Homotopical Combinatorics

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The authors of this piece are organizers of the 2024 AMS Mathematics Research Communities summer conference *Homotopical Combinatorics*, one of four topical research conferences offered this year that are focused on collaborative research and professional development for early-career mathematicians. Additional information can be found at `https://www.ams.org/programs/research-communities/2024MRC-HomotopicalComb`. Applications are open until February 15, 2024.

Homotopical combinatorics is an emerging field that studies combinatorial structures encoding aspects of equivariant homotopy theory, equivariant algebra, and abstract homotopy theory. Its methods — a pleasant mix of enumerative combinatorics, algebraic combinatorics, and order theory — are relatively elementary, but its theorems have deep implications in homotopy theory. The youth and accessibility of homotopical combinatorics should make the field especially attractive to early career researchers, and we hope that this article and the 2024 Mathematics Research Community by the same name welcome mathematicians from many backgrounds into the field.

The central object in homotopical combinatorics is the *transfer system.* These combinatorial gadgets were originally defined in order to encode the homotopy theory of N_∞ operads, which control multiplicative structures in equivariant stable homotopy theory. Special pairs of transfer systems control the structure of bi-incomplete Tambara functors, basic objects of equivariant algebra. In a seemingly unrelated direction, pairs of transfer systems also encode model structures (presentations of $(\infty, 1)$-categories) on posets. Below, we introduce transfer systems in purely combinatorial terms, and then explore their applications.

Transfer systems

Suppose $(P, \le)$ is a finite partially ordered set (poset). A (*categorical*) *transfer system* on $(P, \le)$ is a partial order $\to$ on the set P such that

- $\to$ refines $\le$: $x \to y$ implies $x \le y$, and
- $\to$ is closed under restriction: $x \to y$, $z \le y$, and w maximal among $w' \le x, z$ implies $w \to z$.

In most cases, we restrict attention to finite posets admitting greatest lower bounds (so-called *meet-semilattices*). We write $x \wedge y$ for the greatest lower

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bound (or *meet*) of x, y when it exists. When P is a meet-semilattice, the restriction condition becomes simpler:

- $\diamond$ $x \to y$ and $z \leq y$ implies $x \wedge z \to z$.

Categorically inclined readers will recognize this condition as closure under pullbacks, and it is pleasant to record diagramatically, where solid arrows are relations in the transfer system, dashed arrows represent $\leq$, and the double arrow indicates logical implication; we draw the diagram "oriented upwards" so it is also reminiscent of a Hasse diagram:

$$\begin{tikzcd}[column sep=small] & y & \\ z \arrow[ur, dashed] & & x \arrow[ul] \arrow[ll, phantom, "\Longleftarrow"] \\ & x \wedge z \arrow[ul] \arrow[ur, dashed] & \end{tikzcd}$$

We write $\operatorname{Tr} P$ for the collection of all transfer systems on P. The set $\operatorname{Tr} P$ admits a natural partial order by refinement: $\to \, \leq \, \rightsquigarrow$ if and only if $x \to y$ implies $x \rightsquigarrow y$. If P is a finite lattice (admits least upper and greatest lower bounds), then $\operatorname{Tr} P$ is a finite lattice as well.

One of the fundamental problems of transfer systems is to determine the structure of the lattice $\operatorname{Tr} P$ for a given lattice P or family of lattices. In [BBR21], Balchin–Barnes–Roitzheim achieve this for $P = [n] = \{0 < 1 < \cdots < n\}$ a finite chain. They prove that $\operatorname{Tr} P$ is isomorphic to the famed *Tamari lattice* $\mathcal{A}_{n+1}$ of planar rooted binary trees with $n+2$ leaves; see Figure 1. In particular, transfer systems on $[n]$ are counted by Catalan numbers, with

$$|\operatorname{Tr}[n]| = \operatorname{Cat}(n+1) = \frac{1}{2n+3}\binom{2n+3}{n+1}.$$

There are also some general structural results on transfer systems. In Construction 2.9 of [BMO23a], Balchin–MacBrough–Ormsby give a recursion for $|\operatorname{Tr} P|$ in terms of transfer systems on certain induced subposets. The recursion is based on the notion of the *minimal fibrant* element of a transfer system $\to$, *i.e.*, the (necessarily unique) minimal element m of P such that $m \to \top$, where $\top$ denotes the maximum of P. In [BHK+23], the participants in the 2023 Electronic Computational Homotopy Theory REU follow an idea of Hill to relativize minimal fibrancy, resulting in a *characteristic function* $\chi^{\to}\colon P \to P$ defined by $\chi^{\to}(x) = \min\{y \in P \mid y \to x\}$. This ultimately provides a strong (but far from tight) lower bound on the cardinality of transfer systems. To state the theorem, let $\operatorname{End}^\circ P$ denote the set of *interior operators* on P, that is, order-preserving functions $f\colon P \to P$ that are *contractive* ($f(x) \leq x$) and *idempotent* ($f(f(x)) = f(x)$). We give $\operatorname{End}^\circ P$ the pointwise ordering $f \leq g \iff f(x) \leq g(x)$ for all $x \in P$.

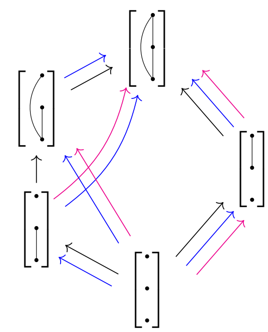

Figure 1: In brackets, we display the five elements of $\operatorname{Tr}[2]$. The elements of $[2]$ are arranged vertically as dots (0 lowest, 2 highest), and each transfer system is depicted by lines indicating relations present in the transfer system, omitting reflexive loops $x \to x$. The black arrows represent the covering (*i.e.*, minimal) relations of $\operatorname{Tr}[2]$; they assemble into a pentagon isomorphic to $\mathcal{A}_3$. The rest of the diagram should be interpreted after the reader engages with the *Model structures on posets* section. The blue arrows correspond to $\preccurlyeq$, and the magenta arrows are the covering relations of $\sqsubseteq$. Counting black, blue, and magenta intervals, we see that $|\operatorname{Pre}[2]| = 13$, $|\operatorname{Pre}^{cc}[2]| = 12$, and $|\operatorname{MS}([2])| = 10$.

Theorem 1 (Theorems 2.8 and 2.12 of [BHK+23])**.** *The assignment*

$$\begin{aligned} \chi\colon \operatorname{Tr} P &\longrightarrow \operatorname{End}(P) \\ \to &\longmapsto \chi^{\to} \end{aligned}$$

is an order-reversing map with image $\operatorname{End}^\circ P$.

While interior operators are hard to enumerate, their asymptotic behavior is understood, and Kleitman [Kle76] proves that the base-2 logarithm of $|\operatorname{End}^\circ([1]^n)|$ grows like $\binom{n}{\lfloor n/2\rfloor}$ (see OEIS A102896).

In order to prepare for applications in equivariant homotopy theory, let G be a finite group. We will take particular interest in the case $P = \operatorname{Sub} G$, the lattice of subgroups of G ordered under inclusion. (Note: If $G = C_{p^n}$, the cyclic group of order p^n, p prime, then $\operatorname{Sub} G \cong [n]$. This is the orginal context of [BBR21].) We will need, though, to introduce one additional axiom in this context: A *G-transfer system* is a categorical transfer system $\to$ on $\operatorname{Sub} G$ such that

- $\to$ is closed under conjugation: $H \to K$ implies ${}^gH \to {}^gK$

where ${}^gH := gHg^{-1}$ is the g-conjugate of H. We write $\operatorname{Tr} G$ for the lattice of G-transfer systems under refinement. Of course, if G is Abelian, then G-transfer systems and categorical transfer systems on $\operatorname{Sub} G$ are identical.

Despite their elementary and relatively natural definition, the authors are not aware of any appearance of such structures on posets prior to [Rub21, BBR21]. If any reader has encountered objects isomorphic to transfer systems in older (presumably combinatorial or order-theoretic) literature, we invite them to contact us.

N_∞ operads

Transfer systems first arose through the work of Blumberg–Hill [BH15] on N_∞ operads. These are equivariant generalizations of E_∞ operads, and their algebras are equipped with both an operation that is associative and commutative up to coherent homotopies (coming from an E_∞ structure) and homotopy coherent multiplicative norm maps (encoded by the fixed points of the spaces in the operad). Ever since their appearance in the Hill–Hopkins–Ravenel [HHR16] solution of the Kervaire invariant one problem, norms have become a critical component of contemporary equivariant homotopy theory. Each N_∞ operad encodes potentially different classes of norms, and thus we need to classify N_∞ operads if we hope to understand what norms might appear in applications.

Let G be a finite group and let $\mathfrak{S}_n$ denote the symmetric group on n letters. A *G-operad $\mathscr{O}$* is a sequence of $G \times \mathfrak{S}_n$-spaces $\mathscr{O}(n)$, $n \geq 0$ along with an identity element $1 \in \mathscr{O}(1)$ fixed by $G = G \times \mathfrak{S}_1$ and a G-equivariant composition map

$$\mathscr{O}(k) \times \mathscr{O}(n_1) \times \cdots \times \mathscr{O}(n_k) \to \mathscr{O}(n_1 + \cdots + n_k)$$

satisfying the standard compatibility conditions for an operad. A *map* of G-operads is a morphism of operads in G-spaces; in particular, at level n it is $G \times \mathfrak{S}_n$-equivariant.

A *G-N_∞ operad* (or just *N_∞ operad* if G is clear from context) is a G-operad such that

- $\mathscr{O}(0)$ is G-contractible,
- the action of $\mathfrak{S}_n = e \times \mathfrak{S}_n$ on $\mathscr{O}(n)$ is free,
- for all $\Gamma \leq G\times\mathfrak{S}_n$, the Γ-fixed point space $\mathscr{O}(n)^\Gamma$ is either contractible or empty, and
- for each n, the collection of $\Gamma \leq G\times\mathfrak{S}_n$ such that $\mathscr{O}(n)^\Gamma \simeq *$ is closed under conjugacy and under passage to subgroups[1] and contains all subgroups of the form $H \times e$.

The category of G-N_∞ operads is denoted $N_\infty\text{-}\mathbf{Op}^G$.

A G-operad map $\varphi\colon \mathscr{O}_1 \to \mathscr{O}_2$ of N_∞ operads is a *weak equivalence* when it induces a weak homotopy equivalence $\mathscr{O}_1(n)^\Gamma \to \mathscr{O}_2(n)^\Gamma$ for all $n \geq 0$ and all $\Gamma \leq G \times \mathfrak{S}_n$. Inverting weak equivalences in $N_\infty\text{-}\mathbf{Op}^G$ produces the *homotopy category of G-N_∞ opeards* $\operatorname{Ho}(N_\infty\text{-}\mathbf{Op}^G)$.

If $H \leq G$ and T is a finite H-set, we say that an N_∞ operad $\mathscr{O}$ *admits a T-norm* when $\mathscr{O}(|T|)^{\Gamma(T)} \simeq *$, where $\Gamma(T) \leq G \times \mathfrak{S}_{|T|}$ is the graph of some permutation representation $H \to \mathfrak{S}_{|T|}$ of T. If X is an $\mathscr{O}$-algebra[2] (say in G-spaces) and $\mathscr{O}$ admits T-norms,

[1] Such a collection $\mathscr{F}$ is called a *family* for the group $G\times\mathfrak{S}_n$; combining the third and fourth criteria implies that $\mathscr{O}(n)$ is a *universal space* for $\mathscr{F}$.

[2] For the operadically uninitiated, the n-th space $\mathscr{O}(n)$ of an operad $\mathscr{O}$ parametrizes n-ary operations. An algebra X over $\mathscr{O}$ comes equipped with maps $\mathscr{O}(n) \times X^n \to X$. Thus for each point of $\mathscr{O}(n)$ we get an n-ary operation on X.

then we get a G-equivariant map

$$G \times_H X^T \longrightarrow X$$

where X^T is the H-space of all functions $f\colon T \to X$ with H acting via $h \cdot f\colon t \mapsto hf(h^{-1}t)$. In particular, if $K \leq H \leq G$, then an H/K-norm induces a 'wrong-way' map

$$X^K \to X^H$$

between fixed point spaces. In an additive setting, these maps are called transfers instead of norms, leading to the nomenclature for transfer systems.

To draw out this connection further, let $\mathscr{O}$ denote a G-N_∞ operad and define a binary relation $\xrightarrow{\mathscr{O}}$ on $\operatorname{Sub} G$ by the rule

$$K \xrightarrow{\mathscr{O}} H \iff K \leq H \text{ and } \mathscr{O}([H:K])^{\Gamma(H/K)} \simeq *.$$

In other words, $K \xrightarrow{\mathscr{O}} H$ if and only if $\mathscr{O}$ admits H/K-norms. Of course, $\xrightarrow{\mathscr{O}}$ turns out to be a G-transfer system, and this assignment is part of a functor from G-N_∞ operads to (the category induced by) the lattie $\operatorname{Tr} G$. Work of many authors [BH15, GW18, BP21, Rub21, BBR21] gives the following theorem:

Theorem 2. *The assignment $\mathscr{O} \mapsto \xrightarrow{\mathscr{O}}$ induces an equivalence of categories*

$$\operatorname{Ho}(N_\infty\text{-}\mathbf{Op}^G) \xrightarrow{\simeq} \operatorname{Tr} G$$

where $\operatorname{Tr} G$ is viewed as the category with objects G-transfer systems and a unique morphism between transfer systems if and only if the source refines the target.

This provides a first and pressing motivation for studying transfer systems: by determining the structure of $\operatorname{Tr} G$, we solve a classification problem for G-N_∞ operads; if we know all the G-transfer systems, then we know exactly which collections of norms are induced by N_∞ operads.

At the time of writing, the full structure of $\operatorname{Tr} G$ is known for the following finite groups G (p, q, r distinct primes): C_{p^n} [BBR21], C_{pq}, $C_2 \times C_2$, Q_8, $\mathfrak{S}_3$ [Rub21], C_{pqr} [BBPR20], and $C_p \times C_p$ [BHK+23]. Additionally, Balchin–MacBrough–Ormsby [BMO23a] determine elaborate interleaved recurrences which effectively compute $|\operatorname{Tr} C_{qp^n}|$ and $|\operatorname{Tr} D_{p^n}|$ but do not give closed forms.

Another motivation for acquiring structural and enumerative knowledge of $\operatorname{Tr} G$ is understanding and describing the complicated behavior of N_∞ structures with respect to localization. While Bousfield and finite localizations of topological spectra preserve E_∞ structures, it is *not* the case that such localizations preserve N_∞ structures. Rather, localization can destroy norms. In [Hil19], Hill has studied certain chromatic localizations of equivariant ring spectra and deduced conditions under which thick subcategories preserve $\mathscr{O}$-algebras (see Theorem 5.2 of *loc. cit.*). Despite this significant progress, much work remains if we are to fully understand how localizations act on $\operatorname{Tr} G$.

Equivariant algebra

Each equivariant commutative ring spectrum R (*i.e.*, representing object for a generalized Bredon-style cohomology on G-spaces) carries a wealth of algebraic data on the level of $\underline{\pi}_0 R$. Here $\underline{\pi}_0 R$ may be viewed as a functor

$$\begin{aligned}(\operatorname{Sub} G)^{\mathrm{op}} &\longrightarrow \mathrm{CRing}\\ H &\longmapsto \pi_0 R^H\end{aligned}$$

where R^H denotes the H-fixed points of R (viewed as a non-equivariant spectrum). The induced homomorphism $\underline{\pi}_0 R(K \leq H) =: r^H_K\colon \pi_0 R^H \to \pi_0 R^K$ is called *restriction* along $K \leq H$. The G-universe over which R is defined (a technical condition regarding which representation spheres R has suspension isomorphisms with respect to) further endows $\underline{\pi}_0 R$ with additive *transfer* maps $t^H_K\colon \pi_0 R^K \to \pi_0 R^H$. These assemble into the data of an $\overset{a}{\dashrightarrow}$ *-Mackey functor*, where $\overset{a}{\dashrightarrow} \in \operatorname{Tr} G$ is a transfer system encoding which transfers are allowed in the Mackey functor. (There are also maps c_g induced by conjugation by group elements, but we omit these from our discussion.) The transfer and restriction maps satisfy compatibility axioms, including an elaborate double coset formula.

Now suppose $\mathscr{O}_m$ is an N_∞ operad with associated transfer system $\xrightarrow{m}$, and that R is an $\mathscr{O}_m$-algebra.

Then the $\overset{a}{\dashrightarrow}$ -Mackey functor $\underline{\pi}_0 R$ also admits multiplicative *norm* maps $n_K^H\colon \pi_0 R^K \to \pi_0 R^H$ for each $K \xrightarrow{m} H$. These maps satisfy further compatibilities involving so-called exponential diagrams which we omit from this discussion. This makes $\underline{\pi}_0 R$ a *bi-incomplete* $(\overset{a}{\dashrightarrow}, \xrightarrow{m})$*-Tambara functor* in the sense of Blumberg–Hill [BH21].[3]

In order to phrase all of the compatibilities between restrictions, transfers, and norms, certain compatibilities are necessary between $\overset{a}{\dashrightarrow}$ and $\xrightarrow{m}$. These are codified in the following theorem of Chan:

Theorem 3 (Theorem 4.10 of [Cha22])**.** *Bi-incomplete Tambara functors with respect to G-transfer systems $(\overset{a}{\dashrightarrow}, \xrightarrow{m})$ are well-defined if and only if $\xrightarrow{m} \leq \overset{a}{\dashrightarrow}$ and the following condition holds:*

- *if $K, L \leq H \leq G$ such that $K \xrightarrow{m} H$ and $K \cap L \overset{a}{\dashrightarrow} K$, then $L \overset{a}{\dashrightarrow} H$.*

We call a pair of transfer systems $(\overset{a}{\dashrightarrow}, \xrightarrow{m})$ satisfying the conditions of the theorem a *compatible pair*. We can record the final compatibility axiom diagrammatically, where the double arrow is logical implication:

$$\begin{tikzcd}
& H & \\
L \arrow[ur, dashed, "a"] & & K \arrow[ul, "m"'] \arrow[ll, phantom, "\Leftarrow"] \\
& K \cap L \arrow[ul, "m"] \arrow[ur, dashed, "a"'] &
\end{tikzcd}$$

(Note that $K \cap L \xrightarrow{m} L$ is forced by the restriction axiom for $\xrightarrow{m}$.) Loosely speaking, we are looking for intervals $\xrightarrow{m} \leq \overset{a}{\dashrightarrow}$ in $\operatorname{Tr} G$ where $\overset{a}{\dashrightarrow}$ satisfies a type of "relative saturation" condition with respect to $\xrightarrow{m}$.

Several authors have undertaken the challenge of enumerating compatible pairs of transfer systems. We highlight the work of Hill–Meng–Li which enumerates compatible pairs for $G = C_{p^n}$ (a cyclic group of order p^n, p prime).

[3] Tambara functors were originally introduced by Tambara in [Tam93], where they were referred to as *TNR-functors* for "Transfer, Norm, Restriction". We note that equivariant ring spectra are not the only source of Tambara functors. They also appear naturally when considering representation rings and other equivariant algebraic structures.

Theorem 4 (Theorem 1.7 of [HML22])**.** *For $G = C_{p^n}$, there are exactly*

$$\frac{1}{3n+4}\binom{3n+4}{n+1}$$

compatible pairs of transfer systems.

The bi-variate sequences $A_n(p,r) := \frac{r}{np+r}\binom{np+r}{n}$ are known as *Fuss–Catalan numbers*. By [BBR21], we have $|\operatorname{Tr} C_{p^n}| = \operatorname{Cat}(n+1) = A_{n+1}(2,1)$, while Theorem 4 says that compatible pairs of transfer systems for C_{p^n} are enumerated by $A_{n+1}(3,1)$. We will enounter the $(3,1)$-Fuss–Catalan numbers once more when considering composition closed premodel structures on $[n] \cong \operatorname{Sub} C_{p^n}$.

Model structures on posets

Thus far, our applications of transfer systems have been equivariant in nature, but these structures also parametrize weak factorization systems on (categories associated with) poset lattices. Compatible pairs of weak factorization systems give rise to model structures, and this provides a link between intervals in $\operatorname{Tr} P$ and abstract homotopy theory.

The role of a weak factorization system is to axiomatize the relationship between acyclic cofibrations and fibrations (or cofibrations and acyclic fibrations) in topology. This is phrased in terms of lifting properties, which we presently define. Given morphisms $i\colon a \to b$ and $p\colon x \to y$ in a category $\mathscr{C}$, we say that i has the *left lifting property* with respect to p, or that p has the *right lifting property* with respect to i, when for all commutative squares of the form

$$\begin{tikzcd}
a \arrow[r] \arrow[d, "i"'] & x \arrow[d, "p"] \\
b \arrow[r] \arrow[ur, dashed, "\exists h"] & y
\end{tikzcd}$$

in $\mathscr{C}$, there exists a morphism $h\colon b \to x$ making the diagram commute. In this situation, we write $i \boxslash p$. Given a class M of morphisms in $\mathscr{C}$, we further define

$$M^{\boxslash} := \{g \in \operatorname{Mor}\mathscr{C} \mid f \boxslash g \text{ for all } f \in M\},$$
$${}^{\boxslash}M := \{f \in \operatorname{Mor}\mathscr{C} \mid f \boxslash g \text{ for all } g \in M\}.$$

A *weak factorization system* on $\mathscr{C}$ is a pair (L, R) of subclasses of $\operatorname{Mor}\mathscr{C}$ such that

- $\diamond$ $R \circ L = \operatorname{Mor}\mathscr{C}$, and
- $\diamond$ $L = {}^{\boxslash}R$ and $R = L^{\boxslash}$.

A *premodel structure* on $\mathscr{C}$ is now a pair of weak factorization systems (L, R), (L', R') such that $R \subseteq R'$ (or equivalently $L' \subseteq L$). A premodel structure is a *model structure* when the morphism set $W := R \circ L'$ satisfies the two-out-of-three property:

- $\diamond$ if f and g are composable morphisms in $\mathscr{C}$ and two of f, g, and $g \circ f$ are in W, then so is the third.

In [JT07], Joyal–Tierney prove that this presentation of a model structure is equivalent to Quillen's, with R' playing the role of fibrations, L cofibrations, and W weak equivalences. The principal role of a model structure is to produce a nice model for the homotopy category $\operatorname{Ho}\mathscr{C} = \mathscr{C}[W^{-1}]$ in which weak equivalences are inverted.

By astounding coincidence, a weak factorization system on a finite lattice P (viewed as a category) is the same thing as a transfer system on P. Let us write $\mathrm{WFS}(P)$ for the collection of weak factorization systems on P ordered by inclusion of right morphism sets.

Theorem 5 (Theorem 4.13 of [FOO$^+$22]). *Let P be a finite poset lattice. Then the assignment*

$$\begin{aligned} \mathrm{WFS}(P) &\longrightarrow \operatorname{Tr} P \\ (L, R) &\longmapsto \xrightarrow{R} \end{aligned}$$

is an isomorphism of posets, where $\xrightarrow{R} \in \operatorname{Tr} P$ is the relation given by

$$x \xrightarrow{R} y \iff (x \to y) \in R.$$

Before considering the ramifications of this theorem for model structures, we note an important corollary regarding self-duality of transfer systems. Suppose that P is a self-dual lattice, *i.e.*, P admits an order-reversing bijection $\nabla \colon P \to P$, or, phrased categorically, ∇ is an isomorphism of categories $P^{\mathrm{op}} \to P$. Importantly, if G is Abelian, then $\operatorname{Sub} G$ is non-canonically self-dual via Pontryagin duality, so this is a case of significant interest in equivariant applications.

Theorem 6 (Theorem 4.21 of [FOO$^+$22]). *If P is a lattice with self-duality ∇, then $\operatorname{Tr} P$ is self-dual with duality*

$$\begin{aligned} \phi\colon \operatorname{Tr} P &\longrightarrow \operatorname{Tr} P \\ \to &\longmapsto \to^{\phi} := ((^{\boxslash}\!\to)^{\mathrm{op}})^{\nabla}. \end{aligned}$$

Moreover, if ∇ is an involution, then so is ϕ.

The proof hinges on the fact that the assignment $({}^{\boxslash}R, R) \mapsto (R^{\mathrm{op}}, ({}^{\boxslash}R)^{\mathrm{op}})$ is an isomorphism $\mathrm{WFS}(P) \to \mathrm{WFS}(P^{\mathrm{op}})$. While it is ultimately possible to construct the duality ϕ without reference to weak factorization systems (see Corollary 4.22 of [FOO$^+$22]), discovering and presenting this duality is much simpler when working with weak factorization systems.

We now turn to the connection between transfer systems and model structures. Any lattice P has an *interval lattice* $\operatorname{Int} P$ whose elements are intervals

$$[x, y] = \{z \in P \mid x \le z \le y\}$$

with $x \le y$; the partial order is defined by $[x, y] \le [x', y']$ if and only if $x \le x'$ and $y \le y'$. (In categorical language, this is the *arrow category* associated with P.) If $\operatorname{Pre} P$ denotes the collection of premodel structures on P, then it follows from Theorem 5 that $\operatorname{Pre} P \cong \operatorname{Int}(\operatorname{Tr} P)$; furthermore, the class $W = R \circ L'$ associated with a premodel structure $(L, R) \le (L', R')$ may be identified with $\xrightarrow{R} \circ {}^{\boxslash}\xrightarrow{R'}$, a formula only involving transfer systems. Thus, in order to enumerate model structures on a finite lattice P, it suffices to find intervals $[\to, \dashrightarrow] \in \operatorname{Int}(\operatorname{Tr} P)$ such that $\to \circ {}^{\boxslash}\!\dashrightarrow$ satisfies the two-out-of-three property.

Balchin–Ormsby–Osorno–Roitzheim solve this problem for $P = [n]$. Let $\mathrm{MS}(P)$ denote the set of model structures on P considered as an induced subposet inside $\operatorname{Pre} P \cong \operatorname{Int}(\operatorname{Tr} P)$.

Theorem 7 (Theorems 4.10 and 4.13 of [BOOR23]). *For $n \geq 0$,*

$$|\operatorname{MS}([n])| = \binom{2n+1}{n}.$$

Each model structure on $[n]$ has homotopy category isomorphic to $[k]$ for some $0 \leq k \leq n$, and the number of model structures on $[n]$ with homotopy category isomorphic to $[k]$ is exactly

$$\frac{2(k+1)}{n+k+2}\binom{2n+1}{n-k}.$$

Despite the simple form of this enumeration, the proof in [BOOR23] passes through a convolution of Catalan numbers and enumeration in terms of north/east paths on an $(n+1) \times (n+1)$ grid with first step north. A more conceptual bijection between model structures on $[n]$ and a certain flavor of tricolored tree is given by Balchin–MacBrough–Ormsby in [BMO23b].

The authors of [BMO23b] achieve their results by considering an intermediate structure between premodel and model structures, which they dub *composition closed premodel structures*. These are pairs of weak factorization systems $(L, R), (L', R')$ with $R \subseteq R'$ and $R \circ L'$ — the putative weak equivalences — closed under composition, but not necessarily fulfilling the full two-out-of-three property required of model structures. It turns out (Theorem 3.8 of [BMO23b]) that for P a finite lattice, there is a refinement $\preccurlyeq$ of the usual order on $\operatorname{WFS}(P)$ such that $(\operatorname{WFS}(P), \preccurlyeq)$ is a lattice and intervals with respect $\preccurlyeq$ are exactly the composition closed premodel structures on P. There is also a partial ordering $\sqsubseteq$ on $\operatorname{WFS}(P)$ further refining $\preccurlyeq$ such that intervals with respect to $\sqsubseteq$ are model structures, but $(\operatorname{WFS}(P), \sqsubseteq)$ is *not* a lattice. The relations $\preccurlyeq$ and $\sqsubseteq$ on $\operatorname{Tr}[2]$ are depicted in Figure 1 in blue and magenta, respectively.

Returning to the case $P = [n]$, where the standard ordering on $\operatorname{WFS}(P) \cong \operatorname{Tr} P$ gives the Tamari lattice, we find (Theorem 4.6 of [BMO23b]) that $(\operatorname{Tr}[n], \preccurlyeq)$ is isomorphic to the *Kreweras lattice* of noncrossing partitions on the set $[n]$, ordered by refinement of partitions. Since Kreweras intervals have already been enumerated, we find there are exactly

$$\frac{1}{3n+4}\binom{3n+4}{n+1}$$

composition closed premodel structures on $[n]$ — the $(3, 1)$-Fuss–Catalan numbers appear again! We rush to note, though, that the intervals encoding composition closed premodel structures on $[n] \cong \operatorname{Sub} C_{p^n}$ are distinct from the intervals encoding compatible pairs for bi-incomplete Tambara functors for C_{p^n}, and thus far no one has constructed a principled bijection between the two structures. For most finite groups G, composition closed premodel structures on $\operatorname{Sub} G$ are not equinumerous with compatible pairs of G-transfer systems.

Since Tamari intervals have also been enumerated [Cha05], we find that the sequences $|\operatorname{MS}([n])| \leq |\operatorname{Pre}^{cc}[n]| \leq |\operatorname{Pre}[n]|$ (where Pre^{cc} denotes composition closed premodel structures) take the form

$$\tbinom{2n+1}{n} \leq \tfrac{1}{3n+4}\tbinom{3n+4}{n+1} \leq \tfrac{2}{(n+1)(n+2)}\tbinom{4n+5}{n}.$$

Asymptotic analysis reveals that model structures on $[n]$ are vanishingly rare among composition closed premodel structures on $[n]$, which are in turn vanishingly rare among premodel structures on $[n]$.

Conclusion

While we have touched on a number of recent advances in homotopical combinatorics, it is not possible in this limited space to cover the entirety of this rapidly growing field. We hope we have conveyed a flavor of work in the area, and want to emphasize that much terrain remains unexplored and there are many ways that researchers from various backgrounds can contribute. (In fact, much of the combinatorial work on transfer systems has been undertaken in collaboration with undergraduates.) To whet the reader's appetite, we provide the following short list of open problems:

1. Explore the combinatorics of the recursive construction of transfer systems from [BMO23a] for new families of lattices/groups.

2. Use multivariable generating functions to convert the recursions of [BMO23a] for $|\operatorname{Tr} D_{p^n}|$ and $|\operatorname{Tr} C_{qp^n}|$ into closed formulæ.

3. Enumerate compatible pairs of transfer systems (in the sense of [Cha22]) for new families of groups.

4. After identifying the lattice of transfer systems for a (family of) poset(s) P, use the methods of [BOOR23, BMO23b] to enumerate $\operatorname{Pre} P$, $\operatorname{Pre}^{cc} P$, and $\operatorname{MS}(P)$.

5. Leverage new structural results on transfer systems to extend the work of [Hil19] on the interaction between localizations and norms.

6. Lift the duality on transfer systems discovered in [FOO+22] to the level of N_∞ operads.

The authors — whose backgrounds are primarily in homotopy theory — are especially eager to see how more advanced tools from algebraic and analytic combinatorics might apply to these problems. We look forward to exploring these topics with participants in our 2024 Mathematics Research Community, and welcome inquiries from potential applicants.

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