The European Journal of Finance

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/rejf20

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To cite this article: Shan Lu (30 Dec 2023): Joint calibration of VIX and VXX options: does volatility clustering matter?, The European Journal of Finance, DOI: 10.1080/1351847X.2023.2297042

To link to this article: https://doi.org/10.1080/1351847X.2023.2297042

Published online: 30 Dec 2023.

THE EUROPEAN JOURNAL OF FINANCE
https://doi.org/10.1080/1351847X.2023.2297042

Joint calibration of VIX and VXX options: does volatility clustering matter?

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ABSTRACT
This paper studies the effects of volatility clustering on the joint calibration of VIX and VXX options. We find that model which incorporates volatility clustering outperforms other models without this feature in joint calibration of VIX and VXX options both in-sample and out-of-sample; the superiority of the model with volatility clustering is statistically significant. Moreover, the information contained in the VXX options is not fully spanned by the VIX options, as a result, one can achieve better joint pricing performance by employing both VIX and VXX derivatives data when calibrating the model, compared to the case when only VIX data are used in calibration.

ARTICLE HISTORY
Received 18 May 2023
Accepted 14 December 2023

KEYWORDS
VIX options; VXX options; VIX exchange traded product; Hawkes process

1. Introduction

Volatility exchange traded products (ETP) and related ETP options have made volatility trading more accessible particularly to non-institutional investors due to their small notional sizes. Among them, the most actively traded ETP is the VXX, an exchange traded note (ETN) on the short-term VIX futures. The VXX tracks the S&P 500 VIX short-term futures index total return (SPVXSTR) whose value represents the performance of a long position in the nearest and second nearest maturing VIX futures contracts that are rebalanced daily to achieve a synthetic 30-day constant maturity. The VXX has flourished since its introduction by Barclays in 2009, this is mainly because to some extent the VXX mimics the movement of the VIX ,[1] thereby providing investors indirect exposure to the equity market volatility, while also offers some highly profitable trading opportunities (Bondonado, Molnár, and Samdal 2017). VXX is initially marked as a diversification tool for equity portfolios, though the effectiveness of volatility ETPs for diversification has been called into question (Alexander and Korovilas 2013; Alexander, Korovilas, and Kapraun 2016; Deng, McCann, and Wang 2012; Whaley 2013). VXX options were subsequently introduced in May 2010 by the Chicago Board Options Exchange (CBOE). Since inception, VXX options have also grown in popularity among investors. The boom of the trading in VXX and VXX options therefore calls for a framework of modelling the VXX and pricing VXX options.

Studies on the VXX modelling and the valuation of VXX options are quite scarce and limited in the literature. In a stand-alone manner, Bao, Li, and Gong (2012) proposes a logarithmic mean reverting stochastic volatility model with jumps, default risk and positive volatility skew to model the VXX, and show that these factors constitute a good model for VXX option pricing; Tan et al. (2021) extend Bao, Li, and Gong's model by adding a second stochastic volatility factor and by modelling the upward and downward jumps separately, and find that the additional stochastic volatility factor has significant impact on the VXX option pricing, and the effects of jumps cannot be eliminated by the additional stochastic volatility factor. Gehricke and Zhang (2020a) studies the implied volatility curves of the VXX options alone, providing a benchmark for VXX option pricing models. In relation to the VXX jump structure, based on a stand-alone log stochastic volatility model, Cao et al. (2021) conduct a comprehensive study on VXX option pricing models under Lévy processes incorporating infinite

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activity jumps with infinite/finite sample paths, and find that infinite activity jump models outperform models with either no jumps or compound poisson jumps with constant intensity. Note that while stand-alone VXX models may provide good option pricing performance, these models do not take into account the link between VIX and VXX, and therefore, the loss in VXX that is driven by the roll yield when the VIX futures is in contango cannot be predicted in these models. Deviating from the stand-alone approach, Grasselli and Wagalath (2020) links the VIX and the VXX in a tractable way under an affine stochastic volatility model of the VIX, but their framework does not allow for an exact pricing formula for the VXX options whose price has to be approximated; Grasselli, Mazzoran, and Pallavicini (2023) explores the ability of multi-factor stochastic local volatility models in joint pricing VIX and VXX options, however, the computation of VXX option prices in these models requires Monte Carlo simulations. Recently, Lin and Zhang (2022) establishes a model-free analytical framework of joint modelling the VIX and VXX consistently, and their framework allows for an exact formula to compute the VXX option prices, and they show that logarithmic stochastic volatility model of the VIX outperforms the constant volatility model in pricing VIX and VXX options. Additionally, it's worth mentioning another line of research focussing on the relationships among the S&P 500, the VIX and the VXX markets within the consistent pricing framework. Eraker and Wu (2017) construct an equilibrium model of the stock price to explain the negative risk premia for the VXX; Gehricke and Zhang (2018, 2020b) models the VXX based on the assumed dynamics of the S&P500, establishing the link between the S&P500, the VIX and the VXX. Yoon, Ruan, and Zhang (2022) studies the relationship between implied volatility smirks of options written on S&P500 and the VIX and the VXX indices.

The joint pricing and calibration framework of VIX and VXX options are important for several reasons. First, while the direct modelling approach offers a flexible and more accurate way of incorporating the empirical facts about the VXX market for VXX option pricing, the consistent/joint pricing framework allows one to establish the link between the VIX and VXX markets, this is because the VXX is a path-dependent products derived from the VIX futures, there must be a theoretical link between the VIX and VXX and subsequently a relationship between the options written on them, which makes it possible to model the two options markets consistently; this was evidenced by the observed high correlation between the VIX and VXX (Alexander, Kapraun, and Korovilas 2015), and the observed relationship between implied volatility smiles of VIX and VXX options (Gehricke and Zhang 2020a; Yoon, Ruan, and Zhang 2022). Besides, the joint pricing framework based on a structural model makes it possible to examine the factors that drive the two markets simultaneously, enabling one to study the drivers of the two options markets. While Gehricke and Zhang (2020a), Yoon, Ruan, and Zhang (2022) present the empirical link between implied volatility smiles of VIX and VXX options, no explanations are provided for the drivers of the two options markets. Although Eraker and Wu (2017), Gehricke and Zhang (2018, 2020b) model the VIX and VXX market in structural models, providing theoretical foundations for observed empirical facts of the two markets, no VXX option pricing formulas are provided. Only recently, Grasselli and Wagalath (2020), Lin and Zhang (2022) and Grasselli, Mazzoran, and Pallavicini (2023) examine whether a stochastic volatility model is sufficient enough to capture the joint dynamics of VIX and VXX implied volatility smiles. Furthermore, the joint pricing of VIX and VXX options has practical implications as well. As VXX is highly positively correlated with the VIX, VXX call options provide an alternative way of hedging against market downturns; although the joint pricing of SPX, VIX and VXX options remains a subject of its own, the joint pricing of VIX and VXX options are a step forward towards a unified framework for markets for options written on the equity index, volatility and volatility ETPs; By using VXX options, investors (especially retail investors) also get the benefit of more cheaply acquiring the underlying to cover their positions in the options market because of the small notional size in the VXX market. In addition, joint calibration of VIX and VXX options leads to a larger data sample for the calibration. Lastly, it should be noted that the joint calibration of the VIX and VXX options are a joint test of model misspecification and how close/integrated the VIX and VXX markets are.

The challenges in joint pricing of VIX and VXX options lie in ensuring the empirical characteristics from the data of the VIX and VXX markets. In this aspect, there are several motivations for our paper. First, jumps are a widely recognised empirical facts of the equity and volatility markets, they are also empirically proven to be important for the VXX options market, see papers which employ a direct approach for VXX option pricing (Bao, Li, and Gong 2012; Cao et al. 2021; Tan et al. 2021); however, the current joint pricing framework does not have jumps included: the joint pricing framework for VIX and VXX options markets developed by Grasselli and Wagalath (2020) and Lin and Zhang (2022) is based on a pure stochastic volatility model. Second, distinct from the pure stochastic volatility models in Grasselli and Wagalath (2020) and Lin and Zhang (2022), Grasselli, Mazzoran, and Pallavicini (2023) attempts a stochastic local-volatility model to the joint pricing problem, but their approach does not allow for a full calibration to be carried out, which is less practically relevant. Third, in addition to jumps, volatility clustering is another empirical characteristics of the volatility market, but the current joint pricing framework does not allow for volatility clustering in the VIX and VXX markets. There is a consensus that equity market volatility is clustered: early work saw the development of ARCH/GARCH models to capture the observed clustering feature of the equity market volatility (see Poon and Granger 2003, for an overview). More recently, Du and Luo (2019) show that clustering of extreme price movements and volatility is important for the valuation of S&P 500 index options; the finding is also supported by Ait-Sahalia, Cacho-Diaz, and Laeven (2015) who show that equity price jumps are clustered and jumps are contagious across different markets. Jing, Li, and Ma (2020) show that volatility clustering is an important factor to be taken into account when pricing VIX options. The clustering of extreme price movements and volatility in aforementioned papers is modelled through a Hawkes jump intensity process.

In light of the above motivations, in this paper, we study the effects of volatility clustering on the joint pricing of VIX and VXX options, by extending the current joint pricing framework to include various jump specifications. Specifically, under the joint pricing framework, our base model for the VIX dynamics is the logarithmic stochastic volatility model with various jumps, and in particular, we incorporate the volatility clustering feature into the assumed VIX dynamics by introducing a self-exciting Hawkes jump intensity process; by using the theoretical link between the VIX and VXX, the resulting implied VXX dynamics inferred from the VIX is shown to also have a Hawkes type of jumps; closed-form formulas for VXX option prices are then obtained via the Levy inversion formula, for both standard-start and forward-start VXX options.

We summarise our contributions to the literature on the pricing of options on VXX or VIX ETPs in general below:

- First, we extend the joint analytical framework of pricing VIX and VXX options of Grasselli and Wagalath (2020), Lin and Zhang (2022) and Grasselli, Mazzoran, and Pallavicini (2023) to include various different jumps, as the current joint framework does not allow for jumps to be included. The resulting VXX option pricing formula is of closed-form, which makes it easy for computation and implementation.
- Second, to our best knowledge, this is the first paper to employ Hawkes processes to the application of valuation of VXX options (or options on VIX ETPs in general), which enables us to examine the effects of volatility clustering on the joint pricing of VIX and VXX options, which has never been done in the literature on VXX option pricing. This paper also contributes to the literature on the applications of Hawkes process in finance.
- Third, this paper is also the first to develop a formula for the VXX forward start options, no prior literature on VXX option pricing has done so.
- Fourth, differing from numerous prior literature on option pricing, we test the significance of volatility clustering in joint pricing VIX and VXX options via a pairwise t-test, in addition to commonly used performance criteria based on error statistics.

In addition to the above summary, we also present a comparison of the existing VXX option pricing models in the literature with the model presented in this paper in Table 1, which further clarify our contributions to the literature.

Through calibrating the model to the real market data, we find that, first, the model with self-exciting Hawkes-type jumps outperforms models with other types of jumps or the pure stochastic volatility model of Grasselli and Wagalath (2020) and Lin and Zhang (2022) in jointly pricing VIX and VXX options both in-sample and out-of-sample; the superiority of the model with volatility clustering is statistically significant. Second, information in the VXX options market is not fully spanned by the VIX options market, VXX options market contain valuable information in addition to those in the VIX options market.

This paper is also related to the literature on the application of the Hawkes process in finance. The Hawkes process of Hawkes (1971) has been widely used in finance to capture the clustering of financial events. For

Table 1. Summary of VXX option pricing models in the literature.

Author(year)	Modelling approach	Model	Jumps included	Type of jumps	VXX vanilla option pricing formula	VXX forward-start option pricing formula
Bao, Li, and Gong (2012)	Direct	Logarithmic stochastic volatility model with jumps and default risk	✓	Compound Poisson	Semi-closed form characteristic function (CF)	✗
Gehricke and Zhang (2020a)	Direct	Quadratic polynomial functions of the VXX implied volatility	n/a	–	Polynomial regressions of implied volatility	✗
Tan et al. (2021)	Direct	Two-factor stochastic volatility jump-to-default model with asymmetric jumps and default risk	✓	Compound Poisson	Semi-closed form CF	✗
Cao et al. (2021)	Direct	Lévy jump processes with stochastic volatility	✓	Infinite-activity Lévy jumps, including normal inverse Gaussian (NIG) and (general) tempered stable (TS)	Semi-closed form CF	✗
Grasselli and Wagalath (2020)	Joint/Consistent	Logarithmic Ornstein–Uhlenbeck model with stochastic volatility	✗	–	Monte Carlo simulation	✗
Lin and Zhang (2022)	Joint/Consistent	Logarithmic Ornstein–Uhlenbeck model with/without stochastic volatility	✗	–	Closed-form CF	✗
Grasselli, Mazzoran, and Pallavicini (2023)	Joint/Consistent	Stochastic local-volatility model	✗	–	Joint fitting the local-volatility function to VIX futures curve and the VXX implied volatility smile involving Monte Carlo simulation	✗
This paper	Joint/Consistent	Logarithmic Ornstein–Uhlenbeck stochastic volatility model with jumps	✓	Self-exciting Hawkes jumps & diffusion-driven stochastic jumps & compound Poisson	Closed-form moment generating function (MGF)	✓ (Semi-closed form MGF)

Notes: This table provides a summary of the development of VXX option pricing models in the literature so far, including this paper. 'Direct' modelling approach means that the dynamics of the VXX (or the VXX option implied volatility) is directly modelled by the proposed model, whereas 'Joint/Consistent' modelling approach means that only the dynamics of the VIX index (or futures) is modelled by the proposed model and the dynamics of the VXX and VXX option prices are inferred from the assumed VIX dynamics. The symbol ✓ means Yes, and the symbol ✗ means No, and n/a means Not applicable.

example, the Hawkes processes are employed for the pricing of variance swaps (Hong and Jin 2022; Liu and Zhu 2019), vulnerable options (Ma, Shrestha, and Xu 2017) and basket-spread options (Li, Tang, and Wang 2022), studying the asset price jump dynamics (Ferriani and Zoi 2022; Yang et al. 2022), modelling limit order books (Kirchner and Vetter 2022), modelling investor sentiment (Yang et al. 2018; Zhang, Wen, and Chen 2023), and for portfolio optimisation (Ait-Sahalia and Hurd 2016). See Chen et al. (2022) and Hawkes (2022) for an overview of Hawkes processes in finance.

The rest of the paper is organised as follows: Section 2 reviews the existing VIX option pricing models in the literature and provides the specifications of VIX models employed in the paper; Section 3 provides the closed-form VXX option pricing formula based on the assumed VIX dynamics with jumps, and also provides the forward start VXX option prices. Data, model calibration and model performance measures are described in Section 4. Section 5 presents the empirical results. Section 6 concludes the paper.

2. VIX models

2.1. Review of VIX models

There are extensive studies on the VIX dynamics and VIX option pricing models. The literature can be divided into two main strands based on the modelling approach: On the one hand, under the consistent pricing framework, the risk-neutral dynamics of S&P 500 and the VIX are jointly modelled, this approach results in consistent pricing of derivatives on the two indices, see e.g. Cont and Kokholm (2013), Gatheral, Jusselin, and Rosenbaum (2020), Guyon (2020), Jeon, Kim, and Huh (2021), Lin and Chang (2009, 2010), Luo, Zhang, and Zhang (2019), Zhou, Xu, and Rubtsov (2022) in continuous-time settings, and Tong and Huang (2021), Cao et al. (2020a) and Guo and Liu (2020) in discrete-time settings; these models are often rigid and less flexible due to that the implied VIX dynamics under these models is usually driven by a single Brownian motion. On the other hand, stand-alone models where (logarithmic) VIX is directly modelled allows a more flexible way to incorporate additional factors which are capable of reproducing the empirical features of the VIX and VIX skew, see e.g. Drimus and Farkas (2013), Goard and Mazur (2013), Jing, Li, and Ma (2020), Kaeck and Seegeer (2020), Mencia and Sentana (2013), Park (2016), Romo (2017), Yuan (2022).

The base model that the VIX dynamics is assumed to follow is the Log Ornstein–Uhlenbeck process with stochastic volatility (LOUSV). This assumption is motivated by the empirical evidence that the LOU specification is better at describing the volatility dynamics of a stock index and pricing VIX derivatives than the square root process (SQR), and that stochastic volatility of the VIX is found to improve the VIX derivatives valuation (Mencia and Sentana 2013). In the next section, specifications of the VIX models considered in the paper are introduced. All the models are based on the LOUSV specification with different jump specifications. It's worth noting that for simplicity, some important factors for VIX dynamics and VIX option pricing are omitted in all the VIX models employed in this paper, including dynamic VIX central tendency (Mencia and Sentana 2013), co-jumps and time-varying mean of the variance of VIX (Yuan 2022). However, the omission of these factors should not impact the relative performance of the models. As our focus is primarily on the impact of different jumps on the VXX option pricing, we keep other factors the same across the models while introducing different jump specifications.

2.2. Specifications of VIX models

In this section, we introduce the specifications of the VIX models.

2.2.1. LOUSV with Hawkes jumps (SVHJ)

We consider a LOUSV model with jumps whose intensity follows a self-exciting Hawkes process. Jing, Li, and Ma (2020) find that the Hawkes jump diffusion provides the better VIX derivatives pricing performance than models with other types of jumps and with multiple variance factors. Let $v_t = \ln(\mathrm{VIX}_t)$. The model specification of the LOUSV model with Hawkes jumps under the risk-neutral measure $\mathbb{Q}$ is as follows:

$$\mathrm{d}v_t = \kappa_v(u - v_t)\,\mathrm{d}t + \sqrt{w_t}\,\mathrm{d}B_t^{v,\mathbb{Q}} + J_t^{\mathbb{Q}}\mathrm{d}N_t^{\mathbb{Q}} - \lambda_t\mu_J\,\mathrm{d}t \tag{1}$$

$$\mathrm{d}w_t = \kappa_w(\bar{w} - w_t)\,\mathrm{d}t + \sigma_w\sqrt{w_t}\,\mathrm{d}B_t^{w,\mathbb{Q}} \tag{2}$$

$$\mathrm{d}\lambda_t = \alpha(\lambda_\infty - \lambda_t)\,\mathrm{d}t + \beta^{\mathbb{Q}}\mathrm{d}N_t^{\mathbb{Q}} \tag{3}$$

where w_t is the variance of v_t. u and $\bar{w}$ and σ_w are the mean levels of v_t and w_t, and the volatility of w_t, respectively. $B_t^{\cdot,\mathbb{Q}}$ are the standard Brownian motions; to account for asymmetric volatility, $\mathrm{d}B_t^{v,\mathbb{Q}}$ and $\mathrm{d}B_t^{w,\mathbb{Q}}$ are correlated with a positive correlation coefficient ρ, which results in an upward VIX skew. The size of VIX jumps $J_t^{\mathbb{Q}}$ is assumed to follow an exponential distribution with a mean of $\mu_J>0$, $J_t^{\mathbb{Q}} \sim \exp(\frac{1}{\mu_J})$; Park (2016) find no significant improving of model performance in pricing VIX futures and options by adding negative jumps, which justifies the assumption of a positive VIX jump size. $N_t^{\mathbb{Q}}$ is a counting process, its jump intensity λ_t follows a self-exciting Hawkes process as in Equation (3): the arrival of a VIX jump instantaneously increases the jump intensity by $\beta^{\mathbb{Q}}$, then the impact of this jump on the system decays exponentially over time at a rate of α; without further stimulus, the intensity λ_t reverts toward its mean level λ_∞.

2.2.2. LOUSV with compound poisson jumps (SVCJ)

We also consider several other models as benchmark models. First, under the risk-neutral measure $\mathbb{Q}$, the logarithm VIX index follows the following LOUSV model with compound Poisson jumps (Mencia and Sentana 2013; Park 2016):

$$\mathrm{d}v_t = \kappa_v(u - v_t)\,\mathrm{d}t + \sqrt{w_t}\,\mathrm{d}B_t^{v,\mathbb{Q}} + J_t^{\mathbb{Q}}\mathrm{d}N_t^{\mathbb{Q}} - \bar{\lambda}\mu_J\,\mathrm{d}t \tag{4}$$

$$\mathrm{d}w_t = \kappa_w(\bar{w} - w_t)\,\mathrm{d}t + \sigma_w\sqrt{w_t}\,\mathrm{d}B_t^{w,\mathbb{Q}} \tag{5}$$

While specifications for v_t and w_t are the same as SVHJ, the jump specification is different here: $N_t^{\mathbb{Q}}$ is a counting of compound Poisson process with constant intensity $\bar{\lambda}$, where $\bar{\lambda}>0$. The SVCJ model nests the LOUSV model (SV henceforth) by setting $\bar{\lambda} = \mu_J = 0$, the SV model is similar to those pure stochastic volatility models in Grasselli and Wagalath (2020) and Lin and Zhang (2022).

2.2.3. LOUSV with stochastic jumps (SVSJ)

In contrary to other VIX models with jumps whose intensity is constant (Mencia and Sentana 2013; Park 2016), Yuan (2022) proposes a model that incorporates co-jumps in the level and variance of VIX with stochastic jump intensity, and find that jumps with stochastic jump intensity are essential to capture the stochastic skew of the VIX option implied volatility smile. Therefore, the second alternative jump specification considered is the Poisson jumps with stochastic jump intensity which follows a CIR process. The model specification of the LOUSV model with stochastic jumps under $\mathbb{Q}$-measure is:

$$\mathrm{d}v_t = \kappa_v(u - v_t)\,\mathrm{d}t + \sqrt{w_t}\,\mathrm{d}B_t^{v,\mathbb{Q}} + J_t^{\mathbb{Q}}\mathrm{d}N_t^{\mathbb{Q}} - \lambda_t\mu_J\,\mathrm{d}t \tag{6}$$

$$\mathrm{d}w_t = \kappa_w(\bar{w} - w_t)\,\mathrm{d}t + \sigma_w\sqrt{w_t}\,\mathrm{d}B_t^{w,\mathbb{Q}} \tag{7}$$

$$\mathrm{d}\lambda_t = \kappa_\lambda(\theta - \lambda_t)\,\mathrm{d}t + \sigma_\lambda\sqrt{\lambda_t}\,\mathrm{d}B_t^{\lambda,\mathbb{Q}} \tag{8}$$

Different from SVHJ and SVCJ, the jump intensity is an independently evolving stochastic process, where κ_λ is the mean reversion rate, θ and σ_λ are the mean and volatility of the intensity process, respectively.

2.3. VIX option prices

Since the semi-closed form characteristic functions are available for the aforementioned models, the VIX option prices can be computed numerically by using the Lévy inversion formula. Let $f(\phi; v_t, \mathcal{X}_t, t, T) = E_t^{\mathbb{Q}}[\,\mathrm{e}^{i\phi v_T}]$ be the characteristic function of v_T, where $\mathcal{X}_t$ is the vector of state variables at time t: $\mathcal{X}_t = \{w_t\}$ for SVCJ, $\mathcal{X}_t = \{w_t, \lambda_t\}$ for SVSJ and SVHJ. The price of a VIX call option with strike price K at time t that matures at T can be computed by using the Lévy inversion formula:

$$C(K, t, T; \mathrm{VIX}_t) = \mathrm{e}^{-r\tau}\left[f(-i; v_t, \mathcal{X}_t, t, T)\Pi_1 - K\Pi_2\right] \tag{9}$$

where

$$\Pi_j(K, \tau; v_t, \mathcal{X}_t, \phi) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \Re\left[\frac{\mathrm{e}^{-i\phi\log(K)}\varphi_j(\phi; v_t, \mathcal{X}_t, t, T)}{i\phi}\right]\mathrm{d}\phi$$

where $\tau = T - t$, r is the continuous-compounding interest rate, $\varphi_1(\phi; v_t, \mathcal{X}_t, t, T) = f(\phi - i; v_t, \mathcal{X}_t, t, T)/f(-i; v_t, \mathcal{X}_t, t, T)$, $\varphi_2(\phi; v_t, \mathcal{X}_t, t, T) = f(\phi; v_t, \mathcal{X}_t, t, T)$; $\tau = T - t$, r is the continuous-compounding interest rate, $\varphi_1(\phi; v_t, \mathcal{X}_t, t, T) = f(\phi - i; v_t, \mathcal{X}_t, t, T)/f(-i;$
$v_t, \mathcal{X}_t, t, T)$, $\Re$ denotes the real part of a complex number. Characteristic functions $f(\phi; v_t, \mathcal{X}_t, \tau)$ for the aforementioned three models are provided in Appendix 1. The above integrals are calculated by using the Gauss-Laguerre quadrature of order 20 for a good precision. VIX put option prices can be then obtained by the put-call parity.

3. VXX option pricing

In this section, we first obtain the implied VXX dynamics based on the assumed VIX dynamics outlined in Section 2, we then follow the approach in e.g. Bates (1996), Heston (1993) to obtain the closed-form formula for VXX vanilla option prices. Second, we also obtain the forward start VXX option prices based on the implied VXX dynamics. Results of various VXX (vanilla and forward start) option price formulas based on the VIX dynamics with various jump specifications in this section are new to the literature.

3.1. VXX option prices

Let $x_t = \ln \mathrm{VXX}_t$, and $C(K, t, T; \mathrm{VXX}_t)$ be the time-t price of a (standard start) VXX call option at time t with strike price K and maturity date T, thus

$$C(K, t, T; \mathrm{VXX}_t) = \mathrm{VXX}_t P_1 - Ke^{-r\tau}P_2 \tag{10}$$

where

$$P_j(K, \tau; x_t, \mathcal{X}_t, \phi) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \Re\left[\frac{\mathrm{e}^{-i\phi\ln K}\psi_j(i\phi; x_t, \mathcal{X}_t, t, T)}{i\phi}\right]\mathrm{d}\phi \tag{11}$$

where $\psi_j(\phi; x_t, \mathcal{X}_t, t, T)$ are moment generating functions (MGFs) underlying $P_j(\ln K, \tau; x_t, \mathcal{X}_t, \phi)$, for $j = 1,2$; and $\psi_2(\phi; x_t, \mathcal{X}_t, t, T) = \mathbb{E}_t^{\mathbb{Q}}[\mathrm{e}^{\phi x_T}]$. The closed form expressions for the moment generating functions under the models are presented in following section. The price of a VXX put option is obtained by the put-call parity.

3.2. Implied VXX dynamics and the moment generating function

3.2.1. SVCJ

Let $f(\phi; v_t, w_t, t, T)$ be the characteristic function of v_T under the SVCJ model in (4)–(5), the VIX futures price at time t that matures at T under the risk-neutral measure $\mathbb{Q}$ can be written as $f(\phi; v_t, w_t, t, T)$ evaluated at $\phi = -i$:

$$F_{t,T} = f(-i; v_t, w_t, t, T) = \mathrm{e}^{A(-i,t,T)+av_t+B(-i,t,T)w_t}$$

where i is the imaginary number, $a = \mathrm{e}^{-\kappa_v(T-t)}$. $A(\phi, t, T)$, and $B(\phi, t, T)$ are specified in Appendix A.1. Let $k_t = \mathrm{e}^{aJ_t^{\mathbb{Q}}} - 1$, $\bar{k} = \mathbb{E}^{\mathbb{Q}}(k_t)$, applying Ito's lemma to $F_{t,T}$, and using the fact that the VIX futures price is a martingale under $\mathbb{Q}$, we have the drift term equal to zero, subsequently we obtain

$$\frac{dF_{t,T}}{F_{t,T}} = a\sqrt{w_t}\, dB_t^{v,\mathbb{Q}} + B(-i,t,T)\sigma_w\sqrt{w_t}\, dB_t^{w,\mathbb{Q}} + k_t dN_t^{\mathbb{Q}} - \bar{k}\bar{\lambda}\, dt \tag{12}$$

According to the results in Cao et al. (2021) and Lin and Zhang (2022),

$$\frac{dVXX_t}{VXX_t} = r\, dt + \left.\frac{dF_{t,T}}{F_{t,T}}\right|_{T=t+\tau_0}$$

is a martingale during a whole trading day with $\tau_0 = 1/12$ (one month). The implied dynamics of VXX with continuous rolls has the following stochastic differential form:

$$\frac{dVXX_t}{VXX_t} = r\, dt + \tilde{\sigma}\sqrt{w_t}dZ_t + k_t dN_t^{\mathbb{Q}} - \bar{k}\bar{\lambda}\, dt \tag{13}$$

$$dw_t = \kappa_w(\bar{w} - w_t)\, dt + \sigma_w\sqrt{w_t}\, dB_t^{w,\mathbb{Q}} \tag{14}$$

where Z_t is a Brownian motion,[2]

$$\tilde{\sigma} = \sqrt{a_0^2 + B(-i,\tau_0)^2\sigma_w^2 + 2\sigma_w\rho a_0 B(-i,\tau_0)} \tag{15}$$

$$dZ_t \cdot dB_t^{w,\mathbb{Q}} = \tilde{\rho}\, dt = \frac{a_0\rho + B(-i,\tau_0)\sigma_w}{\sqrt{a_0^2 + B(-i,\tau_0)^2\sigma_w^2 + 2\sigma_w\rho a_0 B(-i,\tau_0)}}\, dt \tag{16}$$

and $a_0 = e^{-\kappa_v\tau_0}$, $k_t = e^{a_0 J_t^{\mathbb{Q}}} - 1$ in (13). The compound Poisson process $N^{\mathbb{Q}}$ is compensated so that the VXX dynamics in (13) is a martingale. A similar relationship between the VXX and the VIX when the VIX is modelled under a general Lévy jump process can be found in Equation (A.8) of Appendix A in Cao et al. (2021), which justifies Equations (13)–(14). The implied VXX dynamics preserves the positive skew property as $\tilde{\rho}>0$, which is desired. The implied model in (13)–(14) is the Bates model of Bates (1996) but with a different jump size distribution. The MGFs $\psi_j(\phi; x_t, w_t, t, T)$ underlying $P_j(\ln K, \tau; x_t, w_t, \phi)$ in (11) are specified as follows:

$$\psi_j(\phi; x_t, w_t, t, T) = \exp(D_j(\phi,\tau) + \phi x_t + E_j(\phi,\tau) w_t) \tag{17}$$

where $D_j(\phi,\tau)$ and $E_j(\phi,\tau)$ solves the following system of ODEs (Bates 1996):

$$\frac{\partial D_j(\phi,\tau)}{\partial \tau} = \left[r - \bar{k}\bar{\lambda}\right]\phi + \kappa_w\bar{w}E_j(\phi,\tau) + \bar{\lambda}\bar{J}_j$$

$$\frac{\partial E_j(\phi,\tau)}{\partial \tau} = \tilde{\sigma}^2(m_j\phi + \frac{1}{2}\phi^2) + (\tilde{\rho}\tilde{\sigma}\sigma_w\phi - \kappa_j)E_j(\phi,\tau) + \frac{1}{2}\sigma_w^2 E(\phi,\tau)^2$$

subject to boundary conditions $D_j(\phi,0) = E_j(\phi,0) = 0$, where $m_1 = \frac{1}{2}$, $m_2 = -\frac{1}{2}$, $\kappa_1 = \kappa_w - \tilde{\rho}\tilde{\sigma}\sigma_w$, $\kappa_2 = \kappa_w$, $\bar{J}_1 = \frac{1}{1-(1+\phi)a_0\mu_J} - \frac{1}{1-a_0\mu_J}$, and $\bar{J}_2 = \frac{1}{1-\phi a_0\mu_J} - 1$. Then we have the closed-form solutions to the systems of ODEs:

$$D_j(\phi,\tau) = (r - \bar{k}\bar{\lambda})\phi\tau + \bar{\lambda}\bar{J}_j\tau + \frac{\kappa_w\bar{w}}{\sigma_w^2}\left[(\kappa_j - \tilde{\rho}\tilde{\sigma}\sigma_w\phi + d_j)\tau - 2\ln\frac{1 - g_j e^{d_j\tau}}{1 - g_j}\right]$$

$$E_j(\phi,\tau) = \frac{\kappa_j - \tilde{\rho}\tilde{\sigma}\sigma_w\phi + d_j}{\sigma_w^2}\left[\frac{1 - e^{d_j\tau}}{1 - g_j e^{d_j\tau}}\right]$$

$$g_j = \frac{\kappa_j - \tilde{\rho}\tilde{\sigma}\sigma_w\phi + d_j}{\kappa_j - \tilde{\rho}\tilde{\sigma}\sigma_w\phi - d_j}$$

$$d_j = \sqrt{(\tilde{\rho}\tilde{\sigma}\sigma_w\phi - \kappa_j)^2 - \tilde{\sigma}^2\sigma_w^2(2m_j\phi + \phi^2)} \tag{18}$$

3.2.2. SVSJ

Based on the model specification in (6)–(8), we obtain the implied VXX dynamics,

$$\frac{\mathrm{dVXX}_t}{\mathrm{VXX}_t} = r\,\mathrm{d}t + \tilde{\sigma}\sqrt{w_t}\mathrm{d}Z_t + C(-i,\tau_0)\sigma_\lambda\sqrt{\lambda_t}\,\mathrm{d}B_t^{\lambda,\mathbb{Q}} + k_t\mathrm{d}N_t^{\mathbb{Q}} - \bar{k}\lambda_t\,\mathrm{d}t \tag{19}$$

$$\mathrm{d}w_t = \kappa_w(\bar{w} - w_t)\,\mathrm{d}t + \sigma_w\sqrt{w_t}\,\mathrm{d}B_t^{w,\mathbb{Q}} \tag{20}$$

$$\mathrm{d}\lambda_t = \kappa_\lambda(\theta - \lambda_t) + \sigma_\lambda\sqrt{\lambda_t}\,\mathrm{d}B_t^{\lambda,\mathbb{Q}} \tag{21}$$

where $k_t = \mathrm{e}^{a_0 J_t^{\mathbb{Q}}} - 1$, $a_0 = \mathrm{e}^{-\kappa_v\tau_0}$; $\tilde{\sigma}$ is specified in (15); $\mathrm{d}Z_t\,\mathrm{d}B_t^{w,\mathbb{Q}} = \tilde{\rho}\,\mathrm{d}t$, where $\tilde{\rho}$ is in (16). $B(\phi, t, T)$ and $C(\phi, t, T)$ are specified in Appendix A.2. The MGFs underlying P_1 and P_2 in (10) are specified as follows:

$$\psi_j(\phi; x_t, w_t, \lambda_t, t, T) = \exp(D_j(\phi,\tau) + \phi x_t + E_j(\phi,\tau)w_t + F_j(\phi,\tau)\lambda_t) \tag{22}$$

where $D_j(\phi,\tau)$, $E_j(\phi,\tau)$ and $F_j(\phi,\tau)$ solves the following system of ODEs:

$$\begin{aligned}
\frac{\partial D_j(\phi,\tau)}{\partial\tau} &= r\phi + \kappa_w\bar{w}E_j(\phi,\tau) + \kappa_\lambda\theta F_j(\phi,\tau)\\
\frac{\partial E_j(\phi,\tau)}{\partial\tau} &= \tilde{\sigma}^2(m_j\phi + \frac{1}{2}\phi^2) + (\tilde{\rho}\tilde{\sigma}\sigma_w\phi - \kappa_j)E_j(\phi,\tau) + \frac{1}{2}\sigma_w^2 E_j(\phi,\tau)^2\\
\frac{\partial F_j(\phi,\tau)}{\partial\tau} &= -\bar{k}\phi + C(-i,\tau_0)^2\sigma_\lambda^2(m_j\phi + \frac{1}{2}\phi^2) + [C(-i,\tau_0)\sigma_\lambda^2\phi - \gamma_j]F_j(\phi,\tau)\\
&\quad + \frac{1}{2}\sigma_\lambda^2 F_j(\phi,\tau)^2 + \bar{J}_j
\end{aligned}$$

subject to boundary conditions $D_j(\phi,0) = E_j(\phi,0) = F_j(\phi,0) = 0$, where $m_1 = \frac{1}{2}$, $m_2 = -\frac{1}{2}$, $\kappa_1 = \kappa_w - \tilde{\rho}\tilde{\sigma}\sigma_w$, $\kappa_2 = \kappa_w$, $\gamma_1 = \kappa_\lambda - C(-i,\tau_0)\sigma_\lambda^2$, $\gamma_2 = \kappa_\lambda$, $\bar{J}_1 = \frac{1}{1-(1+\phi)a_0\mu_J} - \frac{1}{1-a_0\mu_J}$, and $\bar{J}_2 = \frac{1}{1-\phi a_0\mu_J} - 1$. Proofs are provided in the Appendix 2. Solving the above ODEs leads to the following:

$$\begin{aligned}
D_j(\phi,\tau) &= r\phi\tau + \frac{\kappa_w\bar{w}}{\sigma_w^2}\left[(\kappa_j - \tilde{\rho}\tilde{\sigma}\sigma_w\phi + d_{E,j})\tau - 2\ln\frac{1 - g_{E,j}\,\mathrm{e}^{d_{E,j}\tau}}{1 - g_{E,j}}\right]\\
&\quad\times\frac{\kappa_\lambda\theta}{\sigma_\lambda^2}\left[(\gamma_j - C(-i,\tau_0)\sigma_\lambda^2\phi + d_{F,j})\tau - 2\ln\frac{1 - g_{F,j}\,\mathrm{e}^{d_{F,j}\tau}}{1 - g_{F,j}}\right]\\
E_j(\phi,\tau) &= \frac{\kappa_j - \tilde{\rho}\tilde{\sigma}\sigma_w\phi + d_{E,j}}{\sigma_w^2}\left[\frac{1 - \mathrm{e}^{d_{E,j}\tau}}{1 - g_{E,j}\,\mathrm{e}^{d_{E,j}\tau}}\right]\\
F_j(\phi,\tau) &= \frac{\gamma_j - C(-i,\tau_0)\sigma_\lambda^2\phi + d_{F,j}}{\sigma_\lambda^2}\left[\frac{1 - \mathrm{e}^{d_{F,j}\tau}}{1 - g_{F,j}\,\mathrm{e}^{d_{F,j}\tau}}\right]\\
g_{E,j} &= \frac{\kappa_j - \tilde{\rho}\tilde{\sigma}\sigma_w\phi + d_{E,j}}{\kappa_j - \tilde{\rho}\tilde{\sigma}\sigma_w\phi - d_{E,j}}\\
d_{E,j} &= \sqrt{(\tilde{\rho}\tilde{\sigma}\sigma_w\phi - \kappa_j)^2 - \tilde{\sigma}^2\sigma_w^2(2m_j\phi + \phi^2)}\\
g_{F,j} &= \frac{\gamma_j - C(-i,\tau_0)\sigma_\lambda^2\phi + d_{F,j}}{\gamma_j - C(-i,\tau_0)\sigma_\lambda^2\phi - d_{F,j}}\\
d_{F,j} &= \sqrt{\left[C(-i,\tau_0)\sigma_\lambda^2\phi - \gamma_j\right]^2 - 2\sigma_\lambda^2(-\bar{k}\phi + \bar{J}_j) - C(-i,\tau_0)^2\sigma_\lambda^4(2m_j\phi + \phi^2)}
\end{aligned} \tag{23}$$

3.2.3. SVHJ

Based on the model specification in (1)–(3), the implied VXX dynamics is in the following form:

$$\frac{\mathrm{dVXX}_t}{\mathrm{VXX}_t} = r\,\mathrm{d}t + \tilde{\sigma}\sqrt{w_t}\mathrm{d}Z_t + k_t \mathrm{d}N_t^{\mathbb{Q}} - \bar{k}\lambda_t\,\mathrm{d}t \tag{24}$$

$$\mathrm{d}w_t = \kappa_w(\bar{w} - w_t)\,\mathrm{d}t + \sigma_w\sqrt{w_t}\,\mathrm{d}B_t^{w,\mathbb{Q}} \tag{25}$$

$$\mathrm{d}\lambda_t = \alpha(\lambda_\infty - \lambda_t) + \beta^{\mathbb{Q}}\mathrm{d}N_t^{\mathbb{Q}} \tag{26}$$

where $k_t = \mathrm{e}^{a_0 J_t^{\mathbb{Q}} + C(-i,\tau_0)\beta^{\mathbb{Q}}} - 1$, $a_0 = \mathrm{e}^{-\kappa_v \tau_0}$; $\tilde{\sigma}$ is specified in (15); $\mathrm{d}Z_t\,\mathrm{d}B_t^{w,\mathbb{Q}} = \tilde{\rho}\,\mathrm{d}t$, where $\tilde{\rho}$ is in (16); $B(\phi, t, T)$ and $C(\phi, t, T)$ are specified in Appendix A.3. The MGFs underlying P_1 and P_2 in (10) are specified as follows:

$$\psi_j(\phi; x_t, w_t, \lambda_t, t, T) = \exp(D_j(\phi,\tau) + \phi x_t + E_j(\phi,\tau)w_t + F_j(\phi,\tau)\lambda_t) \tag{27}$$

where $D_j(\phi,\tau)$, $E_j(\phi,\tau)$ and $F_j(\phi,\tau)$ solves the following system of ODEs:

$$\frac{\partial D_j(\phi,\tau)}{\partial \tau} = r\phi + \kappa_w \bar{w} E_j(\phi,\tau) + \alpha\lambda_\infty F_j(\phi,\tau)$$

$$\frac{\partial E_j(\phi,\tau)}{\partial \tau} = \tilde{\sigma}^2\left(m_j\phi + \frac{1}{2}\phi^2\right) + (\tilde{\rho}\tilde{\sigma}\sigma_w\phi - \kappa_j)E_j(\phi,\tau) + \frac{1}{2}\sigma_w^2 E_j(\phi,\tau)^2$$

$$\frac{\partial F_j(\phi,\tau)}{\partial \tau} = -\bar{k}\phi - \alpha F_j(\phi,\tau) + \bar{J}_j$$

subject to boundary conditions $D_j(\phi, 0) = E_j(\phi, 0) = F_j(\phi, 0) = 0$, where $m_1 = \frac{1}{2}$, $m_2 = -\frac{1}{2}$, $\kappa_1 = \kappa_w - \tilde{\rho}\tilde{\sigma}\sigma_w$, $\kappa_2 = \kappa_w$, $\bar{J}_1 = \frac{\mathrm{e}^{(1+\phi)C(-i,\tau_0)\beta^{\mathbb{Q}}}}{1-(1+\phi)a_0\mu_J} - \frac{\mathrm{e}^{C(-i,\tau_0)\beta^{\mathbb{Q}}}}{1-a_0\mu_J}$, and $\bar{J}_2 = \frac{\mathrm{e}^{\phi C(-i,\tau_0)\beta^{\mathbb{Q}}}}{1-\phi a_0\mu_J} - 1$. Proofs are similar to those for the SVSJ model and are therefore omitted in the paper for brevity. We have the following solutions to the above ODEs:

$$D_j(\phi,\tau) = r\phi\tau + \frac{\kappa_w\bar{w}}{\sigma_w^2}\left[(\kappa_j - \tilde{\rho}\tilde{\sigma}\sigma_w\phi + d_j)\tau - 2\ln\frac{1 - g_j e^{d_j\tau}}{1 - g_j}\right]$$

$$+ \lambda_\infty(-\bar{k}\phi + \bar{J}_j)\tau + \frac{\lambda_\infty(-\bar{k}\phi + \bar{J}_j)}{\alpha}(\mathrm{e}^{-\alpha\tau} - 1)$$

$$E_j(\phi,\tau) = \frac{\kappa_j - \tilde{\rho}\tilde{\sigma}\sigma_w\phi + d_j}{\sigma_w^2}\left[\frac{1 - \mathrm{e}^{d_j\tau}}{1 - g_j e^{d_j\tau}}\right]$$

$$F_j(\phi,\tau) = \frac{-\bar{k}\phi + \bar{J}_j}{\alpha}(1 - \mathrm{e}^{-\alpha\tau})$$

$$g_j = \frac{\kappa_j - \tilde{\rho}\tilde{\sigma}\sigma_w\phi + d_j}{\kappa_j - \tilde{\rho}\tilde{\sigma}\sigma_w\phi - d_j}$$

$$d_j = \sqrt{(\tilde{\rho}\tilde{\sigma}\sigma_w\phi - \kappa_j)^2 - \tilde{\sigma}^2\sigma_w^2(2m_j\phi + \phi^2)} \tag{28}$$

3.3. Forward start options

One of the most popular exotic options is the forward start options. Investors sometimes may be interested in European options on the volatility ETP with a strike price that will be determined later. These options are very sensitive to the volatility of the underlying. As an example, the price of forward start options written on the VXX under the SVHJ model is derived below, similar procedures can be used to obtain forward start VXX option prices under SV, SVCJ and SVSJ models. Let $\Psi(\phi; x_0, w_0, \lambda_0, t, T)$ be the characteristic function associated with the forward log-return of the VXX under the SVHJ model,

$$\Psi(\phi; x_0, w_0, \lambda_0, t, T) = \mathbb{E}^{\mathbb{Q}}[\, e^{i\phi(x_T - x_t)}] = \mathbb{E}^{\mathbb{Q}}[\, e^{-i\phi x_t}\mathbb{E}^{\mathbb{Q}}(e^{i\phi x_T})] = \mathbb{E}^{\mathbb{Q}}[\, e^{-i\phi x_t}\psi_2(i\phi; x_t, w_t, \lambda_t, t, T)] \tag{29}$$

where $\psi_2(\phi; x_t, w_t, \lambda_t, t, T)$ is the MGF of x_T under SVHJ. Substituting the expression for $\psi_2(\phi; x_t, w_t, \lambda_t, t, T)$ provided in (27)–(28) into the (29),

$$\Psi(\phi; x_0, w_0, \lambda_0, t, T) = e^{D_2(i\phi,\tau)}\mathbb{E}^{\mathbb{Q}}[\, e^{E_2(i\phi,\tau)w_t + F_2(i\phi,\tau)\lambda_t}] \tag{30}$$

Let $\Psi^L(g, h; x_0, w_0, \lambda_0, t) = \mathbb{E}^{\mathbb{Q}}[\, e^{g w_t + h\lambda_t}]$ be the joint Laplace transform of w_t and λ_t under the risk-neutral measure $\mathbb{Q}$, where $g = E_2(i\phi, \tau)$ and $h = F_2(i\phi, \tau)$. Assuming the solution to $\Psi^L(g, h; x_0, w_0, \lambda_0, t)$ has the following form:

$$\Psi^L(g, h; x_0, w_0, \lambda_0, t) = e^{Q(\phi,t) + i\phi x_0 + R(\phi,t)w_0 + S(\phi,t)\lambda_0} \tag{31}$$

Applying the Feynman–Kac formula, we have

$$\begin{aligned} \frac{\partial \Psi^L}{\partial t} &= (r - \frac{1}{2}\tilde{\sigma}^2 w - \bar{k}\lambda)\frac{\partial \Psi^L}{\partial x} + \kappa_w(\bar{w} - w)\frac{\partial \Psi^L}{\partial w} + \alpha(\lambda_\infty - \lambda)\frac{\partial \Psi^L}{\partial \lambda} + \frac{1}{2}\tilde{\sigma}^2 w \frac{\partial^2 \Psi^L}{\partial x^2} \\ &\quad + \frac{1}{2}\sigma_w^2 w \frac{\partial^2 \Psi^L}{\partial w^2} + \tilde{\rho}\tilde{\sigma}\sigma_w w \frac{\partial^2 \Psi^L}{\partial x \partial w} + \lambda \mathbb{E}^{\mathbb{Q}}[\, e^{i\phi \ln(1+k) + S(\phi,t)\beta^{\mathbb{Q}}} - 1] \end{aligned} \tag{32}$$

Substituting (31) into (32), we have the following system of ODEs:

$$\begin{aligned} \frac{\partial Q(\phi, t)}{\partial t} &= i\phi r + \kappa_w \bar{w} R(\phi, t) + \alpha\lambda_\infty S(\phi, t) \\ \frac{\partial R(\phi, t)}{\partial t} &= -\frac{1}{2} i\phi\tilde{\sigma}^2 - \kappa_w R(\phi, t) - \frac{1}{2}\tilde{\sigma}^2\phi^2 + \frac{1}{2}\sigma_w^2 R(\phi, t)^2 + i\phi\tilde{\rho}\tilde{\sigma}\sigma_w R(\phi, t) \\ \frac{\partial S(\phi, t)}{\partial t} &= -i\phi\bar{k} - \alpha S(\phi, t) + \left(\frac{e^{i\phi C(-i,\tau_0)\beta^{\mathbb{Q}} + S(\phi,t)\beta^{\mathbb{Q}}}}{1 - i\phi a_0 \mu_J} - 1\right) \end{aligned}$$

subject to boundary conditions $Q(\phi, 0) = 0$, $R(\phi, 0) = g$, $S(\phi, 0) = h$. The forward characteristic function is then

$$\Psi(\phi; x_0, w_0, \lambda_0, t, T) = e^{D_2(i\phi,\tau) + Q(\phi,t) + i\phi x_0 + R(\phi,t)w_0 + S(\phi,t)\lambda_0} \tag{33}$$

The time-0 price of a forward start VXX call option with strike price K, determination time t and maturity T is

$$C^F(K, t, T; \mathrm{VXX}_0) = e^{-rT}\mathbb{E}^{\mathbb{Q}}[(e^{x_T - x_t} - e^{\ln K})^+] \tag{34}$$

which can be obtained by using either the Lévy inversion formula like formula (9) or the fast Fourier transform method (see e.g. Carr and Madan 1999). For example, following Carr and Madan (1999),

$$C^F(K, t, T; \mathrm{VXX}_0) = \frac{e^{-(rT + u\ln K)}}{\pi}\int_0^\infty e^{-i\phi\ln K}\frac{\Psi(\phi - i(1+u); x_0, w_0, \lambda_0, t, T)}{(u + 1 + i\phi)(u + i\phi)}\mathrm{d}\phi \tag{35}$$

where the parameter u is introduced in order to have an integrable function; this method only has one integral to calculate. The forward start put option price is then obtained by the put-call parity.

4. Data and model calibration

4.1. Data

Data for VIX futures and options and VXX options are end-of-day quotes obtained from the Chicago Board Options Exchange (CBOE). The sample spans from 10 October 2012 to 10 October 2013. The options data contain the following information: options types, expiration dates, strike prices, bid and ask prices, implied volatilities, underlying asset values, trade volume and open interest, and option Greeks calculated by the CBOE (e.g. delta, vega, etc.). There are two snapshots of the options market in the data, one at 3:45pm E.T. (Eastern Time), and another at end-of-day at 4pm E.T.. We use the data at 3:45pm for both VIX and VXX options. This is because, as noted by the CBOE, the snapshot at 3:45pm E.T. is a more accurate representation of the market liquidity when compared to that at end-of-day.

For VIX options, we use the midpoint of the bid-ask price as the VIX option price; since VXX options are American style ,[3] we obtain the European option prices of VXX options by inserting the quoted implied volatilities into the Black-Scholes model. We use the daily treasury bill rates as the risk-free rate which are obtained from the US treasury. Since the VXX options sample spans between the second and the third VXX reverse split dates, there is no need to adjust the underlying VXX values. To reduce the computational burden of model calibration, we use options data only on Wednesdays in the sample as our in-sample and options data on Thursdays (the next day) as the out-of-sample. This is a common practice in the literature, see e.g. Chernov and Ghysels (2000), Christoffersen, Heston, and Jacobs (2009), Du and Luo (2019), Eraker (2004), Pan (2002). For model estimation, while some literature only use derivatives with a randomly selected maturity, others use all Wednesday data. Here we use all futures and options data on Wednesdays for model calibration.

We apply some commonly used data filters to remove illiquid and likely mispriced options from the sample: (1) Options with bid price smaller than 0.1, and options with zero trade volumes are removed; (2) Only at-the-money (ATM) and out-of-money (OTM) options are used, the moneyness is defined as the logarithm of the ratio between the option strike price and the forward price of the underlying; (3) Options with time-to-maturities smaller than 7 calendar days are removed; and we focus on options with tenor smaller than 1 year. The resulting Wednesday sample consists of 4282 VIX options, 288 VIX futures and 7057 VXX options, and the Thursday sample consists of 4531 VIX options, 299 VIX futures and 7239 VXX options.

Tables 2 and 3 report the summary statistics for VIX derivatives and VXX options in both the Wednesday and Thursday samples. First, the average VIX futures price is an increasing function of the time to maturity, consistent with the fact that VIX futures is usually in contango .[4] Second, the number of both OTM VIX calls and OTM VXX calls is much higher than that of the corresponding OTM puts, implying that investors mostly buy VIX and VXX options as an insurance against a hike in the market volatility. Third, implied volatility from VIX and VXX options increases with moneyness, similar to previous findings of positive volatility skews in the two markets. Fourth, the average implied volatility from VXX options is lower than that from the VIX options, this is probably due to the fact that the VXX is a mix of the near-the-month and the next-month VIX futures, and because VIX futures is usually in contango, the VXX declines in value over time, and VXX options are susceptible to this decrease in value. Lastly, open interest and trade volume in the VXX options market is lower than those in the VIX options market, implying that the VIX options are more liquid than VXX options.

4.2. Model calibration

4.2.1. Calibration to VIX data

First, the models are calibrated by using the Wednesday VIX futures and options data only subject to the following loss function (see e.g. Cao et al. 2020a; Zhou, Xu, and Rubtsov 2022)

$$\begin{aligned}\mathcal{L}_t^{\text{vixall}}(\Theta) &= \mathcal{L}_t^{\text{vix}}(\Theta) + \mathcal{L}_t^{\text{F}}(\Theta)\\ &= \frac{1}{N^{\text{vix}}}\sum_{i=1}^{N^{\text{vix}}}\left(\frac{\widetilde{O}_i^{\text{vix}} - O_i^{\text{vix}}}{O_i^{\text{vix}}}\right)^2 + \frac{1}{N^{\text{F}}}\sum_{n=1}^{N^{\text{F}}}\left(\frac{\widetilde{F}_n^{\text{vix}} - F_n^{\text{vix}}}{F_n^{\text{vix}}}\right)^2\end{aligned} \tag{36}$$

Table 2. Summary statistics for VIX derivatives.

	Wednesday				Thursday			
		Time to maturity τ (days)				Time to maturity τ (days)		
	All	≤ 30	30–90	>90	All	≤ 30	30–90	>90
A. Futures contracts								
Number of contracts	288	36	108	144	299	36	108	155
Average prices	17.9115	15.9171	17.2598	18.8989	17.9142	15.8172	17.1859	18.9088
B. Number of option contracts								
All	4282	455	2014	1813	4531	516	2139	1876
Call	3241	368	1575	1298	3447	426	1679	1342
Put	1041	87	439	515	1084	90	460	534
C. Mean statistics								
Average prices	1.0349	0.5385	0.9019	1.3071	0.9820	0.4719	0.8513	1.2714
Average IV	0.8302	1.0905	0.8944	0.6935	0.8437	1.1401	0.9056	0.6917
Average trade volume	4883	17479	5483	1054	4423	15019	4911	951
Average open interests	44845	135983	55170	10503	46516	134491	56711	10695
D. Average prices by moneyness								
$m \leq -0.2$	0.4849	0.1813	0.3576	0.5654	0.4377	0.1750	0.3058	0.5241
$-0.2<m \leq -0.06$	1.1471	0.4566	0.9332	1.5336	1.0988	0.4340	0.8891	1.4917
$-0.06<m \leq 0.06$	2.0905	1.0446	1.8744	2.6163	2.0756	0.9971	1.8470	2.6107
$0.06<m \leq 0.2$	1.6382	0.7538	1.4982	2.1038	1.6270	0.7225	1.4825	2.0958
$0.2<m \leq 0.4$	0.9930	0.4066	0.9144	1.3459	0.9795	0.3810	0.9185	1.3334
$0.4<m \leq 0.6$	0.5795	0.2494	0.5058	0.7695	0.5619	0.2042	0.5027	0.7726
$m>0.6$	0.3141	0.2000	0.2717	0.3612	0.2819	0.1458	0.2391	0.3465
E. Average IV by moneyness								
$m \leq -0.2$	0.5517	0.7526	0.5926	0.5238	0.5419	0.7598	0.5762	0.5148
$-0.2<m \leq -0.06$	0.6377	0.7610	0.6599	0.5854	0.6308	0.7412	0.6500	0.5826
$-0.06<m \leq 0.06$	0.7138	0.8530	0.7512	0.6355	0.7116	0.8484	0.7522	0.6327
$0.06<m \leq 0.2$	0.8004	1.0121	0.8381	0.6841	0.8038	1.0433	0.8418	0.6799
$0.2<m \leq 0.4$	0.9035	1.2014	0.9385	0.7307	0.9141	1.2312	0.9408	0.7346
$0.4<m \leq 0.6$	0.9615	1.3238	1.0231	0.7793	0.9857	1.3802	1.0309	0.7820
$m>0.6$	0.9714	1.4726	1.1088	0.8134	1.0089	1.4562	1.1354	0.8133

Notes: This table reports the summary statistics for VIX futures and options sampled on each Wednesday and Thursday from 10 October 2012 to 10 October 2013. Option moneyness is defined as the logarithm of the ratio between the strike price of the option and the underlying asset forward price.

where i and n are the option and futures count indices. N^{vix} and N^{F} are the total numbers of VIX options and VIX futures on a single day, respectively. $\widetilde{O}^{\text{vix}}$ and O^{vix} are model-implied and market quoted VIX option prices, respectively; $\widetilde{F}^{\text{vix}}$ and F^{vix} are model-implied and market quoted VIX futures prices, respectively. This loss function avoids allocating high weights to options with particular moneyness.

4.2.2. *Joint calibration to VIX and VXX data*

Second, we calibrate the models to both VIX and VXX data on each Wednesday according to the following loss function :

$$\begin{aligned}\mathcal{L}_t^{\text{all}}(\Theta) &= \mathcal{L}_t^{\text{vix}}(\Theta) + \mathcal{L}_t^{\text{vxx}}(\Theta) + \mathcal{L}_t^{\text{F}}(\Theta) \\ &= \frac{1}{N^{\text{vix}}}\sum_{i=1}^{N^{\text{vix}}}\left(\frac{\widetilde{O}_i^{\text{vix}} - O_i^{\text{vix}}}{O_i^{\text{vix}}}\right)^2 + \frac{1}{N^{\text{vxx}}}\sum_{j=1}^{N^{\text{vxx}}}\left(\frac{\widetilde{O}_j^{\text{vxx}} - O_j^{\text{vxx}}}{O_j^{\text{vxx}}}\right)^2 + \frac{1}{N^{\text{F}}}\sum_{n=1}^{N^{\text{F}}}\left(\frac{\widetilde{F}_n^{\text{vix}} - F_n^{\text{vix}}}{F_n^{\text{vix}}}\right)^2 \end{aligned} \tag{37}$$

where i, j and n are the option and futures count indices. N^{vix}, N^{vxx} and N^{F} are the total numbers of VIX and VXX options and VIX futures on a single day, respectively. $\widetilde{O}$ and O are model-implied and market quoted option prices, respectively; $\widetilde{F}^{\text{vix}}$ and F^{vix} are model-implied and market quoted VIX futures prices, respectively.

Table 3. Summary statistics for VXX options.

	Wednesday				Thursday			
		Time to maturity τ (days)				Time to maturity τ (days)		
	All	≤ 30	30–90	>90	All	≤ 30	30–90	>90
A. Number of option contracts								
All	7057	1565	2515	2977	7239	2063	2429	2747
Call	5076	1095	1892	2089	5152	1460	1783	1909
Put	1981	470	623	888	2087	603	646	838
B. Mean statistics								
Average prices	1.2791	0.5045	0.9816	1.9377	1.1814	0.4985	0.9732	1.8791
Average IV	0.7510	0.7289	0.7741	0.7432	0.7467	0.7285	0.7650	0.7443
Average trade volume	982	1890	1363	183	924	1936	891	193
Average open interests	5918	8560	6271	4231	5896	7904	5525	4717
C. Average prices by moneyness								
$m \leq -0.2$	0.8417	0.1477	0.4183	1.0221	0.8266	0.1682	0.4016	1.0149
$-0.2<m \leq -0.06$	1.3902	0.3488	1.1319	2.6725	1.2782	0.3434	1.0445	2.6824
$-0.06<m \leq 0.06$	2.1572	0.8355	2.3031	4.0753	1.8817	0.8028	2.1505	4.0208
$0.06<m \leq 0.2$	1.6282	0.4779	1.6752	3.3564	1.3783	0.4600	1.6063	3.1408
$0.2<m \leq 0.4$	1.2389	0.2581	0.8503	2.2495	1.1293	0.2737	0.8523	2.1555
$0.4<m \leq 0.6$	0.8550	0.1599	0.3759	1.4336	0.8046	0.1868	0.3791	1.3857
$m>0.6$	0.5728	–	0.2390	0.7133	0.5453	0.1409	0.2286	0.6856
D. Average IV by moneyness								
$m \leq -0.2$	0.6218	0.6310	0.5787	0.6387	0.6200	0.6634	0.5754	0.6373
$-0.2<m \leq -0.06$	0.6215	0.5787	0.6098	0.6753	0.6149	0.5764	0.6034	0.6753
$-0.06<m \leq 0.06$	0.6601	0.6267	0.6703	0.7019	0.6498	0.6174	0.6678	0.7021
$0.06<m \leq 0.2$	0.7379	0.7521	0.7306	0.7240	0.7387	0.7512	0.7283	0.7237
$0.2<m \leq 0.4$	0.8080	0.9059	0.8038	0.7546	0.8137	0.9031	0.7980	0.7567
$0.4<m \leq 0.6$	0.8470	1.0510	0.8882	0.7860	0.8559	1.0486	0.8840	0.7906
$m>0.6$	0.8858	–	0.9869	0.8432	0.8913	1.1792	0.9888	0.8469

Notes: This table reports the summary statistics for VXX options sampled on each Wednesday and Thursday from 10 October 2012 to 10 October 2013. Option moneyness is defined as the logarithm of the ratio between the strike price of the option and the underlying asset forward price.

The starting values of model parameters for the aforementioned calibrations are taken from those reported in Yuan (2022) (for SV, SVCJ and SVSJ models) and Jing, Li, and Ma (2020) (for SVHJ model). A combination of direct search methods (e.g. Nelder–Mead method) and gradient-based methods (e.g. interior-point method) is used to minimise the objective functions, details can be referred to Lin and Zhang (2022), among others.

4.3. Performance measures

Several commonly used performance metrics are calculated to measure the pricing performance of the models in-sample and out-of-sample. These metrics include the Mean Absolute Error (MAE), the Root Mean Squared Error (RMSE), and the Mean Absolute Percentage Error (MAPE):

$$\text{MAE} = \frac{1}{N}\sum_{i=1}^{N}|\widetilde{O}_i - O_i| \tag{38}$$

$$\text{RMSE} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(\widetilde{O}_i - O_i)^2} \tag{39}$$

$$\text{MAPE} = \frac{1}{N}\sum_{i=1}^{N}\frac{|\widetilde{O}_i - O_i|}{O_i} \tag{40}$$

In addition, to assess the significance of the pricing performance of model i over model j, the following t-statistic of the sample differences in daily mean squared errors (MSE) is computed based on Huang and Wu (2004)

$$t = \frac{\overline{MSE}_i - \overline{MSE}_j}{\text{stdev}(\overline{MSE}_i - \overline{MSE}_j)} \tag{41}$$

where the overline denotes the sample average, and stdev denotes the standard error of the sample mean difference. The standard error is adjusted for serial correlation following Newey and West (1987), and the optimal lag number is selected according to Andrews (1991) and an AR(1) specification. A positive value of the t-statistic that is larger than 1.645 indicates model j significantly outperforms model i at 5% significance level.

5. Results

5.1. Parameter estimates

Table 4 reports the mean and standard errors of the daily parameter estimates over the sample period by calibrating the models to the Wednesday data. 'W /O' represents the calibration to the VIX data only, whereas 'W

Table 4. Parameter estimates.

Model	Data	κ_V	κ_W	κ_λ	σ_W	σ_λ	u	$\bar{w}$	θ	α
SV	W/O	7.0709	2.8598	–	2.4658	–	2.9003	3.3108	–	–
		(0.4403)	(0.3458)	–	(0.1130)	–	(0.0169)	(0.2676)	–	–
	W/	6.6351	8.6334	–	5.9571	–	2.9793	1.8016	–	–
		(0.2591)	(0.3616)	–	(0.2880)	–	(0.0085)	(0.0656)	–	–
SVCJ	W/O	5.9999	2.3196	–	1.9085	–	2.9393	4.2269	–	–
		(0.4940)	(0.3384)	–	(0.1466)	–	(0.0277)	(0.4468)	–	–
	W/	5.4740	9.0294	–	3.7263	–	2.9185	1.4286	–	-
		(0.1988)	(0.7116)	–	(0.2990)	–	(0.0076)	(0.2067)	–	–
SVSJ	W/O	4.3637	1.1476	7.5923	1.2981	2.7433	2.9674	3.0118	5.7764	–
		(0.3852)	(0.2939)	(0.9521)	(0.1060)	(0.4524)	(0.0216)	(0.4194)	(0.4998)	–
	W/	5.4584	5.4085	11.6185	1.3104	0.9502	2.9108	2.5831	3.8631	–
		(0.2301)	(0.8019)	(1.6160)	(0.1229)	(0.2561)	(0.0097)	(0.4165)	(0.4444)	–
SVHJ	W/O	2.7733	3.3138	–	1.3813	–	3.0537	0.7127	–	15.0532
		(0.3161)	(0.4433)	–	(0.1767)	–	(0.0294)	(0.0889)	–	(0.9887)
	W/	4.9385	9.5334	–	4.8374	–	2.8933	0.4833	–	19.1922
		(0.1834)	(0.4804)	–	(0.2842)	–	(0.0109)	(0.0374)	–	(1.5719)
Model	Data	ρ	λ_∞	$\beta^{\mathbb{Q}}$	μ_J	$\bar{\lambda}$	w_0	λ_0	$\tilde{\sigma}$	$\tilde{\rho}$
SV	W/O	0.9203	–	–	–	–	0.9376	–	0.6251	0.9332
		(0.0087)	–	–	–	–	(0.0378)	–	(0.0209)	(0.0074)
	W/	0.6674	–	–	–	–	1.0639	–	0.6240	0.5359
		(0.0419)	–	–	–	–	(0.0400)	–	(0.0121)	(0.0312)
SVCJ	W/O	0.7914	–	–	0.2761	2.4567	0.5968	–	0.6674	0.8171
		(0.0181)	–	–	(0.0181)	(0.1921)	(0.0522)	–	(0.0233)	(0.0164)
	W/	0.4118	–	–	0.3040	2.2978	0.5245	–	0.6637	0.4944
		(0.0386)	–	–	(0.0146)	(0.1507)	(0.0380)	–	(0.0105)	(0.0327)
SVSJ	W/O	0.7931	–	–	0.1722	–	0.5062	8.0958	0.7437	0.8047
		(0.0363)	–	–	(0.0134)	–	(0.0502)	(1.1810)	(0.0212)	(0.0351)
	W/	0.7704	–	–	0.2923	–	0.5680	2.0583	0.6600	0.7895
		(0.0414)	–	–	(0.0045)	–	(0.0311)	(0.1249)	(0.0132)	(0.0383)
SVHJ	W/O	0.8190	3.3190	5.8726	0.1535	–	0.1189	15.5097	0.8346	0.8394
		(0.0329)	(0.5965)	(0.5493)	(1.7872e-4)	–	(0.0255)	(0.5782)	(0.0182)	(0.0289)
	W/	0.3801	11.1855	7.2647	0.1605	–	0.5340	7.0667	0.7042	0.4944
		(0.0345)	(0.9501)	(0.7305)	(0.0035)	–	(0.0347)	(0.7872)	(0.0091)	(0.0288)

Notes: This table reports the mean parameter estimates. Robust standard errors are reported in parentheses. Models are calibrated to the Wednesday data sample from 10 October 2012 to 10 October 2013. 'W /O' means calibration to the VIX derivatives data only, 'W /' means calibration to both the VIX and VXX data.

/' means the calibration to both the VIX and VXX data. The mean reversion parameter of the log VIX κ_v is the highest for SV and decreases as jumps are added; κ_v is lower in models with time-varying jump intensity (SVSJ and SVHJ) than in the model with constant jump intensity (SVCJ), and is the lowest in the model with Hawkes jump process (SVHJ). The mean estimate of long term mean u is consistent across the models and data, ranging from 2.8933 to 3.0537. The mean σ_w and $\bar{w}$ decrease as jumps are added to the model. ρ and $\tilde{\rho}$ which represents the leverage factor in the VIX and VXX are both positive, confirming positive volatility skews in both the VIX

Table 5. Pricing performance: Overall.

		VIX futures (Wednesday)		VIX futures (Thursday)	
	Model	W/O (In-sample)	W/ (In-sample)	W/O (Out-of-sample)	W/ (Out-of-sample)
MAE	SV	0.3164	0.6432	0.3719	0.6681
	SVCJ	0.2735	0.5090	0.3550	0.5381
	SVSJ	0.2414	0.4581	0.3265	0.4970
	SVHJ	**0.2018**	**0.3218**	**0.3228**	**0.3715**
RMSE	SV	0.3907	0.7415	0.4620	0.7815
	SVCJ	0.3584	0.6001	0.4498	0.6432
	SVSJ	0.3167	0.5499	0.4475	0.6035
	SVHJ	**0.2738**	**0.4064**	**0.4470**	**0.4680**
MAPE (%)	SV	1.80	3.67	2.09	3.78
	SVCJ	1.55	2.88	1.99	3.03
	SVSJ	1.37	2.57	1.83	2.77
	SVHJ	**1.14**	**1.79**	**1.80**	**2.06**
		VIX options (Wednesday)		VIX options (Thursday)	
	Model	W/O (In-sample)	W/ (In-sample)	W/O (Out-of-sample)	W/ (Out-of-sample)
MAE	SV	0.0753	0.1598	0.0993	0.1657
	SVCJ	0.0602	0.1261	0.0911	0.1332
	SVSJ	0.0560	0.1183	0.0806	0.1273
	SVHJ	**0.0445**	**0.0754**	**0.0792**	**0.0879**
RMSE	SV	0.1013	0.2414	0.1382	0.2504
	SVCJ	0.0870	0.1878	0.1380	0.2031
	SVSJ	0.0809	0.1745	0.1226	0.1920
	SVHJ	**0.0669**	**0.1182**	**0.1196**	**0.1385**
MAPE (%)	SV	12.31	15.44	16.55	18.37
	SVCJ	8.71	12.00	13.04	14.12
	SVSJ	8.53	11.15	12.49	13.70
	SVHJ	**6.59**	**7.67**	**11.74**	**10.58**
		VXX options (Wednesday)		VXX options (Thursday)	
	Model	W/O (Out-of-sample)	W/ (In-sample)	W/O (Out-of-sample)	W/ (Out-of-sample)
MAE	SV	0.3011	0.0980	0.2807	0.1062
	SVCJ	0.2743	0.0845	0.2635	0.0921
	SVSJ	0.2727	0.0775	0.2533	0.0926
	SVHJ	**0.2089**	**0.0426**	**0.2113**	**0.0697**
RMSE	SV	0.4702	0.1537	0.4364	0.1570
	SVCJ	0.3867	0.1415	0.3676	0.1403
	SVSJ	0.3966	0.1272	0.3612	0.1396
	SVHJ	**0.3063**	**0.0622**	**0.3008**	**0.1001**
MAPE (%)	SV	32.86	12.18	33.91	14.64
	SVCJ	32.00	8.45	33.86	11.39
	SVSJ	32.28	7.60	33.00	11.46
	SVHJ	**27.47**	**5.44**	**30.88**	**10.23**

Notes: This table reports the overall model pricing performance. Models are calibrated to the Wednesday data sample from 10 October 2012 to 10 October 2013. Derivative prices are then computed daily by using the calibrated model, the performance metrics, including the Mean Absolute Error (MAE), the Root Mean Squared Error (RMSE), and the Mean Absolute Percentage Error (MAPE) are calculated as in Equations (38)–(40). 'W /O' means calibration to the VIX derivatives data only, 'W /' means calibration to both the VIX and VXX data. Numbers in bold indicate the smallest pricing error.

and VXX options markets; and the mean estimates of ρ and $\tilde{\rho}$ are the highest in the SV model, and decrease as jumps are added.

Regarding the jump dynamics, first, the mean reversion parameters κ_λ and α increase when the VXX data are included in the calibration: mean of κ_λ increases from 7.5923 to 11.6185, and mean of α increases from 15.0532 to 19.1922. Second, the mean jump size parameter μ_J decreases as jump intensity changes from constant to stochastic and to self-exciting. Third, in the SVHJ model, the jump self-excitation parameter $\beta^{\mathbb{Q}}$ increases from 5.8726 to 7.2647, when the VXX data are included in the calibration. Lastly, in the SVSJ and SVHJ models, the mean estimate of daily jump intensity λ_0 is higher when only VIX data are used in calibration, and the inclusion of VXX data in the calibration lowers λ_0; besides, mean estimate of λ_0 is higher in SVHJ model than in SVSJ model, which can be reconciled with the self-excitation feature of the Hawkes jump process.

5.2. Overall pricing performance

Table 5 reports the overall pricing performance of the models in-sample and out-of-sample. Model performance is assessed by employing commonly used performance metrics, including the MAE, RMSE and MAPE. Out-of-sample pricing is conducted by using the Thursday data sample over the same sample period. Numbers in bold indicate the smallest pricing error. In terms of VIX derivatives pricing, the SVHJ model outperforms all other models in-sample and out-of-sample; in general, the pecking order is SVHJ>SVSJ>SVCJ>SV, where > means outperformance. The superiority of the SVHJ model in pricing VIX derivatives has also been documented in Jing, Li, and Ma (2020), and that the model with stochastic jump intensity (SVSJ) outperforms model with constant jump intensity (SVCJ) and no jumps (SV) is consistent with the findings of Yuan (2022). Besides, the VIX pricing error becomes larger for all models when VXX data are included in the calibration, implying that VXX options do not add value to the VIX option pricing. Therefore, it is better to only use VIX data for model calibration when pricing VIX futures and options.

Turning to the VXX option pricing performance, the pecking order of relative model performance does not change: SVHJ still outperforms all other models, model with stochastic jump intensity (SVSJ) is better than model with constant jump intensity (SVCJ), and model with constant jump intensity outperforms model with no jumps (SV). When only VIX data is used in calibration, the (out-of-sample) VXX option pricing error is large; for example, the MAE from SVHJ is 0.2089 for VXX options on Wednesdays, and 0.2113 for Thursday VXX options, and the MAPE is 27.47% and 30.88%, respectively. However, when both the VIX and VXX data are used in calibration, the VXX option pricing error are reduced significantly. For example, the out-of-sample MAE from SVHJ for pricing Thursday VXX options changes from 0.2113 to 0.0697, and MAPE changes from 30.88% to

Table 6. Pairwise comparison of option pricing performance.

Data	In-sample					
W/O	SV-SVCJ	SV-SVSJ	SV-SVHJ	SVCJ-SVSJ	SVCJ-SVHJ	SVSJ-SVHJ
VIX (Wed)	1.5448	**2.3297**	**3.9656**	0.6665	**2.3191**	**1.9218**
	Out-of-sample					
VIX (Thurs)	0.1026	1.0813	**2.3640**	0.8039	**1.8594**	**2.1653**
VXX (Wed)	1.3206	1.1766	**2.3611**	−0.0811	1.4132	1.3126
VXX (Thurs)	1.2161	1.2585	**2.0995**	0.0976	1.2288	1.0642
W/	In-sample					
VIX (Wed)	**4.2871**	**5.1311**	**9.1296**	1.0647	**5.9208**	**4.6286**
VXX (Wed)	0.5197	1.1391	**3.8396**	0.5119	**2.9864**	**3.4960**
	Out-of-sample					
VIX (Thurs)	**3.6006**	**4.2046**	**7.8460**	0.8119	**4.8195**	**3.6766**
VXX (Thurs)	0.8310	0.8933	**3.1046**	0.0196	**2.2773**	**2.4870**

Notes: This table reports the pairwise t-statistics of sample differences in daily mean squared pricing errors. The t-statistic is calculated based on Huang and Wu (2004) as in Equation (41). In the calculation of the t-statistic, the standard error is adjusted for serial correlation following Newey and West (1987), and the optimal lag number is selected according to Andrews (1991) and an AR(1) specification. For a pair of models *i*–*j*, a positive value of the t-statistic that is larger than 1.645 indicates model *j* significantly outperforms model *i* at 5% significance level. Numbers in bold indicate significance at 5% significance level.

10.23%. The results imply that the information contained in the VXX options market is not fully spanned by the VIX options data. This is also evident in the literature, for example, Bergomi (2015) argue that the information on vanillas (VIX) cannot be used to fully recover the information contained in its exotic path-dependent products (VXX).

Table 6 reports the pairwise t-statistic of model comparisons in terms of option pricing. If the t-statistic for model pair i–j is positive, model j outperforms model i; and if the t-statistic is larger than 1.6450, it means the outperformance is significant at 5% significance level, which is shown by numbers in bold in the table. When models are calibrated to only VIX data, regarding the in-sample performance of pricing VIX options, models with time-varying jumps (SVSJ, SVHJ) significantly outperform model with no jumps (SV), whereas the outperformance of SVCJ over SV is insignificant at 5% level; SVHJ also significantly outperforms SVSJ and SVCJ. Turning to the out-of-sample VXX option pricing, significant advantage of SVHJ is found only over SV in the two VXX out-of-samples, but the positive sign of the t-statistics also confirms the relative model performance observed in Table 5. When VXX data are added to the model calibration, SVHJ significantly outperforms all other models, and SVSJ and SVCJ are significantly superior to SV, in pricing VIX options in-sample and out-of-sample; In terms of VXX option pricing, SVHJ is found to significantly outperform not only

Table 7. Overall pricing performance: Information criteria.

		VIX futures (Wednesday)		VIX futures (Thursday)	
	Model	W/O (In-sample)	W/ (In-sample)	W/O (Out-of-sample)	W/ (Out-of-sample)
AIC	SV	−4.9112	−3.9827	−4.5269	−3.8245
	SVCJ	−5.0754	−4.3700	−4.5428	−4.1627
	SVSJ	−5.2648	−4.3434	−4.5605	−4.0916
	SVHJ	**−5.4497**	**−4.7302**	**−4.6506**	**−4.4379**
BIC	SV	−4.8221	−3.8937	−4.3784	−3.7379
	SVCJ	−4.9609	−4.2555	−4.4491	−4.0523
	SVSJ	−5.1122	−4.1908	−4.4521	−3.9431
	SVHJ	**−5.2971**	**−4.5776**	**−4.5061**	**−4.2893**
		VIX options (Wednesday)		VIX options (Thursday)	
	Model	W/O (In-sample)	W/ (In-sample)	W/O (Out-of-sample)	W/ (Out-of-sample)
AIC	SV	−0.5020	−0.6465	0.0297	−0.0962
	SVCJ	−1.1566	−1.2377	−0.3994	−0.6766
	SVSJ	−1.0925	−1.3575	−0.3702	−0.6714
	SVHJ	**−1.5932**	**−1.6712**	**−0.4736**	**−0.8277**
BIC	SV	−0.4916	−0.6361	0.0396	−0.0863
	SVCJ	−1.1432	−1.2243	−0.3867	−0.6638
	SVSJ	−1.0747	−1.3397	−0.3532	−0.6544
	SVHJ	**−1.5753**	**−1.6534**	**−0.4566**	**−0.8107**
		VXX options (Wednesday)		VXX options (Thursday)	
	Model	W/O (Out-of-sample)	W/ (In-sample)	W/O (Out-of-sample)	W/ (Out-of-sample)
AIC	SV	1.4033	−0.5261	1.4950	−0.1549
	SVCJ	1.1241	−1.5021	1.2338	−0.7310
	SVSJ	1.1661	−1.7221	1.2876	−0.7099
	SVHJ	**1.0031**	**−2.1784**	**1.2002**	**−0.7977**
BIC	SV	1.4101	−0.5193	1.5017	−0.1482
	SVCJ	1.1329	−1.4933	1.2423	−0.7196
	SVSJ	1.1777	−1.7104	1.2990	−0.6985
	SVHJ	**1.0148**	**−2.1667**	**1.2116**	**−0.7892**

Notes: This table reports the information criteria for the pricing models in-sample and out-of-sample by using the parameter estimates in Table 4. The information criteria include the Akaike information criterion (AIC) and Bayesian information criterion (BIC). Specifically, $AIC = (2k - 2\ln L)/N$, and $BIC = (k\ln N - 2\ln L)/N$, where k is the number of model parameters, N is the total number of derivatives contracts, $\ln L$ is the value of the log-likelihood function of model pricing errors. We follow Jiang et al. (2022), Ornthanalai (2014), Wang and Wang (2020) to compute the log-likelihood functions of futures and options pricing errors, where the pricing error for the ith derivative contract is defined as the relative pricing error $\varepsilon_i = (\widetilde{O}_i - O_i)/O_i$ or $(\widetilde{F}_i - F_i)/F_i$, where $\widetilde{O}, \widetilde{F}$ are the model-implied price, and O,F are the market quoted price. Numbers in bold indicate the smallest information criteria.

Table 8. Option pricing performance by moneyness and time to maturity: Mean absolute error (MAE).

	VIX futures (Wednesday)								VIX futures (Thursday)							
	W/O (In-sample)				W/ (In-sample)				W/O (Out-of-sample)				W/ (Out-of-sample)			
MAE	SV	SVCJ	SVSJ	SVHJ	SV	SVCJ	SVSJ	SVHJ	SV	SVCJ	SVSJ	SVHJ	SV	SVCJ	SVSJ	SVHJ
$\tau \leq 30$	0.4142	0.3590	0.3688	**0.3500**	0.5723	0.4618	0.4497	**0.3782**	0.4525	0.4084	0.4384	**0.4340**	0.6142	0.5231	0.4978	**0.4271**
$30 < \tau \leq 90$	0.3894	0.3349	0.2891	**0.1931**	0.7384	0.5704	0.4835	**0.3232**	0.4230	0.3951	0.3859	**0.3424**	0.7758	0.5980	0.5120	**0.3520**
$\tau > 90$	0.2372	0.2062	0.1738	**0.1713**	0.5896	0.4747	0.4412	**0.3066**	0.3177	0.3146	**0.2591**	0.2833	0.6056	0.4998	0.4864	**0.3722**
	VIX options (Wednesday)								VIX options (Thursday)							
$m \leq -0.2$	0.0549	0.0635	0.0533	**0.0479**	0.0758	0.0600	0.0563	**0.0417**	**0.0679**	0.0826	0.0740	0.0785	0.0868	0.0655	0.0662	**0.0557**
$-0.2 < m \leq -0.06$	0.0844	0.0815	0.0720	**0.0655**	0.2517	0.1985	0.1681	**0.0977**	**0.1024**	0.1133	0.1132	0.1245	0.2457	0.1891	0.1640	**0.1067**
$-0.06 < m \leq 0.06$	0.1471	0.1288	0.1115	**0.0900**	0.3709	0.2830	0.2558	**0.1716**	0.1779	0.1691	0.1540	**0.1504**	0.3822	0.2980	0.2725	**0.1870**
$0.06 < m \leq 0.2$	0.0754	0.0617	0.0572	**0.0444**	0.2162	0.1549	0.1642	**0.1158**	0.1258	0.1204	0.0973	**0.0939**	0.2412	0.1883	0.1933	**0.1400**
$0.2 < m \leq 0.4$	0.0566	0.0416	0.0391	**0.0282**	0.1281	0.1041	0.1048	**0.0619**	0.0907	0.0852	0.0643	**0.0620**	0.1442	0.1202	0.1230	**0.0822**
$0.4 < m \leq 0.6$	0.0633	0.0381	0.0415	**0.0268**	0.0755	0.0703	0.0632	**0.0318**	0.0791	0.0568	0.0541	**0.0441**	0.0840	0.0754	0.0713	**0.0423**
$m > 0.6$	0.0622	0.0400	0.0418	**0.0378**	0.0472	0.0426	0.0337	**0.0281**	0.0668	**0.0451**	0.0468	0.0503	0.0509	0.0446	0.0386	**0.0340**
$\tau \leq 30$	0.1240	0.0829	0.0850	**0.0600**	0.1121	0.0752	0.0793	**0.0689**	0.1300	0.0899	0.0884	**0.0732**	0.1185	0.0879	0.0901	**0.0757**
$30 < \tau \leq 90$	0.0819	0.0621	0.0593	**0.0437**	0.1126	0.0939	0.0890	**0.0573**	0.1015	0.0895	0.0819	**0.0770**	0.1218	0.1016	0.0970	**0.0680**
$\tau > 90$	0.0557	0.0524	0.0451	**0.0414**	0.2243	0.1747	0.1606	**0.0972**	0.0885	0.0932	**0.0770**	0.0833	0.2288	0.1817	0.1722	**0.1138**
	VXX options (Wednesday)								VXX options (Thursday)							
	W/O (Out-of-sample)				W/ (In-sample)				W/O (Out-of-sample)				W/ (Out-of-sample)			
$m \leq -0.2$	0.1963	0.1941	**0.1932**	0.2256	0.0828	0.0765	0.0670	**0.0466**	0.1997	0.2002	**0.1996**	0.2264	0.1029	0.0941	0.0912	**0.0718**
$-0.2 < m \leq -0.06$	0.2405	0.2340	0.2212	**0.2063**	0.0887	0.0702	0.0627	**0.0430**	0.2425	0.2445	**0.2196**	0.2216	0.1164	0.0941	0.0972	**0.0828**
$-0.06 < m \leq 0.06$	0.2805	0.2612	0.2495	**0.2356**	0.0972	0.0923	0.0830	**0.0471**	0.2731	0.2605	**0.2364**	0.2417	0.1199	0.1079	0.1068	**0.0897**
$0.06 < m \leq 0.2$	0.2924	0.2714	0.2669	**0.2318**	0.0985	0.0853	0.0796	**0.0432**	0.2600	0.2502	0.2454	**0.2411**	0.1024	0.0907	0.0917	**0.0778**
$0.2 < m \leq 0.4$	0.3248	0.3019	0.3040	**0.2165**	0.1026	0.0850	0.0789	**0.0392**	0.2990	0.2868	0.2769	**0.2112**	0.1040	0.0875	0.0890	**0.0627**
$0.4 < m \leq 0.6$	0.3358	0.3003	0.3159	**0.1800**	0.0936	0.0851	0.0783	**0.0394**	0.2988	0.2775	0.2773	**0.1632**	0.0848	0.0776	0.0787	**0.0491**
$m > 0.6$	0.3961	0.3177	0.3142	**0.1544**	0.1162	0.0918	0.0864	**0.0426**	0.3751	0.3022	0.2985	**0.1514**	0.1129	0.0925	0.0922	**0.0483**
$\tau \leq 30$	0.1507	0.1483	**0.1413**	0.1751	0.0543	0.0401	0.0344	**0.0232**	0.1633	0.1671	**0.1479**	0.1864	0.0758	**0.0648**	0.0649	0.0664
$30 < \tau \leq 90$	0.1951	0.2136	0.2074	**0.1545**	0.0605	0.0409	0.0411	**0.0355**	0.1984	0.2218	0.2070	**0.1682**	0.0789	**0.0585**	0.0646	0.0592
$\tau > 90$	0.4697	0.3919	0.3970	**0.2726**	0.1526	0.1447	0.1310	**0.0589**	0.4415	0.3727	0.3734	**0.2682**	0.1530	0.1423	0.1383	**0.0815**

Notes: This table reports the model pricing performance by the moneyness level and time to maturity. Models are calibrated to the Wednesday data sample from 10 October 2012 to 10 October 2013. Derivative prices are then computed daily by using the calibrated model, the performance metrics, the Mean Absolute Error (MAE), is calculated as in Equation (38). Option moneyness is defined as the logarithm of the ratio between the strike price of the option and the underlying asset forward price. 'W /O' means calibration to the VIX derivatives data only, 'W /' means calibration to both the VIX and VXX data. Numbers in bold indicate the smallest pricing error.

Table 9. Option pricing performance by moneyness and time to maturity: Root mean squared error (RMSE).

	VIX futures (Wednesday)								VIX futures (Thursday)							
	W/O (In-sample)				W/ (In-sample)				W/O (Out-of-sample)				W/ (Out-of-sample)			
RMSE	SV	SVCJ	SVSJ	SVHJ	SV	SVCJ	SVSJ	SVHJ	SV	SVCJ	SVSJ	SVHJ	SV	SVCJ	SVSJ	SVHJ
$\tau \leq 30$	0.5153	0.4713	0.4544	**0.4287**	0.6716	0.5831	0.5616	**0.4778**	0.5363	**0.4969**	0.5708	0.5570	0.7355	0.6540	0.6158	**0.5385**
$30 < \tau \leq 90$	0.4527	0.4180	0.3648	**0.2824**	0.7975	0.6352	0.5527	**0.3923**	0.5155	0.5009	0.5228	**0.4992**	0.8460	0.6860	0.5948	**0.4348**
$\tau > 90$	0.2919	0.2654	0.2218	**0.2100**	0.7141	0.5767	0.5447	**0.3972**	0.3998	0.3977	**0.3467**	0.3739	0.7441	0.6089	0.6067	**0.4727**
	VIX options (Wednesday)								VIX options (Thursday)							
$m \leq -0.2$	0.0688	0.0783	0.0649	**0.0606**	0.1032	0.0823	0.0756	**0.0585**	**0.**0893	0.1079	0.0942	0.1041	0.1184	0.0892	0.0901	**0.0763**
$-0.2 < m \leq -0.06$	0.1071	0.1074	0.0975	**0.0885**	0.3170	0.2408	0.2045	**0.1306**	**0.**1335	0.1503	0.1609	0.1746	0.3119	0.2325	0.2015	**0.1417**
$-0.06 < m \leq 0.06$	0.1819	0.1675	0.1497	**0.1280**	0.4489	0.3435	0.3109	**0.2178**	0.2301	0.2312	0.2125	**0.2118**	0.4620	0.3664	0.3405	**0.2457**
$0.06 < m \leq 0.2$	0.0981	0.0811	0.0770	**0.0607**	0.2860	0.2118	0.2194	**0.1554**	0.1765	0.1803	0.1364	**0.1348**	0.3263	0.2653	0.2646	**0.1945**
$0.2 < m \leq 0.4$	0.0742	0.0541	0.0530	**0.0362**	0.1811	0.1529	0.1482	**0.0932**	0.1230	0.1236	**0.0856**	0.0877	0.2024	0.1789	0.1734	**0.1244**
$0.4 < m \leq 0.6$	0.0810	0.0503	0.0558	**0.0344**	0.1016	0.0948	0.0841	**0.0462**	0.0980	0.0779	0.0692	**0.0612**	0.1100	0.1043	0.0967	**0.0597**
$m > 0.6$	0.0736	0.0504	0.0530	**0.0487**	0.0635	0.0586	0.0462	**0.0364**	0.0800	**0.0595**	0.0607	0.0780	0.0660	0.0626	0.0539	**0.0457**
$\tau \leq 30$	0.1479	0.1152	0.1189	**0.0960**	0.1568	0.1270	0.1361	**0.1122**	0.1578	0.1324	0.1379	**0.1245**	0.1743	0.1442	0.1423	**0.1206**
$30 < \tau \leq 90$	0.1081	0.0918	0.0869	**0.0679**	0.1735	0.1426	0.1307	**0.0934**	0.1415	0.1411	0.1252	**0.1247**	0.1828	0.1553	0.1473	**0.1064**
$\tau > 90$	0.0760	0.0720	0.0594	**0.0561**	0.3130	0.2381	0.2197	**0.1420**	0.1281	0.1358	**0.1070**	0.1197	0.3239	0.2577	0.2423	**0.1714**
	VXX options (Wednesday)								VXX options (Thursday)							
	W/O (Out-of-sample)				W/ (In-sample)				W/O (Out-of-sample)				W/ (Out-of-sample)			
$m \leq -0.2$	0.2898	0.2532	**0.2611**	0.3466	0.1216	0.1052	0.0959	**0.0690**	0.2992	**0.2683**	0.2706	0.3503	0.1434	0.1277	0.1335	**0.0991**
$-0.2 < m \leq -0.06$	0.3137	0.3045	**0.2942**	0.3225	0.1246	0.1074	0.0995	**0.0610**	0.3566	0.3200	**0.2924**	0.3385	0.1591	0.1436	0.1485	**0.1174**
$-0.06 < m \leq 0.06$	0.3991	0.3524	**0.3345**	0.3546	0.1511	0.1447	0.1338	**0.0652**	0.3860	0.3459	**0.3147**	0.3424	0.1690	0.1610	0.1550	**0.1267**
$0.06 < m \leq 0.2$	0.4528	0.3691	0.3713	**0.3179**	0.1539	0.1403	0.1278	**0.0608**	0.3901	0.3352	0.3320	**0.3098**	0.1524	0.1342	0.1316	**0.1073**
$0.2 < m \leq 0.4$	0.4927	0.4123	0.4270	**0.2955**	0.1661	0.1492	0.1329	**0.0570**	0.4562	0.3888	0.3871	**0.2829**	0.1606	0.1371	0.1308	**0.0877**
$0.4 < m \leq 0.6$	0.5476	0.4378	0.4734	**0.2725**	0.1542	0.1558	0.1301	**0.0589**	0.4731	0.4034	0.4082	**0.2183**	0.1348	0.1181	0.1237	**0.0714**
$m > 0.6$	0.6378	0.4833	0.5023	**0.2225**	0.1790	0.1593	0.1478	**0.0680**	0.6235	0.4670	0.4739	**0.2450**	0.1711	0.1521	0.1555	**0.0710**
$\tau \leq 30$	0.1926	0.1904	**0.1827**	0.2205	0.0677	0.0584	0.0514	**0.0295**	0.2089	0.2119	**0.1860**	0.2392	0.1003	**0.0978**	0.0980	0.0999
$30 < \tau \leq 90$	0.2539	0.2663	0.2644	**0.2033**	0.0800	0.0561	0.0583	**0.0486**	0.2630	0.2828	0.2674	**0.2190**	0.1054	**0.0845**	0.0945	0.0871
$\tau > 90$	0.6709	0.5248	0.5443	**0.4024**	0.2195	0.2074	0.1847	**0.0820**	0.6388	0.5016	0.5046	**0.3912**	0.2182	0.1958	0.1904	**0.1106**

Notes: This table reports the model pricing performance by the moneyness level and time to maturity. Models are calibrated to the Wednesday data sample from 10 October 2012 to 10 October 2013. Derivative prices are then computed daily by using the calibrated model, the performance metrics, the Root Mean Squared Error (RMSE), is calculated as in Equation (39). Option moneyness is defined as the logarithm of the ratio between the strike price of the option and the underlying asset forward price. 'W /O' means calibration to the VIX derivatives data only, 'W /' means calibration to both the VIX and VXX data. Numbers in bold indicate the smallest pricing error.

Table 10. Option pricing performance by moneyness and time to maturity: Mean absolute percentage error (MAPE).

	VIX futures (Wednesday)								VIX futures (Thursday)							
	W/O (In-sample)				W/ (In-sample)				W/O (Out-of-sample)				W/ (Out-of-sample)			
MAPE (%)	SV	SVCJ	SVSJ	SVHJ	SV	SVCJ	SVSJ	SVHJ	SV	SVCJ	SVSJ	SVHJ	SV	SVCJ	SVSJ	SVHJ
$\tau \leq 30$	2.61	2.26	2.30	**2.17**	3.63	2.88	2.80	**2.36**	2.82	**2.56**	2.68	2.65	3.84	3.24	3.08	**2.63**
$30 < \tau \leq 90$	2.26	1.92	1.66	**1.10**	4.34	3.32	2.81	**1.86**	2.46	2.28	2.21	**1.97**	4.55	3.49	2.99	**2.04**
$\tau > 90$	1.26	1.09	0.92	**0.90**	3.17	2.55	2.32	**1.60**	1.67	1.66	**1.36**	1.48	3.24	2.67	2.54	**1.94**
	VIX options (Wednesday)								VIX options (Thursday)							
$m \leq -0.2$	14.94	17.22	14.92	**13.63**	16.24	12.02	11.73	**9.75**	**20.16**	25.10	23.42	24.11	24.30	17.16	17.84	**16.51**
$-0.2 < m \leq -0.06$	10.36	9.44	9.38	**8.27**	22.41	17.86	15.32	**10.26**	**13.29**	13.67	15.32	15.35	24.33	18.90	16.88	**12.94**
$-0.06 < m \leq 0.06$	8.48	7.42	6.56	**5.39**	18.91	14.36	12.93	**8.94**	10.10	9.27	8.80	**8.56**	19.57	15.20	13.81	**9.85**
$0.06 < m \leq 0.2$	5.55	4.55	4.12	**3.26**	13.22	9.28	9.87	**7.12**	8.68	8.00	6.67	**6.46**	14.81	11.41	11.83	**8.73**
$0.2 < m \leq 0.4$	9.45	6.08	5.49	**3.70**	12.69	9.53	9.71	**6.02**	13.00	10.29	8.18	**7.41**	14.90	11.62	12.07	**8.34**
$0.4 < m \leq 0.6$	16.32	8.62	9.58	**5.60**	13.93	11.38	10.42	**5.73**	20.56	12.44	12.30	**9.25**	16.73	13.05	12.71	**8.16**
$m > 0.6$	23.22	14.29	15.19	**13.18**	15.48	13.25	10.68	**9.85**	29.59	**18.80**	19.81	20.08	20.81	16.08	14.70	**14.79**
$\tau \leq 30$	31.75	18.13	19.29	**12.40**	22.73	12.64	13.07	**12.43**	39.25	22.18	23.86	**18.15**	29.11	18.01	19.04	**16.18**
$30 < \tau \leq 90$	13.16	8.66	8.78	**6.23**	12.78	10.50	9.83	**6.61**	17.20	13.07	12.83	**11.59**	16.20	12.71	12.30	**9.60**
$\tau > 90$	6.48	6.40	5.56	**5.54**	16.56	13.50	12.13	**7.66**	9.57	10.49	**8.97**	10.15	17.88	14.67	13.84	**10.15**
	VXX options (Wednesday)								VXX options (Thursday)							
	W/O (Out-of-sample)				W/ (In-sample)				W/O (Out-of-sample)				W/ (Out-of-sample)			
$m \leq -0.2$	35.52	33.07	**31.04**	33.03	16.31	12.96	10.24	**7.55**	36.69	33.89	**32.01**	34.22	20.76	16.91	15.51	**13.02**
$-0.2 < m \leq -0.06$	28.28	26.46	25.09	**22.52**	13.09	6.76	5.80	**5.27**	31.56	31.50	28.73	**28.23**	18.93	12.63	13.10	**12.40**
$-0.06 < m \leq 0.06$	15.56	14.74	**14.30**	14.57	5.42	5.16	4.33	**3.13**	19.00	18.55	**16.67**	18.75	9.37	8.11	7.96	**7.82**
$0.06 < m \leq 0.2$	22.99	23.47	**22.75**	28.21	8.45	6.36	6.12	**4.12**	**26.15**	27.61	28.55	38.26	11.18	**9.92**	9.97	11.30
$0.2 < m \leq 0.4$	31.04	32.96	35.01	**30.59**	12.09	7.06	6.54	**4.44**	**32.72**	35.44	34.30	32.80	13.75	10.12	10.41	**9.08**
$0.4 < m \leq 0.6$	38.72	41.79	44.78	**30.13**	12.81	9.15	8.55	**6.16**	38.52	42.71	42.95	**30.37**	13.79	10.70	10.84	**8.86**
$m > 0.6$	65.63	55.45	55.10	**32.83**	20.85	14.70	13.68	**9.32**	65.45	55.93	55.60	**33.95**	21.44	16.00	16.75	**11.26**
$\tau \leq 30$	**37.18**	37.33	37.63	46.72	16.93	9.17	7.82	**6.45**	40.55	42.34	**39.92**	50.83	21.25	**15.49**	15.57	16.90
$30 < \tau \leq 90$	29.39	33.48	32.44	**25.50**	10.56	6.61	6.24	**5.78**	28.83	33.72	32.38	**27.32**	12.20	9.08	9.63	**8.82**
$\tau > 90$	33.52	27.94	29.32	**19.01**	11.06	9.63	8.63	**4.62**	33.42	27.61	28.35	**19.05**	11.84	10.36	10.00	**6.46**

Notes: This table reports the model pricing performance by the moneyness level and time to maturity. Models are calibrated to the Wednesday data sample from 10 October 2012 to 10 October 2013. Derivative prices are then computed daily by using the calibrated model, the performance metrics, the Mean Absolute Percentage Error (MAPE), is calculated as in Equation (40). Option moneyness is defined as the logarithm of the ratio between the strike price of the option and the underlying asset forward price. 'W /O' means calibration to the VIX derivatives data only, 'W /' means calibration to both the VIX and VXX data. Numbers in bold indicate the smallest pricing error.

SV but also other models with jumps (SVCJ and SVSJ) both in-sample and out-of-sample, no other significant model pair is found. In addition to the above performance measures, we also report the information criteria for the models in Table 7. The information criteria include the Akaike information criterion (AIC)

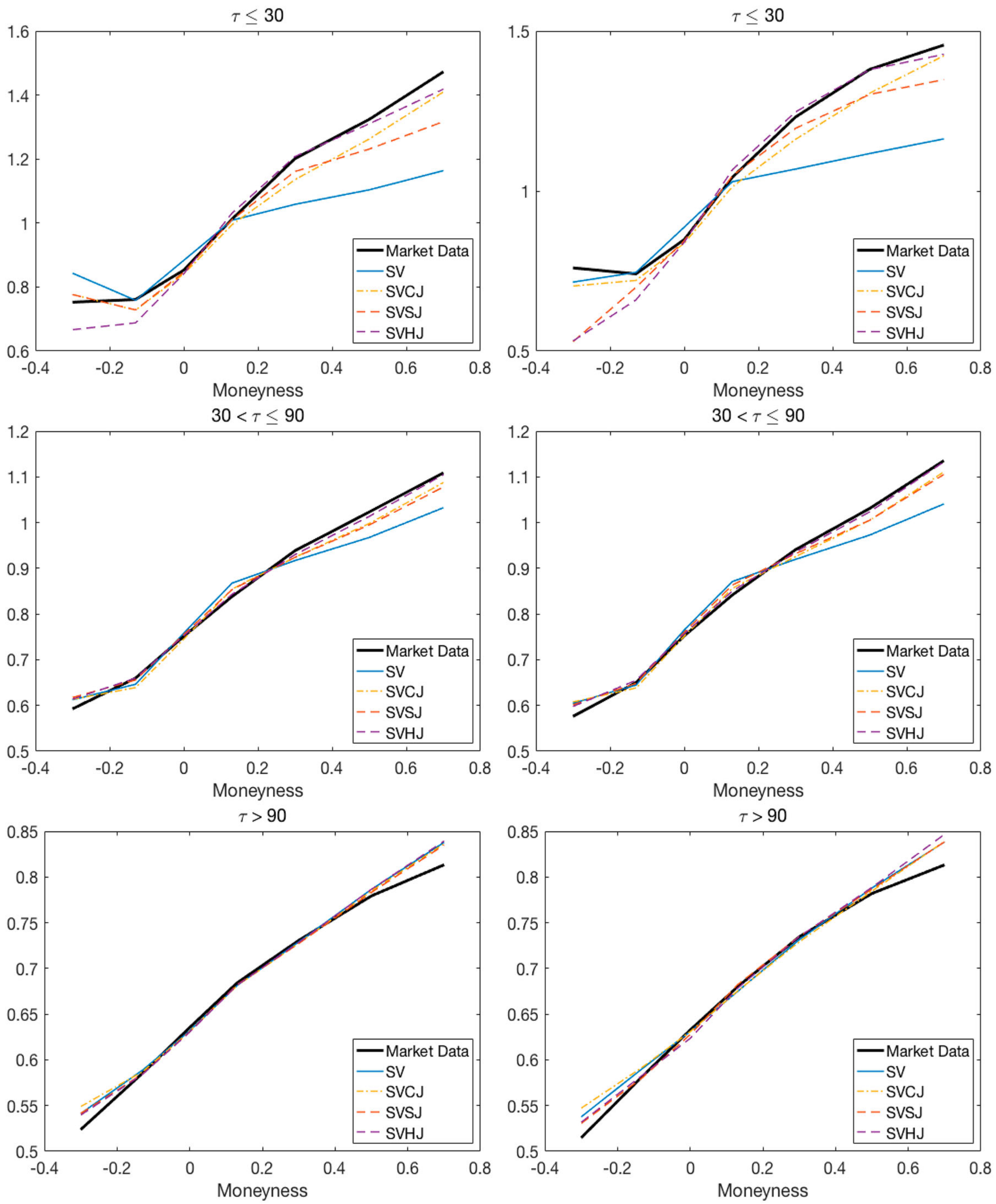

Figure 1. Fit to VIX implied volatility (W/O).

and Bayesian information criterion (BIC). Specifically, $AIC = (2k - 2\ln L)/N$, and $BIC = (k\ln N - 2\ln L)/N$, where k is the number of model parameters, N is the total number of derivatives contracts, $\ln L$ is the value of the log-likelihood function of model pricing errors. We follow Jiang et al. (2022), Ornthanalai (2014), Wang and

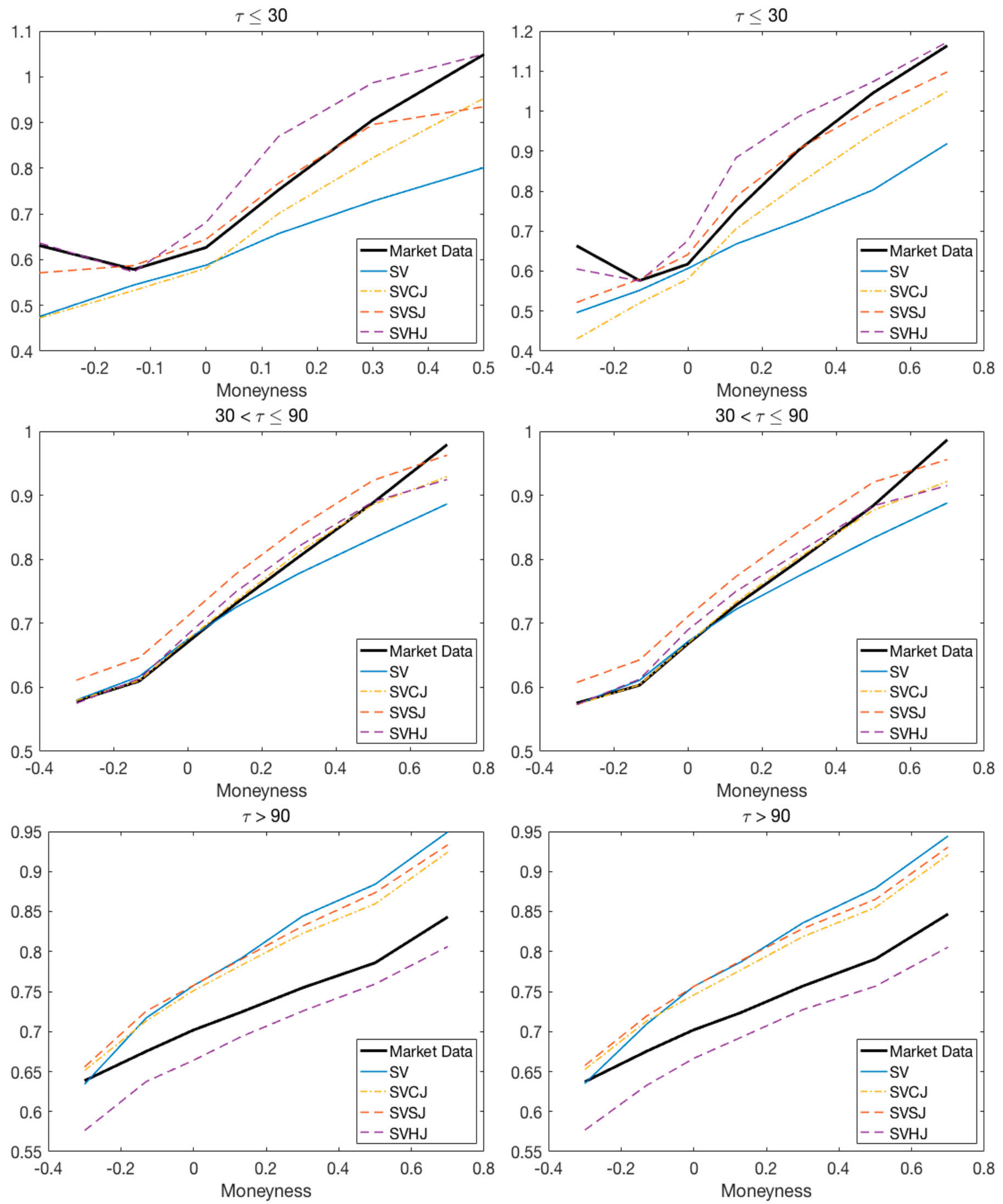

Figure 2. Fit to VXX implied volatility (W/O).

Wang (2020) to compute the log-likelihood functions of futures and options pricing errors, where the pricing error for the ith derivative contract is defined as the relative pricing error $\varepsilon_i = (\widetilde{O}_i - O_i)/O_i$ or $(\widetilde{F}_i - F_i)/F_i$, where $\widetilde{O}, \widetilde{F}$ are the model-implied price, and O,F are the market quoted price. Numbers in bold indicate the

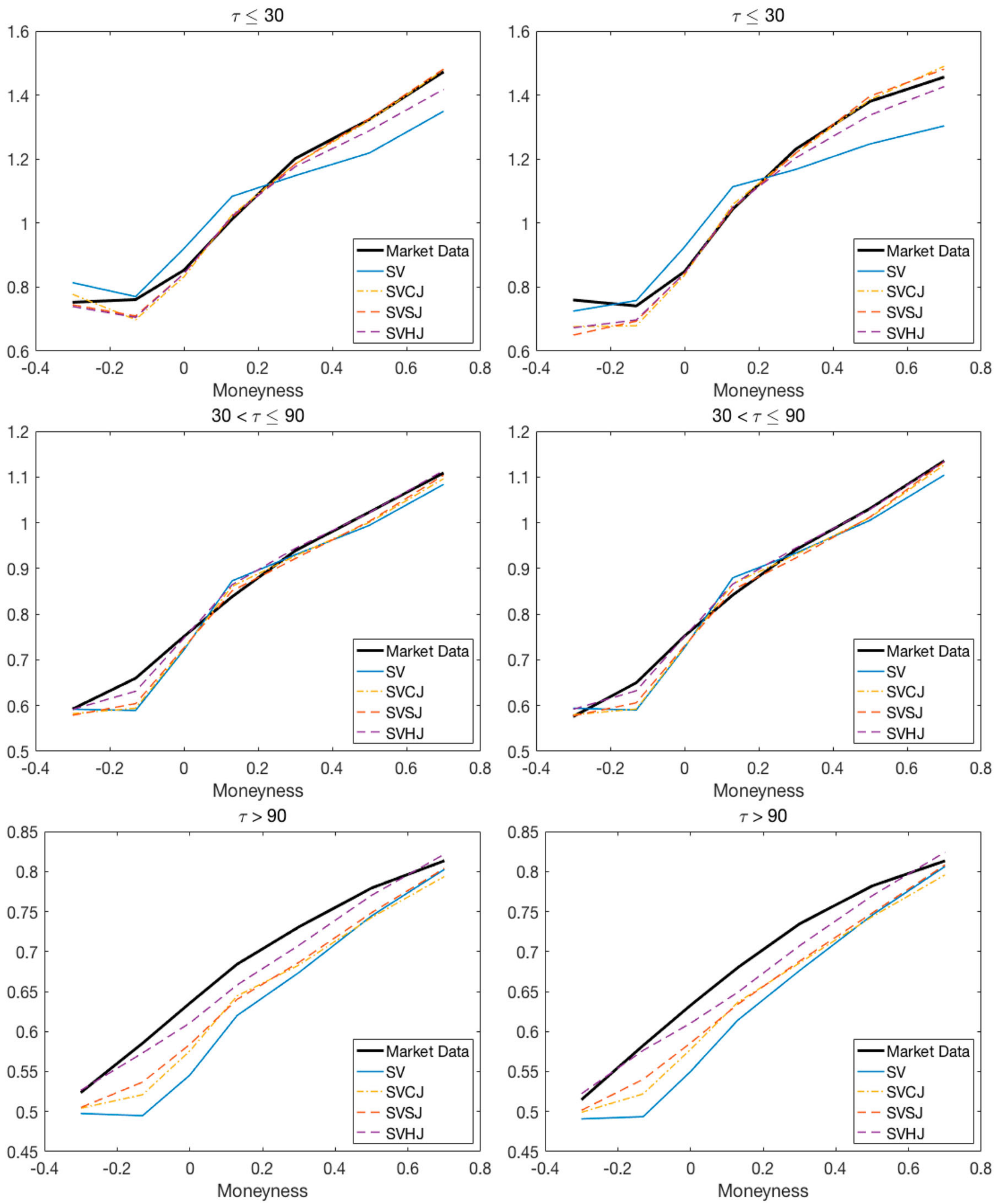

Figure 3. Fit to VIX implied volatility (W/).

smallest information criteria. The patterns in the reported information criteria support our earlier findings that the Hawkes jumps improve the model performance in terms jointly pricing VIX futures and options, and VXX options.

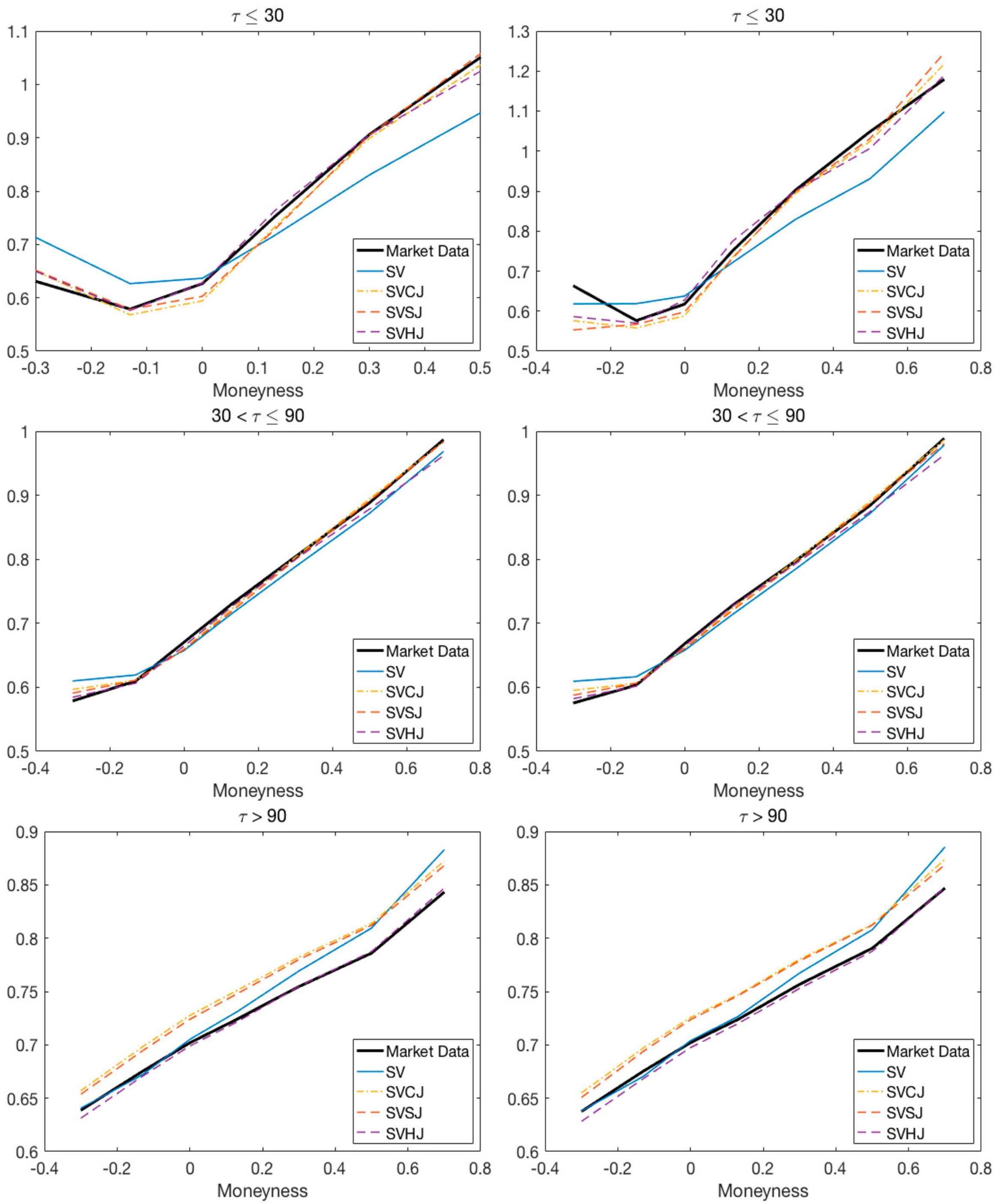

Figure 4. Fit to VXX implied volatility (W/).

5.3. Pricing performance by moneyness and time to maturity

To further analyse the pricing performance, Tables 8–10 report the pricing performance by moneyness and time to maturity. When models are calibrated to VIX data only, regarding the in-sample pricing performance of VIX futures and options, SVHJ model outperforms all other models across all moneyness levels and maturities in pricing VIX futures and options in-sample. Turning to the out-of-sample, in terms of out-of-sample VIX futures pricing, first, SVHJ is the best performing model at short to medium maturities, especially at medium maturities, and SVSJ model performs the best for long maturities; second, models with jumps (SVCJ, SVSJ, SVHJ) are superior to the model without jumps (SV). In terms of VIX option pricing out-of-sample, SVHJ is superior for near-the-money (NTM) options and OTM calls and at short and medium maturities, while SVHJ is inferior to other models for OTM puts and at long maturities. Moving to the out-of-sample VXX option pricing, SVHJ model outperforms all other models for almost all moneyness levels except the deep OTM VXX puts in the Wednesday sample, while only outperforms other models for OTM calls in the Thursday sample; SVHJ is also found to provide better pricing performance at medium and long maturities in both VXX out-of-samples.

When models are calibrated to both VIX and VXX data, SVHJ outperforms all other models across all moneyness levels and at all maturities in terms of pricing VIX futures, VIX and VXX options in-sample. Turning to the out-of-sample pricing, SVHJ is superior to all other models in pricing VIX futures and options across all moneyness levels and maturities, and in pricing VXX options across all moneyness levels and at long maturities.

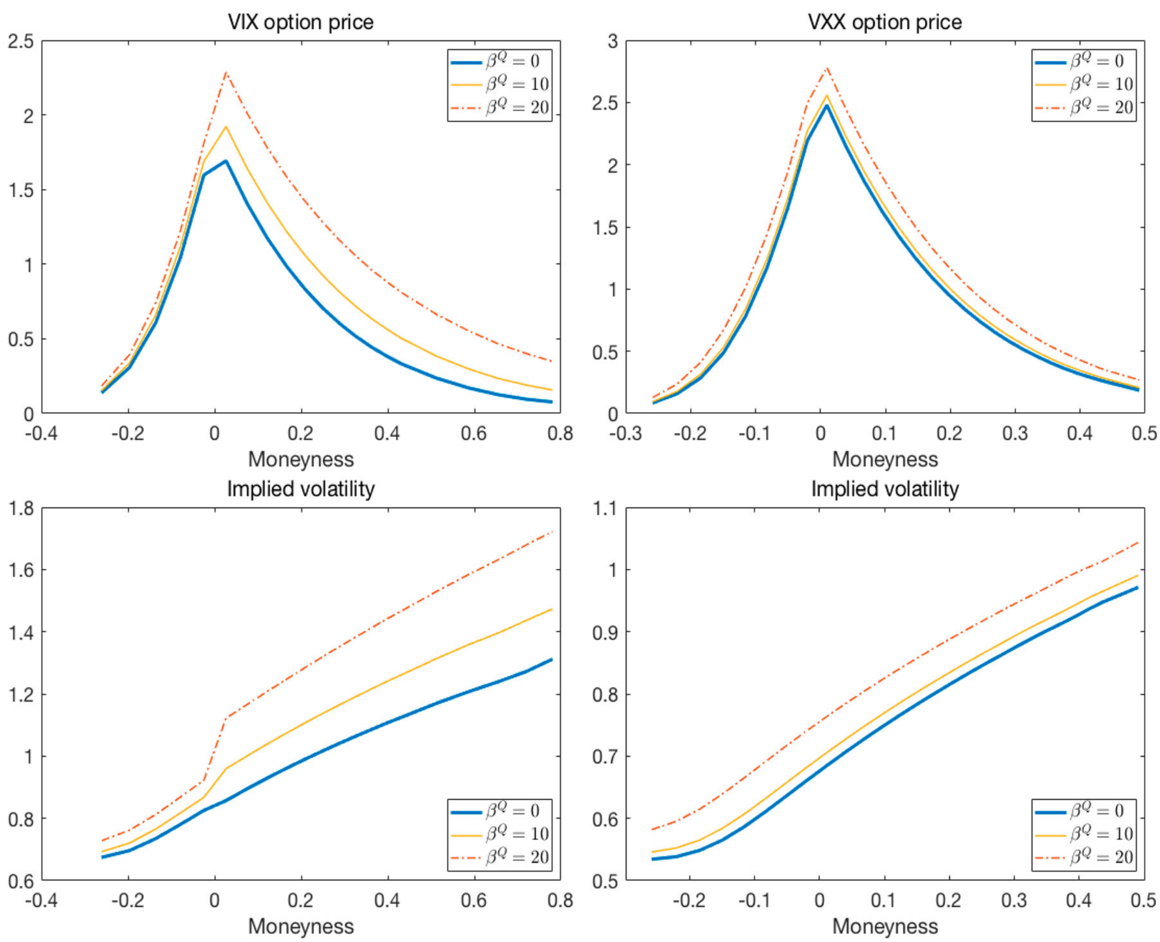

Figure 5. Sensitivity to Hawkes self-excitation parameter $\beta^{\mathbb{Q}}$.

Figure 1 shows the model fit to the average VIX implied volatility skew both in-sample and out-of-sample, and Figure 2 shows the model fit to the average VXX implied volatility skew in the two out-of-samples, when models are calibrated to VIX data only. Figures 3 and 4 show the model fit the average VIX and VXX implied volatility skew both in-sample and out-of-sample, when models are calibrated to both VIX and VXX data. These figures in general confirm our findings. In particular, when VIX and VXX data are both employed in calibration, SVHJ model provides a very good fit to the VXX implied volatility skew, especially at long maturities.

5.4. Sensitivity to Hawkes parameters

Figure 5 shows the sensitivity of VIX and VXX option price and implied volatility to the Hawkes self-excitation parameter $\beta^{\mathbb{Q}}$. The option price and implied volatility are calculated by using the mean parameter estimates for the SVHJ model when the model is calibrated to both VIX and VXX data, while $\beta^{\mathbb{Q}}$ takes on different values. Figure 6 shows the sensitivity of VIX and VXX option price and implied volatility to the Hawkes decaying parameter α. The option price and implied volatility are calculated by using the mean parameter estimates for the SVHJ model when the model is calibrated to both VIX and VXX data, while α takes on different values.

From Figure 5, we can see that, first, as $\beta^{\mathbb{Q}}$ increases (VIX is more clustered), both option price and implied volatility increase as a result; second, $\beta^{\mathbb{Q}}$ has a larger impact on the VIX option price and implied volatility than the VXX option price and implied volatility; third, VIX OTM calls are more sensitive to volatility clustering

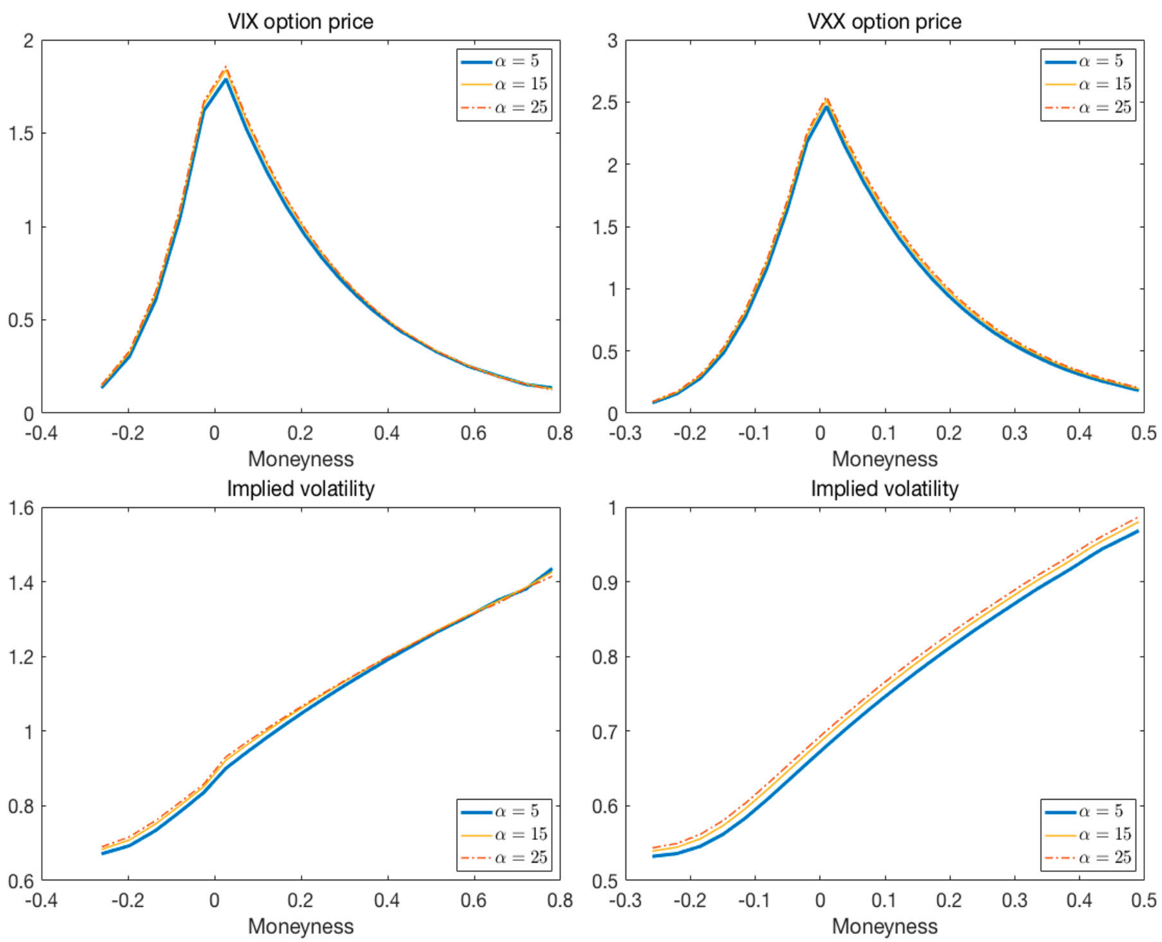

Figure 6. Sensitivity to Hawkes decaying parameter α.

than VIX OTM puts. From Figure 6, we can see that as α increases (the impact of previous jumps decays faster), both option price and implied volatility increase; but the impact of α on the option price and implied volatility is much smaller than that of $\beta^{\mathbb{Q}}$.

6. Conclusion

In this paper, we examine the effects of volatility clustering on the joint calibration of VIX and VXX options. The volatility clustering feature is modelled by a self-exciting Hawkes jump process in the VIX dynamics; under the consistent pricing framework, the resulting implied VXX dynamics inferred from the VIX is shown to also have a Hawkes type of jumps. Closed-form formula for VXX options are obtained via Fourier transform. Calibrating the model to the real market data, we find that: (1) the information contained in the VXX options is not fully spanned by the VIX options, as a result, one can achieve better performance of pricing VXX options when both VIX and VXX derivatives data are used in calibration, compared to the case when only VIX data are used; (2) the model with Hawkes jumps (thereby incorporating the volatility clustering feature) outperforms other models with either constant intensity/stochastic intensity Poisson jumps or no jumps in joint pricing of VIX futures, VIX options and VXX options both in-sample and out-of-sample, the outperformance of the Hawkes jump diffusion is statistically significant based on the pairwise t-test of model comparisons. This paper contributes to the literature that it is the first to apply the Hawkes process in the pricing of VXX options, or more broadly, options written on the VIX exchange traded products, and the formula for forward start VXX options are also obtained. We hope the findings of this paper can shed some light on the joint dynamics of the VIX and VXX and the pricing of options written on them.

This paper relates to a number of studies on the VIX and VXX dynamics. For example, stand-alone models with infinite-activity Lévy jumps are found to be superior to models with poisson jumps with constant intensity in terms of pricing VIX or VXX options (see Cao et al. 2020b, 2021). However, there are no studies that compare these models with the Hawkes jump process, future work could explore the comparative performance of models with these two different jump specifications. Moreover, dynamic VXX options hedging performance of Hawkes jump diffusion model could also be explored, since the VXX option has a tradable underlying asset, the SPVXSTR. Lastly, the approach employed in this paper can be easily extended to include other stylised facts about these two markets such as regime switching.

Notes

1. For example, Alexander, Kapraun, and Korovilas (2015) find that VXX and VIX are highly positively correlated.
2. This is because

$$\mathrm{d}Z_t\mathrm{d}Z_t = \frac{a_0\,\mathrm{d}B_t^{v,\mathbb{Q}} + B(-i,\tau_0)\sigma_w\,\mathrm{d}B_t^{w,\mathbb{Q}}}{\sqrt{a_0^2 + B(-i,\tau_0)^2\sigma_w^2 + 2\sigma_w\rho a_0 B(-i,\tau_0)}} \times \frac{a_0\,\mathrm{d}B_t^{v,\mathbb{Q}} + B(-i,\tau_0)\sigma_w\,\mathrm{d}B_t^{w,\mathbb{Q}}}{\sqrt{a_0^2 + B(-i,\tau_0)^2\sigma_w^2 + 2\sigma_w\rho a_0 B(-i,\tau_0)}} = \mathrm{d}t$$

3. See product specifications at: https://www.cboe.com/exchange_traded_stock/etp_options_spec/.
4. There are only a handful of days where VIX futures is in backwardation from 2006 to 2020. See S&P Global at: https://www.spglobal.com/en/research-insights/articles/the-vix-futures-curve-is-in-backwardation.

Acknowledgements

I thank an anonymous associate editor, and two anonymous referees for their helpful and insightful comments and suggestions which helped improve this paper.

Disclosure statement

No potential conflict of interest was reported by the author(s).

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Appendices

Appendix 1. Characteristic functions

This section contains the characteristic functions of $v_T = \ln \text{VIX}_T$ under the SVCJ, SVSJ and SVHJ models. The characteristic functions can be found in or derived in the same way as in e.g. Jing, Li, and Ma (2020), Park (2016), Yuan (2022). Therefore, derivation is omitted for brevity.

A.1 SVCJ

Based on the model in (4)–(5), the characteristic function has the following form:

$$f(\phi; v_t, w_t, t, T) = \mathrm{e}^{A(\phi,t,T)+i\phi a v_t+B(\phi,t,T)w_t}$$

whose coefficients satisfy the following system of ODEs (see e.g. Yuan 2022):

$$\frac{\partial A(\phi,\tau)}{\partial \tau} = i\phi a\kappa_v u - i\phi a\bar{\lambda}\mu_J + \kappa_w \bar{w} B(\phi,\tau) + \bar{\lambda}\left(\frac{1}{1-i\phi a\mu_J} - 1\right)$$
$$\frac{\partial B(\phi,\tau)}{\partial \tau} = -\kappa_w B(\phi,\tau) - \frac{1}{2}\phi^2 a^2 + \frac{1}{2}\sigma_w^2 B(\phi,\tau)^2 + \rho\sigma_w i\phi a B(\phi,\tau)$$

subject to boundary conditions $A(\phi, 0) = B(\phi, 0) = 0$, $a = \mathrm{e}^{-\kappa_v \tau}$.

A.2 SVSJ

Based on the model in (6)–(8), the characteristic function has the following form:

$$f(\phi; v_t, w_t, \lambda_t, t, T) = \mathrm{e}^{A(\phi,t,T)+i\phi a v_t+B(\phi,t,T)w_t+C(\phi,t,T)\lambda_t}$$

whose coefficients satisfy the following system of ODEs (see e.g. Yuan 2022):

$$\frac{\partial A(\phi,\tau)}{\partial \tau} = i\phi a\kappa_v u + \kappa_w \bar{w} B(\phi,\tau) + \kappa_\lambda \theta C(\phi,\tau)$$
$$\frac{\partial B(\phi,\tau)}{\partial \tau} = -\kappa_w B(\phi,\tau) - \frac{1}{2}\phi^2 a^2 + \frac{1}{2}B(\phi,\tau)^2\sigma_w^2 + i\phi a\rho\sigma_w B(\phi,\tau)$$
$$\frac{\partial C(\phi,\tau)}{\partial \tau} = -i\phi a\mu_J - \kappa_\lambda C(,\phi,\tau) + \frac{1}{2}C(\phi,\tau)^2\sigma_\lambda^2 + \left(\frac{1}{1-i\phi a\mu_J} - 1\right)$$

subject to boundary conditions $A(\phi, 0) = B(\phi, 0) = C(\phi, 0) = 0$, $a = \mathrm{e}^{-\kappa_v \tau}$.

A.3 SVHJ

Based on the model in (1)–(3), the characteristic function has the following form:

$$f(\phi; v_t, w_t, \lambda_t, t, T) = \mathrm{e}^{A(\phi,t,T)+i\phi a v_t+B(\phi,t,T)w_t+C(\phi,t,T)\lambda_t}$$

whose coefficients satisfy the following system of ODEs (see e.g. Jing, Li, and Ma 2020):

$$\frac{\partial A(\phi,\tau)}{\partial\tau} = i\phi a\kappa_v u + \kappa_w\bar{w}B(\phi,\tau) + \alpha\lambda_\infty C(\phi,\tau)$$

$$\frac{\partial B(\phi,\tau)}{\partial\tau} = -\kappa_w B(\phi,\tau) - \frac{1}{2}\phi^2 a^2 + \frac{1}{2}B(\phi,\tau)^2\sigma_w^2 + i\phi a\rho\sigma_w B(\phi,\tau)$$

$$\frac{\partial C(\phi,\tau)}{\partial\tau} = -i\phi a\mu_J - \alpha C(,\phi,\tau) + \left(\frac{\mathrm{e}^{C(\phi,\tau)\beta^{\mathbb{Q}}}}{1-i\phi a\mu_J} - 1\right)$$

subject to boundary conditions $A(\phi,0) = B(\phi,0) = C(\phi,0) = 0$, $a = \mathrm{e}^{-\kappa_v\tau}$.

Appendix 2. Pricing VXX options under SVSJ

This section shows the derivation of the moment generating functions ψ_1 and ψ_2 underlying P_1 and P_2 in (11), based on the implied VXX dynamics in (19)–(21), in order to compute the VXX option prices in (10) under the SVSJ model. Let $x = \ln \mathrm{VXX}$, and let G – a twice-differentiable function of x and state variables w and λ – be the price of a related option contract. First, $G = \mathrm{e}^x P_1$ is the price of an option that pays off VXX_T on maturity T conditional on $\mathrm{VXX}_T > K$ and 0 otherwise. Applying Ito's lemma to G and using the no-arbitrage conditions, we have P_1 satisfy the following partial integro-differential equation (PIDE):

$$\begin{aligned}\frac{\partial P_1}{\partial\tau} &= \left(r + \frac{1}{2}\tilde{\sigma}^2 w + \frac{1}{2}C(-i,\tau_0)^2\sigma_\lambda^2\lambda - \bar{k}\lambda\right)\frac{\partial P_1}{\partial x} + [\kappa_w(\bar{w}-w) + \tilde{\rho}\tilde{\sigma}\sigma_w w]\frac{\partial P_1}{\partial w}\\ &+ [\kappa_\lambda(\theta-\lambda) + C(-i,\tau_0)\sigma_\lambda^2\lambda]\frac{\partial P_1}{\partial\lambda} + \frac{1}{2}(\tilde{\sigma}^2 w + C(-i,\tau_0)^2\sigma_\lambda^2\lambda)\frac{\partial^2 P_1}{\partial x^2}\\ &+ \frac{1}{2}\sigma_w^2 w\frac{\partial^2 P_1}{\partial w^2} + \frac{1}{2}\sigma_\lambda^2\lambda\frac{\partial^2 P_1}{\partial\lambda^2} + \tilde{\rho}\tilde{\sigma}\sigma_w w\frac{\partial^2 P_1}{\partial x\partial w} + C(-i,\tau_0)\sigma_\lambda^2\lambda\frac{\partial^2 P_1}{\partial x\partial\lambda}\\ &+ \lambda\mathbb{E}^{\mathbb{Q}}\{\mathrm{e}^{\ln(1+k)}[P_1(x+\ln(1+k),w,\lambda) - P_1(x,w,\lambda)]\}\end{aligned} \tag{A1}$$

The MGF ψ_1 underlying P_1 must also solve the above equation; Substituting the solution $\psi_1(\phi;x,w,\lambda,t,T) = \exp(D_1(\phi,\tau) + \phi x + E_1(\phi,\tau)w + F_1(\phi,\tau)\lambda)$ to the PIDE in (A1), f_1 solves

$$\begin{aligned}0 &= -\left(\frac{\partial D_1(\phi,\tau)}{\partial\tau} + \frac{\partial E_1(\phi,\tau)}{\partial\tau}w + \frac{\partial F_1(\phi,\tau)}{\partial\tau}\lambda\right) + \left(r + \frac{1}{2}\tilde{\sigma}^2 w + \frac{1}{2}C(-i,\tau_0)^2\sigma_\lambda^2\lambda - \bar{k}\lambda\right)\phi\\ &+ [\kappa_w(\bar{w}-w) + \tilde{\rho}\tilde{\sigma}\sigma_w w]E_1(\phi,\tau) + [\kappa_\lambda(\theta-\lambda) + C(-i,\tau_0)\sigma_\lambda^2\lambda]F_1(\phi,\tau)\\ &+ \frac{1}{2}(\tilde{\sigma}^2 w + C(-i,\tau_0)^2\sigma_\lambda^2\lambda)\phi^2 + \frac{1}{2}\sigma_w^2 wE_1(\phi,\tau)^2 + \frac{1}{2}\sigma_\lambda^2\lambda F_1(\phi,\tau)^2 + \tilde{\rho}\tilde{\sigma}\sigma_w w\phi E_1(\phi,\tau)\\ &+ C(-i,\tau_0)\sigma_\lambda^2\lambda\phi F_1(\phi,\tau) + \lambda\mathbb{E}^{\mathbb{Q}}(\mathrm{e}^{(1+\phi)\ln(1+k)} - \mathrm{e}^{\ln(1+k)})\end{aligned} \tag{A2}$$

subject to the boundary condition $\psi_1|_{\tau=0} = \mathrm{e}^{\phi x}$. Rearranging the above equation, we obtain the system of ODEs in Section 3.2.2 with $j = 1$.

Second, the MGF ψ_2 underlying P_2 can be treated as the current price of an option that pays off $\mathrm{e}^{r\tau+\phi x}$ on maturity. Using the standard conditions for contingent claims and applying the solution to ψ_2 that $\psi_2(\phi;x,w,\lambda,t,T) = \exp(D_2(\phi,\tau) + \phi x + E_2(\phi,\tau)w + F_2(\phi,\tau)\lambda)$, we have that ψ_2 solves

$$\begin{aligned}0 &= -\left(\frac{\partial D_2(\phi,\tau)}{\partial\tau} + \frac{\partial E_2(\phi,\tau)}{\partial\tau}w + \frac{\partial F_2(\phi,\tau)}{\partial\tau}\lambda\right) + \left(r - \frac{1}{2}\tilde{\sigma}^2 w - \frac{1}{2}C(-i,\tau_0)^2\sigma_\lambda^2\lambda - \bar{k}\lambda\right)\phi\\ &+ \kappa_w(\bar{w}-w)E_2(\phi,\tau) + \kappa_\lambda(\theta-\lambda)F_2(\phi,\tau) + \frac{1}{2}(\tilde{\sigma}^2 w + C(-i,\tau_0)^2\sigma_\lambda^2\lambda)\phi^2 + \frac{1}{2}\sigma_w^2 wE_2(\phi,\tau)^2\\ &+ \frac{1}{2}\sigma_\lambda^2\lambda F_2(\phi,\tau)^2 + \tilde{\rho}\tilde{\sigma}\sigma_w w\phi E_2(\phi,\tau) + C(-i,\tau_0)\sigma_\lambda^2\lambda\phi F_2(\phi,\tau) + \lambda[\mathbb{E}^{\mathbb{Q}}(\mathrm{e}^{\phi\ln(1+k)}) - 1]\end{aligned} \tag{A3}$$

subject to the same boundary condition $\psi_2|_{\tau=0} = \mathrm{e}^{\phi x}$. Subsequently, we obtain the system of ODEs in Section 3.2.2 with $j = 2$.