A rate randomized geometric Process with applications

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Abstract

The geometric process has been widely studied in various disciplines and applied in reliability, maintenance and warranty cost analysis, among others. In its applications in maintenance policy optimisation, the geometric process assumes constant repair effectiveness by its process rate. Nevertheless, in practice, maintenance effectiveness may differ from time to time and can therefore be better depicted by a random variable. Motivated by this argument, this paper proposes a new variant of the geometric process, which is referred to as the rate randomized geometric process (RRGP). The probabilistic properties of the RRGP are then investigated. The maximum likelihood method is utilised to estimate the parameters of the RRGP. Numerical examples are given to show its applicability in both maintenance policy optimization and fitting real-world failure datasets.

Keywords: Geometric process, reliability, maintenance, optimisation, stochastic ordering

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1 Introduction

1.1 Motivation

The geometric process (GP), introduced by Lam (1988), defines an extension to the renewal process (RP). A sequence of random variables $\{X_n, n = 1, 2, \dots\}$ is a GP if the cumulative distribution function (cdf) of X_n is given by $F(a^{n-1}t)$ for $n = 1, 2, \dots$ and a is a positive constant. The rate a of the GP can be regarded as the parameter depicting, for example, the effectiveness of the repair after the system under study fails. An interesting observation is that a in the GP is a constant parameter over n, which implies that the effectiveness of the repair is fixed. In the application of reliability and maintenance policy optimisation, the GP assumes that the working times X_n between two adjacent failures become shorter with the increase of n, under which the rate a can be regarded as the effectiveness of repair for failures and the effectiveness can be any value within an interval. That is, a can be regarded as a random variable. Nevertheless, the GP cannot model this randomness, nor can its variants. The motivation behind the research of this paper lies in the inadequacy of the GP and its variants with constant rate to account for the inherent variability in the effectiveness of the repair. Recognizing the limitations of GP models to capture the dynamic nature of repair effectiveness in real-world scenarios, the research endeavours to develop an improved model that can better accommodate and represent the stochastic nature of repair outcomes in the context of reliability and maintenance policy optimization.

1.2 Related literature

Since its introduction, the GP has found various applications, such as reliability analysis and maintenance policy optimization. Among many research works that appeared in the literature, we will mention a few in the following.

Lam et al. (2002) modelled a degenerative $(k + 1)$-states system by a process equivalent to a GP. Tang and Lam (2006) explored a δ-shock maintenance model for a deteriorating system assuming the system is deteriorating so that the successive threshold values are geometrically non-decreasing, and the consecutive repair times after failure constitute an increasing GP. Zhang et al. (2007) studied a deteriorating repairable system with $k+1$ states, in which there are k failure states and one working state, showing that a general monotone process model for the system is equivalent to a GP model. In Zhang and Wang (2007), the authors considered a repair model for a k-dissimilar-component series repairable system for which the successive operating times form a decreasing GP, whereas the consecutive repair times form an increasing GP.

Zhang (2008), using a GP, studied a repairable system consisting of one component and a single repairman with a delayed repair by assuming that the working time, the repair time, and the delayed repair time are all distributed exponentially. Zhang and Wang (2009) considered a deteriorating cold standby repairable system consisting of two distinct components and one repairman. For each component, they assumed that the successive working times form a decreasing GP while the consecutive repair times form an increasing GP. Yuan and Xu (2011) investigated a GP-based replacement policy for a cold standby repairable system consisting of two different components and one repairman. In Aydoğdu et al. (2010), the parameters of a GP are estimated in case the distribution of the first occurrence time of an event is assumed to be the Weibull distribution (see also Aydoğdu et al. (2013)). In Yu et al. (2013), a phase-type GP repair model with spare device procurement lead time and repairman multiple vacations is assessed. Bordes and Mercier (2013) introduced an extended geometric process and studied semi-parametric estimates and some other reliability characteristics of the model. The problem of optimal repair-replacement is explored in Wang and Zhang (2014) for a simple repairable system whose failures are detected only by periodic inspections. In Aydoğdu and Altındağ (2016), numerical approximation and Monte Carlo estimation method are proposed for assessing the mean value and variance functions of a GP. Wang and Yam (2017) proposed two repair-replacement models based on a generalised geometric process in which the geometric rate changes with the number of repairs rather than being a constant. Arnold et al. (2020b) applied an alternating GP for modelling a system's lifetime and repair time and obtained results on the mean and variance functions of corresponding counting processes. Arnold et al. (2022) proposed two new approaches for computing the mean and variance functions of two counting processes related to the alternating GP and applied them in warranty cost analysis. Wang et al. (2022) proposed a single-component repairable system model with a repairman assuming that the successive working time interval of the component and the successive repair time interval after the repair is described by the extended GP. Wu (2022) extended the GP in which the times between events/failures are not necessarily a monotone trend and estimated the model's parameters. Udoh and Effanga (2023) developed a GP-based imperfect preventive maintenance and replacement model for repairable systems that fail due to age and prolonged usage. For more information on the various applications of the GP, we refer to Wu et al. (2020) and Arnold et al. (2020a).

1.3 Contributions of the paper

As we already mentioned, the rate a of the GP can be regarded as the parameter depicting the effectiveness of repair after the system under study fails. In this case, the parameter a in the GP is considered constant over n, which implies that the effectiveness of repair is fixed. Nevertheless, in reality, repair effectiveness may be a random variable. As such, it is more reasonable to assume that the rate a in the GP is a random variable.

The paper's contribution is as follows: We first introduce a new variant of the GP, referred to as the rate-randomized geometric process (RRGP). In the RRGP, it is assumed that the repair effectiveness a in GP, instead of being a constant, is a random variable Z with a known distribution function which is more realistic from a practical point of view. Then we investigate some probabilistic properties of the RRGP. The paper finally provides potential applications in maintenance policy optimization and the maximum likelihood approach to estimating the parameters of the RRGP. The performance of the RRGP is then compared with seven other variants of the GP on two datasets on times between failures. The result showed that the RRGP outperforms seven other variants of the GP.

The proposed variant of the GP can have the strength that it does not need the rate a as needed in the GP and can therefore create a parsimonious model. This is especially helpful as it is notoriously difficult to collect a good number of failure data for reliability modelling. Furthermore, a limitation of the GP is that it can only model recurrent event data (such as failure data) with a monotonous trend. The RRGP can model data with either a monotonous trend or a non-monotonous trend (Wu, 2018).

1.4 Organization of the paper

The remainder of this paper is structured as follows. Section 2 introduces RRGP in two alternative forms. Some interpretations of the RRGP model are presented. We provide two examples in which the distribution function of RRGP has a closed form. Section 3 investigates the probabilistic properties of RRGP. A generalisation of Wald's equation to an RRGP is presented in this section. We compare two RRGPs in the usual stochastic and hazard rate orders. We show that the RRGP is closed under the '*new worse than used in expectation*' (NWUE) class of distributions. Section 4 provides the application of the RRGP in optimising maintenance policies and fits the model on two real-world datasets. We employ an RRGP-based model to maintain a one-component deteriorating (or improving) system on an N replacement policy under which the system is renewed by an identical one at the time Nth failure. In this section, we also compare the performance of the RRGP with other variants of GP models. Section 5 concludes the paper.

2 Rate randomized geometric process

This section introduces a new variant of the GP, for which we introduce a necessary concept, that is, stochastic ordering. Throughout the paper, we assume that when X is a random variable with distribution function $F(x) = P(X \leq x)$, its reliability function is denoted by $\bar{F}(x)$, where $\bar{F}(x) = 1 - F(x)$.

Definition 1. *(Shaked and Shanthikumar, 2007, page 3) Let X and Y be two random variables with reliability functions $\bar{F}$ and $\bar{G}$, respectively, such that $\bar{F}(x) \leq \bar{G}(x)$ for all $x \in (-\infty, +\infty)$, then X is said to be smaller than Y in the usual stochastic order (denoted by $X \leq_{st} Y$).*

Lam (1988) proposes the definition of the GP, as shown below.

Definition 2. (Lam, 1988) Given a sequence of non-negative random variables $\{X_n, n = 1, 2, \ldots\}$, if they are independent and the cdf of $a^{n-1}X_n$, $n = 1, 2, \ldots$, is given by $F(x)$ where a is a positive constant, then $\{X_n, n = 1, 2, \cdots\}$ is called a GP and a is called the rate of the GP.

From Definition 2, the cdf of X_n is given by $F(a^{n-1}x)$.

Remark 1. From Definition 2, we have the following results.

- If $a > 1$, then $\{X_n, n = 1, 2, \cdots\}$ is stochastically decreasing, that is, $X_{n-1} \geq_{st} X_n$.
- If $a < 1$, then $\{X_n, n = 1, 2, \cdots\}$ is stochastically increasing, that is, $X_{n-1} \leq_{st} X_n$.
- If $a = 1$, then $\{X_n, n = 1, 2, \cdots\}$ is a renewal process (RP), that is, $X_{n-1} =_{st} X_n$.

To overcome the drawback that the GP has a constant rate a and therefore cannot depict variable repair effectiveness, we propose a model to extend the GP, where a is a random variable, as defined below. Before giving the definition, we recall that the *support* of the probability distribution of a random variable X is the set of all points whose every open neighbourhood D has the property that $P(X \in D) > 0$.

Definition 3. A sequence of independent non-negative random variables $\{Y_n, n = 1, 2, \ldots\}$ is said to be a *rate-randomized geometric process* (RRGP) if the cdf of Y_n is given as

$$G_n(x) = \int_{\tau_1}^{\tau_2} F\left(\frac{x}{t^{n-1}}\right) dH(t), \quad n = 1, 2, \ldots, \tag{1}$$

where F and H are two cdf's on the positive real line, and H has the support (τ_1, τ_2) with $\tau_1, \tau_2 \geq 0$.

The cdf of G_n can be considered as the cdf of $Y_n \equiv (Z_{n-1})^{n-1}X_n$, $n = 1, 2, \dots$, where X_n's are i.i.d. with cdf F, Z_n's are i.i.d with cdf H, and sequences $\{X_n, n \geq 1\}$ and $\{Z_n, n \geq 1\}$ are independent. This is so because, under the assumptions, we have for any x

$$\begin{aligned} P(Y_n \leq x) &= P((Z_{n-1})^{n-1}X_n \leq x) \\ &= \int_{\tau_1}^{\tau_2} P\left(X_n \leq \frac{x}{t^{n-1}} | Z_{n-1} = t\right) dH(t) \\ &= \int_{\tau_1}^{\tau_2} F\left(\frac{x}{t^{n-1}}\right) dH(t) \\ &= G_n(x) \quad n = 1, 2, \dots. \end{aligned}$$

Thus, Definition 3 can be alternatively defined by Definition 4, as shown below.

Definition 4. *A sequence of independent non-negative random variables $\{Y_n, n = 1, 2, \dots\}$ is said to be a RRGP if $Y_n \equiv (Z_{n-1})^{n-1}X_n$ for $(n = 1, 2, \dots)$, where X_n's are i.i.d. with distribution F, Z_n's are i.i.d with cdf H that has the domain (τ_1, τ_2), and the sequences $\{X_n, n \geq 1\}$ and $\{Z_n, n \geq 1\}$ are independent.*

Remark 2. It should be pointed out that the RRGP defined in Definitions 3 and 4, reduces to GP if the random variables Z_n's are degenerate random variables concentrated on a constant a. Also, we get from the definitions of the RRGP that given independent non-negative random variables Z_n with distribution H, the sequence of independent random variables $\{Y_n; n = 1, 2, \dots\}$, constitutes an RRGP process if $\frac{Y_n}{(Z_{n-1})^{n-1}}$ is a renewal process with cdf F. This is so because one can easily see that

$$P\left(\frac{Y_n}{(Z_{n-1})^{n-1}} < x\right) = F(x), \quad x > 0.$$

The following proposition shows that $G_n(x)$ can be monotone in the sense of the usual stochastic order.

Proposition 1. *Assuming that the support of H is $(0, \tau_2)$, $\tau_2 \leq 1$ or $((\tau_1, \infty), \tau_1 \geq 1))$ the RRGP is decreasing (increasing) in the sense of usual stochastic order, that is, $Y_{n+1} \leq_{st} (\geq_{st}) Y_n$.*

Proof. When the support of H is $(0, \tau_2)$, $\tau_2 \leq 1$, the distribution of G_n is as follows

$$G_n(x) = \int_0^{\tau_2} F\left(\frac{x}{t^{n-1}}\right) dH(t), \quad n = 1, 2, \dots. \tag{2}$$

For any $t \in (0, \tau_2]$ and $x > 0$, $\frac{x}{t^{n-1}} \leq \frac{x}{t^n}$, implying that, for $n = 1, 2, \dots$, $F(\frac{x}{t^{n-1}}) \leq F(\frac{x}{t^n})$. Thus from (2), we have $G_n(t) \leq G_{n+1}(t)$, or equivalently, we have $\bar{G}_{n+1}(t) \leq \bar{G}_n(t)$, for all $t > 0$. This shows that the RRGP is decreasing in the sense of the usual stochastic order when the support of H is $(0, \tau_2)$, $\tau_2 \leq 1$. In the case that the support of H is (τ_1, ∞), $\tau_1 \geq 1$, the RRGP distribution G_n is as follows

$$G_n(x) = \int_{\tau_1}^{\infty} F\left(\frac{x}{t^{n-1}}\right) dH(t), \quad n = 1, 2, \ldots,$$

from which, using the same arguments as used to prove the first part, one can easily see that $\bar{G}_n(t) \leq \bar{G}_{n+1}(t)$, for all $t > 0$. □

Interpretations of RRGP

We can obtain the following interpretations for RRGP.

- Assume that a system with lifetime X is put in operation at time $t = 0$. It is assumed that the system is repaired upon failures. Under the classical model GP, after the nth repair, $n = 1, 2, \ldots$, its lifetime (repair time) becomes $a^n X$, when a is constant on $(0, 1)$ $((1, \infty))$ and the cdf of X is F. In the generalization of GP described in Definition 3, however, it is assumed that after the nth repair, because of maintenance effectiveness, its lifetime (repair time) becomes $(Z_n)^n X$, $n = 1, 2, \ldots$, when Z_n's (instead of a being constant) are i.i.d. random variables on $(0, 1)$ $((1, \infty))$ with common cdf H. After the first repair, its lifetime (repair time) decreases (increases) to a random portion $Z_1 X$ with cdf G_2. After the second failure, it is repaired, and its lifetime (the second repair time) decreases (increases) to $(Z_2)^2 X$ with cdf G_3 and, ..., after the nth failure, it is repaired and its lifetime (the nth repair time) becomes $(Z_n)^n X$ with cdf G_{n+1}, and so on, where the sequences $\{Z_n, n \geq 1\}$, and X are assumed to be independent. Under this scenario, the successive process of the system lifetime (repair time) constitutes an RRGP process.

- Another interpretation of the RRGP may be as follows. Assume that $\{Y_n, n = 1, 2, ...\}$ is an RRGP. Denote the probability density function (pdf) of X by $f(x)$. Then, from the definition of the RRGP, the pdf of Y_n is given by

  $$g_n(x) = \int_{\tau_1}^{\tau_2} \frac{1}{t^{n-1}} f\left(\frac{x}{t^{n-1}}\right) dH(t), \quad n = 1, 2, \ldots.$$

  The pdf $g_n(x)$ can have a Bayesian interpretation. Consider a system with a lifetime X with probability density function (pdf) $f(x)$ and the system starts to work at time $t = 0$. The system is repaired upon failures. As a result of repair, the system lifetime, scaling by factor z, becomes zX. In this case, the pdf of the system in the new cycle would be a conditional pdf given by

  $$f(x|z) = \frac{1}{z} f(\frac{x}{z}).$$

If z is considered as a realization of a random variable Z with pdf $h(z)$ (as a prior distribution) then the joint pdf of X and Z is given as

$$f(x,z) = f(x|z)h(z) = \frac{1}{z}f(\frac{x}{z})h(z).$$

On integrating out z, we obtain the predictor distribution of the lifetime X has a pdf $g(x)$ as

$$g(x) = \int_{\tau_1}^{\tau_2} f(x,z)dz = \int_{\tau_1}^{\tau_2} \frac{1}{z}f(\frac{x}{z})h(z)dz,$$

which is the pdf of ZX. The same argument can be given to interpreting the distribution $g_n(x)$ for all $n = 3, 4, \ldots$. This shows that the RRGP process can be viewed as the prediction of a GP in which the rate constant a is considered as the realization of the random variable with cdf H.

Usually, the cdf $G_n(x)$, may not have a closed form. The following gives two examples in which $G_n(x)$ has closed forms.

Example 1. In a maintenance policy, let X be the lifetime of a system distributed as the power distribution (a special case of beta) with cdf $F(x) = x^\alpha,\ \ 0 < x < 1,\ \alpha > 0$. Assume further that the random rate variable Z has also the power distribution with cdf $H(z) = z^\beta$, $z \in (0,1)$, $\beta > 0$. Then $G_1(x) = F(x)$, and for any $n = 2, \ldots$, we have

$$\begin{aligned}
G_n(x) =& \int_0^1 F\left(\frac{x}{t^{n-1}}\right) dH(t) \\
=& \beta \int_0^{x^{\frac{1}{n-1}}} F(\frac{x}{t^{n-1}})t^{\beta-1}dt + \beta \int_{x^{\frac{1}{n-1}}}^1 F(\frac{x}{t^{n-1}})t^{\beta-1}dt \\
=& \beta \int_0^{x^{\frac{1}{n-1}}} t^{\beta-1}dt + \beta \int_{x^{\frac{1}{n-1}}}^1 (\frac{x}{t^{n-1}})^\alpha t^{\beta-1}dt \\
=& x^{\frac{\beta}{n-1}} + \beta x^\alpha \left(\frac{1 - x^{\frac{\beta-\alpha(n-1)}{n-1}}}{\beta - \alpha(n-1)}\right), \quad \beta \neq \alpha(n-1), \\
=& \frac{\beta}{\beta - \alpha(n-1)} x^\alpha + \left(1 - \frac{\beta}{\beta - \alpha(n-1)}\right) x^{\frac{\beta}{n-1}}.
\end{aligned} \tag{3}$$

Assuming that $\beta > \alpha(n-1)$, this shows that the cdf of the RRGP process for any $n = 1, 2, \ldots,$ is an arithmetic mixture of the distributions of X and Z^{n-1}. If $\beta < \alpha(n-1)$ the mixture distribution is given as

$$G_n(x) = \left(1 - \frac{\alpha(n-1)}{\alpha(n-1) - \beta}\right) x^\alpha + \frac{\alpha(n-1)}{\alpha(n-1) - \beta} x^{\frac{\beta}{n-1}}.$$

Note that

$$\lim_{\alpha\to\frac{\beta}{n-1}} \frac{1-x^{\frac{\beta}{n-1}-\alpha}}{\beta-\alpha(n-1)} = -\frac{1}{n-1}\ln(x).$$

Thus, in this case, we obtain from (3) that when $\alpha \to \frac{\beta}{n-1}$

$$G_n(x) = x^{\frac{\beta}{n-1}} - \frac{\beta}{n-1} x^{\frac{\beta}{n-1}} \ln(x), \quad 0 < x < 1.$$

Obviously, the case of $\beta = 1$, that is, the case that both X and Z are distributed uniformly on (0,1), gives

$$G_n(x) = x^{\frac{1}{n-1}} - \frac{x^{\frac{1}{n-1}}}{n-1} \ln(x), \quad 0 < x < 1.$$

Example 2. Assume that in a maintenance policy, the repair time of the system is X where X is distributed as the Pareto distribution with a reliability function $\bar{F}(x) = \frac{1}{x^\alpha}$, $x > 1$, $\alpha > 0$. Assume further that the random rate variable Z is distributed as the Pareto distribution with reliability function $\bar{H}(x) = \frac{1}{x^\beta}$, $x > 1$, $\beta > 0$. Then for any $n = 2, 3, \ldots$, we have

$$\begin{aligned}
\bar{G}_n(x) &= \int_1^\infty \bar{F}\left(\frac{x}{t^{n-1}}\right) dH(t) \\
&= \int_1^{x^{\frac{1}{n-1}}} \bar{F}\left(\frac{x}{t^{n-1}}\right) dHt) + \int_{x^{\frac{1}{n-1}}}^\infty \bar{F}\left(\frac{x}{t^{n-1}}\right) dH(t) \\
&= \int_1^{x^{\frac{1}{n-1}}} \bar{F}\left(\frac{x}{t^{n-1}}\right) dH(t) + \int_{x^{\frac{1}{n-1}}}^\infty dH(t) \\
&= \frac{\beta}{\beta-\alpha(n-1)} x^{-\alpha} + \left(1 - \frac{\beta}{\beta-\alpha(n-1)}\right) x^{\frac{-\beta}{n-1}}.
\end{aligned}$$

which is an arithmetic mixture of the reliability functions of X and Z^{n-1}. In a similar way as we showed in Example 1, when $\alpha \to \frac{\beta}{n-1}$, we obtain

$$\bar{G}_n(x) = x^{\frac{-\beta}{n-1}} + \frac{\beta}{n-1} x^{\frac{-\beta}{n-1}} \ln(x), \quad x > 1.$$

3 Properties of the RRGP

In this section, we investigate some basic properties of RRGP. As X_n's (Z_{n-1}'s) are assumed to be i.i.d with cdf F (H), in what follows, we use for simplicity, the notation X (Z) instead of X_n (Z_{n-1}). Let μ_n denote the mean of Y_n and μ denote the mean of F, with $\mu < \infty$. Then we have

$$\mu_n = \mathrm{E}(Z^{n-1}X) = \mathrm{E}(Z^{n-1})\mathrm{E}(X) = \mu\mathrm{E}(Z^{n-1}), \quad n = 1, 2, \ldots.$$

If σ_n^2 denotes the variance of Y_n then, using the fact that for any two independent random variables X and Y

$$\text{Var}(XY) = \text{Var}(X)\text{Var}(Y) + \text{Var}(X)(\text{E}(Y))^2 + \text{Var}(Y)(\text{E}(X))^2,$$

it can be easily seen that

$$\begin{aligned}\sigma_n^2 =& \text{Var}(Z^{n-1}X) \\ =& (\text{E}(Z^{n-1}))^2\text{Var}(X) + \text{E}(X^2)\text{Var}(Z^{n-1}), \quad n \geq 1.\end{aligned} \tag{4}$$

Example 3. Assume that Z has the power distribution on $(0, 1)$, with pdf

$$h(t) = \alpha t^{\alpha-1}, \quad t > 0, \ \alpha > 0.$$

Then, we have

$$\begin{aligned}\text{E}(Z^{n-1}) &= \alpha \int_0^1 t^{n+\alpha-2} dt \\ &= \frac{\alpha}{\alpha + n - 1}.\end{aligned} \tag{5}$$

Thus, we get

$$\mu_n = \frac{\alpha}{\alpha + n - 1}\mu.$$

In particular that $\alpha = 1$, that is, Z has uniform distribution on $(0, 1)$, we obtain $\mu_n = \frac{\mu}{n}$. Also, we have

$$\text{E}(Z^{2(n-1)}) = \frac{\alpha}{\alpha + 2(n - 1)}.$$

This implies that

$$\text{Var}(Z^{n-1}) = \frac{\alpha(n - 1)^2}{(2n + \alpha - 2)(\alpha + n - 1)^2}. \tag{6}$$

Thus, Eqs. (4), (5) and (6) the variance σ_n^2 can be calculated. In particular case that Z is uniform on $(0, 1)$, we have

$$\sigma_n^2 = \frac{1}{2n - 1}\left(\sigma^2 + (n - 1)^2\mu^2\right), \quad n \geq 1,$$

where $\sigma^2 = \text{Var}(X)$.

Let $\{Y_n, n = 1, 2, \dots\}$ be a RRGP and define $S_0 = 0$ and $S_n = \sum_{i=1}^{n} Y_n$. If the random variable Z is distributed on $(1, \infty)$, then it is easily seen that $S_n \xrightarrow{a.s} \infty$ as $n \to \infty$. If Z is distributed on $(0, 1)$ then

$$\begin{aligned}\lim_{n\to\infty} \mathrm{E}\left[S_n\right] &= \lim_{n\to\infty} \sum_{m=1}^{n} \mathrm{E}(Z^{m-1}X_m) \\ &= \sum_{m=1}^{\infty} \mathrm{E}(Z^{m-1})\mathrm{E}(X_m) \\ &= \mu \sum_{m=1}^{\infty} \mathrm{E}(Z^{m-1})\end{aligned}$$

There are situations where $\sum_{m=1}^{\infty} \mathrm{E}(Z^{m-1})$ tends to infinity and hence $\lim_{n\to\infty} E\left[S_n\right] = \infty$. For example if $Z \sim U(0, 1)$, uniform distribution, then $\mathrm{E}(Z^{m-1}) = \frac{1}{m}$ and hence

$$\sum_{m=1}^{\infty} \frac{1}{m} = \infty,$$

implying that $\lim_{n\to\infty} E\left[S_n\right] = \infty$. However, there are situations where $\lim_{n\to\infty} E\left[S_n\right] < \infty$.

Let us look at the following example.

Example 4. Let Z be distributed on $(0, 1)$ with a (special case of beta) distribution with pdf

$$h(z) = k(1 - z)^{k-1}, \quad 0 < z < 1, \tag{7}$$

where $k > 1$ is a real number. Then

$$\mathrm{E}(Z^{m-1}) = k \int_0^1 x^{m-1}(1 - x)^{k-1} dx = \frac{\Gamma(k + 1)\Gamma(m)}{\Gamma(m + k)}.$$

It can be shown

$$\begin{aligned}\sum_{m=1}^{\infty} \mathrm{E}(Z^{m-1}) &= \Gamma(k + 1) \sum_{m=1}^{\infty} \frac{\Gamma(m)}{\Gamma(m + k)} \\ &= \Gamma(k + 1) \sum_{m=1}^{\infty} \frac{1}{m(m + 1) \dots (m + k - 1)} = \frac{k}{k - 1}.\end{aligned}$$

This implies that

$$\lim_{n\to\infty} \mathrm{E}\left[S_n\right] = \frac{k}{k - 1}\mu < \infty.$$

Let $N(t)$ be the total number of events in an RRGP in the interval $(0, t]$ and let $S_{N(t)} = \sum_{i=1}^{N(t)} Y_i$. The following theorem gives a generalization of Wald's equation to an RRGP.

Proposition 2. *Assume that* $\{Y_n, n = 1, 2, ...\}$ *is a RRGP with random rate* Z*, and* $E[X_1] = \mu < \infty$*, then for any* $t > 0$*, we have*

$$E[S_{N(t)+1}] = \mu E\left[\sum_{n=1}^{N(t)+1} E(Z^{n-1})\right]. \tag{8}$$

Proof. For any set A let I_A be the indicator function on the set A. Then

$$\begin{aligned}
\mathrm{E}[S_{N(t)+1}] = \mathrm{E}\left[\sum_{n=1}^{N(t)+1} Y_n\right] &= \sum_{n=1}^{\infty} \mathrm{E}[Y_n I_{\{N(t)+1 \geq n\}}] \\
&\overset{(a)}{=} \sum_{n=1}^{\infty} \mathrm{E}[Y_n] P(N(t)+1 \geq n) \\
&= \mu \sum_{n=1}^{\infty} \mathrm{E}[Z]^{n-1} P(N(t)+1 \geq n) \\
&= \mu \sum_{i=1}^{\infty} \left(\sum_{n=1}^{i} \mathrm{E}[Z]^{n-1}\right) P(N(t)+1 = i) = \mu \mathrm{E}\left[\sum_{n=1}^{N(t)+1} \mathrm{E}[Z]^{n-1}\right].
\end{aligned}$$

where equation (a) follows from the fact that, Y_n and $I_{\{S_n \leq t\}} = I_{\{N(t)+1 \geq n\}}$ are independent and the last equality follows from the definition of expectation of a random variable. □

From this proposition, we get the following corollary.

Corollary 1. *Note that if* $Z \in (0, 1)$*, then for all* $n = 1, 2, \ldots$, $E(Z^{n-1}) < 1$ *and if* $Z \in (1, \infty)$*, then* $E(Z^{n-1}) > 1$*. Thus, from equation (8), we can easily get that*

$$E[S_{N(t)+1}] \begin{cases} \geq \mu E[N(t)+1] & \text{if} \quad Z \in (0, 1], \\ \\ \leq \mu E[N(t)+1] & \text{if} \quad Z \in (1, \infty). \end{cases}$$

Note that if Z is a degenerate random variable concentrated on 1, then the RRGP reduces to the renewal process. In this case, we get

$$E[S_{N(t)+1}] = \mu E[N(t)+1].$$

Example 5. Let Z be distributed as beta distribution in (7). Then

$$\mathrm{E}(Z^{n-1}) = \frac{k!(n-1)!}{(n+k-1)!}, \quad n = 1, 2, \ldots.$$

Then

$$\begin{aligned} E[S_{N(t)+1}] =&\mu E\left[\sum_{n=1}^{N(t)+1} E[Z_{n-1}]^{n-1}\right] \\ =&\mu k! E\left[\sum_{n=1}^{N(t)+1} \frac{(n-1)!}{(n+k-1)!}\right]. \end{aligned}$$

In particular that Z is uniform random variable on $(0,1)$, then

$$\mathrm{E}[S_{N(t)+1}] =\mu \mathrm{E}\left[\sum_{n=1}^{N(t)+1} \frac{1}{n}\right] \approx \mu\left(\mathrm{E}\ln(N(t)+1)+\gamma\right), \tag{9}$$

where $\gamma = 0.5772156649$ is the Euler constant.

As $S_{N(t)+1} > t$, from approximation (9), we arrive at the following upper bound.

$$\mathrm{E}\ln(N(t)+1) > \frac{t}{\mu} - \gamma.$$

Stochastic ordering and aging

In this section, we investigate some stochastic ordering and aging properties of RRGP.

Proposition 3. *Let $\{Y_{i,n} \equiv Z_{i,n-1}^{n-1} X_{i,n}, n = 1, 2, \ldots,\}$ be two RRGPs, where $X_{i,n}$'s have cdf's F_i, $i = 1, 2$, and the sequence $\{Z_{i,n}, n \geq 1\}$ (independent of the sequence $\{X_{i,n}, n \geq 1\}$) have cdf's H_i, $i = 1, 2$. If $X_{1,1} \leq_{st} X_{2,1}$, and $Z_{1,1} \leq_{st} Z_{2,1}$, then for all n, $Y_{1,n} \leq_{st} Y_{2,n}$.*

Proof. Assume that $\bar{G}_{i,n}(x)$ denotes the reliability function of $Y_{i,n}$, and $\bar{H}_i$ denotes the reliability function of $Z_{i,n}$, $i = 1, 2$. For $n = 1$, the result is trivial. For $n = 2, \ldots,$ we have

$$\begin{aligned} \bar{G}_{1,n}(x) =& \int_0^\infty \bar{H}_1\left(\left(\frac{x}{t}\right)^{\frac{1}{n-1}}\right) dF_1(t) \\ \leq& \int_0^\infty \bar{H}_1\left(\left(\frac{x}{t}\right)^{\frac{1}{n-1}}\right) dF_2(t) \\ \leq& \int_0^\infty \bar{H}_2\left(\left(\frac{x}{t}\right)^{\frac{1}{n-1}}\right) dF_2(t) \\ =&\bar{G}_{2,n}(x), \end{aligned}$$

where the first inequality follows from the fact that for two random variables T_1 and T_2, $T_1 \leq_{st} T_2$ is equivalent to $E(\eta(T_1)) \leq E(\eta(T_2))$ for any increasing function η; note that here under the assumption $X_{1,1} \leq_{st} X_{2,1}$ and $\bar{H}_1$ is decreasing. The second inequality follows based on the assumption $Z_{1,1} \leq_{st} Z_{2,1}$. □

The GP and some of its variants can only model a monotonic pattern, which may prevent them from wider applications. However, the RRGP proposed in this paper may not be stochastically monotonic and provide wider potential applications.

Proposition 4. *Let* $\{Y_n, n = 1, 2, \ldots,\}$ *be a RRGP. Then* $\{Y_n, n = 1, 2, \ldots\}$ *may be non-monotonic in* n.

To prove Proposition 4, we can simply give an example. For example, consider the sequence $Y_n \equiv (Z_{n-1})^{n-1} X_n$, $n \geq 2$, with $Z_{n-1} = \frac{1}{Z^*_{n-1}}$ where $Z^*_{n-1} \sim H(t) = 1 - e^{-\lambda t}$, $t \in (0, \infty)$, and $F(x) = \frac{x}{b-a}$, $x \in (a, b)$. Then, the cdf of Y_n, $G_n(x)$, is given as

$$\begin{aligned} G_n(x) &= \int_0^{+\infty} F(t^{n-1}x) dH(t) \\ &= \frac{\lambda x}{b-a} \int_0^{+\infty} t^{n-1} e^{-\lambda t} dt \\ &= \frac{x}{(b-a)\lambda^{n-1}} \int_0^{+\infty} (\lambda t)^{n-1} e^{-\lambda t} d(\lambda t) \\ &= \frac{x}{(b-a)\lambda^{n-1}} \Gamma(n). \end{aligned}$$

Then $G_n(x)$ may not be monotonic in n. For example, let $\lambda = 2$, then $G_2(x) < G_1(x) < G_5(x)$.

To investigate further the probabilistic properties of the proposed RRGP, we introduce the concept of hazard rate order.

Definition 5. (Shaked and Shanthikumar, 2007, page 16) Let X and Y be two nonnegative random variables with absolutely continuous cdf's and with hazard rate functions r and q, respectively, such that

$$r(t) \geq q(t), \qquad t \in \mathbb{R}. \tag{10}$$

Then X is said to be smaller than Y in the hazard rate order denoted as $X \leq_{hr} Y$.

In order to give the next result, we need the following lemma, that is, Lemma 1, which was proved by Asadi and Shanbhag (1999, Theorem 4.1).

Lemma 1. *Let* X_1 *and* X_2 *be two random variables such that* $X_2 \leq_{hr} X_1$. *Let* Z *be a continuous random variable independent of* X_1 *and* X_2 *such that* $P(X_1 \geq Z) > 0$*(and hence also such that* $P(X_2 \geq Z) > 0$*). Then*

$$(X_1 | X_1 \geq Z) \geq_{st} (X_2 | X_2 \geq Z).$$

The result of the lemma immediately implies that if the conditions of the lemma are met, then

$$(X_1|ZX_1 \geq z) \geq_{st} (X_2|ZX_2 \geq z), \tag{11}$$

with z arbitrary, provided that $P(ZX_1 \geq z) > 0$ (and hence $P(ZX_2 \geq z) > 0$). Using this result, we have the following proposition.

Proposition 5. *Let $\{Y_{i,n} \equiv (Z_{n-1})^{n-1}X_{i,n}, n = 1, 2, \dots, \}$ be two RRGPs, where $X_{i,n}$ have cdf's F_i, $i = 1, 2$, respectively. Assume that the sequence $\{Z_n, n \geq 1\}$ (independent of $\{X_{i,n}, n \geq 1\}$) has a common increasing hazard rate. If that $X_{2,1} \leq_{hr} X_{1,1}$, then $Y_{2,n} \leq_{hr} Y_{1,n}$.*

Proof. Without loss of generality, we assume that $n = 2$. Assume that $\lambda_i(t)$ denote the hazard rates of Y_i and $r_Z(t)$ denote the common hazard rate of Z_n's. Then we have

$$\begin{aligned}
\lambda_1(x) =& \frac{\int_0^\infty \frac{1}{t} h\left(\frac{x}{t}\right) dF_1(t)}{\int_0^\infty \bar{H}\left(\frac{x}{t}\right) dF_1(t)} \\
=& \frac{\int_0^\infty \frac{1}{t} r_Z\left(\frac{x}{t}\right) \bar{H}\left(\frac{x}{t}\right) dF_1(t)}{\int_0^\infty \bar{H}\left(\frac{x}{t}\right) dF_1(t)} \\
=& E\left(\frac{1}{X_{1,1}} r_Z\left(\frac{x}{X_{1,1}}\right) |ZX_{1,1} > x\right) \\
\leq& \ E\left(\frac{1}{X_{2,1}} r_Z\left(\frac{x}{X_{2,1}}\right) |ZX_{2,1} > x\right) \\
=& \frac{\int_0^\infty \frac{1}{t} r_Z\left(\frac{x}{t}\right) \bar{H}\left(\frac{x}{t}\right) dF_2(t)}{\int_0^\infty \bar{H}\left(\frac{x}{t}\right) dF_2(t)} \\
=& \frac{\int_0^\infty \frac{1}{t} h\left(\frac{x}{t}\right) dF_2(t)}{\int_0^\infty \bar{H}\left(\frac{x}{t}\right) dF_2(t)} = \lambda_2(x),
\end{aligned}$$

where the inequality follows from the assumption that the hazard rate $r_Z(t)$ is increasing, relation (11) and Lemma 1. □

It is well-known that the GP is preserved under the NBUE (NWUE) class of distributions. Recall that a continuous random variable X with survival function $\bar{F}(x) = P(X > x)$ is said to be new better (worse) than used in expectation (NBUE (NWUE)) if

$$\frac{\int_t^\infty \bar{F}(u)du}{\bar{F}(t)} \leq (\geq)\mu, \qquad for\ all\ t > 0,$$

provided that $\bar{F}(t) > 0$, where μ is the expectation of X.

The following proposition shows that the RRGP is preserved under the NWUE property.

Proposition 6. *Let $\{Y_n \equiv (Z_{n-1})^{n-1}X_n, n = 1, 2, \dots, \}$ be a RRGP where the common cdf F of the sequence $\{X_n, n \geq 1\}$ is NWUE. Then Y_n, for any n, is NWUE.*

Proof. We prove, without loss of generality, the result for $n = 2$. Based on the definition, we should prove that, for any $t > 0$,

$$\frac{\int_t^\infty \bar{G}_2(x)dx}{\mu \mathrm{E}(Z_1)} \geq \bar{G}_2(t).$$

We have

$$\begin{aligned}
\frac{\int_t^\infty \bar{G}_2(x)dx}{\mu \mathrm{E}(Z_1)} =& \frac{1}{\mu \mathrm{E}(Z_1)} \int_t^\infty \left(\int_{\tau_1}^{\tau_2} \bar{F}(\frac{x}{z}) dH(z) \right) dx \\
=& \frac{1}{\mu \mathrm{E}(Z_1)} \int_{\tau_1}^{\tau_2} \left(\int_t^\infty \bar{F}(\frac{x}{z}) dx \right) dH(z) \\
=& \frac{1}{\mu \mathrm{E}(Z_1)} \int_{\tau_1}^{\tau_2} \left(z \int_{\frac{t}{z}}^\infty \bar{F}(u) du \right) dH(z) \\
\geq& \frac{1}{\mathrm{E}(Z_1)} \int_{\tau_1}^{\tau_2} z\bar{F}(\frac{t}{z}) dH(z), \qquad (12)
\end{aligned}$$

where the inequality follows from the assumption that F is NWUE. Note that $\bar{F}(\frac{t}{z})$ is an increasing function of z for any values of t, This implies that

$$\mathrm{Cov}\left[Z_1, \bar{F}\left(\frac{t}{Z_1}\right)\right] = \mathrm{E}\left[Z_1 \bar{F}\left(\frac{t}{Z_1}\right)\right] - \mathrm{E}\left[\bar{F}\left(\frac{t}{Z_1}\right)\right] \mathrm{E}(Z_1) \geq 0.$$

This implies that right-hand side of (12) can be written

$$\begin{aligned}
\frac{1}{\mathrm{E}(Z_1)} \int_{\tau_1}^{\tau_2} z\bar{F}(\frac{t}{z}) dH(z) =& \frac{\mathrm{E}\left[Z_1 \bar{F}\left(\frac{t}{Z_1}\right)\right]}{\mathrm{E}(Z_1)} \\
\geq& \mathrm{E}\left[\bar{F}\left(\frac{t}{Z_1}\right)\right] = \int_{\tau_1}^{\tau_2} \bar{F}\left(\frac{t}{z}\right) dH(z) = \bar{G}_2(t).
\end{aligned}$$

This completes the proof of the proposition. □

Remark 3. In Proposition 6, we have demonstrated the closure property of the RRGP under the NWUE class of distributions. As mentioned earlier, the GP is preserved under the NBUE property. An intriguing problem would be to explore the closure property of RRGP under the class of NBUE distributions or to present a counterexample demonstrating that the RRGP is not preserved under NBUE.

4 Applications

4.1 Maintenance models based on RRGP

In the reliability literature, many research works have appeared in the maintenance of one-component systems based on geometric processes (see Lam (2007)). In this section, we employ an

RRGP-based model for the maintenance of a one-component deteriorating (or improving) system adopted by Lam (2007). The policy is an N replacement policy under which the system is renewed by an identical one at the time Nth failure. First, we need to impose the following assumptions (see Lam (2007)) for the deteriorating system:

(i) A new system with lifetime X_1 (and $\mathrm{E}(X_1) = \lambda$) is put in operation and upon the system failure, it gets repaired and the repair makes the system gets back to a new state.

(ii) For $n > 1$, let $Y_{1,n}$ be the system lifetime after the $(n-1)$th repair whose distribution is given by (1), where $Z = Z_1$ with support (0,1) is a random variable independent of $X_1 \sim F_1$. That is, the sequence $\{Y_{1,n},\ n = 1, 2, \dots\}$ form a decreasing RRGP.

(iii) The repair time of the system is denoted X_2 (with $\mathrm{E}(X_1) = \mu$). The repair time after the $(n-1)$th repair is $Y_{2,n}$ whose distribution is given by (1), where $Z = Z_2$ with support $(1, \infty)$ is a random variable independent of $X_2 \sim F_2$. That is the sequence $\{Y_{2,n}, n = 1, 2, \dots\}$ form an increasing RRGP process.

(iv) Let the replacement time be W with $\mathrm{E}(W) = \eta$, the operating reward rate be r, and the cost of repair be c. Moreover, the replacement cost includes two parts: the basic replacement cost denoted by R, and the other is assumed to be proportional to the replacement time W at rate c_p.

Remark 4. In the assumptions (i) and (ii) if we assume that $Z_1 \in (1, \infty)$ and $Z_2 \in (0, 1)$, the RRGP maintenance model can be applied for a reliability improving system.

In this policy, a cycle is the time between two consecutive replacements. Then, the successive cycles together with the costs incurred in each cycle will constitute a renewal reward process. Then, according to the basic renewal reward theorem (see Ross (2016)), the expected cost rate can be obtained as the ratio of the expected cost incurred in a cycle to the expected length of a cycle. Thus, based on the assumptions, the long-run expected cost adopted from Lam (2007) is

$$\begin{aligned} C(N) =& \frac{\mathrm{E}\left(c\sum_{k=1}^{N-1} Y_{2,k} - r\sum_{k=1}^{N} Y_{1,k} + R + c_p W\right)}{\mathrm{E}\left(\sum_{k=1}^{N-1} Y_{2,k} + \sum_{k=1}^{N} Y_{1,k} + W\right)} \\ =& \frac{c\mu\sum_{k=1}^{N-1}\mathrm{E}(Z_2^{k-1}) - r\lambda\sum_{k=1}^{N}\mathrm{E}(Z_1^{k-1}) + R + c_p\eta}{\lambda\sum_{k=1}^{N-1}\mathrm{E}(Z_1^{k-1}) + \mu\sum_{k=1}^{N}\mathrm{E}(Z_2^{k-1}) + \eta}. \end{aligned}$$

The aim here is to find an optimal replacement value N^* that minimizes $C(N)$. As the function $C(N)$, cannot be minimized analytically (see Lam (2007) for some discussion on sufficient conditions on the existence of the optimum value), the value of the optimal N^* may be assessed numerically and graphically.

Let us look at the following example.

Example 6. We consider the data of the example given by Lam (2007, page 162). In this example, parameter values are: $a = 1.05$, $b = 0.95$, $R = 3000$, $\lambda = 40$, $\mu = 15$, $c = 10$, $r = 50$, $c_p = 10$ and $\eta = 10$. Here instead of constant rates, we consider random rates Z_1 and Z_2 as follows: Z_1 is distributed as $h_1(z) = 2z$, $0 < z < 1$ and Z_2 is distributed as $h_2(z) = 1$, $1 < z < 2$. Under these assumptions the cost function $C(N)$ is given as

$$C(N) = \frac{150 \sum_{k=1}^{N-1} \frac{2^k-1}{k} - 2000 \sum_{k=1}^{N} \frac{2}{k+1} + 3000 + 100}{40 \sum_{k=1}^{N} \frac{2}{k+1} + 15 \sum_{k=1}^{N-1} \frac{2^k-1}{k} + 10}.$$

A plot of $C(N)$ is depicted in Figure 1. It is seen that the function $C(N)$ is minimized at $N^* = 4$ at the cost of $C(4) = -8.0399$.

It should be pointed out that Lam (2007) using the rate parameters $a = 1.05$ and $b = 0.95$, for the failure and repair geometric processes, respectively, obtains the optimum value $N^* = 9$ at cost $C(9) = -22.9089$.

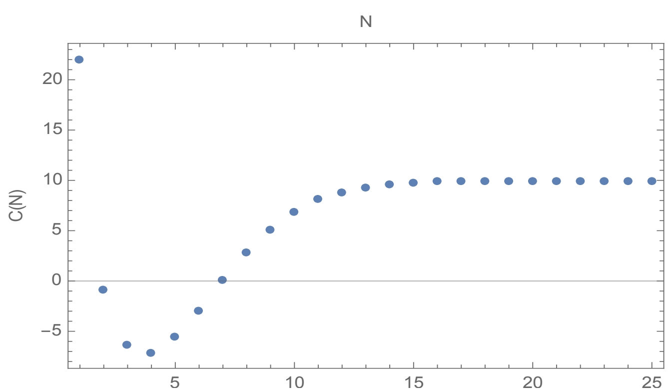

Figure 1: The plot of $C(N)$ in terms of N

Remark 5. To have a more precise approximation for the cdf of Z in the failure RRGP, we should choose the pdf $h(z)$ in such a way that the probability of the values of Z in the interval (0,1) be more concentrated around 1. A justification for choosing such a density function is that after the repair, as the system's lifetime becomes ZX, we expect, from a practical point of view, the values of Z should not be too far from 1. The same argument can be given for the repair RRGP process when $Z \in (1, \infty)$. In this case, although from a theoretical point of view, the random rate Z can be distributed on the whole interval $(1, \infty)$, practically, we should choose the pdf of Z in such a way that the corresponding values of Z be concentrated around 1.

4.2 Numerical example

Suppose one has observed N occurrences of events of a typical type, which may be N failure of a technical system. Assume that the N observed inter-arrival times are denoted by $x_1, x_2, \dots, x_N$. The likelihood function is given by $L(\boldsymbol{\theta}) = \prod_{n=1}^{N} g_n(x_n, \boldsymbol{\theta})$, where $\boldsymbol{\theta}$ is the vector of the parameters of cdf $F(x)$ and the bounds of $H(t)$ can be obtained by maximising $L(\boldsymbol{\theta})$.

Consider a RRGP $Y_n \equiv (Z_{n-1})^{n-1} X_n$, $n \geq 2$, with $Z_{n-1} = \frac{1}{Z^*_{n-1}}$, where Z^*_{n-1} has the uniform distribution on (τ_1, τ_2). Suppose that X_1 obeys the Weibull distribution with cdf $F(x) = 1 - e^{-\left(\frac{x}{\eta}\right)^\beta}$, $x > 0$, $\eta, \beta > 0$. Then one can easily verify that for $n = 1$, the density of Y_1 is $g_1(x) = \beta\eta^{-\beta}x^{\beta-1}e^{-\left(\frac{x}{\eta}\right)^\beta}$, $x > 0$, and for $n > 1$, the density function associated with Y_n is given by

$$
\begin{aligned}
g_n(x) = & \frac{1}{\beta(n-1)^2(\tau_2-\tau_1)} \left(\frac{1}{\eta}\right)^{\frac{1}{1-n}} x^{\frac{n}{1-n}} \left[\gamma\left(\frac{1}{\beta n - \beta}, \left(\frac{x}{\eta}\right)^\beta \tau_2^{\beta n - \beta}\right) - \gamma\left(\frac{1}{\beta n - \beta}, \left(\frac{x}{\eta}\right)^\beta \tau_1^{\beta n - \beta}\right)\right] \\
& - \frac{1}{(n-1)(\tau_2-\tau_1)} x^{-1} \left[\tau_2 e^{-\left(\frac{x}{\eta}\right)^\beta \tau_2^{\beta n-\beta}} - \tau_1 e^{-\left(\frac{x}{\eta}\right)^\beta \tau_1^{\beta n-\beta}}\right], \quad x > 0,
\end{aligned}
\tag{13}
$$

where $\gamma(s, x) = \int_0^x t^{s-1} e^{-t} dt$ is the lower incomplete gamma function. In this case, the process $\{Y_1, Y_2, \dots\}$ does not need a parameter, such as the rate of the geometric process (Lam, 1988), to depict its increasing or decreasing trend. We assume $\tau_1 = 1$ in Equation (13) to ensure that the RRGP is a decreasing process. Substituting $g_n(x)$ in Equation (13), the likelihood function $L(\boldsymbol{\theta})$ can be specified by

$$
\begin{aligned}
L(\boldsymbol{\theta}) &= \prod_{n=1}^{N} g_n(x_n, \boldsymbol{\theta}) \\
&= \beta\eta^{-\beta}x_1^{\beta-1}e^{-\left(\frac{x_1}{\eta}\right)^\beta} \prod_{n=2}^{N} \left\{ \frac{1}{\beta(n-1)^2(\tau_2-1)} \left(\frac{1}{\eta}\right)^{\frac{1}{1-n}} x_n^{\frac{n}{1-n}} \left[\gamma\left(\frac{1}{\beta n-\beta}, \left(\frac{x_n}{\eta}\right)^\beta \tau_2^{\beta n-\beta}\right) \right.\right. \\
&\quad \left.\left. - \gamma\left(\frac{1}{\beta n-\beta}, \left(\frac{x_n}{\eta}\right)^\beta\right)\right] - \frac{1}{(n-1)(\tau_2-1)} x_n^{-1} \left[\tau_2 e^{-\left(\frac{x_n}{\eta}\right)^\beta \tau_2^{\beta n-\beta}} - e^{-\left(\frac{x_n}{\eta}\right)^\beta}\right] \right\}.
\end{aligned}
\tag{14}
$$

Maximising the above likelihood function, we can obtain β, η, and τ_2, that is, $\hat{\boldsymbol{\theta}}$. That is

$$
(\hat{\beta}, \hat{\eta}, \hat{\tau}_2) = \underset{\beta,\eta,\tau_2}{\arg\max}\ \ln(L(\boldsymbol{\theta})). \tag{15}
$$

Lam (2007, Chapter 5) compares the performance of the GP with three other models, that is, the renewal process, the nonhomogeneous Poisson processes (NHPP) with the power-law, and the NHPP with the exponential law, on ten real-world datasets and shows the superiority of the GP.

In the following, we show two cases with the best model performance in terms of the AICc (corrected Akaike Information Criterion), which is conventionally used to measure the goodness of fit of a model on a small dataset. The two cases are built on the two datasets shown in Table 1. We compare the AICc of the RRGP on the two datasets with those of seven other models: GP (geometric process) (Lam, 1988), α-series process (Braun et al., 2005), the extended geometric process (EGP) (Wu and Clements-Croome, 2006), the doubly GP (DGP) (Wu, 2018), and the double ratio GP (DRGR) (Wu, 2022), where the DRGP has three different versions: DRGP-I, DRGP-II, and DRGP-III. The reader is referred to Wu (2022) for the details on the three versions.

Table 2 shows the AICc values of the models. The table shows that the RRGP has the smallest AICc on the two datasets, compared with those of the seven other models. The parameters of the RRGP models are shown in Table 3, where the values in the parentheses are standard errors of the corresponding estimates. Interestingly, on the two models, $\hat{\tau}_2$ is very close to 1, compared to its domain $\tau_2 \in (1, +\infty)$.

It is worth noting that it is possible to obtain different performances when different probability distributions of X_1 in GP-like models are assumed. For instance, Pekalp et al. (2020) show the performance of GP-like models when the distribution of X_1 is the log-normal distribution, and the above numerical examples show that the RRGP outperforms the selected datasets when the probability distribution of X_1 is the Weibull distribution.

Table 1: TBF (Time between failures) datasets.

Dataset No.	Dataset	N	Reference
1	TBF-7912 (Time between failures of an air-conditioner)	30	Proschan (1963)
2	Time between failures (TBF) of a compressor	24	Yanez et al. (2002)

Table 2: AICc

Dataset No.	GP	α-series	EGP	DGP	DRGP-I	DRGP-II	DRGP-III	RRGP
1	307.345	307.456	312.313	310.020	310.022	310.133	309.934	306.807
2	388.064	387.463	394.197	390.557	390.069	390.370	390.249	387.861

Table 3: Parameters

Estimated parameters on Dataset 1			Estimated parameters on Dataset 2		
$\widehat{\beta}$	$\widehat{\eta}$	$\widehat{\tau}_2$	$\widehat{\beta}$	$\widehat{\eta}$	$\widehat{\tau}_2$
0.955	108.293	1.117	0.963	1500.789	1.0747
(0.150)	(40.151)	(0.0639)	(0.148)	(720.391)	(0.0825)

5 Conclusions

This paper proposed a variant of the geometric process (GP), that is, the RRGP (rate-randomized GP), by assuming the rate of the GP is a random variable. It investigated some probabilistic properties of RRGP. It also explored its application in maintenance policy optimization. The RRGP was then fitted on two real-world datasets, which are times between failure data. The result showed that the RRGP outperforms seven other variants of the GP. The proposed variant of the GP can have the strength that it does not require the rate a as needed in the GP and can consequently create a parsimonious model. This fact is particularly useful as it is notoriously difficult to collect enough failure data for reliability modelling and analysis. Furthermore, a restriction of the GP is that it can only model recurrent event data (such as failure data) with a monotonous trend. The RRGP can model data with either monotonous or non-monotonous trends.

Statistical properties of the RRGP were not sufficiently investigated in this paper. For example, the problem of estimating the distribution of an RRGP, and in particular approximating the cdf H, based on real data or simulation study is an interesting problem. This and other related statistical problems will be our future research work.

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