

The automorphism group of the quantum grassmannian *

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Abstract

The automorphism group of a quantised coordinate algebra is usually much smaller than that of its classical counterpart. Nevertheless, these automorphism groups are often very difficult to calculate. In this paper, we calculate the automorphism group of the quantum grassmannian in the case that the deformation parameter is not a root of unity. The main tool employed is the dehomogenisation equality which shows that a localisation of the quantum grassmannian is equal to a skew Laurent extension of quantum matrices. This equality is used to connect the automorphism group of the quantum grassmannian with that of quantum matrices, where the automorphism group is known.

1 Introduction

The quantum grassmannian $\mathcal{O}_q(G(k, n))$ is a noncommutative algebra that is a deformation of the homogeneous coordinate ring of the classical grassmannian of k -planes in n -space. In this paper, we calculate the automorphism group of the quantum grassmannian in the generic case where the deformation parameter q is not a root of unity.

Typically, quantised coordinate algebras are much more rigid than their classical counterparts, in the sense that the automorphism group of the quantum object is much smaller than that of the classical object. Nevertheless, it has proven difficult to calculate these automorphism groups and only a few examples are known where the calculation has been completed, see, for example, [1, 6, 9, 10, 11, 17, 18]. The automorphism group of quantum matrices [10, 17] will prove crucial in our present work.

The quantum grassmannian $\mathcal{O}_q(G(k, n))$ is generated as an algebra by the $k \times k$ quantum minors of the quantum matrix algebra $\mathcal{O}_q(M(k, n))$. These generators are called quantum Plücker coordinates and there is a natural partial order on the quantum Plücker coordinates which is illustrated in the case of $\mathcal{O}_q(G(3, 6))$ in Figure 1. There are two obvious sources of automorphisms for $\mathcal{O}_q(G(k, n))$. The first is by restricting column automorphisms of $\mathcal{O}_q(M(k, n))$ to the subalgebra $\mathcal{O}_q(G(k, n))$, these automorphisms are described in Section 4. The second is via studying certain automorphisms of the (noncommutative) dehomogenisation of $\mathcal{O}_q(G(k, n))$ which is isomorphic to a skew Laurent extension of $\mathcal{O}_q(M(k, p))$, with $p = n - k$, as we shall see in Section 3.

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In Section 4 we study these “obvious” automorphisms of $\mathcal{O}_q(G(k, n))$ and consider the relations between them. We then claim that these provide all of the automorphisms of $\mathcal{O}_q(G(k, n))$, and justify the claim in the following sections. More precisely, we show the following result:

Theorem. 1. The automorphism group $\text{Aut}(\mathcal{O}_q(G(k, n)))$ of $\mathcal{O}_q(G(k, n))$ is isomorphic to $(K^*)^n$ when $2k \neq n$.

2. Assume $2k = n$. There is an automorphism τ of $\mathcal{O}_q(G(k, n))$ which sends the quantum Plücker coordinate $[I]$ to $[w_0(\widehat{I})]$, where \widehat{I} is the complement of I in $\{1, \dots, n\}$ and w_0 is the longest permutation of $\{1, \dots, n\}$. Moreover, the automorphism group $\text{Aut}(\mathcal{O}_q(G(k, n)))$ of $\mathcal{O}_q(G(k, n))$ is isomorphic to $(K^*)^n \rtimes \langle \tau \rangle$.

The quantum grassmannian carries the structure of an \mathbb{N} -graded algebra, with each quantum Plücker coordinate having degree one. In Section 5, we exploit this grading in a series of lemmas to see that we can essentially fix the minimal and maximal elements in the poset after allowing adjustment by the automorphisms that we have found in Section 4.

In Section 6 we study these adjusted automorphisms and show that such an automorphism induces, via the dehomogenisation equality, an automorphism of $\mathcal{O}_q(M(k, n))$. Once this has been done, our main result follows easily in Section 7 from the known structure of the automorphism group of quantum matrices.

2 Basic definitions

Throughout the paper, we work with a field K and a nonzero element $q \in K$ which is not a root of unity.

The algebra of $m \times n$ quantum matrices over K , denoted by $\mathcal{O}_q(M(m, n))$, is the algebra generated over K by mn indeterminates x_{ij} , with $1 \leq i \leq m$ and $1 \leq j \leq n$, which commute with the elements of K and are subject to the relations:

$$\begin{aligned} x_{ij}x_{il} &= qx_{il}x_{ij}, & \text{for } 1 \leq i \leq m, \text{ and } 1 \leq j < l \leq n; \\ x_{ij}x_{kj} &= qx_{kj}x_{ij}, & \text{for } 1 \leq i < k \leq m, \text{ and } 1 \leq j \leq n; \\ x_{ij}x_{kl} &= x_{kl}x_{ij}, & \text{for } 1 \leq k < i \leq m, \text{ and } 1 \leq j < l \leq n; \\ x_{ij}x_{kl} - x_{kl}x_{ij} &= (q - q^{-1})x_{il}x_{kj}, & \text{for } 1 \leq i < k \leq m, \text{ and } 1 \leq j < l \leq n. \end{aligned}$$

It is well known that $\mathcal{O}_q(M(m, n))$ is an iterated Ore extension over K with the x_{ij} added in lexicographic order. An immediate consequence is that $\mathcal{O}_q(M(m, n))$ is a noetherian domain.

When $m = n$, the *quantum determinant* D_q is defined by;

$$D_q := \sum (-q)^{l(\sigma)} x_{1\sigma(1)} \cdots x_{n\sigma(n)},$$

where the sum is over all permutations σ of $\{1, \dots, n\}$ and $l(\sigma)$ denotes the length of the permutation σ . The quantum determinant is a central element in the algebra of quantum matrices $\mathcal{O}_q(M(n, n))$.

If I and J are t -element subsets of $\{1, \dots, m\}$ and $\{1, \dots, n\}$, respectively, then the *quantum minor* $[I \mid J]$ is defined to be the quantum determinant of the $t \times t$ quantum matrix subalgebra generated by the variables x_{ij} with $i \in I$ and $j \in J$.

Assume that $n \geq k$. The *homogeneous coordinate ring of the $k \times n$ quantum grassmannian*, $\mathcal{O}_q(G(k, n))$ (informally known as the *quantum grassmannian*) is the subalgebra of $\mathcal{O}_q(M(k, n))$ generated by the $k \times k$ quantum minors of $\mathcal{O}_q(M(k, n))$, see, for example, [7].

The quantum grassmannian $\mathcal{O}_q(G(1, n))$ is a quantum affine space, and, as such, its automorphism group is known, see [1]; so we will assume throughout this paper that $k > 1$. Also, we will see in Proposition 3.4 that $\mathcal{O}_q(G(k, n)) \cong \mathcal{O}_q(G(n - k, n))$, so in calculating the automorphism group we will assume that $2k \leq n$ (and so $n \geq 4$, as $k \geq 2$).

A $k \times k$ quantum minor of $\mathcal{O}_q(M(k, n))$ must use all of the k rows, and so we can specify the quantum minor by specifying the columns that define it. With this in mind, we will write $[J]$ for the quantum minor $[1, \dots, k \mid J]$, for any k -element subset J of $\{1, \dots, n\}$. Quantum minors of this type are called *quantum Plücker coordinates*. The set of quantum Plücker coordinates in $\mathcal{O}_q(G(k, n))$ is denoted by Π . There is a natural partial order on Π defined in the following way: if $I = [i_1 < \dots < i_k]$ and $J = [j_1 < \dots < j_k]$ then $[I] \leq [J]$ if and only if $i_l \leq j_l$ for each $l = 1, \dots, k$. This partial order is illustrated for the case of $\mathcal{O}_q(G(3, 6))$ in Figure 1. A *standard monomial* in the quantum Plücker coordinates is an expression of the form $[I_1][I_2] \dots [I_t]$ where $I_1 \leq I_2 \leq \dots \leq I_t$ in this partial order. The set of all standard monomials forms a vector space basis of $\mathcal{O}_q(G(k, n))$ over K , see, for example, [7, Proposition 2.8].

3 Dehomogenisation of $\mathcal{O}_q(G(k, n))$

An element a in a ring R is said to be a *normal* element of R provided that $aR = Ra$. If R is a domain then a nonzero normal element a may be inverted, as the Ore condition for the set $S := \{a^n\}$ is easily verified. Standard results for noncommutative noetherian rings can be found in the books by Goodearl and Warfield [5] and McConnell and Robson [16].

Set $u = \{1, \dots, k\}$. Then $[u]$ commutes with all other quantum Plücker coordinates up to a power of q . We need to specify which power occurs. Here, and throughout the paper, for any quantum Plücker coordinate $[I]$ of $\mathcal{O}_q(G(k, n))$, we set $d(I) := \#(I \setminus (I \cap u)) \geq 0$.

Lemma 3.1. *Let $[I]$ be a quantum Plücker coordinate in $\mathcal{O}_q(G(k, n))$. Then $[u][I] = q^{d(I)}[I][u]$.*

Proof. This can be obtained from [8] by combining Lemma 3.7 and Theorem 3.4 of that paper. (Note that [8] uses q^{-1} as the deformation parameter where we use q ; so care must be taken in interpreting their results.) It can also be extracted from [7, Corollary 1.1] by setting $[I] = [u]$ in the statement of the corollary and noting that the summation on the right-hand side of the displayed equation is then empty. \square

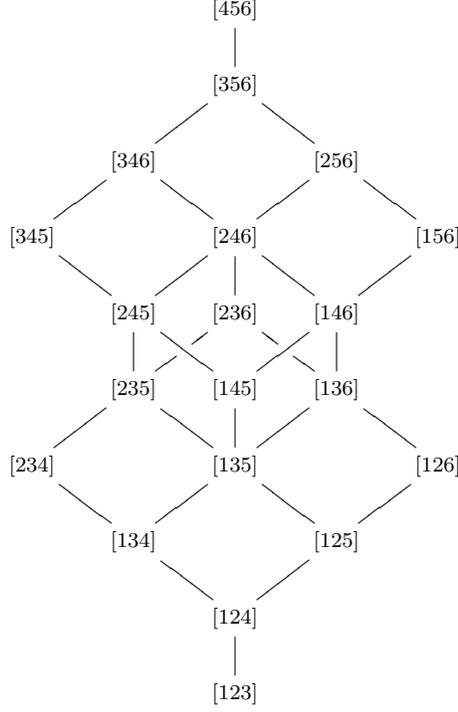


Figure 1: The partial order on Π for $\mathcal{O}_q(G(3, 6))$.

As $\mathcal{O}_q(G(k, n))$ is generated by the quantum Plücker coordinates it follows from the previous lemma that the element $[u]$ is a normal element and so we may invert $[u]$ to obtain the overring $\mathcal{O}_q(G(k, n))[[u]^{-1}]$.

For $1 \leq i \leq k$ and $1 \leq j \leq n - k$, set

$$x_{ij} := [1, \dots, \widehat{k+1-i}, \dots, k, j+k][u]^{-1} \in \mathcal{O}_q(G(k, n))[[u]^{-1}]. \quad (1)$$

The case $a = 1$ of [15, Theorem 3.2] shows that the elements x_{ij} generate an algebra R , say, that sits inside $\mathcal{O}_q(G(k, n))[[u]^{-1}]$ and is isomorphic to $\mathcal{O}_q(M(k, n - k))$. Also,

$$\mathcal{O}_q(G(k, n))[[u]^{-1}] = R[[u], [u]^{-1}].$$

(NB. The way we have fixed things, that really is an equality in the above display, rather than just an isomorphism.) In the rest of this note, we will write $R = \mathcal{O}_q(M(k, p))$ where $p := n - k$ and when we are operating on the right-hand side of this equality, we will write y for $[u]$.

As $d([1, \dots, \widehat{k+1-i}, \dots, k, j+k]) = 1$, it follows from Lemma 3.1 that $[u]x_{ij} = qx_{ij}[u]$ and that $yx_{ij} = qx_{ij}y$.

The equality above says that

$$T := \mathcal{O}_q(G(k, n))[[u]^{-1}] = \mathcal{O}_q(M(k, p))[y, y^{-1}; \sigma], \quad (2)$$

where σ is the automorphism of $\mathcal{O}_q(M(k, p))$ such that $\sigma(x_{ij}) = qx_{ij}$ for each $i = 1, \dots, k$ and $j = 1, \dots, p$. We will refer to Equation (2) as the *dehomogenisation equality*.

The next lemma gives the formulae for passing between quantum minors and Plücker coordinates in the above equality. In the lemma, and elsewhere, we use the following notation: if a is an integer and $S = \{s_1, \dots, s_t\}$ is a set of integers then $a + S := \{a + s_1, \dots, a + s_t\}$ and $a - S := \{a - s_1, \dots, a - s_t\}$.

Lemma 3.2. (i) Let $[I | J]$ be a quantum minor in $R = \mathcal{O}_q(M(k, p))$. Then

$$[I | J] = [\{1 \dots k\} \setminus (k+1 - I) \sqcup (k+J)][u]^{-1} \in \mathcal{O}_q(G(k, n))[[u]^{-1}].$$

(ii) Let $[L]$ be a quantum Plücker coordinate in $\mathcal{O}_q(G(k, n))$ and write $L = L_{\leq k} \sqcup L_{>k}$ where $L_{\leq k} = L \cap \{1, \dots, k\}$ and $L_{>k} = L \cap \{k+1, \dots, n\}$. Then

$$[L] = [(k+1) - (\{1, \dots, k\} \setminus L_{\leq k}) | L_{>k} - k]y \in \mathcal{O}_q(M(k, p))[y, y^{-1}; \sigma].$$

Proof. (i) This formula occurs as the case $a = 1$ of [15, Proposition 4.3] which gives the formula for general quantum minors of $R = \mathcal{O}_q(M(k, p))$ in terms of quantum Plücker coordinates of $\mathcal{O}_q(G(k, n))$.

(ii) Let $[L]$ be a quantum Plücker coordinate in $\mathcal{O}_q(G(k, n))$. Set $I = (k+1) - (\{1, \dots, k\} \setminus L_{\leq k})$ and $J = L_{>k} - k$. Note that $|I| = |J| = |L_{>k}|$ and so we can form the quantum minor $[I | J]$. Apply (i) to $[I | J]$ to see that

$$\begin{aligned} [I | J][u] &= [\{1, \dots, k\} \setminus \{(k+1) - (\{1, \dots, k\} \setminus L_{\leq k})\} \sqcup k + (L_{>k} - k)] \\ &= [\{1, \dots, k\} \setminus (\{1, \dots, k\} \setminus L_{\leq k}) \sqcup L_{>k}] = [L_{\leq k} \sqcup L_{>k}] = [L], \end{aligned}$$

so that (ii) is established. \square

The following corollary to the above lemma will be useful in later calculations.

Corollary 3.3. Suppose that $[L]$ is a quantum Plücker coordinate in $\mathcal{O}_q(G(k, n))$ and that $[L] = [I | J][u]$ for some $[I | J] \in \mathcal{O}_q(M(k, p))$.

(i) Let $i \in \{1, \dots, k\}$. Then $i \in I$ if and only if $(k+1) - i \notin L$.

(ii) Let $j \in \{1, \dots, p\}$. Then $j \in J$ if and only if $j + k \in L$.

We will use the dehomogenisation equality (2) in the next three sections to transfer the problem of finding automorphisms of $\mathcal{O}_q(G(k, n))$ to that of finding automorphisms of $\mathcal{O}_q(M(k, p))$ where the problem has been solved in [10] and [17]. Before doing that, we illustrate the usefulness of dehomogenisation by the following two results, the second of which identifies an extra automorphism of $\mathcal{O}_q(G(k, n))$ in the case that $n = 2k$.

First, note that there is an isomorphism τ between the quantum matrix algebras $\mathcal{O}_q(M(k, n-k)) = K(x_{ij})$ and $\mathcal{O}_q(M(n-k, k)) = K(x'_{ij})$ that sends x_{ij} to x'_{ji} and more generally sends a quantum minor $[I | J]$ of $\mathcal{O}_q(M(k, n-k))$ to the quantum minor $[J | I]$ of $\mathcal{O}_q(M(n-k, k))$. With a slight abuse of notation, we denote quantum minors of $\mathcal{O}_q(M(n-k, k))$ without dashes to differentiate them from quantum minors from $\mathcal{O}_q(M(k, n-k))$. (When $k = n-k$ and $x_{ij} = x'_{ij}$ this is the automorphism of quantum matrices given by transposition and so we will refer to τ as the *transpose isomorphism*.) This automorphism extends to an isomorphism from $\mathcal{O}_q(M(k, n-k))[y, y^{-1}; \sigma]$ to $\mathcal{O}_q(M(n-k, k))[y', y'^{-1}; \sigma']$, where $\sigma(x_{ij}) = qx_{ij}$ and $\sigma'(x'_{ij}) = qx'_{ij}$. We also denote this extension by τ . If L is a subset of $\{1, \dots, n\}$ then \widehat{L} denotes the complement of L in $\{1, \dots, n\}$. We use w_0 to denote the longest element of the symmetric group on $\{1, \dots, n\}$; that is, w_0 reverses the order of $\{1, \dots, n\}$.

Proposition 3.4. $\mathcal{O}_q(G(k, n)) \cong \mathcal{O}_q(G(n-k, n))$ via an isomorphism that sends the quantum Plücker coordinate $[L]$ of $\mathcal{O}_q(G(k, n))$ to the quantum Plücker coordinate $[w_0(\widehat{L})]$ of $\mathcal{O}_q(G(n-k, n))$.

Proof. We may assume that $2k \leq n$. There is an isomorphism

$$\begin{aligned} \mathcal{O}_q(G(k, n))[[u]^{-1}] &= \mathcal{O}_q(M(k, n-k))[y, y^{-1}; \sigma] \cong^1 \\ &\mathcal{O}_q(M(n-k, k)[y', y'^{-1}; \sigma'] \cong \mathcal{O}_q(G(n-k, n))[[u']^{-1}], \end{aligned}$$

where $[u'] = [1, \dots, n-k] \in \mathcal{O}_q(G(n-k, n))$ and $\sigma'(x'_{ij}) = qx'_{ij}$. (Note that \cong^1 is given by applying the transpose isomorphism τ that sends $[I | J]$ to $[J | I]$ and sends $y = [u] = [1, \dots, k]$ to $y' = [u'] = [1, \dots, n-k]$.)

We need to track the destination of an arbitrary quantum Plücker coordinate of $[L]$ of $\mathcal{O}_q(G(k, n))$ under this isomorphism, using the formulae that we have developed above for translating between quantum Plücker coordinates and quantum minors.

$$\begin{aligned} [L] &= [L_{\leq k} \sqcup L_{> k}] \\ &= [(k+1) - (\{1, \dots, k\} \setminus L_{\leq k}) | L_{> k} - k]y \quad (\in \mathcal{O}_q(M(k, n-k))y) \\ &\xrightarrow{1} [L_{> k} - k | (k+1) - (\{1, \dots, k\} \setminus L_{\leq k})]y' \quad (\in \mathcal{O}_q(M(n-k, k))y') \\ &= [\{1, \dots, n-k\} \setminus (n+1 - L_{> k}) \sqcup (n-k) + ((k+1) - (\{1, \dots, k\} \setminus L_{\leq k}))] \\ &= [\{1, \dots, n-k\} \setminus (n+1 - L_{> k}) \sqcup ((n+1) - (\{1, \dots, k\} \setminus L_{\leq k}))] \\ &= [\{1, \dots, n-k\} \setminus w_0(L_{> k}) \sqcup w_0(\{1, \dots, k\} \setminus L_{\leq k})] \\ &= [\{1, \dots, n-k\} \setminus w_0(L_{> k}) \sqcup \{n-k+1, \dots, n\} \setminus w_0(L_{\leq k})] \\ &= [\{1, \dots, n\} \setminus \{w_0(L_{> k}) \sqcup w_0(L_{\leq k})\}] \\ &= [\{1, \dots, n\} \setminus w_0(L)] \\ &= [\widehat{w_0(L)}] = [w_0(\widehat{L})]. \end{aligned}$$

As the quantum Plücker coordinates of $\mathcal{O}_q(G(k, n))$ generate $\mathcal{O}_q(G(k, n))$ as an algebra, and their images generate $\mathcal{O}_q(G(n - k, n))$ as an algebra, this calculation shows that the isomorphism displayed at the beginning of the proof restricts to an isomorphism between $\mathcal{O}_q(G(k, n))$ and $\mathcal{O}_q(G(n - k, n))$. \square

An immediate corollary of this result is the following.

Corollary 3.5. *When $2k = n$, there is an automorphism of $\mathcal{O}_q(G(k, n))$ which sends the quantum Plücker coordinate $[I]$ to $[w_0(\widehat{I})]$.*

Remark 3.6. The automorphism in the previous corollary will be called the *diagram automorphism*. In Figure 1, which shows the standard poset for $\mathcal{O}_q(G(3, 6))$, the effect of this automorphism on the quantum Plücker coordinates is seen by reflection of the poset in the vertical. For example, $[126]$ is sent to $[w_0(\widehat{126})] = [w_0(345)] = [234]$. There is a diagram automorphism for $\mathcal{O}_q(G(k, n))$ only in the case that $n = 2k$. Note that both the diagram automorphism and the transpose automorphism τ extend to $\mathcal{O}_q(G(k, 2k)([u]^{-1}) = \mathcal{O}_q(M(k, k))[y, y^{-1}; \sigma]$ and they agree on this common overring, so we denote the diagram automorphism by τ also.

4 Obvious automorphisms of $\mathcal{O}_q(G(k, n))$

There are two obvious sources of automorphisms of $\mathcal{O}_q(G(k, n))$. The first is via the inclusion $\mathcal{O}_q(G(k, n)) \subseteq \mathcal{O}_q(M(k, n))$. The second is by using the dehomogenisation equality introduced in Section 3:

$$\mathcal{O}_q(M(k, p))[y^{\pm 1}; \sigma] = \mathcal{O}_q(G(k, n))[[u]^{-1}],$$

where $p = n - k$ and $u = \{1, \dots, k\}$ while σ is the automorphism of $\mathcal{O}_q(M(k, p))$ such that $\sigma(x_{ij}) = qx_{ij}$ for each $i = 1, \dots, k$ and $j = 1, \dots, p$.

In this section, we introduce these automorphisms and consider the connections between them.

First, $\mathcal{O}_q(G(k, n))$ is a subalgebra of $\mathcal{O}_q(M(k, n))$ by definition. The torus $\mathcal{H}_0 := (K^*)^n$ acts by column multiplication on $\mathcal{O}_q(M(k, n))$ and this induces an action on $\mathcal{O}_q(G(k, n))$ defined on quantum Plücker coordinates by

$$(\beta_1, \dots, \beta_n) \cdot [i_1, \dots, i_k] = \beta_{i_1} \dots \beta_{i_k} [i_1, \dots, i_k].$$

This is the torus action on $\mathcal{O}_q(G(k, n))$ that is considered in papers such as [12, 14].

Secondly, there is an action of the torus $(K^*)^{k+p}$ on $\mathcal{O}_q(M(k, p))$ which operates by row and column scaling, so that $(\alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_p) \cdot x_{ij} = \alpha_i \beta_j x_{ij}$. As $n = k + p$, we can extend this to an action of the torus $\mathcal{H}_1 := (K^*)^{n+1}$ on $\mathcal{O}_q(M(k, p))[y^{\pm 1}; \sigma]$ by setting

$$(\alpha_0; \alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_p) \cdot x_{ij} = \alpha_i \beta_j x_{ij}, \quad (\alpha_0; \alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_p) \cdot y = \alpha_0 y.$$

Set $h = (\alpha_0; \alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_p) \in \mathcal{H}_1$. It is easy to check that $h \cdot [I | J] = \alpha_I \beta_J [I | J]$, where $\alpha_I := \alpha_{i_1} \dots \alpha_{i_k}$ when $I = [\alpha_{i_1}, \dots, \alpha_{i_k}]$, and $\beta_J = \beta_{j_1} \dots \beta_{j_p}$ when $J = [\beta_{j_1}, \dots, \beta_{j_k}]$.

The dehomogenisation equality induces an action of \mathcal{H}_1 on $\mathcal{O}_q(G(k, n))[[u]^{-1}]$. For $h = (\alpha_0; \alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_p) \in \mathcal{H}_1$ we have $h \cdot [u] = \alpha_0 [u]$. If $[L]$ is any other quantum Plücker coordinate then we may write $[L] = [I | J][u]$, where $I = (k+1) - (\{1, \dots, k\} \setminus L_{\leq k})$ and $J = L_{>k} - k$, by Lemma 3.2. Then

$$h \cdot [L] = h \cdot [I | J] \times h \cdot [u] = \alpha_I \beta_J [I | J] \times \alpha_0 [u] = \alpha_0 \alpha_I \beta_J [I | J][u] = \alpha_0 \alpha_I \beta_J [L].$$

As the quantum Plücker coordinates generate $\mathcal{O}_q(G(k, n))$ and are sent to scalar multiples of themselves by each $h \in \mathcal{H}_1$, such h act as automorphisms of $\mathcal{O}_q(G(k, n))$.

We now consider connections between the actions of \mathcal{H}_0 and \mathcal{H}_1 on $\mathcal{O}_q(G(k, n))$.

Lemma 4.1. *For every automorphism $g \in \mathcal{H}_0$ acting on $\mathcal{O}_q(G(k, n))$ there is an automorphism $f \in \mathcal{H}_1$ which has the same action on $\mathcal{O}_q(G(k, n))$.*

Proof. Let $g = (a_1, \dots, a_n) \in \mathcal{H}_0$. We seek $f = (\alpha_0; \alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_p) \in \mathcal{H}_1$ such that the actions of g and f on $\mathcal{O}_q(G(k, n))$ are the same. As $g \cdot [u] = a_1 \dots a_k [u]$, we may extend g to act on $\mathcal{O}_q(G(k, n))[[u]^{-1}]$. The dehomogenisation equality then transfers this action to $\mathcal{O}_q(M(k, p))[y^{\pm 1}; \sigma]$. We calculate the action of g on the generators x_{ij} and $y^{\pm 1}$ of $\mathcal{O}_q(M(k, p))[y^{\pm 1}; \sigma]$. As y corresponds to $[u]$ in the dehomogenisation equality, $g \cdot y = a_1 \dots a_k y$ and $g \cdot y^{-1} = (a_1 \dots a_k)^{-1} y^{-1}$. Now, $x_{ij} := [1, \dots, \widehat{k+1-i}, \dots, k, j+k][u]^{-1}$; so

$$\begin{aligned} g \cdot x_{ij} &= g \cdot [1, \dots, \widehat{k+1-i}, \dots, k, j+k] \times g \cdot [u]^{-1} \\ &= a_1 \dots a_k a_{k+1-i}^{-1} a_{j+k} [1, \dots, \widehat{k+1-i}, \dots, k, j+k] \times (a_1 \dots a_k)^{-1} [u]^{-1} \\ &= a_{j+k} a_{(k+1)-i}^{-1} [1, \dots, \widehat{k+1-i}, \dots, k, j+k][u]^{-1} \\ &= a_{j+k} a_{(k+1)-i}^{-1} x_{ij}. \end{aligned}$$

We seek an $f \in \mathcal{H}_0$ which has the same effect on the generators x_{ij} and y . Set $f = (a_1 \dots a_k; a_k^{-1}, \dots, a_1^{-1}; a_{k+1}, \dots, a_n) \in \mathcal{H}_0$. Then $f \cdot y = a_1 \dots a_k y = g \cdot y$. Also, the entry in f multiplying the i th row is $a_{(k+1)-i}^{-1}$ and the entry multiplying the j th column is a_{k+j} so $f \cdot x_{ij} = a_{(k+1)-i}^{-1} a_{k+j} x_{ij} = g \cdot x_{ij}$. Hence, f and g agree on the generators x_{ij} and $y^{\pm 1}$ of $\mathcal{O}_q(M(k, p))[y^{\pm 1}; \sigma] = \mathcal{O}_q(G(k, n))[[u]^{-1}]$; so the actions of f and g on $\mathcal{O}_q(G(k, n))$ are the same. \square

The converse question is more delicate, as the following example shows.

Example 4.2. Let K be a field in which there is no element b such that $b^2 = 2$ (eg \mathbb{Q}) and consider $\mathcal{O}_q(G(2, 4))$ over this field. Let $f = (1; 2, 1; 1, 1) \in \mathcal{H}_1$. Then there is no element $g = (a_1, a_2, a_3, a_4) \in \mathcal{H}_0$ whose action on $\mathcal{O}_q(G(2, 4))$ coincides with the action of f on $\mathcal{O}_q(G(2, 4))$. To verify this claim, we use the formulae in Lemma 3.2 to see that

$$[12] = [u], [13] = x_{11}[u], [14] = x_{12}[u], [23] = x_{21}[u], [24] = x_{22}[u], [34] = [12 | 12][u].$$

These equations lead to the following actions of f on the quantum Plücker coordinates:

$$f \cdot [12] = [12], f \cdot [13] = 2[13], f \cdot [14] = 2[14], f \cdot [23] = [23], f \cdot [24] = [24], f \cdot [34] = 2[34].$$

Next, for $g = (a_1, a_2, a_3, a_4) \in \mathcal{H}_0$ we see that

$$\begin{aligned} g \cdot [12] &= a_1 a_2 [12], g \cdot [13] = a_1 a_3 [13], g \cdot [14] = a_1 a_4 [14], \\ g \cdot [23] &= a_2 a_3 [23], g \cdot [24] = a_2 a_4 [24], g \cdot [34] = a_3 a_4 [34]; \end{aligned}$$

so for this g to act in the same way as f we require that

$$a_1 a_2 = 1, a_1 a_3 = 2, a_1 a_4 = 2, a_2 a_3 = 1, a_2 a_4 = 1, a_3 a_4 = 2.$$

The fourth and fifth equations on the line above show that we need $a_3 = a_4$ ($= b$ say) and then the sixth equation gives $b^2 = 2$, which is not possible for any $b \in K$.

In view of this example, it is appropriate to assume that K is algebraically closed in the following result.

Lemma 4.3. *Suppose that K is algebraically closed. Consider $\mathcal{O}_q(G(k, n))$ and $\mathcal{O}_q(M(k, p))$ over K . For every automorphism $f \in \mathcal{H}_1$ acting on $\mathcal{O}_q(G(k, n))$ there is an automorphism $g \in \mathcal{H}_0$ which has the same action on $\mathcal{O}_q(G(k, n))$.*

Proof. Consider the set S of elements $f = (\alpha_0; \alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_p) \in \mathcal{H}_1$ that are equal to 1 in all positions except one. The set S generates \mathcal{H}_1 so it is enough to show that the action of each member of S can be realised via the action of an element of \mathcal{H}_0 .

Let $[L] \neq [u]$ be a quantum Plücker coordinate in $\mathcal{O}_q(G(k, n))$ with $[L] = [I | J][u]$ for some quantum minor $[I | J] \in \mathcal{O}_q(M(k, p))$. Then

$$f \cdot [L] = (f \cdot [I | J])(f \cdot [u]) = \alpha_0 (f \cdot [I | J])[u].$$

We look at three cases separately.

The first case we consider is when one of the β_i terms is not equal to 1. Suppose that the element β_j in position $k+1+j$ of f is not equal to 1 but that all other positions of f contain the element 1. Note that $\alpha_0 = 1$ and so $f \cdot [u] = [u]$. Thus, $f \cdot [L] = (f \cdot [I | J])[u]$ so that $f \cdot [L] = [L]$ when $j \notin J$ and $f \cdot [L] = \beta_j [L]$ when $j \in J$.

By using Corollary 3.3(ii) we see that $f \cdot [L] = [L]$ if $j+k \notin L$ and that $f \cdot [L] = \beta_j [L]$ when $j+k \in L$. An element $g \in \mathcal{H}_0$ that has the same effect on quantum Plücker coordinates is $g = (g_1, \dots, g_n)$ where $g_{j+k} = \beta_j$ while all other $g_i = 1$.

The next case that we consider is when α_i is not equal to 1 for some $i \in \{1, \dots, k\}$ but all other entries of f are equal to 1. Again, $f \cdot [u] = [u]$. Then $f \cdot [L] = (f \cdot [I | J])[u]$ so that $f \cdot [L] = [L]$ when $i \notin I$ and $f \cdot [L] = \alpha_i [L]$ when $i \in I$. By using Corollary 3.3(i) we see that $f \cdot [L] = [L]$ if $(k+1) - i \in L$ and $f \cdot [L] = \alpha_i [L]$ when $(k+1) - i \notin L$. Let b be

an element in K such that $b^k = \alpha_i$. Let $g = (g_1, \dots, g_n) \in \mathcal{H}_0$ be such that $g_{k+1-i} = b^{-(k-1)}$ while every other entry $g_j = b$. Let $[L] = [l_1, \dots, l_k]$. Then $g \cdot [L] = g_{l_1} \dots g_{l_k} [L]$. Suppose that $k+1-i \in L$. Then one of g_{l_1}, \dots, g_{l_k} is equal to $b^{-(k-1)}$ while the other $k-1$ are equal to b . Thus, $g_{l_1} \dots g_{l_k} = b^{k-1} b^{-(k-1)} = 1$ and so $g \cdot [L] = [L]$. Now assume that $k+1-i \notin L$. Then each of the g_{l_j} is equal to b and so $g \cdot [L] = b^k [L] = \alpha_i [L]$. This shows that the action of g coincides with the action of f , as required.

The final case to consider is when α_0 is not equal to 1 but all other entries are equal to 1. Thus, $f \cdot [u] = \alpha_0 [u]$ and $f \cdot [I \mid J] = [I \mid J]$ for all quantum minors $[I \mid J]$. Hence, $f \cdot [L] = \alpha_0 [L]$ for all quantum minors $[L]$ of $\mathcal{O}_q(G(k, n))$. Let $b \in K$ be such that $b^k = \alpha_0$ and set $g = (b, \dots, b)$. Then $g \cdot [L] = b^k [L] = \alpha_0 [L]$ for all $[L]$ and the actions of f and g coincide. \square

The action of \mathcal{H}_1 on $\mathcal{O}_q(G(k, n))$ is not faithful, as we will see in the next proposition. Let \mathcal{H} denote \mathcal{H}_1 factored by this kernel of this action. Then \mathcal{H} acts faithfully on $\mathcal{O}_q(G(k, n))$. The next result shows that \mathcal{H} is isomorphic to a torus $(K^*)^n$.

Proposition 4.4. *The group \mathcal{H} is isomorphic to a torus $(K^*)^n$.*

Proof. The kernel of the action of \mathcal{H}_1 on $\mathcal{O}_q(G(k, n))$ is the same as the kernel of the action of \mathcal{H}_1 on $\mathcal{O}_q(G(k, n))[[u^{-1}]] = \mathcal{O}_q(M(k, p))[y^{\pm 1}]$. Recall from earlier in the section that the action of \mathcal{H}_1 on the right-hand side of this equality is given by

$$(\alpha_0; \alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_p) \cdot x_{ij} = \alpha_i \beta_j x_{ij}, \quad (\alpha_0; \alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_p) \cdot y = \alpha_0 y.$$

Using the right-hand side, it is easy to check that this kernel is $\{(1; \lambda, \dots, \lambda; \lambda^{-1}, \dots, \lambda^{-1}) \mid \lambda \in K^*\}$. Hence, choosing $\lambda = \beta_p$, we see that $h = (\alpha_0; \alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_p) \in \mathcal{H}_1$ has the same action as $h' := (\alpha_0; \alpha_1 \beta_p, \dots, \alpha_k \beta_p; \beta_1 \beta_p^{-1}, \dots, \beta_{p-1} \beta_p^{-1}, 1) \in \mathcal{H}_1$. It is also easy to check that two distinct elements in \mathcal{H}_1 that each have 1 in the final place act differently on $\mathcal{O}_q(M(k, p))[y^{\pm 1}]$, and so the claim is established. \square

In view of this result, we refer to the action on $\mathcal{O}_q(G(k, n))$ provided by \mathcal{H} as *the torus automorphisms of $\mathcal{O}_q(G(k, n))$* .

In the case that $n = 2k$, so that $k = p$, the dehomogenisation equality (2) states that

$$T = \mathcal{O}_q(M(k, k))[y^{\pm 1}; \sigma] = \mathcal{O}_q(G(k, n))[[u^{-1}]].$$

In this case, a simple analysis using the formula $[L] = [I \mid J][u]$ shows that the extra automorphism of $\mathcal{O}_q(M(k, k))$ given by transposition of the x_{ij} variables extends to an automorphism of T which, when restricted to $\mathcal{O}_q(G(k, n))$, gives rise to the diagram automorphism τ of Corollary 3.5. We will denote this automorphism by τ for each of the three algebras $\mathcal{O}_q(G(k, n))$, $\mathcal{O}_q(M(k, k))$ and T .

In the case where $2k = n$, let $(\alpha_0; \alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_k) \in \mathcal{H}_1$. It is easy to check that

$$\tau \circ (\alpha_0; \alpha_1, \dots, \alpha_k; \beta_1, \dots, \beta_k) \circ \tau = (\alpha_0; \beta_1, \dots, \beta_k; \alpha_1, \dots, \alpha_k) \in \mathcal{H}_1,$$

so that $\langle \tau \rangle$ acts on \mathcal{H}_1 . Also, if $(1; \lambda, \dots, \lambda; \lambda^{-1}, \dots, \lambda^{-1})$ is in the kernel of the action of \mathcal{H}_1 on $\mathcal{O}_q(G(k, n))[[u]^{-1}]$ then

$$\begin{aligned} \tau \circ (1; \lambda, \dots, \lambda; \lambda^{-1}, \dots, \lambda^{-1}) \circ \tau &= (1; \lambda^{-1}, \dots, \lambda^{-1}; \lambda, \dots, \lambda) \\ &= (1; \lambda^{-1}, \dots, \lambda^{-1}; (\lambda^{-1})^{-1}, \dots, (\lambda^{-1})^{-1}) \end{aligned}$$

is also in the kernel of this action and so $\langle \tau \rangle$ acts on \mathcal{H} .

Definition 4.5. Set $\mathcal{A} := \mathcal{H}$ when $2k \neq n$ and $\mathcal{A} := \mathcal{H} \rtimes \langle \tau \rangle$ when $2k = n$.

The analysis above shows that the elements of \mathcal{A} act naturally as automorphisms of $\mathcal{O}_q(G(k, n))$ via the dehomogenisation equality.

Claim 4.6. The automorphism group of $\mathcal{O}_q(G(k, n))$ is \mathcal{A} .

We will prove this claim in the following sections.

5 Adjusting automorphisms

The quantum grassmannian $\mathcal{O}_q(G(k, n))$ carries the structure of an \mathbb{N} -graded algebra generated in degree one when we give degree one to each of the quantum Plücker coordinates. In addition, [13, Theorem 5.3] shows that $\mathcal{O}_q(G(k, n))$ is a unique factorisation domain in the sense of Chatters [3]. According to Chatters, an element p of a noetherian domain R is said to be prime if (i) $pR = Rp$, (ii) pR is a height one prime ideal of R , and (iii) R/pR is an integral domain. A noetherian domain R is then said to be a *unique factorisation domain (UFD)* if R has at least one height one prime ideal, and every height one prime ideal is generated by a prime element.

In this section, we exploit these properties of $\mathcal{O}_q(G(k, n))$ in a series of results to see that given an arbitrary automorphism of $\mathcal{O}_q(G(k, n))$ we can essentially fix the minimal and maximal elements in the poset after allowing adjustment of the automorphism by elements of \mathcal{H} .

Lemma 5.1. *Let $A = \bigoplus_{i=0}^{\infty} A_i$ be a graded algebra that is a domain with A_0 equal to the base field and A generated in degree one. Suppose that $a = a_1 + \dots + a_m$ is a normal element with $a_i \in A_i$ for each i . Then a_1 is a normal element.*

Proof. If $a_1 = 0$ then there is nothing to prove; so assume that $a_1 \neq 0$. As A is generated in degree one, it is enough to check normality with respect to homogeneous elements of degree one; so suppose that $b \in A_1$. Then $ba = ba_1 + \dots + ba_m = ac = (a_1 + \dots + a_m)(c_0 + c_1 + \dots + c_t)$ for some $c = c_0 + c_1 + \dots + c_t \in A$ with $c_i \in A_i$. Comparing degree one terms gives $0 = a_1 c_0$; so $c_0 = 0$. The degree two terms then show that $ba_1 = a_1 c_1 \in a_1 A$, and this demonstrates that a_1 is normal. \square

Lemma 5.2. *Let $A = \bigoplus_{i=0}^{\infty} A_i$ be a graded algebra that is a domain with A_0 equal to the base field. Suppose also that A is a UFD.*

Let a be a homogeneous element of degree one that is normal. Then a generates a prime ideal of height one.

Proof. Let P be a prime that is minimal over the ideal aR . By the noncommutative principal ideal theorem [16, Theorem 4.1.11], the height of P is one. Hence, $P = pA$ for some normal element p , as A is a UFD. Thus, a is a (right) multiple of p . By degree considerations, p must have degree one and a must be a scalar multiple of p . Thus, a and p generate the same ideal, which is the prime ideal P . This establishes the claim. \square

The remaining results in this section all deal with $\mathcal{O}_q(G(k, n))$. As in earlier sections, let $[u] = [1, \dots, k]$.

Lemma 5.3. *Suppose that $a = \sum a_I [I] \neq 0$, with $a_I \in K$, is a linear combination of quantum Plücker coordinates that is a normal element. Then $d(I)$ is the same for each I that has $a_I \neq 0$.*

Proof. Since $a = \sum a_I [I]$ has degree one, a is irreducible, as well as being normal. Hence, the ideal P generated by a is a height one prime ideal of $\mathcal{O}_q(G(k, n))$, by Lemma 5.2.

If $a = a_I [I]$ for some $[I]$ then the result holds. Thus we may assume that at least two scalars a_I are nonzero. In particular, $aK \neq [u]K$, and so $[u] \notin a\mathcal{O}_q(G(k, n)) = P$.

Let J be such that $d(J)$ is as small as possible among those $d(I)$ for which $a_I \neq 0$. We will show that $d(I) = d(J)$ for all I such that $a_I \neq 0$.

Now

$$[u]a = [u] \left(\sum_{I \neq J} a_I [I] + a_J [J] \right) = \left(\sum_{I \neq J} a_I q^{d(I)} [I] + a_J q^{d(J)} [J] \right) [u],$$

by using Lemma 3.1, and so

$$\left(\sum_{I \neq J} a_I q^{d(I)} [I] + a_J q^{d(J)} [J] \right) [u] - q^{d(J)} \left(\sum_{I \neq J} a_I [I] + a_J [J] \right) [u] = [u]a - q^{d(J)} a [u] \in P;$$

that is,

$$b := \left(\sum_{I \neq J} (q^{d(I)} - q^{d(J)}) a_I [I] \right) [u] \in P.$$

As $[u] \notin P$ this gives

$$b = \left(\sum_{I \neq J} (q^{d(I)} - q^{d(J)}) a_I [I] \right) \in P = a\mathcal{O}_q(G(k, n)).$$

Thus, there is a scalar $\lambda \in K$ with $b = \lambda a$. If $\lambda \neq 0$ then this is a contradiction, as $[J]$ occurs nontrivially in λa and not in b . Therefore, $b = 0$ and so each $q^{d(I)} - q^{d(J)} = 0$. This forces $d(I) = d(J)$, since q is not a root of unity. \square

Before proceeding further with our analysis, we recall a result from [10] that we need. The notation is adapted from that paper.

Proposition 5.4. *Let $A = \bigoplus_{i \in \mathbb{N}} A_i$ be an \mathbb{N} -graded (connected) K -algebra that is a domain. Suppose that A is generated as an algebra by x_1, \dots, x_t and that $A_1 = Kx_1 \oplus \dots \oplus Kx_t$. Set $A_{>d} := \bigoplus_{i>d} A_i$. Assume that for all $i \in \{1, \dots, t\}$ there exists j and $1 \neq q_{ij} \in K$ with $x_i x_j = q_{ij} x_j x_i$. Let σ be an automorphism of A and $0 \neq x \in A_d$. Then $\sigma(x) = y_d + y_{>d}$, where $y_d \in A_d \setminus \{0\}$ and $y_{>d} \in A_{>d}$.*

Proof. This is [10, Proposition 4.2]. □

In order to prove the next lemma, we need to check that we can apply this proposition to the quantum grassmannian. In order to do this, we need to observe that for each quantum Plücker coordinate $[L]$ there exists a quantum Plücker coordinate $[L']$ such that $[L][L'] = \alpha[L'][L]$ for some $1 \neq \alpha \in K$. We use Lemma 3.1 to do this. If $[L] = [u] = [1, \dots, k]$ then set $[L'] = [1, \dots, k-1, k+1]$ so that $[L][L'] = q[L'][L]$ while if $[L] \neq [u]$ set $[L'] = [u]$ so that $[L][L'] = q^{-d(L)}[L'][L]$.

Lemma 5.5. *Let ρ be an automorphism of $\mathcal{O}_q(G(k, n))$. Then $\rho([u])_1 = \lambda[u]$, for some $\lambda \in K^*$.*

Proof. The previous proposition shows that $\rho([u])_0 = 0$. Note that $\rho([u])$ is a normal element of $\mathcal{O}_q(G(k, n))$; so Lemma 5.1 applied to $\rho([u])$ shows that $\rho([u])_1$ is a normal element. Assume that $\rho([u])_1 \notin K^*[u]$. Then $\rho([u])_1 = \sum a_I [I]$ for some $[I]$ with each $d(I) = d > 0$, by Lemma 5.3.

Let $f \in \mathcal{O}_q(G(k, n))$ be such that $\rho(f) = [u]$, or, equivalently, $\rho^{-1}([u]) = f$. Note that the degree zero term of f must be zero. Write $f = f_1 + f_{>1}$. As $\rho(f_{>1})_1 = 0$, by [10, Proposition 4.2], we see that $[u] = \rho(f) = \rho(f_1) + \rho(f_{>1})$; so that $[u] = \rho(f)_1 = \rho(f_1)_1$. If $f_1 = \lambda[u]$ then $[u] = \rho(f_1)_1 = \rho(\lambda[u])_1 = \lambda \rho([u])_1$ and so $\rho([u])_1 = \lambda^{-1}[u]$, a contradiction; so we have $f_1 \neq \lambda[u]$, for any $\lambda \in K^*$. By applying Lemma 5.3 to f_1 , we may write $f_1 = \sum b_J [J]$ for some $[J]$ with each $d(J) = e > 0$, say. Hence, $[u]f_1 = q^e f_1 [u]$.

Consider the degree two term in $\rho([u]f)$. We know that $\rho([u]f) = \rho([u])\rho(f)$ and that the degree zero terms of $\rho([u])$ and $\rho(f)$ are both zero, by [10, Proposition 4.2]. Hence,

$$\rho([u]f)_2 = \rho([u])_1 \rho(f)_1 = \left(\sum a_I [I] \right) [u] = q^{-d}[u] \left(\sum a_I [I] \right).$$

On the other hand, $\rho([u]f) = \rho([u]f_1 + [u]f_{>1}) = \rho([u]f_1) + \rho([u]f_{>1})$. Now, $[u]f_{>1}$ has no term in degree less than three. Hence, the same is true for $\rho([u]f_{>1})$, by [10, Proposition 4.2], and, in particular, $\rho([u]f_{>1})_2 = 0$. Thus, $\rho([u]f)_2 = \rho([u]f_1)_2$. However, $\rho([u]f_1) = \rho(q^e f_1 [u]) = q^e \rho(f_1) \rho([u])$. Therefore,

$$\rho([u]f)_2 = \rho([u]f_1)_2 = \rho(q^e f_1 [u])_2 = q^e \rho(f_1)_1 \rho([u])_1 = q^e [u] \left(\sum a_I [I] \right).$$

The two expressions we have obtained for $\rho([u]f)_2$ must be equal; so $q^{-d}[u](\sum a_I[I]) = q^e[u](\sum a_I[I])$. Hence, $q^{-d} = q^e$; so that $q^{e+d} = 1$. As q is not a root of unity and $e + d > 0$, this is a contradiction, and our lemma is proved. \square

Lemma 5.6. *Let ρ be an automorphism of $\mathcal{O}_q(G(k, n))$. Then $\rho([u]) = \lambda[u]$, for some $\lambda \in K^*$.*

Proof. The element $\rho([u])$ is a normal element, and the degree zero term of $\rho([u])$ is equal to 0, by [10, Proposition 4.2]. Suppose that the degree of $\rho([u])$ is t and that $\rho([u]) = a_1 + \cdots + a_t$ with $\deg(a_i) = i$. Recall that $a_1 = \lambda[u]$, for some $\lambda \in K^*$, by Lemma 5.5. There is an element $r \in \mathcal{O}_q(G(k, n))$ with $[u]\rho([u]) = \rho([u])r$. The degree of r must be one. Assume $r = r_0 + r_1$ with r_i having degree i . Thus,

$$[u](\lambda[u] + a_2 + \cdots + a_t) = (\lambda[u] + a_2 + \cdots + a_t)(r_0 + r_1).$$

As there is no term in degree one on the left hand side of the above equation, we must have $r_0 = 0$. Looking at terms in degree two, we then see that $\lambda[u]^2 = \lambda[u]r_1$; so that $r_1 = [u]$ and $[u]\rho([u]) = \rho([u])[u]$.

Write $\rho([u])$ in terms of the standard basis for $\mathcal{O}_q(G(k, n))$, as in [7], say $\rho([u]) = \sum \alpha_i S_i$, where α_i is in the field and each S_i is a standard monomial. If $S = [I_1] \cdots [I_m]$ is such a standard monomial, then set $d(S) := \sum d(I_i)$ and note that each $d(I_i) \geq 0$ with $d(I_i) = 0$ if and only if $[I_i] = [u]$. Then, $S[u] = q^{d(S)}[u]S$, and so $S[u] = [u]S$ if and only if $d(S) = 0$ (in which case $S = [u]^m$ for some m).

In any case, note that $[u]S$ is a standard monomial, as $[u]$ is the unique minimal quantum Plücker coordinate. Hence,

$$\sum \alpha_i [u]S_i = [u]\rho([u]) = \rho([u])[u] = \left(\sum \alpha_i S_i \right) [u] = [u] \left(\sum \alpha_i q^{d(S_i)} S_i \right) = \sum \alpha_i q^{d(S_i)} [u]S_i.$$

As the extreme left and right terms in the above display are in the standard basis, this forces $d(S_i) = 0$ whenever $\alpha_i \neq 0$. Hence, $\rho([u])$ must be a polynomial in $[u]$.

The same argument applies to the automorphism ρ^{-1} ; so $\rho^{-1}([u])$ must also be a polynomial in $[u]$.

Suppose that $\rho([u]) = \sum_{i=1}^t \alpha_i [u]^i$ with $\alpha_t \neq 0$, and, similarly, suppose that $\rho^{-1}([u]) = \sum_{i=1}^s \beta_i [u]^i$ with $\beta_s \neq 0$

Then,

$$[u] = \rho^{-1}\rho([u]) = \rho^{-1} \left(\sum_{i=1}^t \alpha_i [u]^i \right) = \sum_{i=1}^t \alpha_i \rho^{-1}([u])^i = \alpha_1 \beta_1 [u] + \cdots + \alpha_t \beta_s [u]^{st}.$$

Therefore, $s = t = 1$, and $\rho([u]) = \lambda[u]$, for some $\lambda \in K^*$, as required. \square

The above result refers to $[u] = [1, \dots, k]$, the extreme leftmost quantum Plücker coordinate. We want to establish a similar result for $[w] := [n - k + 1, \dots, n]$, the extreme

rightmost quantum Plücker coordinate. In order to do this we employ an antiautomorphism of $\mathcal{O}_q(G(k, n))$ which we now describe.

Let w_0 denote the longest element on the symmetric group on n elements; that is, $w_0(i) = n + 1 - i$. The discussion immediately before Proposition 2.12 of [2] shows that the map $\theta : \mathcal{O}_q(G(k, n)) \rightarrow \mathcal{O}_q(G(k, n))$ given by $\theta([I]) = [w_0(I)]$ for each quantum Plücker coordinate $[I]$ is an antiautomorphism. Note that $\theta([u]) = \theta([1, \dots, k]) = [n + 1 - k, \dots, n] = [w]$.

Corollary 5.7. *Let ρ be an automorphism of $\mathcal{O}_q(G(k, n))$. Then there exists $h \in \mathcal{H}$ such that $(h \circ \rho)([u]) = u$ and $(h \circ \rho)([w]) = [w]$.*

Proof. The map $\theta\rho\theta$ is an automorphism of $\mathcal{O}_q(G(k, n))$. By Lemma 5.6, there is an element $\mu \in K^*$ such that $\theta\rho\theta([u]) = \mu[u]$. Apply θ to both sides of this equality to obtain $\rho\theta([u]) = \mu\theta([u])$; that is, $\rho([w]) = \mu[w]$. We also know that $\rho([u]) = \lambda[u]$ for some $\lambda \in K^*$. Set $h := (\lambda^{-1}, 1, \dots, 1, \mu^{-1}) \in \mathcal{H}_0$. Then $(h \circ \rho)([u]) = [u]$ and $(h \circ \rho)([w]) = [w]$. Lemma 4.1 shows that the action of an element of \mathcal{H}_0 is realised by the action of an element of \mathcal{H}_1 and hence of \mathcal{H} , so the result follows. \square

In what follows, we will often replace the original automorphism ρ by $h \circ \rho$ so that we may assume that $\rho([u]) = [u]$ and $\rho([w]) = [w]$ in calculations.

Remark 5.8. While the assumption that q is not a root of unity was not necessary in Sections 2, 3 and 4, it was used in a number of places in this section.

6 Transfer to quantum matrices

Recall from the discussion in Section 2 that when discussing $\mathcal{O}_q(G(k, n))$ we are assuming that $1 < k$ and that $2k \leq n$. We also continue to use the notation $p := n - k$.

Let ρ be an automorphism of $\mathcal{O}_q(G(k, n))$. Recall from earlier that $[u] = [1, \dots, k]$ and $[w] = [n - k + 1, \dots, n]$. By using Corollary 5.7, at the expense of adjusting ρ by an element of \mathcal{H} , we can, and will, assume that $\rho([u]) = [u]$ and $\rho([w]) = [w]$. The automorphism ρ now extends to $T = \mathcal{O}_q(G(k, n))[[u]^{-1}] = \mathcal{O}_q(M(k, p))[y^{\pm 1}; \sigma]$ and we know that $\rho(y) = y$.

We will show that such a ρ sends $\mathcal{O}_q(M(k, p))$ to itself. Once we have done this, we will know how ρ acts on each quantum minor in $\mathcal{O}_q(M(k, p))$ as we know the automorphism group of $\mathcal{O}_q(M(k, p))$. We can then calculate how ρ acts on arbitrary quantum Plücker coordinates of $\mathcal{O}_q(G(k, n))$, by using the formulae of Lemma 3.2.

From Equation (1) in Section 3, we know that the quantum matrix generators x_{ij} are defined by

$$x_{ij} := [1, \dots, \widehat{k + 1 - i}, \dots, k, j + k][u]^{-1},$$

for $1 \leq i \leq k$ and $1 \leq j \leq p$. In the following calculations, all quantum minors $[- | -]$ are formed from the generators x_{ij} of $\mathcal{O}_q(M(k, p))$.

As $2k \leq n$, we know that $k \leq n - k = p$. In this case, the quantum minor $[I | J] := [1, \dots, k | p + 1 - k, \dots, p]$ is defined (we are using all the rows of $\mathcal{O}_q(M(k, p))$ and the last k columns (and there are at least k columns, by the assumption)).

Now, Lemma 3.2 shows that $[I | J] = [p + 1, \dots, n][1, \dots, k]^{-1} = [w][u]^{-1}$, and it follows from this that $\rho([I | J]) = [I | J]$ for any element $\rho \in \mathcal{H}$ such that $\rho([u]) = [u]$ and $\rho([w]) = [w]$.

We can calculate how $[I | J]$ commutes with $[u] = [1, \dots, k]$. Note that $k < n - k + 1$, as $2k \leq n$. Thus the index sets $\{1, \dots, k\}$ and $\{p + 1, \dots, n\}$ do not overlap, and

$$[u][I | J] = [u][w][u]^{-1} = q^k[w][u][u]^{-1} = q^k[w][u]^{-1}[u] = q^k[I | J][u],$$

where the second equality comes from Lemma 3.1.

Also, we know how $[I | J]$ commutes with the x_{ij} by the following lemma.

Lemma 6.1. (i) If $j \geq n + 1 - 2k = p + 1 - k$ then $x_{ij}[I | J] = [I | J]x_{ij}$.
(ii) If $j < n + 1 - 2k = p + 1 - k$ then $x_{ij}[I | J] = q[I | J]x_{ij}$.

Proof. (i) In this case, x_{ij} is in the quantum matrix algebra determined by the rows from I and the columns from J and $[I | J]$ is the quantum determinant of this algebra, so the claim follows as $[I | J]$ is central in this algebra.

(ii) This result is obtained from the first equation in E(1.3c) in [4, Section 1.3]. \square

We define two gradings on $T := \mathcal{O}_q(M(k, p))[y^{\pm 1}; \sigma] = \mathcal{O}_q(G(k, n))[[u]^{-1}]$ which grade T according to how elements commute with $y = u$ and $[I | J] = [1, \dots, k | p + 1 - k, \dots, p]$.

First, set $T_i := \{a \in T \mid yay^{-1} = q^i a\}$.

Lemma 6.2. (i) $T = \bigoplus_{i=1}^{\infty} T_i$,
(ii) $\rho(T_i) = T_i$.

Proof. Note that T is generated by $y^{\pm 1}$ and the x_{ij} , and that $y^{\pm 1} \in T_0$, while $x_{ij} \in T_1$, as $yx_{ij} = qx_{ij}y$. As $\mathcal{O}_q(M(k, p))$ is an iterated Ore extension with the elements x_{ij} added lexicographically, the elements of the form $x_{11}^{a_{11}} \dots x_{kp}^{a_{kp}} y^s$ with $a_{ij} \geq 0$ and $s \in \mathbb{Z}$ form a basis for T . Part (i) now follows as $yx_{ij}y^{-1} = qx_{ij}$. Part (ii) follows from the fact that $\rho(y) = y$. \square

Next, set $T^{(i)} := \{a \in T \mid [I | J]a[I | J]^{-1} = q^{-i}a\}$. The commutation rules given in Lemma 6.1 show that $x_{ij} \in T^{(0)} \cup T^{(1)}$. Also, note that $y[I | J] = [u][I | J] = q^k[I | J][u] = q^k[I | J]y$, so $y = u \in T^{(k)}$ and $y^{-1} \in T^{(-k)}$.

Lemma 6.3. (i) $T = \bigoplus_{i \in \mathbb{Z}} T^{(i)}$,
(ii) $\rho(T^{(i)}) = T^{(i)}$.

Proof. Part (i) is proved as in the previous lemma, and Part (ii) follows from the fact that $\rho([I | J]) = [I | J]$. \square

Lemma 6.4. $(T^{(0)} \cup T^{(1)}) \cap T_1 \subseteq \mathcal{O}_q(M(k, p))$.

Proof. Suppose that $a \in (T^{(0)} \cup T^{(1)}) \cap T_1$. Then a is a sum of scalar multiples of monomials of the form $m := x_{11}^{a_{11}} \dots x_{kp}^{a_{kp}} y^s$ with each $a_{ij} \geq 0$ and $s \in \mathbb{Z}$. Such a monomial is in T_f , where $f = \sum a_{ij}$ and so we must have $\sum a_{ij} = 1$, as $a \in T_1$. Thus, only one x_{ij} can occur in each monomial and a is a sum of scalar multiples of monomials of the form $m := x_{ij} y^b$. Such an m is in $T^{(e+bk)}$, where $e = 0$ or $e = 1$. As each m must be in $T^{(0)} \cup T^{(1)}$ we must have $e + bk = 0$ or $e + bk = 1$. The only possible solutions to these restrictions is that $b = 0$, as $k \geq 2$. Hence, a is a sum of scalar multiples of monomials of the form x_{ij} which means that $a \in \mathcal{O}_q(M(k, p))$. \square

Theorem 6.5. *Let ρ be an automorphism of $\mathcal{O}_q(G(k, n))$ with $1 < k$ and $2k \leq n$ that satisfies $\rho([u]) = [u]$ and $\rho([w]) = [w]$. Then ρ extends naturally to an automorphism of $T = \mathcal{O}_q(G(k, n))[[u]^{-1}] = \mathcal{O}_q(M(k, p))[y^{\pm 1}; \sigma]$ such that $\rho(y) = y$ and $\rho(\mathcal{O}_q(M(k, p))) = \mathcal{O}_q(M(k, p))$.*

Proof. We know that $\rho(y) = y$ and so only need to show that $\rho(\mathcal{O}_q(M(k, p))) = \mathcal{O}_q(M(k, p))$. To do this, it is sufficient to prove that $\rho(x_{ij}) \in \mathcal{O}_q(M(k, p))$ for each generator x_{ij} . Note that $x_{ij} \in (T^{(0)} \cup T^{(1)}) \cap T_1$ for each i, j , and so $\rho(x_{ij}) \in (T^{(0)} \cup T^{(1)}) \cap T_1$ for each i, j . Hence, $\rho(x_{ij}) \in \mathcal{O}_q(M(k, p))$ by the previous lemma. \square

In this section, we have again made crucial use of the assumption that q is not a root of unity.

7 The automorphism group of $\mathcal{O}_q(G(k, n))$

In Definition 4.5 and Claim 4.6 we identified a group of automorphisms of $\mathcal{O}_q(G(k, n))$ that we claimed would give us all the automorphisms. We are now in a position to justify this claim.

Recall from Definition 4.5 that $\mathcal{A} = \mathcal{H}$ when $2k \neq n$ and $\mathcal{A} := \mathcal{H} \rtimes \langle \tau \rangle$ when $2k = n$.

Theorem 7.1. *The automorphism group $\text{Aut}(\mathcal{O}_q(G(k, n)))$ of $\mathcal{O}_q(G(k, n))$ is isomorphic to \mathcal{A} .*

Proof. Let ρ be an arbitrary automorphism of $\mathcal{O}_q(G(k, n))$. By Corollary 5.7, there is an automorphism $h \in \mathcal{H}$ such that $h \cdot \rho([u]) = [u]$ and $h \cdot \rho([w]) = [w]$. It is enough to prove that this adjusted automorphism is in \mathcal{H} or in $\mathcal{H} \rtimes \langle \tau \rangle$; so we may assume that $\rho([u]) = [u]$ and $\rho([w]) = [w]$. This adjusted automorphism satisfies the hypothesis for Theorem 6.5 and so ρ extends naturally to an automorphism of $T = \mathcal{O}_q(G(k, n))[[u]^{-1}] = \mathcal{O}_q(M(k, p))[y^{\pm 1}; \sigma]$ such that $\rho(y) = y$ and $\rho(\mathcal{O}_q(M(k, p))) = \mathcal{O}_q(M(k, p))$. Given this, the action of ρ is completely determined by its restriction to $\mathcal{O}_q(M(k, p))$; so it is enough to show that ρ restricted to $\mathcal{O}_q(M(k, p))$ is realised by an element of \mathcal{H} or, if $2k = n$, that either ρ or $\rho \circ \tau$ is realised by an element of \mathcal{H} .

If $2k \neq n$, then by [10, Corollary 4.11 and its proof] ρ is determined on $\mathcal{O}_q(M(k, p))$ by row and column operations and so is in \mathcal{H} , as required.

If $2k = n$, then by [17, Theorem 3.2] either ρ or $\rho \circ \tau$ is determined by row and column operations and so is in \mathcal{H} . In either case, $\rho \in \mathcal{H} \times \langle \tau \rangle$, as required. \square

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