_i') + \sum_{j \notin I} d_j(b_j, x_j) + h_{y_j}(x_j) < \infty, \end{aligned}$$

where the last inequality follows from the assumption that h_i and h_i' are in the same part of the boundary for each $i \in I$. A symmetrical argument shows similarly that $H(h', h) < \infty$, so we conclude that h and h' lie in the same part of the horofunction boundary.

Conversely, assume that h and h' are of the form (4.1), and lie in the same part. Thus $H(h, h') < \infty$ and $H(h', h) < \infty$. We can assume that $h_{x^n} \to h$ and $h_{y^n} \to h'$ for almost-geodesic sequences (x^n) and (y^n). The proof of Theorem 1.2 shows that (x_i^n) and (y_i^n) are almost-geodesic sequences for $i \in I$. Moreover, $x_j^n \to x_j$ and $y_j^n \to y_j$ for $j \notin I$. Using Lemma 2.4,

$$H(h,h') = \lim_{n\to\infty} \sum_{i\in I\cap I'} \big(d_i(b_i,x_i^n) + h_i'(x_i^n)\big) + \sum_{j\in I\setminus I'} \big(d_j(b_j,x_j^n) + h_{y_j}(x_j^n)\big) + \sum_{i\in I'\setminus I} \big(d_i(b_i,x_i^n) + h_i'(x_i^n)\big) + \sum_{j\notin I\cup I'} \big(d_j(b_j,x_j^n) + h_{y_j}(x_j^n)\big).$$

The sum of the two terms in each of the above four sums is non-negative for all indices, so for $H(h,h')$ to be finite it is required that the limit of each sum of two terms within the sums is finite. For $i \notin I$, (x_i^n) is bounded, so we need only be concerned with instances where $i \in I$. In particular, we require that

$$\sum_{j\in I\setminus I'} d_j(b_j,x_j^n) + h_{y_j}(x_j^n) < \infty,$$

but $h_{y_j}(x_j^n) \geq -d_j(b_j,y_j)$ for each $j \in I\setminus I'$, whereas $d_j(b_j,x_j^n) \xrightarrow{n} \infty$, so this is only possible if $I \subseteq I'$. A symmetrical argument applied to $H(h',h)$ shows that $I' \subseteq I$, meaning that $I = I'$. Thus,

$$H(h,h') = \lim_{n\to\infty} \sum_{i\in I} d_i(b_i,x_i^n) + h_i'(x_i^n) + \sum_{j\notin I} d_j(b_j,x_j^n) + h_{y_j}(x_j^n), \quad \text{and}$$

$$H(h',h) = \lim_{n\to\infty} \sum_{i\in I} d_i(b_i,y_i^n) + h_i(y_i^n) + \sum_{j\notin I} d_j(b_j,y_j^n) + h_{x_j}(y_j^n).$$

Using Lemma 2.4 again we find that

$$\delta(h,h') = \sum_{i\in I} \delta(h_i,h_i') + \sum_{j\notin I} \big(d_j(b_j,x_j) + h_{y_j}(x_j) + d_j(b_j,y_j) + h_{x_j}(y_j)\big)$$
$$= \sum_{i\in I} \delta(h_i,h_i') + 2\sum_{j\notin I} d(x_j,y_j),$$

which is finite if and only if $\delta(h_i,h_i')$ is finite for each $i \in I$, completing the proof.

□

Example 4.2. The two horofunctions of $(\mathbb{R},|\cdot|)$ are both Busemann points. Theorem 1.2 thus allows us to immediately conclude that the horofunctions of $\ell^1(n,\mathbb{R})$ as described in Example 3.2 are all Busemann points. Furthermore, we see from Theorem 4.1 that two horofunctions $h^{I^1}_{\epsilon^1,z^1}, h^{I^2}_{\epsilon^2,z^2} \in \partial\overline{(\mathbb{R}^n,\|\cdot\|_1)}^h$ are in the same part if and only if $I^1 = I^2$ and $\epsilon^1 = \epsilon^2$, and for two such horofunctions we see that

$$\delta(h^I_{\epsilon,z^1}, h^I_{\epsilon,z^2}) = 2\sum_{j\notin I} |z_j^1 - z_j^2|,$$

which is interesting, because we see that the parts of the boundary inherit the metric structure of the original space.

Example 4.3. Let us now consider an example from the field of several complex variables. Recall that, for any convex domain $D \subseteq \mathbb{C}^n$, the *Kobayashi distance*, κ_D, is defined as follows: if Δ is the open disk in $\mathbb{C}$, ρ is the hyperbolic metric on Δ, and $\mathrm{hol}(\Delta,D)$ is the set of holomorphic functions from Δ to D, then

$$\kappa_D(w,z) = \inf\{\rho(\zeta,\eta) : \exists\ f \in \mathrm{hol}(\Delta,D) \text{ with } f(\zeta) = w, \text{ and } f(\eta) = z\}.$$

Recall that we can express ρ as

$$\rho(w,z) = 2\tanh^{-1}\left(1 - \frac{(1-|w|^2)(1-|z|^2)}{|1-w\overline{z}|^2}\right).$$

If D is a bounded convex domain it is known that (D,κ_D) is a proper geodesic metric space [1]. On the open Euclidean ball $B^n \subseteq \mathbb{C}^n$ it is known that

$$\kappa_{B^n}(w,z) = 2\tanh^{-1}\left(1 - \frac{(1-\|w\|^2)(1-\|z\|^2)}{|1-\langle w,z\rangle|^2}\right),$$

and the horofunction compactification of (B^n,κ_{B^n}), with basepoint 0, is known to consist entirely of the functions h^ξ, for $\xi \in \partial B^n$, where

$$h^\xi(z) = \log\frac{|1-\langle z,\xi\rangle|^2}{1-\|z\|^2}.$$

Each horofunction is a Busemann point, as it arises as the limit of the geodesic $t \mapsto \frac{e^t-1}{e^t+1}\xi$. If $B = \prod_{i=1}^m B^{n_i}$ is a product of open balls, the product property [20, Theorem 3.1.9] tells us that, for any $w=(w_1,\dots,w_m)\in B$ and $z=(z_1,\dots,z_m)\in B$,

$$\kappa_B(w,z) = \max_{i\in\{1,\dots,m\}} \kappa_{B^{n_i}}(w_i,z_i).$$

Instead of the Kobayashi metric on $B = \prod_{i=1}^m B^{n_i}$ we can consider the ℓ_1-metric,

$$d^1(w,z) = \sum_{i=1}^m \kappa_{B^{n_i}}(w_i,z_i).$$

Theorems 1.1 and 1.2 immediately show that each horofunction, h, of (B,d^1) is a Busemann point, and there exists a non-empty $I \subseteq \{1,\dots m\}$, $\xi_i \in \partial B^{n_i}$ for each $i\in I$, and $z_j \in B^{n_j}$ for each $j\notin I$, such that for all $w\in B$,

$$h(w) = \sum_{i\in I} h^{\xi_i}(w_i) + \sum_{j\notin I} h_{z_j}(w_j). \tag{4.4}$$

Moreover, if $h' \in \partial\overline{B}^h$ is a horofunction given by

$$h'(w) = \sum_{i\in I'} h^{\xi'_i}(w_i) + \sum_{j\notin I'} h_{z'_j}(w_j),$$

then h' is in the same part as h if, and only if, $I=I'$ and $\xi_i=\xi'_i$ for all $i\in I$ by Theorem 4.1. In that case, the detour distance is given by

$$\delta(h,h') = 2\sum_{i\notin I} \kappa_{B^{n_i}}(z_i,z'_i).$$

It is interesting to note that $\overline{B}^h$ is homeomorphic to $\mathrm{cl}(B)$. Indeed, it can be shown that $\Psi\colon \overline{B}^h \to \mathrm{cl}(B)$ given by $\Psi(h)_i=\xi_i$ for $i\in I$ and $\Psi(h)_j = z_j$ for $j\notin I$, where h is given by (4.4), is a homeomorphism. It maps the set of internal points onto B and each part of the horofunction boundary onto the relative interior of a boundary face of $\mathrm{cl}(B)$. So, in this case the horofunction compactification coincides with the bounded symmetric domain compactification of B.

5. ℓ^1-sums of normed spaces

There has recently been interest in understanding the global topology and geometry of the horofunction compactification of finite dimensional normed spaces, which appears to be connected to the closed unit ball in the dual normed space. In [16, Question 6.18] Kapovich and Leeb asked the question whether, for each finite dimensional normed space $(X, \|\cdot\|)$ the horofunction compactification $\overline{X}^h$ is homeomorphic to the closed unit ball in the dual space, where the homeomorphism maps horofunctions $h \sim h'$ (see (2.2)) into the relative interior of the same boundary face of the dual ball and the internal point onto the interior of the dual ball. Recall that in a finite dimensional normed vector space, the closed unit ball can be partitioned into the disjoint collection of the relative interior of each of its non-empty faces [31, Theorem 18.2].

Ji and Schilling answered this question affirmatively in [15] for all normed spaces with a polyhedral unit ball. In addition, Schilling showed in [32] that such homeomorphisms exist for all finite dimensional normed spaces where the norm is smooth and uniformly convex, and for all two dimensional normed spaces where the unit ball only has finitely many connected components that are smooth. Further evidence was provided in [7, 22, 25] where finite dimensional normed spaces associated to Jordan algebras are considered. It should, however, be noted that in all these cases each horofunction is a Busemann point. It could well be the case that the relation with the dual ball may fail if there are non-Busemann points. A characterisation of the finite dimensional normed spaces for which all horofunctions are Busemann points was given by Walsh in [33], which contains an explicit example with non-Busemann points.

The duality phenomenon is also known to exist for a variety of symmetric spaces of non-compact type equipped with an invariant Finsler metric [7, 22, 25]. In that setting the horofunction compactification is homeomorphic to the closed unit ball of the dual of the Finsler norm in the tangent space at the basepoint.

The results of this paper allow us to say something about the duality phenomenon in ℓ^1-sums of finite dimensional normed spaces. More specifically we have the following result.

Theorem 5.1. *Let $(X_i, \|\cdot\|_i)$ be a finite dimensional normed space for $i \in \{1, \ldots, m\}$. Suppose for each i that all horofunctions of $\overline{X_i}^h$ are Busemann points, and that there exists a homeomorphism, $\psi_i \colon \overline{X}_i^h \to B_{X_i^*}$, mapping parts of the horofunction boundary onto the relative interior of the boundary faces of the closed dual unit ball $B_{X_i^*}$, and the set of internal points onto the interior of $B_{X_i^*}$.*

If $X = \prod_{i=1}^m X_i$ is the ℓ^1-product, so $\|x\| = \sum_{i=1}^m \|x_i\|_i$, then there exists a homeomorphism $\Psi \colon \overline{X}^h \to B_{X^*}$ that maps the parts onto the relative interiors of the boundary faces of the closed dual unit ball B_{X^*}, and the set of internal points onto the interior of B_{X^*}.

Proof. It is well known that the dual of $(X, \|\cdot\|)$ is the space $(X^*, \|\cdot\|_\infty)$, where $X^* = \prod_{i=1}^m X_i^*$, and $\|(x_1^*, \ldots, x_m^*)\|_\infty = \max_{i\in\{1,\ldots,m\}} \|x_i^*\|_i^*$, and has the unit ball $B_{X^*} = \prod_{j=1}^m B_{X_j^*}$. Define $\Psi \colon \overline{X}^h \to B_X^*$ using the homeomorphism $\varphi \colon \overline{X}^h \to \prod_{i=1}^m \overline{X}_i^h$ given in Corollary 3.3 as follows:

$$\Psi(h) = (\psi_1(\varphi(h)_1), \ldots, \psi_m(\varphi(h)_m)).$$

As φ is a homeomorphism, and each ψ_i is a homeomorphism onto $B_{X_i^*}$, Ψ is a homeomorphism onto B_{X^*}. Every face $F \subseteq B_{X^*}$ is of the form $\prod_{i=1}^m F_i$, where each $F_i \subseteq X_i$ is a face of $B_{X_i^*}$. Furthermore, it follows from [31, Theorem 6.4] that $\mathrm{relint}(F) = \prod_{i=1}^m \mathrm{relint}(F_i)$.

To show that Ψ maps parts onto the relative interiors of the boundary faces of B_{X^*} let h and h' be Busemann points in the same part of $\overline{X}^h$. As a consequence of Theorem 4.1, there exists an $I \subseteq \{1,\dots,m\}$ nonempty, $h_i, h_i' \in \partial\overline{X}_i^h$ with h_i and h_i' are in the same part for all $i \in I$, and $x_j, y_j \in X_j$ for all $j \notin I$, such that

$$\Psi(h)_i = \begin{bmatrix} \psi_i(h_i) & \text{for } i \in I \\ \psi_i(h_{x_i}) & \text{otherwise} \end{bmatrix} \quad \text{and} \quad \Psi(h')_i = \begin{bmatrix} \psi_i(h_i') & \text{for } i \in I \\ \psi_i(h_{y_i}) & \text{otherwise.} \end{bmatrix}$$

By assumption, each coordinate homeomorphism ψ_i maps each part of the boundary $\partial\overline{X}_i^h$ bijectively onto the relative interior of a proper face F_i of $B_{X_i^*}$, and maps the set of internal points bijectively onto the interior of $B_{X_i^*}$. Thus, $\Psi(h), \Psi(h') \in \prod_{i=1}^m \mathrm{relint}(F_i)$, where F_i is a proper face of $B_{X_i^*}$ for all $i \in I$, and $F_i = B_{X_i^*}$ for all $i \notin I$, hence Ψ maps elements of the same part into the relative interior of the same face.

Conversely, if h and h' are Busemann points in different parts of $\overline{X}^h$, then there exists an i such that $\Psi(h)_i$ and $\Psi(h')_i$ lie in the relative interiors of different faces of $B_{X_i^*}$, hence $\Psi(h)$ and $\Psi(h')$ lie in the relative interiors of different faces of B_{X^*}. To complete the proof note that Ψ maps the set of internal points onto the interior of B_{X^*}. □

Combining Theorem 5.1 with the results from [15, 32] we see that the horofunction compactification of any finite ℓ^1-product of finite dimensional normed spaces with smooth or polyhedral norms is homeomorphic to the closed unit ball of the dual space. We point out that this is a different setting to the one studied by Schilling [32], where finite dimensional normed spaces $(X, \|\cdot\|)$ are considered with $\|x\| = \|x\|_s + \|x\|_p$, with $\|\cdot\|_s$ a smooth norm and $\|\cdot\|_p$ a polyhedral norm. Our result covers the product space $X \times X$ with norm $\|(x,y)\| = \|x\|_s + \|y\|_p$.

As an example we consider $X = \mathbb{R}^2 \times \mathbb{R}$ with norm $\|(x,y,z)\| = \sqrt{x^2+y^2} + |z|$, making X the ℓ^1-sum of the Euclidean plane and the real line. The unit ball is the bicone as depicted in Figure 1. The dual space $(X^*, \|\cdot\|_*)$ has norm $\|(x,y,z)\|_* = \max\{\sqrt{x^2+y^2}, |z|\}$, and B_{X^*} is the cylinder shown in Figure 2.

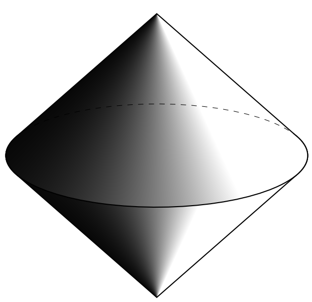

FIGURE 1. The closed unit ball of X.

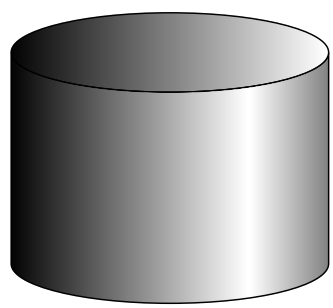

FIGURE 2. The closed unit ball of X^*.

It is well known that the horocompactification of $\mathbb{R}^2$ with the Euclidean norm consists of all the functions h^z, for z in the unit sphere of $\mathbb{R}^2$, where $h^z(x) = -\langle x, z\rangle$ for all $x \in \mathbb{R}^2$, and the horofunction compactification of $\mathbb{R}$ consists solely of the functions h^ϵ, where $\epsilon \in \{-1,1\}$, and $h^\epsilon(x) = -\epsilon x$. It is also known that all of these horofunctions are Busemann points of their respective spaces and each part is a singleton. Each point in the boundary of the Euclidean unit balls in $\mathbb{R}^2$ and in $\mathbb{R}$ is a face, as each point is an extreme point. In this case the maps $\psi_1 \colon \overline{\mathbb{R}^2}^h \to B_{\mathbb{R}^2}$ and

$\psi_2 \colon \overline{\mathbb{R}}^h \to [-1,1]$, defined by

$$\psi_1(h) = \left[\begin{array}{ll} -\tanh(\|x\|_2)\frac{x}{\|x\|_2} & \text{if } h = h_x \text{ for } x \in \mathbb{R}^2\backslash\{0\} \\ 0 & \text{if } h = h_0 \\ z & \text{if } h = h^z \end{array} \right.$$

and

$$\psi_2(h) = \left[\begin{array}{ll} -\tanh(x) & \text{if } h = h_x \text{ for } x \in \mathbb{R} \\ \epsilon & \text{if } h = h^\epsilon. \end{array} \right.$$

are homeomorphisms mapping parts of the horofunction boundaries onto the relative interiors of the faces of the respective dual balls. Theorem 5.1 thus tells us that $\overline{X}^h$ is homeomorphic to B_{X^*}, and the parts of the boundary can be identified precisely with the relative interiors of the faces of the cylinder in Figure 2.

To conclude the paper, we note that in [21] a partial characterisation of the horofunction compactification of the ℓ^∞-product of proper geodesic metric spaces was given. The results of this paper completely characterise the horofunction compactification of the ℓ^1-products of proper geodesic metric spaces. It would be interesting to know whether a similar characterisation exists for other ℓ^p-products for $1 < p < \infty$. Unfortunately, the techniques used here rely heavily on linear structure of the ℓ^1-metric, and there is no direct way to generalise the arguments to the case $1 < p < \infty$. In fact, it is not even clear if there is a simple way to express the horofunction compactification of an ℓ^2-product of finite dimensional normed spaces in terms of the horofunctions of the normed spaces making up the product.

References

[1] Marco Abate. *Iteration theory of holomorphic maps on taut manifolds.* Research and Lecture Notes in Mathematics. Complex Analysis and Geometry. Mediterranean Press, Rende, 1989.

[2] Zoltán M. Balogh and Mario Bonk. Gromov hyperbolicity and the Kobayashi metric on strictly pseudoconvex domains. *Comment. Math. Helv.*, 75(3):504–533, 2000.

[3] A. F. Beardon. The dynamics of contractions. *Ergodic Theory Dynam. Systems*, 17(6):1257–1266, 1997.

[4] Filippo Bracci and Hervé Gaussier. Horosphere topology. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 20(1):239–289, 2020.

[5] Filippo Bracci, Hervé Gaussier, and Andrew Zimmer. Homeomorphic extension of quasi-isometries for convex domains in $\mathbb{C}^d$ and iteration theory. *Math. Ann.*, 379(1-2):691–718, 2021.

[6] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, Berlin, 1999.

[7] Cho-Ho Chu, María Cueto-Avellaneda, and Bas Lemmens. Horofunctions and metric compactification of noncompact hermitian symmetric spaces. *Ann. Mat. Pura Appl. (4)*, 203(4):1703–1751, 2024.

[8] Moon Duchin and Nate Fisher. Stars at infinity in Teichmüller space. *Geom. Dedicata*, 213:531–545, 2021.

[9] Sébastien Gouëzel and Anders Karlsson. Subadditive and multiplicative ergodic theorems. *J. Eur. Math. Soc. (JEMS)*, 22(6):1893–1915, 2020.

[10] Mark Greenfield and Lizhen Ji. Metrics and compactifications of Teichmüller spaces of flat tori. *Asian J. Math.*, 25(4):477–504, 2021.

[11] M. Gromov. *Hyperbolic Manifolds, Groups and Actions*, pages 183–214. Princeton University Press, Princeton, 1981.

[12] Armando W. Gutiérrez. The horofunction boundary of finite-dimensional ℓ_p spaces. *Colloq. Math.*, 155(1):51–65, 2019.

[13] Armando W. Gutiérrez. On the metric compactification of infinite-dimensional ℓ_p spaces. *Canad. Math. Bull.*, 62(3):491–507, 2019.

[14] Thomas Haettel, Anna-Sofie Schilling, Cormac Walsh, and Anna Wienhard. Horofunction compactifications of symmetric spaces. arXiv:1705.05026, 2018.

[15] Lizhen Ji and Anna-Sofie Schilling. Toric varieties vs. horofunction compactifications of polyhedral norms. *Enseign. Math.*, 63(3-4):375–401, 2017.

[16] Michael Kapovich and Bernhard Leeb. Finsler bordifications of symmetric and certain locally symmetric spaces. *Geom. Topol.*, 22(5):2533–2646, 2018.

[17] Anders Karlsson. Non-expanding maps and Busemann functions. *Ergodic Theory Dynam. Systems*, 21(5):1447–1457, 2001.

[18] Anders Karlsson. Two extensions of Thurston's spectral theorem for surface diffeomorphisms. *Bull. Lond. Math. Soc.*, 46(2):217–226, 2014.

[19] John L. Kelley. *General topology*, volume No. 27 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1975. Reprint of the 1955 edition [Van Nostrand, Toronto, Ont.].

[20] Shoshichi Kobayashi. *Hyperbolic complex spaces*, volume 318 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, Berlin, 1998.

[21] Bas Lemmens. A metric version of Poincaré's theorem concerning biholomorphic inequivalence of domains. *J. Geom. Anal.*, 32(5):Paper No. 160, 20, 2022.

[22] Bas Lemmens. Horofunction compactifications of symmetric cones under Finsler distances. *Ann. Fenn. Math.*, 48(2):729–756, 2023.

[23] Bas Lemmens, Brian Lins, and Roger Nussbaum. Detecting fixed points of nonexpansive maps by illuminating the unit ball. *Israel J. Math.*, 224(1):231–262, 2018.

[24] Bas Lemmens, Brian Lins, Roger Nussbaum, and Marten Wortel. Denjoy-Wolff theorems for Hilbert's and Thompson's metric spaces. *J. Anal. Math.*, 134(2):671–718, 2018.

[25] Bas Lemmens and Kieran Power. Horofunction compactifications and duality. *J. Geom. Anal.*, 33(5):Paper No. 154, 57, 2023.

[26] Bas Lemmens, Mark Roelands, and Marten Wortel. Hilbert and Thompson isometries on cones in JB-algebras. *Math. Z.*, 292(3-4):1511–1547, 2019.

[27] Bas Lemmens and Cormac Walsh. Isometries of polyhedral Hilbert geometries. *J. Topol. Anal.*, 3(2):213–241, 2011.

[28] Hideki Miyachi. Extremal length boundary of the Teichmüller space contains non-Busemann points. *Trans. Amer. Math. Soc.*, 366(10):5409–5430, 2014.

[29] Athanase Papadopoulos. *Metric spaces, convexity and nonpositive curvature*, volume 6 of *IRMA Lectures in Mathematics and Theoretical Physics*. European Mathematical Society (EMS), Zürich, 2005.

[30] Marc A. Rieffel. Group C^*-algebras as compact quantum metric spaces. *Doc. Math.*, 7:605–651, 2002.

[31] R. Tyrrell Rockafellar. *Convex analysis*, volume No. 28 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1970.

[32] Anna-Sofie Schilling. *The Horofunction Compactification of Finite-Dimensional Normed Vector Spaces and of Symmetric Spaces*. PhD thesis, Heidelberg Germany, 1996.

[33] Cormac Walsh. The horofunction boundary of finite-dimensional normed spaces. *Math. Proc. Cambridge Philos. Soc.*, 142(3):497–507, 2007.

[34] Cormac Walsh. The horoboundary and isometry group of Thurston's Lipschitz metric. In *Handbook of Teichmüller theory. Vol. IV*, volume 19 of *IRMA Lect. Math. Theor. Phys.*, pages 327–353. Eur. Math. Soc., Zürich, 2014.

[35] Cormac Walsh. Gauge-reversing maps on cones, and Hilbert and Thompson isometries. *Geom. Topol.*, 22(1):55–104, 2018.

[36] Cormac Walsh. Hilbert and Thompson geometries isometric to infinite-dimensional Banach spaces. *Ann. Inst. Fourier (Grenoble)*, 68(5):1831–1877, 2018.

[37] Andrew M. Zimmer. Gromov hyperbolicity and the Kobayashi metric on convex domains of finite type. *Math. Ann.*, 365(3-4):1425–1498, 2016.