Computational Methods for Pricing and Hedging Derivatives

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Declaration

I hereby declare that except where specific reference is made to the work of others, the contents of this dissertation are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other university.

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Abstract

In this thesis, we propose three new computational methods to price financial derivatives and construct hedging strategies under several underlying asset price dynamics. First, we introduce a method to price and hedge European basket options under two displaced processes with jumps, which are capable of accommodating negative skewness and excess kurtosis. The new approach uses Hermite polynomial expansion of a standard normal variable to match the first m moments of the standardised basket return. It consists of Black-and-Scholes type formulae and its improvement on the existing methods is twofold: we consider more realistic asset price dynamics and we allow more flexible specifications for the basket.

Additionally, we propose two methods for pricing and hedging American options: one quasi-analytic and one numerical method. The first approach aims to increase the accuracy of almost any existing quasi-analytic method for American options under the geometric Brownian motion dynamics. The new method relies on an approximation of the optimal exercise price near the beginning of the contract combined with existing pricing approaches. An extensive scenario-based study shows that the new method improves the existing pricing and hedging formulae, for various maturity ranges, and, in particular, for long-maturity options where the existing methods perform worst.

The second method combines Monte Carlo simulation with weighted least squares regressions to estimate the continuation value of American-style derivatives, in a similar framework to the one of the least squares Monte Carlo method proposed by Longstaff and Schwartz. We justify the introduction of the weighted least squares regressions by numerically and theoretically demonstrating that the regression estimators in the least squares Monte Carlo method are not the best linear unbiased estimators (BLUE) since there is evidence of heteroscedasticity in the regression errors. We find that the new method considerably reduces the upward bias in pricing that affects the least squares Monte Carlo algorithm. Finally, the superiority of our new two approaches for American options are also illustrated over real financial data by considering S&P 100™ options and LEAPS®, traded from 15 February 2012 to 10 December 2014.

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Nomenclature

General Symbols

b Cost of carry

δ Dividend yield

T Maturity date

$N_i(\cdot)$ Cumulative density function of a i-dimensional normal variable. $N(\cdot)$ is the short for $N_1(\cdot)$

$n(\cdot)$ Probability density function of a standard normal variable

t_0 Current time

$\mathbb{P}$ Physical probability measure

$\mathbb{Q}$ Risk-neutral probability measure

r Risk-free interest rate

S Underlying spot price

$\mathrm{std}[\cdot]$ Standard deviation of a random variable

K Strike price

$\mathrm{Var}[\cdot]$ Variance of a random variable

W Wiener process under $\mathbb{P}$

$\tilde{W}$ Wiener process under $\mathbb{Q}$

$E_t(\cdot)$ Expectation operator under the risk-neutral probability measure at time t. If the subscript t is omitted, then $t = t_0$

$E_{\mathbb{P}}[\cdot]$ Expectation operator under the physical probability measure

Acronyms / Abbreviations

aOEP Asymptotic expansion of the optimal exercise price near expiration

*m*GA Exact moment-matching method that matches the first *m* moments of the basket returns. The moments are rescaled so that the first one is equal to 1

*m*GAB Exact moment matching method that combines *m*GA and *m*GB

mGB Exact moment-matching method that matches the first m moments of the basket returns. The moments are rescaled so that the first one is equal to 0

BAW Quadratic method by Barone-Adesi and Whaley (1987)

CS# Static-replicating portfolio method in Chung and Shih (2009) with # time-steps

CZ Asymptotic expansion of the optimal exercise price by Cheng and Zhang (2012)

CZ-P Asymptotic expansion of the optimal exercise price by Cheng and Zhang (2012) with Pade's approximation

CZ-P-m Asymptotic expansion of the optimal exercise price by combination of CZ-P and EKK

EKK Asymptotic expansion of the optimal exercise price by Evans et al. (2002)

GJ# Compound-option method by Geske and Johnson (1984) with # time-steps

ICLS Inequality constrained least squares Monte Carlo method by Létourneau and Stentoft (2014)

JZ Quadratic method by Ju and Zhong (1999)

K# Integral method by Kim (1990) with # time-steps

LI Interpolation method by Li (2010b)

LSMC Least squares Monte Carlo method by Longstaff and Schwartz (2001)

OEP Optimal exercise price

OLS Ordinary least squares regression method

WLS Weighted least squares regression method

wLSMC Weighted least squares Monte Carlo method

ZL Asymptotic expansion of the optimal exercise price by Zhang and Li (2010)

BPW Shifted log-normal method for basket options by Borovkova et al. (2007)

Symbols for the 'exact moment matching' method for basket options (Chapter 2)

B_t Value of the basket at time t

φ_l Coefficient of the l-th order Hermite-polynomial, calculated by moment matching

γ_{ij} Correlation coefficient between asset i and j in the basket

β_i, $\tilde{\beta}_i$ Expected jump size for asset i under the physical and risk-neutral measure, respectively

λ_i, $\tilde{\lambda}_i$ Intensity of the Poisson process $\left\{N_t^{(i)}\right\}_{t\geq t_0}$ under the physical and risk-neutral measure, respectively

$Y_j^{(i)}$ Amplitude of the j-th jump of asset i

η_i Expected jump amplitude for asset i under the risk-neutral measure

υ_i Standard deviation of the jump amplitude for asset i under the risk-neutral measure

Υ Number of assets in the basket

n_w Number of Wiener processes

h_1 Parameter equal to 0 for the method *mGA* and equal to 1 for the method *mGB*

h_2 Parameter equal to $\operatorname{sgn}(B_0^*)$

$f^{(i)}(y), \tilde{f}^{(i)}(y)$ Probability density function of the jump amplitude for asset i under the physical and risk-neutral measure, respectively

α_i Expected rate of return of the shifted asset i under the physical probability measure

$\delta_t^{(i)}$ Shift applied to the price of asset i at time t

B_t^* Value of the shifted basket at time t

K^* Shifted strike price

$\left\{\mathrm{Q}_t^{(i)}\right\}_{t\geq t_0}$ Compound Poisson process driving the jump component of asset i

$\left\{N_t^{(i)}\right\}_{t\geq t_0}$ Poisson process driving the jump arrival of asset i

Symbols for the 'Extension'-method (Chapter 4)

$\Delta_{t_x,T}$ Step-size used for the short-maturity option

Λ Flat approximation of optimal exercise price between $[t_0, t_x]$

$P_t(S_t, T, K|t_x, \Lambda)$ Time-t price of an American put option (long-maturity) with maturity at T and optimal exercise price approximation between $[t_0, t_x]$ equal to Λ

$\tilde{P}_{t_x}(S, T, K)$ Time-t_x price of an American put option (short-maturity) with maturity at T

$S_f^{(E)}(t)$ Optimal exercise price for the long-maturity option

$S_{f_x}(t)$ Shifted optimal exercise price for the short-maturity option (defined on $[t_x, T]$)

$S_f(t)$ Optimal exercise price for the short-maturity option defined on $[t_0, t_1]$

t Generic time in $[t_0, T]$

t_1 Maturity date of the short-maturity option

t_x Splitting point of the time to maturity. It is where $S_f^{(E)}$ is divided into two parts.

Symbols for the 'weighted least squares Monte Carlo' method (Chapter 5)

$\hat{C}_i(\cdot)$ Estimated continuation value function of an American option at time-step i via a regression method

$C_i(\cdot)$ Continuation value function of an American option at time-step i

Δ_t Time-step length

$\psi_l(\cdot)$	l-th basis function (we assume $\psi_l(\mathscr{X}) = \mathscr{X}^l$)
$\mathtt{n_S}$	Number of simulated paths
$V_i(\cdot)$	American option price function at time step i
$h_i(\cdot)$	Payoff function at time-step i
$r_{0,i}$	Discount factor from time t_i to t_0 (i.e. $r_{0,i} = e^{-ri\Delta_t}$)
u_i	Error for the least squares regression at time-step i
$\mathscr{X}$	Generic state value
$\beta^w_{i,l}$	Weighted least squares estimator of the coefficient of $\psi_l(\cdot)$ at time step i
$\beta_{i,l}$	Ordinary least squares estimator of the coefficient of $\psi_l(\cdot)$ at time step i
M	Number of basis functions
m	Number of time-steps

Chapter 1

Introduction

Financial derivatives are contracts whose price depends upon or derives from one or more underlying assets. Existence of rudimentary derivatives date back to 2000 BC in Mesopotamia, where contracts for future delivery of goods were written in cuneiform script on clay tablets (Weber, 2009). For centuries, derivatives have been exchanged in one-to-one transactions in over-the-counter markets, but it was only in 1973, with the creation of the Chicago Board of Options Exchange, (CBOE), that this industry experienced rapid growth. Nowadays, the list of available contracts in organised markets is diverse and traders can buy and sell derivatives on many asset classes such as currencies, stock indices, bonds, energy and commodities, interest rates or even other derivatives. Symmetrically, trading in the over-the-counter markets is also very active and has flourished to meet investors' needs, which are not satisfied by the standard products traded in the exchanges.

In recent years, financial derivatives often made headlines for causing substantial losses to a number of globally renowned companies and financial institutions. High profile cases include Procter & Gamble (in 1994), Amaranth Advisors (in 2006), Société Générale (in 2008), Morgan Stanley (in 2008) and JPMorgan Chase (in 2012), all of whom lost billions through derivatives trading. Additionally, in the United States, the derivatives created from the subprime mortgages such as asset-backed securities and collateralised debt obligations played a major role in the subprime mortgage crisis. For example, in 2008, American International Group (AIG), a multinational insurance corporation with headquarters in the United States, faced a credit downgrade for losses in credit default swaps on collateralised debt obligations. Consequently, AIG was required to post additional collaterals. In order to fulfil these requirements, the beleaguered company borrowed $85 billion from the Federal Reserve Bank. However, most of the time derivatives do help companies, banks and other economic agents in running their businesses. Derivatives are employed to lower costfunding, increase rates of return and manage risks and, consequently, they are traded by many economic actors (Beder and Marshall, 2011).

One of the most well-known and traded classes of derivatives is options. An option is a financial contract that gives its holder the right, but not the obligation, to purchase or sell a prescribed asset (known as the underlying asset) for a prescribed amount (known as exercise price or strike price, K). There are two main exercise styles: American, if the option can be exercised any time before or at a prescribed time in the future (known as the expiry date or maturity date, T); and, European, if it can be exercised only at the maturity date.

Speculation and hedging are the main two reasons for trading options. Moreover, embedded option features exist in many derivative contracts and this makes options a 'special' derivative class. Since the opening of the CBOE, the number of these contracts traded annually in this market has increased 200 times, from 5.6 million to 1,119 million contracts and CBOE is now the largest option exchange in the world (CBOE Holdings Inc., 2010). Furthermore, the World Federation of Exchanges, an international federation composed of 54 of the most active options exchanges, reported for 2010 an options market of six hundred trillion USD and a total of approximately 11 billion traded contracts (World Federation of Exchange, 2010). Table 1.1 summarises details concerning the option market size for different underlying assets, and Table 1.2 presents the option market divided into the three main geographic regions, the Americas (North America, South America and associated islands), Asia Pacific (East Asia, South Asia, South-East Asia, Oceania), and EMEA (Europe, the Middle East and Africa).

In this thesis, we propose three new computational methods to price and hedge two types of option contracts. In the section to follow, we outline the original contributions and describe the structure of the thesis.

Table 1.1 Options market sizes: Classification for underlying asset class

Underlying	**Volume traded (No. of contracts)**	**Notional value (USD millions)**	**Volume traded % (No. of contracts)**	**Notional value %**
Stock index options	5,019,127,872	124,551,764.71	45.10%	20.31%
Single stock options	3,904,583,092	13,580,748.65	35.09%	2.21%
ETF options	1,242,256,285	2,074,949.93	11.16%	0.34%
STIR options	491,065,415	414,361,194.02	4.41%	67.57%
Commodity options	194,138,505	10,247.08	1.74%	0.00%
LTIR options	156,821,953	55,358,193.88	1.41%	9.03%
Currency options	55,799,674	3,148,247.39	0.50%	0.51%
Others options	64,152,131	149,638.60	0.58%	0.02%
Total	11,127,944,927	613,234,984.25	100.00%	100.00%

Note: Elaboration of the 2010 Annual Report of World Federation of Exchange (2010). In the table, the acronym *ETF* stands for exchange-traded fund, *STIR* for short term interest rate, *LTIR* for long term interest rate and "other options" are options on environmental commodities and VIX.

Table 1.2 Options market sizes: Classification for region of trading

		Volume traded (No. of contracts)	Notional value (USD millions)	Volume traded % (No. of contracts)	Notional value % (USD millions)
Stock index options	Americas	266,701,307	28,942,701.02	5.31%	23.24%
	Asia Pacific	4,215,121,087	76,140,263.65	83.98%	61.13%
	EMEA	537,305,478	19,468,800.04	10.71%	15.63%
Single stock options	Americas	3,197,186,575	12,421,641.98	81.88%	84.40%
	Asia Pacific	106,296,615	679,265.90	2.72%	4.62%
	EMEA	601,099,902	1,616,702.07	15.39%	10.98%
ETF options	Americas	1,241,616,618	2,070,376.64	99.95%	99.78%
	Asia Pacific	497,949	3,350.51	0.04%	0.16%
	EMEA	141,718	1,222.78	0.01%	0.06%
STIR options	Americas	300,001,865	190,923,909.62	61.09%	46.08%
	Asia Pacific	42,040	38,565.27	0.01%	0.01%
	EMEA	191,021,510	223,398,719.13	38.90%	53.91%
Commodity options	Americas	183,390,620	750.76	94.46%	7.33%
	Asia Pacific	93,042	5,602.61	0.05%	54.68%
	EMEA	10,654,843	3,893.72	5.49%	38.00%
LTIR options	Americas	86,736,965	41,421,903.10	55.31%	74.83%
	Asia Pacific	5,001,474	3,499,149.77	3.19%	6.32%
	EMEA	65,083,514	10,437,141.01	41.50%	18.85%
Currency options	Americas	38,232,235	3,057,458.27	68.52%	97.12%
	Asia Pacific	6,277,165	6,270.70	11.25%	0.20%
	EMEA	11,290,274	84,518.42	20.23%	2.68%
Other options	Americas	62,768,313	147,518.73	97.84%	98.58%
	Asia Pacific	-	0.00	0.00%	0.00%
	EMEA	1,383,818	2,119.87	2.16%	1.42%

Note: Elaboration of the 2010 Annual Report of World Federation of Exchange (2010). In the table, Americas stands for North America, South America and associated islands, Asia Pacific for the part of the world in or near the Pacific ocean, EMEA for Europe, the Middle East and Africa.

1.1 Original contributions and structure of the thesis

This thesis investigates computational methods for pricing option contracts and finding the corresponding hedging strategy. In particular, the main focus will be on numerical and quasi-analytic methods to price and hedge European basket options and American options under several underlying asset price dynamics. The material presented in this thesis represents the author's original contribution and covers the following three topics:

1. the pricing and hedging of basket options with an exact moment-matching procedure;
2. an improved quasi-analytic method for pricing and hedging long-dated American options under the geometric Brownian motion;
3. a simulation-based method that corrects the least squares Monte Carlo valuation method developed by Longstaff and Schwartz (2001) for heteroscedasticity.

In Chapter 2, we present new quasi-analytic formulae to price and hedge basket options, (i.e., call and put options written on a group of assets). Pricing and hedging these options is not straightforward since many computational difficulties arise from the fact that the distribution of the basket return at the maturity date is not known in closed form, nor even for simple dynamics such as the geometric Brownian motion. Many of the existing methods resort to two assumptions: the price of each asset in the basket is log-normally distributed, and the weighted sum of log-normal random variables is approximated by a (modified) log-normal random variable. Our new methodology consists of Black-and-Scholes' formulae for both pricing and hedging and improves the results in the literature in both directions. First, we model the asset prices with dynamics that (by comparison to the geometric Brownian motion), fit the empirical evidence in the financial markets better. Indeed, we employ the displaced jump-diffusion process of Câmara et al. (2009), which accounts for the negative skewness and excess kurtosis that characterise equity stocks, and the shifted asymmetric jump-diffusion of Ramezani and Zeng (2007), where two independent sources of jumps are considered. Second, we employ a Hermite polynomial expansion of a standard normal variable to approximate the basket return at maturity by matching its first m moments. We demonstrate via an extensive scenario-based comparison, that by matching only the first four moments, one obtains satisfactory results, which, in most cases, are better than those found by existing techniques in the literature.

The other two topics of this thesis concern the American option pricing and hedging problem. This problem is intrinsically more complex than the corresponding one for European-style options, since it requires the solution of an optimal stopping time problem and, consequently, the selection of the optimal exercise price together with the valuation of the contract. This double selection makes it a stochastic optimisation problem. In Chapter 3, we review the theoretical properties and characteristics of, and the literature on, American option pricing and hedging. Additionally, in this chapter, we introduce the notation that we use in Chapters 4 and 5, where we present two new solution approaches addressing the American option pricing and hedging problem.

Chapter 4 concentrates on a quasi-analytic technique that potentially improves on any pricing and hedging method for long-maturity American options on a log-normally distributed asset. Based on an extensive numerical study, also confirmed by other research in the literature (Broadie and Detemple (1996), AitSahlia and Carr (1997) and Kim et al. (2013)), we identify that the existing pricing and hedging methods in the literature perform very well for short-term options but their performance worsens the longer the time-to-maturity. The new technique relies on an approximation of the optimal exercise price near the beginning of the contract, combined with existing pricing approaches so that the maturity range for which small errors are attained is extended. Additionally, the new approach retains the quasi-analytic nature of the methods it improves and, consequently, we derive generic quasi-analytic formulae for the price of an American option as well as for its delta parameter. The method is shown to provide improved pricing and hedging performances compared to the well-known methods of Geske and Johnson (1984), Barone-Adesi and Whaley (1987), Kim (1990), Ju and Zhong (1999), Chung and Shih (2009) and Li (2010b). This improvement is substantial over a large range of maturities and, in particular, for long-maturity options. Moreover, we show that the new method also works well in 'extending' many asymptotic expansions of the optimal exercise price. These asymptotic expansions are methods that approximate the optimal exercise price near the maturity date and are recognised as cutting-edge methods for American options as a result of their precise and fast estimation of the optimal exercise price. However, these methods only work properly for maturities of a few months, and in a few cases up to two years and cannot be employed for longer maturities. By 'extending' with our new technique the asymptotic expansion methods of Evans et al. (2002), Zhang and Li (2010) and Cheng and Zhang (2012), we manage to obtain good pricing and hedging performance for options with maturities as long as 5 years.

Chapter 5 describes a new numerical algorithm for pricing American-style derivatives based on employing Monte Carlo simulations together with the weighted least squares regression method. The new technique, which we call the weighted least squares Monte Carlo, is an improvement on the least squares Monte Carlo algorithm (LSMC) originally proposed by Longstaff and Schwartz (2001), one of the best-known methods for pricing American-style derivatives. The main contribution of Chapter 5 is that we demonstrate that the errors of the ordinary least squares regressions of the LSMC algorithm are heteroscedastic, (i.e. the conditional variance of these errors with respect to the regressors is not constant). This heteroscedasticity, which we prove to exist in three different ways (numerically, graphically and theoretically), makes the ordinary least squares estimators not the best linear unbiased estimators (BLUE), as proved by the Gauss-Markov theorem (Section 5.2.5). In order to retain the BLUE property when the errors are heteroscedastic, our new weighted least squares Monte Carlo method follows the same steps as the LSMC method but substitutes the ordinary least squares estimators with the weighted least squared ones. We show that not accounting for heteroscedasticity (as in LSMC) causes upward bias in option prices. Our weighted least squares Monte Carlo is shown to provide remarkable reduction of this bias over both American call and put options under several price dynamics. In particular, we numerically examine the pricing problem under the following price dynamics: geometric Brownian motion, exponential Ornstein-Uhlenbeck process, log-normal jump-diffusion process, and dual exponential jump-diffusion process.

To understand the importance of a newly introduced technique for pricing derivatives, one needs to evaluate the performance of this method when applied to real financial data. In Chapter 6, we complement the extensive scenario-based performance comparisons in Chapters 4 and 5 by considering a performance comparison in pricing S&P 100™ options and LEAPS® (i.e., Long-term Equity AnticiPation Securities™) traded in the CBOE from February 2012 to December 2014. The results are reassuring and indicate that both methods improve on their direct competitors. Finally, Chapter 7 concludes and proposes ideas and directions for further research.

Chapter 2

European Basket Options: Pricing and Hedging via Moment Matching

In this chapter, we describe a new methodology to price and hedge European basket options under the displaced log-normal process with jumps, which is capable of accommodating negative skewness and excess kurtosis, which are well known to characterise equities. Our technique involves Hermite polynomial expansions that can match exactly the first m moments of the model-implied basket return and it is shown to provide superior results, not only with respect to pricing, but also for hedging. This chapter is structured as follows: Sections 2.1 introduces the basket-option pricing and hedging problem; Section 2.2 provides an extensive review of the literature; Section 2.3 describes the pricing and hedging methodology; Section 2.4 compares the pricing and hedging performance of the new methodology with the methodology in Borovkova et al. (2007) over a large set of option scenarios; Section 2.5 provides another application of our new method to basket options whose assets follow shifted asymmetric jump-diffusion processes. Finally, Section 2.6 concludes.

2.1 Introduction

Basket options are contingent claims on a group of assets such as equities, commodities, currencies and even other vanilla derivatives. They are commonly traded to hedge away exposure to correlation or contagion risk and hedge-funds also use them for investment purposes, to combine diversification with leverage. Basket options are also usually cheaper than individual options on each asset in the basket (Dionne et al., 2006), and are used in situations when one does not need to hedge against movements of a single asset price but, rather, one is interested in the relative movements of the prices of a group of assets.

These contracts are mainly traded over-the-counter but some spread options, i.e. basket options on the price difference between two or more assets, are also traded in some exchanges. For example, the New York Mercantile Exchange, NYMEX, lists crack spread options and calendar spread options. The former aim to solve the problems faced by refinery managers in the petroleum industry. These managers are concerned about the price differences of their inputs and their outputs. One example of a contract of this type is the "NY Harbor ULSD crack spread option", which is the spread between NY Harbor ULSD futures (HO) and Light Sweet Crude Oil futures (CL). On the other hand, calendar spread options are options on the difference between futures on the same underlying asset but with different maturities. An example is the "WTI crude oil 1 month calendar spread option", which is an option on the spread between the first and the second expiring Light Sweet Crude Oil futures. They are largely traded by storage facility managers and other players in the refining supply chain of oil or other energy products. On NYMEX there are also traded spread options on assets of the same type but with different quality and/or point of production/extraction. An example is the "WTI-Brent crude oil spread option", i.e. an option on the spread between the NYMEX Light Sweet Crude Oil futures first nearby contract and the Brent Crude Oil (ICE) futures first nearby contract.

Additionally, the Chicago Board of Trade, CBOT mainly lists agricultural spread options. Typical products are: inter-commodity spread options and crush spread options. The former are contracts on the difference in value between two or more futures on different asset classes. An example is the "Chicago SRW wheat-corn inter-commodity spread option", which is an option on the spread between Chicago SRW Wheat futures and Corn futures with the same maturity date. The crush spread options are spread on inputs and outputs of the soybean. They are used for the same purpose as the crack spread options discussed above. An example of these contracts is the "soybean crush option", which is written on the spread

$$\begin{gathered} +0.022 \text{ Soybean Meal in \$/ton} \\ +11 \text{ Soybean Oil in ¢/lb} \\ -1 \text{ Soybeans in \$/bu.} \end{gathered}$$

Since the compositions of the underlying baskets are very heterogeneous and may consist of several assets (especially in over-the-counter markets), the pricing and hedging of these derivatives ought to be carried out with multidimensional models that are able to describe the empirical characteristics of the assets in the basket. Many pricing models that seem to work well for single assets cannot be easily extended to a multidimensional set-up, mainly due to computational difficulties. Hence, practitioners usually resort to classical multidimensional geometric Brownian motion type models, to keep the modelling framework as simple as possible. However, by doing so, the empirical characteristics of the assets are simply overlooked. In particular, the negative skewness and excess kurtosis, which characterise equity stocks (Bakshi et al., 1997, 2003), cannot be captured properly by these simple models because they produce a limited range of values for these statistics.

Ideally, one would like the best of both worlds, realistic modelling and precise calculations. In this chapter, we present a general computational solution to the problem of multidimensional models lacking closed-form formulae or requiring burdensome numerical procedures. The purpose of this chapter is to provide a robust and precise methodology for pricing and hedging basket options when the price of each of the assets in the basket follows a model that is able to accommodate the empirical characteristics of the assets. One such model is the displaced jump-diffusion, which will be used as a test subject to show the superiority of the presented methodology. Câmara et al. (2009) priced options on a single asset under this model; however, expanding the set-up to a basket of assets leads to computational problems related to the calculation of the probability distribution of the basket price at expiration. Therefore, we circumvent this problem by employing a Hermite polynomial expansion that matches exactly the first m moments of the model-implied basket return.

The pricing and hedging methodology we propose consists of quasi–analytical formulae for both pricing and hedging. They are Black-and-Scholes type formulae and some of their inputs are calculated as the solution of a system of m equations in m unknowns (the moment-matching system). The main advantages of the new methodology are threefold:

- it incurs low computational cost compared to numerical methods, especially when one prices a portfolio of options written on the same basket with different strikes and/or payoffs, since the moment-matching procedure needs to be carried out only once;
- it provides precise calculations and availability of formulae for the hedging parameters. The estimations of these parameters are arguably even more important than the calculation of prices since despite the latter being available in the market, the former are not. The availability of formulae for the hedging parameters is an important advantage of our methodology over numerical methods such as the Monte Carlo simulation method, for which the calculation of these parameters is usually computationally intensive and less precise;[1]
- the only prerequisite of our method is the existence of the moments of the basket and,

[1] See Glasserman (2003), Chapter 7, for detailed discussion on Monte Carlo methods for the estimation of price sensitivities.

consequently, it is applicable to situations in which some assets in the basket follow one diffusion model and other assets follow a different diffusion model.

2.2 Literature review

The number of papers covering basket options has increased considerably in the last three decades. Carmona and Durrleman (2003) reviewed the main contributions available in the literature on solution approaches for the pricing problem. The available methods can be classified into two categories: those that are numerical, and those that are (quasi)-analytical, which includes methods based on various expansions and moment-matching techniques. Our method belongs to the second category. In what follows, we review the main contributions to the two categories.

2.2.1 Numerical methods

When analytical formulae are difficult to derive under a particular model and/or payoff structure, it is common, in the finance industry, to resort to numerical methods to approximate the solution of a pricing problem. Among these methods, we enumerate lattice methods and Monte Carlo simulations.

Within the first group, Boyle (1988) introduced a binomial three method for pricing contracts on two assets and Boyle et al. (1989) propose an extension of their method to price contracts on $\Upsilon \geq 2$ assets. Kamrad and Ritchken (1991) presented a multi-asset trinomial tree which generalises the binomial tree methods in Boyle (1988) and Boyle et al. (1989). However, the computational effort required by these three methods is prohibitive in real word applications, since the number of nodes in the trees grows exponentially with the number of underlying assets. This problem is usually defined as the "curse of dimensionality". Borovkova et al. (2012), under the assumption of a log-normally distributed basket return at maturity, priced American style options using a standard binomial tree; for general payoffs, Leccadito et al. (2012) employed Hermite polynomials technique to price European and American basket options in the context of jump-diffusion and stochastic volatility frameworks.

Unlike the lattice methods above, the Monte Carlo simulation methods do not suffer from the curse of dimensionality and can be used efficiently in multi-dimensional pricing frameworks. Boyle (1977) was the first to introduce this technique to solve option pricing problems and his standard method has been improved in several ways. Barraquand (1995) used a Monte Carlo simulation method with an error reduction technique, called quadratic re-sampling, which is based on the mean and variance-covariance matrix of the underlying assets prices. Pellizzari (2001) proposed the mean Monte Carlo method, i.e. a control variate technique for pricing multidimensional contingent claims based on the property that if all but one of the asset prices are fixed at their mean value, the corresponding uni-dimensional option price can often be calculated analytically.[2] Korn and Zeytun (2013) proposed a control variate technique for pricing basket options that is based on the price of the corresponding geometric-basket option formula, which is given in exact closed-form under the geometric Brownian motion.

While Monte Carlo methods offer a feasible solution, analytical formulae are usually preferred by practitioners. Hence, the majority of the literature on basket option pricing focuses on approximation methods that circumvent the numerical problems generated by the high dimensionality of basket models.

2.2.2 Analytic and quasi-analytic methods

By analogy to early papers on pricing Asian options, Gentle (1993) proposed pricing basket options by approximating the arithmetic weighted average with its geometric counterpart so that a Black-and-Scholes type formula could be applied. Korn and Zeytun (2013) improved this approximation using the fact that, if the spot prices of the assets in the basket are shifted by a large scalar constant C, their arithmetic and geometric means converge asymptotically. They consider log-normally distributed assets and approximate the C-shifted distribution with standard log-normal distributions. Kirk (1995) developed a technique for pricing a spread option by coupling the asset with negative weight with the strike price, considering their combination as one asset with a shifted distribution and then employing the Margrabe (1978) formula for options on the exchange of two assets. The methods in Li et al. (2008, 2010) extended the procedure proposed in Kirk (1995) to the case of more than two assets with positive and negative weights. Venkatramanan and Alexander (2011) and Alexander and Venkatramanan (2012) replicated European and American style spread options using a portfolio of compound exchange options,[3] under local volatility and local correlation frameworks.

Curran (1994) priced basket options with only positive weights by conditioning on the geometric basket value: the resulting formula is given as an exact term plus an approximated term. Deelstra et al. (2004, 2010) extended on Curran (1994) and obtained lower and upper bounds for the prices of basket options and Asian basket options, respectively. Similarly, Xu

[2]In Appendix 2.B, we describe the mean Monte Carlo method and how we apply this methodology for asset price dynamics following the displaced jump-diffusion process in Section 2.3.

[3]Compound exchange options are options to exchange one option with another.

and Zheng (2009) derived bounds for basket options on assets following a jump-diffusion model with idiosyncratic and systematic jumps. Bertsimas and Popescu (2002), Hobson et al. (2005a,b), and Laurence and Wang (2004, 2005) proposed model-free bounds based on the prices of the European options, each on a single asset in the basket. However, in general, they found that these bounds are not very tight.

Remarkably, Dempster and Hong (2000) employed the fast Fourier transform in Carr and Madan (1999) to price spread options on two assets following models for which the characteristic functions are known in closed form. Subsequently, Hurd and Zhou (2010) extended this to more general spread options and also calculated the hedging parameters.

A vast amount of research has been carried out on approximate closed-form formulae for basket options that involve moment-matching procedures to overcome the problem of not knowing analytically the probability density function for the basket. Levy (1992) approximated the distribution of a basket of log-normal assets by matching its first two moments with the moments of a log-normal density function, and then derived a Black-and-Scholes type pricing formula. A long series of papers have improved on this approximation, either by allowing more flexible basket densities or by allowing for more realistic asset price dynamics. Among the former group, Huynh (1994) employed the Edgeworth expansion around the log-normal distribution that matches the first two moments of the basket; Milevsky and Posner (1998a,b) used the reciprocal gamma distribution to approximate the probability density function of a basket by matching its first two moments; Posner and Milevsky (1998) and Dionne et al. (2011) employed the log-normal and the unbounded Johnson density functions, and matched the first four moments of the basket; and, Ju (2002) employed Taylor expansions to replicate the basket density. However, the methods above cannot be employed when the asset weights are negative since the basket values can be negative, while the supports of the considered distributions are positive.

A remarkable improvement in pricing basket options with generic asset weights has been proposed by Borovkova et al. (2007). Their methodology can incorporate negative skewness while still retaining analytical tractability, under a shifted log-normal distribution, by considering the entire basket as one single asset. This strong assumption allows the derivation of closed-form formulae for basket option pricing.[4] Zhou and Wang (2008) advocated a method similar to that of Borovkova et al. (2007), selecting the log-extended-skew-normal as the approximating distribution. They also obtained a Black-and-Scholes type pricing formula, where the standard extended-skew-normal cumulative distribution function replaces the normal one.

[4]In Appendix 2.C we describe in detail this methodology and in Section 2.4 we compare its performance with the one of our new methodology.

On the other hand, some other research has priced basket options whose asset dynamics are more appropriate to accommodate the empirical characteristics of the asset returns. Flamouris and Giamouridis (2007) priced basket options on assets following a Bernoulli jump-diffusion process using the Edgeworth expansion in Huynh (1994); Wu et al. (2009) assumed that asset prices follow the multivariate normal inverse Gaussian model (mNIG) and employed the fast Fourier transform together with the methodology outlined by Milevsky and Posner (1998b) to approximate the sum of assets following the mNIGs model as a mNIG; Bae et al. (2011) priced basket options (with positive weights) on assets following a jump-diffusion process by using the Taylor expansion method of Ju (2002).

Our methodology improves on both research directions. First, we consider that assets in the basket follow the displaced jump-diffusion process in Câmara et al. (2009), which is shown to produce large ranges of skewness and kurtosis. Second, we employ a Hermite polynomial expansion of a standard normal variable to match the first m moments of the standardised basket returns. Our methodology works for any m and we shall show in Section 2.4, using a scenario-based comparison, that matching higher-order moments considerably improves the pricing and hedging performances.

An approximation similar to ours has been proposed by Necula and Farkas (2014), who priced basket and spread options for assets whose joint characteristic function is known in closed-form. Their methodology is based on the results of Necula et al. (2013), who employed the physicists' Hermite polynomials to expand the risk-neutral density of the terminal underlying asset value. Their methodology is very precise being able to match any number of moments. In order for their expansion to be a proper density (total mass equal to 1), high-order truncations are required and, therefore, the computational cost may be occasionally high, especially when the number of assets in the basket is relatively high. The main differences with our method are that we use probabilists' Hermite polynomials rather than the physicists' ones, and we employ these polynomials to expand the random variable that represents the basket return at maturity rather than its distribution, so we can truncate our expansion at any order. Additionally, we also provide a quasi-analytic formula for the delta-hedging parameters, which we employ to implement a delta-hedging strategy for a set of option scenarios and it is shown to provide a better performance than those of the existing methodologies. This mainly depends on the fact that, since our method is capable of matching a higher number of moments, it employs a better approximation of the basket dynamics than most of the existing methods. Furthermore, we provide a general formula for any hedging parameter. The calculation of each hedging parameter requires the solution of a system of m equations in m unknowns. Consequently, other hedging strategies can be easily implemented.

2.3 The "exact moment-matching" method for basket options

This section describes our new pricing and hedging methodology for European basket options. Without loss of generality, henceforth we fix the current valuation time at $t_0 = 0$ and we indicate, with T, the time-to-maturity. The payoff at maturity of this type of option is:

$$\begin{cases} (B_T - K)^+, & \text{for call options} \\ (K - B_T)^+, & \text{for put options} \end{cases} \tag{2.1}$$

driven by the underlying variable

$$B_t = \sum_{i=1}^{\Upsilon} a_i S_t^{(i)}, \tag{2.2}$$

which is the value of the basket at time t, where Υ is the number of assets in the basket, K is the strike price, $\boldsymbol{a} = (a_1, \ldots, a_\Upsilon)'$ is the vector of non-stochastic basket weights, which could be positive or negative, and $(x)^+ = \max\{0, x\}$.

Under the majority of models applied in practice, and under the processes we assume for the asset price dynamics in this chapter (i.e the displaced jump-diffusion process described in Section 2.3.1 and the displaced asymmetric jump-diffusion described in Section 2.5), the probability density of the basket value B_t cannot usually be obtained in closed-form and, consequently, for pricing and hedging purposes, researchers and practitioners usually resort to approximations. The methodology proposed in this chapter circumvents this problem by approximating the standardised return of the basket by a polynomial transformation of a standard normal random variable. This approximation is constructed in such a way as to match exactly the first m moments of the model implied risk-neutral return.

In what follows, Section 2.3.1 defines the modelling framework, and Section 2.3.2 describes the moment-matching technique and provides the analytic formulae for the moments. Finally, Sections 2.3.3 and 2.3.4 depict the pricing and hedging methodologies, respectively.

An alternative approach for pricing basket options is to model the dynamics of the entire basket directly as a single asset and employ existing pricing and hedging formulae for vanilla call/put options. However, by doing so, one would lose many of the advantages that comes when modelling each asset separately. In particular, one can carry out much deeper sensibility analysis over each of the parameters of the asset price dynamics and their correlations and, consequently, more robust results can be achieved for risk management purposes.

2.3.1 Modelling framework

Consider the filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ under the objective probability measure $\mathbb{P}$ and, on this space, the financial market consisting of the assets

$$S^{(i)}, i = 1, \cdots, \Upsilon$$

and the bank account

$$M_t = e^{rt}$$

that can be used to borrow and deposit money with a continuously compounded interest rate $r \geq 0$, assumed constant over time.

From a modelling point of view, it would be appropriate for the assets $S^{(i)}$ to follow models that are capable of generating negative skewness and excess kurtosis reflecting the empirical evidence in equity markets. One such flexible model is the correlated displaced (or shifted) jump-diffusion, that is a jump-diffusion process for the displaced or shifted asset value, introduced by Câmara et al. (2009). In what follows, we assume that the asset price dynamics follow this jump-process, which under the objective probability measure $\mathbb{P}$ is defined by:

$$\begin{aligned} d\left(S_t^{(i)} - \delta_t^{(i)}\right) &= (\alpha_i - \beta_i \lambda_i)\left(S_t^{(i)} - \delta_t^{(i)}\right)dt + \left(S_t^{(i)} - \delta_t^{(i)}\right)\sum_{j=1}^{n_w} \gamma_{ij} dW_t^{(j)} \\ &\quad + \left(S_{t^-}^{(i)} - \delta_t^{(i)}\right)dQ_t^{(i)}, \quad i = 1, \cdots, \Upsilon, \end{aligned} \tag{2.3}$$

where: α_i is the expected rate of return on the shifted asset i; $\left\{W_t^{(j)}\right\}_{t\geq 0}$ are n_w mutually independent Wiener processes; γ_{ij} defines the correlation among assets i and j; $\left\{Q_t^{(i)}\right\}_{t\geq 0}$ are independent compound Poisson processes formed from some underlying Poisson processes $\left\{N_t^{(i)}\right\}_{t\geq 0}$ with intensity $\lambda_i \geq 0$; and $\delta_t^{(i)} = \delta_0^{(i)} e^{rt}$ is the shift applied to $S_t^{(i)}$. Additionally, we indicate with $Y_j^{(i)}$ the amplitude of the j-th jump (of the shifted process) of $N_t^{(i)}$ for any

$i = 1, \cdots, \Upsilon$. The jumps are i.i.d. random variables with probability density function[5]

$$f^{(i)}(y) : [-1, +\infty) \to [0, 1]$$

having an expected value under the physical measure $\mathbb{P}$

$$\beta_i = E_{\mathbb{P}}[Y^{(i)}] = \int_{-1}^{\infty} y f^{(i)}(y) dy.$$

Jumps for different assets are assumed to be independent.

Câmara (1999) studied the relationship between the shift and the probability density function of the displaced log-normal process (i.e. process (2.3) without jumps).[6] In particular, Câmara found that a positive (resp. negative) value of $\delta_0^{(i)}$ is associated with a more positively (negatively) skewed and leptokurtic (mesokurtic) distribution. In order to replicate the empirical evidence in the financial markets, it would be necessary in a model with no-jump to consider negative values of δ_0. However, negative values of δ_0 lead to negative stock prices with positive probability. On the other hand, by introducing jumps as in (2.3) and in Câmara et al. (2009), one can capture the empirical properties of stocks for $\delta_0 \geq 0$ and also guarantee that the stock prices will always be positive. For this reason, in what follows, we assume $\delta_0^{(i)} \geq 0$, for any $i = 1, \ldots, \Upsilon$.

For pricing and hedging purposes, a change of measure is applied to the dynamics (2.3). The analytical framework in Shreve (2004), Chapter 11.5, for standard multidimensional jump-diffusion models, is adapted here to deal with shifted assets by simply noticing that, for $\delta_t^{(i)} = \delta_0^{(i)} e^{rt}$, the martingale conditions for the discounted asset prices are easily verified.

The solution of SDE (2.3) under the risk-neutral pricing measure $\mathbb{Q}$ is carried out via Itô's lemma for jump-diffusion and reads

$$S_t^{(i)} = \left(S_0^{(i)} - \delta_0^{(i)}\right) e^{\left(r - \tilde{\beta}_i \tilde{\lambda}_i - \frac{1}{2}\sum_{j=1}^{n_w} \gamma_{ij}^2\right)t + \sum_{j=1}^{n_w} \gamma_{ij} \tilde{W}_t^{(j)}} \prod_{l=1}^{N_t^{(i)}} \left(Y_l^{(i)} + 1\right) + \delta_0^{(i)} e^{rt} \tag{2.4}$$

where the intensity of the Poisson process $\left\{N_t^{(i)}\right\}_{t \geq 0}$ is $\tilde{\lambda}_i$, the expected value of $Y_l^{(i)}$ for any $l = 1, \ldots, N_t^{(i)}$ is

$$\tilde{\beta}_i = E[Y_l^{(i)}] = \int_{-1}^{+\infty} y \tilde{f}^{(i)}(y) dy,$$

[5] Note that we do not specify any probability density function here but we rather leave $f^{(i)}(\cdot)$ unspecified, because what follows holds for general densities. Later, for exemplification purposes, we will first consider the jump sizes to be log-normally distributed and, in Section 2.5, jump sizes will be assumed Pareto and Beta distributed.

[6] This model was first introduced by Rubinstein (1981).

the probability density function of $Y_j^{(i)}$ is $\tilde{f}^{(i)}(y) : [-1,+\infty) \to [0,1]$ and $\left\{\tilde{W}_t^{(j)}\right\}_{t\geq 0}$ are independent Wiener processes under the martingale measure $\mathbb{Q}$. We indicate with E the expectation operator under the risk-neutral measure $\mathbb{Q}$.

For the model with SDE (2.4) not to introduce arbitrage, the parameters $\tilde{\beta}_1, \cdots, \tilde{\beta}_\Upsilon$, $\tilde{\lambda}_1, \cdots, \tilde{\lambda}_\Upsilon$, and $\theta_1, \cdots, \theta_{n_w}$ need to satisfy the system of equations

$$\alpha_i - \beta_i \lambda_i - r = \sum_{j=1}^{n_w} \gamma_{ij} \theta_j - \tilde{\beta}_i \tilde{\lambda}_i, \quad i = 1, \cdots, \Upsilon. \tag{2.5}$$

The solution to (2.5) is, in general, not unique so we are in incomplete markets. Nevertheless, we assume that one solution of the system (2.5) is selected and a pricing measure $\mathbb{Q}$ is fixed. There is a large amount of literature devoted to the issue of selecting a pricing measure[7] and we do not analyse this topic further but, henceforth, we assume the risk-neutral measure $\mathbb{Q}$ as given: in particular, the model we consider is the displaced jump-diffusion process with unsystematic jump risk in Câmara et al. (2009). We consider the jumps of each asset price i.i.d. log-normally distributed such that

$$E[\log(Y_j^{(i)} + 1)] = \eta_i$$

$$\mathrm{Var}[\log(Y_j^{(i)} + 1)] = \upsilon_i^2.$$

Henceforth, in order to simplify the notation, we denote $V_t^{(i)} = \sum_{j=1}^{n_w} \frac{\gamma_{ij}}{\sigma_i} \tilde{W}_t^{(j)}$ where $\sigma_i^2 = \sum_{j=1}^{n_w} \gamma_{ij}^2$. Thus $\left\{V_t^{(i)}\right\}_{t\geq t_0}$ are dependent standard Brownian motions with correlation

$$\rho_{l_1 l_2} = \mathrm{corr}\left(V_t^{(l_1)}, V_t^{(l_2)}\right) = \frac{1}{\sigma_{l_1}\sigma_{l_2}} \sum_{j=1}^{n_w} \gamma_{l_1 j}\gamma_{l_2 j}, \tag{2.6}$$

and, consequently, (2.4) can be rewritten as

$$S_t^{(i)} = \left(S_0^{(i)} - \delta_0^{(i)}\right) e^{\left(r - \tilde{\beta}_i \tilde{\lambda}_i - \frac{1}{2}\sigma_i^2\right)t + \sigma_i V_t^{(i)}} \prod_{l=1}^{N_t^{(i)}} \left(Y_l^{(i)} + 1\right) + \delta_0^{(i)} e^{rt}. \tag{2.7}$$

All of the results below are stated for the dynamics in (2.7). Additionally, we point out that the shifted jump-diffusion model will encompass three sub-cases:

- multidimensional geometric Brownian motion, GBM, when $\delta_0^{(i)} = 0$ and $\tilde{\lambda}_i = 0$ for each asset i;

[7] For a review, see Frittelli (2000) and references within.

- shifted GBM when $\tilde{\lambda}_i = 0$ for each asset i;
- standard jump-diffusion models when $\delta_0^{(i)} = 0$ for each asset i[8].

In Section 2.5, we also describe a different displaced jump-diffusion process, i.e. the shifted asymmetric jump-diffusion process, which includes two sources of jumps. The new pricing method is then employed to price basket options on assets following this process, as in Paletta et al. (2014).

2.3.2 Moment-matching procedure

The key feature of the proposed new methodology is the use of the variable

$$J(Z) = \sum_{k=0}^{m-1} \varphi_k H_k(Z), \tag{2.8}$$

to approximate the standardised basket return[9]

$$X_T = \frac{B_T^*}{B_0^* e^{rT}} - h_1 \tag{2.9}$$

by matching its first m moments.[10] Above, $H_k(x)$ denotes the kth-order probabilists' Hermite polynomial (see Appendix 2.A for a description of these polynomials)

$$H_k(x) = \frac{(-1)^k}{n(x)} \frac{\partial^k n(x)}{\partial x^k},$$

$n(\cdot)$ is the standard normal density function and Z is a standard normal random variable. Additionally, we employed the two shifted quantities: the *shifted strike price*

$$K^* = K - \sum_{i=1}^{\Upsilon} a_i \delta_0^{(i)} e^{rT}; \tag{2.10}$$

[8] For example, the log-normal jump-diffusion process of Merton (1976) and the dual-exponential jump-diffusion process of Kou (2002) are within this category.

[9] The shifted basket value at time 0, B_0^*, is assumed to be different from 0.

[10] In (2.9), we consider the standardized basket return rather than the basket value for computational reasons. Indeed, the moments of the standardized return X_T usually have the same order of magnitude. On the contrary, for the basket value B_T^*, different-order moments can also differ by several orders of magnitude and this may cause numerical inefficiencies in the solution of the moment-matching system of equation (2.12).

and the *shifted basket value* at time T

$$B_T^* = B_T - \sum_{i=1}^{\Upsilon} a_i \delta_0^{(i)} e^{rT}. \tag{2.11}$$

Then, we consider two values for the parameter h_I, and this leads to two variants of our new methodology:

***m*GA** indicates a moment matching procedure with $h_I = 0$ and, consequently, the first moment of X_T (under the risk-neutral measure $\mathbb{Q}$) is equal to 1;

***m*GB** indicates a moment matching procedure with $h_I = 1$ and, consequently, the first moment of X_T (under the risk-neutral measure $\mathbb{Q}$) is equal to 0;

where the mnemonics driven by m stand for the number of moments matched and G highlights that a transformation of the Gaussian distribution is considered.

We also consider a hybrid methodology spanned by the two methods *m*GA and *m*GB, which henceforth will be called ***m*GAB**. This hybrid method returns the solution of the method that correctly matches the moments if one of *m*GA and *m*GB works properly and takes into account the worst error between the two variants if both correctly match the moments. Consequently, we consider the worst scenario that can happen in choosing either one or other method. The reason for doing so is that, while the fair benchmark prices are known (as given by the method of Pellizzari (2001)) in the scenario-comparison we will carry out in Section 2.4, when the *m*GAB method is applied in the financial markets, there is no benchmark price, since the actual market price is unknown and has to be calculated. By using the worst error in the scenario-based study, we find the worst case we may have in employing *m*GAB (under the assumption of a market following the model we are considering).

The methods *m*GA, *m*GB and *m*GAB expand on the Hermite tree method for pricing financial derivatives proposed in Leccadito et al. (2012). The idea of their method is to match the moments of the underlying asset log-return with the moments of a discrete random variable. The methodology in this chapter extends on Leccadito et al. (2012) to deal with baskets that may take on negative values and replaces the binomial distribution that they use with the asymptotically equivalent Gaussian distribution. Consequently, our new methodology consists of quasi-analytic pricing and hedging formulae, which do not employ a tree or lattice method.

The basis of the methodologies mGA, mGB and mGAB is the calculation of the parameters φ_k in (2.8) by solving the following system of m equations in m unknowns:

$$\begin{cases} E[J] = E[X_T] \\ E[J^2] = E[X_T^2] \\ \quad\ldots \\ E[J^m] = E[X_T^m] \end{cases} \tag{2.12}$$

The solution of this system is carried out numerically. In the following, in Proposition 2.3.1 and formula (2.17), we calculate the k-th moment of X_T and J, respectively.

Proposition 2.3.1 (Basket's moments)**.** *The k-th moment of the standardised return X_T in formula (2.9), under $\mathbb{Q}$, is given by*

$$E[X_T^k] = E\left[\left(\frac{B_T^*}{B_0^* e^{rT}} - h_I\right)^k\right] = \sum_{i=0}^{k} \binom{k}{i} \frac{(-h_I)^i}{(B_0^* e^{rT})^{k-i}} E[B_T^{*\,k-i}], \tag{2.13}$$

where

$$\begin{aligned} E[B_t^{*k}] \quad = \quad & \sum_{i_1=1}^{\Upsilon} \cdots \sum_{i_k=1}^{\Upsilon} a_{i_1}\left(S_0^{(i_1)} - \delta_0^{(i_1)}\right) e^{(r+\omega_{i_1})t} \times \ldots \\ & \ldots \times a_{i_k}\left(S_0^{(i_k)} - \delta_0^{(i_k)}\right) e^{(r+\omega_{i_k})t} \operatorname{mgf}(\boldsymbol{e}_{i_1} + \ldots + \boldsymbol{e}_{i_k}), \end{aligned} \tag{2.14}$$

$\omega_j = -\tilde{\beta}_j \tilde{\lambda}_j - \frac{1}{2}\sigma_j^2$, $\boldsymbol{e}_j \in \Re^{\Upsilon}$ *is the vector having 1 in position j and 0 elsewhere. Furthermore, the moment generation function of* $\sigma_i V_t^{(i)} + \sum_{l=1}^{N_t^{(i)}} \log(Y_l^{(i)} + 1)$ *is given by*

$$\operatorname{mgf}(\boldsymbol{u}) = \exp\left\{t\boldsymbol{u}'\boldsymbol{\Sigma}\boldsymbol{u}/2\right\} \prod_{i=1}^{\Upsilon} \operatorname{mgf}_{N_t^{(i)}}\left(\eta_i u_i + \upsilon_i^2 u_i^2/2\right) \tag{2.15}$$

where $\boldsymbol{\Sigma}$ denotes the covariance matrix of $\boldsymbol{V} = \left(V_t^{(1)}, \cdots, V_t^{(\Upsilon)}\right)'$*, and*

$$\operatorname{mgf}_{N_t^{(i)}}(u) = \exp(t\tilde{\lambda}_i(e^u - 1)). \tag{2.16}$$

Proof. Formulae (2.13) and (2.14) are derived by exponentiation of formulae (2.9) and (2.11), respectively and the linear property of the expectation operator. Additionally, the moment generation function of $\sigma_i V_t^{(i)} + \sum_{l=1}^{N_t^{(i)}} \log(Y_l^{(i)} + 1)$ in (2.15) is calculated by conditioning with respect to $N_t^{(i)}$. □

On the other hand, since (1) Hermite polynomials are orthogonal with respect to the standard normal probability density function (see formula (2.37)), (2) the product between two Hermite polynomials is still a polynomial (although non-Hermitian) and (3) the expected value is a linear operator, then the k-th moment of $J(Z)$, under $\mathbb{Q}$, is calculated as:

$$E[J^k] = \sum_{i_1=0}^{m-1} \cdots \sum_{i_k=0}^{m-1} \varphi_{i_1} \ldots \varphi_{i_k} E[H_{i_1}(Z) \ldots H_{i_k}(Z)]. \tag{2.17}$$

which admits a closed-form solution as a weighted sum of the moments of the standard normal variable Z for any polynomial order m and moment order k.

For example, for $m = 4$, the first 4 moments of $J(Z)$ are[11]

$$\begin{aligned}
E[J\,] &= \varphi_0 \\
E[J^2] &= \sum_{i=0}^{m-1} i!\varphi_i^2 \\
E[J^3] &= \varphi_0^3 + (3\varphi_1^2 + 6\varphi_2^2 + 18\varphi_3^2)\varphi_0 + 6\varphi_1^2\varphi_2 + 36\varphi_1\varphi_2\varphi_3 + 8\varphi_2^3 + 108\varphi_2\varphi_3^2 \\
E[J^4] &= (6\varphi_1^2 + 12\varphi_2^2 + 36\varphi_3^2)\varphi_0^2 + (24\varphi_1^2\varphi_2 + 144\varphi_1\varphi_2\varphi_3 + 32\varphi_2^3 + 432\varphi_2\varphi_3^2)\varphi_0 \\
&\quad + 3\varphi_1^4 + 24\varphi_1^3\varphi_3 + 60\varphi_1^2\varphi_2^2 + 252\varphi_1^2\varphi_3^2 + 576\varphi_1\varphi_2^2\varphi_3 + 1296\varphi_1\varphi_3^3 \\
&\quad + 60\varphi_2^4 + 2232\varphi_2^2\varphi_3^2 + 3348\varphi_3^4 + \varphi_0^4.
\end{aligned} \tag{2.18}$$

In the following, we assume $\varphi_0, \ldots, \varphi_{m-1}$ given and we state our main results for pricing and hedging basket options.

2.3.3 Pricing methodology

Let us consider a basket call option with the payoff as in (2.1). The price of this option is given as the expected discounted payoff at maturity. The mechanism of shifting the basket and strike price in formulae (2.10) and (2.11) allows us to write the pricing formula in two equivalent ways:

$$c_0(B_0, T, K) = e^{-rT} E[(B_T - K)^+] = e^{-rT} E[(B_T^* - K^*)^+] = c_0(B_0^*, T, K^*). \tag{2.19}$$

By using (2.19), the next proposition provides a quasi-analytic pricing formula for the European basket call option when the Hermite polynomial approximation is employed.

Proposition 2.3.2 (Pricing formula)**.** *The price of a European basket call option with the*

[11] The following calculations are carried out by the symbolic-calculus tool in Matlab R2013a by using formulae (2.37) and (2.38).

Hermite expansion variant mGA or mGB is given by:

$$c_0(B_0,T,K) = B_0^*[(\varphi_0 + h_1)N(-h_2\tilde{z}) + h_2 g_H(\tilde{z})] - K^* e^{-rT} N(-h_2\tilde{z}) \tag{2.20}$$

where

$$g_H(\tilde{z}) = n(\tilde{z}) \sum_{k=0}^{m-2} \varphi_{k+1} H_k(\tilde{z}), \tag{2.21}$$

K^* is the shifted strike price, B^* is the shifted basket, $h_1 = 0$ for the variant mGA and $h_1 = 1$ for the variants mGB, $h_2 = \operatorname{sgn}(B_0^*)$, $\tilde{z}$ is the solution of $[J(\tilde{z}) + h_1]B_0^* e^{rT} = K^*$, $n(\cdot)$ is the standard normal density function, $N(\cdot)$ is the standard normal cumulative distribution function and the parameters φ_k are calculated by matching the first m moments of the standardised return quantity X_T.

Proof. Let us consider the approximation of X_T by the random variable $J(Z)$ (see Section 2.3.2) via the solution of the moment-matching procedure in the system of equations in (2.12). Consequently,

$$B_T^* \approx B_0^* e^{rT}(J(Z) + h_1) \tag{2.22}$$

and substituting it into the equality (2.19) we have:

$$\begin{aligned} c_0(B_0^*,T,K^*) &= e^{-rT}E[(B_T^* - K^*)^+] \approx e^{-rT}\int_{l_1}^{l_2}\left[B_0^* e^{rT}(J(z)+h_1) - K^*\right] n(z)\,dz \\ &= B_0^*\int_{l_1}^{l_2} J(z)n(z)\,dz + (h_1 B_0^* - K^* e^{-rT})N(-h_2\tilde{z}) \end{aligned} \tag{2.23}$$

where, for $B_0^* > 0$, $l_1 = \tilde{z}$ and $l_2 = +\infty$ and, for $B_0^* < 0$, $l_1 = -\infty$ and $l_2 = \tilde{z}$.

For the calculation of the integral

$$\int_{l_1}^{l_2} J(z)n(z)\,dz,$$

the results in formulae (2.39) and (2.40) are employed for $B_0^* > 0$ and $B_0^* < 0$, respectively. Formula (2.20) is then proved by rearranging the terms. □

On the other hand, let us consider a basket put option. Since the pricing formula (2.20) works for both positive and negative basket values, the pricing of the put option can be achieved via the call-pricing formula as shown by the following chain of equalities

$$p_0(B_0,T,K) = e^{-rT}E[(K - B_T)^+] = e^{-rT}E[(K^* - B_T^*)^+] = e^{-rT}E[\{(-B_T^*) - (-K^*)\}^+].$$

Alternatively, the put price can be calculated directly using formula (2.20) and using the put-call parity relation for basket options:

$$p_0(B_0,T,K) = c_0(B_0,T,K) + Ke^{-rT} - B_0, \tag{2.24}$$

which follows from the no-arbitrage principle.

Therefore, in what follows, we will focus exclusively on European basket call options.

We also note that the methodology described above holds for general asset dynamics and that the assumptions in Section 2.3.1 impacts only on the φs, which are calculated by solving the system of equations (2.12).

2.3.4 Hedging methodology

This section reports the formula for the hedging parameter with respect to the variable u, which can be any of the involved quantities, such as $S_0^{(i)}$, B_0, σ_i, r, T, a_i, $\tilde{\lambda}_i$, $\delta_0^{(i)}$, $\tilde{\beta}_i$, η_i or υ_i.

Proposition 2.3.3 (Hedging parameters)**.** *For h_1, h_2, $\tilde{z}$, $g_H(\cdot)$, $n(\cdot)$ and $N(\cdot)$ defined in Proposition 2.3.2, the hedging parameter of a European basket call option, with respect to the variable u, under the Hermite expansion variant mGA or mGB, is given by*

$$\begin{aligned}\frac{\partial c_0}{\partial u} &= c_0 e^{rT}\frac{\partial e^{-rT}}{\partial u} + B_0^*\left[h_2 {g_H}'(\tilde{z}) + \frac{\partial \varphi_0}{\partial u} n(-h_2\tilde{z})\right] + h_2 e^{-rT} N(\tilde{z})\frac{\partial K^*}{\partial u} \\ &\quad + e^{-rT}\frac{\partial (B_0^* e^{rT})}{\partial u}\left[h_2 g_H(\tilde{z}) + \varphi_0 n(-h_2\tilde{z}) + h_1\left(-h_2 N(\tilde{z}) + \frac{h_2+1}{2}\right)\right]\end{aligned} \tag{2.25}$$

where

$${g_H}'(\tilde{z}) = n(\tilde{z})\sum_{k=0}^{m-2}\frac{\partial \varphi_{k+1}}{\partial u} H_k(\tilde{z}) \tag{2.26}$$

and c_0 is the short for $c_0(B^*,T,K^*)$.

Proof. The calculation of the hedging parameter can be achieved by direct differentiation using Leibniz' rule of the pricing formula (2.19) considered together with approximation (2.22), as follows:

$$\begin{aligned}\frac{\partial c_0}{\partial u} &= \frac{\partial e^{-rT}}{\partial u}\int_{l_1}^{l_2}\left[B_0^* e^{rT}(J(z)+h_1) - K^*\right] n(z)\,dz \\ &\quad + e^{-rT}\int_{l_1}^{l_2}\frac{\partial\left[B_0^* e^{rT}(J(z)+h_1) - K^*\right]}{\partial u} n(z)\,dz = \\ &= c_0 e^{rT}\frac{\partial e^{-rT}}{\partial u} + e^{-rT}\int_{l_1}^{l_2}\frac{\partial\left[B_0^* e^{rT}(J(z)+h_1) - K^*\right]}{\partial u} n(z)\,dz.\end{aligned}$$

Additionally, since the Hermite polynomials do not depend on u,

$$\frac{\partial J(z)}{\partial u} = \sum_{k=0}^{m-1} \frac{\partial \varphi_k}{\partial u} H_k(z),$$

and, consequently, formulae (2.39) and (2.40) in Appendix 2.A can also be used for the integral

$$\int_{l_1}^{l_2} \frac{\partial J(z)}{\partial u} n(z)\, dz$$

where l_1 and l_2 are as defined for function (2.23). Formula (2.25) is then proved by rearranging the terms. □

In Proposition 2.3.3, the terms

$$\frac{\partial \varphi_k}{\partial u}, k = 0, \ldots, m-1$$

are needed and are calculated with the methodology outlined in Borovkova et al. (2007). In what follows, for the sake of completeness, we summarise and particularize this methodology for the price dynamics considered in this chapter. Consider the "moment-matching" system of equations in (2.12) and differentiate both sides of each equation with respect to u. The quantities $\frac{\partial \varphi_k}{\partial u}$ are given by the solution of the new system of equations

$$\begin{cases} \left.\frac{\partial E[J]}{\partial u}\right|_{\varphi} = \frac{\partial E[X_T]}{\partial u} \\ \left.\frac{\partial E[J^2]}{\partial u}\right|_{\varphi} = \frac{\partial E[X_T^2]}{\partial u} \\ \qquad \cdots \\ \left.\frac{\partial E[J^m]}{\partial u}\right|_{\varphi} = \frac{\partial E[X_T^m]}{\partial u} \end{cases} \tag{2.27}$$

where $\varphi = [\varphi_0, \ldots, \varphi_{m-1}]$ is the vector of the coefficients of the Hermite polynomials obtained as a solution of the system of equations (2.12), $\left.\frac{\partial E[J^k]}{\partial u}\right|_{\varphi}$ is the first derivative of the k-moment of J (formula (2.17)) with respect to u at φ and $\frac{\partial E[X_T^k]}{\partial u}$ is the first derivative of the k-moment of X_T (formula (2.13)) with respect to u.

In Section 2.4.2, our method is compared with the method of Borovkova et al. (2007) using the delta-hedging performance as a yardstick. For that exercise, it is particularly important to apply formula (2.25) for the calculation of the delta parameter. Below, we show how the calculations are carried out in this case.

Delta parameter For the case when $u = B_0$, by employing the chain rule and the inverse-function rule of differentiation we have:

$$\frac{\partial c_0}{\partial B_0} = \frac{\partial c_0}{\partial S_0^{(1)}} \frac{\partial S_0^{(1)}}{\partial B_0} = \frac{\partial c_0}{\partial S_0^{(1)}} \frac{1}{a_1}, \tag{2.28}$$

where

$$\begin{aligned} \frac{\partial c_0}{\partial S_0^{(1)}} &= B_0^* \left[h_2 g_H{}'(\tilde{z}) + \frac{\partial \varphi_0}{\partial S_0^{(1)}} n(-h_2 \tilde{z}) \right] \\ &+ a_1 \left[h_2 g_H(\tilde{z}) + \varphi_0 n(-h_2 \tilde{z}) + h_1 \left(-h_2 N(\tilde{z}) + \frac{h_2 + 1}{2} \right) \right] \end{aligned} \tag{2.29}$$

which follows from Proposition 2.3.3 for $u = S_0^{(1)}$. For the calculation of $\frac{\partial \varphi_k}{\partial S_0^{(1)}}$, as stated before, the first derivatives with respect to $S_0^{(1)}$ of the moments of X_T are needed. By using (2.13), the first derivative of the k-th moment of X_T with respect to $S_0^{(1)}$ is given as:

$$\frac{\partial E[X_T^k]}{\partial S_0^{(1)}} = \sum_{i=0}^{k} \binom{k}{i} \frac{(-h_1)^i}{(B_0^* e^{rT})^{k-i}} \left(\frac{\partial E[B_T^{*\,k-i}]}{\partial S_0^{(1)}} - a_1 \frac{(k-i) E[B_T^{*\,k-i}]}{B_0^*} \right), \tag{2.30}$$

where $\frac{\partial E[B_t^*]}{\partial S_0^{(1)}} = a_1$ and for $k > 1$

$$\begin{aligned} \frac{\partial E[B_t^{*k}]}{\partial S_0^{(1)}} &= \frac{\partial E[B_t^k]}{\partial S_0^{(1)}} = k a_1 e^{(r+\omega_1)t} \sum_{i_1=1}^{\Upsilon} \cdots \sum_{i_{k-1}=1}^{\Upsilon} \left(a_{i_1} S_0^{(i_1)} e^{(r+\omega_{i_1})t} \right) \times \cdots \\ &\cdots \times \left(a_{i_{k-1}} S_0^{(i_{k-1})} e^{(r+\omega_{i_{k-1}})t} \right) \mathrm{mgf}(\boldsymbol{e}_1 + \boldsymbol{e}_{i_1} + \ldots + \boldsymbol{e}_{i_{k-1}}), \end{aligned}$$

and $\mathrm{mgf}(\cdot)$ is defined in (2.15).

2.4 Numerical Study

The usefulness of a newly proposed method can be gauged by comparing it with other established methods in the literature. The variants mGA, mGB and mGAB of our Hermite approximation approach are compared on a large set of simulated option scenarios with the method described in Borovkova et al. (2007), which is capable of matching quite large ranges of skewness and kurtosis and is supported by a Black-and-Scholes type pricing formula. In Appendix 2.C, we provide a detailed description of the method of Borovkova, Permana and Weide.

Both pricing and hedging performances are studied below.

2.4.1 Pricing performance

In this section, we study the pricing performance of each method with respect to the "exact" fair benchmark prices, which are calculated using the mean Monte Carlo algorithm outlined in Pellizzari (2001). This algorithm is described in Appendix 2.B, where we also adapt Pellizzari's method to basket options whose assets' price dynamics follow a displaced diffusion with jumps. The pricing performance is determined considering the following two measures of error:

- the percentage of "good prices", *%-Good*, defined as the percentage of scenarios for which the absolute percentage error of the considered method is below 2%;[12]
- the mean absolute percentage error, *MAPE*, calculated relative to the scenarios for which the method was able to find a numerical solution.

A moment-matching pricing method finds the price of an option scenarios whenever the system of m equations in (2.12) admits a solution. The moments of the basket return may be outside the domain of the Hermite polynomial expansions (for mGA or mGB) and/or of the log-normal density (for the method of Borovkova et al. (2007)) and, consequently, the method employed may not be able to price and hedge that option scenario. In our numerical study, we find that our methods mGA and mGB found a solution in more than 90% of the option scenarios. Additionally, we found that this percentage is very close to that for the method of Borovkova et al., although in the analysis we require our methods to match a higher number of moments ($m = 4$ or $m = 6$ moments versus only $m = 3$ for the method of Borovkova et al. (2007)). Not surprisingly, the hybrid mGAB has a higher percentage of solutions.

In what follows, we carry out two different performance comparisons. For the first, we consider the six option scenarios in Borovkova et al. (2007), while for the second we consider a larger set made up of 2,000 randomly simulated option scenarios.

[12]The 2% threshold was selected a posteriori as the smallest integer greater than the average MAPEs in Tables 2.3 and 2.4. We also carried out a similar analysis for different threshold values and the analyses returned qualitatively similar conclusions, with our mGA, mGB and mGAB significantly outperforming the method of Borovkova et al. (2007).

Comparison under the option scenarios in Borovkova et al. (2007)

This section directly compares the Hermite expansion methodologies introduced in this chapter with the method of Borovkova et al. (2007) on the six basket option scenarios that they considered in their study.

It is assumed that the i-th asset in the basket follows the process described by SDE (2.7) with $\tilde{\lambda}_i = 0$ and $\delta_0^{(i)} = 0$ (i.e. the assets following correlated geometric Brownian motions) and the other parameters being as in Table 2.1. The results are depicted in Table 2.2: the prices of the option scenarios calculated using the method of Borovkova et al. (2007) and presented in this table differ from those they showed in their research article because, to be consistent with the other results considered in this chapter, we price basket options on equities and not on forward contracts, as they did in their study.

The numerical results indicate that our 4GA and 4GB return, for these six scenarios, exactly the same prices (and consequently also 4GAB), and the two methods, appear to be as good as the method of Borovkova et al. (2007) according to the %-Good criterion and outperform it according to the MAPE criterion. The methods 6GA and 6GB underperform the other methods and, consequently, for the baskets analysed here, there is very little advantage in matching six moments, as the Hermite approximation method works better overall when only the first four moments are matched.

Comparison under a set of simulated scenarios

A general comparison is performed considering 2,000 simulated option scenarios. In the first 1,000 scenarios (henceforth “Set 1”) each asset in the baskets follows the displaced jump-diffusion model with dynamics given by SDE (2.7) where the parameters are drawn based on the following specifications: all σ_i are independently uniformly distributed between 0.1 and 0.6; the spot prices $S_0^{(i)}$ are uniformly distributed between 70 and 130; the shifts $\delta_0^{(i)}$ range uniformly between 0 and 20; the intensities of the Poisson processes $\tilde{\lambda}_i$ are uniformly distributed between 0 and 0.2; the average jump size (η_i) is uniformly distributed between -0.3 and 0; and the volatility (υ_i) is uniformly distributed between 0 and 0.3. Furthermore, the number of assets in the basket in each scenario is uniformly distributed between 2 and 15, the risk-free rate r is uniformly distributed between 0.0 and 0.1, T is uniformly distributed between 0.1 and 1 year, the weights a_i of the assets in the basket are uniformly distributed between -1 and 1, the ratios K/B_0 are uniformly distributed between 0.8 and 1.2, and the correlation matrix among assets is randomly generated satisfying the semi-positiveness condition (as in Hardin et al. (2013)). The second 1,000 scenarios (hence-

Table 2.1 Specification of the basket option scenarios in Borovkova et al. (2007)

	Basket 1	Basket 2	Basket 3	Basket 4	Basket 5	Basket 6
Stock Prices	[100,120]	[150,100]	[110,90]	[200,50]	[95,90,105]	[100,90,95]
Volatilities	[0.2,0.3]	[0.3,0.2]	[0.3,0.2]	[0.1,0.15]	[0.2,0.3,0.25]	[0.25,0.3,0.2]
Weights	[-1,1]	[-1,1]	[0.7,0.3]	[-1,1]	[1,-0.8,-0.5]	[0.6,0.8,-1]
Correlation(s)	$\rho_{1,2}=0.9$	$\rho_{1,2}=0.3$	$\rho_{1,2}=0.9$	$\rho_{1,2}=0.8$	$\rho_{1,2}=0.9$, $\rho_{2,3}=0.9$ $\rho_{1,3}=0.8$	$\rho_{1,2}=0.9$, $\rho_{2,3}=0.9$ $\rho_{1,3}=0.8$
Strike price	20	-50	104	-140	-30	35

Notes: Other relevant parameters are $r=3\%$, 1-year maturity, $\tilde{\lambda}_i=0$ and $\delta_0^{(i)}=0$. The first row indicates the stock prices $S_0^{(i)}$, the second the volatilities σ_i, the third the weights a_i of the assets in the basket, the fourth the correlation $\rho_{i,j}$ for each couple (i,j) of assets and the fifth the strike K. The only difference compared with the scenarios in Borovkova et al. (2007) is that they price options on baskets of forward contracts, while we price options on baskets of equities.

Table 2.2 Comparison on the option scenarios in Borovkova et al. (2007)

# Basket	MC (SD)	BPW	4GA	4GB	6GA	6GB
1	8.2263 (0.0031)	8.2442	8.1977	8.1977	8.2222	8.2222
2	16.47 (0.0052)	16.6215	16.4424	16.4424	16.4631	16.3654
3	12.5887 (0.0005)	12.5911	12.5695	12.5695	12.5888	12.5888
4	1.1459 (0.0008)	1.1456	1.1453	1.1453	1.0938	1.1162
5	7.4681 (0.0027)	7.4951	7.4563	7.4563	7.4555	7.4555
6	9.7767 (0.0030)	9.7989	9.7628	9.7628	9.7856	9.7856
%-Good		100.00%	100.00%	100.00%	83.33%	83.33%
MAPE		0.30%	0.17%	0.17%	0.82%	0.59%

Notes: This table reports the comparison on the six basket option scenarios in Borovkova et al. (2007) (see Table 2.1). The second column shows the prices (standard deviation in bracket) calculated using the Monte Carlo method (MC) with the control variate in Pellizzari (2001) with 10^6 simulations that are considered as benchmarks. The third column shows the prices calculated using the method in Borovkova et al. (2007), BPW in the table. The last four columns contain the prices under the methods mGA and mGB when $m=4$ and $m=6$. The last two rows show the pricing performance: %-Good is the percentage of absolute percentage errors smaller than 2% ("good price"), and MAPE is the mean absolute percentage error.

forth "Set 2") are identical to the scenarios in Set 1, except for the average jump size (η_i), which is uniformly distributed between -0.3 and 0.3.

The number of simulations used when applying the mean Monte Carlo method in Pellizzari (2001) is between 10^5 and 10^6. In particular, for each scenario, this number is selected in such a way that the standard deviations of the option prices are smaller than 0.01.

The results with respect to Set 1 and Set 2 are summarised in Tables 2.3 and 2.4, respectively. The two tables show similar results. Overall, methods 4GA and 4GB have analogous performance in terms of %-Good and MAPE criteria, with 4GB performing slightly better than 4GA. 4GA outperforms 4GB only for longer maturities (greater than 0.5 years) scenarios under the %-Good measure and for near-the-money scenarios under MAPE. Both 4GA and 4GB are robust to a change in the risk-free rate. However, the performance of both methods improves for longer-maturities under %-Good and worsens under MAPE. Comparing our two Hermite expansion methodologies with the Borovkova-Permana-Weide method, it is clear that the latter is not as good as the former at matching the model-implied characteristics and that the fourth moment is necessary for pricing basket options. Although not shown in the tables, both 4GA and 4GB show greater improvement on the method of Borovkova et al. (2007) the greater the basket size. Finally, method 4GAB outperforms the two methods under %-Good, performing almost as well under MAPE. Consequently, one can use this hybrid method for practical purposes.

A cross analysis of Tables 2.3 and 2.4 shows that changes in the expected jump intensity impact on the performances of the Hermite-approximant methods that are slightly better for $\eta_i \in [-0.3, 0]$.

Additionally, the two tables show the pricing performance when methods mGA and mGB are used for $m = 6$ moments. Method 6GB outperforms all of the others while 6GA also outperforms the other methods under %-Good but underperforms 4GA and 4GB under MAPE. Finally, also considering 6GB, the increase in computational time (about 10 times) for the method may not be sustainable in real life applications and, therefore, we suggest using 4GAB for a good trade-off in performance versus computational effort.

2.4.2 Delta-hedging performance

A comparison of delta-hedging performance between our formula (2.29) and the formula proposed in Borovkova et al. (2007) is illustrated in this section.

A sample of $\mathtt{n_S} = 1{,}000$ simulated paths with a 1-week-interval hedging rolling frequency is generated for six basket option scenarios. All of the scenarios have: $\Upsilon = 2$, $\sigma_1 = 0.3$, $\sigma_2 = 0.2$, $T = 0.5$ years, $S_0^{(1)} = 110$, $S_0^{(2)} = 90$, $a_1 = 0.7$, $a_2 = 0.3$, $\delta_0^{(1)} = \delta_0^{(2)} = 20$, $\tilde{\lambda}_1 = \tilde{\lambda}_2 = 0.2$, and $\upsilon_1 = \upsilon_2 = 0.2$. Additionally, we consider for three of the scenarios

Table 2.3 Pricing performance comparison: Set 1 (negative average jump-sizes)

		r		T		$\frac{K}{B_0}$			
		≤ 0.05	>0.05	≤ 0.5	>0.5	≤ 0.98	$(0.98,1.02]$	>1.02	Total
%-Good	BPW	58.86%	61.38%	69.72%	50.79%	64.82%	66.98%	53.62%	60.10%
	4GA	76.77%	74.39%	69.51%	81.50%	74.12%	74.53%	77.38%	75.60%
	4GB	77.36%	75.20%	72.97%	79.53%	74.34%	76.42%	78.28%	76.30%
	4GAB	82.48%	80.89%	80.08%	83.27%	79.42%	81.13%	84.16%	81.70%
	6GA	85.46%	84.75%	85.68%	84.58%	81.93%	88.30%	87.56%	85.11%
	6GB	88.99%	90.58%	86.37%	92.93%	89.60%	86.17%	90.80%	89.78%
MAPE	BPW	1.44%	1.29%	1.00%	1.73%	1.15%	1.35%	1.59%	1.37%
	4GA	0.42%	0.39%	0.29%	0.50%	0.46%	0.43%	0.34%	0.40%
	4GB	0.41%	0.39%	0.29%	0.50%	0.44%	0.45%	0.35%	0.40%
	4GAB	0.43%	0.38%	0.30%	0.50%	0.44%	0.44%	0.36%	0.41%
	6GA	0.58%	0.58%	0.54%	0.61%	0.56%	0.55%	0.60%	0.58%
	6GB	0.36%	0.38%	0.42%	0.32%	0.34%	0.64%	0.33%	0.37%
No. of Options		508	492	492	508	452	106	442	1,000

Notes: This table contains a summary of the performances of several methods for pricing options in Set 1. The assets follow equation (2.7) where the parameters are randomly generated and uniformly distributed in the following ranges: $\Upsilon \in [2,15]$, $r \in (0;0.1]$, $\sigma_i \in [0.1;0.6]$, $T \in [0.1;1]$, $S_0^{(i)} = [70;130]$, $a_i \in [-1;1]$, $\frac{K}{B} \in [0.8;1.2]$, $\delta_0^{(i)} \in [0;20]$, $\tilde{\lambda}_i \in [0;0.2]$, $\eta_i \in [-0.3;0]$ and $\upsilon_i \in [0;0.3]$ for all $i = 2,\cdots,\Upsilon$. In each row the results per method are shown: BPW stands for the method in Borovkova et al. (2007), mGA and mGB are the Hermite approximation methods matching the first m moments of X_T with $m \in \{4,6\}$. Furthermore, 4GAB is a mixture of 4GA and 4GB and returns the solution of the method that correctly matches the moments if only one of 4GA and 4GB works properly, or the solution of the method that is the worst out of the two. The mean Monte Carlo method (MC) of Pellizzari (2001) is the benchmark price.

Table 2.4 Pricing performance comparison: Set 2 (positive and negative average jump-sizes)

		r		T		$\frac{K^*}{B_0^*}$			
		≤ 0.05	>0.05	≤ 0.5	>0.5	≤ 0.98	$(0.98,1.02]$	>1.02	Total
%-Good	BPW	52.95%	57.72%	65.04%	45.87%	59.73%	57.55%	50.23%	55.30%
	4GA	72.24%	72.56%	65.65%	78.94%	70.80%	68.87%	74.89%	72.40%
	4GB	74.02%	71.75%	68.70%	76.97%	71.46%	70.75%	74.89%	72.90%
	4GAB	78.15%	77.24%	74.80%	80.51%	75.88%	73.58%	80.54%	77.70%
	6GA	82.68%	84.96%	80.08%	87.40%	82.74%	83.96%	84.84%	83.80%
	6GB	88.19%	90.04%	83.33%	94.69%	87.83%	88.68%	90.50%	89.10%
MAPE	BPW	1.59%	1.42%	1.18%	1.84%	1.26%	1.55%	1.74%	1.51%
	4GA	0.59%	0.54%	0.46%	0.65%	0.64%	0.61%	0.49%	0.57%
	4GB	0.57%	0.55%	0.46%	0.65%	0.62%	0.66%	0.48%	0.56%
	4GAB	0.59%	0.55%	0.48%	0.66%	0.62%	0.68%	0.49%	0.57%
	6GA	0.76%	0.70%	0.71%	0.75%	0.67%	0.82%	0.77%	0.73%
	6GB	0.50%	0.52%	0.53%	0.49%	0.46%	0.62%	0.53%	0.51%
No. of Options		508	492	492	508	452	106	442	1,000

Notes: This table contains the summary of the performances of several methods for pricing options in Set 2. The assets follow equation (2.7) where the parameters are randomly generated and uniformly distributed in the following ranges: $\Upsilon \in [2,15]$, $r \in (0;0.1]$, $\sigma_i \in [0.1;0.6]$, $T \in [0.1;1]$, $S_0^{(i)} = [70;130]$, $a_i \in [-1;1]$, $\frac{K}{B} \in [0.8;1.2]$, $\delta_0^{(i)} \in [0;20]$, $\tilde{\lambda}_i \in [0;0.2]$, $\eta_i \in [-0.3;0.3]$ and $\upsilon_i \in [0;0.3]$ for all $i = 2,\cdots,\Upsilon$. For other information see Table 2.3.

$r = 2\%$ and $\eta_1 = \eta_2 = -0.3$ and for the other three $r = 5\%$ and $\eta_1 = \eta_2 = 0.3$. The strikes considered are $K = \{100, 104, 110\}$.

For each path, the option deltas are calculated at each time-step using the methods 4GA, 4GB and the methodology outlined in Borovkova et al. (2007). The evaluation of the performances for the delta-hedged portfolios is carried out via two error measures:

- the average hedging error among all of the simulations, *AHE*;
- the average quadratic hedging error, *AQHE*

where the hedging error is defined as the difference in values between the hedged portfolio at the maturity date and the option's payoff.

The results for the hedging performances are reported in Table 2.5. Methods 4GA and 4GB produce good results and their performances are virtually identical on the six scenarios considered. For $\eta_i = -0.3$, the two Hermite expansion methods tend to super-hedge, as the measure AHE indicates, although the average errors are almost negligible. However, when $\eta_i = 0.3$, the hedging error is negative on average, showing a sub-hedge that is caused by the high average jump size.

The method of Borovkova et al. also performs fairly well for the three scenarios with $\eta_i = -0.3$ with virtually the same performances as 4GA and 4GB under the measure AQHE. However, under these three scenarios, the new methods have much better performances than the method of Borovkova et al. (2007) under the AHE measures (a remarkable reduction of more than 25% is reached). When one considers the positive average jump size ($\eta_i = 0.3$), the method of Borovkova et al. (2007) also sub-hedges under each scenario and both its AHE and AQHE measures of error are worse than those of 4GA and 4GB.

2.5 Additional application: The shifted asymmetric jump-diffusion process

In this section, we propose another application of the pricing formula in Proposition 2.3.2, when the underlying assets follow more general price dynamics. The content presented in this section summarises the results in Paletta et al. (2014).

Under the modelling framework in Section 2.3.1, let us consider the financial market consisting of Υ assets $S^{(i)}$ for any $i = 1, \ldots, \Upsilon$, with dynamics given by:

$$d\left(S_t^{(i)} - \delta_t^{(i)}\right) = \left(\alpha_i - \sum_{q=\{U,D\}} \lambda_{i,q}\beta_{i,q}\right)\left(S_t^{(i)} - \delta_t^{(i)}\right)dt + \left(S_t^{(i)} - \delta_t^{(i)}\right)\sum_{j=1}^{n_w} \gamma_{ij} dW_t^{(j)} + \left(S_{t^-}^{(i)} - \delta_t^{(i)}\right)\sum_{q\{U,D\}} d\mathrm{Q}_t^{(i,q)} \tag{2.31}$$

Equation (2.31) describes the shifted asymmetric jump-diffusion process. This process is a generalisation of the shifted jump-diffusion process in (2.3) and includes two sources of jumps. We indicate by $q \in \{U, D\}$ each source of jumps. The parameters in the stochastic differential equation (2.31) are the same of those in (2.3), with the addition of the index q. In particular, $\left\{\mathrm{Q}_t^{(i,q)}\right\}_{t\geq 0}$ are independent compound Poisson processes formed from some underlying Poisson processes $\left\{N_t^{(i,q)}\right\}_{t\geq 0}$ with intensity $\lambda_{i,q} \geq 0$ and driven by the source of jumps q; $Y_j^{(i,q)}$ represents the jump amplitude of the j-th jump of $N_t^{(i,q)}$ for any $i = 1, \ldots, \Upsilon$; and the jumps $Y_j^{(i,q)}$ for any $i = 1, \ldots, \Upsilon$ are independent and identically distributed random variables with probability density function $f^{(i,q)}(y) : [-1, \infty) \to [0, 1]$ and expected value under the physical measure $\mathbb{P}$

$$\beta_{i,q} = E_{\mathbb{P}}[Y^{(i,q)}] = \int_{-1}^{\infty} y f^{(i,q)}(y)dy.$$

Following the same approach as in Section 2.3.1, if a solution $(\theta, \tilde{\beta}_U, \tilde{\lambda}_U, \tilde{\beta}_D, \tilde{\lambda}_D)$ of the system

$$\alpha_i - \lambda_{i,U}\beta_{i,U} - \lambda_{i,D}\beta_{i,D} - r = \sum_{j=1}^{n_w} \gamma_{ij}\theta_j - \tilde{\lambda}_{i,U}\tilde{\beta}_{i,U} - \tilde{\lambda}_{i,D}\tilde{\beta}_{i,D}, \quad i = 1, \ldots, \Upsilon \tag{2.32}$$

does exist and is selected in association with the risk-neutral pricing measure $\mathbb{Q}$, then, under this risk-neutral measure, the solution of (2.31) is:

$$S_t^{(i)} = \delta_0^{(i)} e^{rt} + (S_0^{(i)} - \delta_0^{(i)}) e^{\left(r - \sum_{q=\{U,D\}} \tilde{\lambda}_{i,q}\tilde{\beta}_{i,q} - \frac{1}{2}\sum_{j=1}^{n_w} \gamma_{ij}^2\right)t + \sum_{j=1}^{n_w} \gamma_{ij}\tilde{W}_t^{(j)}} \prod_{q=\{U,D\}} \prod_{l=1}^{N_t^{(i,q)}} (Y_l^{(i,q)} + 1). \tag{2.33}$$

Because the solution to (2.32) is, in general, not unique, as for (2.5), we assume that one solution is selected and a pricing measure $\mathbb{Q}$ is fixed. Under this $\mathbb{Q}$-measure, the intensity of the Poisson process $\left\{N_t^{(i,q)}\right\}_{t\geq t_0}$ for the i-th asset in the basket is $\tilde{\lambda}_{i,q}$ and

$$\tilde{\beta}_{i,q} = E[Y^{(i,q)}] = \int_{-1}^{+\infty} y\tilde{f}^{(i,q)}dy.$$

An example of the distributions for the two jumps can be found in Ramezani and Zeng (2007) where they choose for the U-jumps the Pareto distribution and for the D-jumps the Beta distribution. The two sources respectively represent the arrival of good and bad news in the market, which cause upward and downward jumps in prices, respectively.

The model implied k-th moment of B_t^* under $\mathbb{Q}$, after the changing of variables $\sigma_i^2 = \sum_{j=1}^{n_w}\gamma_{ij}^2$ and $V_t^{(i)} = \sum_{j=1}^{n_w}\frac{\gamma_{ij}}{\sigma_i}\tilde{W}_t^{(j)}$ as in (2.7), is:

$$E[B_t^{*k}] = \sum_{i_1=1}^{\Upsilon}\cdots\sum_{i_k=1}^{\Upsilon}\text{mgf}(\mathbf{e}_{i_1}+\ldots+\mathbf{e}_{i_k})\prod_{l=1}^{k}a_{i_l}\left(S_0^{(i_l)}-\delta_0^{(i_l)}\right)e^{(r+\omega_{i_l})t} \tag{2.34}$$

as in (2.14), where $\omega_j = -\tilde{\lambda}_{j,U}\tilde{\beta}_{j,U} - \tilde{\lambda}_{j,D}\tilde{\beta}_{j,D} - \frac{1}{2}\sigma_j^2$, $\mathbf{e}_j$ is the vector having 1 in position j and zero elsewhere, the moment generation function (mgf) of

$$\sigma_i V_t^{(i)} + \sum_{q=\{U,D\}}\sum_{l=1}^{N_t^{(i,q)}}\log\left(Y_l^{(i,q)}+1\right)$$

is given by:

$$\text{mgf}(\mathbf{u}) = \exp\left\{t\mathbf{u}'\Sigma\mathbf{u}/2\right\}\prod_{q=\{U,D\}}\prod_{i=1}^{\Upsilon}\text{mgf}_{N_t^{(i,q)}}\left(cgf_{\log(Y_{i,q}+1)}(u_i)\right) \tag{2.35}$$

where Σ denotes the covariance matrix of $V = \left(V_t^{(1)},\cdots,V_t^{(\Upsilon)}\right)'$,

$$\text{mgf}_{N_t^{(i,q)}}(u) = \exp(t\tilde{\lambda}_{i,q}(e^u-1))$$

and $\text{cgf}_{\log(Y_{i,q}+1)}(u_i)$ is the cumulant-generating function of $\log(Y_{i,q}+1)$. The approximation in Section 2.3 is then employed for pricing and hedging purposes.

Additionally, we benchmark our method 4GA with the method in Borovkova et al. (2007) over three basket put option scenarios, where the "true" fair no-arbitrage prices are calculated using the standard Monte Carlo methodology with 10^6 simulated terminal values of the underlying asset prices. The prices for the basket put option scenarios are calculated by the put-call parity in (2.24) and the call pricing formula (2.20). Table 2.6

contains the prices for the basket put option scenarios and also, on the last two rows, the error measures %-Good and MAPE. The three scenarios are identical, with $\Upsilon = 6$ assets that follow the shifted asymmetric jump-diffusion process with $S_0^{(i)} = i$, $\tilde{\lambda}_{i,U} = 0.4$, $\tilde{\lambda}_{i,D} = 0.04$, $Y_l^{(i,U)} + 1 \sim \text{Pareto}(60)$, $Y_l^{(i,D)} + 1 \sim \text{Beta}(70,1)$ and $\delta_0^{(i)} = e^{-rT}$. Furthermore, $K = 21$ (the scenarios are at-the-money options), $r = 4\%$ and the variance-covariance matrix is simulated using the algorithm in Hardin et al. (2013). The entries of the upper triangular part of this matrix are in the order: 0.1037, 0.7620, 0.8570, 0.8790, 0.6970, -0.3040; 0.1657, 0.4050, 0.5108, 0.4331, 0.1701; 0.2124, 0.9067, 0.8829, -0.6870; 0.1431, 0.7479, -0.5702; 0.0468, -0.7557; 0.2136. The three scenarios differ only for time-to-maturity with $T = \{1,2,3\}$ years.

The performances of our method 4GA and the method of Borovkova et al. (2007) are summarised in Table 2.6, where it is shown that for the three scenarios considered, 4GA outperforms the latter under both measures of error. The comparison shows that our methodology also works effectively in pricing basket options whose assets follow more general dynamics.

2.6 Conclusions

Recent techniques for pricing and hedging basket options have imposed strong assumptions on the overall evolution dynamics of the basket, searching for closed-form solutions and repackaging log-normal Black-and-Scholes type pricing formulae. However, by doing so, the empirical characteristics of historical prices are not taken into consideration. In this chapter, we have highlighted a new methodology that can handle baskets of assets following more realistic diffusions.

In particular, we considered the correlated shifted log-normal diffusion with jumps in Câmara et al. (2009), which is capable of accounting for negative skewness and excess kurtosis, which characterize equity stocks. We demonstrated with numerical comparisons that our Hermite expansion approach provides pricing and hedging results for basket options that are as good as the competing methods, and in many cases superior. The improved results emphasized in the chapter are not surprising since the technique is fundamentally based on matching additional moments under the model specification. Thus, we allow granular specification of dynamics for each asset but then we only need to determine the moments of the basket. We also provided an additional application of our methodology over the shifted asymmetric jump-diffusion process. This application shows that the new method works fairly well even for more general dynamics.

Our methodology has evident advantages over numerical methods since it consists of a

Table 2.5 Delta-hedging performance comparison

Scenario		AHE	AQHE	Scenario		AHE	AQHE	Scenario		AHE	AQHE	
	$r = 2\%$	0.021	0.102		$r = 2\%$	0.041	0.110		$r = 2\%$	0.066	0.121	BPW
1	$\eta_i = -0.3$	0.014	0.101	2	$\eta_i = -0.3$	0.034	0.108	3	$\eta_i = -0.3$	0.059	0.119	4GA
	$K = 100$	0.014	0.101		$K = 104$	0.034	0.108		$K = 110$	0.059	0.119	4GB
	$r = 5\%$	-0.663	0.589		$r = 5\%$	-0.642	0.558		$r = 5\%$	-0.618	0.521	BPW
4	$\eta_i = 0.3$	-0.337	0.460	5	$\eta_i = 0.3$	-0.317	0.455	6	$\eta_i = 0.3$	-0.293	0.449	4GA
	$K = 100$	-0.337	0.460		$K = 104$	-0.317	0.455		$K = 110$	-0.293	0.449	4GB

Notes: This table contains a summary of the delta-hedging performances of three methods: BPW stands for the method in Borovkova et al. (2007) and 4GA and 4GB are the Hermite approximation methods matching the first 4 moments of X_T. The measures of error considered are: AHE– average error, AQHE– average quadratic hedging error. The six scenarios considered are: $\Upsilon = 2$, $\sigma_1 = 0.3$, $\sigma_2 = 0.2$, $T = 0.5$ years, $S_0^{(1)} = 110$, $S_0^{(2)} = 90$, $a_1 = 0.7$, $a_2 = 0.3$, $\delta_0^{(1)} = \delta_0^{(2)} = 20$, $\tilde{\lambda}_i = 0.2$, and $\upsilon_1 = \upsilon_2 = 0.2$ and the other parameter values are in the 'Scenario' columns.

Table 2.6 Pricing performance comparison: The shifted asymmetric jump-diffusion process

Maturity	MC	BPW	4GA
$T = 1$	0.5473 (0.0040)	0.6078	0.5125
$T = 2$	0.6382 (0.0051)	0.7162	0.6402
$T = 3$	0.6791 (0.0058)	0.7465	0.6784
%-Good		0.00%	66.67%
MAPE		11.07%	2.26%

Notes: This table shows the results for three put basket option scenarios under the shifted asymmetric jump-diffusion. The scenarios are described in Section 2.5. The last two rows indicate the two measures of error in Section 2.4.1.

Black-and-Scholes type formula that only requires the solution of a system of equations for the matching of the moments. Computationally, the effort required is relatively small and our method is even more advantageous when one prices and hedges a portfolio of options written on the same basket, since the solution of the system for the moment matching is carried out only once.

While this chapter focused on equity baskets, it is clear that the same methodology can be applied for mixtures of assets and models, as long as the moments can be calculated easily.

2.6.1 Further research

Currently, the method we presented in this chapter has been employed exclusively for European basket options. In the financial markets, a large number of American-style basket options is traded, so it would be useful to extend our Hermite expansion method to include the pricing and hedging of these derivatives. As discussed in Section 2.2, there exist some methodologies for pricing American-style derivatives that use moment matching procedures together with binomial tree methodologies and some quasi-analytic approximations for spread options are also available. Further research would be on quasi-analytic pricing formulae for American-style basket options that use the methodology presented here.

Another direction for the research will be to employ the Hermite expansion approach for different payoff functions and/or asset price models. This will be researched in particular, together with future research that we will conduct on the weighted least squares Monte Carlo method described in Chapter 5. In particular, this new methodology improves on the performances of the least squares Monte Carlo method of Longstaff and Schwartz (2001) by employing the weighted least squares regression method. In Chapter 5, we will show that a good approximation of the weighting function is given as the difference in price between two European options. Consequently, we will conduct further research on this topic to find an efficient way to price these options for multidimensional payoffs, which could be done by employing the methodology outlined in this chapter.

Appendix

Appendix 2.A Results for the probabilists' Hermite polynomials

Throughout this chapter we use the probabilists' Hermite polynomials which are defined as:

$$H_k(x) = \frac{(-1)^k}{n(x)} \frac{\partial^k n(x)}{\partial x^k}.$$

We report in the following the first six polynomials:

$$\begin{aligned}
H_0(x) &= 1 \\
H_1(x) &= x \\
H_2(x) &= x^2 - 1 \\
H_3(x) &= x^3 - 3x \\
H_4(x) &= x^4 - 6x^2 + 3 \\
H_5(x) &= x^5 - 10x^3 + 15x
\end{aligned}$$

Among the properties of the Hermite polynomials we enumerate:

- the recursive relation

$$H_k(z) = zH_{k-1}(z) - H'_{k-1}(z) \quad k = 1, 2, \ldots \tag{2.36}$$

  where $H'_k(\cdot)$ is the first derivative of $H_k(\cdot)$ with respect to z;

- the orthogonality property

$$\int_{-\infty}^{+\infty} H_m(x) H_n(x) n(x)\, dx = 0, m \neq n \tag{2.37}$$

$$\int_{-\infty}^{+\infty} H_n(x)H_n(x)n(x)\,dx = n!, m \neq n \tag{2.38}$$

i.e. the probabilists' Hermite polynomials are orthogonal w.r.t. the standard normal probability density function.

In the following, we show some useful results. Following from (2.36), for $k \geq 1$

$$\int_{\tilde{z}}^{+\infty} H_k(z)n(z)\,dz = \int_{\tilde{z}}^{+\infty} zH_{k-1}(z)n(z)\,dz - \int_{\tilde{z}}^{+\infty} H'_{k-1}(z)n(z)\,dz.$$

Solving the second integral by parts and using $n'(z) = -zn(z)$,

$$\begin{aligned}\int_{\tilde{z}}^{+\infty} H_k(z)n(z)\,dz &= \int_{\tilde{z}}^{+\infty} zH_{k-1}(z)n(z)\,dz \\ &\quad - \left[+H_{k-1}(z)n(z)|_{\tilde{z}}^{+\infty} + \int_{\tilde{z}}^{+\infty} zH_{k-1}(z)n(z)\,dz \right] = \\ &= H_{k-1}(\tilde{z})n(\tilde{z})\,.\end{aligned}$$

Additionally $\int_{\tilde{z}}^{+\infty} H_0(z)n(z)\,dz = N(-\tilde{z})$, so

$$\int_{\tilde{z}}^{+\infty} J(z)n(z)\,dz = g_H(\tilde{z}) + \varphi_0 N(-\tilde{z})\,. \tag{2.39}$$

where $g_H(\cdot)$ is defined in formula (2.21).

Given the orthogonality property of these polynomials,

$$\int_{-\infty}^{\tilde{z}} H_k(z)n(z)\,dz = -H_{k-1}(\tilde{z})n(\tilde{z})$$

and, consequently,

$$\int_{-\infty}^{\tilde{z}} J(z)n(z)\,dz = -g_H(\tilde{z}) + \varphi_0 N(\tilde{z})\,. \tag{2.40}$$

In the proofs of Propositions 2.3.2 and 2.3.3, formula (2.39) and formula (2.40) are used for $B_0^* > 0$ and $B_0^* < 0$, respectively.

Appendix 2.B The mean Monte Carlo method (Pellizzari, 2001)

The fair benchmark prices in this chapter are calculated via the Monte Carlo method with control variates in Pellizzari (2001). The key idea is to employ the closed-form pricing formula in the univariate case to reduce the variance of the Monte Carlo simulation method for pricing multivariate payoffs.

In particular, the method considers a general European-style derivative, which pays out at maturity

$$\mathtt{h}_T^P(S_T^{(1)},\ldots,S_T^{(\Upsilon)}) \tag{2.41}$$

and considers the control variates

$$M_T(i) = \mathtt{h}_T^P\left(E[S_T^{(1)}],\ldots,E[S_T^{(i-1)}],S_T^{(i)},E[S_T^{(i+1)}],\ldots,E[S_T^{(\Upsilon)}]\right). \tag{2.42}$$

Consequently, the Monte Carlo estimate is obtained by

$$MC_{(i)} = e^{-rT}\frac{1}{\mathtt{ns}}\sum_{j=1}^{\mathtt{ns}}\left[\mathtt{h}_{T_{(j)}}^P - M_{T(j)}(i) + E[M_T(i)]\right] \tag{2.43}$$

where $\mathtt{ns}$ is the number of simulations, $\mathtt{h}_{T_{(j)}}^P$ is the payoff function (2.41) evaluated for the j-th path, $M_{T(j)}(i)$ is the control variates function evaluated for the j-th path and $E[M_T(i)]$ is the analytical price for the univariate case.

The method is quite general and also works for the basket options considered in this chapter. In particular, if the payoff (2.41) is given as the payoff of a basket call option, then (2.42) corresponds to the price of an option on the single asset $S_T^{(i)}$, with payoff function at maturity

$$a_i \max\left\{0, S_T^{(i)} - \frac{K - a_1E[S_T^{(1)}] - \ldots - a_{i-1}E[S_T^{(i-1)}] - a_{i+1}E[S_T^{(i+1)}] - \ldots - a_\Upsilon E[S_T^{(\Upsilon)}]}{a_i}\right\} \tag{2.44}$$

i.e. the price of a call option with strike price

$$\frac{K - a_1E[S_T^{(1)}] - \ldots - a_{i-1}E[S_T^{(i-1)}] - a_{i+1}E[S_T^{(i+1)}] - \ldots - a_\Upsilon E[S_T^{(\Upsilon)}]}{a_i}.$$

For $a_i < 0$, formula (2.42) becomes the payoff of a vanilla put option on $S_T^{(i)}$. In both cases, under the modelling framework in Section 2.3.1, considering the mechanism of shifting the strike price and basket value, these options can be priced using the option pricing formula in Merton (1976) for a (non-shifted) jump-diffusion process.

Appendix 2.C The shifted log-normal method in Borovkova et al. (2007)

The method of Borovkova et al. (2007) uses a moment-matching technique to price basket options and calculate the hedging parameters. The idea is to approximate the value of the basket at maturity via a generalisation of the log normal distribution to cover baskets that can take over negative values. This method assumes basket options whose assets follow correlated geometric Brownian motions and we adapt it so that is also works for assets following the shifted jump-diffusion model in Section 2.3.1.

The probability density function they consider is:

$$f(u) = \frac{1}{\sqrt{2\pi}\sigma_s(b_h u - \tau_s)} e^{-\frac{1}{2\sigma_s^2}(\log(b_h u - \tau_s) - \mu_s)^2}, u < -\tau_s \tag{2.45}$$

where μ_s is the scale, σ_s is the shape, τ_s is the location parameter and $b_h = \{-1, 1\}$. The first three moments of a random variable with the probability density function in (2.45) are:

$$M_1 = b_h \left[\tau_s + e^{\mu_s + \frac{1}{2}\sigma_s^2}\right] \tag{2.46}$$

$$M_2 = \tau_s^2 + 2\tau_s e^{\mu_s + \frac{1}{2}\sigma_s^2} + e^{2\mu_s + 2\sigma_s^2} \tag{2.47}$$

$$M_3 = b_h \left[\tau_s^3 + 3\tau_s^3 e^{\mu_s + \frac{1}{2}\sigma_s^2} + 3\tau_s e^{2\mu_s + 2\sigma_s^2} + e^{3\mu_s + \frac{9}{2}\sigma_s^2}\right]. \tag{2.48}$$

Given the skewness of the basket at the maturity time,

$$skew_T = \frac{E[B_T^* - E[B_T^*]]^3}{\left[E[{B_T^*}^2] - E[B_T^*]^2\right]^{3/2}}, \tag{2.49}$$

for $E[{B_T^*}^k]$ as in (2.14), Borovkova et al. (2007) select $b_h = -1$ when $skew_T < 0$ and $b_h = 1$ for $skew_T > 0$. The moment matching is then carried out by solving a system of three equations in three unknowns, similar to that in (2.12).

Since the density (2.45) is a generalisation of the log-normal density, Borovkova et al. carried out the pricing of a basket option using the Black-and-Scholes formula. In particular, for the cases when $b_h = 1$ and $\tau_s = 0$, the option pricing is carried out by the standard Black-and-Scholes pricing formula, while for the other cases they consider an adjusted version, where the shift τ_s is coupled with the strike price. Specifically, considering B_1, a log-normally distributed basket with $b_h = 1$ and $\tau_s = 0$ with parameters μ_s and σ_s, then the basket

$$B_2 = B_1 + \tau_s \tag{2.50}$$

is a log-normally distributed with parameters μ_s, σ_s and τ_s. On the other hand, the basket

$$B_3 = -B_1 + \tau_s \tag{2.51}$$

is negative shifted log-normal with parameters μ_s, σ_s and τ_s. Consequently, in all of the three cases above, one can still employ the Black-and-Scholes formula.

Chapter 3

American Options: Problem Formulations and Existing Methodologies

This chapter presents the American option pricing problem and it is a preamble to Chapters 4 and 5 where we describe two new methodologies for pricing and hedging American options. The chapter is structured as follows: Section 3.1 describes the market settings, provides mathematical formulations of the problem and introduces the notation for the following chapters; Section 3.2 looks over the main properties and characteristics of the pricing formula and the optimal exercise price; Section 3.3 reviews existing solution approaches and classifies them into quasi-analytic methods and numerical methods; and, Section 3.4 details the six quasi-analytic methodologies we improve with our new 'extension' method in Chapter 4. Finally, Section 3.5 overviews the contributions we will provide in the next three chapters.

3.1 Problem formulations

Consider[1] a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq t_0}, \mathbb{Q})$ representing a financial market consisting of three assets: (1) a bank account $dM_t = rM_t dt$, where the risk-free interest rate r is assumed constant over time, (2) a risky asset with the dynamics $\{S_t\}_{t\geq t_0}$ given under the risk-neutral measure $\mathbb{Q}$ as $S_t = S_{t_0} e^{s_t}$, where $S_{t_0} > 0$ and $\{s_t\}_{t\geq t_0}$ is a Markovian process with $s_{t_0} = 0$ and (3) an American-style derivative written on the risky asset with payoff function from exercise at time t in time-t_0 dollars to the holder indicated by $h_t(\cdot)$, and maturity at

[1]The notation we introduce here follows Glasserman (2003), chapter 8.

time T. For example, for the state $S_t = \mathscr{X}$, the vanilla put option has the payoff function:

$$h_t(\mathscr{X}) = e^{-r(t-t_0)} \max\{0, K - \mathscr{X}\} \tag{3.1}$$

and, for a call option, the payoff function is:

$$h_t(\mathscr{X}) = e^{-r(t-t_0)} \max\{0, \mathscr{X} - K\}, \tag{3.2}$$

where, as before, K is the strike price.

McKean (1967) and Van Moerbeke (1974) formulated the American option pricing problem as a free boundary problem. Following a no-arbitrage argument, Bensoussan (1984) and Karatzas (1988) expressed the American option pricing problem as the problem of finding the optimal expected discounted payoff under the risk-neutral measure $\mathbb{Q}$:

$$V_{t_0}(S_{t_0}) = \sup_{t^* \in \Gamma} E\left[h(S_{t^*})|S_{t_0}\right], \tag{3.3}$$

where Γ is the class of admissible stopping times in $(t_0, T]$. In particular, when considering an American put option, the supremum in formula (3.3) is achieved by the optimal stopping time:

$$t^* = \inf\left\{\inf_{t \in [t_0, \infty)} \{S_t \leq S_f^{(E)}(t)\}, T\right\} \tag{3.4}$$

where $S_f^{(E)}(t)$ is the optimal exercise price. On the other hand, for an American call option, the optimal stopping time assumes the form:

$$t^* = \inf\left\{\inf_{t \in [t_0, \infty)} \{S_t \geq S_f^{(C,E)}(t)\}, T\right\} \tag{3.5}$$

where equivalently, $S_f^{(C,E)}$ is the optimal exercise price for call option.[2] Since the optimal exercise price is not known in advance, in the following formulation we make explicit that this function has to be determined together with the option price. Let us define the set $\Pi = \{f_S(t) : (t_0, T] \mapsto \Re^+\}$ of all real functions. Then, the problem (3.3) for an American put option corresponds to:[3]

$$V_{t_0}(S_{t_0}) = \sup_{f_S \in \Pi} E\left[h\left(S_{t^*(f_S)}\right)|S_{t_0}\right], \tag{3.6}$$

[2] In equation (3.31), we report the relation between $S_f^{(E)}$ and $S_f^{(C,E)}$ for American options under geometric Brownian motion, which is commonly defined as the put-call symmetry for optimal exercise prices.

[3] See, among others, Garcia (2003), for a parametric version of formulation (3.6).

where, as before, $t^*(f_S) = \inf\left\{\inf_{t\in[t_0,\infty)}\{S_t \leq f_S(t)\}, T\right\}$. Equivalently, for American call options, the optimal stopping time is $t^*(f_S) = \inf\left\{\inf_{t\in[t_0,\infty)}\{S_t \geq f_S(t)\}, T\right\}$.

Numerical solutions for problem (3.3) usually restrict the pricing of American options to contracts that can be exercised only to a fixed set of exercise opportunities $t_1 < t_2 < \ldots < t_m = T$ and t_0, the time of evaluation, is not usually part of this set. Without loss of generality, we assume that $\Delta_{t_i} = t_{i+1} - t_i = T/m = \Delta_t$, for any $i = 0, \ldots, m-1$. This time discretisation leads to the solution of the pricing problem for the so-called Bermudan contracts. Henceforth, we will refer to these contracts simply as American options.

Additionally, to simplify the notation under the discrete-time settings, we denote: the underlying asset price at the ith exercise opportunity (the one at time t_i) by S_i; the logarithmic return over the period (t_i, t_{i+1}) by $s_{i+1} = \log\left(\frac{S_{i+1}}{S_i}\right)$; the payoff function in time-t_0 dollars for exercise at time t_i when the current state of the underlying asset is $S_i = \mathscr{X}$ by $h_i(\mathscr{X})$; the discount factor from time t_i to time t_0 by:

$$r_{0,i} = e^{-ri\Delta_t}; \tag{3.7}$$

the value in time-t_0 dollars of the American-style derivative at time t_i given $S_i = \mathscr{X}$ (assuming the option has not been exercised previously) by $V_i(\mathscr{X})$; the continuation value of the American-style derivative measured in time-t_0 dollars conditional on the current state $\mathscr{X}$ by:

$$C_i(\mathscr{X}) = E_{t_i}\left[V_{i+1}(S_{i+1})|S_i = \mathscr{X}\right], \tag{3.8}$$

where $E_{t_i}[\cdot]$ is the expectation operator under the risk-neutral measure $\mathbb{Q}$ with respect to the filtration $\mathscr{F}_{t_i}$; and, the optimal exercise price at time t_i by S_{f_i}, which is defined as the underlying asset price $\mathscr{X}$ such that:

$$C_i(\mathscr{X}) = h_i(\mathscr{X}), \tag{3.9}$$

i.e. the underlying asset price for which it is indifferent to exercise the option or keep it alive.

Under these discrete settings, problem (3.3) is often solved via the following equivalent dynamic programming formulation:

$$\begin{cases} V_m(\mathscr{X}) = h_m(\mathscr{X}) & (3.10) \\ V_i(\mathscr{X}) = \max\{h_i(\mathscr{X}), C_i(\mathscr{X})\}, i = 0, \cdots, m-1 & (3.11) \end{cases}$$

where one is ultimately interested in $V_0(S_{t_0})$ and time t_0 is excluded from the set of exercise opportunities by simply choosing $h_0(S_{t_0}) = 0$.

Moreover, considering the definition of $S_f^{(E)}(t)$, problem (3.10)-(3.11) for American put options can also be formulated as:

$$\begin{cases} V_m(\mathscr{X}) = h_m(\mathscr{X}) & (3.12) \\ V_i(\mathscr{X}) = \begin{cases} h_i(\mathscr{X}) & \text{if } \mathscr{X} \leq S_{f_i} \\ C_i(\mathscr{X}) & \text{if } \mathscr{X} > S_{f_i} \end{cases}, \quad i = 0, \ldots, m-1 & (3.13) \end{cases}$$

and, equivalently, for American call options as:

$$\begin{cases} V_m(\mathscr{X}) = h_m(\mathscr{X}) & (3.14) \\ V_i(\mathscr{X}) = \begin{cases} h_i(\mathscr{X}) & \text{if } \mathscr{X} \geq S_{f_i} \\ C_i(\mathscr{X}) & \text{if } \mathscr{X} < S_{f_i} \end{cases}, \quad i = 0, \ldots, m-1. & (3.15) \end{cases}$$

In Chapter 4, we employ formulation (3.3) and a decomposition of the time-to-maturity into two components to derive the pricing formula of our new quasi-analytic pricing and hedging method. In Chapter 5, we use formulation (3.10)-(3.11) for the iterations of our new weighted least squares Monte Carlo algorithm. We use the two formulations (3.12)-(3.13) and (3.14)-(3.15) to prove our main result in Chapter 5, that is the existence of heteroscedasticity in the least squares Monte Carlo method by Longstaff and Schwartz (2001), and to provide an approximation of the weighting function in the regressions we employ for our new pricing method.[4]

3.1.1 Underlying asset price dynamics

In this section, we specify the underlying asset price dynamics we consider in the following chapters, under the risk-neutral measure $\mathbb{Q}$. The new method in Chapter 4 aims to improve any quasi-analytic methodology for pricing and hedging American options under the geometric Brownian motion dynamics. The notation we use is illustrated below.

Geometric Brownian motion (GBM): If a stochastic process $\{S_t\}_{t \geq t_0}$ is a geometric Brownian motion then it satisfies the following stochastic differential equation (SDE):

$$dS_t = (r - \delta) S_t dt + \sigma S_t d\tilde{W}_t \tag{3.16}$$

[4] We detail our algorithm in Section 5.3.

where $\tilde{W}_t$ is a standard Wiener process under $\mathbb{Q}$, σ is the volatility parameter and δ is the dividend yield. The solution to (3.16) is:

$$S_t = S_{t_0} e^{(r-\delta-\frac{\sigma^2}{2})(t-t_0)+\sigma\tilde{W}_{t-t_0}}. \tag{3.17}$$

The arithmetic version of (3.16), $dS_t = (r-\delta)dt + \sigma d\tilde{W}_t$, was studied by Bachelier (1900) in his doctoral thesis and then detailed by Albert Einstein in 1905.

Additionally, in Chapter 5, we propose a new numerical algorithm, the weighted least squares Monte Carlo, which intends to solve the upper bias of one of the most employed regression-based methods, the least squares Monte Carlo method of Longstaff and Schwartz (2001). In order to prove the superiority of our new method, we will carry out a scenario-based comparison (Section 5.3.2) on American options under the geometric Brownian motion or any of the following processes.

Exponential Ornstein-Uhlenbeck process: If a stochastic process $\{S_t\}_{t\geq t_0}$ is following the exponential Ornstein-Uhlenbeck process then it satisfies the following SDE:

$$dS_t = \eta\left(\mu - \log S_t\right) S_t dt + \sigma S_t d\tilde{W}_t \tag{3.18}$$

where μ is the logarithmic long-term mean under $\mathbb{Q}$, $\eta > 0$ is the speed of mean-reversion and σ is the volatility parameter. This process is discussed in Brigo et al. (2009) and corresponds to the exponentiation of the logarithm-return process $\{s_t\}_{t\geq t_0}$:

$$ds_t = \eta(\theta - s_t)d_t + \sigma d\tilde{W}_t \tag{3.19}$$

where $\theta = \mu - \frac{\sigma^2}{2\eta}$. Process (3.18) has the advantage over (3.19) of providing only positive values.

Log-normal jump-diffusion process: If a stochastic process $\{S_t\}_{t\geq t_0}$ follows the log-normal jump-diffusion then it satisfies the following SDE:

$$dS_t = (r-\delta-\lambda\kappa)S_{t-}dt + \sigma S_{t-}d\tilde{W}_t + S_{t-}d\sum_{l=1}^{Q_t^M}(Y_l^M - 1) \tag{3.20}$$

where r, δ and σ are as above, $J_t^M = \sum_{l=1}^{Q_t^M}(Y_l^M - 1)$ is a compound Poisson process driven by the Poisson process Q_t^M with intensity λ, the jump sizes Y_l^M are i.i.d with $\log(Y_l^M) \sim N(\alpha_J, \sigma_J^2)$ and $\kappa = E\left[Y_l^M - 1\right] = e^{\alpha_J+\sigma_J^2/2} - 1$. All three random sources are assumed to be independent. This process was introduced in finance by Merton (1976) and its solution is:

$$S_t = S_{t_0} e^{(r-\delta-\lambda\kappa-\frac{\sigma^2}{2})(t-t_0)+\sigma\tilde{W}_{t-t_0}} \prod_{l=1}^{Q_t^M} Y_l^M. \tag{3.21}$$

Double exponential jump-diffusion process: If a stochastic process $\{S_t\}_{t\geq t_0}$ is following the double exponential jump-diffusion then it satisfies the following SDE:

$$dS_t = (r-\delta-\lambda\kappa)S_{t-}dt + \sigma S_{t-}d\tilde{W}_t + S_{t-}d\sum_{l=1}^{Q_t^K}(Y_l^K - 1) \tag{3.22}$$

where r, δ and σ are as above, $J_t^K = \sum_{l=1}^{Q_t^K}(Y_l^K - 1)$ is a compound Poisson process driven by the Poisson process Q_t^K with intensity λ, the jump sizes Y_l^K are i.i.d with:

$$\log(Y_l^K) = \begin{cases} x^+, & \text{with probability } q, \\ -x^-, & \text{with probability } 1-q \end{cases} \tag{3.23}$$

where x^+ and x^- are exponential random variables with mean $1/\eta_1$ and $1/\eta_2$ respectively ($\eta_1 > 1$ and $\eta_2 > 0$), and $\kappa = (1-q)\frac{\eta_2}{\eta_2+1} + q\frac{\eta_1}{\eta_1-1} - 1$. All random sources are assumed to be independent. This process was introduced in finance by Kou (2002) and its solution is:

$$S_t = S_{t_0} e^{(r-\delta-\lambda\kappa-\frac{\sigma^2}{2})(t-t_0)+\sigma\tilde{W}_{t-t_0}} \prod_{l=1}^{Q_t^K} Y_l^K. \tag{3.24}$$

All of the processes above are of the type $S_t = S_{t_0}e^{s_t}$ and, consequently, fulfil the assumptions of Propositions 5.2.1 and 5.2.2 in Chapter 5. Additionally, for the four dynamics above, there exist closed-form pricing formulae for European options, which we employ in Section 5.3.1 to estimate the weights of the weighted least squares regression method.

3.2 Theoretical properties

Some important seminal works on the American pricing problem are Samuelson (1965), McKean (1967) and Van Moerbeke (1974). The problem was later studied from different perspectives and particular attention was focused on the optimal exercise price. In this section, we review the main results on the option price function and the optimal exercise price for American options. The main focus will be on options under the geometric Brownian motion since, in Chapter 4, we use this process to derive some theoretical results of our new quasi-analytic 'extension' method as well as to motivate the new methodology. For completeness, we also look over the main properties under other dynamics.

3.2.1 Properties of the option price function

We start by considering an American put (resp. call) option under the geometric Brownian dynamics in (3.16).[5] The pricing function, $V_t(\mathscr{X})$, of this option is continuous on $t \in [t_0, T]$ and, for all $t \in [t_0, T]$, it is non-increasing (non-decreasing) and convex on $\Re^+$.

Additionally, the option pricing formula, for all $\mathscr{X} \in \Re^+$, is non-increasing on $[t_0, T]$, i.e. it has decreasing time-value. Let us also define the delta parameter as $\Delta_t(\mathscr{X}) = \frac{\partial V_t(\mathscr{X})}{\partial \mathscr{X}}$. For an American put option (resp. call), it holds that $\Delta_t(\mathscr{X}) \in [-1, 0]$ ($\Delta_t(\mathscr{X}) \in [0, -1]$) for any $\mathscr{X} \in \Re^+$ and all $t \in [t_0, T]$.

An important property that links put and call options under geometric Brownian motion is the put-call parity (McDonald and Schroder, 1998).[6] Let $P_t(S_t, K, r, \delta, T)$ and $C_t(S_t, K, r, \delta, T)$ denote the time-t price functions of American put and call options respectively, where we explicitly indicate the inputs in order: underlying spot price, strike price, risk-free rate, dividend yield and maturity date. Then, it holds true that:

$$P_t(S_t, K, r, \delta, T) = C_t(K, S_t, \delta, r, T) \tag{3.25}$$

and, consequently, the problem of pricing American call options can be reduced to that of pricing American puts by reversing the order of S_t and K, and r and δ.

Moreover, in the case of infinite maturity, $T \to +\infty$, McKean (1967) and Merton (1973) priced American options by exact-closed formulae. These options are called perpetual and their pricing formula for put options is:

$$V_{t_0}^{\infty}(S_{t_0}) = \alpha(S_f^{\infty}) S_{t_0}^{\beta} \tag{3.26}$$

where

$$\alpha(S_f^{\infty}) = (K - S_f^{\infty})(S_f^{\infty})^{-\beta}, \tag{3.27}$$

[5] An overview of some of these properties is also in Detemple (2005), Chapter 4.

[6] McDonald and Schroder (1998) derived formula (3.25) by expanding on the result by Grabbe (1983) who worked for options on foreign exchange.

$$\beta = \left(\frac{1}{2} - \frac{r-\delta}{\sigma^2}\right) - \sqrt{\left(\frac{1}{2} - \frac{r-\delta}{\sigma^2}\right)^2 + 2\frac{r}{\sigma^2}} \tag{3.28}$$

and the optimal exercise price is a flat function equal to:

$$S_f^\infty = K\frac{\beta}{\beta-1}. \tag{3.29}$$

The call pricing formula is derived by put-call symmetry. Additionally, Mordecki (2002) and Boyarchenko and Levendorskii (2002) priced perpetual options under various Lévy models. Although perpetual options are not traded, they are useful to price real options and/or to approximate finite-maturity options as we show in Chapter 4, where we employ formula (3.26) to prove our new pricing formula (Proposition 4.2.1).

3.2.2 Properties of the optimal exercise price

We next review a number of documented properties of the optimal exercise price of an American put, which will be useful in Chapter 4 for our 'extension' method. Again, let us assume the geometric Brownian motion dynamics (3.18) and an American put option with maturity T and strike price K. Jacka (1991) proved that the optimal exercise price $S_f^{(E)}(t)$ is a continuous function, non-decreasing with respect to time, with the limiting value

$$\lim_{t\to T} S_f^{(E)}(t) = \min\{K, rK/\delta\} \tag{3.30}$$

when dividends are paid at the rate δ, and bounded below by the optimal exercise price of the perpetual put option in (3.29). Ekstrom (2004) and Chen et al. (2008) showed that the optimal exercise price is a convex function when no dividends are paid. Chen et al. (2013) extended the proof to dividend-paying assets and demonstrated that for a dividend yield very close but slightly greater than the risk-free rate, i.e. $0 < \delta - r \ll 1$, the optimal exercise price loses convexity near maturity. Additionally, Van Moerbeke (1974) showed that the optimal exercise price approaches the maturity date with infinite speed.[7]

On the other hand, moving away from maturity, the optimal exercise price has 'nicer' behaviour: Chen et al. (2011) for dividend-paying options and Xie et al. (2011) for the non-dividend paying case calculated an upper bound of the optimal exercise price and proved that

[7] A vast amount of the literature has studied the behaviour of the optimal exercise price near expiration. In Section 4.4, we review the contributions that provide the asymptotic behaviour of the optimal exercise price near the maturity of the contract.

this bound converges quickly (more than exponentially in time) to the flat function in (3.29). Consequently, being reasonably far from maturity, one can expect that the optimal exercise price is almost flat.

The results we just presented for times close to maturity and times far away from it (i.e. times close to the beginning of the financial contract) are the starting points of our methodology in Chapter 4. As we discuss there, the new method is based on splitting the maturity time into two regions, one close-to-maturity and the other further from it, and using two different approximations in order to focus any computational effort in the estimation of the optimal exercise price close-to-maturity, since it changes slope rapidly and can be difficult to estimate, and a simpler approximation near the beginning of the contract.

Additionally, our new method takes advantage of existing pricing functions[8] to estimate the optimal exercise price in the second region of time-to-maturity and this is made feasible by the well-known property discussed in Geske and Johnson (1984), Kim (1990) and Basso et al. (2004), whereby, under the Black-and-Scholes model, the optimal exercise price does not depend on the current spot price. As a consequence, it is possible to employ the optimal exercise price of a shorter maturity option to build part of the optimal exercise price of an American option written on the same asset, with the same strike price but with a longer maturity. This is the basis of the methodology in Chapter 4.

In addition, the optimal exercise prices of call and put options with identical characteristics are linked via a put-call symmetry relation. Let us define $S_f^{(E)}(t,K,r,\delta,T)$ and $S_f^{(C,E)}(t,K,r,\delta,T)$ the optimal exercise prices of the put and call respectively, where t is the valuation time and the other inputs are as in (3.25), then:

$$S_f^{(E)}(t,K,r,\delta,T) = \frac{K^2}{S_f^{(C,E)}(t,K,\delta,r,T)}. \tag{3.31}$$

Most of the results above have been generalised to other dynamics. For dividend yield, risk-free rate and volatility given as a function of time and current underlying asset price, Jacka and Lynn (1992) and Detemple and Tian (2002) proved that the optimal exercise price of an American call option is unique (the exercise region is up-connected), non-increasing and right-continuous. They also proved that the optimal exercise price for this dynamics is not stochastic and is a function of only time-to-maturity, extending the results of Geske and

[8]The 'extension' method works in extending almost any method for pricing and hedging American options under the geometric Brownian motion dynamics. In this thesis, we will incorporate the methodologies reviewed in Section 3.4. In Chapter 4 and 6, we show numerically that our methodology improves on these existing methodologies in most cases.

Johnson (1984), Kim (1990) and Basso et al. (2004). Assuming a constant risk-free rate, they proved that the optimal exercise price is continuous. Bayraktar and Xing (2009) proved that the optimal exercise price for American put options under various jump-diffusion processes is continuously differentiable everywhere, except at maturity, and it is strictly increasing. Chiarella et al. (2014) derived in implicit form the limiting value of the optimal exercise price for jump-diffusions for $t \to T$. It has a similar structure to the limiting value (3.30), and it is expressed as an integral equation which involves the probability density function of the jumps. Lamberton and Mikou (2008) proved continuity of the optimal exercise price for general Lévy processes.

3.3 Review of the existing methodologies

The American option pricing problem has been studied in great depth by researchers from different disciplines and backgrounds. As shown above, the main challenge consists in the fact that the American optionality requires the selection of the optimal exercise price together with the valuation of the contingent claim, and this selection makes it a stochastic optimisation problem. The solution of the problem is useful not only from an academic point of view, but also because it is particularly relevant for practitioners who operate in the market. Neglecting the right exercise-decision process in option markets may cause severe losses to option holders. Among others, Barraclough and Whaley (2012) analysed the exercise of American put options on stocks over the period January 1996 through to September 2008, and found that over 3.96 million contracts remained unexercised although it was optimal to do so, with a lost profit to the contract holders of nearly \$1.9 billion. Similarly, Pool et al. (2008) estimated a lost profit to call option holders of over \$491 million during the period January 1996 through to April 2006. Both pieces of research justify these losses partially with the existence of exercise/trading costs, but both state that the bulk of the losses is not rationally justifiable. From a theoretical point of view, the same problem was investigated by Ibáñez and Paraskevopoulos (2011) and Chockalingam and Feng (2015), who derived upper bounds for the profit lost in the case of a suboptimal exercise strategy.

It is only in the last 10 years that exact formulae for pricing these financial instruments have come out: using homotopy-analysis, Zhu (2006) provided a Taylor's expansion to price American options under the standard geometric Brownian motion dynamics for a non-paying dividend asset and Zhao and Wong (2012) extended Zhu's result to a general diffusion process with local volatility and deterministic dividend yield, by providing a Maclaurin series pricing formula. However, the two formulae are expressed as infinite series so they are sometimes not recognized as exact closed-form pricing formulae and they do not have a clear advantage over pure numerical methods from a computational point of view. On the other hand, several approximating methods have been proposed so far and below we review the main contributions.

Reviews of existing solution approaches are in Broadie and Detemple (1996, 2004), Glasserman (2003), Barone-Adesi (2005) and Pressacco et al. (2008).

3.3.1 Quasi-analytic methods

The methods reviewed in this section consist of analytic formulae that require at most a reasonably small number of numerical solutions of (integral) equations. One of the first methods in this category for pricing an American call option on a stock paying a single (known) dividend is the Roll–Geske–Whaley formula (Roll (1977), Geske (1979, 1981) and Whaley (1981)). Whaley (1982), by using weekly closing prices for the call options traded on the Chicago Board of Options Exchange (CBOE) and written on all 91 dividend-paying stocks over the period 1975-1978, show the superiority of this formula over the Black-and-Scholes formula modified by Black (1975) to include the early exercise feature.[9] Cassimon et al. (2007) generalised the Roll–Geske–Whaley formula to the case of multiple discrete (known) dividends.

Assuming a constant dividend yield, Geske and Johnson (1984) used a portfolio of compound European options to replicate the early exercise feature of American options. However, their method includes high-dimension multivariate normal probabilities, whose calculation quickly becomes cumbersome. The method of Geske and Johnson (1984) was then studied and extended in many directions: Shastri and Tandon (1986) adapted it to options on futures but found significant deviations from market prices; Bunch and Johnson (1992) optimally located the exercise points by maximizing the option value, i.e. they searched for the best lower bound for the option pricing, and showed that most of the time only two – and in a few cases for deep-in-the-money options only three – early-exercise dates including maturity are required; Ho et al. (1997) included a stochastic interest rate economy; Gukhal (2004) extended the method to the log-normal jump-diffusion process in Merton (1976); and Prekopa and Szantai (2010) provided an exponential smoother Richardson extrapolation, which was found to provide biased prices by Joshi and Staunton (2012).

A remarkable technique is the quadratic approximation in Barone-Adesi and Whaley (1987), which gives in closed form an approximated solution of the Black-and-Scholes partial differential equation with dividend by generalising the method in MacMillan (1986).

[9] Black (1975) priced American call options on a underlying asset paying one (known) dividend as the higher of the prices of a European call option where the underlying spot price is net the dividend, and a European call option where the time to ex-dividend is substituted for the time-to-maturity.

Carr and Faguet (1996) proved that this method is the first-order expansion of the option price function. The quadratic approximation, which is extremely fast and accurate for very short and very long maturities, has been refined by Ju and Zhong (1999) including a second-order extension that improves accuracy for middle-term maturities. Subsequently, Li (2010a) further refined this second-order expansion by improving the accuracy of the estimation of the optimal exercise price, although the complexity of the method increases significantly without real improvements on the pricing performances. Kou and Wang (2004) extended Barone-Adesi and Whaley (1987) to the double exponential jump-diffusion model (see SDE (3.22)). However, the approximations of Barone-Adesi and Whaley (1987), Ju and Zhong (1999), Kou and Wang (2004) and Li (2010a) have the limitation that the error cannot be controlled.

An important step in the American option pricing literature is the result of Kim (1990), who derived an implicit-form integral equation for the optimal exercise price. Hence, the pricing of American options can be reduced to identifying the optimal exercise price efficiently. Additionally, they provided an American option pricing formulae that is given as the European option price plus a correction term, which is named the early exercise premium. Jacka (1991) and Carr et al. (1992) independently derived the same price decomposition. Several subsequent papers focused on improving the computational performance of the integral method. Among them, Sullivan (2000) employed Chebyshev polynomials and Gaussian quadrature; Kallast and Kivinukk (2003) used the trapezoidal rule and the Newton-Raphson method; Ibáñez (2003) modified the method to guarantee that the prices monotonically get closer to the true price when the number of steps increases; and Kim et al. (2013), based on an idea from Little et al. (2000), transformed the integral equation into a numerical functional form with respect to the optimal exercise boundary, and subsequently constructed an iterative method to calculate the boundary as a fixed point of the functional. Additionally, Kim and Yu (1996) derived the integral-equation under a local volatility process, and Detemple and Tian (2002) further generalised it to diffusions with stochastic volatility and interest rate.

Moreover, Johnson (1983) proposed an analytic formula for non-dividend-paying stock options as a weighted average of a lower and an upper bound for the option prices. The disadvantage of this method is that the weights in the sum are found by regressing on the option prices in Parkinson (1977) and consequently they are not reliable when the conditions change. Blomeyer (1986) improved on it by including one known dividend. Li (2010b), further improved on Johnson (1983) by deriving closed-form approximations of the weight parameter. Other approaches are based on approximation of the optimal exercise price: among them, Bjerksund and Stensland (1993) used a flat approximation of the optimal exercise price; Bjerksund and Stensland (2002) generalised on Bjerksund and Stensland (1993) using a two-step function; Omberg (1987) approximated the optimal exercise price with an exponential function; Ju (1998) proposed a piece-wise exponential function for the optimal exercise price; and, Gutierrez (2013) generalised the approach to any function by approximating the first-passage density of a Brownian motion to a curved barrier. Carr (1998) priced American options from a series of random maturity options but this method appears to be quite slow, as pointed out by Sullivan (2000). Chung and Shih (2009) proposed a static replicating portfolio of European options with different strikes and maturities in order to price American options under the geometric Brownian motion dynamics and the constant elasticity of variance (CEV) model. Ruas et al. (2013) extended Chung and Shih (2009) to the jump-to-default extended CEV model in Carr and Linetsky (2006).

Finally, a vast branch of the literature focuses on the calculation of lower (LB) and upper (UB) bounds of the option price. Broadie and Detemple (1996) presented both a LB and a UB: the latter is based on a lower bound of the optimal exercise price while the LB relies on the approximation of the optimal exercise price via a flat function, in a similar way to Bjerksund and Stensland (1993). Chung et al. (2010) tightened the bounds in Broadie and Detemple (1996) by assuming an exponential approximation of the optimal exercise price. Chen and Yeh (2002) and Chung and Chang (2007) proposed analytical upper bounds for American options under stochastic interest rate, stochastic volatility and jumps. Laprise et al. (2006) calculated the UB and LB by a series of replicating portfolios of European-style options and two interpolation methods: the secant-line interpolation is used to calculate the UB while the LB is calculated by the tangent-line interpolation.

3.3.2 Numerical methods

Although the quasi-analytic methods are preferable because they are fast and robust and allow precise estimation of the Greek parameters, in many cases, the lack of flexibility to adapt to different payoffs and/or underlying asset price dynamics makes these methods less attractive. On the other hand, a vast family of methods that work extremely well for exotic options and also for multidimensional cases, is numerical methods. In the following, we review the main categories of numerical methods.

Lattice methods

Lattice methods proceed by a discretisation of the state space and the time space. Among them, the finite difference method is a flexible technique that can price different derivatives with the underlying asset prices following different stochastic processes (Tavella and

Randall, 2000). The literature on this method is vast since it has been employed over many different research areas such as physics and engineering. The most commonly used methods in the financial industry are the explicit method by Schwartz (1977), the first-order implicit method by Brennan and Schwartz (1978) and the Crank-Nicolson second-order implicit method Courtadon (1982).

Another branch of lattice methods is the subfamily of binomial tree methods. This was introduced by Sharpe (1978) and further developed by Cox et al. (1979) to price options under the geometric Brownian motion. The method was further developed by many other researchers: Trigeorgis (1991) proposed the additive binomial tree, i.e. a binomial tree for the log-transformation of the stock price, for additional stability and efficiency; Tian (1993) imposed a condition to match up to the third moment in the tree, reaching higher convergence. Additionally, Breen (1991) used Richardson's extrapolation of prices to speed up convergence of the binomial trees; Broadie and Detemple (1996) proposed to price the options at the nodes of the tree just before maturity with the Black-and-Scholes formulae for European options; and, Chen and Joshi (2012) proposed truncation techniques to accelerate the binomial trees. Further studies were devoted to extending the binomial tree to dynamics other than standard geometric Brownian motion: among them, Nelson and Ramaswamy (1990) studied the binomial tree for general diffusion; and Amin (1993) and Hilliard and Schwartz (2005) studied the binomial tree method under jump-diffusion processes. A performance comparison of binomial trees is provided in Leisen (1998), where it is shown that the methods of Cox et al. (1979) and Tian (1993) have similar performance.

Additionally, Heston and Zhou (2000) proved that given a tree with m successors for each node and a payoff function differentiable $2m$ times, the tree can match the first m moments of the underlying asset price process and converges at the rate $O\left(\Delta_t^{-\frac{m-1}{2}}\right)$ where Δ_t is the width of the time-step. This suggests that it is preferable to select higher orders for the trees. Heston and Zhou (2000) and Alford and Webber (2001) proposed a m-nomial tree and found that the heptanomial tree ($m = 7$) is the most competitive method when the trade-off between speed of convergence and error is considered. Moreover, various lattice methods have been proposed for multi-asset derivatives (see Section 2.2) although they are usually not very useful for high-dimension problems since these methods suffer from the so-called "curse of dimensionality".

Another branch of the literature on lattice methods is that constituted by the 'smile-consistent' methods. This class of methods takes the prices of traded European options as given and builds lattice models that are consistent with the volatility smiles, which are empirically observed for options written on several underlying asset types (Derman and Kani, 1998; Rubinstein, 1994, among others). Derman and Kani (1994) proposed a recombining binomial implied tree method that captures both the maturity and the strike dimension of the implied volatility curve. Barle and Cakici (1995) modified the binomial tree of Derman and Kani (1994) and worked with futures prices rather than spot prices, consequently ensuring positive transition probabilities in the tree. In contrast to the previous two methodologies, Rubinstein (1994) proposed a binomial implied tree that uses only backward induction (as the standard binomial tree in Cox et al. (1979)) and does not require the initial forward induction step that the previous two methods require. However, the method of Rubinstein includes information only from European options with maturity at T, which corresponds with the maturity of the tree. To overcome this limitation, Jackwerth (1997) generalised Rubinstein (1994) by including in the tree the prices of European options with shorter maturities. The novelty of Jackwerth's method is that it does not require that paths ending in the same node of the tree to have equal probabilities. Derman et al. (1996) proposed a trinomial implied tree method that works in a similar way to the method in Derman and Kani (1994) and solves the problem of negative transition probabilities encountered by the latter. For an extensive literature review on 'smile-consistent' methods see Skiadopoulos (2001).

Several empirical studies have been carried out to evaluate the pricing performance of these 'smile-consistent' methods. Brandt and Wu (2002) compared the implied tree of Derman and Kani (1994) with the standard binomial tree of Cox et al. (1979) (calculated from at-the-money implied volatility) and the ad-hoc binomial tree method.[10] The study was conducted employing both European and American options of the FTSE 100 index traded from October 1995 to September 1997 on the London International Financial Futures and Options Exchange (LIFFE). It was found that the implied tree method outperforms the standard binomial tree but the performance is comparable with that of the ad-hoc method. These results are similar to that in Dumas et al. (1998) where the test was carried for out-of-sample options. Similarly, Lim and Zhi (2002) used daily prices of FTSE 100 index options from January to November 1999 and tested the generalised binomial tree of Jackwerth (1997), Derman and Kani (1994) and the standard binomial tree of Cox et al. (1979). They found that the implied tree of Derman and Kani (1994) is highly sensitive to the interpolation techniques employed and performs better for call than for put options. Additionally the implied tree of Derman and Kani (1994) outperforms both the methods of Jackwerth (1997) and Cox et al. (1979) for short maturity options while the generalised method produces better results for at-the-money options. Linaras and Skiadopoulos (2005) compared the implied trees of Derman and Kani (1994) and Barle and Cakici (1995) under several specifications of the interpolation function,[11] the standard tree of Cox et al. (1979) and the ad-hoc method.

[10] The ad-hoc method is a standard binomial tree (Cox et al., 1979) built from the implied volatility surface. This method is ad-hoc because it is internally inconsistent.

[11] For both the implied tree methods, either the linear or the cubic spline interpolation method was employed.

They employed daily prices of American options written on the S&P 100$^{\text{TM}}$ index traded from 15 August 2001 to 21 July 2003. They found that linear interpolation is preferable to the cubic spline interpolation. Additionally, the ad-hoc method performs better than the others, the standard tree method of Cox et al. has the worst performance, and method of Barle and Cakici (1995) is preferable to the tree of Derman and Kani (1994) since it has a smaller pricing error. In Chapter 6,[12] we carry out a similar analysis to that in Linaras and Skiadopoulos (2005) over the pricing methods we propose in Chapters 4 and 5.

Monte Carlo methods

For a long time, it was believed that Monte Carlo simulation methods were suitable only for pricing European style derivatives. It was Tilley (1993) who applied Monte Carlo methods to American options, although his methodology is ad-hoc for plain American options and difficult to adapt to multi-assets contracts.

The Monte Carlo methods consider the American option pricing problem from alternative paradigms. One of these is the approximation of the optimal exercise price. Garcia (2003) and Ibáñez and Zapatero (2004) presented a method that involves two steps: a first stage to find an approximation of the optimal exercise price and a valuation stage in which the optimal exercise price is used in the actual pricing. The two methods differ in the first step: Garcia performs an optimisation step where he calibrates a parametric optimal exercise price, while Ibáñez and Zapatero use a backward iterative procedure and a trial-and-error methodology that employs Monte Carlo simulations to determine the continuation value at each time-step. Both use Monte Carlo simulations for the second step.

Another approach is the random tree method in Broadie and Glasserman (1997). This involves dynamic backward programming as in the standard lattice method but the successors of each node are simulated by a Monte Carlo method. A biased high and a biased low approximation are proposed. The complexity of this method is exponential in the number of exercise dates and consequently it cannot be implemented in the case of many exercise opportunities. Similarly, Broadie and Glasserman (2004) introduced the stochastic mesh method. The main difference between this and the random tree method is that for the pricing of the option at each node, the stochastic mesh method considers all of the nodes at the following time-step, not only the successor nodes as the random tree method does. Consequently, the number of nodes per time-step is kept constant rather than growing exponentially. Also for the stochastic mesh method, Broadie and Glasserman proposed both

[12] A review of other empirical comparisons on the pricing performance of other methods/underlying asset models is provided in Section 6.1.

biased high and low methodologies and proved that both approximations are asymptotically unbiased.

On the other hand, Barraquand and Martineau (1995) proposed a state-space partitioning method called stratified state aggregation along the payoff (SSAP). While the random tree method is based on sampled states, in SSAP the states are defined in advance and the transition probabilities from one state partition to the other are estimated by a Monte Carlo simulation method. This method is very competitive for a high number of exercise dates but its complexity is exponential in the number of assets/stochastic factors. Another paradigm is the dual formulation as discussed independently in Rogers (2002) and Haugh and Kogan (2004). This is based on the duality formulations of the pricing problem and it generates upper bounds for American option prices through a minimisation problem. Broadie and Cao (2008) proposed techniques to improve the lower and upper bounds for American options: they use the concept of "distance to the exercise boundary" to discern between paths that are more likely to be exercised and that consequently require further analysis than the others. Their methodology positively impacts on the computational effort required by the existing methods and also reduces the pricing variances.

Finally, a well-known class of Monte Carlo methods includes regression-based methods. These methods, introduced by Carriere (1996), Tsitsiklis and Van Roy (2001) and Longstaff and Schwartz (2001), use regression methodology to estimate the continuation value of an American-style option from simulated paths. This approach is equivalent to the stochastic mesh method where each regression corresponds to the selection of the weight for that time-step. Both the regression based methods and the stochastic mesh method are the most powerful techniques for problems with many exercise dates and many stochastic factors. In Chapter 5, we present our new weighted least squares Monte Carlo method, which aims to correct the least squares Monte Carlo method of Longstaff and Schwartz for heteroscedasticity in the regression errors, a condition that makes their regressions not BLUE, i.e. their estimators are not the best linear unbiased estimator. In that chapter, we also provide an expanded review of this family of methods (Section 5.1.1).

We point out that Alcock and Carmichael (2008), Alcock and Auerswald (2010) and Yu and Xie (2015) also employed a weighted least squares Monte Carlo procedure to price American style options by using market data. Their methods are different from the one that we present in Chapter 5 since the weights they use are a change of measure from the real-world measure (inferred from market prices) to the risk-neutral measure needed for pricing purposes. In the light of what we will illustrate in Chapter 5, our weights are a correction to the regressions in the algorithm by Longstaff and Schwartz (2001) to guarantee the BLUE condition and it may be useful to consider them in the weighted least squares methods of Alcock and Carmichael, Alcock and Auerswald and Yu and Xie.

Other methods

In this section we review the numerical methods that do not fit in the previous two categories. Carr and Faguet (1996) employed the analytical method of horizontal lines (also known as Rothe's method), which is similar to the finite difference but only considers a time discretisation. Consequently, this method is less computational intensive. Hon and Mao (1997), Hon (2002) and Rad et al. (2015) used radial basis functions to price American options under the geometric Brownian motion, and Chan and Hubbert (2014) applied the radial basis function interpolation scheme to jump-diffusions. The radial basis function method consists of solving the partial differential equation governing the evolution of the option by polynomial expansions of the option pricing. Kim et al. (2014) modified the radial basis function method and have a fast and robust estimation of high-order Greeks. Very recently, Muthuraman (2008), Chockalingam and Muthuraman (2010, 2015) employed the approximate moving boundaries method which iteratively finds an approximation of the optimal exercise price. This method converts the problem in (3.3) where the optimal exercise price has to be found together with the option price in a sequence of problems with a given approximation of the optimal exercise price. Medvedev and Scaillet (2010) employed the moneyness for standard deviations, $\theta = \frac{\ln(K/S)}{\sigma\sqrt{\tau}}$, and exercised the American option as soon as θ hit a flat barrier. The method is based on the solution of the partial differential equation driving the option pricing for the given exercise barrier, which is achieved by an asymptotic expansion of the option price near maturity. Their method has a comparable performance with the method in Bunch and Johnson (2000) and Broadie and Detemple (1996) although it is slower in the computations. The main advantage of their method is that it can be easily extended to stochastic volatility and stochastic interest rate regimes.

A very powerful solution approach is the fast Fourier transform (FFT) introduced by Carr and Madan (1999). Its main advantage is that the FFT solution is given in terms of a general initial underlying asset price and payoff function, and consequently can handle a broad class of American-style derivatives. Chiarella et al. (2014) priced American options under the stochastic volatility model of Heston (1993) with jumps in the returns by using Fourier and Laplace transforms.

Fang and Oosterlee (2008) developed the Fourier cosine expansion approach (COS) for Bermudan options. This method works iteratively and, at any time-step, employs the cosine series expansions of the option pricing function at the next time-step and the underlying price density function. Chiarella et al. (2014) extended COS for American options under the Heston stochastic volatility model.

3.3.3 Other applications of option-pricing theory

Options are a 'special' class of derivatives since many other financial products have embedded some degree of optionality. Consequently, option-theory and more specifically some of the methods reviewed in the sections above are useful, not only to price and hedge single options but also more complex derivatives that can be decomposed in a portfolio of vanilla options. Merton (1998) provided a short review of applications of the option-pricing theory. Among others, he enumerated the valuation of employee stock options, loan prepayments, deposit insurances, student loan guarantees, patents and government policies. A vast amount of the literature studies the valuation of these composite contracts by adapting some of the methods above.

For example, the borrowers in a mortgage-backed security (i.e. a claim to the payments generated by a pool of mortgages) have the optionality to prepay their loads. This optionality substantially complicates the valuation of the mortgage-backed security since the prepayments may be triggered by many factors and, consequently, the borrowers may return the notional at any time before the expiration. The approaches currently employed in the literature evaluate these contracts by assuming that the borrowers aim to minimize the present value of their overall cash flows related to the mortgage (Dunn and Spatt, 2005; Longstaff, 2005; Stanton and Wallace, 1998). In particular, Longstaff (2005) evaluated mortgage-backed securities using a modified version of the least squares Monte Carlo method.

Also life insurance contracts usually offer policyholders several types of optionalities (Smith, 1982; Walden, 1985). These options allow the policyholders to vary the insurance contract at any time before expiration (American-style optionality) or only at the contract expiration (European-style optionality). A common optionality embedded in life insurance contract is the "surrender option". This option has American exercise-style and gives the policyholder the right to terminate the contract and receive the "surrender value", which is a predetermined amount of cash. This option can be exercised only upon the survival of the policyholder and, consequently, it is a knock-out American barrier options, whose evaluation requires the analysis of financial factors as well as demographic risk factors. Andreatta and Corradin (2003), Baione et al. (2006) and Bacinello et al. (2010) employed the least squares Monte Carlo method to determine the contract premium.

Additionally, the "abandonment option", i.e. the option held by investors to abandon the business for the assets' exit value, is an American real option with both a stochastic underlying asset price (the value of future cash flows) and a stochastic strike price (the exit value) (Berger et al., 1996). Other implicit embedded options are in the callable U.S. Treasury bond, which is a standard coupon bond and a (short) call option on the coupon that can be exercised by the U.S. Treasury during the last 5 years before expiration (Longstaff, 1993). A swaption gives the holder the right to enter into a swap contract, which is a contract in which two counterparties agree to exchange cash flows (Schrager and Pelsser, 2006). Finally, Ingersoll (2002); Murphy (2000) prices incentive stock options. These options differ from standard equity options since the holders (employees, managers and executives) have portfolios that are not differentiated but are principally made up of their own company. Consequently, these options have less value than if they were part of a diversified portfolio.

3.4 Details on selected quasi-analytic methods

In Chapter 4, we describe the 'extension' method, i.e. a methodology that aims to improve the pricing and hedging performance of almost any quasi-analytic method in the literature. In this section, we review the six quasi-analytic methods used to test our new methodology: Geske and Johnson (1984), Barone-Adesi and Whaley (1987), Ju and Zhong (1999), Kim (1990), Li (2010b) and Chung and Shih (2009). Given the put-call symmetry in (3.25), we consider only the put pricing formulae. In Chapter 4, we differentiate between a long-maturity option and a short-maturity option. With regard to the former we mean the option we are about to price and with regard to the latter we mean an option that has shorter maturity, which is used to build part of the price formula for the former (more details in Section 4.2.1). Here, as in the following chapter, $t_x \in [t_0, T]$ is the intermediary date, where we split the maturity of the long-maturity option and it is consequently the time where the short option is priced; $S_{f_x}(\cdot)$ is the optimal exercise price of the short-maturity starting life at t_x and with maturity at T; and, $\tilde{P}_{t_x}(S_{t_x}, T, K)$ is the time-t_x price of an American option contingent on an underlying asset with the dynamics specified in equation (3.16), with time-to-maturity $\tau = T - t_x$, strike price K and underlying price S_{t_x}.

Before describing each method, we enumerate some common factors underpinning the methodologies below. First, the optimal exercise price of the option is calculated by solving equation (3.9), which corresponds to finding $S_{f_x}(t)$ by solving:

$$K - S_{f_x}(t) = \tilde{P}_t(S_{f_x}(t), T, K), \forall t \in [t_x, T]. \tag{3.32}$$

In particular, for Barone-Adesi and Whaley (1987), Ju and Zhong (1999) and Li (2010b), one needs only $S_{f_x}(t_x)$, while for Geske and Johnson (1984), Kim (1990) and Chung and Shih (2009) the calculations of $S_{f_x}(\cdot)$ for intermediate times are also required. In Chapter 4, we apply the compound-option (Geske and Johnson, 1984) methodology, as well as the integral method (Kim, 1990) and static replication portfolio methodology (Chung and Shih, 2009), with three as well as two steps. However, in the interest of brevity, we only present the formulae for two time-steps in this section.

Moreover, the price of a European put option is given by the well-known Black-and-Scholes formula for dividend-paying assets (Black and Scholes, 1973; Merton, 1973):

$$p_{t_x}(S_{t_x},T,K) \quad = \quad Ke^{-r\tau}N(-d_2(S_{t_x},K,\tau)) - S_{t_x}e^{-\delta\tau}N(-d_1(S_{t_x},K,\tau)) \tag{3.33}$$

where for the generic time-to-maturity τ_m:

$$d_1(S,q,\tau_m) = \frac{\log\left(\frac{S}{q}\right) + \left(r - \delta + \frac{\sigma^2}{2}\right)\tau_m}{\sigma\sqrt{\tau_m}}, \quad d_2(S,q,\tau_m) = d_1(S,q,\tau_m) - \sigma\sqrt{\tau_m}. \tag{3.34}$$

The following shorter notations for d_1 will also be used: $d_1(S) = d_1(S,K,\tau)$ or $d_1(S,q) = d_1(S,q,\tau)$, and equivalently for d_2. Furthermore, $N(\cdot)$ is the cumulative distribution function (cdf) of a standard normal variable and $N_2(x,y,\rho)$ is the bivariate standard normal cdf with correlation ρ. Finally, we denote by $\Delta_{t_x,T} = \frac{T-t_x}{2}$ the time-step size and report only the formulae for $S_{t_x} > S_{f_x}(t_x)$, because for $S_{t_x} \leq S_{f_x}(t_x)$, the option price is simply its immediate exercise $\tilde{P}_{t_x}(S_{t_x},T,K) = K - S_{t_x}$.

3.4.1 Compound-option method (GJ)

This method was outlined by Geske and Johnson (1984) and it is called compound-option method because the price of an American option is given as a portfolio of options on options. The method discretises the time-to-maturity. At each time-step, the option is exercised if it has not been exercised at the previous time-steps, and if the current underlying asset price is below the optimal exercise price. If, at any time-step, the option is not exercised, then it is equivalent to have another option for the next time-steps. For the first step (i.e. the one closest to the beginning of the contract), the probability that the option has been already exercised is zero and, consequently, the expected payoff at that time-step in time-t_x dollars corresponds to the price of an European option with strike price equal to the optimal exercise price. At the second time-step, the option can be exercised only if it was not exercised at the first time-step and if the current spot price is below the optimal exercise price. Consequently, the probability to exercise the option at the second time-step is given in terms of the bivariate normal cumulative distribution function. Equivalently, for the third time-step, the option is exercised if the current spot price is below the optimal exercise price and if the underlying asset prices at the previous two time-steps were above the optimal exercise prices. In this case, the trivariate normal cumulative distribution function is needed to calculate the probability of exercise. The main drawback of this method is that, if m is the number of time-steps considered, the pricing formula includes m-variate cumulative distribution functions, whose calculations are computationally intensive.

In what follows, we report the pricing formula of the compound-option method with two time-steps for dividend-paying assets:

$$\tilde{P}_{t_x}(S_{t_x},T,K) = KU_2(S_{t_x}) - S_{t_x}W_2(S_{t_x}), \text{ for } S_{t_x} > S_{f_x}(t_x) \tag{3.35}$$

where:

$$\begin{aligned} U_2(S) &= e^{-r\Delta_{t_x,T}}N(-d_2(S,q_1,\Delta_{t_x,T})) \\ &\quad + e^{-2r\Delta_{t_x,T}}N_2\left(d_2(S,q_1,\Delta_{t_x,T}),-d_2(S,K,2\Delta_{t_x,T}),-\frac{1}{\sqrt{2}}\right), \\ W_2(S) &= e^{-\delta\Delta_{t_x,T}}N(-d_1(S,q_1,\Delta_{t_x,T})) \\ &\quad + e^{-2\delta\Delta_{t_x,T}}N_2\left(d_1(S,q_1,\Delta_{t_x,T}),-d_1(S,K,2\Delta_{t_x,T}),\frac{1}{\sqrt{2}}\right), \end{aligned}$$

$q_1 = S_{f_x}(t_x+\Delta_{t_x,T})$ solves $K - q_1 = p_{t_x+\Delta_{t_x,T}}(q_1,T,K)$ and $S_{f_x}(t_x)$ solve (3.32).

3.4.2 Quadratic methods (BAW and JZ)

The quadratic method in Barone-Adesi and Whaley (1987) corresponds to a first-order approximation of the option pricing formula (Carr and Faguet, 1996). The method starts from the partial differential equation for the price function of American options under the geometric Brownian motion. When one writes it for the (short-maturity) American put option at time t_x, the formula reads:[13]

$$\frac{1}{2}\sigma^2 S_{t_x}^2 \frac{\partial^2 \tilde{P}_{t_x}}{\partial S_{t_x}^2} - r\tilde{P}_{t_x} + (r-\delta)S_{t_x}\frac{\partial \tilde{P}_{t_x}}{\partial S_{t_x}} + \frac{\partial \tilde{P}_{t_x}}{\partial t_x} = 0. \tag{3.36}$$

Since the differential operator is linear, the partial differential equation (3.36) also holds true for the early exercise premium:

$$\varepsilon_{t_x} = \tilde{P}_{t_x} - p_{t_x}. \tag{3.37}$$

[13] In the partial differential equations below, we drop the input parameters of the price functions to avoid cumbersomeness.

The technique used by Barone-Adesi and Whaley is to consider

$$\varepsilon_{t_x} = (1-e^{-r-\tau}) f_{BAW}(S_{t_x}, T, K),$$

and consequently, after some algebraic manipulations, (3.36) can be rewritten as:

$$S_{t_x}^2 \frac{\partial^2 f_{BAW}}{\partial S_{t_x}^2} + \frac{2(r-\delta)}{\sigma^2} S_{t_x} \frac{\partial f_{BAW}}{\partial S_{t_x}} - \frac{2r}{\sigma^2(1-e^{r\tau})} f_{BAW} - \frac{2re^{-r\tau}}{\sigma^2} \frac{\partial f_{BAW}}{\partial(1-e^{-r\tau})} = 0. \quad (3.38)$$

Barone-Adesi and Whaley (1987) assume

$$\frac{2re^{-r\tau}}{\sigma^2} \frac{\partial f_{BAW}}{\partial(1-e^{-r\tau})} \approx 0, \quad (3.39)$$

which is a good approximation for short and long time-to-maturities but works less efficaciously for medium-term maturities. Simplifying (3.38) by using the approximation above, it becomes a second-order differential equation with two-linear independent solutions of the form $AS_{t_x}^B$. Consequently, solving (3.38) for $f_{BAW} = A_1 S_{t_x}^{\beta_1} + A_1^{(2)} S_{t_x}^{\beta_1^{(2)}}$ and setting[14] $A_1^{(2)} = 0$, one has the Barone-Adesi-Whaley pricing formula:

$$\tilde{P}_{t_x}(S_{t_x}, T, K) = p_{t_x}(S_{t_x}, T, K) + A_1 \left(\frac{S_{t_x}}{S_{f_x}(t_x)} \right)^{\beta_1} \quad \text{for } S_{t_x} > S_{f_x}(t_x) \quad (3.40)$$

where $A_1 = -\frac{S_{f_x}(t_x)}{\beta_1} \left[1 - e^{-\delta\tau} N\left(-d_1(S_{f_x}(t_x))\right) \right]$, $\beta_1 = \frac{1}{2} - \frac{r-\delta}{\sigma^2} - \sqrt{\left(\frac{1}{2} - \frac{r-\delta}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2 h}}$, $h = 1 - e^{-r\tau}$ and $S_{f_x}(t_x)$ solves (3.32).

Ju and Zhong (1999) improved the approximation in Barone-Adesi and Whaley (1987) by proposing a second-order expansion. They considered

$$\varepsilon_{t_x} = (1-e^{-r-\tau}) \left(f_{BAW}(S_{t_x}, T, K) + f_{JZ}(S_{t_x}, T, K) \right),$$

[14] This is a consequence of the fact that $\beta_1^{(2)} > 0$, but for $S_{t_x} \to +\infty$ it has to be $\varepsilon_{t_x} \to 0$.

and (3.36), after some algebraic manipulations, becomes

$$\begin{aligned}
0 &= S_{t_x}^2 \frac{\partial^2 f_{BAW}}{\partial S_{t_x}^2} + \frac{2(r-\delta)}{\sigma^2} S_{t_x} \frac{\partial f_{BAW}}{\partial S_{t_x}} - \frac{2r}{\sigma^2 h} f_{BAW} \\
&+ S_{t_x}^2 \frac{\partial^2 f_{JZ}}{\partial S_{t_x}^2} + \frac{2(r-\delta)}{\sigma^2} S_{t_x} \frac{\partial f_{JZ}}{\partial S_{t_x}} - \frac{2r}{\sigma^2 h} f_{JZ} \\
&- \frac{(1-h)2r}{\sigma^2} \left(\frac{\partial f_{BAW}}{\partial h} + \frac{\partial f_{JZ}}{\partial h} \right).
\end{aligned} \tag{3.41}$$

As in Barone-Adesi and Whaley (1987), the sum of the first three terms in the right-hand side of (3.41) is fixed to zero. The remaining of (3.41) is solved by using the approximation $f_{JZ} = \frac{\chi}{1-\chi} f_{BAW}$. After some algebraic manipulations, the pricing formula proposed by Ju and Zhong (1999) is:

$$\tilde{P}_{t_x}(S_{t_x}, T, K) = p_{t_x}(S_{t_x}, T, K) + \frac{A_2(S_{f_x}(t_x)) \left(\frac{S_{t_x}}{S_{f_x}(t_x)} \right)^{\lambda_1}}{1-\chi} \text{ for } S_{t_x} > S_{f_x}(t_x) \tag{3.42}$$

where:

$$\begin{aligned}
A_2(S_{f_x}(t_x)) &= K - S_{f_x}(t_x) - p_{t_x}(S_{f_x}(t_x), T, K), \\
\alpha_1 &= \frac{2r}{\sigma^2},\ \alpha_2 = \frac{2(r-\delta)}{\sigma^2},\ \alpha_3 = \frac{(1-h)\alpha_1 \lambda_1'}{2(2\lambda_1 + \alpha_2 - 1)}, \\
\alpha_4 &= -\frac{(1-h)\alpha_1}{2\lambda_1 + \alpha_2 - 1} \left(\frac{1}{A_2} \frac{\partial p_{t_x}(S_{f_x}(t_x), T, K)}{\partial h} + \frac{1}{h} + \frac{\lambda_1'}{2\lambda_1 + \alpha_2 - 1} \right), \\
\chi &= \alpha_3 \left(\log \left(\frac{S_{t_x}}{S_{f_x}(t_x)} \right) \right)^2 + \alpha_4 \left(\log \left(\frac{S_{t_x}}{S_{f_x}(t_x)} \right) \right), \\
\lambda_1 &= \frac{-(\alpha_2 - 1) - \sqrt{(\alpha_2 - 1)^2 + \frac{4\alpha_1}{h}}}{2},\ \lambda_1' = \frac{\alpha_1}{h^2 \sqrt{(\alpha_2 - 1)^2 + \frac{4\alpha_1}{h}}}, \\
\frac{\partial p_{t_x}(S_{f_x}(t_x), T, K)}{\partial h} &= \frac{\sigma S_{f_x}(t_x) n\left(d_1(S_{f_x}(t_x))\right)}{2r\sqrt{\tau} e^{-(r-\delta)\tau}} + \frac{\delta S_{f_x}(t_x) N\left(-d_1(S_{f_x}(t_x))\right)}{r e^{-(r-\delta)\tau}} \\
&\quad - KN\left(-d_2(S_{f_x}(t_x))\right)
\end{aligned}$$

and $S_{f_x}(t_x)$ solves

$$1 - e^{-\delta\tau} N\left(-d_1(S_{f_x}(t_x))\right) + \frac{\lambda_1 A_2(S_{f_x}(t_x))}{S_{f_x}(t_x)} = 0.$$

The method in Li (2010a), a further modification of the quadratic method, is not considered here, since it has almost the same pricing performance of Ju and Zhong (1999) and is more computationally intensive. Its only advantage is in the estimation of the optimal exercise price, which is more precise.

3.4.3 Integral method (K)

Kim (1990) discretised the option life into a finite number of time-steps and works backwardly by recursively finding the values of the optimal exercise price functions. At the last time-step (i.e. the one before maturity), the American option is equal to the equivalent European option. Moving one time-step backward, the option value consists of two components: the expected immediate exercise at the last time-step before maturity, conditional on the underlying asset price being below the optimal exercise price; and the value of the American option starting life at the next time-step, conditional on the underlying asset price being above the optimal exercise price at that step. Moving backward and considering $\Delta_{t_x,T} \to 0$, the option price is given as:

$$\begin{aligned}\tilde{P}_{t_x}(S_{t_x},T,K) &= p_{t_x}(S_{t_x},T,K)+\\ &+\int_0^{\tau}\Big[rKe^{-r(\tau-t)}N\left(-d_2(S_{t_x},S_{f_x}(t_x+\tau-t),\tau-t)\right)+\\ &-\delta S_{t_x}e^{-\delta(\tau-t)}N\left(-d_1(S_{t_x},S_{f_x}(t_x+\tau-t)),\tau-t\right)\Big]\,dt,\ \text{for } S_{t_x}>S_{f_x}(t_x)\end{aligned} \tag{3.43}$$

where $S_{f_x}(t)$ solves (3.32). For implementation purposes, formula (3.43) is discretised: when 2 time-steps of equal length $\Delta_{t_x,T}$ are considered, formula (3.43) becomes:

$$\begin{aligned}\tilde{P}_{t_x}(S_{t_x},T,K) &= p_{t_x}(S_{t_x},T,K)+\\ &+\Delta_{t_x,T}\sum_{j=1}^{2}\Big[rKe^{-r\Delta_t^j}N\left(-d_2(S_{t_x},S_{f_x}(t_x+\Delta_t^j),\Delta_t^j)\right)\\ &\qquad -\delta S_{t_x}e^{-\delta\Delta_t^j}N\left(-d_1(S_{t_x},S_{f_x}(t_x+\Delta_t^j),\Delta_t^j)\right)\Big]\end{aligned} \tag{3.44}$$

where $\Delta_t^j = \Delta_{t_x,T}(2-j)$.

We consider the basic integral method only and not any of the improvements proposed in the literature (Kallast and Kivinukk, 2003; Kim et al., 2013; Sullivan, 2000), since only few iterations (early exercise dates) will be considered in the computational comparison in Section 4.3 and, consequently, the computational improvements do not have a big impact.

However, it is worth pointing out that our 'extension' method can also be connected to these improved methodologies.

3.4.4 Interpolation method (LI)

The interpolation method in Li (2010b) is based on two bounds of the price of an American put option:

$$p_{t_x}(S_{t_x}, T, K) \leq \tilde{P}_{t_x}(S_{t_x}, T, K) \leq p_{t_x}(S_{t_x}, T, Ke^{r\tau}) \tag{3.45}$$

where the lower bound is the correspondent European option, while the upper bound is an European option with strike price $Ke^{r\tau}$. This method is a generalization of the interpolation method of Johnson (1983). The pricing formula is given as a weighted sum between the lower and upper bounds as in Johnson (1983):

$$\tilde{P}_{t_x}(S_{t_x}, T, K) = A_3\, p_{t_x}(S_{t_x}, T, Ke^{r\tau}) + (1 - A_3)\, p_{t_x}(S_{t_x}, T, K), \text{ for } S_{t_x} > S_{f_x}(t_x). \tag{3.46}$$

The advantage of Li's method is that he provides an analytic formula for the weight (A_3) rather than a parameter based on empirical fitting as in Johnson (1983). The weight A_3 in formula (3.46) is

$$A_3 = A_4 \left(\frac{S_{t_x}}{S_{f_x}(t_x)} \right)^{q(r-\delta, \frac{r}{\Phi})} \tag{3.47}$$

where:

$$\begin{aligned} q\left(r - \delta, \frac{r}{\Phi}\right) &= \frac{1}{2} - \frac{r - \delta}{\sigma^2} - \frac{1}{2\sigma^2} \sqrt{(\sigma^2 - 2(r - \delta))^2 + 8\frac{r}{\Phi}\sigma^2}, \\ A_4 &= \frac{e^{\delta\tau} - N(-d_1(S_{f_x}(t_x), K))}{\left[N(-d_1(S_{f_x}(t_x), Ke^{r\tau})) - N(-d_1(S_{f_x}(t_x), K))\right] - \frac{q(b, \frac{r}{\Phi}) D(S_{t_x}, \tau, K)}{S_{f_x}(t_x) e^{-\delta\tau}}} \\ \Phi &= 1 - e^{-r\tau}, D(S_{t_x}, \tau, K) = p_{t_x}(S_{t_x}, T, Ke^{r\tau}) - p_{t_x}(S_{t_x}, T, K). \end{aligned}$$

This analytic formula is based on an approximating solution of the partial differential equation (3.36) using an approximation similar to that of Barone-Adesi and Whaley (1987). This method has the advantage of being employable also for other underlying asset price dynamics. In their research paper, Li also applied the interpolation method to price American options under the Heston's stochastic volatility model (Heston, 1993).

3.4.5 Static-replicating portfolio method (CS)

The idea of this method is to create a portfolio of standard European options in such a way that the value of this portfolio matches the payoff of the American option at maturity and along the optimal exercise price. The advantage of this method is twofold. First, it allows the construction of a static hedging strategy that is cheaper than the dynamic hedging strategy. Second, the calculation of the American option in future dates is simple since there is no need to find the static hedge portfolio again.

The method we study in Chapter 4 is that of Chung and Shih (2009). They construct a portfolio of European options of different strikes and different maturities. Previous studies proposed similar methods. Carr et al. (1998) used a portfolio of European options with same maturity and different strikes, while Derman et al. (1995) used a portfolio of European options with same strike and different maturities. The method of Chung and Shih (2009) is a combination of these two methods. The advantage of the Chung-Shih method over the other two is that, by using both different strikes and different maturities, the estimation of the optimal exercise price is more precise, since this function is time variant.

When one considers only two time-steps, the static-replicating portfolio method approximates an American option as a portfolio of three European options, where the number of contracts held (w_0 and w_1) is calculated recursively. The pricing formula is:

$$\tilde{P}_{t_x}(S_{t_x},T,K) = p_{t_x}(S_{t_x},T,K) + Z_2(S_{t_x},T,K) \text{ for } S_{t_x} > S_{f_x}(t_x) \tag{3.48}$$

where

$$Z_2(S_{t_x},T,K) = w_1 p_{t_x}(S_{t_x},T,S_{f_x}(t_x+\Delta_{t_x,T})) + w_0 p_{t_x}(S_{t_x},t_x+\Delta_{t_x,T},S_{f_x}(t_x)) \tag{3.49}$$

and w_1, $S_{f_x}(t_x+\Delta_{t_x,T})$ and w_0, $S_{f_x}(t_x)$ are backwardly determined as the solutions to the smooth-pasting condition and the value matching condition in (3.32).

3.5 Conclusions and outline of next chapters

In this chapter, we reviewed the main contributions with regard to the American pricing problem. In particular, after a summary of mathematical formulations of the pricing problem and a description of the market, we focused on a review of the main characteristics of the option price function and the optimal exercise price. Then, in Section 3.3, we carried out an extensive review of the literature of the solution approaches. In conducting this review, we found two gaps in the literature that lead to the two new methodologies that we are presenting in this thesis: the first is a quasi-analytic method (Chapter 4) and the second is a numerical method (Chapter 5).

The main problem we found among many quasi-analytic methods for pricing and hedging American options under the geometric Brownian motion is their poor performance over long-maturity options. The cause of this problem is, in many cases, the approximation of an American option as Bermudan option. Indeed, for any given number m of early exercise dates t_i, the time gaps between them, $t_i - t_{i-1}$, are larger the longer the time-to-maturity. As we will show in detail in Chapter 4, the majority of quasi-analytic method perform well for short maturities but, when time-to-maturity is longer than one year, their performance deteriorates and the pricing and hedging errors become so significant that these methods cannot be employed in many cases. However, options with maturity longer than one year are traded on a large scale in the financial markets: for example, the Chicago Board of Options Exchange (CBOE) lists LEAPS®, i.e. long-term options with maturity of up to 39 months. Consequently, improving on these methods is useful not only from an academic point of view, but particularly from a practitioners' perspective. In Chapter 4, we describe a new methodology that can potentially improve on any quasi-analytic method for American options under the geometric Brownian motion and, as an exemplification via a scenario-based comparison, we show that the improvements over the methods outlined in Section 3.4 are also sizeable for short maturities.

On the other hand, in Chapter 5, we present our new numerical algorithm, the weighted least squares Monte Carlo method that can be categorized under the regression-based methods. The aim of our new method is an upper-bias reduction to the algorithm outlined by Longstaff and Schwartz (2001). In the chapter, we demonstrate in three different ways (numerical, graphical and theoretical) that the regressions carried out in Longstaff and Schwartz' algorithm do not satisfy one of the assumptions of the ordinary least squares regression, namely the homoscedasticity of the errors, and, consequently, their estimators are not BLUE, i.e. the best linear unbiased estimators. Using a scenario-based comparison, we show that, under the four dynamics in Section 3.1.1, by substituting the ordinary least squares regressions with the weighted version, which is BLUE even for non-homoscedastic errors, we reduce the upper bias of the algorithm by Longstaff and Schwartz (2001). Additionally, we show that our method outperforms the inequality constrained least squares algorithm of Létourneau and Stentoft (2014), who aim to reduce the upper bias by imposing constraints on the ordinary least squares estimators.[15]

[15] The constraints they impose are monotonicity and convexity of the continuation value function as reviewed in Section 3.2.1. Additional details on this method can be found in Appendix 5.A.

Finally, in Chapter 6, we test our two new methods over LEAPS® options on the S&P 100™ stock index, traded on the CBOE from 15 February 2012 to 10 December 2014. For the new quasi-analytic method, the comparison shows that improvements are found for wide ranges of maturities, and, in particular, for long-maturity options. On the other hand, the new numerical method outperforms its competitors (the methods of Longstaff and Schwartz (2001) and Létourneau and Stentoft (2014)) over all of the maturity ranges and almost all of the error measures.

Chapter 4

American Options: An Improved Method for Pricing and Hedging

In this chapter, we introduce a method to increase the accuracy of almost any existing quasi-analytic methodology for pricing and hedging American options under the geometric Brownian motion dynamics. Based on an extensive numerical comparison, we show that the improvements are indeed sizeable over many methodologies, for all of the maturities considered and, particularly, for long-maturity options when the existing methodologies perform poorly. This chapter is structured as follows. Section 4.1 justifies the introduction of the new method. Section 4.2 describes the pricing and hedging methodology together with a convergence result. Section 4.3 numerically evaluates the performance of the new method by comparing it with many existing quasi-analytic methodologies and Section 4.4 shows how well the new method works together with the asymptotic expansions of the optimal exercise price near maturity. Finally, Section 4.5 concludes.

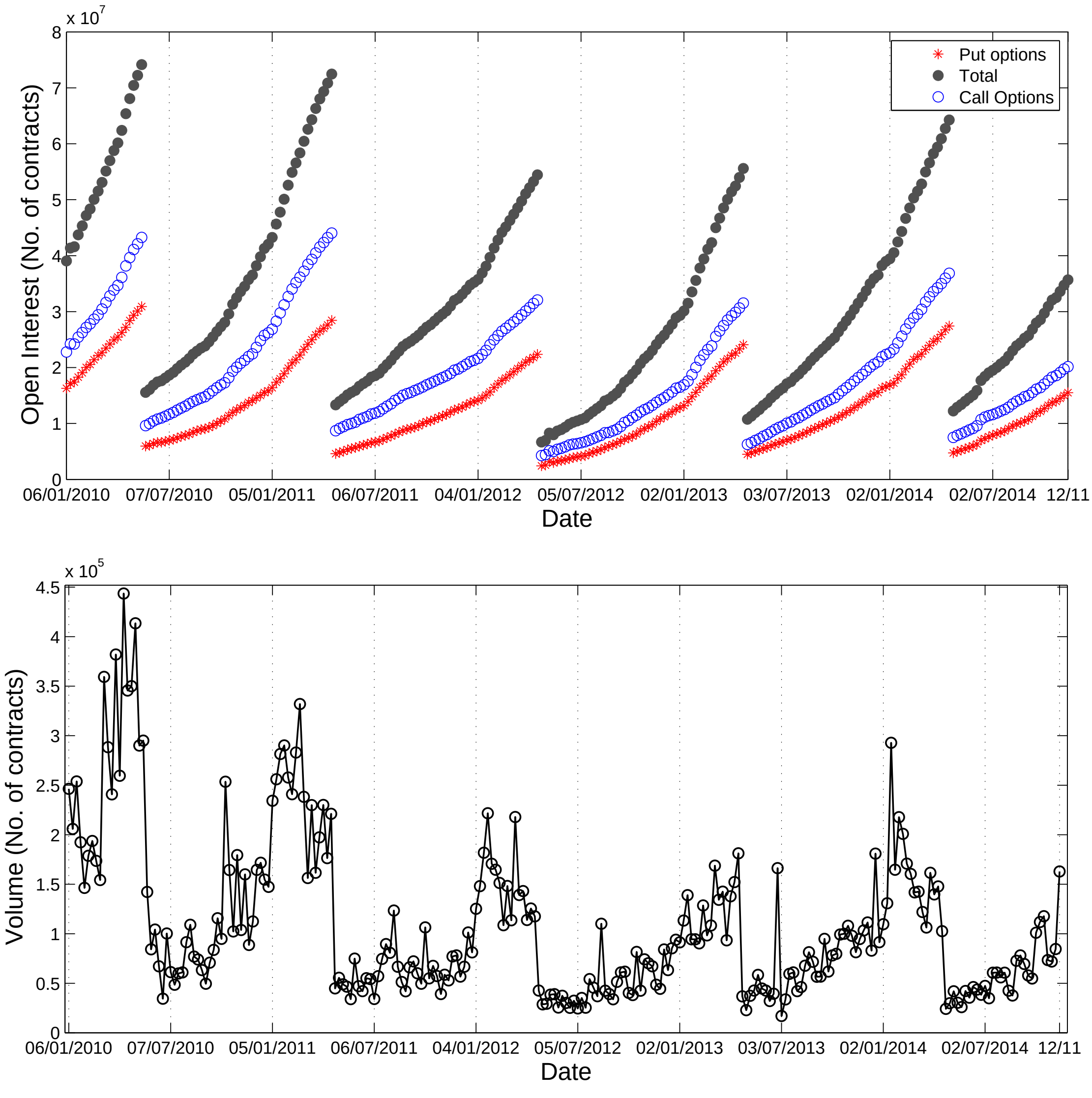
4.1 Introduction

In 1990, the Chicago Board of Options Exchange (CBOE) listed the Long-term Equity AnticiPation Securities™ (LEAPS®), which are American-style long-term option contracts with expiration of up to 39 months from the date of the initial listing, currently listed on about 500 equities[1] and on the Standard & Poor's 100 Index (S&P 100™ OEX®).

Many researchers have carried out performance comparisons over several pricing methods. Among them, Broadie and Detemple (1996) in their Figures 3 and 4 (p. 1227), Ait-

[1] Only the equity stock options with a maturity of 3 years have been considered. Considering shorter maturities (above two years) the number is much higher.

Figure 4.1 Equity LEAPS® market: Open interest and volume (Jan 2010 - Nov 2014)



Note: This figure illustrates daily data on the Wednesdays from 6 January 2010 to 12 November 2014, for a total of 254 observations for equity LEAPS® traded in the CBOE. The upper plot shows the open interest for call options, put options and the total; the other shows the daily volume of call and put options together. The values in the plots indicate the number of contracts.

Sahlia and Carr (1997) in their Tables[2] 2a-2e and 3a-3e (pp. 76-85), Ju and Zhong (1999) in their Exhibits 3 and 5, Li (2010b) in their Tables 3, 4 and 5 (pp. 91-93), Kallast and Kivinukk (2003) in their Figures 4. and 5. (pp. 373-374) and Kim et al. (2013) in their Tables 4 and 5 (p. 7) show that many of the pricing methods perform well for short-maturity options and that when they are employed for long-maturity options, their pricing performance is worse.

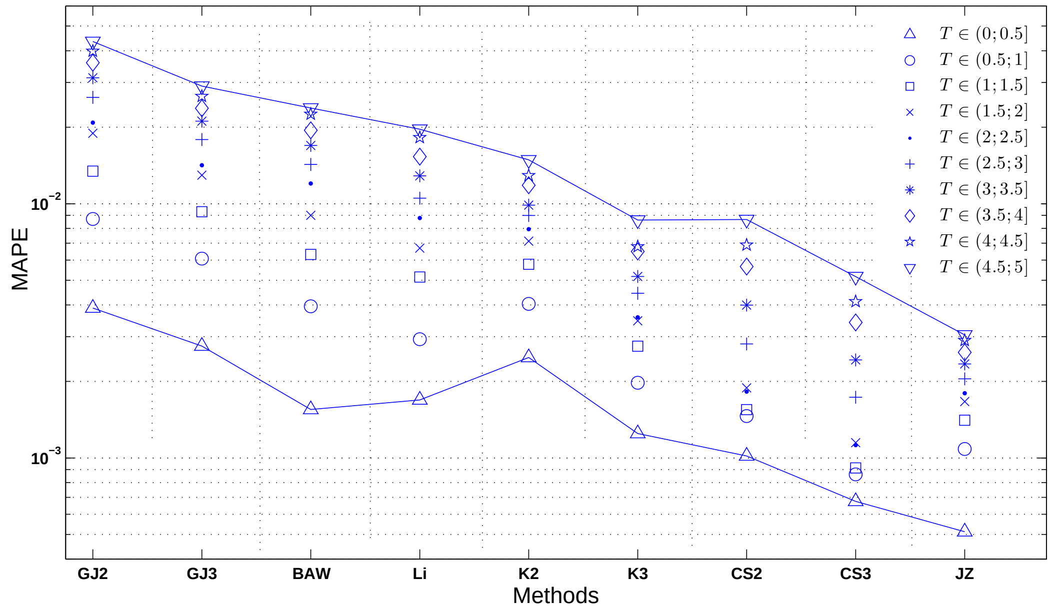
In the interest of facilitating a deeper understanding of the change in performance of the existing methodologies over options with different maturities, we consider a detailed numerical study based on 10,000 put option scenarios with maturities ranging from a few days to 5 years, divided into 10 sets (A,..., J) of 1,000 scenarios each. The purpose is to extend the results of former pricing and hedging performance analyses, which usually restrict their studies to options with maturities of up to 24 months. All of the other parameters are randomly drawn as in Broadie and Detemple (1996) and, for this chapter to be self-contained, they are described in Section 4.3. Figure 4.2 shows for six quasi-analytic methods the mean absolute percentage error (MAPE) with respect to the benchmark prices given by the binomial tree in Cox et al. (1979). The studied methods (details in Section 3.4) are the compound-option method in Geske and Johnson (1984), the quadratic method in Barone-Adesi and Whaley (1987), the interpolation method in Li (2010b), the integral method in Kim (1990), the static-replicating portfolio method in Chung and Shih (2009), and the improved quadratic method in Ju and Zhong (1999).

As in the previous research, we found that the existing methods perform less efficaciously for long-maturity options than for short-maturity ones. Figure 4.2 shows that the difference in MAPEs between options with a maturity shorter than 6 months and options with maturities longer than 4.5 years is more than one order of magnitude (for example, the Geske-Johnson method with 2 exercise dates produces a MAPE of 0.39% for maturities below 6 months and 4.35% for maturities above 4.5 years; the interpolation method produces respectively 0.17% and 1.96%). Moreover, the figure shows that the performance over the 10 sets (A, ..., J) monotonically decreases with time-to-maturity. Similar results are obtained for the hedging performance (see Table 4.4).

In most of the cases, the causes of this phenomenon are in the approximations employed for the solution of the American option pricing problem. Most of the existing pricing methods approximate American options via the equivalent Bermudan options; that is, a discrete set of exercise dates is considered rather than a continuum exercise-interval. These approximations lead to solutions of the pricing problem that converge to the benchmarks when the

[2] The results are for options on a non-dividend paying asset for short time-to-maturity and on a dividend paying asset for long time-to-maturity options.

Figure 4.2 Comparison of quasi-analytic methods for different maturities.



Note: This plot presents the mean absolute percentage error (MAPE) for six quasi-analytic methods. The methods studied are: (GJ) the compound-option method in Geske and Johnson (1984) with two and three exercise dates, (BAW) the quadratic method in Barone-Adesi and Whaley (1987), (LI) the interpolation method in Li (2010b), (K) the integral method in Kim (1990) with two and three exercise dates, (CS) the static-replicating portfolio method in Chung and Shih (2009) with two and three exercise dates, and (JZ) the improved quadratic method in Ju and Zhong (1999). Ten ranges of maturities (in years) are considered: $(0;0.5]$ (*A*), $(0.5;1]$ (*B*), $(1;1.5]$ (*C*), $(1.5;2]$ (*D*), $(2;2.5]$ (*E*), $(2.5;3]$ (*F*), $(3;3.5]$ (*G*), $(3.5;4]$ (*H*), $(4;4.5]$ (*I*), $(4.5;5]$ (*J*). The results are based on 10,000 put option scenarios (simulated as in Broadie and Detemple (1996) and as described in Section 4.3).

number of early exercise dates increases or, equivalently, when the size of the time-steps decreases. Consequently, for any given number of time-steps, the performance of these approximation methods is better for short-maturity options than for long-maturity options and more time-steps are required for the latter to have the same performance level as the former. However, an increase in the number of time-steps rapidly makes these methods inefficient and not applicable in real life situations where traders find themselves pricing many hundreds of options simultaneously and, consequently, they require higher performance methods.

In this chapter, we introduce a new pricing and hedging methodology with the primary purpose of increasing the accuracy of existing quasi-analytic methods for long-maturity options. Our new method resorts to an approximation of the optimal exercise price near the beginning of the contract combined with existing pricing approaches. In particular, as we will show in Section 4.2, we divide each option's time-to-maturity into two components according to the closeness to the beginning of the contract and we approximate the optimal exercise price (OEP) separately in each of them. For the component closest to the beginning of the contract, we use a constant function to approximate the OEP, while, for the second component (the closest to maturity), we employ existing pricing methods and their associated estimation of the OEPs. Additionally, this method retains the quasi-analytic nature of the methods it improves on and, consequently, we derive generic quasi-analytic formulae for the price of an American put as well as for its delta parameter.

Since, the new approach extends the range of maturities for which an existing quasi-analytic method returns good results, we call it the **‘extension’ method**. Henceforth, the word ‘extended’ means the opposite of ‘standard’: the latter refers to each method as found in the literature (and reviewed in Section 3.4) and the former is given as in formulae (4.4) and (4.17), for pricing and hedging purposes respectively.

The applications of the ‘extension’ method are not limited to standard quasi-analytic methodologies. The new method also works successfully with asymptotic expansions of the OEP near maturity (see Section 4.4). They consist of closed-form formulae that precisely estimate the OEPs. However, their application is limited to option contracts with short time left to maturity. By incorporating them into the ‘extension’ method, we show that good pricing and hedging results are reached even for maturities as long as 5 years.

4.2 The ‘extension’ method

Let us consider the Black-and-Scholes market described in Section 3.1 so that, under the risk-neutral measure $\mathbb{Q}$, the dynamics of the underlying asset price S is given by:

$$dS_t = (r-\delta)S_t dt + \sigma S_t d\tilde{W}_t, t \geq t_0$$

where r is the (constant) risk-free rate, δ is the dividend yield and $\{\tilde{W}_t\}_{t\geq t_0}$ is a Wiener process under the martingale measure $\mathbb{Q}$. For simplicity, we denote the difference $r-\delta$ by b, the cost of carry.

Since, via the put-call symmetry of McDonald and Schroder (1998),[3] the pricing problem for American call options can be reduced to that for an American put, without any loss of generality, we will present the ‘extension’ method exclusively for American put options.

[3] We reviewed this property of American call and put options in Section 3.2, in equations (3.25) and (3.31).

4.2.1 Idea

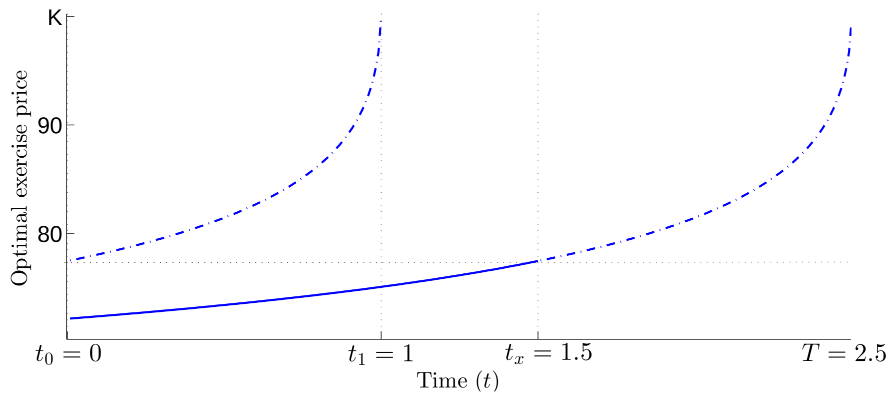
In Section 3.2.2, we reviewed the properties of the OEP of an American put option under geometric Brownian motion. Among those results, there are two properties that, to the best of our knowledge, have not been used together before for pricing purposes: close to the beginning of the contract, the OEP is almost constant; and the OEP is independent of the current underlying asset price and it is non-stochastic (i.e. it depends only on the risk-free rate, strike price, volatility and time to maturity).

Taking advantage of these two properties and considering also that the estimation of the OEP is more complex near maturity, it seems convenient to employ any computational effort in the estimation of the OEP near the maturity date, and to use a simple approximation of the OEP near the beginning of the contract. Therefore, the essential idea underpinning our approach is to break down the option life into two parts, one closest to the beginning of the contract and one close to maturity, and employ existing pricing methods and their corresponding estimation of the OEP in the latter, while using a flat approximation of the OEP for the former. Consequently, any option priced by the ‘extension’ method is constructed over a shorter-maturity option, which is alive near its expiration. Henceforth, the second option is simply called the “*short-maturity option*” while the initial option is called the “*long-maturity option*” to highlight that the former is part of the latter.

Figure 4.3 delineates the intuition behind our method: for a given set of parameter values ($\sigma = 20\%$, $\delta = 5\%$, $r = 8\%$ and $K = 100$), the OEPs for two American put options, written on the same underlying asset, with maturities $t_1 = 1$ year and $T = 2.5$ years are considered at the current time $t_0 = 0$. Let us also define the intermediary time point $t_x = t_0 + (T - t_1) = 1.5$ years. Because the two options are identical apart from their different maturity dates, and because the OEP does not depend on the spot price at t_0, the OEPs for the two options will coincide whenever the options have the same time-to-maturity. In particular, for any time t in the interval $[t_x, T]$, the OEP of the option maturing at T (long-maturity option) will be the same as the OEP of the option maturing at t_1 (short-maturity option), which is defined on $[t_0, t_1]$. In the figure, the continuous line represents the OEP of the long-maturity option and the dash-dot lines represent the OEPs of the short-maturity option. The left-most dash-dot line is the ‘original’ OEP and the other is its translation over the continuous line to show that they coincide on the interval $[t_x, T]$. Consequently, we can think of the short-maturity option as either starting life at t_0 and having maturity date t_1, or as starting life at t_x and having maturity date T. In either case, at the onset, the shorter maturity option has $t_1 - t_0 = T - t_x$ time (years) to maturity.

The ‘extension’ method splits the long-maturity option’s life into two components at time t_x: for the first part, i.e. the one closest to t_0, we approximate the OEP as a constant

Figure 4.3 Example of 'extension' method mechanism.



Note: The optimal exercise prices of two American put options are considered in the figure. The two options are written on the same underlying asset with $\sigma = 20\%$, $\delta = 5\%$, $r = 8\%$ and $K = 100$. One option has maturity $t_1 = 1$ year and the other $T = 2.5$ years. The continuous line represents the optimal exercise price of the option with maturity T and the dash-dot lines represent the optimal exercise of the option with maturity t_1. In particular, the left-most dash-dot line is the 'original' function and the other is its translation over the continuous line to show that they coincide in the interval $[t_x, T]$ where $t_x = t_0 + (T - t_1) = 1.5$ years represents the size of the translation. The OEPs are calculated by the integral method in Kim (1990).

Λ, while for the second part, i.e. the one closest to T, the OEP together with the time-t_x pricing formula for the short maturity option (time to maturity $T - t_x$) is provided by the pricing method we are 'extending'. Consequently, we assume that the OEP of an American put option with maturity T and starting life at t_0, is given by:

$$S_f^{(E)}(t) = \begin{cases} \Lambda & \text{for } t \in [t_0, t_x) \\ S_f\left(t - (T - t_1)\right) & \text{for } t \in [t_x, T] \end{cases} \tag{4.1}$$

where $t_x \in [t_0, T]$ is the break-point and $S_f(\cdot)$, the OEP of the shorter maturity option, will be estimated via any existing quasi-analytic method in the literature (e.g. one of the methods described in Section 3.4). The selection of t_x is discussed in Section 4.2.3 together with the selection of the parameter Λ.

Additionally, we note that when selecting $t_x \to t_0$ in formula (4.1), the option price obtained with the 'extension' method converges to the price obtained via the standard method that we are extending, and when $t_x \to T$, the 'extension' method price converges to the price of an American put option with a flat OEP equal to Λ. In particular, the method converges to the method of Bjerksund and Stensland (1993).[4] The numerical study in Section 4.3 shows

[4]In Section 4.2.4 we show that if we apply our 'extension' method to the method of Bjerksund and Stensland (1993), we obtain the pricing methodology in Bjerksund and Stensland (2002). Consequently, our method can be considered a generalisation of the methodology of Bjerksund and Stensland (2002).

that for intermediate values of t_x, each ‘extended’ method provides better prices (according to a number of criteria) than the corresponding ‘standard’ version.

In employing the ‘extension’ method for pricing purposes, the advantage is threefold:

- the estimation of the OEP near maturity is more precise: given a number of time-steps, the standard pricing methods use them across the interval $[t_0, T]$ while the ‘extension’ method only employs them across the shorter interval $[t_x, T]$. Consequently, in the part of the option’s life close to maturity, the length of the time-step is much smaller for the ‘extension’ method and the Bermudan approximation is closer to the American price;
- the existing methods are used where they have a better performance (comparative advantage): even for those methods not based on the explicit calculation of the OEP (examples are Barone-Adesi and Whaley (1987), Ju and Zhong (1999) and Li (2010b)), the ‘extension’ method resorts to a standard method to price a short-maturity option, which is where the latter performs better;
- finally, the ‘extension’ method requires very low computational effort for the part of the option’s life close to the beginning of the contract, where the theory suggests that the OEP is ‘nicer’. Indeed, as Proposition 4.2.1 below shows, the expected payoff from exercising the option between $(t_0, t_x]$ can be calculated in closed-form and consequently the computational time is negligible.

4.2.2 Pricing and hedging methodology

Pricing method

With the optimal exercise price approximated as in (4.1) and assuming that t_x and Λ are known,[5] the price of the long-maturity American put option is calculated as the sum of the expected discounted payoff (between t_0 and t_x), assuming that the option is exercised as soon as the spot price hits Λ, and the expected discounted payoff from the short-maturity American option (between t_x and T) starting at time t_x conditional on the underlying asset price being above the barrier Λ between t_0 and t_x. Proposition 4.2.1 derives the pricing formula of the ‘extension’ method, where the following notation will be used:

- $P_{t_0}(S_{t_0}, T, K | t_x, \Lambda)$ is the time-t_0 price of the long-maturity option with maturity at T when the underlying asset price is S_{t_0} and the OEP is given by (4.1).

[5] Section 4.2.3 discusses the selection of the two parameters.

- $\tilde{P}_{t_x}(S_{t_x}, T, K)$ is the time-t_x price of the short-maturity option with time to maturity $T - t_x$ for a value of the underlying asset price of S_{t_x};

- $S_{f_x}(\cdot)$ is the OEP of the short maturity option. It is a function defined on $[t_x, T]$ and is the translation of the function $S_f(\cdot)$, the latter being defined on $[t_0, t_1]$. In Figure 4.3, $S_{f_x}(\cdot)$ is the right-most dash-dot line and $S_f(\cdot)$ is the left-most dash-dot line;

We also define the expectation term:

$$\begin{aligned}
\varphi(\gamma, H) &= \varphi_{t_0}^{P}(S_{t_0}, t_x|\gamma, H, \Lambda) = \\
&= E_{t_0}\left[e^{-rt_x} S_{t_x}^{\gamma} I(S_{t_x} > H) I\left(\inf_{t\in[t_0,t_x)} S_t > \Lambda\right)\right] = \\
&= e^{\lambda t_x} S_{t_0}^{\gamma}\left[N\left(d_{\varphi,1}(H)\right) - \left(\frac{\Lambda}{S_{t_0}}\right)^{\kappa} N\left(d_{\varphi,2}(H)\right)\right] \qquad (4.2)
\end{aligned}$$

the derivation of which is provided in detail in Appendix 4.A (see equation (4.31)) together with the definitions of λ, κ, $d_{\varphi,1}(H)$ and $d_{\varphi,2}(H)$. Additionally, as in Ingersoll (1987), page 352, we define the probability density function of an arithmetic Brownian motion at time-t_x with a positive initial value z_{t_0}, drift parameter $b_1 = b - \frac{1}{2}\sigma^2$, volatility parameter σ and an absorbing barrier at zero as:

$$f_0(z) = \frac{n\left(\frac{z - z_{t_0} - b_1(t_x - t_0)}{\sigma\sqrt{t_x - t_0}}\right) - e^{-\frac{2b_1 z_{t_0}}{\sigma^2}} n\left(\frac{z + z_{t_0} - b_1(t_x - t_0)}{\sigma\sqrt{t_x - t_0}}\right)}{\sigma\sqrt{t_x - t_0}}. \qquad (4.3)$$

With the notation introduced above, the new pricing formula is given in the next result.

Proposition 4.2.1 (Pricing formula). *Assuming Black-and-Scholes dynamics, the time-t_0 price of an American put option with strike price K and maturity T, based on the ‘extension’ of the standard method with pricing function $\tilde{P}_{t_x}(S_{t_x}, T, K)$, is given by:*

$$\begin{aligned}
P_{t_0}(S_{t_0}, T, K|t_x, \Lambda) \;=\; & e^{rt_0}\Big\{\alpha(\Lambda)\left[S_{t_0}^{\beta} e^{-rt_0} - \varphi(\beta, \Lambda)\right] - \varphi(1, \Lambda) + \varphi(1, S_f^{(E)}(t_x)) \\
& + K\left[\varphi(0, \Lambda) - \varphi(0, S_f^{(E)}(t_x))\right]\Big\} + \int_B^{+\infty} g(z)dz, \qquad (4.4)
\end{aligned}$$

where

$$g(z) = e^{-r(t_x - t_0)}\tilde{P}_{t_x}(\Lambda e^z, T, K) f_0(z), \qquad (4.5)$$

$B = \ln \frac{S_f^{(E)}(t_x)}{\Lambda}$, $\alpha(\Lambda) = (K - \Lambda)\Lambda^{-\beta}$ *and* $\beta = \left(\frac{1}{2} - \frac{b}{\sigma^2}\right) - \sqrt{\left(\frac{1}{2} - \frac{b}{\sigma^2}\right)^2 + 2\frac{r}{\sigma^2}}.$

Proof. Let us define the stopping time corresponding to the OEP in (4.1) for any $0 < \Lambda \leq$

$S_{f_x}(t_x)$ as:

$$t^* = \inf\left\{\inf_{t\in[t_0,\infty)}\{S_t \le S_f^{(E)}(t)\}, T\right\} = \inf\left\{t_0^*(\Lambda), t_x^*(S_{f_x}(t)), T\right\}$$

where $t_u^*(x) = \inf_{t\in[t_u,\infty)}\{S_t \le x\}$. Additionally, we indicate with $I(\cdot)$ the indicator function. The American put price is then calculated as:

$$\begin{aligned} P_{t_0}(S_{t_0},T,K|t_x,\Lambda) &= E_{t_0}\left[e^{-r(t^*-t_0)}(K-S_{t^*})^+\right] = e^{rt_0}E_{t_0}\left[e^{-rt^*}(K-S_{t^*})^+\right] \\ &= e^{rt_0}\left\{E_{t_0}\left[e^{-rt^*}(K-\Lambda)I(t_0 \le t^* < t_x)\right]\right. \\ &\quad +E_{t_0}\left[e^{-rt^*}(K-S_{t_x})I(t^*=t_x)\right] \\ &\quad \left.+E_{t_0}\left[e^{-rt^*}(K-S_f^{(E)}(t^*))^+I(t_x < t^* \le T)\right]\right\}. \end{aligned} \tag{4.6}$$

In what follows, we will calculate separately the three expectations in (4.6). We use the results in Appendix 4.A.1 and the function $\varphi_{t_0}^P$, as in (4.2).

The first expectation in equation (4.6) is given as:

$$\begin{aligned} &E_{t_0}\left[e^{-rt^*}(K-\Lambda)I(t_0\le t^*<t_x)\right] = E_{t_0}\left[e^{-rt_0^*(\Lambda)}(K-\Lambda)I\left(\inf_{t\in[t_0,t_x)} S_t<\Lambda\right)\right] \\ &= E_{t_0}\left[e^{-rt_0^*(\Lambda)}(K-\Lambda)\left(1-I\left(\inf_{t\in[t_0,t_x)} S_t>\Lambda\right)\right)\right] \\ &\overset{(e.1)}{=} \alpha(\Lambda)S_{t_0}^\beta e^{-rt_0} - E_{t_0}\left[e^{-rt_0^*(\Lambda)}(K-\Lambda)I\left(\inf_{t\in[t_0,t_x)} S_t>\Lambda\right)\right] \\ &\overset{(e.2)}{=} \alpha(\Lambda)S_{t_0}^\beta e^{-rt_0} - E_{t_0}\left[e^{-rt_x}E_{t_x}\left[e^{-r(t_x^*(\Lambda)-t_x)}(K-\Lambda)\right]I\left(\inf_{t\in[t_0,t_x)} S_t>\Lambda\right)\right] \\ &\overset{(e.3)}{=} \alpha(\Lambda)S_{t_0}^\beta e^{-rt_0} - \alpha(\Lambda)E_{t_0}\left[e^{-rt_x}S_{t_x}^\beta I(S_{t_x}\ge\Lambda)I\left(\inf_{t\in[t_0,t_x)} S_t>\Lambda\right)\right] \\ &= \alpha(\Lambda)S_{t_0}^\beta e^{-rt_0} - \alpha(\Lambda)\varphi_{t_0}^P(S_{t_0},t_x|\beta,\Lambda,\Lambda) \end{aligned} \tag{4.7}$$

where equivalences $(e.1)$ and $(e.3)$ follow from formula (4.30) and equivalence $(e.2)$ follows from the definition of the stopping time and the indicator function.

The second expectation in equation (4.6) can be calculated as:

$$
\begin{aligned}
&E_{t_0}\left[e^{-rt^*}(K-S_{t_x})I\left(t^*=t_x\right)\right]\\
&=E_{t_0}\left[e^{-rt_x}(K-S_{t_x})I\left(\Lambda\le S_{t_x}\le S_f^{(E)}(t_x)\right)I\left(\inf_{t\in[t_0,t_x)}S_t>\Lambda\right)\right]\\
&\overset{(e.4)}{=}E_{t_0}\left[e^{-rt_x}(K-S_{t_x})\left[I\left(S_{t_x}\ge\Lambda\right)-I\left(S_{t_x}\ge S_f^{(E)}(t_x)\right)\right]I\left(\inf_{t\in[t_0,t_x)}S_t>\Lambda\right)\right]\\
&=KE_{t_0}\left[e^{-rt_x}I\left(S_{t_x}\ge\Lambda\right)I\left(\inf_{t\in[t_0,t_x)}S_t>\Lambda\right)\right]\\
&\quad-KE_{t_0}\left[e^{-rt_x}I\left(S_{t_x}\ge S_f^{(E)}(t_x)\right)I\left(\inf_{t\in[t_0,t_x)}S_t>\Lambda\right)\right]\\
&\quad-E_{t_0}\left[e^{-rt_x}S_{t_x}I\left(S_{t_x}\ge\Lambda\right)I\left(\inf_{t\in[t_0,t_x)}S_t>\Lambda\right)\right]\\
&\quad+E_{t_0}\left[e^{-rt_x}S_{t_x}I\left(S_{t_x}\ge S_f^{(E)}(t_x)\right)I\left(\inf_{t\in[t_0,t_x)}S_t>\Lambda\right)\right]\\
&=K\left[\varphi_{t_0}^P(S_{t_0},t_x|0,\Lambda,\Lambda)-\varphi_{t_0}^P(S_{t_0},t_x|0,S_f^{(E)}(t_x),\Lambda)\right]\\
&\quad-\left[\varphi_{t_0}^P(S_{t_0},t_x|1,\Lambda,\Lambda)-\varphi_{t_0}^P(S_{t_0},t_x|1,S_f^{(E)}(t_x),\Lambda)\right].
\end{aligned}\tag{4.8}
$$

where equivalence $(e.4)$ follows from:

$$I\left(\Lambda\le S_{t_x}\le S_f^{(E)}(t_x)\right)=I\left(S_{t_x}\ge\Lambda\right)-I\left(S_{t_x}\ge S_f^{(E)}(t_x)\right).$$

The third expectation in equation (4.6) is:

$$
\begin{aligned}
&E_{t_0}\left[e^{-rt^*}(K-S_f^{(E)}(t^*))^+I\left(t_x<t^*\le T\right)\right]\\
&=e^{-rt_x}E_{t_0}\left[E_{t_x}\left[e^{-r(t_x^*(S_f^{(E)}(t))-t_x)}(K-S_f^{(E)}(t^*))^+\right]I\left(t_x<t^*\le T\right)\right]\\
&\overset{(e.5)}{=}e^{-rt_x}E_{t_0}\left[\tilde{P}_{t_x}(S_{t_x},T,K)I\left(S_{t_x}>S_f^{(E)}(t_x)\right)I\left(\inf_{t\in[t_0,t_x)}S_t>\Lambda\right)\right]\\
&\overset{(e.6)}{=}e^{-rt_x}E_{t_0}\left[\tilde{P}_{t_x}(\Lambda e^{z_{t_x}},T,K)I\left(z_{t_x}>B\right)I\left(\inf_{t\in[t_0,t_x]}z_t>0\right)\right]\\
&=e^{-rt_x}\int_B^{+\infty}\tilde{P}_{t_x}(\Lambda e^z,T,K)f_0(z)dz
\end{aligned}\tag{4.9}
$$

where equivalence $(e.5)$ follows because the inner expectation is the time-t_x price of an option with maturity T and strike price K, i.e. what we defined before the short-maturity option, and the equivalence $(e.6)$ from $z_t=\ln\frac{S_t}{\Lambda}$. By replacing the three expectations in equations (4.7), (4.8) and (4.9) within equation (4.6), we get the pricing formula (4.4). This concludes the proof. □

The proof above extends the proof that Bjerksund and Stensland (2002) provide for their pricing formula. In Section 4.2.4 we show that the pricing method in Bjerksund and Stensland (2002) can be seen as a particular case of the pricing method in Proposition 4.2.1, when the OEP of the short-maturity option is approximated by a step-function as well. As for the pricing formula of Bjerksund and Stensland (2002), also the pricing formula in Proposition 4.2.1 depends on two parameters t_x and Λ. In Section 4.2.3, we will show a methodology to select the optimal values for these two parameters in order to minimize the pricing errors with respect to the fair benchmark prices calculated by the binomial tree of Cox et al. (1979). In Chapter 6, we employ the parameters selected in Section 4.2.3 to real financial options.

Additionally, we note that all of the terms in (4.4) except the last, i.e. the integral term, are independent of the selection of the pricing method for the short-maturity option, $\tilde{P}_{t_x}(S_{t_x}, T, K)$. The calculation of the last addend depends on the selected method and, in many cases, the function $g(\cdot)$ in (4.5) can be analytically integrated: examples are the quadratic method of Barone-Adesi (2005), the integral method of Kim (1990) and the static-replicating portfolio method of Chung and Shih (2009).

The following result concerns the asymptotic behaviour of the pricing formula in Proposition 4.2.1 when the time-to-maturity becomes arbitrarily large.

Proposition 4.2.2 (Convergence to perpetual option)**.** *For any $t_x \in (t_0, T]$, any $0 < \Lambda < S_{f_x}(t_x)$ and any pricing formula for the short-maturity option $\tilde{P}_{t_x}(S_{t_x}, T, K)$, when $T \to +\infty$, the price*

$$P_{t_0}(S_{t_0}, T, K|t_x, \Lambda)$$

given in Proposition 4.2.1 converges to the price of the perpetual option written on the same underlying asset, with the same strike price and which is exercised as soon as the underlying asset price goes below Λ.

Proof. The proof consists in showing that:

$$\lim_{T \to +\infty} P_{t_0}(S_{t_0}, T, K|t_x, \Lambda) = \alpha(\Lambda) S_{t_0}^{\beta}, \tag{4.10}$$

for any selection of $t_x \in (t_0, T]$ and Λ (i.e. the option price converges to the pricing formula in equation (3.26)). For $t_x = t_0 + \vartheta(T - t_0)$ with $\vartheta \in (0, 1]$, we prove that the result in (4.10) is independent of ϑ. We discuss separately the cases for different values of γ, which can be any of 0, 1 and β.

When $\gamma = \beta$, for positive risk-free rate r,

$$\gamma = \beta = \left(\frac{1}{2} - \frac{b}{\sigma^2}\right) - \sqrt{\left(\frac{1}{2} - \frac{b}{\sigma^2}\right)^2 + 2\frac{r}{\sigma^2}} < \frac{1}{2} - \frac{b}{\sigma^2} \tag{4.11}$$

and, consequently, since $b + (\gamma - \frac{1}{2})\sigma^2 < 0$,

$$\lim_{T \to +\infty} d_{\varphi,1}(H) = \lim_{T \to +\infty} d_{\varphi,2}(H) = -\infty$$

and

$$\lim_{T \to +\infty} \varphi^P_{t_0}(S_{t_0}, t_x | \beta, \Lambda, \Lambda) = 0, \tag{4.12}$$

for any ϑ. We note that the result (4.12) also holds when $\lambda > 0$, since l’Hôpital’s rule guarantees that:

$$\lim_{T \to +\infty} e^{\lambda \vartheta T} N\left(d_{\varphi,1}(H)\right) = \lim_{T \to +\infty} e^{\lambda \vartheta T} N\left(d_{\varphi,2}(H)\right) = 0.$$

On the other hand, for $\gamma = 0$ or $\gamma = 1$, from the definition of λ in (4.34), we have:

$$\lambda = -r + \gamma b + \frac{1}{2}\gamma(\gamma - 1)\sigma = -r + \gamma b \leq 0$$

and

$$\lim_{T \to +\infty} d_{\varphi,1}(H) = \lim_{T \to +\infty} d_{\varphi,2}(H) = \mathrm{v},$$

with v independent from H. Therefore, the limit,

$$\lim_{T \to +\infty} \varphi^P_{t_0}(S_{t_0}, t_x | \gamma, H, \Lambda) \tag{4.13}$$

is finite for any positive and finite H and, since it does not depend on the selection of H, we have

$$\lim_{T \to +\infty} \left[\varphi^P_{t_0}(S_{t_0}, t_x | \gamma, H_1, \Lambda) - \varphi^P_{t_0}(S_{t_0}, t_x | \gamma, H_2, \Lambda)\right] = 0 \tag{4.14}$$

for any finite H_1 and H_2. Finally,

$$\lim_{T \to +\infty} \int_B^{+\infty} g(z) dz = \lim_{T \to +\infty} \int_B^{+\infty} e^{-r(t_x - t_0)} \tilde{P}_{t_x}(\Lambda e^z, T, K) f_0(z) dz = 0 \tag{4.15}$$

since $\lim_{T \to +\infty} f_0(z) = 0$, $\lim_{T \to +\infty} e^{-r(t_x - t_0)} = \lim_{T \to +\infty} e^{-r\vartheta(T - t_0)} = 0$, and

$$0 \leq \tilde{P}_{t_x}(\Lambda e^z, T, K) \leq K$$

following the non-arbitrage condition. Since the quantities $\alpha(\Lambda)$ and β are time invariant, limit (4.10) is proved. This concludes the proof.

□

In particular, selecting $\Lambda = S_f^{\infty}$ where S_f^{∞} is the optimal exercise price of the corresponding perpetual option in formula (3.29), the option price calculated via formula (4.4) corresponds to the price of the perpetual option in McKean (1967) and Merton (1973) (see formula (3.26)).

Delta parameter

In any financial markets, the calculation of the delta parameter is as important as the pricing of the option. The proposition below provides an analytic formula for the calculation of the delta parameter of an American put option by the 'extension' method. In the following, we use the notation:

$$\varphi'(\gamma, H) = \varphi'_{t_0, S_{t_0}}(S_{t_0}, t_x | \gamma, H, \Lambda) \tag{4.16}$$

to denote the partial derivative of formula (4.2) w.r.t. S_{t_0}. Its formula and that for $f_0'(z) = \frac{\partial f_0(z)}{\partial S_{t_0}}$ are provided in Appendix 4.A.

Proposition 4.2.3 (Delta parameter)**.** *Under the same conditions as stated in Proposition 4.2.1, the delta parameter is given by the following formula:*

$$\begin{aligned} \Delta_{t_0} \;=\; & e^{rt_0}\Big\{\alpha(\Lambda)\left[\beta S_{t_0}^{\beta-1} e^{-rt_0} - \varphi'(\beta,\Lambda)\right] - \varphi'(1,\Lambda) + \varphi'(1, S_f^{(E)}(t_x)) \\ & + K\left[\varphi'(0,\Lambda) - \varphi'(0, S_f^{(E)}(t_x))\right]\Big\} + \int_B^{+\infty} g'(z)dz, \end{aligned} \tag{4.17}$$

where

$$g'(z) = e^{-r(t_x - t_0)}\tilde{P}_{t_x}(\Lambda e^z, T, K) f_0'(z).$$

The result above is an application of Leibniz's derivation formula to function (4.4), taking into account the results $\frac{\partial B}{\partial S_{t_0}} = 0$, $\frac{\partial \tilde{P}_{t_x}(\Lambda e^z, T, K)}{\partial S_{t_0}} = 0$, $\frac{\partial \Lambda}{\partial S_{t_0}} = 0$ that follow from the independence of the OEP from S_{t_0}. By applying once more Leibniz's derivative formula to function (4.17), the Gamma parameter (i.e. the second derivative of the option price w.r.t. S_{t_0}) can also be calculated in closed-form.

The pricing formula and the delta parameter in Propositions 4.2.1 and 4.2.3 respectively, work under any specification for the pricing formula of the short-maturity option. Choosing any of the standard methods only changes the last addends (i.e. the integrals) of formulae (4.4) and (4.17). This makes the 'extension' method very flexible and, therefore, it can potentially improve the performance of any quasi-analytic method.

4.2.3 Selection of the parameters

Up to now, we have assumed that the two parameters Λ and t_x are known. In this section, we will detail how they should be selected.

One way to select the best values for the couple (t_x, Λ) is via the solution of the following problem:

$$(t_x^*, \Lambda^*) = \underset{(t_x,\Lambda)\in[t_0,T]\times[S_f^\infty,K]}{\operatorname{argmax}} P_{t_0}(S_{t_0}, T, K | t_x, \Lambda), \tag{4.18}$$

i.e. t_x and Λ are the values corresponding to the best lower bound (see, among others, Bjerksund and Stensland (1993, 2002)), where S_f^∞ is in (3.29). Problem (4.18) follows from formulation (3.6), under the assumption that the standard method we incorporate into the ‘extension’ method returns a lower bound of the short maturity option’s price.

However, since the solution of problem (4.18) could be computationally intensive because it consists of a non-linear optimization problem, and the standard method could, in general, not be a lower bound, we adopt a different approach that also leads to good results, as the numerical comparison in Section 4.3 shows.

In particular, we assume:

$$\Lambda = S_{f_x}(t_x), \tag{4.19}$$

so that the approximation of the OEP in (4.1) satisfies the continuity property, which theoretical characterizes the OEP (see Section 3.2.2) also for $t = t_x$.[6]

All of the numerical results in Section 4.3 are calculated for this equality holding. Exceptions are the results for the compound-option method (see Section 3.4.1) and the integral method (see Section 3.4.3), which are obtained for Λ equal to the OEP at time t_x calculated by the quadratic method in Barone-Adesi and Whaley (1987) (pricing formula (3.40)). This is because the calculations of $S_{f_x}(t_x)$ by either of the two methods are poor when only a few early-exercise dates are considered. However, for these two methods, other choices of Λ could have been made: among others, we point out the initial guess in Barone-Adesi and Whaley (1987),

$$\Lambda = S_f^\infty + [K - S_f^\infty]e^{y_1}, \text{ with } y_1 = K\frac{b(T-t_0) - 2\sigma\sqrt{T-t_0}}{K - S_f^\infty} \tag{4.20}$$

and that in Bjerksund and Stensland (1993),

$$\Lambda = S_f^\infty + [S_f(T^-) - S_f^\infty]e^{y_2}, \text{ with } y_2 = S_f(T^-)\frac{b(T-t_0) - 2\sigma\sqrt{T-t_0}}{S_f(T^-) - S_f^\infty} \tag{4.21}$$

[6] We note that the pricing and hedging functions are much simpler since the second expectation in formula (4.6) is equal to zero.

where S_f^∞ is as in (3.29) and $S_f(T^-) = \min\{K, \frac{r}{\delta}K\}$.[7]

On the other hand, in order to select t_x without loss of generality we choose $t_0 = 0$, and we express t_x as a percentage of time-to-maturity $t_x = \vartheta T$. Then, for any standard method considered for the short-maturity options in the set,

$$\mathscr{M} \in \{\text{GJ2, GJ3, BAW, LI, K2, K3, CS2, CS3, JZ}\}$$

we numerically solve the problem:

$$\vartheta^{*,\mathscr{M}} = \underset{\vartheta\in\boldsymbol{\vartheta}}{\operatorname{argmin}}\ MAPE^{(\mathscr{M})}(\vartheta) = \frac{1}{|O|}\underset{\vartheta\in\boldsymbol{\vartheta}}{\operatorname{argmin}}\ \sum_{l\in O} APE^{(l,\mathscr{M})}(\vartheta) \tag{4.22}$$

where O is the set of option scenarios with cardinality $|O| = 10{,}000$ and $\boldsymbol{\vartheta} = \{0.05 + 0.1j | j = 0,\ldots,9\}$ is the discrete version of the interval $(t_0, T]$. MAPE is the mean absolute percentage error and APE, the absolute percentage error for option l and method $\mathscr{M}$ is calculated as:

$$APE^{(l,\mathscr{M})}(\vartheta) = \left|\frac{P_{t_0}^{(E)}\left(S_{t_0}^{(l)}, T^{(l)}, K^{(l)}\right) - P_{t_0}\left(S_{t_0}^{(l)}, T^{(l)}, K^{(l)} \middle| \vartheta T^{(l)}, \Lambda^{(l)}\right)}{P_{t_0}^{(E)}\left(S_{t_0}^{(l)}, T^{(l)}, K^{(l)}\right)}\right| \tag{4.23}$$

where $P_{t_0}^{(E)}(\cdot)$ is the benchmark price, $P_{t_0}(\cdot)$ is the pricing function in Proposition 4.2.1, and all of the other quantities are as above. We explicitly indicate, by the superscript (l), which option scenario they refer to.

Figure 4.4 illustrates the mean absolute percentage error produced by the ‘extension’ method over the 10,000 scenarios in Figure 4.2 (Section 4.3) as a function of the ratio $\vartheta = t_x/T$. It is clear from the plots in the figure that the ratio t_x/T, for any of the methods considered in Section 3.4, can be selected in a wide range, and the pricing performance of the ‘extension’ method is still better than the standard method (the dash-dot lines). All of the results in Section 4.3 are calculated for $\vartheta = \vartheta^{*,\mathscr{M}}$, for any of the considered method $\mathscr{M}$. Additionally, the plots in Figure 4.4 show that the optimal solution of problem (4.22), for almost any of the methods considered, is reached for

$$\vartheta^{*,\mathscr{M}} \approx 0.5, \tag{4.24}$$

i.e. half way through the time-to-maturity. Consequently, we suggest using the value 0.5 whenever the solution of problem (4.22) is not available.

[7] $S_f(T^-)$ corresponds to the limiting value in (3.30).

Besides, Figure 4.4 can be used to analyse the performance of the standard methods, similarly to Figure 4.2. In particular, the position of the best ratio t_x/T in any of the plots gives information on how the corresponding standard method performs for long maturities: the smaller the optimal ratio, the better the standard method we are ‘extending’ performs for long-maturity options. Considering the two extreme situations, the method of Ju and Zhong has the best ratio at about 0.35 while the method of Geske and Johnson with two time-steps has the best ratio at about 0.6. This means that the former gives more weight to the standard method than the flat function Λ; on the other hand, the latter considers the approximation Λ for a longer interval. This implies that the standard Geske-Johnson method with two time-steps performs more poorly than the Ju-Zhong method does.

4.2.4 Similarities with Bjerksund and Stensland (2002)

As discussed extensively in the previous sections, the ‘extension’ method divides the option life into two parts and uses an existing standard pricing technique in the part closest to maturity. Suppose that the short-maturity option price is given by the method in Bjerksund and Stensland (1993), which approximates the OEP as a flat function x and has a time-t_x pricing formula for an American put option with maturity at time T equal to:

$$\begin{aligned} \tilde{P}_{t_x}(S_{t_x},T,K) \quad = \quad & e^{rt_x}\left\{\alpha(\Lambda)\left[S_{t_x}^{\beta}e^{-rt_x}-\varphi_{t_x}^{P}(S_{t_x},T|\beta,x,x)\right]-\varphi_{t_x}^{P}(S_{t_x},T|1,x,x)\right. \\ & \left.+\varphi_{t_x}^{P}(S_{t_x},T|1,K,x)+K\left[\varphi_{t_x}^{P}(S_{t_x},T|0,x,x)-\varphi_{t_x}^{P}(S_{t_x},T|0,K,x)\right]\right\} \end{aligned} \tag{4.25}$$

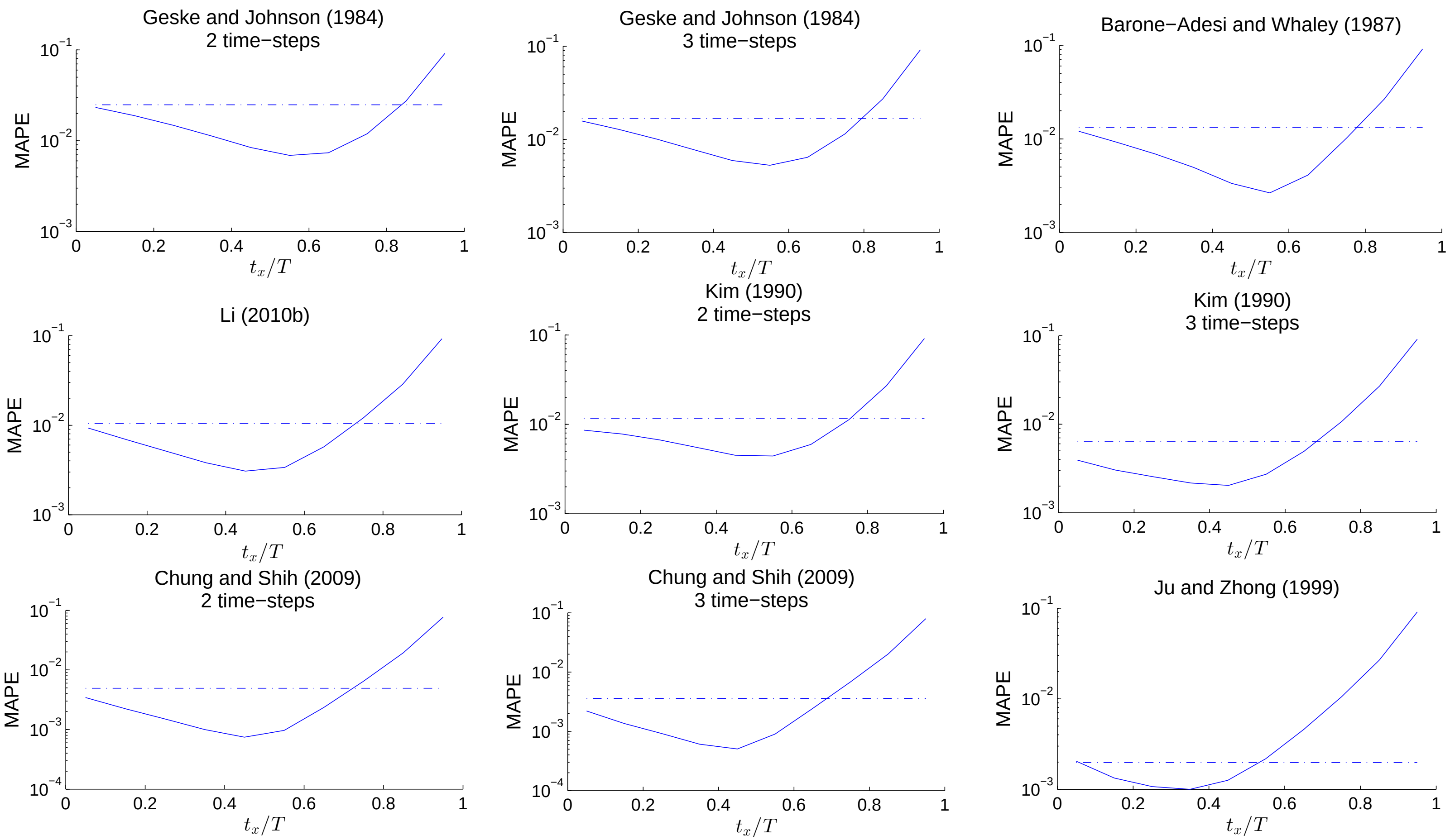
where $\varphi_{t_x}^{P}$ is as defined in (4.2).

When formula (4.25) is used in the pricing formula in Proposition 4.2.1, formula (4.4) is given in closed form and corresponds to the pricing formula in Bjerksund and Stensland (2002). Indeed, Bjerksund and Stensland (2002) also divided the life of the option into two parts (t_0,t_x) and $[t_x,T]$, but the OEP is approximated by two flat functions as:

$$S_f^{(E)}(t)=\begin{cases}\Lambda & \text{for } t\in(t_0,t_x)\\ x & \text{for } t\in[t_x,T].\end{cases} \tag{4.26}$$

Consequently, the ‘extension’ method can be considered as a generalisation over the method in Bjerksund and Stensland (2002): the latter works by ‘extending’ only the standard method in Bjerksund and Stensland (1993), while the former works for any quasi-analytic pricing method.

Figure 4.4 Optimal selection of the ratio $\vartheta = t_x/T$ (quasi-analytic methods).



This figure shows the ranges of ratios t_x/T for which the 'extended' version (solid lines) outperforms the 'standard' version (dash-dot lines) for each method when problem (4.22) is considered. The methods considered are those in Section 3.4 and the results are shown for all maturities ($\leq$ 5 years). The minima of the solid lines correspond to the values shown in Table 4.1 in column A-J.

4.3 Numerical Study

The aim of this section is to show the usefulness of the 'extension' method. To this end, we consider some of the most well-known and applied methods for pricing American put options ('standard' methods), compare the performance of each of them with the 'extended' version and, then, highlight that the performance improves considerably when the new method is employed.

Within the broad class of approximation methods, we focus our study on the quasi-analytic methods, i.e. those consisting of analytic formulae that require at most a reasonably small number of numerical solutions of integral equations. Among these methods, we overlook those that depend on an optimization stage and/or parameters found by an intermediary regression step and, consequently we exclude methods such as Johnson (1983), Blomeyer (1986) and Broadie and Detemple (2004). We finish by selecting the following six methods:

GJ the compound-option method in Geske and Johnson (1984) with two and three exercise dates (see Section 3.4.1);

BAW the quadratic method in Barone-Adesi and Whaley (1987) (see Section 3.4.2);

JZ the improved quadratic method in Ju and Zhong (1999) (see Section 3.4.2);

K the integral method in Kim (1990) with two and three exercise dates (see Section 3.4.3);

LI the interpolation method in Li (2010b) (see Section 3.4.4);

CS the static-replicating portfolio method in Chung and Shih (2009) with two and three exercise dates (see Section 3.4.5).

GJ, K and CS will be followed in the tables and figures by the number of time-steps employed (for example GJ2 is the compound-option method with 2 time-steps). Additionally, although most of these methods, in their original definition, include an extrapolation step (mainly Richardson's extrapolation) to speed up the convergence to the true price, we do not consider any extrapolations since we focus on the improvement of the method for a specific number of early-exercise dates.

Finally, the focus of this study is exclusively on the improvement in accuracy since the computational effort required by the 'extension' method is only slightly higher than that of the standard methods and, in most cases, the additional computational time is negligible. Indeed, it should be noted that:

- for any given number of time-steps, any standard method and our extended version calculate the same number of value points of the OEP. Moreover, for those standard methodologies that do not explicitly calculate the entire OEP but only require the calculation of the OEP at the initial time $S_f(t_0)$ (see for example Barone-Adesi and Whaley (1987), Ju and Zhong (1999) and Li (2010b)), the 'extension' method also requires only the calculation of the OEP at time t_x, $S_{f_x}(t_x)$;

- $\varphi(\gamma, \Lambda)$ is provided in closed-form (see formula (4.2)) as is its derivative w.r.t. the underlying spot price (see formula (4.16)) and, consequently, the pricing formula (4.4) and the hedging formula (4.17) can be calculated quickly;
- the integral $\int_B^{+\infty} g(z)dz$ can, in many cases, be calculated analytically. For those cases where an analytic formula does not exist its numerical solution is much faster than the solution of the integral equation required for the calculation of the OEP.

In the next two subsections, we study the pricing and the hedging performance of our new method.

4.3.1 Pricing performance

The pricing-performance study is constructed from a total of 10,000 randomly generated option scenarios. In particular, the parameter values of the underlying price dynamics and the characteristics of the options (strike prices and maturity dates) are drawn as in Broadie and Detemple (1996):

- the volatility σ is distributed uniformly between 0.1 and 0.6;
- the initial asset price S_{t_0} is fixed at 100;
- the strike price K is distributed uniformly between 70 and 130;
- the dividend rate δ is distributed uniformly between 0.0 and 0.10 with probability 0.8 and equal to 0.0 with probability 0.2;
- the risk-free interest rate is uniformly distributed between 0.0 and 0.1.

Without loss of generality, we assume $t_0 = 0$. Given the importance of the time-to-maturity to establish the usefulness of the 'extension' method, as stated previously, we divide the 10,000 option scenarios into 10 sets of equal cardinality. The sets are named $A, \ldots, J$ and the options are divided according to their time-to-maturity in ranges of 6 months. So, for example, set A contains options with a maturity of between of 1 day and 6 months, set B options have maturities of between 6 months and 1 year, and so on, up to set J, which contains options with maturities of between 4.5 years and 5 years. The "exact" fair price (benchmark) is the binomial tree price in Cox et al. (1979) with 15,000 time-steps. As in Broadie and Detemple (1996), options with benchmark prices smaller than 0.5 are discarded.

We use four measures of error to compare each method with its 'extended' version:

- mean absolute percentage error, *MAPE* (results in Table 4.1);
- percentage of good solution, *% good*, defined as the percentage of option scenarios for which the relative error committed by the method considered is below 1% (results in Table 4.2);[8]
- maximum relative error, *Max Error* (results in Table 4.3);
- number of best solutions found, *% best*, defined as the number of option scenarios for which the relative error of the 'extension' method is smaller than that of the standard method

and all of the results in the Tables are for Λ and t_x as selected in Section 4.2.3.

For each 'standard' method considered, the 'extension' method increases its pricing performance remarkably. Table 4.1 shows that the 'extension' method has the advantage of evening out the performance of quasi-analytic methods across maturities, as shown by the shrinking of the range of MAPEs. For example, the standard GJ2 goes from about 0.39% to 4.35% when the maturity range goes from less than 6 months to above 4.5 years, while for our 'extended' version the MAPE goes from 0.25% to only 0.85%. This is true for all of the quasi-analytic methods considered. Moreover, for some methods (BAW, CS2 and CS3) the 'extension' method achieves remarkable reductions in MAPE of over 80%. The 'extension' method also works efficiently for option scenarios with maturities of below 6 months: for any of the standard methods considered, we assist in a reduction of the MAPE even over the scenarios in set A, with the only exception being K3, whose MAPE is virtually identical to that of our extended version.

Table 4.2 shows that the improvements of the 'extension' method are consistent over the entire set of considered options. The % good measure indicates how reliable a method is in pricing American put options. Across all of the maturity ranges, the 'extension' method substantially increases the reliability of all of the standard methods. One noteworthy example is for BAW. As we were expecting, this method has a very good performance for short maturity and the performance is very poor when the time-to-maturity is longer, as shown by

[8] As for the "exact moment-matching" method in Chapter 2 (see footnote 12 at page 26), the 1% threshold is selected a posteriori as the smallest integer greater than the average MAPEs in Table 4.1. In selecting this value, we consider only options with time-to-maturity shorter than one year (i.e. maturity ranges A and B).

Table 4.1 Comparison of quasi-analytic ‘standard’ methods and our ‘extended’ versions: MAPE

		A	B	C	D	E	F	G	H	I	J	A-J
GJ2	S	0.389%	0.871%	1.343%	1.893%	2.084%	2.622%	3.131%	3.587%	3.984%	4.350%	2.482%
	E	0.244%	0.433%	0.552%	0.653%	0.682%	0.751%	0.816%	0.823%	0.845%	0.843%	0.675%
GJ3	S	0.276%	0.608%	0.930%	1.295%	1.417%	1.788%	2.112%	2.376%	2.648%	2.902%	1.673%
	E	0.195%	0.328%	0.436%	0.511%	0.523%	0.573%	0.617%	0.623%	0.650%	0.668%	0.521%
BAW	S	0.155%	0.395%	0.632%	0.900%	1.202%	1.429%	1.696%	1.945%	2.252%	2.380%	1.331%
	E	0.119%	0.140%	0.192%	0.237%	0.239%	0.268%	0.288%	0.298%	0.337%	0.337%	0.249%
LI	S	0.169%	0.293%	0.515%	0.669%	0.879%	1.053%	1.288%	1.532%	1.821%	1.965%	1.043%
	E	0.087%	0.131%	0.205%	0.247%	0.256%	0.320%	0.359%	0.393%	0.395%	0.410%	0.285%
K2	S	0.248%	0.404%	0.578%	0.713%	0.794%	0.899%	0.988%	1.182%	1.291%	1.488%	1.169%
	E	0.180%	0.271%	0.348%	0.404%	0.450%	0.476%	0.500%	0.503%	0.535%	0.534%	0.426%
K3	S	0.125%	0.198%	0.275%	0.346%	0.358%	0.445%	0.518%	0.649%	0.678%	0.862%	0.634%
	E	0.128%	0.129%	0.158%	0.177%	0.189%	0.203%	0.217%	0.229%	0.249%	0.256%	0.196%
CS2	S	0.102%	0.146%	0.155%	0.189%	0.182%	0.281%	0.400%	0.566%	0.688%	0.867%	0.493%
	E	0.063%	0.084%	0.079%	0.069%	0.058%	0.051%	0.057%	0.069%	0.069%	0.069%	0.067%
CS3	S	0.067%	0.086%	0.091%	0.115%	0.112%	0.173%	0.243%	0.341%	0.412%	0.517%	0.358%
	E	0.040%	0.049%	0.046%	0.040%	0.035%	0.036%	0.038%	0.044%	0.056%	0.062%	0.045%
JZ	S	0.051%	0.108%	0.141%	0.167%	0.180%	0.205%	0.234%	0.260%	0.290%	0.305%	0.198%
	E	0.044%	0.076%	0.096%	0.105%	0.105%	0.106%	0.112%	0.108%	0.108%	0.108%	0.098%
No. of scenarios		783	941	963	978	986	991	990	992	990	993	9,607

Note: This table presents the mean absolute percentage errors (MAPE) for six quasi-analytic methods (indicated as ‘S’ for standard) and our ‘extended’ versions (indicated as ‘E’), i.e. when the ‘extension’ method in Proposition 4.2.1 is applied to them. The methods considered are: (GJ) the compound-option method in Geske and Johnson (1984) with two and three exercise dates; (BAW) the quadratic method in Barone-Adesi and Whaley (1987); (LI) the interpolation method in Li (2010b); (K) the integral method in Kim (1990) with two and three exercise dates; (CS) the static-replicating portfolio method in Chung and Shih (2009) with two and three exercise dates and (JZ) the improved quadratic method in Ju and Zhong (1999). Ten ranges of maturity (in years) are considered: $(0;0.5]$ (*A*), $(0.5;1]$ (*B*), $(1;1.5]$ (*C*), $(1.5;2]$ (*D*), $(2;2.5]$ (*E*), $(2.5;3]$ (*F*), $(3;3.5]$ (*G*), $(3.5;4]$ (*H*), $(4;4.5]$ (*I*), $(4.5;5]$ (*J*). The results are based on 1,000 simulated scenarios for each maturity range drawn from the distribution indicated in Broadie and Detemple (1996) and summarised in Section 4.3. In the last row, for each maturity range, there is the number of options with a fair price above 0.5.

a '% good' measure of only about 28% for maturities of between 4.5 and 5 years. When the 'extension' method is considered together with BAW, the percentage of good solutions becomes 95%. For all of the standard methods considered with the exclusion of GJ2 and GJ3, the percentage of good solutions found with the 'extension' method is above 90%. For GJ2 and GJ3, the performance is lower but the 'extension' method still improves over the standard methods.

Furthermore, as indicated in Table 4.3, the 'extension' method substantially reduces the maximum relative error of the standard methods. This provides evidence that the new method solves some of the problems encountered by standard methods, which, in many cases, are caused by a low-volatility regime. This is confirmed by Broadie and Detemple (1996) who, in their Plots 7 and 8, show that the performance of standard pricing methods deteriorates when low-volatility values are considered.

Finally, the percentage of 'best solutions' for the 'extended' methods is always above 99%. The only exception is K3 for short-maturity options (set A): in this case, the percentage is only 38.1% with the standard method outperforming the 'extended' version.

4.3.2 Hedging performance

The numerical study on hedging performance is based on the implementation of a delta-hedging strategy and the analysis of the hedging errors. Delta-hedging is a commonly used strategy to hedge a short position in a put option through a varying short position (equal to the delta parameter of the option at that time) in the underlying asset and a varying position in the risk-less asset. The performance evaluation is carried out according to the *average quadratic hedging error* where the hedging error is defined as the difference in values between the hedging portfolio at the exercise date and the option's payoff. This error measure has been used by Schweizer (1995) and Remillard et al. (2012) for European and American-style options, respectively.

We consider a set of 15 option scenarios with strike price $K = 100$, maturity T (in years) in the set $\{1,2,3,4,5\}$, written on underlying assets with volatility $\sigma = 0.4$, dividend yield $\delta = 0.04$ and initial spot price $S_0 = 100$. Three different underlying assets are considered. The stochastic differential equation describing these assets, under the objective-probability measure $\mathbb{P}$ is:

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{4.27}$$

and μ is one of the elements of the set $\{0.05, 0.06, 0.07\}$. The risk-free rate is $r = 0.05$. For each of the three underlying assets, we simulate 1,000 paths (under the objective measure), and for each of them, we set-up the one-month-rolling delta-hedging strategy deriving from

Table 4.2 Comparison of quasi-analytic ‘standard’ methods and our ‘extended’ version: Percentage of ‘good’ solutions

		A	B	C	D	E	F	G	H	I	J	A-J
GJ2	S	86.973%	66.525%	56.906%	47.444%	44.118%	42.482%	38.384%	37.702%	32.525%	33.333%	47.694%
	E	95.402%	84.910%	76.428%	71.575%	68.763%	65.893%	60.909%	57.157%	58.586%	57.301%	69.033%
GJ3	S	93.231%	76.939%	65.317%	55.317%	50.811%	47.629%	44.242%	42.742%	38.384%	37.261%	54.221%
	E	97.318%	93.836%	86.085%	80.982%	80.122%	75.782%	74.747%	74.496%	74.040%	73.212%	80.629%
BAW	S	97.573%	89.586%	80.270%	68.200%	55.375%	48.335%	42.121%	36.694%	32.121%	27.795%	56.698%
	E	98.978%	98.512%	97.404%	97.137%	96.755%	95.964%	95.455%	96.472%	94.646%	95.368%	96.607%
LI	S	98.212%	94.899%	85.358%	75.767%	66.531%	59.132%	50.505%	44.657%	37.980%	34.542%	63.797%
	E	100.00%	99.469%	98.546%	95.706%	96.755%	94.753%	92.626%	89.516%	90.707%	90.030%	94.660%
K2	S	95.66%	90.86%	83.70%	75.15%	68.97%	64.88%	60.10%	54.33%	48.89%	47.43%	68.252%
	E	98.851%	98.300%	96.781%	95.910%	93.813%	94.248%	93.232%	95.363%	93.737%	93.051%	95.233%
K3	S	99.49%	98.41%	95.95%	94.48%	93.20%	92.13%	89.39%	85.69%	84.95%	81.47%	91.288%
	E	98.978%	100.00%	99.896%	99.489%	99.797%	99.495%	99.495%	99.194%	99.394%	98.892%	99.469%
CS2	S	100.00%	99.68%	99.58%	97.85%	97.77%	93.95%	88.99%	81.45%	76.67%	71.60%	90.465%
	E	100.00%	100.00%	99.896%	99.693%	100.00%	100.00%	99.899%	99.698%	99.899%	100.00%	99.906%
CS3	S	100.00%	100.00%	99.69%	99.08%	99.39%	97.58%	95.45%	92.64%	88.99%	83.69%	95.514%
	E	100.00%	100.00%	99.896%	100.00%	100.00%	100.00%	100.00%	99.899%	99.899%	100.00%	99.969%
JZ	S	100.00%	99.575%	99.896%	99.284%	98.884%	98.184%	97.475%	96.371%	95.960%	95.569%	98.064%
	E	100.00%	100.00%	99.896%	99.796%	99.899%	99.899%	99.495%	99.698%	99.394%	99.496%	99.750%

Note: This table presents for six quasi-analytic methods (indicated as ‘S’ for standard) and our ‘extended’ versions (indicated as ‘E’) the percentage of options for which the relative error they commit is below 1%. For other information, see the caption of Table 4.1.

Table 4.3 Comparison of quasi-analytic ‘standard’ methods and our ‘extended’ versions: Maximum relative error

		A	B	C	D	E	F	G	H	I	J	A-J
GJ2	S	4.866%	10.754%	24.208%	32.925%	17.259%	29.542%	27.883%	43.737%	35.698%	34.591%	43.737%
	E	1.971%	3.458%	5.222%	4.838%	5.176%	5.221%	5.898%	3.838%	3.832%	4.052%	5.898%
GJ3	S	3.106%	6.751%	15.898%	22.218%	12.231%	19.700%	18.470%	30.427%	30.663%	23.497%	30.663%
	E	1.425%	3.204%	5.434%	3.464%	3.680%	3.711%	4.181%	4.208%	4.315%	4.272%	5.434%
BAW	S	2.866%	7.260%	5.773%	11.854%	11.289%	12.121%	15.509%	14.205%	13.830%	14.592%	15.509%
	E	1.317%	2.133%	1.556%	3.748%	2.258%	2.562%	3.201%	3.117%	2.887%	2.849%	3.748%
LI	S	1.669%	2.747%	9.944%	8.912%	34.387%	7.363%	8.746%	11.622%	11.578%	14.179%	34.387%
	E	0.752%	1.241%	4.055%	3.053%	2.542%	2.435%	3.122%	3.265%	2.831%	2.816%	4.055%
K2	S	3.706%	3.306%	11.376%	9.403%	9.544%	13.658%	10.227%	18.407%	17.647%	21.612%	21.612%
	E	1.319%	1.385%	1.275%	1.510%	1.990%	2.082%	1.707%	1.847%	1.816%	1.515%	2.082%
K3	S	1.670%	2.384%	10.140%	10.705%	4.135%	10.923%	8.444%	16.484%	15.517%	18.420%	18.420%
	E	1.315%	0.745%	3.234%	2.095%	1.262%	2.120%	1.485%	2.778%	2.773%	1.858%	3.234%
CS2	S	0.852%	1.245%	6.315%	9.556%	3.786%	7.127%	8.374%	18.115%	12.442%	15.902%	18.115%
	E	0.561%	0.701%	2.093%	1.238%	0.538%	0.872%	1.007%	2.237%	1.209%	0.701%	2.237%
CS3	S	0.824%	0.874%	3.511%	6.153%	2.644%	4.477%	5.895%	11.622%	6.820%	9.524%	11.622%
	E	0.428%	0.442%	1.244%	0.733%	0.453%	0.531%	0.601%	1.328%	1.342%	0.827%	1.342%
JZ	S	0.438%	1.521%	2.682%	2.856%	2.912%	3.028%	3.863%	3.815%	3.624%	3.701%	3.863%
	E	0.260%	0.665%	1.459%	1.359%	1.053%	1.049%	1.455%	1.440%	1.513%	1.410%	1.513%

Note: This table presents for six quasi-analytic methods (indicated as ‘S’ for standard) and our ‘extended’ versions (indicated as ‘E’) the maximum relative error. For other details see the caption of Table 4.1.

Table 4.4 Delta-hedging comparison for different maturities: Quasi-analytic methods

		$\mu = r = 5\%$					$\mu = 6\%$					$\mu = 7\%$				
Time-to-maturity		1 (year)	2	3	4	5	1	2	3	4	5	1	2	3	4	5
GJ2	S	0.551	0.792	1.001	1.195	1.356	0.631	0.889	1.101	1.279	1.403	0.562	0.810	1.032	1.218	1.368
	E	0.470	0.705	0.918	1.124	1.318	0.468	0.664	0.858	1.069	1.214	0.479	0.726	0.946	1.150	1.331
GJ3	S	0.553	0.796	1.009	1.207	1.371	0.633	0.895	1.111	1.293	1.419	0.563	0.814	1.039	1.229	1.382
	E	0.470	0.705	0.917	1.123	1.317	0.468	0.663	0.857	1.069	1.213	0.479	0.725	0.945	1.150	1.330
BAW	S	0.554	0.800	1.016	1.217	1.383	0.639	0.907	1.127	1.312	1.438	0.564	0.818	1.046	1.240	1.394
	E	0.470	0.704	0.916	1.121	1.314	0.468	0.663	0.856	1.066	1.210	0.479	0.725	0.944	1.147	1.327
LI	S	0.554	0.800	1.016	1.218	1.384	0.639	0.907	1.128	1.313	1.439	0.565	0.818	1.047	1.241	1.395
	E	0.458	0.666	0.866	1.072	1.256	0.445	0.632	0.818	1.020	1.160	0.438	0.679	0.891	1.092	1.263
K2	S	0.551	0.786	0.986	1.169	1.320	0.635	0.892	1.097	1.264	1.376	0.561	0.804	1.017	1.192	1.332
	E	0.470	0.705	0.917	1.123	1.317	0.468	0.663	0.857	1.068	1.213	0.479	0.725	0.945	1.150	1.330
K3	S	0.554	0.791	0.992	1.177	1.328	0.639	0.897	1.103	1.272	1.384	0.564	0.809	1.023	1.200	1.340
	E	0.550	0.673	0.838	1.020	1.172	0.578	0.672	0.820	0.986	1.098	0.402	0.618	0.814	0.998	1.144
CS2	S	0.555	0.802	1.020	1.224	1.394	0.639	0.908	1.131	1.319	1.447	0.565	0.820	1.051	1.247	1.404
	E	0.461	0.659	0.861	1.065	1.241	0.479	0.636	0.820	1.014	1.146	0.424	0.662	0.876	1.073	1.236
CS3	S	0.555	0.802	1.019	1.223	1.392	0.639	0.908	1.130	1.318	1.446	0.565	0.820	1.050	1.246	1.403
	E	0.485	0.672	0.869	1.072	1.248	0.504	0.649	0.829	1.022	1.154	0.427	0.664	0.878	1.075	1.239
JZ	S	0.554	0.801	1.018	1.221	1.388	0.639	0.908	1.129	1.316	1.442	0.565	0.819	1.048	1.243	1.398
	E	0.470	0.704	0.916	1.122	1.315	0.468	0.663	0.857	1.067	1.212	0.479	0.725	0.944	1.148	1.329

Note: This table presents the average quadratic hedging error for six quasi-analytic methods (indicated as ‘S’ for standard), our ‘extended’ versions (indicated as ‘E’). The quasi-analytic methods considered are: (GJ) the compound-option method in Geske and Johnson (1984) with two and three exercise dates; (BAW) the quadratic method in Barone-Adesi and Whaley (1987); (LI) the interpolation method in Li (2010b); (K) the integral method in Kim (1990) with two and three exercise dates; (CS) the static-replicating portfolio method in Chung and Shih (2009) with two and three exercise dates and (JZ) the improved quadratic method in Ju and Zhong (1999). The results are based on three sets of 1,000 simulated paths, each with a different expected log-return for the underlying asset: 0.05, 0.06 or 0.07. The parameters are $r = 0.05$, $\delta = 0.04$, $K = 100$, $\sigma = 0.4$ and $S_0 = 100$. The analysis is based on a monthly hedging rolling frequency. The results are presented for five different maturities from 1 to 5 years.

each of the methods in the study. As in Section 4.3.1, the 'exact' fair option price is chosen to be the 15,000 time-step binomial-tree price. Table 4.4 summarises the average quadratic hedging error over the 15 option scenarios.

From the table, we notice that the different standard methods have very similar hedging performances and that there is no clear correspondence between pricing performance and hedging performance: for example, GJ2 is the worst standard method for pricing purposes but for hedging purposes, it is one of the best. When the 'extension' method is considered, across all of the different maturities and the three drifts μ, the quadratic errors are reduced and, consequently, the 'extension' method also outperforms the standard methods when it comes to delta-hedging strategies.

4.4 Additional application: Asymptotic expansions of the OEP

The applications of the 'extension' method are not limited to quasi-analytic methods. This section shows how well the new method performs when used together with asymptotic expansion of the optimal exercise price near the expiration of the option (aOEP).

Barone-Adesi (2005) and Chung and Shih (2009) consider the aOEP as one of the main research streams in pricing methods for American-options. As stated in Chapter 3, the OEP has to be calculated together with the American option price and this makes the American option pricing problem more complex than the corresponding one for European options. The aOEP literature is interested in finding analytic approximations of the OEP that are valid close to expiration. The main drawback of these methods is that most of them work effectively only up to a few months to maturity and only a few methods can be employed for maturities of between 1 and 2 years. Therefore the aOEPs cannot be employed to price long-maturity options.

Evans et al. (2002) suggest using the aOEPs together with numerical methods to overcome this deficiency, and then price long-maturity options:

> "Therefore the results of this paper (i.e. the aOEPs) are complementary to the numerical methods (they refer mainly to binomial tree methods), since they provide values near expiry that can be combined with the numerical methods to calculate the option values and the optimal exercise boundary away from expiry."
>
> (p. 200. The notes in brackets were added)

However, in doing so, the advantages deriving from having analytic formulae for the aOEPs are simply lost. In what follows, we show that by employing the aOEPs and the new 'extension' method together, we preserve the analytic formulation of the former and we are able to price American options with a maturity as long as five years. In what follows, Section 4.4.1 describes the existing contributions in the aOEP literature and Section 4.4.2 numerically evaluates the pricing and hedging performance of the 'extension' method when it is employed together with the aOEPs.

4.4.1 Literature Review

The seminal papers on the approximation of the OEP near expiration are McKean (1967) and Van Moerbeke (1974). They analysed the American call option case and found that the OEP is parabolic near expiration. Traditionally, the approximations have been based on expansion or asymptotic methods. Barles et al. (1995) attained the first-term expansion of the aOEP for American put options on non-dividend paying assets. Kuske and Keller (1998) found an aOEP that agrees to the leading order to that of Barles et al. (1995) and proposed, by starting from the partial differential equation of Black-and-Scholes, an option pricing formula that works only close-to-expiration. Stamicar et al. (1999) derived an integral equation for the OEP that asymptotically behaves as the aOEP of Kuske and Keller (1998). Also, Alobaidi and Mallier (2001) found an aOEP equal to the leading order to the Kuske-Keller aOEP, by expanding the OEP over the term $[(T-t)\ln(T-t)]$. Chung et al. (2011) found results consistent with Barles et al. (1995) by exploring the relationship between the OEP and the gamma of the American put on non-dividend paying stock. Lauko and Sevcovic (2010) compare the approximations in Evans et al. (2002), Stamicar et al. (1999) and Zhou (2006) for non-dividend paying stock options. The results show that the first and second methods have the same asymptotic behaviour while the third, obtained by the projected successive over relaxation method, differs by a logarithmic factor.

Evans et al. (2002), henceforth **EKK**, generalised the aOEP in Kuske and Keller (1998) for dividend-paying stocks, showing that the asymptotic behaviour of the OEP depends on the ratio δ/r. The formula they propose is:

$$S_{f_x}(t) \approx \begin{cases} K - K\sigma\sqrt{(T-t)\ln\frac{\sigma^2}{8\pi(T-t)(r-\delta)^2}} & \text{if } 0 \le \delta < r \\ K - K\sigma\sqrt{2(T-t)\ln\frac{1}{4\sqrt{\pi}\delta(T-t)}} & \text{if } \delta = r \\ \frac{r}{\delta}K\left(1-\sigma\alpha_0\sqrt{2(T-t)}\right) & \text{if } \delta > r \end{cases} \tag{4.28}$$

where $\alpha_0 \approx 0.4517$. Independently, Lamberton and Villeneuve (2003) proved similar results.

Zhang and Li (2010), henceforth **ZL**, generalised Evans et al. (2002) by using the perturbation methodology in Chen and Chadam (2007). Their formula, which corresponds to a fourth order expansion, is:

$$S_{f_x}(t) \approx \begin{cases} Ke^{-\sqrt{2\sigma^2(T-t)u(\xi)}} & \text{if } 0 \leq \delta < r \\ Ke^{-\sqrt{2\sigma^2(T-t)v(\eta)}} & \text{if } \delta = r \\ \frac{r}{\delta}Ke^{-2\sqrt{\tau^*}w(\sqrt{\tau^*})} & \text{if } \delta > r \end{cases} \tag{4.29}$$

where

$$\begin{aligned}
u(\xi) &= -\xi - \frac{1}{2\xi} + \frac{1}{8\xi^2} + \frac{11}{24\xi^3} + O\left(\frac{1}{\xi^4}\right), \xi = \ln\sqrt{8\pi b^2(T-t)/\sigma^2} \\
v(\eta) &= -\eta - \frac{1}{2}\ln(-\eta) - \frac{1}{4\eta}\ln(-\eta) - \frac{1-\frac{5}{4\sqrt{2\pi}}}{\eta} + o\left(\frac{1}{\eta}\right) \\
\eta &= \ln(4\sqrt{\pi}r(T-t)), \alpha_1 = \frac{2r}{\sigma^2}, \alpha_5 = \frac{2\delta}{\sigma^2}, \tau^* = \frac{1}{2}\sigma^2(T-t) \\
w(\sqrt{\tau^*}) &= \beta_0 + \beta_1\sqrt{\tau^*} + \beta_2\tau^* + \beta_3\tau^{*3/2} + O(\tau^{*2}) \\
\beta_0 &= 0.451723, \beta_1 = 0.144914(\alpha_1 - \alpha_5) \\
\beta_2 &= -0.009801 - 0.041764(\alpha_1 + \alpha_5) + 0.014829(\alpha_1 - \alpha_5)^2 \\
\beta_3 &= -0.000618 - 0.002087(\alpha_1 - \alpha_5) - 0.015670(\alpha_1^2 - \alpha_5^2) \\
&\quad -0.001052(\alpha_1 - \alpha_5)^3.
\end{aligned}$$

For $\delta = 0$, the formula corresponds to that in Chen and Chadam (2007). Formula (4.29) returns an accurate OEP for small values of $\sigma(T-t)$ and usually works well for maturities of up to 2 months.

All of the methods above are designed for a short time-to-maturity of, at most, a few months. Recently, using the homotopy analysis method, researchers have presented aOEPs that can also cover up to about a two-year maturity. Liao (2012) derived an analytic formula for the OEP for an American option on non-dividend paying stock. Cheng and Zhang (2012), henceforth CZ, provided an explicit approximated formula for the OEP function that covers the case of dividends as an extension of the result in Cheng et al. (2010). Their analytic approximations are in essence an expansion in terms of powers of $\sqrt{\frac{1}{2}\sigma^2(T-t)}$, and, since the formulae are rather long, they will not be reproduced here. They provide three alternative methods:

CZ the simple analytic formula calculated by the homotopy analysis method, which is given in their formula [8];

CZ-P the expansion of CZ calculated by Pade' approximation as in their Appendix [C];

CZ-P-m a correction of CZ-P by formula (4.28) for $\delta > r$. This method is introduced by Cheng and Zhang (2012) since both CZ and CZ-P are calculated under the assumption that $\delta < r$.

A list of further references on aOEPs can be found in Chen and Chadam (2007), Byun (2011) and Liao (2012).

4.4.2 Performance analysis

Among the methods described above, in what follows, we study five aOEP methods for American options on dividend-paying stocks: EKK, ZL, CZ, CZ-P and CZ-P-m. As in Zhang and Li (2010), we will use these 5 aOEPs together with the integral method in Kim (1990) (see formula (3.43)) to find the time-t_x price of the short-maturity options with maturity at T and strike price K.

In order to establish the usefulness of the 'extension' method in 'extending' aOEPs, we carry out a similar study to that in Section 4.3.

Pricing performance

The pricing performance analysis is based on the 10,000 option scenarios employed in Section 4.3.1.

We fix Λ to be equal to the OEP of the quadratic method (see Section 3.4.2) at time t_x: the performances of CZ, CZ-P and CZ-P-m do not change when Λ is selected to be equal to $S_{f_x}(t_x)$, while methods EKK and ZL perform poorly under this approximation for long maturity options. Therefore, we report all of the results for Λ equal to the OEP at t_x calculated by the quadratic method in Barone-Adesi and Whaley (1987).

On the other hand, for the selection of t_x, we proceed as in Section 4.2.3 by numerically solving the optimization problem (4.22) for $t_0 = 0$. Unlike in the case of the quasi-analytic methods, the difference $T - t_x$ is relevant here rather than the ratio t_x/T since the feasibility of the asymptotic expansions depends exclusively on the time-to-maturity, in absolute terms. The optimization is carried out for a series of values $\zeta = T - t_x$ at intervals of 2 weeks. The maximal expansion considered is two years (as suggested by Cheng and Zhang (2012)) and, consequently, 52 points were considered overall:

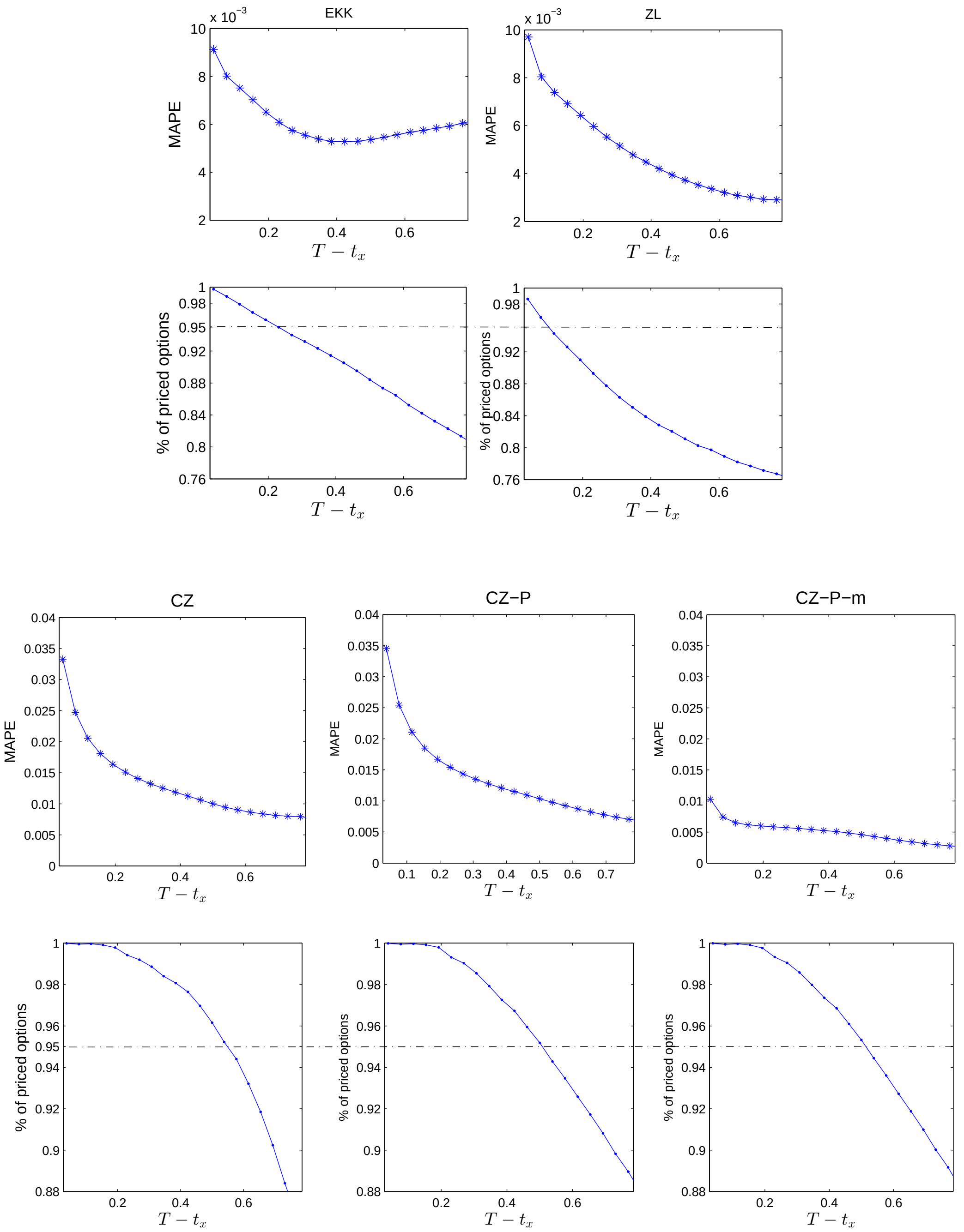
$$\boldsymbol{\zeta} = \{2 \text{ weeks}, 4 \text{ weeks}, \ldots, 2 \text{ years}\}.$$

For this reason, options with maturities shorter than 2 weeks have been discarded. The calculation is carried out for seven exercise days per week.

The solutions of the optimization problem are in Figure 4.5 where, together with the MAPE corresponding to each value $\zeta = T - t_x$, we show the percentage of options we could price, excluding those for which the aOEP does not satisfy the theoretical properties. In particular, we considered as non-valid aOEPs the methods that return negative values, values above the strike price, non-monotonic or non-real functions (see Section 3.2.2)). In order to guarantee the applicability of the 'extension' method, we selected the best ζ among the only values for which at least 95% of the options could be priced. The figure shows that the first two methods, EKK and ZL, work up to 2-3 month maturities. However, CZ, CZ-P and CZ-P-m work properly up to longer maturities and with the restriction we impose of 95%, they work for about 6 months.

Table 4.5 shows the pricing performance of the 'extension' method via three of the measures defined in Section 4.3.1: MAPE, Max Error and % Good. The % best has been excluded since the standard aOEP methods cannot price options with maturities longer than 2-3 years and, consequently, a comparison cannot be carried out. When the 'extension' method is employed together with aOEPs, the MAPEs are remarkably low even for maturities longer than 4.5 years (set J), with an average error over the 10,000 options of well below 1%. Comparing these results with those in Table 4.1, on average lower MAPEs are attained than those for most standard methods. However, the errors of the 'extended' aOEPs are higher than the MAPEs attained when the 'extension' method was employed together with the standard methods. Also, the Max Error measure shows impressive results since it is significantly reduced. This shows that the new method solves many of the problems that the standard methods encounter. The % Good measure shows that, unsurprisingly, the aOEP methods are less reliable for long maturity options. However, considering MAPE and % Good, together we see that, although the number of options with errors of less than 1% is quite low for options above 3.5 years (sets H, I, J), the errors are usually close to 1%. Overall, the 5 methods also perform very well for long maturity options. The CZ-P-M method outperforms all of the others but remarkable results are obtained by EKK, the performance of which is comparable with CZ-P-M but it is much simpler to implement and has a shorter computational time.

Figure 4.5 Optimal selection of the difference $\zeta = T - t_x$ (asymptotic expansions of the OEP near maturity)



Note: This figure shows that the MAPEs (the starred lines) change for different values of ζ (in years) for the following asymptotic expansion methods: (EKK) the method in Evans et al. (2002); (ZL) the method in Zhang and Li (2010); (CZ),(CZ-P)-(CZ-P-m) the method in Cheng and Zhang (2012) basic, with Pade' approximation and with Pade' approximation corrected for Evans et al. (2002). Moreover, the figure shows for each method the percentage of options for which a feasible aOEP is obtained (% of priced options). The threshold at 95% indicates the value above which the value ζ is accepted. The results are based on the 10,000 option scenarios in Section 4.3.1.

Table 4.5 Pricing performance of ‘extended’ asymptotic expansions of the OEP near maturity

		A	B	C	D	E	F	G	H	I	J	A-J
MAPE	EKK	0.143%	0.322%	0.487%	0.732%	0.568%	0.502%	0.505%	0.703%	0.897%	1.172%	0.617%
	ZL	0.208%	0.496%	0.656%	0.788%	0.686%	0.660%	0.741%	0.942%	1.209%	1.446%	0.800%
	CZ	0.323%	0.499%	0.611%	0.954%	0.796%	0.679%	0.720%	0.925%	1.057%	1.448%	0.816%
	CZ-P	0.347%	0.604%	0.800%	1.188%	0.943%	0.831%	0.807%	1.042%	1.256%	1.527%	0.952%
	CZ-P-m	0.121%	0.181%	0.242%	0.480%	0.371%	0.229%	0.261%	0.468%	0.759%	1.037%	0.424%
Max Error	EKK	3.123%	4.612%	5.096%	6.717%	5.801%	5.293%	4.354%	4.221%	8.214%	8.184%	8.214%
	ZL	4.456%	6.624%	8.681%	9.104%	7.988%	6.936%	5.943%	5.265%	5.414%	6.393%	9.104%
	CZ	3.528%	4.256%	4.566%	5.415%	5.837%	5.166%	4.663%	5.061%	4.785%	5.323%	5.837%
	CZ-P	3.577%	4.898%	5.290%	6.960%	6.516%	6.320%	5.357%	5.917%	6.110%	5.461%	6.960%
	CZ-P-m	4.242%	4.150%	4.776%	2.543%	2.269%	2.103%	2.871%	2.920%	4.054%	5.461%	5.461%
% Good	EKK	98.239%	90.848%	82.552%	74.094%	78.730%	83.140%	81.529%	69.442%	61.123%	50.424%	76.371%
	ZL	96.006%	85.588%	80.128%	77.601%	77.345%	77.754%	69.675%	58.176%	45.864%	40.381%	70.078%
	CZ	92.000%	84.298%	80.413%	73.155%	78.427%	79.788%	76.030%	65.572%	58.643%	37.014%	71.934%
	CZ-P	90.400%	82.927%	77.609%	63.695%	72.892%	75.661%	73.913%	61.734%	50.790%	35.406%	67.831%
	CZ-P-m	99.620%	99.104%	99.128%	86.373%	92.537%	96.614%	93.242%	81.751%	67.158%	52.892%	86.402%
No. of scenarios		743	941	963	978	986	991	990	992	990	993	9,567

Note: This table presents for five asymptotic expansions methods (coupled with the ‘extension’ method) the mean absolute percentage error (MAPE), the maximum relative error and the percentage of ‘good’ solutions. The methods considered are: (EKK) the method in Evans et al. (2002); (ZL) the method in Zhang and Li (2010); (CZ),(CZ-P)-(CZ-P-m) the method in Cheng and Zhang (2012) basic, with Pade’ approximation and with Pade’ approximation corrected for Evans et al. (2002), respectively. Ten ranges of maturities (in years) are considered: $(0;0.5]$ (A), $(0.5;1]$ (B), $(1;1.5]$ (C), $(1.5;2]$ (D), $(2;2.5]$ (E), $(2.5;3]$ (F), $(3;3.5]$ (G), $(3.5;4]$ (H), $(4;4.5]$ (I), $(4.5;5]$ (J). The results are based on 1,000 simulated scenarios for each maturity range drawn from the distribution indicated in Broadie and Detemple (1996) and summarised in Section 4.3. In the last row, for each maturity range, there is the number of options with an ‘exact’ price above 0.5 and a maturity of longer than 2 weeks. The values for the ‘extension’ method are calculated for the value $T - t_x$ which has the smallest MAPE linked to it. All of the results are calculated considering one exercise date per day (7 days a week).

Table 4.6 Delta-hedging comparison for different maturities: 'Extended' asymptotic expansions of the OEP

ttm	1 (year)	2	3	4	5
		$\mu = 0.05$			
EKK	0.47352	0.76189	1.00068	1.20457	1.36923
ZL	0.55815	0.80576	1.02811	1.24195	1.39007
CZ	0.39186	0.72313	0.97802	1.19328	1.36048
CZ-P CZ-P-m	0.43306	0.74524	0.99166	1.20065	1.36234
		$\mu = 0.06$			
EKK	0.56	0.86987	1.11097	1.29713	1.42159
ZL	0.64413	0.91436	1.14728	1.34103	1.44488
CZ	0.49471	0.8289	1.08657	1.28004	1.40973
CZ-P CZ-P-m	0.526	0.84986	1.10056	1.29017	1.41367
		$\mu = 0.07$			
EKK	0.47593	0.77894	1.02059	1.21979	1.37429
ZL	0.56951	0.82384	1.06247	1.26565	1.39759
CZ	0.39673	0.73723	0.99649	1.20358	1.36231
CZ-P CZ-P-m	0.43222	0.75699	1.00973	1.21248	1.36434

Note: This table presents the average quadratic hedging error for five 'extended' asymptotic-expansion methods: (EKK) the method in Evans et al. (2002); (ZL) the method in Zhang and Li (2010); (CZ), (CZ-P) and (CZ-P-m) the method in Cheng and Zhang (2012) basic, with Pade' approximation and with Pade' approximation corrected for Evans et al. (2002), respectively. CZ-P and CZ-P-m are considered together since $\delta < r$. Details on the analysis are in the caption of Table 4.4.

Hedging performance

Table 4.6 summarises the performance of the five 'extended' aOEP methods when employed to hedge the 15 option scenarios introduced in Section 4.3.2. Also for this study, the error measure is the average quadratic hedging error.

As for the standard methods, as for the extended aOEPs, the hedging performance is very similar across the different methods with ZL having the worst and CZ having the best performance across the 5 methods studied. Overall, comparing Table 4.4 and Table 4.6, the hedging performances of the aOEPs are better than the ones of the standard methods. However, the five aOEPs are generally worse that the 'extended' methods across all of the 5 maturities with the exception of the one-year-maturity options, where the aOEP methods perform better.

Analysis

Overall, we think that the performance of the aOEP methods are impressive. Although their performance is not as good as that of the 'extended' methods for standard quasi-analytic methodologies, they are very precise in pricing and hedging American options and they have the advantage of being fast since they do not require the solution of integral equations as many standard methods do.

4.5 Conclusions

Most of the quasi-analytic methods currently used for pricing and hedging American options are more likely to perform better for short-maturity options than for long-maturity ones. This is because of the nature of the approximation algorithms available in the literature, which achieve higher pricing and hedging performance, the smaller the time-step size employed (i.e. the higher the number of exercise-dates considered).

In this chapter, we proposed a quasi-analytic method, which we referred to as the 'extension' method, which has the potential to improve the performance of any quasi-analytic pricing and hedging method for long-maturity options under geometric Brownian motion. The idea underpinning our new method is to use a flat approximation of the optimal exercise price near the beginning of the contract, where the theory suggests that this function is almost constant, combined with existing pricing approaches near the contract expiration. Our extensive scenario-based study shows the usefulness of the 'extension' method in improving the pricing and hedging performance of 6 well-known quasi-analytic methodologies where the fair benchmark prices are calculated by the binomial tree method of Cox et al. (1979): Geske and Johnson (1984), Barone-Adesi and Whaley (1987), Kim (1990), Ju and Zhong (1999), Chung and Shih (2009) and Li (2010b). This study shows that the 'extension' method achieves a remarkable reduction in the pricing and hedging errors over each of the 6 existing methods for a wide range of time-to-maturities. In particular, the new method leads to sizeable improvements for long-maturity options, where the existing methodologies usually fail, although it also improves on these standard methods for maturities shorter than 6 months.

Additionally, we showed that our method can successfully incorporate methods that calculate asymptotic expansions of the optimal exercise price near maturity. These are the cutting-edge methods for American options pricing (Barone-Adesi, 2005) and provide precise estimations of the optimal exercise price for maturities of the order of months. Currently, much research incorporates these methods into numerical algorithms (mainly the binomial tree method of Cox et al. (1979)) for pricing and hedging longer-maturity options. However, in doing so, the advantage of having an analytic formula for the optimal exercise price is lost because the method of Cox et al. (1979) and the other numerical methods require intense calculation and consequently the pricing process is very time consuming. In this chapter, we used these asymptotic formulae into our new method. We studied 3 asymptotic-expansion methods (Evans et al. (2002), Zhang and Li (2010) and Cheng and Zhang (2012)) and our scenario-based study showed that, if these 3 asymptotic-expansion methods are incorporated into our 'extension' method, good pricing and hedging performances are reached for options with a time-to-maturity as long as 5 years, with the advantage of retaining their analytic nature.

In Chapter 6, we strengthen these results for both standard quasi-analytic and asymptotic expansion methods by showing that the 'extension' method also improves on them when employed over real financial data. For this purpose, we will use options and LEAPS® on the S&P 100™ equity index traded from 15 February 2012 to 10 December 2014.

In what follows, we discuss some improvements to the 'extension' method that we will leave for future research.

4.5.1 Further research

At the current stage, the 'extension' method has been applied exclusively to the pricing and hedging of American options written on assets whose price dynamics follow geometric Brownian motion. This model, although still largely used in the context of American option pricing by both practitioners and academics as the recent literature shows, have many drawbacks in representing the empirical evidence on the markets (see among other Fama (1965), Bakshi et al. (1997), Bates (2000), Pan (2002) and Broadie et al. (2007)).

Further research will involve applying the idea in Section 4.2.1 to pricing long-maturity options written on underlying assets following different dynamics than the geometric Brownian motion. As in this chapter, the three key elements for this idea to be applicable are:

- the OEP of the American option has to be independent of the filtration at time t_0;
- the availability of good approximations of (quasi)-analytic pricing formulae and/or optimal exercise prices for the short-maturity options;
- the availability in closed-form of the probability density function of the underlying asset price at time $t_x \in (t_0, T]$ conditional on not hitting the flat barrier Λ between t_0 and t_x.

At the current stage, the log-normal jump-diffusion process of Merton (1976) (see SDE (3.20)) and the dual exponential jump-diffusion process of Kou (2002) (see SDE (3.22)) appear to be possible candidates. As in the case of geometric Brownian motion, since all of the parameters of the two jump-diffusion are time independent, it is shown that the OEPs are also time independent; consequently, one can employ the OEP of a short-maturity option to construct the OEP of a longer-maturity option, as we did in this chapter for options under geometric Brownian motion. Moreover, several quasi analytic formulae are available for pricing these contracts (see among others, the methods in Kou and Wang (2004) for the dual exponential diffusion process, and in Gukhal (2004) for the log-normal jump-diffusion process). Finally, several studies (e.g. Kou and Wang (2003)) have investigated the first passage times to flat boundaries for jump-diffusion processes; consequently the extension mechanism can work.

Another possible improvement over the 'extension' method is attainable by modifying the approximation of the OEP near the beginning of the contract (t_0, t_x). In particular, we think that changing the flat approximation Λ to some other functions may improve the results we have. One suitable function seems to be $\Lambda e^{x(t-t_0)}$, which corresponds to the approximation introduced by Omberg (1987) and Ju (1998). However, the degree of complexity of the new approximations and the possible performance improvement should be traded off with the loss in computational efficiency with respect to the flat approximation.

Appendix

Appendix 4.A Useful results for the proofs

We collected here some results used in the proofs of Propositions 4.2.1 and 4.2.3.

4.A.1 Results for Proposition 4.2.1

The first result is the time-u price of a perpetual put option starting at time u:

$$E_u\left[e^{-r(t_u^*(x)-u)}(K-x)\right]=\alpha(x)S_u^{\beta} \tag{4.30}$$

where $\alpha(x)$ and β are given in Proposition 4.2.1. Formula (3.26) corresponds to the formula above for $u=t_0$.

The second result is the following expectation:[9]

$$\begin{aligned}
\varphi_{t_0}^P(S_{t_0},t_x|\gamma,H,\Lambda) &= E_{t_0}\left[e^{-rt_x}S_{t_x}^{\gamma}I\left(S_{t_x}>H\right)I\left(\inf_{t\in[t_0,t_x)}S_t>\Lambda\right)\right]=\\
&= \Lambda^{\gamma}E_{t_0}\left[e^{-rt_x}e^{z_{t_x}}I\left(z_{t_x}>B_H\right)I\left(\inf_{t\in[t_0,t_x)}z_t>0\right)\right]=\\
&= \Lambda^{\gamma}\int_{B_H}^{+\infty}e^{-rt_x}e^{z}f_0(z,\gamma)dz=\\
&= e^{\lambda t_x}S_{t_0}^{\gamma}\left[N\left(d_{\varphi,1}(H)\right)-\left(\frac{\Lambda}{S_{t_0}}\right)^{\kappa}N\left(d_{\varphi,2}(H)\right)\right]
\end{aligned} \tag{4.31}$$

where $B_H=\gamma\ln\frac{H}{\Lambda}$, $z_{t_x}=\gamma\ln\frac{S_{t_x}}{\Lambda}$, $N(\cdot)$ is the standard normal cumulative distribution function (cdf),

$$d_{\varphi,1}(H) = \frac{\ln\frac{S_{t_0}}{H}+(b+(\gamma-\frac{1}{2})\sigma^2)(t_x-t_0)}{\sigma\sqrt{t_x-t_0}}, \tag{4.32}$$

[9] We prove here the result for $\gamma>0$. It is straightforward to prove that formula (4.31) also holds for $\gamma=0$.

$$d_{\varphi,2}(H) = \frac{\ln\frac{\Lambda^2}{S_{t_0}H} + (b + (\gamma - \frac{1}{2})\sigma^2)(t_x - t_0)}{\sigma\sqrt{t_x - t_0}}, \tag{4.33}$$

$$\lambda = -r + \gamma b + \frac{1}{2}\gamma(\gamma - 1)\sigma^2 \tag{4.34}$$

and $\kappa = \frac{2b}{\sigma^2} + (2\gamma - 1)$. In the above derivation we also used the expression for $f_0(z,\gamma)$, the probability density function of an arithmetic Brownian motion at time t_x with positive initial value z_{t_0}, drift parameter $b_2 = \gamma b_1$, volatility parameter $\sigma_1 = \gamma\sigma$ and an absorbing barrier at 0 (Ingersoll, 1987, p. 352):

$$f_0(z,\gamma) = \frac{n\left(\frac{z - z_{t_0} - b_2(t_x - t_0)}{\sigma_1\sqrt{t_x - t_0}}\right) - e^{-\frac{2b_2 z_{t_0}}{\sigma_1^2}} n\left(\frac{z + z_{t_0} - b_2(t_x - t_0)}{\sigma_1\sqrt{t_x - t_0}}\right)}{\sigma_1\sqrt{t_x - t_0}}. \tag{4.35}$$

For $\gamma = 1$, the formula above is equal to formula (4.3), i.e.

$$f_0(z) = f_0(z,1).$$

4.A.2 Results for Proposition 4.2.3

Defining $n(\cdot)$ as the standard normal density function, we have:

$$\begin{aligned} f_0'(z) = {} & \frac{1}{S_{t_0}\sigma\sqrt{t_x - t_0}} \\ & \times \left[e^{-\frac{2b_1 z_{t_0}}{\sigma^2}} n\left(\frac{z + z_{t_0} - b_1(t_x - t_0)}{\sigma\sqrt{t_x - t_0}}\right)\left(\frac{z + z_{t_0} - b_1(t_x - t_0)}{\sigma^2(t_x - t_0)} + \frac{2b_1}{\sigma^2}\right) \right. \\ & \left. + \frac{z - z_{t_0} - b_1(t_x - t_0)}{\sigma^2(t_x - t_0)} n\left(\frac{z - z_{t_0} - b_1(t_x - t_0)}{\sigma\sqrt{t_x - t_0}}\right) \right] \end{aligned}$$

and

$$\begin{aligned} \varphi'_{t_0,S_{t_0}}(S_{t_0},t_x|\gamma,H,\Lambda) = {} & \frac{\partial\varphi^P_{t_0}(S_{t_0},t_x|\gamma,H,\Lambda)}{\partial S_{t_0}} = e^{\lambda t_x}\left[\gamma S_{t_0}^{\gamma-1} N\left(d_{\varphi,1}(H)\right)\right. \\ & + \frac{S_{t_0}^{\gamma-1}}{\sigma\sqrt{t_x - t_0}} n\left(d_{\varphi,1}(H)\right) - \Lambda^{\kappa}(\gamma - \kappa) S_{t_0}^{\gamma-\kappa-1} N\left(d_{\varphi,2}(H)\right) \\ & \left. + \frac{\Lambda^{\kappa}}{S_{t_0}^{\kappa-\gamma+1}\sigma\sqrt{t_x - t_0}} n\left(d_{\varphi,2}(H)\right)\right]. \end{aligned}$$

Chapter 5

American Options: The Weighted Least Squares Monte Carlo Valuation Method

The least squares Monte Carlo algorithm introduced by Longstaff and Schwartz (2001) is one of the most widely applied numerical methods for pricing American-style derivatives. This chapter examines the regression step of this algorithm and proves that, although unbiased, the estimators calculated by the ordinary least squares regression method are not the best linear unbiased estimators because there is evidence of heteroscedasticity. We propose a new pricing method to account for heteroscedasticity and we demonstrate numerically that the new method reduces the upper bias that is well-known to characterise the algorithm of Longstaff and Schwartz. The chapter is structured as follows. Section 5.1 reviews the algorithm of Longstaff and Schwartz and justifies the introduction of our new pricing method. Section 5.2 is the main contribution of the chapter. It shows the existence of heteroscedasticity in each regression step of the least squares Monte Carlo method for a set of scenarios and then proves it theoretically for a wide class of underlying asset price dynamics. Section 5.3 describes our new method, numerically evaluates its pricing performance by comparing it with existing methodologies and shows that it reduces the upward pricing bias significantly. Section 5.4 concludes.

5.1 Introduction

One of the most powerful families of methods for pricing American-style options, when many exercise opportunities and many random sources are involved, is the family of the regression-based methods (Glasserman, 2003, p. 478). Unlike many other numerical methods, regression-based methodologies do not suffer from the so-called "curse of dimensionality", i.e. the exponential growth of the computational time with an increase in the number of random sources. The key feature of these methodologies is the use of Monte Carlo simulations together with a regression method to estimate numerically the continuation value of the priced contract. This turns out to be one of the main advantages of the regression-based methods since the computations are reduced to simple linear algebraic operations.

A long series of papers can be classified within this family of methods and the seminal contributions are Carriere (1996), Tsitsiklis and Van Roy (2001) and Longstaff and Schwartz (2001). While they all use regressions in a dynamic programming context, they have distinctive features: Carriere (1996) estimates the continuation value along each simulated path by employing spline regressions and regressions with a local polynomial smoother; Tsitsiklis and Van Roy (2001) similarly to Carriere (1996) also estimate the continuation value along each path but they do so by using the ordinary least squares regression (OLS) method; Longstaff and Schwartz (2001) also adopt ordinary least squares calculus but their method proceeds by finding the optimal stopping time.

In the literature, the regression-based methods for pricing American options are centred on the algorithm formulated by Longstaff and Schwartz (2001), the least squares Monte Carlo (LSMC). A justification for the great success of the LSMC is given by Stentoft (2014), who justifies its widespread use by noting that it has the smallest absolute bias and less error accumulation when compared to the methods of Carriere and Tsitsiklis and Van Roy.

In this chapter, we study in detail the regression step of the LSMC algorithm. We proceed by proving that the standard assumption of homoscedasticity does not hold for the regressions in the LSMC when it is applied to price American call and put options because the variance of the errors conditional on the explanatory variables is a function of these variables. Consequently, the errors of the regressions in the LSMC are heteroscedastic, a condition that makes the OLS estimators not the best linear unbiased estimators, BLUE (see Section 5.2). Additionally, we propose a new pricing method, which we refer to as the weighted least squares Monte Carlo (wLSMC). The new method, which is similar in structure to the LSMC, employs the weighted least squares regression (WLS) method instead of the OLS method. Consequently, we are able to retain the BLUE condition, even in the presence of heteroscedasticity, as shown by Aitken's theorem (see Section 5.2.5). In order to apply the WLS regression method, an estimation of the variance of the errors is necessary for the weights of the new method. In Section 5.3, together with the explanation of the wLSMC, we provide a closed-form approximation for the weights.

Additionally, via an extensive numerical comparison (see Section 5.3.2), we show that the LSMC tends to exhibit a large pricing bias because the OLS estimators are more prone to overfitting the continuation value curve. Finally, we show that correcting for heteroscedasticity via the wLSMC also corrects for curve overfitting problems and leads to a much smaller upward bias.

5.1.1 Literature review on the least squares Monte Carlo method

Many researchers have examined the LSMC algorithm, from both theoretical and computational perspectives.

On the theoretical side, the convergence of the LSMC has been studied in Longstaff and Schwartz (2001), Clement et al. (2002) and Stentoft (2004b). Longstaff and Schwartz (2001) proved the convergence for problems with one state variable and only one exercise date (except maturity). Clement et al. (2002) showed that, for a given set of basis functions, the error resulting from Monte Carlo simulations converges to zero when the number of simulated paths goes to infinity. Within a multi-dimensional and multi-period setting, Stentoft (2004b) proved convergence as the number of basis functions M and number of paths $\mathtt{n_S}$ go to infinity with $M^3/\mathtt{n_S} \to 0$. Egloff (2005) and Zanger (2009, 2013) improved on these convergence results by using statistical learning. Additionally, Mostovyi (2013) studied the stability of the LSMC near the beginning of the contract, for a time-step size approaching zero and found that the regression problem is ill-posed and thus the LSMC is unstable in these settings. Klimek and Pitera (2014) extended the results of Clement et al. (2002) to non-Markovian processes and path-dependent payoffs.

On the computational side, Moreno and Navas (2003), Stentoft (2004a) and Areal et al. (2008) assessed the pricing performances of the LSMC under different numbers of simulated paths, payoff structures and polynomial families in the regressions. They provide evidence that the performance of the LSMC is virtually the same for vanilla options when different polynomial families are employed and that their selection has a major impact in the case of exotic options. Bolia et al. (2004), Lemieux and La (2005), Areal et al. (2008) and Juneja and Kalra (2009), among others, tested different variance-reduction techniques and quasi-random sequences for the LSMC algorithm. Rasmussen (2002) used the dispersed path technique to improve the estimation of the optimal exercise price in the LSMC. Similarly, Wang and Caflisch (2010) modified the LSMC to straightforwardly calculate the delta and gamma parameters of the options. Remarkably, Kan et al. (2009) proposed an asymptotic correction for the pricing bias in the LSMC (and other Monte Carlo methods) and successfully employed it for multi-asset options. Stentoft (2005) adapted the LSMC to price American options under GARCH models. AitSahlia et al. (2010) priced American options under the Heston model by modifying the LSMC algorithm: their method estimates the optimal exercise surface (the equivalent of the optimal exercise price for bi-dimensional processes) via the LSMC and then employs it in the early exercise decomposition derived by Chiarella and Ziogas (2006).

Particularly relevant to the topic of this chapter are the studies carried out on the regression step of the LSMC algorithm and the price bias it may cause. The literature has shown that the pricing bias in the LSMC is a combination of the downward bias caused by the approximation of this curve by a finite low-dimensional polynomial, and the upward bias caused by using the same paths to estimate the optimal stopping time, and is consequently linked to overfitting the regression curve (see, among others, Létourneau and Stentoft (2014)).

Glasserman and Yu (2004) proposed the "regression later" algorithm (as opposed to the "regression now" algorithm in Longstaff and Schwartz (2001)) for American option pricing. The "regression later" algorithm regresses the continuation value on functions of the spot price at the current time rather than on values at the previous time-step as the LSMC does. Although it has the advantage of having estimators with smaller variances, it requires martingale basis functions that may be difficult to find in multidimensional settings. Tompaidis and Yang (2014) investigated different regression techniques as alternatives to the OLS method. They examined quantile regression, Tikhonov regularization, matching projection pursuit, classification and regression trees and a non-parametric method. However, by doing so, they lost the unbiasedness property of ordinary least squares estimators, which we are able to retain by employing the wLSMC. Létourneau and Stentoft (2014) employed the linear inequality constrained least squares method (ICLS) to impose monotonicity and convexity properties on the continuation value curve as the theoretical results reviewed in Section 3.2.1 suggest (for an outline of the ICLS methodology see Appendix 5.A).[1] In a numerical study, Létourneau and Stentoft showed that the ICLS algorithm is less prone to curve-overfitting than the standard LSMC and that, consequently, the upward pricing bias is significantly reduced. In Section 5.3.2, we numerically compare the wLSMC method with the ICLS and show that they both succeed in reducing the bias in the LSMC and that, in many cases, our methodology outperforms the inequality constrained least squares Monte Carlo method.

Finally, we enumerate some applications that have successfully employed the LSMC algorithm in fields other than the American option pricing problem: Longstaff (2005) employed the LSMC to value mortgage-backed securities in a multifactor framework; Sabour and Poulin (2006) adapted LSMC to value real capital investments with examples in the valuation of copper-extraction projects; Bacinello et al. (2010) priced life insurance con-

[1]One can show, equivalently to what we prove in this chapter, that also the ICLS method is affected by heteroscedasticity, while the "regression later" approach by Glasserman and Yu (2004) is not affected since the continuation value is regressed on the current spot price.

tracts with surrender guarantees; Carmona and Ludkovski (2010) adapted the LSMC to optimal switching models with inventory to evaluate energy storage facilities (natural gas dome storage and hydroelectric pumped storage); Jarrow et al. (2010) priced callable bonds via the LSMC and showed that the same technique can be applied to mortgage-backed securities. Recently, Carmona and Hinz (2011) studied a storage management problem, where they controlled over time the level of the commodity stored in the facility to maximize the returns.

5.1.2 Description of the least squares Monte Carlo method

Let us consider on the filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{Q})$ the financial market consisting of three assets as described in Section 3.1: a bank account with constant risk-free interest rate r; a risky asset with price dynamics $\{S_t\}_{t\geq t_0}$; and, an American-style derivative written on the risky asset. Then, we restrict the pricing of American options to contracts that can be exercised only to a fixed set of exercise opportunities $t_1 < t_2 < \ldots < t_m = T$ and $t_0 = 0$, the time of evaluation, is not usually part of this set. Henceforth, we borrow the notation introduced in Section 3.1, which we briefly summarise below: S_i is the underlying asset price at the ith exercise opportunity (the one at time t_i); s_{i+1} is the logarithmic return over the period (t_i, t_{i+1}); $h_i(\mathscr{X})$ is the payoff function in time-0 dollars for exercise at time t_i when the current state of the underlying asset is $S_i = \mathscr{X}$; $r_{0,i}$ is the discount factor as defined in formula (3.7); $V_i(\mathscr{X})$ is the value in time-0 dollars of the American-style derivative at time t_i given $S_i = \mathscr{X}$ (assuming the option has not been previously exercised); $C_i(\mathscr{X})$ is the continuation value of the American-style derivative measured in time-0 dollars conditional on the current state $\mathscr{X}$ (see formula (3.8)); and, S_{f_i} is the optimal exercise price at time t_i.

The LSMC method solves the dynamic programming problem in (3.10)-(3.11), by combining Monte Carlo simulations and the OLS regression method. In particular, given a set of $\mathtt{n_S}$ simulated paths of the Markovian process $\{S_t\}_{t\geq 0}$, a set of $M+1$ basis functions[2] $\psi_l(\cdot) : \Re \mapsto \Re$ and a set of $M+1$ parameters $\beta_{i,l} \in \Re$, $l = 0, \cdots, M$, for any time t_i with $i = 1, \ldots, m-1$, Longstaff and Schwartz employ the following expression for the continuation value of an American-style derivatives (see equation (3.8)):

$$\hat{C}_i(\mathscr{X}) = \sum_{l=0}^{M} \beta_{i,l} \psi_l(\mathscr{X}), \tag{5.1}$$

[2]It is usually required that $\psi_0(\cdot) = 1$.

and the OLS method to calculate the parameters $\beta_{i,l}$ from the observations of pairs

$$\left(S_{i(j)}, V_{i+1}\left(S_{i+1(j)}\right)\right), j = 1 \ldots, \mathtt{ns}$$

where $S_{i(j)}$ indicates the value of process $\{S_t\}_{t\geq 0}$ at time t_i for the simulated path j-th. In particular, the LSMC algorithm goes through the steps in the box below (LSMC).

LSMC Steps of the (ordinary) least squares Monte Carlo algorithm

1. Simulate $\mathtt{ns}$ independent paths $\{S_{1(j)}, \cdots, S_{m(j)}\}, j = 1, \cdots, \mathtt{ns}$,
2. Set the option terminal-value equal to $V_m\left(S_{m(j)}\right) = h_m\left(S_{m(j)}\right)$, $j = 1, \ldots, \mathtt{ns}$,
3. Using backward dynamic programming for $i = m-1, \cdots, 1$,
   (a) Select the set $\tilde{J}_i$ of in-the-money paths at time-step i, i.e. $\tilde{J}_i = \{j | h_i\left(S_{i(j)}\right) > 0\}$,
   (b) Run an OLS regression on the pairs $\left(S_{i(j)}, V_{i+1}\left(S_{i+1(j)}\right)\right)$ for $j \in \tilde{J}_i$, with basis functions $\psi_l(\cdot)$, to determine $\beta_{i,l}$,
   (c) For each $j \in \tilde{J}_i$ set

$$V_i\left(S_{i(j)}\right) = \begin{cases} h_i\left(S_{i(j)}\right), & h_i\left(S_{i(j)}\right) \geq \hat{C}_i\left(S_{i(j)}\right); \\ V_{i+1}\left(S_{i+1(j)}\right) & h_i\left(S_{i(j)}\right) < \hat{C}_i\left(S_{i(j)}\right) \end{cases} \tag{5.2}$$

   with $\hat{C}_i(\cdot)$ as in (5.1) and $\beta_{i,l}$ found in step 3.(b). For $j \in \{1, \cdots, \mathtt{ns}\} \setminus \tilde{J}_i$ (out-of-the money paths), set $V_i\left(S_{i(j)}\right) = V_{i+1}\left(S_{i+1(j)}\right)$,
4. Set $V_0(S_0) = \frac{1}{\mathtt{ns}} \sum_{j=1}^{\mathtt{ns}} V_1\left(S_{1(j)}\right)$.

5.2 Heteroscedastic errors in the least squares Monte Carlo method

Let us consider the OLS regression at any time-step $i = 1, \ldots, m-1$ of step 3b in the LSMC algorithm. Additionally, let us define u_i as the error of the time-t_i regression given the current price S_i:

$$u_i = V_{i+1}(S_{i+1}) - C_i(S_i), \tag{5.3}$$

which is a random variable dependent on $S_{i+1} = S_i e^{s_{i+1}}$. These errors are homoscedastic if

$$\mathrm{Var}[u_i | S_i = \mathscr{X}] = c,\, c \in \Re^+, \forall \mathscr{X} \in \Re^+. \tag{5.4}$$

We note that in (5.4) it is required that the variance of the errors u_i is equal to a constant c for any value of the underlying asset price $\mathscr{X}$.

The aim of this section is to show that there exist some underlying asset prices $\mathscr{X}_1 \in \Re^+$ and $\mathscr{X}_2 \in \Re^+$ such that

$$\mathrm{Var}[u_i | S_i = \mathscr{X}_1] \neq \mathrm{Var}[u_i | S_i = \mathscr{X}_2].$$

This means that the conditional variance of errors changes with the underlying spot price, a condition that is usually defined as heteroscedasticity of the errors.

In what follows, first, we demonstrate numerically via three statistical tests (see Section 5.2.2), and graphically (see Section 5.2.3) the presence of heteroscedasticity for a set of 320 option scenarios (described in Section 5.2.1) with an underlying asset price following any of the four well-known dynamics reviewed in Section 3.1.1. Together with these scenario-based proofs, we also provide a formal proof of heteroscedasticity (see Section 5.2.4) for an underlying asset whose price follows the general dynamics $S_t = S_0 e^{s_t}$ where $\{s_t\}_{t \geq t_0}$ is a Markovian process. In what follows, we make use of the equality

$$\mathrm{Var}[u_i | S_i = \mathscr{X}] = \mathrm{Var}[V_{i+1}(S_{i+1}) | S_i = \mathscr{X}], \tag{5.5}$$

which follows from (5.3).

5.2.1 Description of the option scenarios

Let us describe the option scenarios we will use in Sections 5.2.2 and 5.2.3 and for the numerical comparison in Section 5.3.2. The option scenarios are American put and call options on an underlying asset with one of the four price dynamics described mathematically in Section 3.1.1: the geometric Brownian motion with SDE (3.16), the exponential Ornstein-Uhlenbeck process with SDE (3.18), the log-normal jump-diffusion process with SDE (3.20), and the double exponential jump-diffusion process with SDE (3.22).

Option scenarios under different models

For our numerical exercises, we construct two sets, each consisting of 160 scenarios. The scenarios in the first group are American put options and the other are American call options.

They are all written on assets whose risk-neutral price dynamics is one of the four above. The scenarios in the sets are spanned by the values for the parameters associated with the underlying processes.

For both call and put options, under geometric Brownian motion, the study is carried out on the 20 scenarios (rescaled for the strike price) described in Table 1 of Longstaff and Schwartz (2001). The parameters are $S_0 \in \{0.9, 0.95, 1, 1.05, 1.1\}$, $\sigma \in \{0.2, 0.4\}$, $T = \{1, 2\}$ year(s), $r = 6\%$ and $K = 1$. For the other three processes, we use the following additional parameters: the exponential Ornstein-Uhlenbeck process has $\eta \in \{0.15, 0.3\}$, $\mu = \{0, \log(0.9)\}$ and $T = 1$ year; the log-normal jump-diffusion process has $\lambda \in \{0.5, 1\}$, $\alpha_J \in \{-0.25, 0.25\}$, $\sigma_J \in \{0.2, 0.4\}$ and $T = 1$ year, and the double exponential jump-diffusion process has $q = 0.5$, $\lambda = 0.5$, $(\eta_1, \eta_2) \in \{(2,3),(4,6)\}$ and $T = 1$ year. Additionally, for the American put case we consider options on non-dividend paying assets $(\delta = 0\%)$ as in Longstaff and Schwartz (2001); on the other hand, we assume a value for the dividend yield of $\delta = 3\%$ for the American call options since an American call option for a non-dividend paying asset is equivalent to an European option (Merton, 1973) and, consequently, there is no need to apply these numerical algorithms.

Overall, for each of the two sets, we consider 20 option scenarios under geometric Brownian motion, 40 scenarios under the exponential Ornstein-Uhlenbeck process, 80 scenarios under the log-normal jump-diffusion process and 40 scenarios under the double exponential jump-diffusion process.

5.2.2 Statistical tests for heteroscedasticity

In this section, we run three alternative tests to detect heteroscedasticity in the regressions 3b in the LSMC (for all time-step $i = m-1, \dots, 1$), for each of the 320 scenarios described above. In particular, we consider:

- Park's test (see Park (1966));
- White's general heteroscedasticity test (see White (1980));
- Breusch-Pagan-Godfrey test (BPG, see Breusch and Pagan (1979), Godfrey (2008)),

which are briefly described in Appendix 5.B and are based on the residuals

$$\hat{u}_{i,j} = V_{i+1}\left(S_{i+1(j)}\right) - \hat{C}_i\left(S_{i(j)}\right), j \in \tilde{J}_i, \tag{5.6}$$

where as before the index j indicates the j-th simulated path and $\tilde{J}_i$ is the set of the in-the-money paths at time-step i.

For each of the 320 scenarios in the two sets described above, we implemented the LSMC method with 50 time-steps per year ($\Delta_T = 0.02$ year), basis functions

$$\psi_l(\mathscr{X}) = \mathscr{X}^l, l = 0, \ldots, 3 \tag{5.7}$$

and $\mathtt{n_S} = 5,000$ to estimate the continuation value in equation (5.1). Overall, for each of the two sets of scenarios we have:

- 1,480 regressions under geometric Brownian motion (10 scenarios with $T = 1$ year $\times$ (50-1) regressions + 10 scenarios with $T = 2$ year $\times$ (100-1) regressions);
- 1,960 regressions under the exponential Ornstein-Uhlenbeck process (40 scenarios with $T = 1$ year $\times$ (50-1) regressions);
- 3,920 regressions under the log-normal jump-diffusion process (80 scenarios with $T = 1$ year $\times$ (50-1) regressions);
- 1,960 regressions under the double exponential jump-diffusion process (40 scenarios with $T = 1$ year $\times$ (50-1) regressions).

American put options In Table 5.1, we present the percentage of regressions for which it was not possible to reject homoscedasticity for the American put option scenarios.

Table 5.1 Statistical tests for heteroscedasticity: American put options

	Statistical test					
	Park's		**White's**		**BPG**	
Critical Value	1%	5%	1%	5%	1%	5%
GBM	2.30%	2.03%	4.05%	2.84%	9.19%	7.36%
Exp. Ornstein-Uhlenbeck	4.18%	3.27%	5.77%	3.93%	13.01%	11.02%
Log-normal jumps	2.19%	1.43%	10.38%	6.61%	9.54%	7.17%
Double Exp. jumps	0.56%	0.36%	12.45%	6.79%	6.94%	4.08%

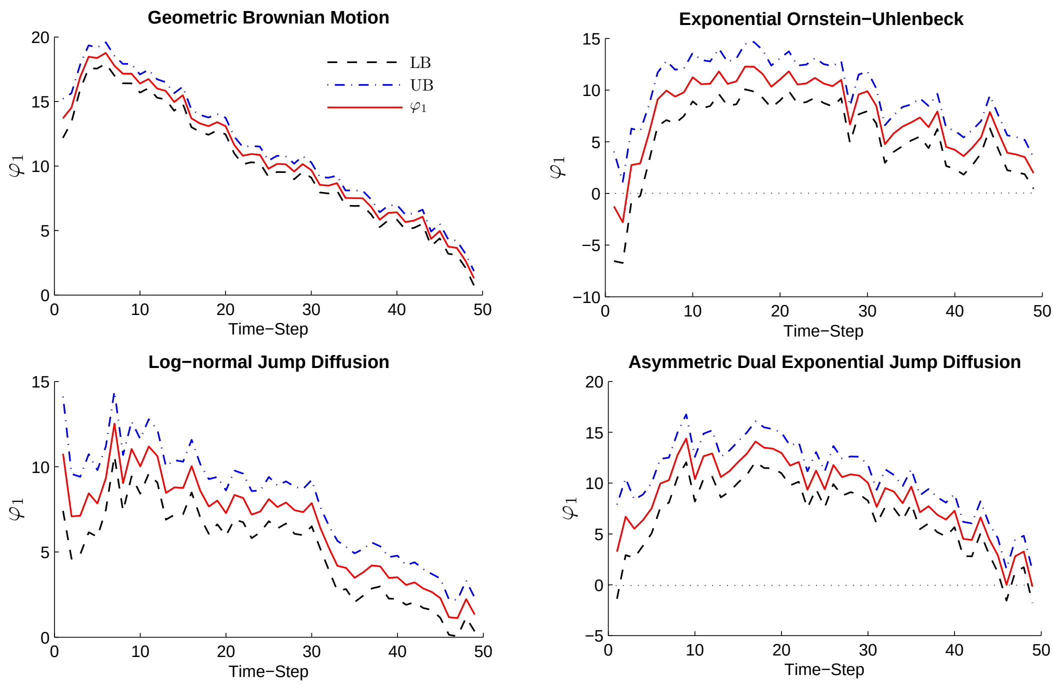
Note: The entries in the table are the percentage of time for which it is not possible to reject the null hypothesis of homoscedasticity for the regressions in the LSMC algorithm for the 160 put option scenarios considered. A low percentage indicates serious evidence of heteroscedascity among the option scenarios considered.

Although these three tests, to some extent, returned different results, it is clear from Table 5.1 that heteroscedasticity is an issue in the vast majority of the regressions for the 160 scenarios considered. The majority of the percentages shown in the table are below 10% and this proves numerically that for the scenarios considered, homoscedasticity is rejected in about 90% of the cases. Moreover, as an example, in Figure 5.1 we provide the result of Park's test over each regression of the LSMC algorithm for four of the options in the set. In particular, we plot the value of the parameter φ_1 in the equation

$$\log \hat{\boldsymbol{u}}_i^2 = \varphi_0 + \varphi_1 \log \boldsymbol{S}_i,$$

which, as we review in Appendix 5.B, are the ordinary least squares estimators over the residual $\hat{u}_i$ of a LSMC regression. Additionally, we plot the confidence interval for φ_1 for a critical level of 1%. Heteroscedasticity characterizes the regressions for which φ_1 is significantly different from zero, i.e. regressions with confidence intervals that do not contain zero. The four option scenarios considered have $S_0 = 0.9$, $\sigma = 20\%$, $r = 6\%$, $T = 1$ and $K = 1$. Moreover, the exponential Ornstein-Uhlenbeck process has $\eta = 0.15$ and $\mu = 0$; the log-normal jump-diffusion has $\alpha_J = -0.25$, $\sigma_J = 0.2$ and $\lambda = 0.5$; and the double exponential jump-diffusion has $\lambda = 0.5$, $\eta_1 = 2$, $\eta_2 = 3$ and $q = 0.5$.

Figure 5.1 Park's test for heteroscedasticity: American put options



Note: The four plots report the coefficient φ_1 (solid lines), and the extrema of the confidence interval when using a critical level of 1% for each of the 49 regressions ($T = 1$ year and $\Delta_T = 0.02$ years) $\log \hat{\boldsymbol{u}}_i^2 = \varphi_0 + \varphi_1 \log \boldsymbol{S}_i$ (see Appendix 5.B) for the LSMC algorithm applied to four underlying price dynamics. The option scenarios considered in the plot have $S_0 = 0.9$, $\sigma = 20\%$, $r = 6\%$, $T = 1$ and $K = 1$. Moreover, the exponential Ornstein-Uhlenbeck process has $\eta = 0.15$ and $\mu = 0$; the log-normal jump-diffusion has $\alpha_J = -0.25$, $\sigma_J = 0.2$ and $\lambda = 0.5$; the double exponential jump-diffusion has $\lambda = 0.5$, $\eta_1 = 2$, $\eta_2 = 3$ and $q = 0.5$.

American call options On the other hand, Table 5.2 presents the percentage of regressions where homoscedasticity could not be rejected for the 160 American call option scenarios. Although, as for the put options, there are some differences in the results among the three statistical tests, the table shows that there is evidence of heteroscedasticity in most of the regressions of the LSMC when it is applied to price American call options. An exception is the double exponential jump-diffusion process, where White's test does not reject homoscedasticity in the majority of cases. However, Park's test and the Breusch-Pagan-Godfrey test support the hypothesis of the existence of heteroscedasticity with percentages comparable with the other three dynamics. Additionally, as for the American put scenarios, Figure 5.2 shows the values of φ_1 (see also Section 5.B) for the same options in Figure 5.1 with underlying spot price $S_0 = 1.1$.

Table 5.2 Statistical tests for heteroscedasticity: American call options

	Statistical test					
	Park's		**White's**		**BPG**	
Critical Value	1%	5%	1%	5%	1%	5%
GBM	21.62%	16.15%	7.97%	4.86%	4.32%	2.97%
Exp. Ornstein-Uhlenbeck	13.78%	10.77%	10.66%	7.81%	11.12%	7.96%
Log-normal jumps	21.33%	16.45%	26.20%	19.67%	3.04%	2.04%
Double Exp. jumps	29.74%	23.42%	65.20%	59.85%	6.58%	3.98%

Note: The entries in the table are the percentage of time for which it is not possible to reject the null hypothesis of homoscedasticity for the regressions in the LSMC algorithm for the 160 call option scenarios considered. A low percentage indicates serious evidence of heteroscedascity among the option scenarios considered.

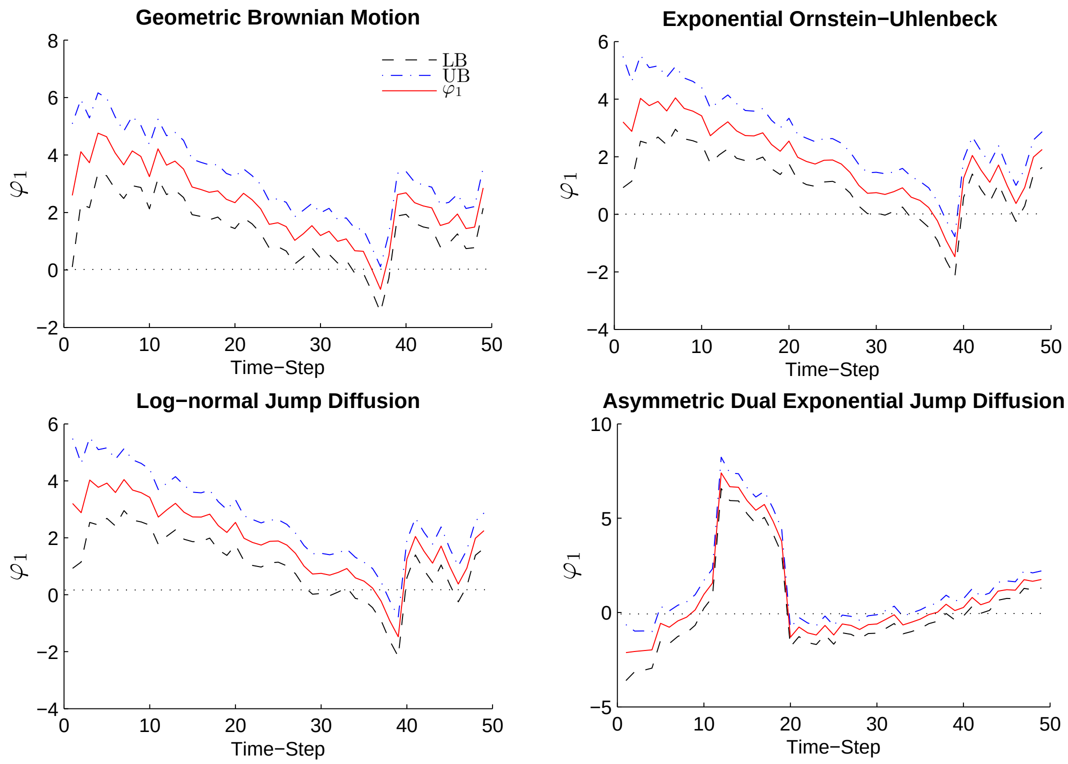
5.2.3 Graphical proof of heteroscedasticity

This section provides a new numerical technique that shows graphically the conditional standard deviation of the errors as a function of the values of the underlying asset prices and, consequently, provides evidence of the existence of heteroscedasticity for any regression in the LSMC algorithm. The reason for considering such an alternative method is that the statistical tests considered above are based on the residuals in (5.6) rather than the (theoretical) errors in (5.3) and one may erroneously conclude that heteroscedasticity depends on the selection of the basis functions $\psi_l(\cdot)$ for the regressions at step 3b.

We use the equivalent of formula (5.5) for the standard deviation,

$$\mathrm{std}[u_i|S_i = \mathscr{X}] = \mathrm{std}[V_{i+1}(S_{i+1})|S_i = \mathscr{X}], \tag{5.8}$$

Figure 5.2 Park's test for heteroscedasticity: American call options



Note: The four plots report the coefficient φ_1 (solid lines), and the extrema of the confidence interval when using a critical level of 1% for each of the 49 regressions ($T = 1$ year and $\Delta_T = 0.02$ years) $\log \hat{\boldsymbol{u}}_i^2 = \varphi_0 + \varphi_1 \log \boldsymbol{S}_i$ for the LSMC algorithm applied to four underlying price dynamics, with $S_0 = 1.1$, $\sigma = 20\%$, $r = 6\%$, $T = 1$ and $K = 1$. For other information, see Figure 5.1.

and estimate the conditional standard deviation on the right-hand side via the Monte Carlo simulation technique. The calculations we carry out here differ from those in Section 5.2.2, since we estimate the conditional standard deviation of the errors for each regression without making use of the LSMC algorithm and, therefore, we circumvent the selection of the basis functions, as shown below.

For this graphical proof, we consider the time-steps $i = 1, \ldots, T/\Delta_t$ with $\Delta_t = 0.1$ years and discretise[3] the underlying spot price range as $S = 0, \Delta_S, \ldots, K$ for the put options and as $S = K, K+\Delta_S, \ldots, 2K$ for the call options with step size $\Delta_S = 0.05$. For each point on the grid (i, S), we simulate $N_h = 100$ stock prices at the next time-step $(i+1)$ conditional on the spot price at the current time-step (S). We then price the American options for each of the N_h underlying spot prices at time t_{i+1} using the binomial tree method: for the geometric Brownian motion case we use the binomial tree method in Cox et al. (1979) with 1,000 steps, for the exponential Ornstein-Uhlenbeck case we employ the binomial tree in Nelson and Ramaswamy (1990) with 1,000 time-steps and for the two jump processes (log-normal and double exponential) we apply the binomial tree method in Amin (1993) with 250 time-steps. Then, we calculate the standard deviation of the N_h prices described above.

Figures 5.3 and 5.4 illustrate the standard deviations of the errors for the same four American options considered in Figures 5.1 and 5.2. Each cross section along the i axis is the standard deviation of the regression errors conditional on the underlying asset price being S at time-step i. The plots in Figures 5.3 and 5.4 indicate that the conditional standard deviation changes with the level of price S and that consequently, for the eight selected options, the errors are heteroscedastic. These patterns can be observed for all of the other scenarios and, consequently, there is graphical evidence of heteroscedasticity for the 320 option scenarios considered.

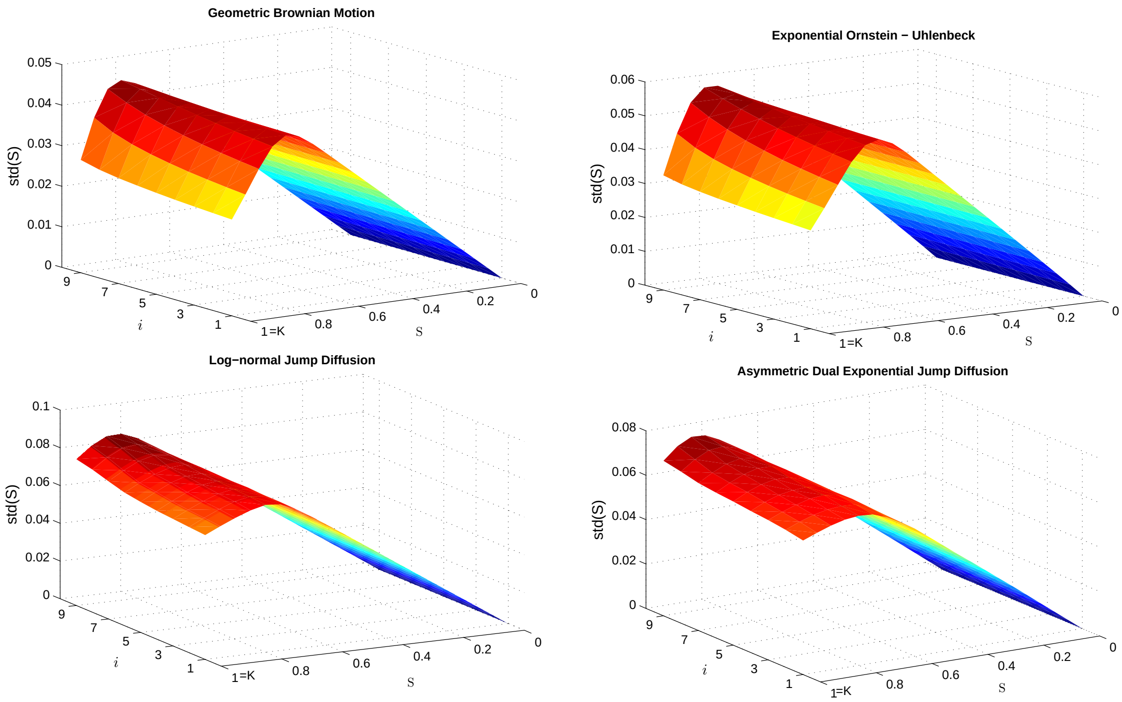
5.2.4 Formal proof of heteroscedasticity

In this section, we generalise the results in the two previous sections. In particular, we prove that, for well-known underlying price dynamics, there is heteroscedasticity of errors in the regressions of the LSMC algorithm when it is employed to price American call and put options.

First, let us consider the American put option written on the risky asset whose price dynamics is $S_t = S_0 e^{s_t}$, for which the first two conditional moments are finite and which is defined for $S_0 > 0$ and $\{s_t\}_{t \geq 0}$ being a Markovian process with $s_0 = 0$. Then, we can prove the following result.

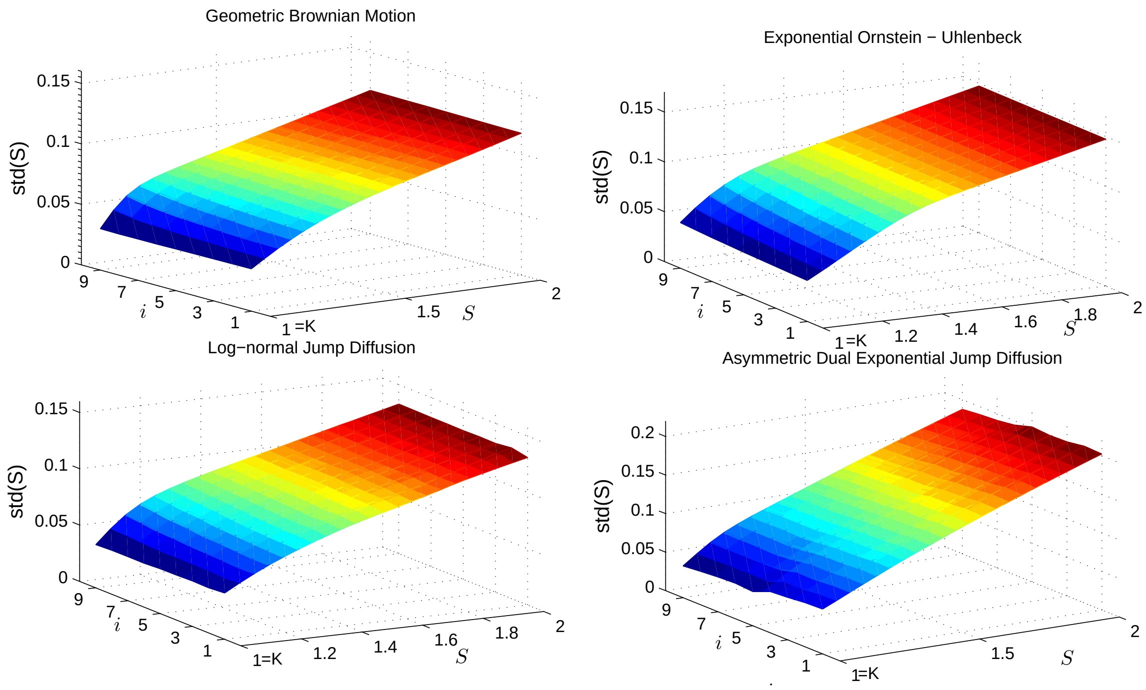
[3] As in the LSMC algorithm, we consider in-the-money paths only.

Figure 5.3 Heteroscedasticity in the regressions of the LSMC algorithm via simulation: American put options



Note: The plots report the conditional standard deviations $\text{std}(S) = \sqrt{\text{Var}[u_i|S_i = S]}, i = 1,\ldots,9$ calculated using formula (5.5) for American put options under four price dynamics. All price dynamics have $\sigma = 20\%$, $r = 6\%$, $K = 1$ and $T = 1$. Moreover, the exponential Ornstein-Uhlenbeck process has $\eta = 0.15$ and $\mu = 0$; the log-normal jump-diffusion has $\alpha_J = -0.25$, $\sigma_J = 0.2$ and $\lambda = 0.5$; the double exponential jump-diffusion has $\lambda = 0.5$, $\eta_1 = 2$, $\eta_2 = 3$ and $q = 0.5$. The plots are created for a grid with $\Delta_S = 0.05$ and $\Delta_T = 0.1$ years. For each point on the grid (i, S), $N_h = 100$ (50+50 antithetic) simulations of S_{i+1} (conditional on $S_i = S$) are calculated together with the price of the option with time-to-maturity $T - t_{i+1}$ and underlying spot price S_{i+1}. The option prices are calculated using the binomial tree method. The plotted points represent the standard deviation of the N_h prices.

Figure 5.4 Heteroscedasticity in the regressions of the LSMC algorithm via simulation: American call options



Note: The plots report the conditional standard deviations $\text{std}(S) = \sqrt{\text{Var}[u_i|S_i = S]}, i = 1,\ldots,9$ calculated using formula (5.5) for American call options under four price dynamics. For other information, see Figure 5.4.

Proposition 5.2.1 (American put options)**.** *The errors of the regressions in the LSMC algorithm for the American put option are heteroscedastic.*

Proof. Let us consider the regression at any given time-step $i = 1,\dots,m-1$. We proceed by showing that for any[4] $\mathscr{X}_2 \in (0,K]$ there exists $\mathscr{X}_1 \in (0,\mathscr{X}_2)$ such that

$$\mathrm{Var}[u_i|S_i = \mathscr{X}_1] < \mathrm{Var}[u_i|S_i = \mathscr{X}_2]$$

or, equivalently (see (5.5)),

$$\mathrm{Var}[V_{i+1}(S_{i+1})|S_i = \mathscr{X}_1] < \mathrm{Var}[V_{i+1}(S_{i+1})|S_i = \mathscr{X}_2] \tag{5.9}$$

holds. The proof consists of showing that $\mathscr{X}_1$ and $\mathscr{X}_2$ are such that a lower bound of $\mathrm{Var}[V_{i+1}(S_{i+1})|S_i = \mathscr{X}_2]$ is strictly greater than an upper bound of $\mathrm{Var}[V_{i+1}(S_{i+1})|S_i = \mathscr{X}_1]$ and consequently the two conditional variances satisfy (5.9). For an illustration of the proof see Figure 5.5.

Let us define the random variable X_{i+1} as:

$$X_{i+1} = \begin{cases} 1 & \text{if } h_{i+1}(S_{i+1}) \geq C_{i+1}(S_{i+1}); \\ 0 & \text{if } h_{i+1}(S_{i+1}) < C_{i+1}(S_{i+1}), \end{cases} \tag{5.10}$$

where

$$h_i(\mathscr{X}) = r_{0,i} \max\{0, K - \mathscr{X}\}$$

and $C_{i+1}(\cdot)$ is as in (3.8). The random variable X_{i+1} indicates whether the option is exercised at the next time-step $i+1$, or, in other words, whether $S_{i+1} \leq S_{f_{i+1}}$ where S_{f_i} is the optimal exercise price at time i (see formulation (3.12)-(3.13)). Since the discounted option price $V_{i+1}(\cdot)$ is an integrable random variable, by the law of total variance, we have:

$$\begin{aligned}
&\mathrm{Var}[u_i|S_i = \mathscr{X}_2] = \mathrm{Var}[V_{i+1}(S_{i+1})|S_i = \mathscr{X}_2] \\
&= E\left[\mathrm{Var}[V_{i+1}(S_{i+1})|S_i = \mathscr{X}_2, X_{i+1}]\right] + \mathrm{Var}[E\left[V_{i+1}(S_{i+1})|S_i = \mathscr{X}_2, X_{i+1}\right]] \\
&= \mathrm{Var}[V_{i+1}(S_{i+1})|S_i = \mathscr{X}_2, X_{i+1} = 1] P_{\mathbb{Q}}(X_{i+1} = 1|S_i = \mathscr{X}_2) \\
&\quad + \mathrm{Var}[V_{i+1}(S_{i+1})|S_i = \mathscr{X}_2, X_{i+1} = 0]\left(1 - P_{\mathbb{Q}}(X_{i+1} = 1|S_i = \mathscr{X}_2)\right) \\
&\quad + \mathrm{Var}\left[E\left[V_{i+1}(S_{i+1})|S_i = \mathscr{X}_2, X_{i+1}\right]\right]
\end{aligned}$$

and, since $P_{\mathbb{Q}}(X_{i+1} = 1|S_i = \mathscr{X}_2) \in (0,1)$ (it is the conditional probability of exercising the

[4] Since the LSMC algorithm restricts the regressions to only in-the-money paths ($\tilde{J}_i$), we will prove the following result for $S_i \in (0,K]$, although our result is also true for out-of-the-money paths.

option at the next time-step),

$$\mathrm{Var}[V_{i+1}(S_{i+1})|S_i = \mathscr{X}_2, X_{i+1} = 0]$$

and

$$\mathrm{Var}\left[E\left[V_{i+1}(S_{i+1})|S_i = \mathscr{X}_2, X_{i+1}\right]\right]$$

are non-negative by definition, and $S_{i+1} = S_i e^{s_{i+1}}$, it follows that

$$\begin{aligned}
\mathrm{Var}[u_i|S_i = \mathscr{X}_2] &\geq \mathrm{Var}[V_{i+1}(S_{i+1})|S_i = \mathscr{X}_2, X_{i+1} = 1]P_{\mathbb{Q}}(X_{i+1} = 1|S_i = \mathscr{X}_2) \\
&= \mathrm{Var}[h_{i+1}(S_{i+1})|S_i = \mathscr{X}_2, X_{i+1} = 1]P_{\mathbb{Q}}(X_{i+1} = 1|S_i = \mathscr{X}_2) \\
&= \mathrm{Var}[r_{0,i}(K - S_{i+1})|S_i = \mathscr{X}_2, X_{i+1} = 1]P_{\mathbb{Q}}(X_{i+1} = 1|S_i = \mathscr{X}_2) \\
&= r_{0,i}^2 \mathrm{Var}[S_{i+1}|S_i = \mathscr{X}_2, X_{i+1} = 1]P_{\mathbb{Q}}(X_{i+1} = 1|S_i = \mathscr{X}_2) \\
&= r_{0,i}^2 \mathscr{X}_2^2 \mathrm{Var}\left[e^{s_{i+1}}|S_i = \mathscr{X}_2, X_{i+1} = 1\right] P_{\mathbb{Q}}(X_{i+1} = 1|S_i = \mathscr{X}_2).
\end{aligned} \tag{5.11}$$

Furthermore,[5] let us define, for the generic $S_{i+1} > 0$, the functions

$$f(S_{i+1}) = V_{i+1}(S_{i+1}) - r_{0,i}K$$

and $g(S_{i+1}) = \frac{f(S_{i+1})}{S_{i+1}}$. The function f is non-increasing with $f(0) = 0$ since $V_{i+1}(S_{i+1})$ is a non-increasing function and $g(S_{i+1}) \in [\alpha_1, \alpha_2]$ with $\alpha_2 = \lim_{S_{i+1}\to+\infty} g(S_{i+1}) = 0$ and

$$\alpha_1 = \lim_{S_{i+1}\to 0} g(S_{i+1}) = \lim_{S_{i+1}\to 0} \frac{f(S_{i+1})}{S_{i+1}} = \lim_{S_{i+1}\to 0} \frac{h_{i+1}(S_{i+1}) - r_{0,i}K}{S_{i+1}} = -r_{0,i}.$$

Moreover, since

$$\begin{aligned}
E\left[f^2(S_{i+1})|S_i = \mathscr{X}_1\right] &= E\left[S_{i+1}^2 g^2(S_{i+1})|S_i = \mathscr{X}_1\right] \\
&\leq E\left[S_{i+1}^2|S_i = \mathscr{X}_1\right]\left(\lim_{S_{i+1}\to 0} g(S_{i+1})\right)^2 \\
&= r_{0,i}^2 E\left[S_{i+1}^2|S_i = \mathscr{X}_1\right] = r_{0,i}^2 \mathscr{X}_1^2 E\left[e^{2s_{i+1}}\right]
\end{aligned}$$

and $E\left[f(S_{i+1})|S_i = \mathscr{X}_1\right]^2 \geq 0$, then

$$\begin{aligned}
\mathrm{Var}[u_i|S_i = \mathscr{X}_1] &= \mathrm{Var}[V_{i+1}(S_{i+1})|S_i = \mathscr{X}_1] \\
&= \mathrm{Var}[f(S_{i+1})|S_i = \mathscr{X}_1] \leq r_{0,i}^2 \mathscr{X}_1^2 E[e^{2s_{i+1}}].
\end{aligned} \tag{5.12}$$

Imposing the condition that the lower bound in (5.11) is strictly greater than the upper bound

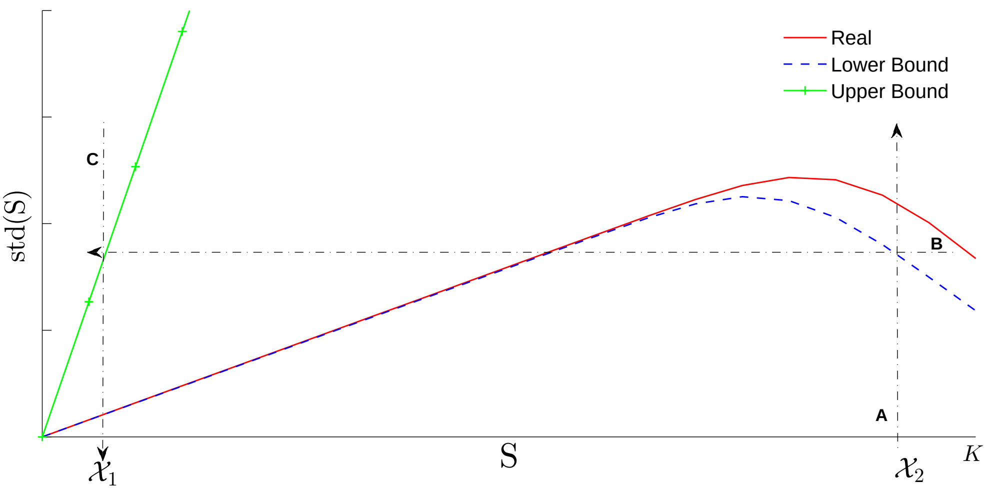
[5] The following bound modifies for left-bounded intervals $[0,\infty)$ the bound derived in Goldstein (1974), Theorem 2.

in (5.12), we prove that for any couple $(\mathscr{X}_1, \mathscr{X}_2)$ which satisfies

$$\mathscr{X}_1^2 < \mathscr{X}_2^2 \left\{ P_{\mathbb{Q}}(X_{i+1} = 1 | S_i = \mathscr{X}_2) \frac{\mathrm{Var}\left[e^{s_{i+1}} | S_i = \mathscr{X}_2, X_{i+1} = 1\right]}{E\left[e^{2s_{i+1}}\right]} \right\}, \tag{5.13}$$

relationship (5.9) holds. Since the quantity in curly brackets in (5.13) is strictly positive, for any $\mathscr{X}_2$ there exists $\mathscr{X}_1 \in (0, \mathscr{X}_2)$ that satisfies (5.13). Therefore, the proposition is proved. □

Figure 5.5 Exemplification of the proof of heteroscedastic errors in LSMC



Note: The figure exemplifies the proof of Proposition 5.2.1. The solid line is the real standard deviation $\mathrm{std}(S) = \sqrt{\mathrm{Var}[u_i | S_i = S]}$. It is unknown for the generic time-step i but it is given in exact closed form as in Formula (5.23) for the last time-step, the one we are plotting in this figure. The dashed line is the lower bound and the line marked with $+$ is the upper bound. The steps of the proofs are represented in order by the three letters A, B and C. The steps are: (A) fix $\mathscr{X}_2$, (B) find the value of the lower bound of the standard deviation at $\mathscr{X}_2$, (C) find $\mathscr{X}_1$ such that the upper bound of the standard deviation of the errors is equal to the lower bound previously found. Condition (5.9) holds for any $S < \mathscr{X}_1$.

Second, let us consider the American call option written on the same risky asset with price dynamics $\{S_t\}_{t \geq 0}$. Then, the following result holds.

Proposition 5.2.2 (American call options)**.** *The errors of the regressions in the LSMC algorithm for the American call option are heteroscedastic.*

Proof. We proceed by showing that for any $\mathscr{X}_1 \geq 0$ there exists $\mathscr{X}_2 \in (0, \mathscr{X}_1)$ such that

$$\mathrm{Var}[V_{i+1}(S_{i+1}) | S_i = \mathscr{X}_2] < \mathrm{Var}[V_{i+1}(S_{i+1}) | S_i = \mathscr{X}_1] \tag{5.14}$$

holds. By using similar calculations as in formula (5.11), we derive:

$$\mathrm{Var}[u_i|S_i = \mathscr{X}_1] \geq r_{0,i}^2 \mathscr{X}_1^2 Var\left[e^{s_{i+1}}|S_i = \mathscr{X}_1, X_{i+1} = 1\right] P_{\mathbb{Q}}(X_{i+1} = 1|S_i = \mathscr{X}_1) \tag{5.15}$$

where X_{i+1} is equivalent to (5.10) with

$$h_{i+1}(\mathscr{X}) = r_{0,i} \max\{0, \mathscr{X} - K\}.$$

In other words, $X_{i+1} = 1$ if and only if $S_{i+1} \geq S_{f_{i+1}}$ (see formulation (3.14)-(3.15)).

Furthermore, let us define

$$f^C(S_{i+1}) = -V_{i+1}(S_{i+1})$$

and $g^C(S_{i+1}) = \frac{f^C(S_{i+1})}{S_{i+1}}$. The function f^C is non-increasing in S_{i+1}, while $g^C \in [-r_{0,i}, 0]$ since $V_{i+1}(S_{i+1}) \leq r_{0,i} S_{i+1}$. Consequently, since

$$\begin{aligned}
E\left[\left(f^C(S_{i+1})\right)^2 |S_i = \mathscr{X}_2\right] &= E\left[S_{i+1}^2 \left(g^C(S_{i+1})\right)^2 |S_i = \mathscr{X}_2\right] \\
&\leq E\left[S_{i+1}^2|S_i = \mathscr{X}_2\right] \left(\sup_{S_{i+1}} g^C(S_{i+1})\right)^2 \\
&= r_{0,i}^2 E\left[S_{i+1}^2|S_i = \mathscr{X}_2\right] \\
&= r_{0,i}^2 \mathscr{X}_2^2 E\left[e^{2s_{i+1}}\right]
\end{aligned} \tag{5.16}$$

and $E\left[f^C(S_{i+1})|S_i = \mathscr{X}_2\right]^2 \geq 0$, then

$$\begin{aligned}
Var[u_i|S_i = \mathscr{X}_2] &= Var[V_{i+1}(S_{i+1})|S_i = \mathscr{X}_2] \\
&= Var[f(S_{i+1})|S_i = \mathscr{X}_2] \leq r_{0,i}^2 \mathscr{X}_2^2 E[e^{2s_{i+1}}].
\end{aligned} \tag{5.17}$$

As in the case of American put options, imposing that the lower bound in (5.15) is strictly greater than the upper bound in (5.17), we prove that for any couple $(\mathscr{X}_1, \mathscr{X}_2)$ which satisfies

$$\mathscr{X}_2^2 < \mathscr{X}_1^2 \left\{ P_{\mathbb{Q}}(X_{i+1} = 1|S_i = \mathscr{X}_1) \frac{Var\left[e^{s_{i+1}}|S_i = \mathscr{X}_1, X_{i+1} = 1\right]}{E[e^{2s_{i+1}}]} \right\}, \tag{5.18}$$

the relationship (5.14) holds. Since the quantity in curly brackets in (5.18) is positive, for any $\mathscr{X}_1$ there exists $\mathscr{X}_2 < \mathscr{X}_1$ satisfying (5.18). Therefore, the proposition is proved. □

5.2.5 Violation of the homoscedasticity assumption and its implications

The three proofs above show the heteroscedasticity of the errors of the regressions in the LSMC algorithm. We summarise here its impact on the regression estimators and how we can correct for it.

Let us assume the following regression model:[6]

$$y_l = \sum_{i=0}^{k} \beta_i x_{i,l} + \varepsilon_l, \tag{5.19}$$

where y_l and $x_{i,l}, i = 0, \ldots, k$ are the l-th observation in the sample, and ε_l is the random error. Let us define $X = [\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_k]$ as the sample matrix, $\boldsymbol{x}_i$ as the column vector of the components $x_{i,l}$ and $\bar{\boldsymbol{x}}_l$ as the l-th column of matrix X, then the classical linear regression assumptions for (5.19) are:

Linearity in the parameters Equation (5.19) is a linear combination of the parameters β_i;

Full rank The vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_k$ are linearly independent;

Exogeneity of the independent variables $E[\varepsilon_l | \bar{\boldsymbol{x}}_{l_1}] = 0$, for any l_1, i.e. the independent variables will not carry useful information for prediction of ε_l;

Homoscedasticity and non-autocorrelation Each error ε_l has the same finite variance and is not correlated with any other error, ε_{l_1}, conditional on X;

For the standard regression (5.19), the following result holds:

Theorem 5.2.3 (Gauss-Markov theorem)**.** *Under the classical linear regression assumptions, the OLS estimators are the best linear unbiased estimators (BLUE), i.e. among the unbiased linear ones, the OLS estimators have the minimum variance.*

It can be shown that, if all of the assumptions of the classical linear regression model hold except for homoscedasticity of the errors, the following properties are true:

- the regression method is still useful since the violation of the homoscedasticity assumption does not affect the unbiasedness;
- the OLS estimators may not have minimum variances and, consequently, they are not BLUE.

[6] The content of this section is based on Gujarati and Porter (2008) and Greene (2012).

In order to account for the heteroscedasticity in the regressions, it is necessary to employ the weighted least squares regression (WLS) method instead of the OLS. In doing so, we retain the BLUE property as shown by the following result:

Theorem 5.2.4 (Aitken's theorem)**.** *Under all classical linear regression assumptions with the exception of homoscedasticity of error, the WLS estimators are the best linear unbiased estimators (BLUE).*

This theorem lays the ground for us to introduce, in Section 5.3, a new pricing method that retains the BLUE property of the regression estimators in the LSMC algorithm even in the presence of heteroscedastic errors. Section 5.3.2 shows the positive effect of this new pricing method over the upward pricing bias.

Additionally, it is commonly assumed that in an ordinary least squares regression the errors are normally distributed. Although this assumption has no impact on the BLUE property of the estimators (Gujarati and Porter, 2008), in Appendix 5.D, we show that it does not hold for the regressions in the LSMC algorithm.

5.3 The weighted least squares Monte Carlo method

The proofs outlined in Section 5.2 and Aitken's theorem justify the introduction of the wLSMC algorithm that is detailed here. The new method is equivalent to the LSMC (see page 115) whose step 3b is substituted by

wLSMC Step of the weighted least squares Monte Carlo method

$3b^w$ Run a WLS regression on the pairs $\left(S_{i(j)}, V_{i+1}\left(S_{i+1(j)}\right)\right)$ for $j \in \tilde{J}_i$, with basis functions $\psi_l(\cdot)$, to determine $\beta^w_{i,l}$.

Running a WLS regression corresponds to computing an OLS regression for the transformed variables

$$\frac{\psi_l\left(S_{i(j)}\right)}{\mathrm{std}(S_{i(j)})} \to \psi^w_l\left(S_{i(j)}\right), \qquad \frac{V_{i+1}\left(S_{i+1(j)}\right)}{\mathrm{std}(S_{i(j)})} \to V^w_{i+1}\left(S_{i+1(j)}\right) \tag{5.20}$$

where

$$\mathrm{std}(S_{i(j)}) = \sqrt{\mathrm{Var}[u_i | S_i = S_{i(j)}]}$$

is the conditional standard deviation of the errors and $\psi_l(\cdot)$ are the basis functions in (5.1).

Thus, the continuation value is now calculated as:

$$\hat{C}_i(\mathscr{X}) = \sum_{l=0}^{M} \beta_{i,l}^{w} \psi_l(\mathscr{X}).$$

Theorem 5.2.4 ensures that the WLS regression method is BLUE even for heteroscedastic errors.

5.3.1 Approximation of the standard deviation of the errors

For the use of the wLSMC algorithm, as shown in (5.20), the availability of the conditional standard deviation std$(\cdot)$ is required, that is the weighting function for the weighted least squares regressions. We propose here an analytical formula for the approximation of the standard deviation. In the following, we work with variances rather than standard deviations to avoid carrying the square root notation.

For the time-step just before maturity, since the American option is equal to the corresponding European option, we can express the conditional variance of the errors in exact closed-form:

$$\begin{aligned}
\mathrm{Var}[V_m(S_m)|S_{m-1} = S_{m-1(j)}] &= \mathrm{Var}[h_m(S_m)|S_{m-1} = S_{m-1(j)}] \\
&= E\left[h_m(S_m)^2|S_{m-1} = S_{m-1(j)}\right] \\
&\quad -E\left[h_m(S_m)|S_{m-1} = S_{m-1(j)}\right]^2. \qquad (5.21)
\end{aligned}$$

In order to calculate the conditional variance of the errors at the generic time-step i, we assume a similar structure to the conditional variance in (5.21) and consider the following approximation:

$$\mathrm{Var}[V_{i+1}(S_{i+1})|S_i = S_{i(j)}] \approx \mathrm{Var}[h_{i+1}(S_{i+1})X_{i+1}|S_i = S_{i(j)}] \qquad (5.22)$$

where X_i is as in (5.10). This approximation allows one to preserve the evidence that:

- for American put options, the standard deviation of the errors (and consequently the variance) peaks near the optimal exercise price and has smaller values far away from it (see Figure 5.3);
- for American call options, the standard deviation of the errors is monotonically increasing (see Figure 5.4);

By employing approximation (5.22), the conditional variance of the errors of the regression at any time-step i is then approximated as:

$$\begin{aligned}\mathrm{Var}[u_i|S_i = S_{i(j)}] \;\approx\; & E\left[h_{i+1}(S_{i+1})^2 X_{i+1}|S_i = S_{i(j)}\right] \\ & -E\left[h_{i+1}(S_{i+1})X_{i+1}|S_i = S_{i(j)}\right]^2, i = 1,\dots,m-1. \end{aligned} \tag{5.23}$$

Formula (5.23) corresponds to the pricing of two European style options: a vanilla option with payoff function in time-0 dollars $h_{i+1}(S_{i+1})$ and an option with payoff $h_{i+1}(S_{i+1})^2$.

In order to simplify the calculations for American put options, we write the two expectations in formula (5.23) as:

$$\begin{aligned} E\left[h_{i+1}(S_{i+1})^2 X_{i+1}|S_i = S_{i(j)}\right] &= r_{0,i}^2\left[K^2 E_{X,0} - 2KS_{i(j)}E_{X,1} + S_{i(j)}^2 E_{X,2}\right] \\ E\left[h_{i+1}(S_{i+1})X_{i+1}|S_i = S_{i(j)}\right] &= r_{0,i}\left[KE_{X,0} - S_{i(j)}E_{X,1}\right] \end{aligned} \tag{5.24}$$

where

$$E_{X,l} = E\left[e^{ls_{i+1}}X_{i+1}|S_i = S_{i(j)}\right] = \int_{-\infty}^{\log\frac{S_{f_{i+1}}}{S_{i(j)}}} e^{ly} f(y)dy \tag{5.25}$$

with $f(\cdot)$ denoting the probability density function of the logarithmic return over the period (t_i, t_{i+1}).[7] In the following we calculate formula (5.25) for the price dynamics in Section 5.2.1. The proofs are in the Appendix 5.C.

Formula (5.23) can be calculated in closed-form for many other underlying price dynamics since it is the difference between the prices of two European-style options. However, instead of calculating formula (5.23), one can always resort to the two-step algorithm in Greene (2012) to estimate the standard deviation via OLS regression as a function of the residuals. This algorithm first computes the conditional standard deviation, usually using the ordinary least squares residuals; then, in the second step, it employs this standard deviation as weights in the weighted least squares regression.

Geometric Brownian motion Assuming that the process $\{S_t\}_{t\geq 0}$ follows the dynamics in (3.16), then the $E_{X,l}$ is:

$$E_{X,l} = e^{l\mu_s + l^2\frac{\sigma_s^2}{2}} N\left(-d(S_{f_{i+1}}) - l\sigma_s\right) \tag{5.26}$$

where $d(S) = \frac{-\log\frac{S}{S_{i(j)}} + \mu_s}{\sigma_s}$, $\mu_s = (r - \delta - \frac{\sigma^2}{2})\Delta_t$ and $\sigma_s = \sigma\sqrt{\Delta_t}$.

[7] As shown in Létourneau and Stentoft (2014), the LSMC does not guarantee monotonicity and convexity of the continuation value functions and, consequently, more than one price S may satisfy $h_i(S) = \hat{C}_i(S)$. In (5.25), if more than one such S does exist, we consider S_{f_i} equal to the arithmetic average between the smallest and the biggest S satisfying the condition.

Exponential Ornstein-Uhlenbeck process Assuming that the process $\{S_t\}_{t\geq 0}$ follows the dynamics in (3.18), then $E_{X,l}$ is as in (5.26) and $\mu_s = \tilde{\mu}(1-e^{-\eta\Delta_t})$, $\sigma_s^2 = \frac{\sigma^2}{2\eta}(1-e^{-2\eta\Delta_t})$ and $\tilde{\mu} = \mu - \frac{\sigma^2}{2\eta}$.

Log-normal jump-diffusion process Assuming that the process $\{S_t\}_{t\geq 0}$ follows the dynamics in (3.20), then $E_{X,l}$ is

$$E_{X,l} = (1-\lambda\Delta_t)e^{l\mu_s + l^2\frac{\sigma_s^2}{2}}N\left(-d(S_{f_{i+1}}) - l\sigma_s\right) + \lambda\Delta_t e^{l\mu_{s_J}+l^2\frac{\sigma_{s_J}^2}{2}}N\left(-d_J(S_{f_{i+1}}) - l\sigma_{s_J}\right), \tag{5.27}$$

where $d(S) = \frac{-\log\frac{S}{S_{i(j)}}+\mu_s}{\sigma_s}$, $d_J(S) = \frac{-\log\frac{S}{S_{i(j)}}+\mu_{s_J}}{\sigma_{s_J}}$, $\mu_s = (r-\delta-\lambda\kappa-0.5\sigma^2)\Delta_t$, $\sigma_s = \sigma\sqrt{\Delta_t}$, $\mu_{s_J} = (r-\delta-\lambda\kappa-0.5\sigma_{s,J}^2+\frac{\gamma}{\Delta_t})\Delta_t$, $\sigma_{s_J}^2 = \sigma^2\Delta_t + \sigma_J^2$ and $\gamma = \alpha_J + \frac{\sigma_J^2}{2}$.

Double exponential jump-diffusion process Assuming that the process $\{S_t\}_{t\geq 0}$ follows the dynamics in (3.22) then $E_{X,l}$ is:

$$E_{X,l} = (1-\lambda\Delta_t)e^{l\mu_s + l^2\frac{\sigma_s^2}{2}}N\left(-d(S_{f_{i+1}}) - l\sigma_s\right) + \lambda\Delta_t\int_{-\infty}^{\log\frac{S_{f_{i+1}}}{S_{i(j)}}} e^{ly}f_1(y)dy, \tag{5.28}$$

where $d(\cdot)$, σ_s and μ_s are defined as in the log-normal jump-diffusion process, $f_1(y) = f_Z(y+(r-\delta-\lambda\kappa-0.5\sigma^2)\Delta_t)$ and $f_Z(\cdot)$, the density function of the variable $Z_t = \sigma\tilde{W}_t + J_t^K$ is

$$\begin{aligned} f_Z(y) &= q\eta_1 e^{0.5\sigma^2\Delta_t\eta_1^2}e^{-y\eta_1}N\left(\frac{y}{\sigma\sqrt{\Delta_t}} - \sigma\eta_1\sqrt{\Delta_t}\right) \\ &\quad +(1-q)\eta_2 e^{y\eta_2}e^{0.5\sigma^2\Delta_t\eta_2^2}N\left(-\frac{y}{\sigma\sqrt{\Delta_t}} - \sigma\eta_2\sqrt{\Delta_t}\right). \end{aligned} \tag{5.29}$$

On the other hand, we note that the formulae in (5.24) also hold for American call options where (5.25) is substituted by

$$E_{X,l} = E\left[e^{ls_{i+1}}X_{i+1}|S_i = S_{i(j)}\right] = \int_{\log\frac{S_{f_{i+1}}}{S_{i(j)}}}^{+\infty} e^{ly}f(y)dy, \tag{5.30}$$

formula (5.26) becomes

$$E_{X,l} = e^{l\mu_s + l^2\frac{\sigma_s^2}{2}}N\left(d(S_{f_{i+1}}) + l\sigma_s\right) \tag{5.31}$$

and formulae (5.27) and (5.28) change accordingly.

5.3.2 Numerical study on pricing performance

In this section, we evaluate the pricing performance of the new weighted least squares Monte Carlo method by comparing it with the performance of the least squares Monte Carlo algorithm of Longstaff and Schwartz (2001) and the performance of the inequality constrained least squares Monte Carlo method of Létourneau and Stentoft (2014). The study is carried out for the option scenarios described in Section 5.2.1 under any of the following asset price dynamics: geometric Brownian motion, the exponential Ornstein-Uhlenbeck process, the log-normal jump-diffusion process and the dual-exponential jump-diffusion process.

In the comparison, each algorithm $\mathscr{M}$ (any among LSMC, ICLS and wLSMC) calculates the price of each option scenario by using $\mathtt{n_S}$ simulated paths of the underlying asset price. In particular, we consider $\mathtt{n_S}$ to take values in the set $\{1,000; 2,000; 5,000; 10,000\}$, as in Létourneau and Stentoft (2014) and we report two measures of error based on the mean over 100 independent repetitions of each method $\mathscr{M}$. The two measures of error we consider are the *root mean squared relative error* (RMSE) and the *mean relative error* (MRE). The former is a measure of dispersion of prices around the "exact" fair price and is calculated as

$$\mathrm{RMSE}^{(\mathscr{M})} = |\mathbf{O}|^{-0.5} \sqrt{\sum_{o\in\mathbf{O}} \left(\frac{1}{100} \sum_{l=1}^{100} \left(\frac{V_{l,o}^{(\mathscr{M})} - V_o^{(E)}}{V_o^{(E)}} \right)^2 \right)}, \tag{5.32}$$

while the latter is a measure of pricing bias and is computed as

$$\mathrm{MRE}^{(\mathscr{M})} = \frac{1}{|\mathbf{O}|} \sum_{o\in\mathbf{O}} \frac{\bar{V}_o^{(\mathscr{M})} - V_o^{(E)}}{V_o^{(E)}}, \tag{5.33}$$

where $V_{l,o}^{(\mathscr{M})}$ is the price of option o in the set of all the options $\mathbf{O}$ ($|\mathbf{O}|$ is its cardinality) calculated via method $\mathscr{M}$ with the l-th set of the $\mathtt{n_S}$ simulated paths, $\bar{V}_o^{(\mathscr{M})} = \frac{\sum_{l=1}^{100} V_{l,o}^{(\mathscr{M})}}{100}$ is the mean price of option o calculated via method $\mathscr{M}$, and $V_o^{(E)}$ is the "exact" fair price, calculated via the binomial tree methods of Cox et al. (1979) (for the scenarios under geometric Brownian motion), Nelson and Ramaswamy (1990) (under the exponential Ornstein-Uhlenbeck process) and Amin (1993) (under the jump-diffusion processes).

Additionally, for the regression step of each of the three methods (LSMC, ICLS and wLSMC), we employ the basis function

$$\psi_l(\mathscr{X}) = \mathscr{X}^l, l = 1, \ldots, M$$

for M taking values in the set $\{2, \ldots, 7\}$, and we fix the time-step length $\Delta_t = 0.02$ years (i.e. 50 exercise dates per year) as in Longstaff and Schwartz (2001). In the implementation of the ICLS of Létourneau and Stentoft (2014) we impose monotonicity and convexity constraints over a grid of six points (see Appendix 5.A for more details), as suggested by Létourneau and Stentoft.

American put options In this section we compare the pricing performance of the three Monte Carlo regression methods under the 160 American put option scenarios in Section 5.2.1.

The first comparison is carried out for $\mathtt{n_S} = 1,000$ simulated paths. As in Létourneau and Stentoft (2014), we use only 1,000 paths to study the behaviour of the three methods in reducing different levels of upper bias in pricing. Indeed, Stentoft (2004b) proved that both number of simulations $\mathtt{n_S}$ and number of regressors M have to grow indefinitely for the LSMC method to converge to the fair benchmark price, and, by considering a fixed number of simulation $\mathtt{n_S}$ and an increasing number of regressors, one experiences different degrees of biasedness. The results of this analysis are summarised in Tables 5.3-5.6.

For the LSMC method, both the RMSEs and the MREs monotonically increase with the polynomial order. Consequently, when only $\mathtt{n_S}$ =1,000 simulated scenarios are used, the upward bias caused by overfitting the continuation value curve dominates the downward bias caused by the approximation over finite low-order polynomials. This confirms the results found in Létourneau and Stentoft (2014) for price dynamics following geometric Brownian motion. Moreover, the LSMC's MREs are positive under all of the price dynamics considered, indicating that this method tends to overestimate the option prices.

Under all of the considered price dynamics, by imposing structure on the estimators, the ICLS method has a direct impact on reducing the upward bias over the estimated continuation value, as shown in Létourneau and Stentoft (2014). ICLS not only reduces the spread around the exact fair price (shown by the RMSEs) but also leads to a reduction in the upward bias as shown by the diminution of the MREs. However, under the four price dynamics considered, ICLS is still affected by upward pricing bias as shown by the all positive MREs.

By correcting for the heteroscedasticity via the wLSMC, as proposed in this thesis, the main advantage is that the errors across the polynomial orders are evened out, i.e. the errors for high-order polynomials (seriously affected by upward bias in the LSMC algorithm) are closer to the errors for low-order polynomials. This phenomenon is linked to a reduction in curve overfitting. Moreover, two other elements support the reduction in curve overfitting. First, the increasing RMSE patterns over polynomial orders are lost in some circumstances (for example, for $\sigma = 0.4$ in Table 5.3, for $S_0 = 1$ in Table 5.4, for $\kappa > 0$ in Table 5.5, the minimum RMSE is reached when the polynomial order is 3 while for $\kappa = 0.375$ in Table 5.6 the minimum is reached for the fourth order). This suggests that by increasing the polynomial order, the upward bias does not necessarily monotonically increase but other combinations of upward and downward bias can become more favourable for the pricing performances. Secondly, for low-order polynomials the MREs of wLSMC are in many cases negative. This suggests that the wLSMC is effective in reducing the upward bias and, for low-order polynomials, the downward bias dominates the other.

Overall, considering Tables 5.3-5.6, the wLSMC outperforms both the LSMC and the ICLS in pricing American put options. Both the RMSEs and the MREs are remarkably smaller when wLSMC is employed, especially for high-order polynomials. Only in a few cases does the ICLS perform slightly better for second-order polynomials than the wLSMC relative to the RMSE measure. However, the wLSMC has much smaller MREs even in those few cases when its RMSE is larger. The only exceptions are for those scenarios with $\sigma = 20\%$ in Table 5.5 where the ICLS has smaller MREs for second and third-order polynomials than wLSMC, and those with $S_0 = 1$ in Table 5.4 where ICLS has smaller (in absolute terms) MREs for second-order polynomials.

For the second comparison, as in Létourneau and Stentoft (2014), we fixed the number of basis functions equal to $M = 5$ and we let the number of paths increase with values: $\mathtt{n_S} \in \{1,000; 2,000; 5,000; 10,000\}$. The results are shown in Figures 5.6-5.8. For the comparison, the weights for the regression in the wLSMC algorithm are calculated using the two-step algorithm in Greene (2012). In particular, in order to estimate the variance of the errors, we consider the simple regression formula:

$$\hat{\boldsymbol{u}}_i^2 = \varphi_0 + \varphi_1 \boldsymbol{S}_i + \varphi_2 \boldsymbol{S}_i^2 \tag{5.34}$$

where $\hat{\boldsymbol{u}}_i$ is the vector of the residuals of an ordinary least squares regression, $\hat{\boldsymbol{u}}_i^2$ is used as a proxy of the variances of the residuals, and $\boldsymbol{S}_i$ is the vector of the underlying asset price values. This comparison is carried out by using the two-step algorithm in Greene (2012) rather than the estimation of the weighting function in Section 5.3.1 because we are mainly interested in the effect of the heteroscedasticity on the upper bias of the LSMC algorithm, and the use of the approximation may distort the analysis. We also carried out the same comparison by considering the approximations in Section 5.3.1 and, for the put option

Table 5.3 Pricing performance comparison for American put options under geometric Brownian motion

		RMSE						MRE					
Polynomial Order		2	3	4	5	6	7	2	3	4	5	6	7
$S_0 < 1$	LSMC	2.92%	3.09%	3.21%	3.37%	3.56%	3.78%	0.97%	1.44%	1.74%	2.03%	2.30%	2.63%
	ICLS	2.89%	2.91%	3.00%	3.12%	3.32%	3.49%	0.27%	0.90%	1.25%	1.56%	1.93%	2.20%
	wLSMC	2.84%	2.85%	2.87%	2.94%	2.99%	3.07%	-0.02%	0.37%	0.76%	0.97%	1.18%	1.34%
$S_0 = 1$	LSMC	3.87%	4.12%	4.33%	4.63%	4.90%	5.25%	1.39%	2.05%	2.46%	2.93%	3.35%	3.78%
	ICLS	3.74%	3.86%	3.99%	4.18%	4.47%	4.72%	0.60%	1.29%	1.75%	2.14%	2.64%	3.06%
	wLSMC	3.74%	3.72%	3.83%	3.91%	4.07%	4.20%	0.14%	0.73%	1.12%	1.46%	1.76%	1.98%
$S_0 > 1$	LSMC	5.01%	5.47%	5.87%	6.28%	6.77%	7.22%	2.14%	2.98%	3.62%	4.20%	4.81%	5.37%
	ICLS	4.74%	4.96%	5.17%	5.39%	5.85%	6.30%	1.26%	2.00%	2.48%	2.96%	3.64%	4.22%
	wLSMC	4.70%	4.81%	4.97%	5.16%	5.36%	5.55%	0.58%	1.22%	1.77%	2.15%	2.50%	2.85%
$\sigma = 20\%$	LSMC	4.41%	4.80%	5.18%	5.57%	5.99%	6.42%	1.71%	2.45%	3.01%	3.53%	4.04%	4.56%
	ICLS	4.18%	4.36%	4.55%	4.77%	5.19%	5.59%	0.79%	1.54%	2.03%	2.48%	3.10%	3.60%
	wLSMC	4.13%	4.25%	4.42%	4.63%	4.85%	5.08%	0.70%	1.27%	1.75%	2.13%	2.52%	2.86%
$\sigma = 40\%$	LSMC	3.66%	3.90%	4.06%	4.27%	4.53%	4.79%	1.33%	1.91%	2.26%	2.62%	2.99%	3.35%
	ICLS	3.57%	3.67%	3.77%	3.91%	4.15%	4.37%	0.67%	1.30%	1.66%	2.00%	2.42%	2.76%
	wLSMC	3.56%	3.54%	3.56%	3.60%	3.64%	3.67%	-0.19%	0.29%	0.72%	0.95%	1.13%	1.28%
$T = 1$	LSMC	4.29%	4.70%	4.99%	5.35%	5.75%	6.14%	1.80%	2.52%	3.02%	3.51%	3.98%	4.46%
	ICLS	4.07%	4.24%	4.41%	4.61%	5.00%	5.37%	1.02%	1.67%	2.11%	2.51%	3.08%	3.56%
	wLSMC	4.00%	4.09%	4.23%	4.37%	4.56%	4.74%	0.55%	1.13%	1.54%	1.86%	2.20%	2.45%
$T = 2$	LSMC	3.81%	4.03%	4.29%	4.54%	4.83%	5.14%	1.24%	1.85%	2.25%	2.65%	3.05%	3.45%
	ICLS	3.69%	3.80%	3.93%	4.10%	4.38%	4.64%	0.44%	1.16%	1.58%	1.96%	2.44%	2.80%
	wLSMC	3.71%	3.72%	3.79%	3.90%	4.00%	4.10%	-0.05%	0.44%	0.93%	1.22%	1.45%	1.69%
Overall	LSMC	4.05%	4.38%	4.65%	4.96%	5.31%	5.66%	1.52%	2.18%	2.64%	3.08%	3.52%	3.96%
	ICLS	3.89%	4.03%	4.18%	4.36%	4.70%	5.02%	0.73%	1.42%	1.84%	2.24%	2.76%	3.18%
	wLSMC	3.86%	3.91%	4.01%	4.14%	4.29%	4.43%	0.25%	0.78%	1.23%	1.54%	1.82%	2.07%

Note: The comparison is based on the 20 scenarios also used in Longstaff and Schwartz (2001), Table 1 ($r = 6\%$, $\delta = 0$, $K = 1$, $S_0 \in \{0.9, 0.95, 1, 1.05, 1.1\}$, $\sigma \in \{0.2, 0.4\}$ and $T = \{1, 2\}$ year(s)) and compares the LSMC by Longstaff and Schwartz (2001), the ICLS by Létourneau and Stentoft (2014) and the new method wLSMC. The root mean squared relative errors (RMSE) and the mean relative errors (MRE) are based on the mean over 100 independent simulations as in formulae (5.32) and (5.33). For the three algorithms, b =1,000 simulated paths of the underlying asset and 50 exercise dates per year are used. The basis functions are $\psi_l(S) = S^l$ with $l = 0, \ldots, M$ and $2 \leq M \leq 7$. The binomial tree in Cox et al. (1979) with 15,000 time-steps is used to calculate the benchmark prices.

Table 5.4 Pricing performance comparison for American put options under the exponential Ornstein-Uhlenbeck process

		RMSE						MRE					
Polynomial Order		2	3	4	5	6	7	2	3	4	5	6	7
$S_0 < 1$	LSMC	2.82%	3.02%	3.15%	3.30%	3.50%	3.66%	1.09%	1.52%	1.80%	2.05%	2.34%	2.61%
	ICLS	2.69%	2.82%	2.94%	3.07%	3.25%	3.39%	0.63%	1.09%	1.39%	1.66%	2.00%	2.23%
	wLSMC	2.66%	2.73%	2.73%	2.78%	2.85%	2.88%	-0.42%	0.87%	0.99%	1.18%	1.35%	1.39%
$S_0 = 1$	LSMC	3.46%	3.63%	3.77%	4.00%	4.28%	4.52%	1.05%	1.58%	1.94%	2.34%	2.74%	3.10%
	ICLS	3.40%	3.48%	3.55%	3.70%	3.93%	4.14%	0.55%	1.02%	1.41%	1.70%	2.17%	2.51%
	wLSMC	3.48%	3.38%	3.42%	3.49%	3.50%	3.62%	-0.80%	0.77%	0.95%	1.15%	1.39%	1.51%
$S_0 > 1$	LSMC	4.50%	4.81%	5.10%	5.43%	5.77%	6.11%	1.86%	2.53%	3.03%	3.50%	4.02%	4.48%
	ICLS	4.35%	4.45%	4.65%	4.82%	5.12%	5.47%	1.24%	1.76%	2.18%	2.57%	3.10%	3.61%
	wLSMC	4.08%	4.31%	4.34%	4.49%	4.62%	4.69%	-0.20%	1.46%	1.69%	1.98%	2.26%	2.37%
$\sigma = 20\%$	LSMC	3.94%	4.23%	4.47%	4.78%	5.10%	5.38%	1.48%	2.08%	2.51%	2.95%	3.41%	3.79%
	ICLS	3.82%	3.90%	4.09%	4.25%	4.53%	4.83%	0.92%	1.39%	1.81%	2.14%	2.66%	3.06%
	wLSMC	3.63%	3.78%	3.86%	4.00%	4.12%	4.23%	-0.24%	1.16%	1.43%	1.75%	2.01%	2.17%
$\sigma = 40\%$	LSMC	3.45%	3.63%	3.80%	3.99%	4.22%	4.44%	1.30%	1.79%	2.12%	2.44%	2.77%	3.12%
	ICLS	3.30%	3.44%	3.55%	3.68%	3.89%	4.08%	0.79%	1.30%	1.61%	1.93%	2.28%	2.61%
	wLSMC	3.26%	3.34%	3.28%	3.35%	3.39%	3.41%	-0.58%	1.01%	1.10%	1.23%	1.44%	1.44%
$e^{\mu} = 1$	LSMC	3.73%	3.97%	4.17%	4.42%	4.67%	4.94%	1.35%	1.90%	2.30%	2.66%	3.06%	3.43%
	ICLS	3.60%	3.69%	3.85%	3.98%	4.23%	4.48%	0.79%	1.29%	1.67%	1.98%	2.44%	2.81%
	wLSMC	3.51%	3.58%	3.64%	3.77%	3.84%	3.87%	-0.46%	1.07%	1.27%	1.50%	1.75%	1.82%
$e^{\mu} = 0.9$	LSMC	3.67%	3.91%	4.12%	4.38%	4.68%	4.93%	1.43%	1.97%	2.33%	2.72%	3.12%	3.48%
	ICLS	3.54%	3.66%	3.81%	3.97%	4.21%	4.46%	0.93%	1.39%	1.75%	2.09%	2.51%	2.86%
	wLSMC	3.39%	3.55%	3.53%	3.60%	3.71%	3.81%	-0.36%	1.10%	1.26%	1.49%	1.69%	1.80%
$\eta = 0.15$	LSMC	3.72%	3.97%	4.21%	4.48%	4.77%	5.05%	1.39%	1.97%	2.37%	2.79%	3.21%	3.60%
	ICLS	3.60%	3.71%	3.87%	4.00%	4.25%	4.51%	0.84%	1.34%	1.71%	2.03%	2.51%	2.89%
	wLSMC	3.50%	3.63%	3.64%	3.75%	3.89%	3.93%	-0.48%	1.13%	1.32%	1.58%	1.84%	1.90%
$\eta = 0.3$	LSMC	3.68%	3.91%	4.09%	4.32%	4.58%	4.81%	1.40%	1.91%	2.26%	2.59%	2.98%	3.32%
	ICLS	3.54%	3.64%	3.79%	3.95%	4.19%	4.43%	0.87%	1.34%	1.71%	2.03%	2.43%	2.79%
	wLSMC	3.40%	3.49%	3.52%	3.62%	3.65%	3.75%	-0.34%	1.04%	1.21%	1.41%	1.61%	1.71%
Overall	LSMC	3.70%	3.94%	4.15%	4.40%	4.68%	4.94%	1.39%	1.94%	2.32%	2.69%	3.09%	3.46%
	ICLS	3.57%	3.68%	3.83%	3.97%	4.22%	4.47%	0.86%	1.34%	1.71%	2.03%	2.47%	2.84%
	wLSMC	3.45%	3.56%	3.58%	3.69%	3.77%	3.84%	-0.41%	1.09%	1.26%	1.49%	1.72%	1.81%

Note: The comparison is based on 40 scenarios. The parameters considered are $r = 6\%$, $\delta = 0$, $K = 1$, $S_0 \in \{0.9, 0.95, 1, 1.05, 1.1\}$, $\sigma \in \{0.2, 0.4\}$, $\eta = \{0.15, 0.3\}$, $\mu = \{0, \log(0.9)\}$ and $T = 1$ year. The study compares the performances of the LSMC by Longstaff and Schwartz (2001), the ICLS by Létourneau and Stentoft (2014) and the new method wLSMC. The binomial tree in Nelson and Ramaswamy (1990) with 15,000 time-steps is used to calculate the benchmark prices. For other information, see Table 5.3.

Table 5.5 Pricing performance comparison for American put options under the log-normal jump diffusion process

		RMSE						MRE					
Polynomial Order		2	3	4	5	6	7	2	3	4	5	6	7
$S_0 < 1$	LSMC	2.72%	2.89%	3.01%	3.18%	3.37%	3.53%	0.86%	1.35%	1.66%	1.94%	2.24%	2.47%
	ICLS	2.63%	2.72%	2.81%	2.90%	3.11%	3.22%	0.32%	0.78%	1.13%	1.41%	1.80%	2.00%
	wLSMC	2.64%	2.66%	2.72%	2.75%	2.83%	2.89%	-0.02%	0.51%	0.76%	0.98%	1.20%	1.37%
$S_0 = 1$	LSMC	3.36%	3.56%	3.70%	3.92%	4.07%	4.22%	1.04%	1.64%	2.00%	2.36%	2.65%	2.85%
	ICLS	3.27%	3.37%	3.43%	3.56%	3.72%	3.94%	0.47%	0.94%	1.34%	1.66%	2.04%	2.34%
	wLSMC	3.32%	3.34%	3.35%	3.42%	3.54%	3.61%	0.12%	0.70%	0.95%	1.27%	1.50%	1.68%
$S_0 > 1$	LSMC	3.88%	4.14%	4.36%	4.62%	4.87%	5.07%	1.37%	2.00%	2.43%	2.81%	3.15%	3.51%
	ICLS	3.77%	3.88%	3.96%	4.12%	4.45%	4.56%	0.80%	1.25%	1.62%	1.98%	2.49%	2.73%
	wLSMC	3.74%	3.78%	3.88%	3.94%	4.07%	4.15%	0.25%	0.90%	1.21%	1.51%	1.80%	1.96%
$\sigma = 20\%$	LSMC	3.80%	4.01%	4.15%	4.37%	4.52%	4.67%	1.00%	1.56%	1.96%	2.30%	2.60%	2.88%
	ICLS	3.74%	3.82%	3.85%	3.96%	4.19%	4.27%	0.35%	0.78%	1.18%	1.51%	1.97%	2.20%
	wLSMC	3.74%	3.80%	3.87%	3.90%	4.03%	4.11%	0.42%	0.88%	1.14%	1.43%	1.67%	1.87%
$\sigma = 40\%$	LSMC	2.84%	3.06%	3.27%	3.50%	3.77%	3.97%	1.21%	1.77%	2.12%	2.45%	2.77%	3.05%
	ICLS	2.69%	2.82%	2.96%	3.11%	3.41%	3.59%	0.74%	1.22%	1.56%	1.87%	2.27%	2.52%
	wLSMC	2.67%	2.66%	2.73%	2.82%	2.91%	2.97%	-0.19%	0.52%	0.81%	1.08%	1.33%	1.46%
$\kappa < 0$	LSMC	3.84%	4.10%	4.27%	4.52%	4.73%	4.92%	0.99%	1.72%	2.15%	2.57%	2.91%	3.24%
	ICLS	3.75%	3.85%	3.93%	4.09%	4.41%	4.50%	0.27%	0.79%	1.30%	1.73%	2.30%	2.52%
	wLSMC	3.71%	3.78%	3.89%	3.96%	4.11%	4.20%	0.15%	0.78%	1.12%	1.42%	1.74%	1.98%
$\kappa > 0$	LSMC	2.79%	2.94%	3.11%	3.30%	3.50%	3.66%	1.22%	1.61%	1.93%	2.18%	2.46%	2.69%
	ICLS	2.67%	2.78%	2.86%	2.93%	3.11%	3.29%	0.82%	1.21%	1.44%	1.65%	1.94%	2.20%
	wLSMC	2.71%	2.69%	2.70%	2.74%	2.79%	2.84%	0.08%	0.62%	0.84%	1.08%	1.26%	1.35%
$\lambda = 0.5$	LSMC	3.48%	3.72%	3.90%	4.15%	4.36%	4.56%	1.14%	1.75%	2.14%	2.53%	2.86%	3.16%
	ICLS	3.35%	3.47%	3.55%	3.68%	3.99%	4.11%	0.53%	1.02%	1.42%	1.75%	2.22%	2.48%
	wLSMC	3.35%	3.41%	3.49%	3.59%	3.71%	3.82%	0.18%	0.86%	1.14%	1.46%	1.75%	1.91%
$\lambda = 1$	LSMC	3.23%	3.41%	3.56%	3.76%	3.95%	4.10%	1.07%	1.58%	1.94%	2.22%	2.51%	2.76%
	ICLS	3.16%	3.24%	3.31%	3.43%	3.64%	3.77%	0.55%	0.99%	1.32%	1.62%	2.02%	2.24%
	wLSMC	3.15%	3.15%	3.20%	3.21%	3.31%	3.34%	0.05%	0.55%	0.81%	1.04%	1.25%	1.42%
Overall	LSMC	3.36%	3.57%	3.74%	3.96%	4.16%	4.34%	1.10%	1.66%	2.04%	2.37%	2.69%	2.96%
	ICLS	3.26%	3.36%	3.43%	3.56%	3.82%	3.94%	0.54%	1.00%	1.37%	1.69%	2.12%	2.36%
	wLSMC	3.25%	3.28%	3.35%	3.40%	3.51%	3.59%	0.11%	0.70%	0.98%	1.25%	1.50%	1.67%

Note: The comparison is based on 80 scenarios. The parameters considered are $r = 6\%$, $\delta = 0$, $K = 1$, $S_0 \in \{0.9, 0.95, 1, 1.05, 1.1\}$, $\sigma \in \{0.2, 0.4\}$, $\lambda \in \{0.5, 1\}$, $\alpha_J \in \{-0.25, 0.25\}$, $\sigma_J \in \{0.2, 0.4\}$ and $T = 1$ year. The study compares the performances of the LSMC by Longstaff and Schwartz (2001), the ICLS by Létourneau and Stentoft (2014) and the new method wLSMC. The binomial tree in Amin (1993) with 1,000 time-steps is used to calculate the benchmark prices. For other information, see Table 5.3.

Table 5.6 Pricing performance comparison for American put options under the double exponential jump diffusion process

		RMSE						MRE					
Polynomial Order		2	3	4	5	6	7	2	3	4	5	6	7
$S_0 < 1$	LSMC	2.43%	2.61%	2.77%	2.95%	3.15%	3.40%	1.09%	1.46%	1.74%	2.04%	2.32%	2.64%
	ICLS	2.27%	2.37%	2.49%	2.67%	2.87%	3.08%	0.50%	1.02%	1.30%	1.60%	1.91%	2.20%
	wLSMC	2.33%	2.33%	2.34%	2.37%	2.46%	2.62%	-0.18%	0.21%	0.61%	0.91%	1.20%	1.50%
$S_0 = 1$	LSMC	2.95%	3.20%	3.51%	3.79%	4.20%	4.47%	1.49%	1.98%	2.43%	2.83%	3.32%	3.69%
	ICLS	2.76%	2.96%	3.09%	3.25%	3.58%	3.93%	0.83%	1.40%	1.78%	2.09%	2.56%	3.02%
	wLSMC	2.79%	2.80%	2.83%	2.94%	3.22%	3.38%	-0.10%	0.50%	0.98%	1.35%	1.81%	2.17%
$S_0 > 1$	LSMC	4.08%	4.46%	4.92%	5.37%	5.89%	6.35%	2.16%	2.81%	3.44%	4.05%	4.63%	5.20%
	ICLS	3.73%	4.00%	4.24%	4.45%	4.92%	5.33%	1.40%	1.99%	2.42%	2.80%	3.47%	3.98%
	wLSMC	3.64%	3.67%	3.89%	4.23%	4.56%	4.95%	0.17%	0.94%	1.61%	2.16%	2.80%	3.28%
$\sigma = 20\%$	LSMC	3.58%	3.86%	4.26%	4.63%	5.09%	5.48%	1.69%	2.20%	2.74%	3.21%	3.71%	4.18%
	ICLS	3.29%	3.49%	3.68%	3.87%	4.25%	4.62%	1.03%	1.55%	1.91%	2.27%	2.77%	3.25%
	wLSMC	3.23%	3.33%	3.51%	3.77%	4.08%	4.43%	0.28%	0.85%	1.45%	1.87%	2.36%	2.84%
$\sigma = 40\%$	LSMC	2.95%	3.25%	3.50%	3.79%	4.10%	4.42%	1.51%	2.01%	2.38%	2.79%	3.17%	3.57%
	ICLS	2.74%	2.94%	3.10%	3.28%	3.60%	3.89%	0.82%	1.42%	1.77%	2.08%	2.56%	2.90%
	wLSMC	2.76%	2.68%	2.72%	2.83%	3.00%	3.17%	-0.33%	0.27%	0.73%	1.13%	1.56%	1.85%
$\kappa = 0.375$	LSMC	2.97%	3.26%	3.50%	3.75%	4.07%	4.35%	1.49%	1.94%	2.37%	2.73%	3.12%	3.50%
	ICLS	2.80%	2.94%	3.07%	3.22%	3.49%	3.80%	0.98%	1.42%	1.69%	1.99%	2.40%	2.79%
	wLSMC	2.83%	2.83%	2.79%	2.88%	3.05%	3.21%	-0.32%	0.21%	0.71%	1.06%	1.50%	1.86%
$\kappa = 0.0952$	LSMC	3.56%	3.85%	4.26%	4.66%	5.12%	5.53%	1.70%	2.27%	2.76%	3.27%	3.76%	4.25%
	ICLS	3.24%	3.49%	3.70%	3.92%	4.35%	4.69%	0.87%	1.55%	1.99%	2.37%	2.93%	3.36%
	wLSMC	3.17%	3.20%	3.45%	3.74%	4.05%	4.40%	0.27%	0.91%	1.46%	1.94%	2.42%	2.83%
Overall	LSMC	3.28%	3.57%	3.90%	4.23%	4.62%	4.98%	1.60%	2.11%	2.56%	3.00%	3.44%	3.88%
	ICLS	3.03%	3.22%	3.40%	3.59%	3.94%	4.27%	0.93%	1.49%	1.84%	2.18%	2.67%	3.08%
	wLSMC	3.00%	3.02%	3.14%	3.34%	3.58%	3.85%	-0.03%	0.56%	1.09%	1.50%	1.96%	2.34%

Note: The comparison is based on 40 scenarios. The parameters considered are $r = 6\%$, $\delta = 0$, $K = 1$, $S_0 \in \{0.9, 0.95, 1, 1.05, 1.1\}$, $\sigma \in \{0.2, 0.4\}$, $q = 0.5$, $\lambda = 0.5$, $(\eta_1, \eta_2) =\in \{(2,3), (4,6)\}$ and $T = 1$ year. The study compares the performances of the LSMC by Longstaff and Schwartz (2001), the ICLS by Létourneau and Stentoft (2014) and the new method wLSMC. The binomial tree in Amin (1993) with 1,000 time-steps is used to calculate the benchmark prices. For other information, see Table 5.3.

scenarios considered, the performances of the two alternatives are very similar. However, since for American call options (see next paragraph), our variance approximations are not as good as the put' ones, for consistency, here we also illustrate the results when the algorithm of Greene is employed.

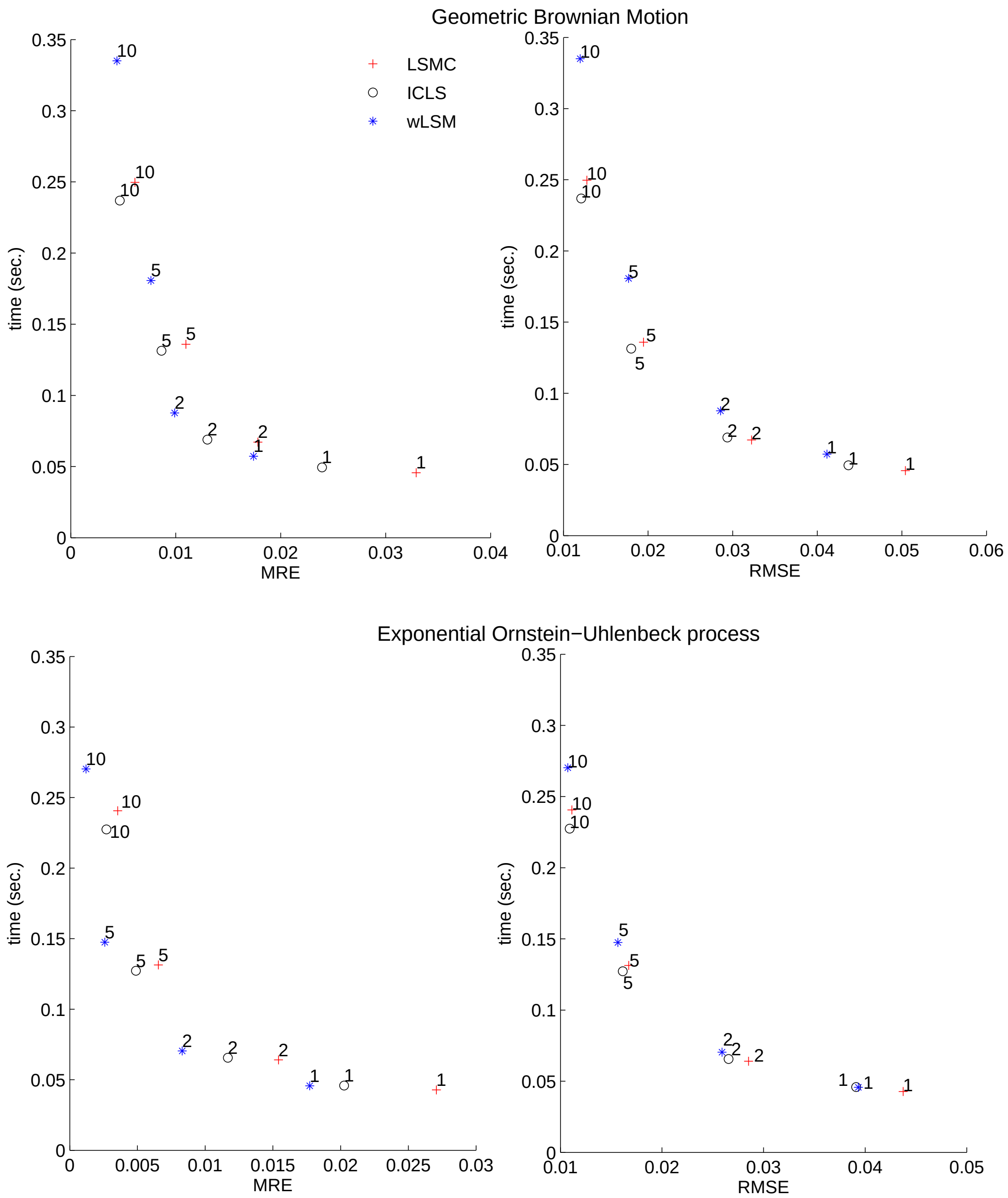
Figures 5.6-5.7 plot the MRE and RMSE against the running time of the three algorithms, for the four dynamics. The three methods have similar running times for $\mathtt{n_S} = 1{,}000$, $2{,}000$ and $5{,}000$ paths and the wLSMC method is slightly slower for $\mathtt{n_S} = 10{,}000$. Considering the trade-off of pricing error versus running time, the plots show that the wLSMC algorithm for $\mathtt{n_S} = 1{,}000$, $2{,}000$ and $5{,}000$ paths outperforms the other two algorithms.

For each of the four price dynamics considered, the upper plots in Figure 5.8 detail the MRE measure as a function of the number of paths. They show how the bias in price changes with the change in the number of paths. For the scenarios considered in this study, the wLSMC always has lower upper bias in option pricing. Not surprisingly, the improvements of the new method are more significant for a low number of paths, although the wLSMC algorithm also outperforms the other two methods for a high number of paths.

Additionally, it is common in studies that use the Monte Carlo method to carry out a diagnostic test for the convergence of a simulation technique (Broadie and Glasserman, 1997; Longstaff and Schwartz, 2001). Basically, each algorithm is executed twice, each over a different set of simulated paths: the exercise rule is estimated from one set of paths (in sample) and then applied to another set of paths (out-of-sample). For the LSMC algorithm, Longstaff and Schwartz proposed to implement this technique by estimating the regression estimators from the in-sample set of paths and then apply these regression functions to the out-of-sample set of paths. While the prices based on in-sample sets of paths are affected by upper bias, the out-of-sample prices are affected by low bias. For each of the four underlying asset dynamics, the lower plots of Figure 5.8 represent the MRE for the average between the in-sample and out-of-sample prices. In most of the cases (exceptions are the cases with $\mathtt{n_S} = 1{,}000$ for the exponential Ornstein-Uhlenbeck process and the dual exponential jump-diffusion process) the MRE of this average for the wLSMC method is negative, this being further proof of upper price reduction. Additionally, in most cases, the MRE of these estimations is much closer to zero than the estimators of the LSMC and ICLS. Exceptions are for $\mathtt{n_S} = 5{,}000$ and $10{,}000$ for the jump-diffusion processes, where the LSMC and ICLS are slightly better.

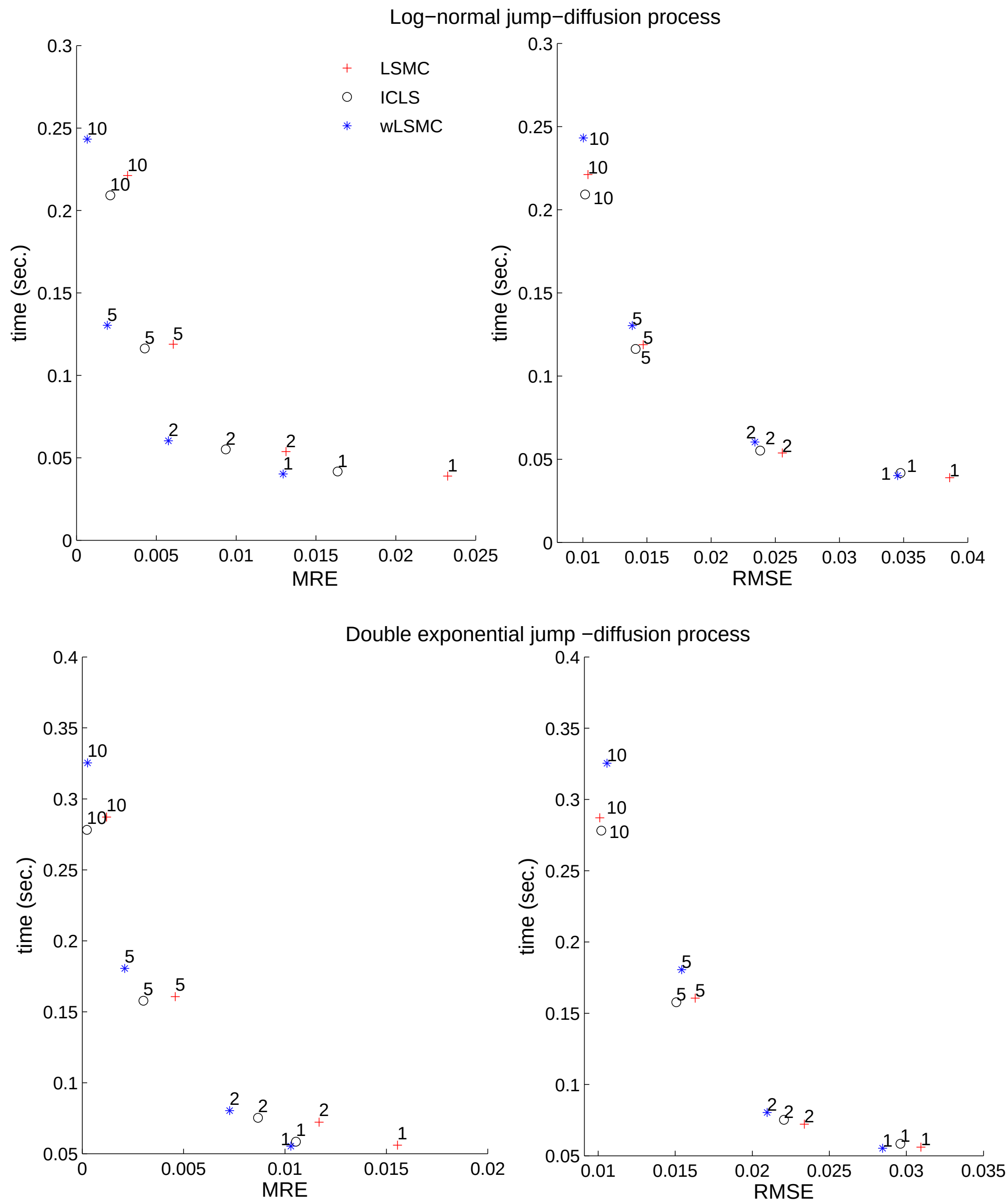
Overall, for the scenarios considered, the best method is our wLSMC with $2{,}000$ paths (in-sample), since it has almost the same running time as the ICLS and LSMC, slightly smaller RMSE and a much smaller upper bias, as shown by the MRE. However, when considering out-of sample sets, the best performance is that of our wLSMC with $\mathtt{n_S} = 5{,}000$.

Figure 5.6 Pricing comparison for American PUT option scenarios: error measures versus average running time (1)



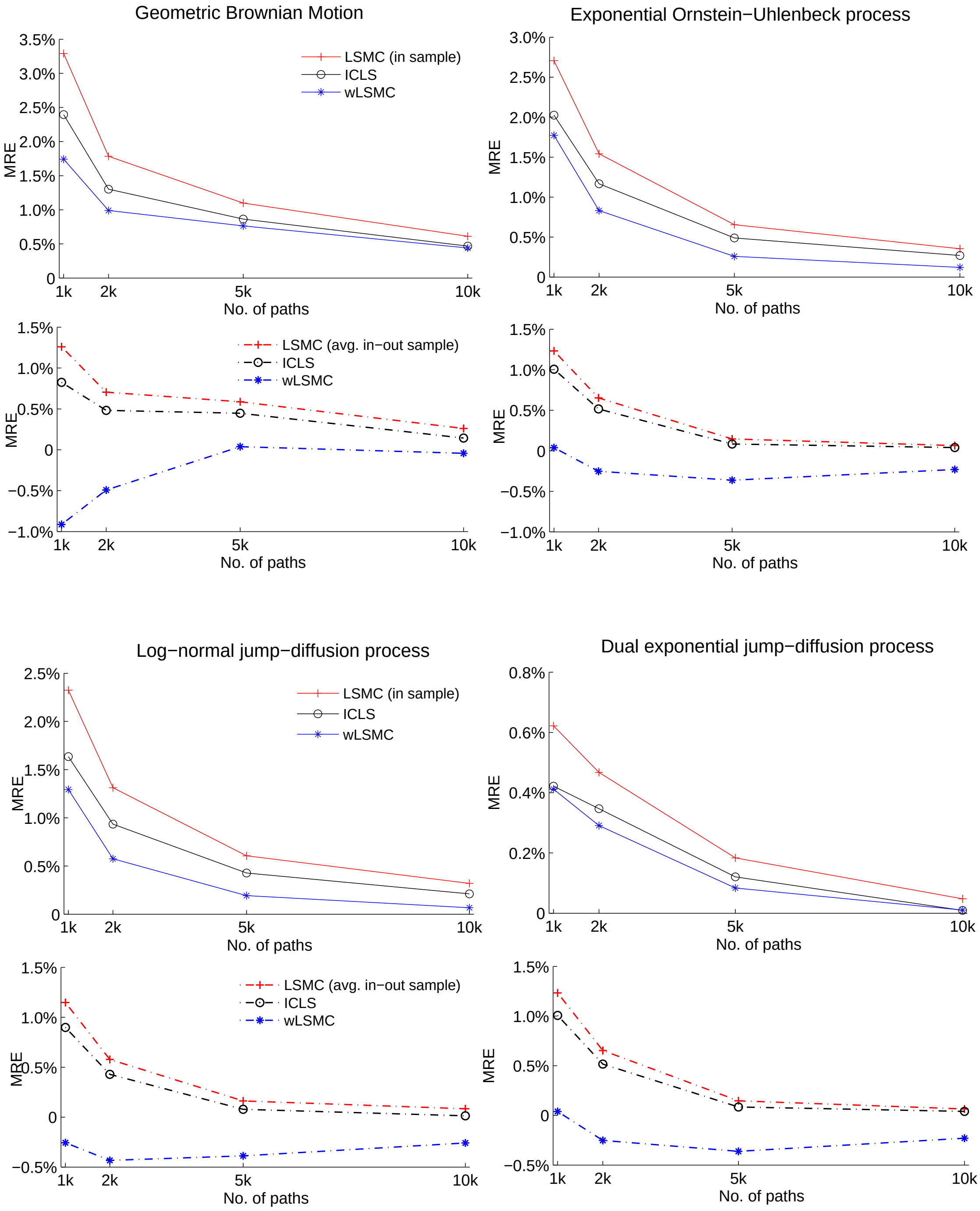
Note: The comparison in the upper plots is based on the 20 scenarios under geometric Brownian motion, which are also used in Longstaff and Schwartz (2001), Table 1 ($r = 6\%$, $\delta = 0$, $K = 1$, $S_0 \in \{0.9, 0.95, 1, 1.05, 1.1\}$, $\sigma \in \{0.2, 0.4\}$ and $T = \{1, 2\}$ year(s)). The comparison in the lower plots is based on the 40 scenarios under the exponential Ornstein-Uhlenbeck process with $r = 6\%$, $K = 1$, $S_0 \in \{0.9, 0.95, 1, 1.05, 1.1\}$, $\sigma \in \{0.2, 0.4\}$, $T = 1$ year, $\eta = \{0.2, 0.4\}$, $\mu = \{0, \log(0.9)\}$. The methods compared are the least squares Monte Carlo (LSMC), the inequality constrained least squares (ICLS) and the weighted least squares Monte Carlo (wLSMC). The root mean squared relative errors (RMSE) and the mean relative errors (MRE) are based on the mean over 100 independent simulations as in formulae (5.32) and (5.33). Fifty exercise dates per year are used and the label of each data point indicates the number of paths $\mathtt{n_S}$ (in thousands). This is a similar analysis to that carried out in Table 2.2. The main difference is that here we used the two-step algorithm in Greene (2012), as in formula (5.34).

Figure 5.7 Pricing comparison for American PUT option scenarios: error measures versus average running time (2)



Note: The comparison in the upper plots is based on the 80 scenarios under the log-normal jump-diffusion process ($r = 6\%$, $\delta = 0$, $K = 1$, $S_0 \in \{0.9, 0.95, 1, 1.05, 1.1\}$, $\sigma \in \{0.2, 0.4\}$, $\lambda = \{0.5, 1\}$, $\alpha_J = \{-0.25, 0.25\}$, $\sigma_J = \{0.2, 0.4\}$ and $T = 1$ year). The comparison in the lower plots is based on the 40 scenarios under the exponential Ornstein-Uhlenbeck process with $q = 0.5$, $\lambda = 0.5$ and $(\eta_1, \eta_2) \in \{(2,3), (4,6)\}$. For other information, see Figure 5.6.

Figure 5.8 Pricing comparison for American PUT options: MRE as a function of number of simulated paths n_S



Note: The comparison is carried out under the 160 option scenarios in Section 5.2.1. For each of the four price dynamics, the upper plot represents the mean relative error (MRE) for the in-sample methods, while the lower plot represents the MRE for the average price between the in-sample method and the out-of-sample method.

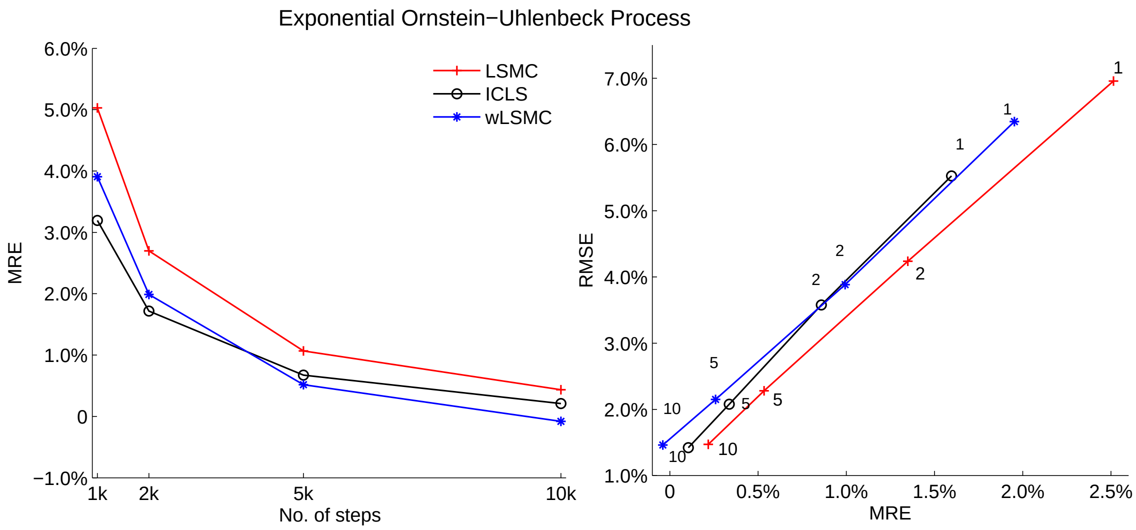
American call options In this section, we further analyse the weighted least squares Monte Carlo method by pricing American call option scenarios. We consider the option scenarios in Section 5.2.1 and the additional scenarios created by varying the dividend yield parameter. In particular we assume for the dividend yield δ any of the values in the set $\{3\%, 5\%, 7\%\}$. As for the put options, we fix $M = 5$ and let the number of paths $\mathtt{n_S}$ change in the set $\{1,000; 2,000; 5,000; 10,000\}$. The weights for the regression in the wLSMC are calculated using the two-step algorithm in Greene (2012), by using (5.34).

The results of the analysis are shown in Figures 5.9-5.11. Figure 5.11 and the left-hand side plot in Figure 5.9 show that the wLSMC algorithm also works efficaciously in reducing the upper bias of the LSMC for American call options under the four dynamics studied in this chapter. Remarkable results are obtained for $\mathtt{n_S}$ equal to $\{1,000; 2,000; 5,000\}$ simulated paths, for which the MREs are much smaller (in absolute terms) than those of the LSMC and ICLS. The improvements are consistent across the different dividends. However, for $\mathtt{n_S} = 10,000$ simulated paths the wLSMC algorithm performs slightly more poorly than the ICLS method although better than the LSMC. The cause of this may be the approximation in (5.34), which, when the number of paths is high, does not properly estimate the standard deviation of the errors. Finally, Figure 5.11 and the right-hand side of Figure 5.9 show the evolution of the MRE measure and the RMSE measure across the different numbers of simulated paths. The RMSEs of the three methods are similar and, as for the MRE measure, the wLSMC has a slightly higher value than the other two methods for $\mathtt{n_S} = 10,000$. In the overall analysis, as with put option scenarios, the best method is the wLSMC algorithm with $2,000$ paths, which has a very small MRE (under almost all underlying dynamics MRE is lower than 1% in absolute values) compared to the other methods and similar RMSE.

5.4 Conclusions

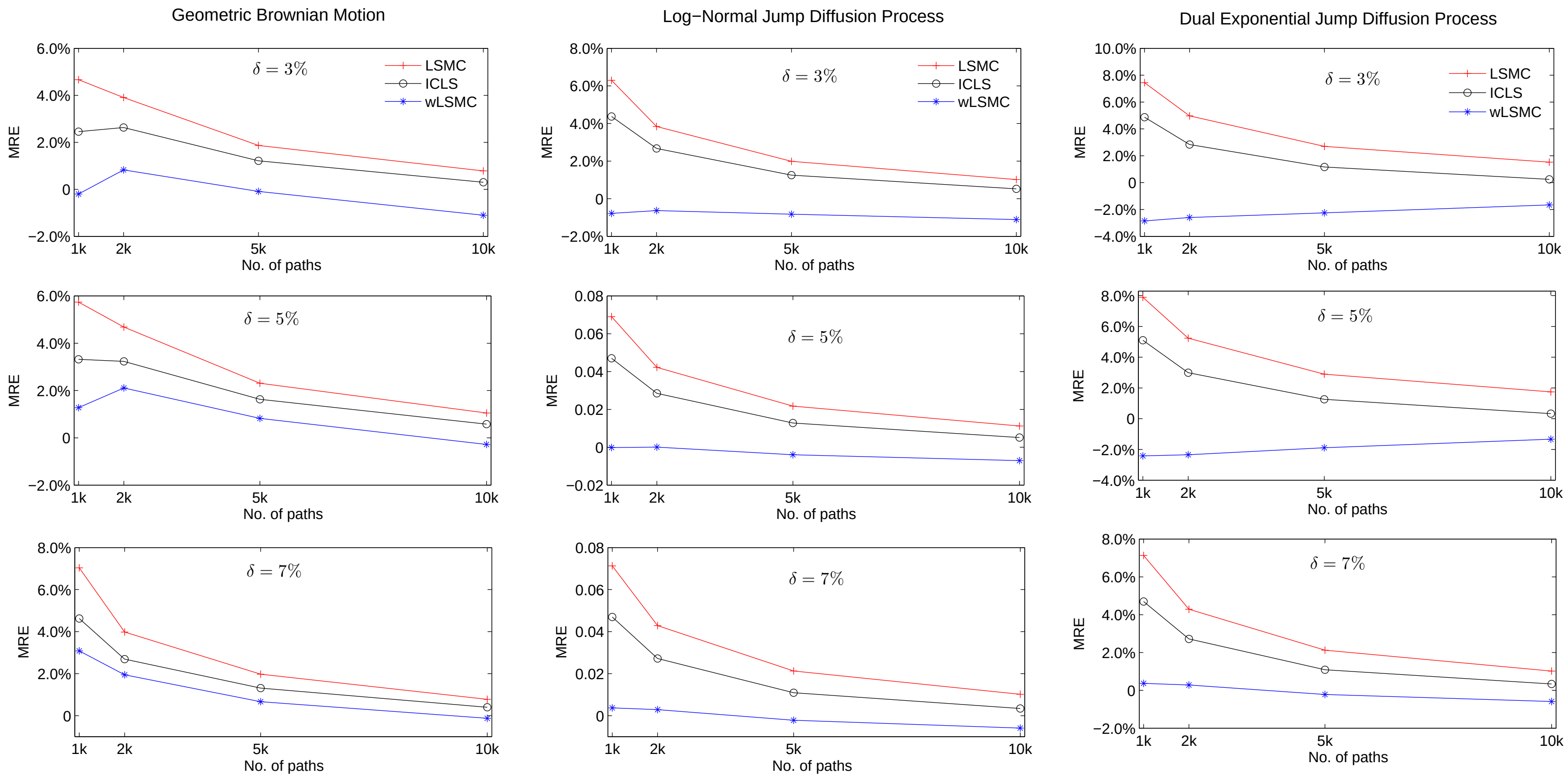
In this chapter, we studied in detail the regression step of the least squares Monte Carlo algorithm for pricing American call and put options. We showed both numerically and theoretically that there exists heteroscedasticity in the regressions performed for pricing American options, for several well-known models of the underlying asset prices. Therefore, we showed that the least squares estimators are not the best in terms of variances among the linear unbiased estimators. Additionally, we also illustrated via an extensive numerical study that the failure to account for this heteroscedasticity results in curve overfitting problems (especially for high-order polynomial basis functions in the regressions) and consequently in an upward bias in the option prices.

Figure 5.9 Pricing comparison for American CALL option scenarios under the exponential Ornstein-Uhlenbeck process



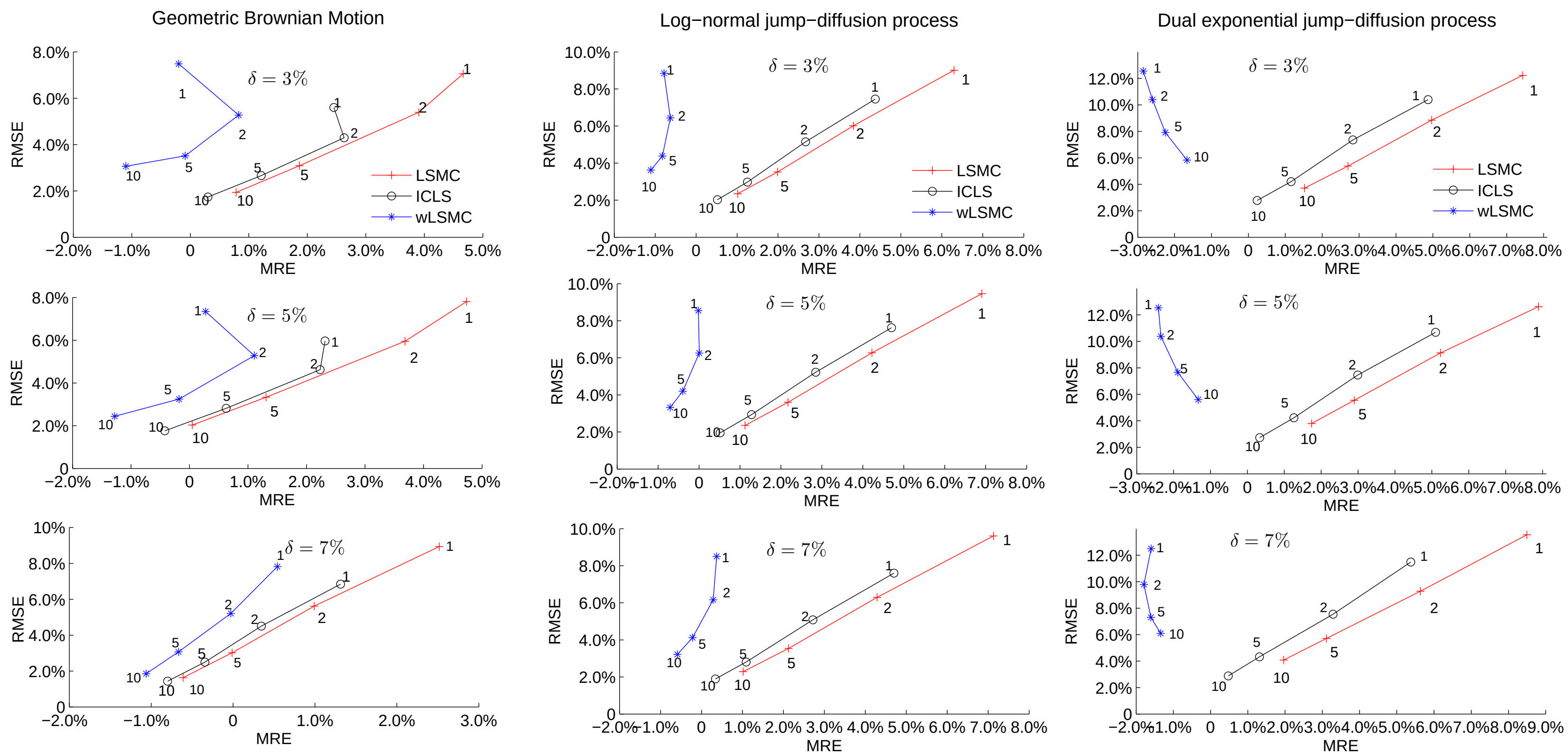
Note: The comparison is based on the 40 scenarios under the exponential Ornstein-Uhlenbeck process ($r = 6\%$, $K = 1$, $S_0 \in \{0.9, 0.95, 1, 1.05, 1.1\}$, $\sigma \in \{0.2, 0.4\}$ and $T = 1$ year, $\eta = \{0.2, 0.4\}$, $\mu = \{0, \log(0.9)\}$. The methods compared are the least squares Monte Carlo (LSMC), the inequality constrained least squares (ICLS) and the weighted least squares Monte Carlo (wLSMC). The root mean squared relative errors (RMSE) and the mean relative errors (MRE) are based on the mean over 100 independent simulations as in formulae (5.32) and (5.33). Fifty exercise dates per year are used and the label of each data point indicates the number of paths $\mathtt{n_S}$ (in thousands).

Figure 5.10 Pricing comparison for American CALL option scenarios: MRE as a function of number of simulated paths n_S



Note: The comparison in the first row is based on the scenarios in Section 5.2.1 (for a dividend yield of $\delta = 3\%$). Additionally in the second and third rows, we consider the same scenarios in the first row with other dividend yields ($\delta = 5\%$ and $\delta = 7\%$), respectively. The methods compared are the least squares Monte Carlo (LSMC), the inequality constrained least squares (ICLS) and the weighted least squares Monte Carlo (wLSMC). The mean relative errors (MRE) are based on the mean over 100 independent simulations as in formula (5.33). Fifty exercise dates per year are used.

Figure 5.11 Pricing comparison for American CALL options: MRE versus RMSE as a function of number of simulated paths $\mathtt{n_S}$



Note: The comparison is based on the scenarios in Section 5.2.1 (for a dividend yield $\delta = 3\%$). The methods compared are the least squares Monte Carlo (LSMC), the inequality constrained least squares (ICLS) and the weighted least squares Monte Carlo (wLSMC). The mean relative errors (MRE) and the root mean squared relative errors (RMSE) are based on the mean over 100 independent simulations as in formulae (5.32) and (5.33). Fifty exercise dates per year are used.

As a solution to this problem, we proposed the weighted least squares Monte Carlo method. It retains all of the original steps of the (ordinary) least squares Monte Carlo method described in Longstaff and Schwartz (2001) but substitutes the ordinary least squares regression with its weighted version in order to account for heteroscedasticity.

In our numerical study, we found that for each of the four considered underlying price dynamics (geometric Brownian motion, exponential Ornstein-Uhlenbeck process, log-normal jump-diffusion process and double exponential jump-diffusion process), the wLSMC produces a much smaller pricing error than the LSMC. We also compared the wLSMC with the inequality constrained least squares method proposed by Létourneau and Stentoft (2014), which, by using a different approach, seeks to reduce the pricing bias in the LSMC. The results showed that both methods reduce the curve-overfitting problem (and consequently the upward bias) and that, in many cases, the wLSMC outperforms the ICLS.

5.4.1 Further research

This chapter mainly discussed the existence of heteroscedasticity when the LSMC is applied to price American call and put options on a single asset. However, we strongly believe that heteroscedasticity can be generalised to many other payoff structures, even multi-assets. Consequently, one direction for future research is formal proofs that generalise propositions 5.2.1 and 5.2.2 for generic square-integrable payoff functions, also considering multi-asset derivatives. Preliminary scenario-based proofs for some other derivative-types have been carried out and they are summarised in Appendix 5.E: for American spread options, American basket options and American options on the minimum of two assets, there is evidence of heteroscedasticity. This supports the need to conduct this future research.

Moreover, an additional field that requires further investigation is the estimation of the standard deviation of the errors that is used within the wLSMC algorithm to correct for heteroscedasticity. The results we have for American put options are already satisfactory, and it will be interesting to study approximations for the other payoff structures. It will be of particular interest to approximate the standard deviations for multi-asset options. In particular, we will investigate the use of the moment-matching methodology in Chapter 2 to estimate this standard deviation since, as stated, for these approximations the price of two European style derivatives is required.

Finally, additional study will be carried over the errors of the regression, and in particular on the bounds derived in Appendix 5.D.

Appendix

Appendix 5.A The linear inequality constrained least squares MC method (ICLS) of Létourneau and Stentoft (2014)

Létourneau and Stentoft (2014) refine the regression step 3b in the box LSMC on page 115. The aim of their method is to reduce the upward bias in the valuation of American options and they achieve this by substituting the OLS regression with a linear inequality constrained least squares (studied by Liew (1976)). In particular, they impose structure on the estimated continuation value function $\hat{C}_i(\cdot)$, by considering two types of constraints:

- convexity of the continuation value with respect to the underlying asset price;
- boundedness of the continuation value slope with respect to the underlying asset price (delta parameter),

which reflect the theoretical properties of the continuation value discussed in Section 3.2.1.

These two types of constraints are imposed on a grid of k points for the underlying asset values (see also Beresteanu (2007)). Overall, Létourneau and Stentoft impose k constraints: $k-2$ convexity constraints and just 2 slope constraints, since the convexity constraints ensure the monotonicity of the slopes and, consequently, the slope constraints are required for only the smallest and largest points of the grid. As they point out, imposing constraints on a discrete grid does not guarantee that the function $\hat{C}_i(\cdot)$ respects the constraint over its whole domain. However, they choose $k=6$ and suggest increasing the value of k and consequently they have a finer grid, which reduces the chances to break the constraints where they are not imposed. In the numerical study we carry out in Section 5.3.2, we also impose $k=6$.

To summarise, the ICLS algorithm is equal to the LSMC where step 3b in LSMC is substituted with the step $3b^{ICLS}$ below.

ICLS Steps of the linear inequality constrained least squares Monte Carlo method (ICLS)

$3b^{ICLS}$ Run a linear inequality constrained least squares regression on the pairs $\left(S_{i(j)}, V_{i+1}\left(S_{i+1(j)}\right)\right)$ for $j \in \tilde{J}_i$, with basis functions $\psi_l(\cdot)$, to determine $\beta_{i,l}^{ICLS}$;

Appendix 5.B Review of the heteroscedasticity tests employed

In the numerical proof in Section 5.2.2, for each of the regressions 3b in LSMC on page 115, we consider the following heteroscedasticity tests:

Park's test The test checks whether, in the regression

$$\log \hat{\boldsymbol{u}}_i^2 = \varphi_0 + \varphi_1 \log \boldsymbol{S}_i, \tag{5.35}$$

the coefficient φ_1 is different from 0, where $\hat{\boldsymbol{u}}_i$ and $\boldsymbol{S}_i$ are column vectors made by $\hat{u}_{i,j}$ and $S_{i(j)}$, $j \in \tilde{J}_i$, respectively and the squared residuals are used as a proxy of the variances.

White's general heteroscedasticity test The test checks whether, in the regression

$$\hat{\boldsymbol{u}}_i^2 = \varphi_0 + \varphi_1 \hat{C}_i(\boldsymbol{S}_i) + \varphi_2 \hat{C}_i^2(\boldsymbol{S}_i), \tag{5.36}$$

the coefficients φ_l, $l = 1,2$ are all different from 0, where $\hat{C}_i(\cdot)$ is as in (5.1).

Breusch-Pagan-Godfrey test Given the regression

$$\hat{\boldsymbol{u}}_i^2 = \varphi_0 + \sum_{l=1}^{k} \varphi_l \psi_l(\boldsymbol{S}_i), \tag{5.37}$$

the test checks whether the coefficients φ_l, $l = 1,\ldots,k$ are different from 0, where $k \leq M$, M is the number of basis function considered in the regression (5.1) and $\psi_l(\cdot)$ are the basis functions.

More details can be found in Gujarati and Porter (2008) and Greene (2012).

Appendix 5.C Derivation of the approximations for the conditional variances of the errors

In this section, we prove formulae (5.26)-(5.29). Firstly, we calculate formula (5.26) when y is normally distributed with $y \sim N(\mu_N, \sigma_N^2)$:

$$\begin{aligned}
E_{X,l} &= E\left[e^{ls_{i+1}} X_{i+1} | S_i = S_{i(j)}\right] = \int_{-\infty}^{+\infty} e^{ly} I\left(s_{i+1} < \frac{S_{f_{i+1}}}{S_{i(j)}}\right) f(y)dy \\
&= \int_{-\infty}^{\log \frac{S_{f_{i+1}}}{S_{i(j)}}} e^{ly} f(y)dy = \int_{-\infty}^{\log \frac{S_{f_{i+1}}}{S_{i(j)}}} e^{ly} \frac{1}{\sqrt{2\pi}\sigma_N} e^{-\frac{(y-\mu_N)^2}{2\sigma_N^2}} dy \\
&= e^{l\mu_N + 0.5l^2\sigma_N^2} \int_{-\infty}^{\frac{1}{\sigma_N}\left(\log \frac{S_{f_{i+1}}}{S_{i(j)}} - \mu_N\right)} \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-l\sigma_N)^2}{2}} dy
\end{aligned} \tag{5.38}$$

where $I(x)$ is the indicator function with value 1 if x is true (see also the definition of X_{i+1} in (5.10)). Formula (5.26), for both geometric Brownian motion and exponential Ornstein-Uhlenbeck process, follows directly from (5.38) since the log-returns are normally distributed with $s_{i+1} \sim N(\mu_s, \sigma_s^2)$. Formula (5.27) follows from (5.38) by conditioning first on the event of not having a jump and then on the event of having a jump. In both cases the log-returns are normally distributed and $s_{i+1}|_{\text{no-jumps}} \sim N(\mu_s, \sigma_s^2)$ and $s_{i+1}|_{\text{no-jumps}} \sim N(\mu_{sJ}, \sigma_{sJ}^2)$. We assume that the probability of multiple jumps in the interval Δ_t is null.

Additionally, the first addend of formula (5.28) also follows from (5.38) and the density function f_Z of the random variable $Z_t = \sigma \tilde{W}_t + J_t^K$ is given by the convolution of the density of $\sigma_N \tilde{W}_t$ ($f_{\tilde{W}}(\cdot)$) and the density of J_t^K ($f_J(\cdot)$) where:

$$\begin{aligned}
f_{\tilde{W}}(x_1) &= \frac{1}{\sigma_Z\sqrt{2\pi}} e^{-\frac{x_1^2}{2\sigma_Z^2}}, \text{ with } \sigma_Z = \sigma\sqrt{\Delta_T} \\
f_J(x_2) &= q\eta_1 e^{-\eta_1 x_2} I(x_2 \geq 0) + (1-q)\eta_2 e^{\eta_2 x_2} I(x_2 < 0).
\end{aligned}$$

In particular, we have:

$$\begin{aligned}
f_Z(y) &= \int_{-\infty}^{+\infty} f_{\tilde{W}}(y - x_2) f_J(x_2) dx_2 = \int_{-\infty}^{+\infty} f_{\tilde{W}}(x_2 - y) f_J(x_2) dx_2 \\
&= \int_0^{+\infty} \frac{1}{\sigma_Z\sqrt{2\pi}} e^{-\frac{(x_2-y)^2}{2\sigma_Z^2}} q\eta_1 e^{-\eta_1 x_2} dx_2 \\
&\quad + \int_{-\infty}^0 \frac{1}{\sigma_Z\sqrt{2\pi}} e^{-\frac{(x_2-y)^2}{2\sigma_Z^2}} (1-q)\eta_2 e^{\eta_2 x_2} dx_2
\end{aligned}$$

$$
\begin{aligned}
&= q\eta_1 e^{0.5\sigma_Z^2\eta_1^2 - y\eta_1} \int_{-\frac{y}{\sigma_Z}}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(z+\eta_1\sigma_Z)^2}{2}} dx_2 \\
&\quad + (1-q)\eta_2 e^{0.5\sigma_Z^2\eta_2^2 + y\eta_2} \int_{-\infty}^{-\frac{y}{\sigma_Z}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-\eta_2\sigma_Z)^2}{2}} dx_2.
\end{aligned} \tag{5.39}
$$

Formula (5.29) follows directly from (5.39).

Equivalently, for call options we have:

$$
\begin{aligned}
E_{X,l} &= E\left[e^{ls_{i+1}} X_{i+1} | S_i = S_{i(j)}\right] = \int_{\log \frac{s_{f_{i+1}}}{S_{i(j)}}}^{+\infty} e^{ly} f(y) dy \\
&= e^{l\mu_N + 0.5l^2\sigma_N^2} \int_{\frac{1}{\sigma_N}\left(\log \frac{s_{f_{i+1}}}{S_{i(j)}} - \mu_N\right)}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-l\sigma_N)^2}{2}} dy
\end{aligned} \tag{5.40}
$$

and formula (5.31) follows directly.

Appendix 5.D Non-normality of the regression errors

This section shows some additional results on the errors of the regressions at step 3b in the LSMC method. Let us consider the financial market defined in Section 5.1.2, then we can prove the following result.

Proposition 5.D.1 (Bounds for the errors: put option)**.** *The error u_i of the regression at time-step i in the LSMC algorithm for an American put option is bounded by:*

$$
C_i(S_i) \leq u_i \leq r_{0,i}K - C_i(S_i). \tag{5.41}
$$

Proof. Starting from the definition of the error in equation (5.3), we prove the lower bound in (5.41) by using the fact that the American put option price has a lowest value equal to 0, when the spot price goes to infinity. Furthermore, we prove the upper bound by using the fact that the strike price K is an upper bound of the option price. In (5.41), we discount the strike price K since $V_{i+1}(\cdot)$ is given in time-0 dollars. □

Equivalently, for an American call options we have:

$$
u_i \geq -C_i(S_i) \tag{5.42}
$$

since the lowest attainable value for $V_{i+1}(S_{i+1})$ is zero.

As an exemplification, in Figure 5.D.1 we plot the residuals of one of the regressions in the LSMC for American call and put options with $r = 0.06$, $K = 1$ and underlying asset price dynamics following geometric Brownian motion with $\sigma = 0.2$ and $S_0 = 0.9$. Furthermore, we assume $\delta = 0$ for the put option and $\delta = 4\%$ for the call option.

Finally, by considering Proposition 5.D.1, the following additional remark can be done.

Remark 5.D.2. *The errors of the regressions in the LSMC algorithm are not normally distributed, since they are bounded.*

Appendix 5.E Graphical proof of heteroscedasticity for other payoffs

The results in Section 5.2 are not limited to standard American call and put options. In this section, we consider the graphical proof carried out in Section 5.2.3 for options written on two assets: a spread option, a basket put option and an option on the minimum of two assets.

In particular, we consider the price dynamics of the two assets $\left\{S_t^{(1)}\right\}_{t\geq t_0}$ and $\left\{S_t^{(2)}\right\}_{t\geq t_0}$, to be two dependent geometric Brownian motions with $\rho = corr(\tilde{W}_t^{(1)}, \tilde{W}_t^{(2)}) = 0.3$ where $\tilde{W}_t^{(1)}$ and $\tilde{W}_t^{(2)}$ are the Wiener processes driving the two assets, respectively. Additionally, the volatility is 0.2 for the first asset and 0.4 for the second, the strike price is $K = 1$ and time-to-maturity is $T = 1$ year. The plots in Figure 5.E.1 are the conditional standard deviations

$$\mathrm{std}(S^{(1)}, S^{(2)}) = \sqrt{\mathrm{Var}\left[u_i | S_i^{(1)} = S^{(1)}, S_i^{(2)} = S^{(2)}\right]} \tag{5.43}$$

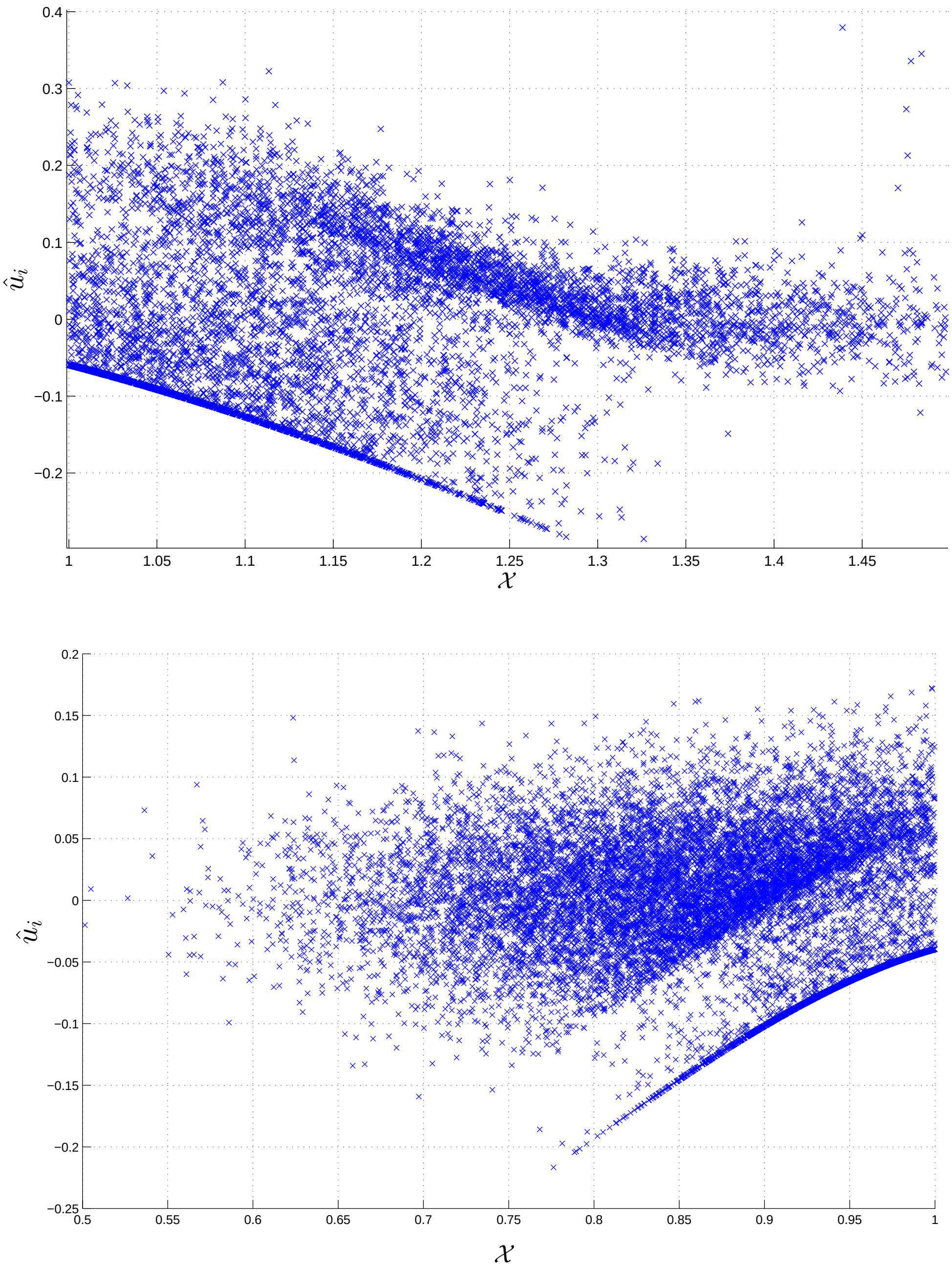
for

$$u_i = V_{i+1}\left(S_{i+1}^{(1)}, S_{i+1}^{(2)}\right) - C_i\left(S_i^{(1)}, S_i^{(2)}\right) \tag{5.44}$$

and $V_i(\cdot,\cdot)$ and $C_i(\cdot,\cdot)$ being the bi-dimensional versions of the value option and the continuation value of the option at time t_i, respectively. The three plots refer to time-step $i = 5$ i.e. $t_5 = 0.5$ ($\Delta_t = 0.1$ years) years with a time-to-maturity of 0.5 years. The same grid as in Section 5.2.3 has been considered for the simulation exercise ($\Delta_S = 0.05$).

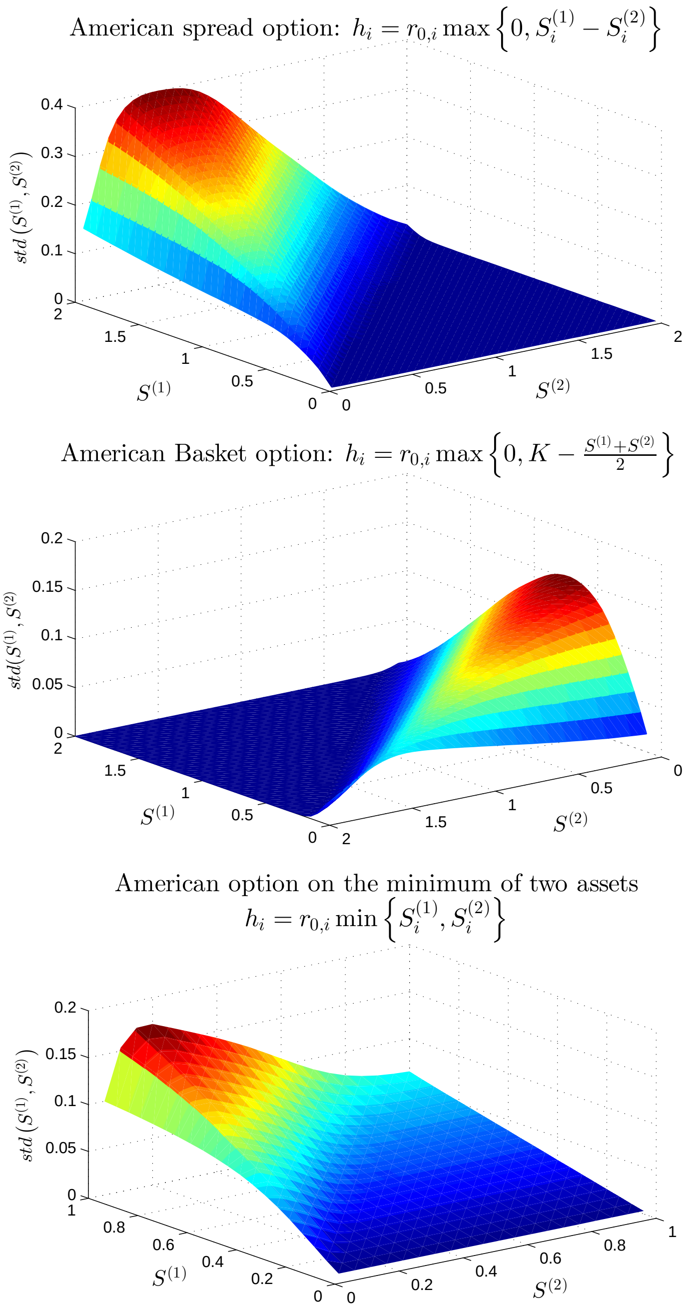
For the options considered, the plots in the figure show the existence of heteroscedasticity since for each cross-section, the conditional standard deviation changes with the underlying spot price.

Figure 5.D.1 Regression residuals in the algorithm of Longstaff and Schwartz (2001)



Note: This figure plots for an American call option (above) and an American put option (below) the residuals of the OLS regression for one time-step. The underlying asset price dynamics is the geometric Brownian motion with $r = 0.06$, $K = 1$, $\sigma = 0.2$ and $S_0 = 0.9$. Additionally, for the call option $\delta = 3\%$, while for the put option $\delta = 0$.

Figure 5.E.1 Heteroscedasticity in the regressions of the LSMC algorithm via simulations: multi-asset options



Note: The three plots show the conditional standard deviation of the errors as a function of the two underlying asset values, $S_i^{(1)} = S^{(1)}$ and $S_i^{(2)} = S^{(2)}$. The plots are created as described in Section 5.2.3.

Chapter 6

Empirical study of pricing performances: Options and LEAPS® on the S&P 100™ index

In this chapter, we examine the performances of the pricing methods introduced in Chapters 4 and 5 over real financial data. To this end, we consider American put and call options and LEAPS® on the S&P 100™ index traded from February 2012 to December 2014. This empirical performance study is complementary to the scenario-based studies carried out in the previous two chapters and shows that, in many cases, the 'extension' method and the weighted least squares Monte Carlo method both outperform the corresponding competitor methodologies when real financial data is considered. The chapter is structured as follows: Section 6.1 reviews other research on S&P 100™ options; Section 6.2 describes the dataset and the methodology we employ to carry out the empirical study; Section 6.3 summarises the results; and Section 6.4 concludes.

6.1 Introduction

In the previous two chapters, we introduced two methodologies to price American options and we carried out scenario-based comparisons,[1] which confirmed that our new methods outperform the existing methodologies when one employs different measures of error with respect to the fair benchmark prices calculated via binomial tree methodologies. In this chapter, we consider a complementary comparison and investigate how the 'extension' method and the weighted least squares Monte Carlo method perform over real financial

[1] See Sections 4.3.1, 4.4.2 and 5.3.2.

data. For this purpose, we employ options on the S&P 100™ stock index traded from 15 February 2012 to 10 December 2014.

We chose to work with the S&P 100™ index for two main reasons. First, this index is the only one that has exchange-listed options of American (with ticker OEX®) and European (with ticker XEO®) exercise types. This is an essential feature for the implementation of the comparison: we employ the European contracts to estimate the unknown quantities, volatility and dividend yield of the underlying asset (as shown in Section 6.2.2), and then we price the corresponding American contracts by using these implied parameters as inputs. Additionally, the CBOE lists LEAPS® on the S&P 100™ index and we can consequently also evaluate our 'extension' method for long-term options, where the existing methodologies are known to perform badly.

S&P 100™ options have been used by many researchers to evaluate models and methods for American options.[2] However, the empirical studies on American options are relatively fewer than the correspondent ones for European options.[3] Among them, Evnine and Rudd (1985) examined the OEX® options using hourly data from June to August 1984 and found that 2.7% of the S&P 100™ call option prices violated the non-arbitrage bounds although, since the underlying indexes are not traded contracts, these arbitrage opportunities were not easily exploitable. Whaley (1982), Sterk (1983) and Geske and Roll (1984) studied the performance of the correction[4] proposed by Black (1975) to price American call option by correcting the European option prices. They found that this formula is biased and that the Roll-Geske-Whaley formula (see Section 3.3.1) performs better. Sheikh (1991) found that the XEO® implied standard deviation distributions over the period 1983 to 1985 were skewed and leptokurtic. Day and Lewis (1992) found that the XEO® implicit volatilities were an almost unbiased and informative forecast of the subsequent weekly volatility (data from November 1983 to December 1989). In contrast, Canina and Figlewski (1993) studied the performance of the XEO® implied volatilities from March 1983 to March 1987 in forecasting future realized volatility and found that they did not have any forecasting power. Fleming et al. (1996) studied how the option market forecasts the underlying asset price and found that XEO® options anticipated changes in the underlying stock index value by about 5 minutes (data from January 1988 to March 1991). Harvey and Whaley (1992a,b) used daily prices of OEX® options from March 1983 to December 1989. By employing a dividend-adjusted binomial tree method, they found significantly higher implied volatil-

[2]Literature reviews on this topic can be found in Bates (1996) and references within.

[3]For a list of references see Bakshi et al. (1997) and reference within.

[4]Black (1975) priced American call options on a underlying asset paying one (known) dividend as the higher of the prices of a European call option where the underlying spot price is net the dividend, and a European call option where the time to ex-dividend is substituted for the time-to-maturity.

ities for put options than call options. Additionally they found that the early exercises of American option were mainly driven by dividends. Linaras and Skiadopoulos (2005) examined the pricing performance of several implied tree methods by using S&P 100™ option prices.[5] Stentoft (2005, 2008) employed the least squares Monte Carlo method of Longstaff and Schwartz (2001) to price American options on assets whose price dynamics follow a Gaussian GARCH model and a normal inverse Gaussian (NIG) GARCH model, respectively. They use weekly observations on the S&P 100™ options traded[6] from 1991 to 1995 and the results show that both the model outperform the Black-and-Scholes method (with constant volatility) and that the NIG GARCH model outperform the Gaussian GARCH, particularly for out-of-the-money options. Broadie et al. (2000a,b) consider end-of-the-day daily data of OEX® options from 3 January 1984 to 30 March 1990, together with the observations of the exercise decisions of investor data. They estimate how the optimal exercise price is perceived by the market participants using non-parametric methods. They found that the exercise behaviours before and after the crash of 1987 are similar and this goes against the part of literature that suggests the changes occurred in option market after the crisis of 1987.

In the following, we employ a methodology similar to that used by Linaras and Skiadopoulos (2005) and we evaluate our two techniques: the 'extension' method and the wLSMC algorithm. In Section 6.2, we describe the comparison methodology and the dataset we employ in the comparison and, in Section 6.3, we summarise our findings.

6.2 Dataset description and comparison methodology

We consider both OEX® (American) and XEO® (European) options written on the S&P 100™ index traded on the Wednesdays between 15 February 2012 and 10 December 2014.[7] Overall, we consider options data for 148 days, spanned over the three years. We retrieved the data from the Bloomberg database: for both the XEO® and the OEX®, we collected ask and bid prices, open interest, volume and the contract specifications (strike, maturity date and exercise-style). Additionally, we collected the last prices of the S&P 100™ index and the US dollar Libor for seven days, one month, three months, six months and one year. We use the Libor rates as proxy of the risk-free rate and we employ linear interpolation for maturities shorter than one year and linear regression for maturities longer than one year.

[5] See also Section 3.3.2 for more details.

[6] The same test was carried for options on General Motors (GM), International Business Machines (IBM), and Merck & Company, Inc. (MRK).

[7] Three Wednesdays, namely 4$^{\text{th}}$ July 2012, 25$^{\text{th}}$ December 2013 and 1$^{\text{st}}$ January 2014, were closing days of the CBOE and consequently, the data were collected for the previous day.

6.2.1 Data screening

In order to reduce possible sources of error, we filtered the data using a methodology similar to that outlined by Linaras and Skiadopoulos (2005). From the XEO® options (European), we discarded options with either (1) a zero volume and zero open interest, (2) a premium smaller than \$0.5, (3) a maturity smaller than 50 trading days and (4) a negative implied dividend yield (see Section 6.2.2) since that corresponds to the existence of arbitrages. Given the implied dividend yield, we also discard options whose prices did not satisfy the static bounds. The filters we applied are slightly different than those in Linaras and Skiadopoulos (2005) since we are interested in longer maturity options, i.e. LEAPS® to test the improvement introduced by the 'extension' method. Then, we discarded the OEX® options (American) for which there is no respective XEO®, for which the early exercise premium (American-option price minus European-option price) is negative, and for which ask or bid prices are unavailable. Table 6.1 details the number of contracts left after the application of each filter: from the initial 84,394 contracts, after the screening we were left with 7,320 options (3,562 calls and 3,758 puts). Table 6.2 summarises the statistics of the retained options. These options are classified with respect to time-to-maturity as:

- Short-maturity options with maturities from 50 to 250 trading days;
- Medium-maturity options with maturities from 250 to 500 trading days;
- Long-maturity options with maturities of more than 500 trading days.

With the term LEAPS®, one usually refers to the third category. S&P 100™ LEAPS® are usually traded for hedging and investment purposes over the entire American market for a time that can be measured in years rather that months as for standard options. Although these contracts are traded with lower frequency (lower liquidity) than standard options since they have very long maturities, they account for about 5/10% of the S&P 100™ option market and their pricing is interesting for large part of option traders.

The next section shows how XEO®(European) options are employed to imply some of the parameters that will then be used in the pricing of the corresponding American options.

6.2.2 Calculation of implied parameters

S&P 100™ is a dividend-paying asset and, consequently, in order to price the OEX® options, we proceed by estimating the dividend yield over the life of each contract. As in Aït-Sahalia and Lo (2002) and Linaras and Skiadopoulos (2005), we imply the dividend

Table 6.1 Screening of S&P 100™ options data

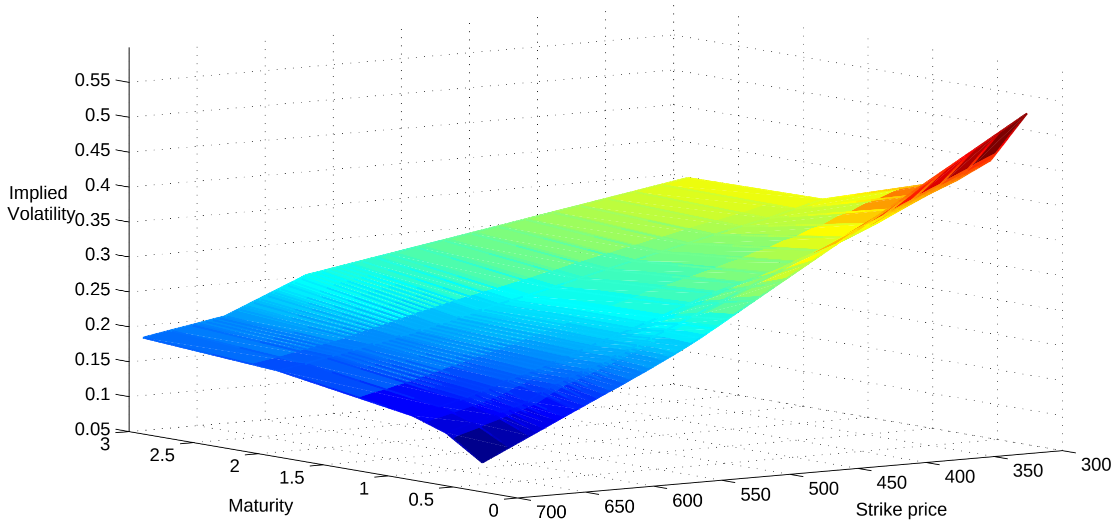
Filters	No. of contracts
Initial contracts in dataset	84,394
- Zero volume and open interest (XEO®)	27,149
- Premium XEO® smaller than $0.5	24,146
- Maturities shorter than 50 trading days	13,245
- Negative implied dividend yield or no enough near-the-money options	12,279
- Bounds not satisfied (XEO®)	11,976
- OEX® with no correspondence to XEO®	11,881
- Negative early exercise premium (OEX®)	10,574
- Missing ask or bid price (OEX®)	10,148
- Zero volume and zero open Interest (OEX®)	7,320
Call Options	3,562
Put options	3,758

Note: This table indicates the number of options left in the dataset after the application of each filter. The data was retrieved from the Bloomberg database every Wednesday over the period 15 February 2012 to 10 December 2014. XEO® indicates the European style contracts and OEX® indicates the American style contracts.

Table 6.2 Summary statistics for the OEX®(American) options on S&P 100™

	Short			Medium			Long		
	Call	Put	Both	Call	Put	Both	Call	Put	Both
Mean Mid-price	91.57	20.57	57.26	145.80	29.73	78.90	87.23	43.57	65.30
Mean bid-ask spread	3.20	2.15	2.70	6.53	3.60	4.84	7.32	4.33	5.82
Mean $\frac{\text{Ask}-\text{Bid}}{\text{Mid}}$ spread	9.23%	23.14%	15.95%	5.36%	21.83%	14.85%	8.89%	11.82%	10.36%
Mean maturity	0.49	0.51	0.50	1.45	1.49	1.47	2.45	2.37	2.41
No. of retained options	2124	1987	4111	909	1237	2146	529	534	1063

Figure 6.1 Implied volatility surface for the S&P 100™ options on 14 March 2012



yield for each trading day and maturity date from the put-call parity

$$p_t(S_t, T, K) + S_t e^{-\delta(T-t)} = c_t(S_t, T, K) + K e^{-r(T-t)} \tag{6.1}$$

where p_t and c_t are the XEO® (European) put and call prices of at-the-money options and all the other quantities are as defined in the previous chapters. The prices of at-the-money options were calculated from the prices of the nearest-the-money call and put options by linear interpolation. Consequently maturities with less than two call options or two put options are discarded. Additionally, since negative implied dividend yields suggest the existence of arbitrage opportunities, we also discarded options with this characteristic.

The second input we imply from the XEO® market prices is the volatility of the underlying asset. We follow Skiadopoulos et al. (1999), Brandt and Wu (2002), Panigirtzoglou and Skiadopoulos (2004) and Linaras and Skiadopoulos (2005) and we employ out-the-money and at-the-money XEO® (European) call and put option prices. The in-the-money options are not employed because: they are illiquid and consequently carry higher measurement errors; and they are redundant and can be calculated from out-the-money options via the put-call parity.

Figure 6.1 shows an example of volatility surface for the 14 March 2012: the volatility surface is a function of strike price and maturity, which decreases with strike price and evens out as maturity increases, or in other words, the volatility skew decreases. Similar shapes are found for the other dates.

6.2.3 Measures of pricing error

For our comparison, we employ the six error measures in Brandt and Wu (2002) and Linaras and Skiadopoulos (2005). These are:

- average difference between each model price and the OEX® mid-price. This measure is defined as MVE, the mean valuation error and it is positive when the model overprices the market on average;
- square-root of the average squared difference between the OEX® model price and the mid-price. This measure is defined as RMSVE, the root mean squared valuation error;
- percentage of time the model price is included within the market bid-ask spread. This measure is defined as FIBA, the frequency in bid-ask;
- average error outside of the bid-ask spread, which is named the MOE, mean outside error. The error outside the spread is defined as the model price minus the bid (resp. ask quote) if the model price is below (above) the bid price (ask price) and is fixed to zero for the cases when the model price falls within the bid-ask spread. This is a measure of symmetry in model errors and of bias;
- variability of the errors outside the bid-ask spread, which is defined as the root mean squared outside error, RMSOE;
- average outside error divided by the market price, which corresponds to the MOE but in percentage terms. This is defined as the MROE, mean relative outside error.

6.3 Pricing performance

This section constitutes the main contribution of the chapter. It is divided into two parts: in the first part, we show the pricing performances of the 'extension' method (Chapter 4) and its competitors; the second part shows the pricing performances of the weighted least squares Monte Carlo method (Chapter 5) compared to the LSMC by Longstaff and Schwartz (2001) and ICLS by Létourneau and Stentoft (2014). Additionally, in the first part we report the pricing performances of the asymptotic expansions of the optimal exercise price as in Section 4.4.

6.3.1 Results for the ‘extension’ method (Chapter 4)

Quasi-analytic methods

As in Chapter 4, we studied the ‘extension’ method when it is employed together with the compound method in Geske and Johnson (1984), the quadratic method in Barone-Adesi (2005), the interpolation method in Li (2010b), the integral method in Kim (1990), the static-replicating portfolio method in Chung and Shih (2009), and the improved quadratic method in Ju and Zhong (1999). Tables 6.3, 6.4 and 6.5 summarise the performances respectively for long, medium and short maturities. Considering the analysis in Section 4.2.3, we select the same ratio $\vartheta^{*,\mathscr{M}} = t_x/T$ for each method. This ratio is 0.2 for short-maturity options, 0.3 for medium-maturity options and 0.5 for long-maturity options.

First, from the three tables we notice that for many of the existing methodologies the most challenging maturity range, when pricing real financial options on the S&P 100™ index, is that between 250 and 500 trading days. We also note that the nine methods have comparable performances and there are not such big differences as we found in the scenario-based comparison (see Figure 4.2).

The ‘extension’ method has been introduced in Chapter 4 to solve the problems of the existing quasi-analytic methodologies in pricing long-term options. Considering the results for long maturities (LEAPS®), we see that the ‘extended’-version of many methods (BAW, LI, K3, CS2, CS3 and JZ) outperforms the standard version over all six error measures. Overall, each extended method price is closer to the benchmark OEX® mid-price and more condensed around it, and the prices outside the bid-ask spread are more symmetric and smaller in size. Remarkable improvements are also reached over the methods K2 where the ‘extended’ version outperforms the standard one over five error measures with the exception of MVE. The extended versions of GJ2 and GJ3 perform slightly worse than their standard versions for the measures FIBA, MOE, RMSOE and MROE while they outperform the respective standard version over the MVE and RMSVE measures. Our extended version of GJ2 reaches an extraordinary MVE of 0.005, which is by far the best result over long maturities.

For medium-term options, the ‘extension’ method also significantly improves on the methods GJ2, GJ3, BAW, LI, K2 over all six measures. The (standard) methods K3, CS2, CS3 and JZ perform very well for this maturity range and the extension method, with the selected ratio $\vartheta^{*,\mathscr{M}}$, is underperforming by comparison with the corresponding standard method. However, the ‘extension’ method outperforms these 4 methods over the MOE and MROE measures, being then less biased outside the bid-ask spread.

Not surprisingly, the ‘extension’ method also outperforms GJ2, BAW, LI, K2, K3 over

Table 6.3 Empirical performances on S&P 100™ LEAPS®: Standard versus 'extended' pricing methods over LONG maturities

		MVE	RMSVE	FIBA	MOE	RMSOE	MROE
GJ2	S	-0.575	1.573	96.049%	0.041	0.644	0.044%
	E	0.005	1.446	94.450%	0.087	0.719	0.099%
GJ3	S	-0.232	1.412	95.767%	0.064	0.676	0.072%
	E	-0.059	1.410	95.014%	0.077	0.696	0.087%
BAW	S	0.562	1.742	91.063%	0.165	0.882	0.192%
	E	0.281	1.567	92.380%	0.123	0.796	0.143%
LI	S	0.446	1.651	91.816%	0.143	0.844	0.165%
	E	0.202	1.529	92.756%	0.112	0.778	0.130%
K2	S	0.139	1.496	93.039%	0.106	0.746	0.122%
	E	0.162	1.488	93.133%	0.103	0.743	0.118%
K3	S	0.414	1.679	91.722%	0.150	0.827	0.170%
	E	0.293	1.570	92.286%	0.126	0.795	0.145%
CS2	S	0.381	1.646	91.910%	0.140	0.816	0.159%
	E	0.360	1.612	92.098%	0.136	0.816	0.156%
CS3	S	0.422	1.667	91.816%	0.146	0.831	0.166%
	E	0.400	1.640	92.098%	0.142	0.828	0.162%
JZ	S	0.336	1.581	92.380%	0.127	0.793	0.145%
	E	0.254	1.550	92.474%	0.120	0.788	0.139%

Note: This table shows the pricing performances for the methods GJ2 and GJ3 by Geske and Johnson (1984), BAW by Barone-Adesi and Whaley (1987), LI by Li (2010b), K2 and K3 by Kim (1990), CS2 and CS3 by Chung and Shih (2009) and JZ by Ju and Zhong (1999). 'S' stands for standard and 'E' for extended, i.e. respectively the existing methods in the literature and our corresponding 'extended' versions via the 'extension' method. The results are for options with maturities above 500 trading days. The six measures of error are detailed in Section 6.2.3.

the six measures while the performances of the standard GJ3, CS2, CS3 and JZ and our extended versions are virtually equal.

These results indicate that, as shown by the scenario-based comparison in Section 4.3.1, the 'extension' method can be applied over many maturity ranges with remarkable improvements over the existing methodologies.

Asymptotic expansions of the optimal exercise price

In addition, we carried out a similar analysis employing asymptotic expansions of the optimal exercise price. As stated in Chapter 4, many of these methods provide the optimal exercise price for only short maturities and consequently they cannot be implemented straightforwardly for long maturity options such as OEX® LEAPS®. We employ the integral equation method by Kim (1990) together with the 'extension' method and we can price options

Table 6.4 Empirical performances on S&P 100™ options: Standard versus 'extended' pricing methods over MEDIUM maturities

		MVE	RMSVE	FIBA	MOE	RMSOE	MROE
GJ2	S	-1.061	2.941	75.582%	-0.324	1.655	-0.234%
	E	-0.678	2.467	78.611%	-0.088	1.314	-0.032%
GJ3	S	-0.732	2.567	78.052%	-0.142	1.404	-0.085%
	E	-0.783	2.567	77.866%	-0.135	1.358	-0.062%
BAW	S	-0.276	2.188	83.970%	0.073	1.220	0.095%
	E	-0.242	2.179	84.296%	0.081	1.222	0.100%
LI	S	-0.324	2.218	83.085%	0.054	1.231	0.077%
	E	-0.292	2.209	83.411%	0.064	1.231	0.085%
K2	S	-0.629	2.550	78.658%	-0.111	1.439	-0.017%
	E	-0.626	2.430	79.031%	-0.061	1.294	-0.005%
K3	S	-0.215	2.212	83.737%	0.078	1.249	0.109%
	E	-0.475	2.323	80.475%	0.006	1.256	0.048%
CS2	S	-0.136	2.149	86.114%	0.102	1.224	0.114%
	E	-0.351	2.251	82.246%	0.052	1.241	0.081%
CS3	S	-0.125	2.153	85.927%	0.107	1.228	0.119%
	E	-0.281	2.216	83.178%	0.074	1.236	0.097%
JZ	S	-0.206	2.185	84.436%	0.083	1.233	0.097%
	E	-0.588	2.411	79.310%	-0.043	1.287	0.013%

Note: This table shows the results for options with maturities of between 250 and 500 trading days. See Table 6.3 for additional notes.

Table 6.5 Empirical performances on S&P 100™ options: Standard versus 'extended' pricing methods over SHORT maturities

		MVE	RMSVE	FIBA	MOE	RMSOE	MROE
GJ2	S	-0.697	1.248	83.946%	-0.104	0.503	-0.167%
	E	-0.664	1.178	84.456%	-0.090	0.436	-0.135%
GJ3	S	-0.618	1.116	86.621%	-0.073	0.426	-0.127%
	E	-0.698	1.250	83.508%	-0.109	0.479	-0.152%
BAW	S	-0.517	0.976	90.027%	-0.046	0.358	-0.088%
	E	-0.508	0.960	90.124%	-0.044	0.353	-0.085%
LI	S	-0.530	0.992	89.832%	-0.049	0.368	-0.094%
	E	-0.523	0.979	89.638%	-0.047	0.361	-0.091%
K2	S	-0.638	1.168	85.989%	-0.099	0.476	-0.124%
	E	-0.533	0.985	89.297%	-0.048	0.362	-0.094%
K3	S	-0.504	0.957	89.443%	-0.048	0.354	-0.083%
	E	-0.478	0.923	90.854%	-0.038	0.340	-0.079%
CS2	S	-0.459	0.906	91.121%	-0.036	0.338	-0.080%
	E	-0.459	0.908	91.194%	-0.036	0.339	-0.079%
CS3	S	-0.450	0.900	91.340%	-0.035	0.337	-0.077%
	E	-0.451	0.901	91.340%	-0.035	0.338	-0.077%
JZ	S	-0.462	0.916	91.146%	-0.037	0.344	-0.079%
	E	-0.467	0.920	90.902%	-0.038	0.345	-0.079%

Note: This Table shows the results for options with maturities of between 50 and 250 trading days. See Table 6.3 for additional notes.

Table 6.6 Empirical performances on S&P 100™ options: 'Extended' asymptotic expansions of the OEP

			MVE	RMSVE	FIBA	MOE	RMSOE	MROE
Short	EKK	E	-0.684	1.442	85.892%	-0.147	0.751	-0.208%
	ZL	E	-0.940	2.019	80.564%	-0.337	1.256	-0.417%
	CZ	E	-0.542	1.027	88.640%	-0.060	0.388	-0.117%
	CZ-P	E	-0.565	1.017	88.494%	-0.057	0.368	-0.109%
Medium	EKK	E	-2.003	4.428	70.363%	-1.017	2.883	-0.725%
	ZL	E	-3.215	6.442	66.729%	-2.050	4.905	-1.669%
	CZ	E	-0.586	2.289	81.221%	0.027	1.243	0.062%
	CZ-P	E	-1.066	2.630	76.701%	-0.149	1.373	-0.062%
Long	EKK	E	-0.270	3.999	76.858%	-0.223	2.202	0.036%
	ZL	E	-2.163	7.511	70.931%	-1.747	5.526	-1.192%
	CZ	E	0.155	1.896	89.087%	0.152	0.881	0.180%
	CZ-P	E	-0.103	2.225	83.067%	0.135	0.903	0.168%

Note: This table shows the pricing performances for the 'extended' asymptotic expansion methods EKK by Evans et al. (2002), ZL by Zhang and Li (2010), and CZ and CZ-P by Cheng and Zhang (2012). The options are classified as Short (50-250 trading days), Medium (250-500 trading days) and Long (above 500 trading days) maturities. The six measures of error are detailed in Section 6.2.3.

with maturities as long as 5 years; this is more than the longest maturity listed today on S&P 100™ options. For the selection of t_x, we employ Figure 4.5 as discussed in Section 4.4.2. The performances are summarised in Table 6.6. On average, the performances of the five 'extended' asymptotic methods are better than some standard methods but are usually worse than the 'extended' versions.[8] Besides, the 'extended' versions of CZ, CZ-P and CZ-P-m are much better than that of CZ over almost any measure and any maturity range. Finally, the EKK method also has very good performance when considering real financial data. Its performance measures are close to the values of CZ, CZ-P and CZ-P-m but it is much shorter running time and it is much easier to implement.

6.3.2 Results for the weighted least squares Monte Carlo method (Chapter 5)

Finally, we carry out a similar analysis for the Monte Carlo methodologies in Chapter 5. We compare the new weighted least squares Monte Carlo method with the least squares Monte Carlo by Longstaff and Schwartz (2001) and the inequality constrained least squares Monte Carlo method by Létourneau and Stentoft (2014).

We considered for each method $\mathscr{M}$ (LSMC, ICLS and wLSMC) $M = 5$ basis functions.

[8] In Table 6.6, the CZ-P and CZ-P-m methods are presented together because they have virtually the same performances over the set of options considered.

Additionally, we let the number of paths increase with the values $\mathtt{n_S} \in \{1,000; 2,000; 10,000\}$. For each of the options considered, the model price is assumed to be equal to the mean over 100 independent simulations. Figures 6.2-6.3 summarise the six error measures in Section 6.2.3, for the three methods.

For short maturities, the LSMC and wLSMC methods perform similarly for $\mathtt{n_S} = 2,000$ and $10,000$ number of paths. However, our new wLSMC method performs better than LSMC when only $1,000$ paths are employed, as shown by the reduction in all of the error measures considered in Section 6.2.3. On the other hand, the performance of the ICLS is slightly worse than the other two under all of the measures for $\mathtt{n_S} = 2,000$ and $10,000$. However, the ICLS for $\mathtt{n_S} = 1,000$ performs very well and outperforms the other two methods under all the error measures.

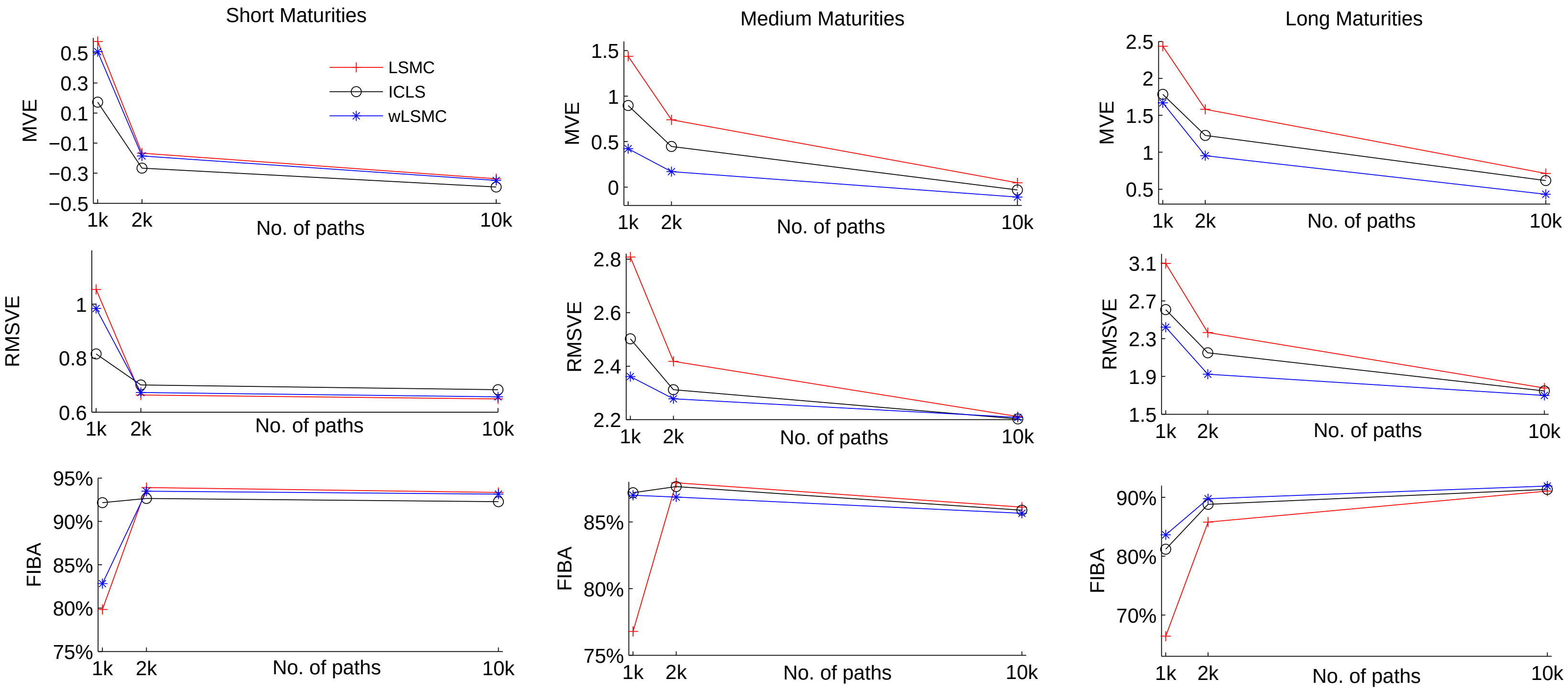
For medium and long maturities, the wLSMC improvements over the other two methods are more evident. The wLSMC algorithm outperforms the other two under all performance measures. The only exception is for medium-maturity options, under the FIBA measure, where our new method performs slightly worse than the other two for $\mathtt{n_S} = 2,000$ and $5,000$ steps. However, this is not of great importance for the valuation of the performance of the wLSMC, since on average the measures of error outside the bid-ask spread (i.e. MOE, RMSOE and MROE) are substantially smaller for the wLSMC than for the other two methods.

6.4 Conclusions

When introducing a new method for pricing derivatives, it is necessary to assess its performances with respect to those of the established methods for that particular class of financial instruments. In the previous two chapters, we tested the 'extension' method and the wLSMC method over a wide set of option scenarios, and we showed that, under the assumption of a specific model for the underlying asset, our two methodologies outperform the existing methodologies since our model prices are 'closer' to the fair benchmark price (calculated via a binomial tree method). However, the relevance of the method is established considering how close the model prices are to the market prices since it is there that the methodologies are used and, consequently, the market always has the last word. In this chapter, we considered options on the S&P 100™ index over a period of almost three years and we conclude that the methodologies discussed in Chapters 4 and 5 achieved the aims for which they were introduced.

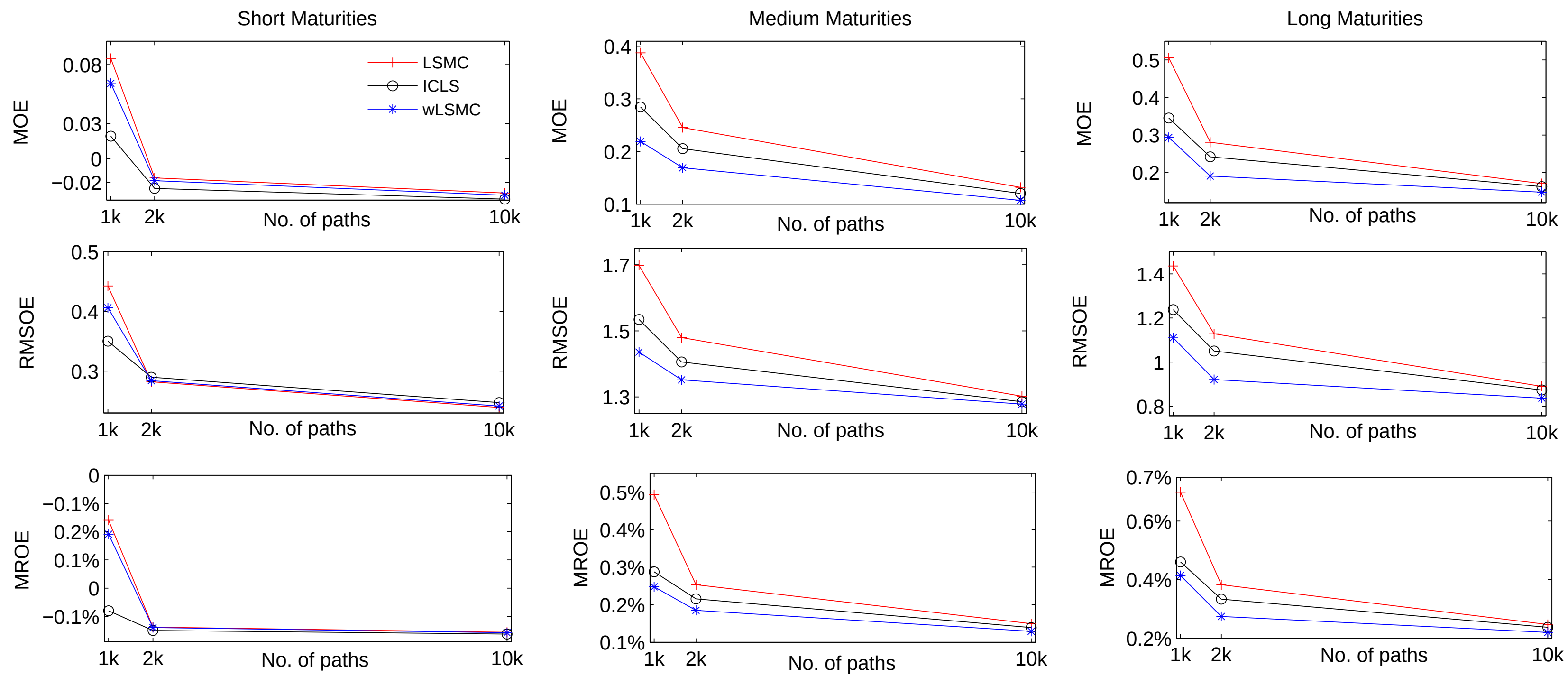
The 'extension' method is introduced in Chapter 4 to correct any quasi-analytic method for long-maturity options. In this chapter we show that it provides superior results compared to its competitors over long maturity options, i.e. OEX® LEAPS®. The only (partial) ex-

Figure 6.2 Empirical performances of the S&P 100™ options and LEAPS®: Monte Carlo simulation methods (1)



Note: This figure illustrates the pricing performance for the least squares Monte Carlo method (LSMC) of Longstaff and Schwartz (2001), the inequality constrained least squares Monte Carlo (ICLS) of Létourneau and Stentoft (2014) and the weighted least squares Monte Carlo in Chapter 5 (wLSMC). The options are classified as Short (50-250 trading days), Medium (250-500 trading days) and Long (above 500 trading days) maturities. The three measures of error are detailed in Section 6.2.3.

Figure 6.3 Empirical performances of the S&P 100$^{\text{TM}}$ options and LEAPS$^{®}$: Monte Carlo simulation methods (2)



Note: This figure illustrates the pricing performance for the least squares Monte Carlo method (LSMC) of Longstaff and Schwartz (2001), the inequality constrained least squares Monte Carlo (ICLS) of Létourneau and Stentoft (2014) and the weighted least squares Monte Carlo in Chapter 5 (wLSMC). The options are classified as Short (50-250 trading days), Medium (250-500 trading days) and Long (above 500 trading days) maturities. The three measures of error are detailed in Section 6.2.3.

ception is the method in Geske and Johnson (1984): the ‘extension’ method also improves on GJ2 and GJ3 (respectively their method with two and three time-steps) for the measures MVE and RMSVE but its performances are slightly worse than the standard versions although still better than the other seven methodologies. Therefore, the extended version of GJ3 is probably the best method (among those considered) to price long maturity options. Besides, the ‘extension’ method also performs very well for short and medium term options, outperforming in many cases the existing methodologies. Additionally, we considered the asymptotic expansions of the optimal exercise price. Also in this case, employing the ‘extension’ method we see good performance across different maturities.

Furthermore, the comparison over the S&P 100™ options shows that the weighted least squares Monte Carlo in many cases remarkably improves on the two best least squares Monte Carlo regression methods considered: the LSMC algorithm by Longstaff and Schwartz (2001) and the ICLS by Létourneau and Stentoft (2014). Our new method reduces pricing bias as shown by the MVE, which is similar to the mean relative error (MRE) employed in Chapter 5, and the price dispersion around the OEX® (American) mid-prices. Consequently, the new method constitutes a valid alternative to the other two and also creates smaller errors outside the bid-ask spread. Future work will be carried out to test these two methods over other asset classes and over an extended period of time.

Chapter 7

Conclusions

This thesis set out to explore new computational methods to price and hedge financial derivatives when several models for the underlying asset price dynamics are considered. We mainly focused on two types of derivatives, namely European basket options and American options, and we proposed three new methodologies: (1) an exact moment-matching procedure for the pricing and hedging of basket options for assets under displaced jump-diffusion processes; (2) a quasi-analytic method with the potential to improve almost any existing quasi-analytic method for pricing and hedging long-dated American options under the assumption of log-normal returns and (3) a simulation-based method that improves on the regression step of the least squares Monte Carlo method developed by Longstaff and Schwartz (2001) by correcting for heteroscedasticity.

The first method we proposed employs Hermite polynomial expansions and prices and hedges European basket options via a moment-matching technique. Many of the existing approaches impose strong assumptions either in terms of the price dynamics of the assets in the basket, or on the overall evolution of the basket. On the other hand, our new technique allows granular specification of price dynamics for each asset, and assumes more realistic models such as the displaced jump-diffusion process of Câmara et al. (2009) and the displaced version of the asymmetric jump-diffusion process of Ramezani and Zeng (2007), which account for negative skewness and excess kurtosis known to characterise equity stocks. Then, without assuming any dynamics for the overall basket value, we employ an expansion of a standard normal random variable to replicate the random variable representing the standardised basket return at maturity by a moment-matching technique. Using the properties of these polynomials, we obtain Black-and-Scholes type pricing and hedging formulae. As shown by an extensive scenario-based comparison, the new method produces small pricing errors as well as precise calculation of the Greek parameters for hedging purposes. Finally, we point out that the new method can be applied to other price dynamics as far as the moments of the overall basket can be calculated in closed form.

Second, we proposed a new quasi-analytic method to price American options under the geometric Brownian motion dynamics. The aim of our new approach is to improve existing methods available for pricing long-maturity options. We justified the introduction of this new technique by observing that the pricing approaches available in the literature work properly for short-medium maturity options, but their performance worsens the longer the time-to-maturity. The idea of our method for pricing a given option is to split its time to maturity into two parts that are priced separately: the one closest to the beginning of the contract is priced by approximating the optimal exercise price as a flat function, and the second part (i.e. the one closest to the expiration date) is priced by employing any existing pricing method. We term our new approach the ‘extension’ method. It can be considered a technique that works in conjunction with any existing method in the literature and extends the maturity range for which it attains small errors.

There are several advantages of this new technique. The first is that the new method achieves a more precise estimation of the optimal exercise price near expiration, since we focus any computational effort in the part where theory suggests that the optimal exercise price is more complicated to estimate (see Jacka (1991) and Chen et al. (2013)). The second advantage is that it incorporates the existing approaches to price options with shorter maturities, which is where they perform better. The third is that very low additional computational effort is required for the ‘extension’ interval, (i.e., for the first part of option life), since we provide an analytic closed-form formula. The fourth advantage is that the method facilitates precise and fast estimation of the delta and the gamma parameters of the options via a quasi-analytic formula. Additionally, we provided a convergence result of the new pricing technique to the perpetual option price when time-to-maturity approaches infinity, and we showed that the applications of the new pricing and hedging formula are not limited to standard quasi-analytic methods but it can also be employed to extend any asymptotic expansion of the optimal exercise price. These expansion techniques are currently the cutting-hedge methods for American option pricing and consist of a closed-form approximation of the optimal exercise price. However, although they perform very precisely for short-maturity options, they cannot be employed for long-maturity options. With the approach we proposed, one can price options with maturities as long as five years with pricing and hedging performances that are usually better than standard methods and much faster, since they consist of fully-analytic formulae. Finally, we tested our new method together with six standard quasi-analytic methods and three asymptotic expansions of the optimal exercise price, and we achieved improvements for a wide range of maturities (from few days to 5 years) and especially for long maturities, where the existing methods perform worst.

The last major contribution of this thesis is a new algorithm to improve on one of the most applied numerical methods for pricing American-style derivatives, the least squares regression method proposed by Longstaff and Schwartz (2001). The algorithm of Longstaff and Schwartz employs Monte Carlo simulations and ordinary least squares regressions for the estimation of the continuation value of an American-style derivative. The method we introduced, termed the weighted least squares Monte Carlo, follows the same steps as those in the Longstaff-Schwartz algorithm, but it substitutes the ordinary least squares regressions with the weighted version. Our main contribution is that we prove that the ordinary least squares regressions carried out in the algorithm of Longstaff and Schwartz do not provide the best linear unbiased estimator when employed for the pricing of American call and put options, since there is evidence of heteroscedasticity. First, we numerically and graphically demonstrated the existence of heteroscedastic errors for four price dynamics (namely, geometric Brownian motion, exponential Ornstein- Uhlenbeck process, log-normal jump-diffusion process and asymmetric dual exponential jump-diffusion), and then we provided a theoretical proof that generalises the results to other Markovian processes. The existence of this heteroscedasticity justifies the correction of the least squares Monte Carlo method by considering the weighted least squares regressions that account for heteroscedasticity and provide the best linear unbiased estimators. Furthermore, via an extensive scenario-based study we demonstrated that our new algorithm is effective in reducing the upward bias of the Longstaff-Schwartz prices, at the small cost of little additional computational time.

Finally, the performances of the 'extension' method and the weighted least squares Monte Carlo method are compared against their major competitors using real financial data. We considered S&P 100™ options and LEAPS® traded from 15 February 2012 to 10 December 2014. The empirical comparison showed that the two methods in most of the cases outperform the existing approaches using real financial data, making these methods preferable to existing ones.

7.1 Further Research

The contributions in this thesis can be further expanded in several ways. Major further research directions involve the application of the new methods to price and hedge options written on assets under price dynamics different than the ones considered and/or with other payoff structures. In particular, the moment-matching method introduced in Chapter 2 can be easily adapted to price path-dependent options such as Asian basket options via a double moment-matching procedure, as in the work of Borovkova and Permana (2007). However, it would be more beneficial to apply the new method to price and hedge American-style basket options since, as we outlined in Chapter 2, many basket options traded in organised exchanges are American type.

Furthermore, it would be valuable to expand on the 'extension' method. Currently, this method works exclusively for geometric Brownian motion dynamics, but it is well-known that this process has several drawbacks in replicating the empirical evidence in the financial markets (Bakshi et al., 1997; Fama, 1965, among others). Consequently, it would be useful to apply the same approach of dividing the time-to-maturity into two parts to other price dynamics. In order to apply the 'extension' method to other dynamics, it is necessary that under this new dynamics: the optimal exercise price is independent of the filtration at time of evaluation; a good approximation for the optimal exercise price and/or the option price is available for short maturity options; and, the probability density function of the underlying asset price conditional on not hitting a flat barrier is obtainable in closed-form. Preliminary research indicates that the jump-diffusion models of Merton (1976) and Kou (2002) satisfy the three conditions above and consequently, we will further explore them in future.

It would prove useful to conduct further research on the heteroscedasticity we identified in Chapter 5 for the least squares Monte Carlo method. Preliminary results (Appendix 5.E) show that heteroscedasticity of the errors also characterises the regression steps in the algorithm of Longstaff and Schwartz (2001) for many multiasset payoffs and it would be worth generalising the proof of heteroscedasticity that we presented in this thesis to general square-integrable payoffs. Together with the derivation of these theoretical results, we will also investigate approximations of the conditional variance of the regression errors (which correspond to the weighting function of the weighted least squares method) for other payoff structures. Following the approximations of the conditional variance for single asset options that we considered in Chapter 5, we will investigate approximations which involve price spreads between two European-style derivatives and we will most likely employ some of the results presented in Chapter 2 to calculate these prices. Finally, it would be useful to study additional properties of the errors of the regressions in the Longstaff-Schwartz algorithm, to take advantage of their bounds for pricing purposes (Appendix 5.D). To complement this, another line of research is the application of the weighted least squares regression method to general optimal-control problems in the field of real options, optimal investments, management and control, such as in the work of Carmona and Hinz (2011), Sabour and Poulin (2006) and Carmona and Durrleman (2003).

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