# Decentralised State Feedback Tracking Control for Large-Scale Interconnected Systems Using Sliding Mode Techniques

**Yueheng Ding**

Supervisor: Dr. Xinggang Yan

Division of Computing, Engineering and Mathematical Sciences

This thesis is submitted to the
University of Kent
for the degree of
*Doctor of Philosophy*

School of Engineering August 2023

*To my parents and grandparents, who have been loving me for the past 26 years*
*to acquaintances and strangers, who gave me pain and strength*
*and to the one, who completes me.*

知不可乎骤得
托遗响于悲风

# Declaration

I hereby declare that except where specific reference is made to the work of others, the contents of this thesis are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other university. This thesis is my own work and contains nothing which is the outcome of work done in collaboration with others, except as specified in the text and Acknowledgements. This thesis contains fewer than 65,000 words including appendices, bibliography, footnotes, tables and equations and has fewer than 150 figures.

Yueheng Ding
August 2023

# Acknowledgements

This thesis represents the culmination of four years of research conducted at the Instrumentation, Control & Embedded Systems research group, School of Engineering, University of Kent. I would like to express my gratitude to the individuals who have provided invaluable support and assistance throughout this endeavour.

First of all, I would like to express my heartfelt gratitude to my supervisor, Dr. Xinggang Yan, for his unwavering support, invaluable suggestions, and kind guidance throughout all stages of my work. From the very beginning of my enrolment, Dr. Yan played a crucial role in my academic journey. During the initial transitional period, he provided me with valuable guidance that facilitated my swift adaptation to the new learning environment. Throughout the subsequent years, I consistently benefited from Dr. Yan's assistance, which not only fostered my intellectual growth but also instilled in me the ability to conduct independent research. His patience, motivation, enthusiasm, and extensive knowledge in his field greatly enhanced my learning process and made my research experience enjoyable and memorable. I am sincerely grateful for Dr. Yan's continuous encouragement and professional advice, which have significantly shaped the outcomes of this thesis. His unwavering support and confidence in my abilities have been instrumental in overcoming challenges and pushing the boundaries of my research.

I would also like to extend my appreciation to the China Scholarship Council (CSC) and the University of Kent for providing me with a scholarship that supported my overseas studies and enabling me to pursue my research goals and navigate through the challenges posed by the COVID-19 pandemic. What's more, I want to acknowledge and thank the members of the Instrumentation, Control & Embedded Systems research group for their valuable insights, stimulating discussions, and collaborative spirit. Their contributions have enriched my research and further expanded my understanding of the subject matter.

Last but not least, I am deeply thankful to my family and friends for their unwavering support, encouragement, and understanding throughout this academic journey. Their love and belief in me have been a constant source of inspiration and motivation.

Once again, I express my heartfelt appreciation to everyone who has contributed to the completion of this thesis. Your support and guidance have been instrumental in my academic and personal growth.

# Abstract

A class of large-scale interconnected systems with matched and unmatched uncertainties is studied in this thesis, with the objective of proposing a controller based on diffeomorphisms and some techniques to deal with the tracking problem of the system. The main research developed in this thesis includes:

- Large-scale interconnected system is a complex system consisting of several semi-independent subsystems, which are typically located in distinct geographic or logical locations. In this situation, decentralised control which only collects the local information is the practical method to deal with large-scale interconnected systems. The decentralised methodology is utilised throughout this thesis, guaranteeing that systems exhibit essential robustness against uncertainty.
- Sliding mode technique is involved in the process of controller design. By introducing a nonsingular local diffeomorphism, the large-scale system can be transformed into a system with a specific regular form, where the matched uncertainty is completely absent from the subspace spanned by the sliding mode dynamics. The sliding mode based controller is proposed in this thesis to successfully achieve high robustness of the closed-loop interconnected systems with some particular uncertainties.
- The considered large-scale interconnected systems can always track the smooth desired signals in a finite time. Each subsystem can track its own ideal signal or all subsystems can track the same ideal signal. Both situations are discussed in this thesis and the results are mathematically proven by introducing the Lyapunov theory, even when operating under the presence of disturbances.

At the end of each chapter, some simulation examples, like a coupled inverted pendulums system, a river pollution system and a high-speed train system, are presented to verify the correctness of the proposed theory. At the conclusion of this thesis, a bried summary of the research findings has been provided, along with a mention of potential future research directions in tracking control of large-scale systems, like more general boundedness of interconnections,

possibilities of distributed control, collaboration with intelligent control and so on. Some mathematical theories involved and simulation code are included in the appendix section.

# Table of contents

# List of figures

# Nomenclature

### Roman Symbols

| | |
|---|---|
| $L$ | Lipschitz constant |

### Greek Symbols

| | |
|---|---|
| $i$ | unit imaginary number $\sqrt{-1}$ |
| $\pi$ | $\simeq 3.14\ldots$ |

### Superscripts

| | |
|---|---|
| $A^{-1}$ | the inverse of matrix $A$ |
| $A^{\top}$ | the transpose of matrix $A$ |
| $\mathbb{R}^{+}$ | Nonnegative real numbers |
| $\mathbb{R}^{n}$ | $n$ dimensional Euclidean space |
| $\mathbb{R}^{n\times m}$ | $n \times m$ matrices with elements in $\mathbb{R}$ |
| $x^{\circ}$ | equilibrium point |

### Subscripts

| | |
|---|---|
| $\bar{\mathscr{B}}_r$ | the closure of $\mathscr{B}_r$ |
| $\mathscr{B}_r$ | the ball with radius $r$ |
| $I_n$ | the identity matrix with dimension $n$ |
| $\lambda_{\min}(A)$ | the minimum eigenvalue of the square matrix $A$ |
| $x_0$ | the initial state |

**Other Symbols**

$A < 0$ $A$ is a symmetric negative definite matrix

$A > 0$ $A$ is a symmetric positive definite matrix

$\mathbb{C}$ Complex numbers

$col(x_1, x_2, ..., x_n)$ the coordinates $[x_1, x_2, ..., x_n]^\top$

$\text{diag}\{A_1, A_2, ..., A_N\}$ a block-diagonal matrix with diagonal elements $A_1, A_2, ..., A_N$

$\Omega \backslash \{0\}$ domain $\Omega$ except 0

$\exists$ for "there exists"

$\forall$ to mean "for any"

$\Rightarrow$ for "implies that"

$A := B$ $A$ is defined by $B$

$\in$ for "in the set"

$L_f^r h(x)$ the $r$th order Lie derivate of the mapping $h(x)$, along the vector field $f(x)$

$\|\cdot\|$ the Euclidean norm or its induced norm

$\mathbb{R}$ Real numbers

$sgn(\cdot)$ signum function

**Acronyms / Abbreviations**

e.g. Latin exempli gratia, or 'for example,'

i.e. Latin id est, or 'that is,'

LTI Linear Time-Invariant

MIMO Multiple-input and Multiple-output

PID Proportional-Integral-Derivative control

SISO Single-input Single-output

SMC Sliding Mode Control

UUB Uniformly Ultimately Bounded

# Chapter 1

# Introduction

Control theory, which primarily focuses on control systems, has become a crucial field in modern technical society since the early twentieth century. A control system is designed to enhance the performance of a specific process and achieve the desired objectives to meet user's requirements. An automatic control system refers to a self-regulating system that operates under certain environment without direct human intervention. As our understanding of the world has deepened, there has been a shift from studying and controlling individual systems to investigating and managing large-scale systems. Large-scale interconnected systems are pervasive in various domains, such as ecological systems, power systems, aerospace, robotics, etc. Control of large-scale interconnected systems has become an irreversible trend. Subsequent advancements in control technologies, such as sliding mode control, intelligent control, and adaptive control, have significantly contributed to the stabilisation of large-scale interconnected systems. However, with society progresses and technology advances, merely achieving stability in large-scale interconnected systems is no longer sufficient to meet current demands. Desired signal tracking in large-scale interconnected systems has emerged as a new research direction. In this thesis, the focus is on studying the problem of signal tracking of large-scale interconnected systems in the presence of disturbances.

## 1.1 Background and Motivation

In today's modern society, there is a growing demand for the control of complex systems. As a result, researchers have been increasingly focusing on advanced control technologies to effectively manage these complex systems. One specific type of complex systems is large-scale interconnected systems, which are often composed of interconnected subsystems, and the dynamics of each subsystem can be influenced by others due to the interconnections between subsystems. These systems can be found in various areas, including engineering [39], biology [37], economics [8], social networks [61, 46], and transportation systems [51, 72], etc.

The background of large-scale systems can be traced back to several fundamental concepts and theories:

- Systems Theory: Systems theory [41, 34, 43] provides the foundation for understanding and analysing the behaviour of interconnected systems. It encompasses concepts such as system dynamics, system structure, feedback loops, and system interconnections. Systems theory [3] allows the study of how individual components are interacted with each others and affect the overall behaviour of the system.

- Control Theory: Control theory [34, 14] deals with the design and analysis of control systems that can be influenced or regulated to achieve desired objectives. In the context of large-scale systems, control theory provides the tools and methodologies [59, 115] to design control strategies that ensure stability, performance, and robustness in the presence of complex interconnections and uncertainties using decentralised or distributed control strategies.

- Network Theory: Network theory is concerned with the study of interconnected networks and their properties. Large-scale systems can often be represented as networks [57], where nodes represent individual components, and edges represent interconnections between them. Network theory provides insights into the structural properties, connectivity, and dynamics of large-scale systems.

- Optimization Theory: Optimization theory [89] plays a crucial role in large-scale systems by addressing challenges of resource allocation, coordination, and decision-making. Optimization techniques [5] are used to find optimal or sub-optimal solutions [99, 110, 109] to problems such as task allocation, scheduling, routing, and resource allocation in large-scale systems.

- Complex Systems Theory: Complex systems theory focuses on understanding the behaviour of systems [73] with a large number of interacting components. It explores emerging properties, self-organization, and the dynamics of complex systems. Large-scale systems often exhibit complex behaviour and properties that require a multidisciplinary approach drawing from physics, mathematics, computer science, and other fields.

Advancement in computing power, communication technologies, and data analysis techniques has also contributed to the increased interest in large-scale systems. The advancement enables the collection and analysis of vast amounts of data from interconnected systems, leading to improved performance, modelling, and control of large-scale systems.

A notable characteristic of these systems is their wide spatial distribution [63, 121]. Consequently, the control design of such systems needs to consider robustness against potential data loss during data transfer, which may occur due to broken or unknown interconnections and poor communication. The control problem for large-scale interconnected systems is inherently challenging.

Decentralised control strategies only require local information from individual subsystems, eliminating the need for extensive data transfer between subsystems. This characteristic makes decentralised control more robust in cases where interconnections are disrupted or communication is unreliable. Therefore, decentralised control has become a popular choice for controlling large-scale interconnected systems, allowing them to exhibit the required level of robustness.

Overall, the background of large-scale systems encompasses foundational theories and concepts from systems theory, control theory, network theory, optimization theory, and complex systems theory. It aims to provide a comprehensive understanding of the behaviour, dynamics, and control of complex interconnected systems, with applications in diverse fields.

### 1.1.1 Main streams of nonlinear approach

The nonlinear approach to dealing with large-scale systems encompasses several main streams, each offering distinct methodologies and techniques. These streams can be broadly categorized as follows:

- Nonlinear Control Strategies: This stream focuses on the development and application of nonlinear control techniques to manage the complexity of large-scale systems. Methods such as feedback linearisation, sliding mode control, and adaptive control fall under this category.

- Decentralised Control and Coordination: Decentralised approaches distribute control tasks among subsystems, promoting scalability and reducing the burden on a central controller. Techniques like decentralized control, consensus algorithms, and game theory are explored to enhance coordination in large-scale systems.

- Distributed Optimization: Addressing optimisation problems in a distributed manner is crucial for large-scale systems. This stream involves algorithms and approaches that enable distributed optimization, including convex optimization, distributed model predictive control, and consensus-based optimization.

- Machine Learning and Data-Driven Methods: Leveraging machine learning and data-driven methods has become increasingly popular. This stream involves using techniques such as neural networks, reinforcement learning, and data-driven modelling to capture and control the nonlinear dynamics of large-scale systems.

- Complex Network Theory: Large-scale systems can often be modelled as complex networks. This stream explores nonlinear approaches based on network theory, examining concepts like network controllability, observability, and resilience to enhance the understanding and control of these systems.

- Adaptive and Robust Control: Dealing with uncertainties and variations in large-scale systems is a key focus of this stream. Adaptive and robust control methods are employed to handle the nonlinearities and disturbances, ensuring the system's stability and performance under varying conditions.

These main streams collectively contribute to the diverse toolkit available for addressing the challenges posed by nonlinearities in large-scale systems. Researchers and practitioners often integrate elements from multiple streams to develop comprehensive solutions tailored to specific applications.

### 1.1.2 History of Tracking Control

System tracking control has been widely applied in practical engineering. The history of tracking control can be traced back to the early development of control theory. A brief overview of the key milestones in the history of tracking control is presented as follows:

- Development of Feedback Control Theory (1940s-1950s): The foundation of modern control theory [23, 4] was laid in the 1940s and 1950s with the development of feedback control concepts. The introduction of proportional-integral-derivative (PID) control and the Nyquist stability criteria, provided the fundamental theory for tracking control [128, 71, 130].

- Model-Based Control (1960s-1970s): In the 1960s and 1970s, researchers began to explore model-based control techniques for achieving accurate tracking. Model predictive control [87, 82, 53, 54] and optimal control methods were developed, which involved formulating dynamic models of the system and designing control strategies based on optimization principles.

- Sliding Mode Control (1970s-1980s): Sliding mode control (SMC) emerged as a robust control technique [90, 85] in the 1970s and 1980s. SMC provides robustness against uncertainties and disturbances by ensuring that the system trajectory reaches and stays on a predefined sliding surface. It has been widely applied in tracking control for various systems [55, 21, 13].

- Adaptive Control (1980s-1990s): Adaptive control techniques [42] gained attention in the 1980s and 1990s, focusing on developing controllers that could adapt to varying system dynamics and parameters. Adaptive control algorithms incorporate online parameter estimation and adaptation mechanisms to achieve accurate tracking performance [104, 19] in the presence of changing conditions.

- Robust Control (1990s-2000s): The development of robust control methods in the 1990s and 2000s addressed the challenges of tracking control [76, 15] in the presence of uncertainties and disturbances. Robust control techniques, such as H-infinity control and mu-synthesis, provide guarantees on system performance and stability even in the presence of model uncertainties and external disturbances.

- Intelligent Control (2000s-present): With the development of artificial intelligence and computational intelligence techniques, intelligent control methods have been applied to tracking control problems. Fuzzy logic control [98], neural network control [25, 26], and evolutionary algorithms [84] have been widely used to handle complex systems with nonlinearities and uncertainties.

Currently, system tracking control has received great attention in both control theory and control engineering. Researchers have focused on improving the accuracy, robustness, and adaptability of control systems to achieve accurate tracking for desired reference signals in various applications.

### 1.1.3 Sliding Mode-based Tracking Control

At the early stages of control theory, the focus was primarily on single-input single-output (SISO) systems, where control techniques such as PID control were developed. The PID control algorithm, which emerged in the 1940s, can be tuned to regulate a system's response to track desired reference signals. Then, PID control was widely used in various industries and applications [65, 38]. In the 1960s-1970s, the focus shifted towards stabilisation of SISO systems. Classical control techniques, including PID control and frequency-domain methods, were widely applied to stabilise control systems [20]. With advancement of control theory, the control of multiple-input multiple-output (MIMO) systems gained more attention [18]. This enabled better coordination and control of complex systems including large-scale systems [36, 106, 2]. In parallel, sliding mode control emerged as a robust control technique in the late 1950s and early 1960s. The concept was introduced by the Russian mathematician Lev S. Pontryagin and his colleagues in the field of optimal control theory. However, the practical implementation [100, 81] of sliding mode control was not realised until the 1970s when advancement in control theory and power electronics made it feasible.

The development [32, 95] of robust control techniques for large-scale systems began to emerge in the late 1980s and early 1990s. Robust control theory is a branch of control theory that focuses on designing controllers that can effectively handle uncertainties and disturbances existing in a system. Control techniques such as adaptive control [33], optimal control [47], robust control, and intelligent control [134] have been applied to address the challenges of controlling large-scale systems. These techniques aim to enhance the performance, robustness, and adaptability of control systems by specifically focusing on uncertainties, disturbances, and nonlinearities. The 1980s and 1990s also witnessed significant advancement [40, 12] in sliding mode control theory, including the development of higher-order sliding modes [48], reaching laws, and chattering reduction techniques [9]. However, sliding mode techniques did not be primarily used for the control of large-scale systems. On the contrary, the focus of control theory during this period was mainly for stabilisation control [106, 17] and analysis [27] of large-scale systems. Researchers were exploring decentralised control methods and coordination strategies to ensure stability in interconnected systems. And the emphasis on tracking specific reference signals was not as a prominent problem during this period.

### 1.1.4 Motivation of this thesis

Sliding mode control is very popular in dealing with complex systems with uncertainties due to its unique control characteristics ([119, 125]). On the one hand, the sliding mode dynamics are often composed of a reduced-order system when compared with the original system, which may simplify the corresponding system analysis and design. On the other hand, sliding mode control is totally robust to matched uncertainty and disturbances. This has resulted in the sliding mode control method being widely applied to deal with tracking problems, and standing out among numerous technologies.

While the research on the stability of large-scale systems has already reached a high level of maturity, the need for accurate tracking of reference signals in large-scale systems increased greatly since 2000s. And then, researchers started developing tracking control techniques specifically tailored for large-scale systems. However, there is still a huge gap between large-scale systems' tracking and stabilisation, especially combing with sliding mode techniques and some other techniques to deal with the uncertainties and boost robustness.

Motivated by this gap, this thesis focuses on addressing the challenge of tracking control for large-scale systems with uncertainties using decentralised-based sliding mode control, which can effectively handle inner uncertainties and improve the overall system performance.

## 1.2 Contributions and Thesis Organisation

This thesis focuses on tracking control of large-scale systems. The approaches developed in this thesis are applied to various practical systems through case studies, including inverted pendulum, river quality control systems, and high-speed trains. The key contributions of this thesis can be summarised as follows:

- This is the first time to deal with the tracking problem of large-scale systems using diffeomorphism-based sliding mode techniques. Various non-singular local diffeomorphism are applied to the considered large-scale systems, enabling the transformed systems to have a required form to facilitate the analysis and design. This opens up possibilities for the use of sliding mode techniques. Additionally, the proposed local diffeomorphisms also convert the tracking problem into the problem of system stabilisation which has a more mature theory.

- Under some proposed decentralised control strategies in this thesis, different types of large-scale systems are demonstrated to have a good tracking performance for desired signals. Moreover, by utilizing sliding mode techniques, the effect of matched disturbances can be completely rejected even using decentralised strategy. In this thesis, the desired signals are time-varying rather than being limited to constant signals, which is new compared to the existed works. For the specific large-scale systems, it is possible to achieve the tracking for the same desired signal by each subsystem, as well as the tracking of different desired signals by different subsystems.

- A solution is proposed to address the inevitable external disturbances in the control process of large-scale systems for tracking problem of large-scale interconnected systems. The solution assumes that the upper bound of the unknown external disturbance is known and utilizes this known upper bound to aid in the design of the controller. The innovation is that the asymptotically tracking results can be guaranteed by using the proposed controllers, even in the presence of the internal and external disturbances.

The remaining structure of this thesis is as follows:

**Chapter 2** provides an overview of the control theory necessary for the subsequent analysis and design conducted in this thesis. It covers essential definitions, including the concept of a local diffeomorphism, which is a fundamental mathematical tool in the study of nonlinear systems. This chapter also explores the key principles and basic results of sliding mode techniques, which are widely applied in control systems. Additionally, the elementary theory of Lyapunov stability is discussed, as it plays a crucial role in analysing the stability properties of control systems in this thesis.

In **Chapter 3**, a decentralised controller is proposed for a class of nonlinear interconnected systems with uncertainties using the sliding mode technique. The scheme considers both matched nonlinear uncertainty and mismatched interconnections. To facilitate system analysis, a geometric transformation is applied to transfer the interconnected system into a new nonlinear system with a special structure. And a set of conditions is developed to ensure that the output tracking errors converge to equilibrium point asymptotically and state variables are bounded as well.

We focus on a class of interconnected systems in **Chapter 4** where the isolated subsystems are described by nonlinear systems with linear components. We propose a state feedback decentralised control strategy based on sliding mode techniques. The objective is to ensure that the outputs of the controlled interconnected system track the desired signals uniformly ultimately, even when the desired reference signals are time-varying.

In **Chapter 5**, some proper coordinate transformations are introduced to explore the properties of system structure. Then, a system structure-based analysis is presented to decompose the high-speed train models into large-scale interconnected systems with special structure where both internal and external uncertainties are considered to reflect the practical situation. A decentralised sliding mode controller is designed such that the output tracking errors are uniformly bounded with bounded states. Simulations are performed at the end of each chapter to demonstrate the effectiveness and feasibility of the proposed control strategies.

**Chapter 6** provides a summary of the key findings and main conclusions derived from this thesis. It outlines the significant contributions made in the thesis and discusses the implications of the results. Furthermore, this chapter includes a discussion on potential areas for future research and explores possible directions for further investigation.

Finally, the **Appendix** section provides an explanatory example of the SIMULINK software used in this thesis. It shows how simulation results are obtained for all practical and numerical examples included in the study. Additionally, the Appendix including a mathematical supplement, provides further details and supporting information for the research work conducted in the thesis.

### 1.2.1 An Aerial View

Table 1.1 (See, next page) provides readers with a macro-level perspective to explore subtle differences in the research subject and research methods across the main chapters. It also highlights some commonalities among these five chapters as well as the unique aspects of each chapter.

Table 1.1 Main Chapters Comparison

| Properties \ Chapters | Chapter 3 | Chapter 4 | Chapter 5 |
|---|---|---|---|
| Nominal Isolated Subsystems | Nonlinear | Mainly Linear | Linear |
| Dimension of Subsystems | SISO | MIMO(Square) | Single Input |
| Properties of Interconnections | Known Upper Bound | | |
| Desired signals | Smooth functions with some specific conditions | | |
| Tracking Targets | Output Tracking | | |
| Technology 1 | Sliding Mode Technique | | |
| Technology 2 | Decentralised Control | | |

| | |
|---|---|
| Objective of all Chapters | Sliding mode based tracking control of large-scale systems |
| Common of all Chapters | Switch tracking problem to system stabilisation by using proper coordinate transformation |

| | |
|---|---|
| Uniqueness of Chapter 3 | A **WEAK** decentralised control for a local diffeomorphism-based interconnected systems |
| Uniqueness of Chapter 4 | Sliding mode controllers using **OUTPUT INFORMATION** |
| Uniqueness of Chapter 5 | Tracking control of **UNDERACTUATED** large-scale systems |

# Chapter 2

# Fundamental Knowledge and Basic Concepts

## 2.1 State-Space Systems

### 2.1.1 Linear Time-Invariant (LTI) Control Systems

Consider a continuous-time LTI control system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{2.1}$$

$$y(t) = Cx(t) + Du(t) \tag{2.2}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ with $m \leq p < n$ are the state, input and output of the system. (2.1) is called "state equation" and (2.2) is called "output equation". The matrices $A, B, C, D$ are constant $\forall t \geq 0$.

The equations (2.1)-(2.2) imply an input-output relationship between the input $u(t)$ and the output $y(t)$. For a given input $u(t)$, the state $x(t)$ can be determined by (2.1) and then determines the output $y(t)$ through (2.2).

**Remark 2.1.1** *The matrix $D$ in (2.2) is assumed to be zero matrix in this thesis, which means the input signal $u(t)$ does not exist in the output channel.*

### Coordinate Transformations

For the system (2.1)-(2.2), we introduce a nonsingular coordinate transformation $z(t) = Tx(t)$ and then the transformed system in the new coordinates $z$ can be described by

$$\dot{z}(t) = \bar{A}z(t) + \bar{B}u(t) \tag{2.3}$$
$$y(t) = \bar{C}z(t) + \bar{D}u(t) \tag{2.4}$$

where the relationship between (2.1)-(2.2) and (2.3)-(2.4) is described by

$$\bar{A} = TAT^{-1},\ \bar{B} = TB \tag{2.5}$$
$$\bar{C} = CT^{-1}, \quad \bar{D} = D \tag{2.6}$$

## 2.1.2 Nonlinear Single-Input Single-Output (SISO) Systems

### Local State Transformations

The purpose of this section is to show how to use a proper diffeomorphism to transform a single-input single-output nonlinear system to a required form, a "normal form" of special interest, on which several important properties can be elucidated.

Consider a continuous-time single-input single-output nonlinear system

$$\dot{x}(t) = f(x) + g(x)u(t) \tag{2.7}$$
$$y(t) = h(x) \tag{2.8}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$ and $y \in \mathbb{R}$ are the input, state and output of the system. $f(x), g(x)$ and $h(x)$ are corresponding system, input and output distribution functions which are assumed to be smooth enough in their arguments to guarantee the existence and uniqueness of the system solutions.

**Definition 2.1.1** *System (2.7)-(2.8) is said to have relative degree $r$ at a point $x^\circ$ ($r \leq n$), if*

- $L_g L_f^k h(x) = 0$ *for all $x$ in a neighbourhood of $x^\circ$ and all $k < r - 1$.*
- $L_g L_f^{r-1} h(x^\circ) \neq 0$. [1]

[1] $L_f^k h(x)$ here means the $k$th order Lie derivative of the mapping $h(x)$ along the vector field $f(x)$. More details of Lie derivative are available in Appendix A.4.

Then, introduce a local diffeomorphism[2] $\Phi(x)$ as:

$$z(t) = \Phi(x(t)) = \begin{bmatrix} \Phi_1(x) \\ \Phi_2(x) \\ \vdots \\ \Phi_r(x) \\ \Phi_{r+1}(x) \\ \vdots \\ \Phi_n(x) \end{bmatrix} = \begin{bmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{r-1} h(x) \\ \Phi_{r+1}(x) \\ \vdots \\ \Phi_n(x) \end{bmatrix} \tag{2.9}$$

If $r$ is strictly less than $n$, it is always possible to find $n-r$ more functions $\Phi_{r+1}(x),\ldots,\Phi_n(x)$ such that the local diffeomorphism $\Phi(x)$ has a Jacobian matrix which is nonsingular at $x^\circ$ and therefore qualifies as a local state transformation in a neighbourhood of $x^\circ$. The value of these additional functions at $x^\circ$ can be fixed arbitrarily. Actually, it is always possible to choose $\Phi_{r+1}(x),\ldots,\Phi_n(x)$ in such a way that

$$L_g \Phi_i(x) = 0 \quad \text{for all } r+1 \le i \le n \text{ and all } x \text{ around } x^\circ. \tag{2.10}$$

Thus, in summary, the state-space description of the systems (2.1) in the new coordinates $z$ defined in (2.9) will be shown as follows:

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ &\cdots \\ \dot{z}_{r-1} &= z_r \\ \dot{z}_r &= b(z) + a(z)u \\ \dot{z}_{r+1} &= q_{r+1}(z) \\ &\cdots \\ \dot{z}_n &= q_n(z) \end{aligned} \tag{2.11}$$

where

$$a(z) = L_g L_f^{r-1} h(\Phi^{-1}(z)) \tag{2.12}$$

$$b(z) = L_f^r h(\Phi^{-1}(z)) \tag{2.13}$$

[2] More details about diffeomorphisms are available in Appendix A.5.

With the condition in (2.10), the partial functions $q_i(z)$ in (2.11) are given by:

$$q_i(z) = L_f \Phi_i(\Phi^{-1}(z)) \quad \text{for all } r+1 \leq i \leq n. \tag{2.14}$$

Note that at the point $z^\circ = \Phi(x^\circ)$, $a(z) \neq 0$ by definition. Thus, the coefficient $a(z)$ is non-zero for all $z$ in a neighbourhood of $z^\circ$.

As for the output equation $y = h(x)$ in (2.8), it is easy to check that in the new coordinates $z$,

$$y = z_1 \tag{2.15}$$

The structure of (2.11) and (2.15) is illustrated in the block diagram depicted in Fig. 2.1. The equations thus defined are said to be in normal form, which facilities to understand the structure of the considered system.

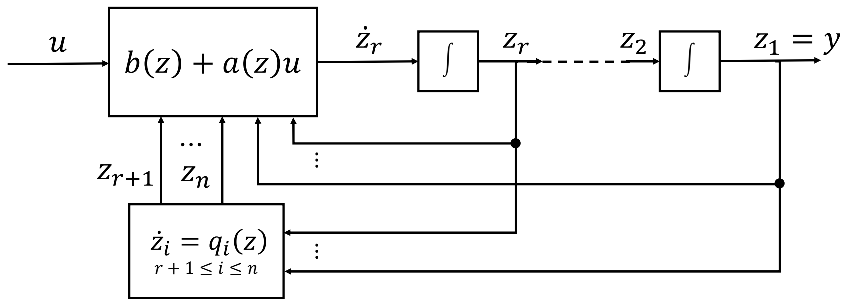


Fig. 2.1 Block diagram of the transferred system (2.11)-(2.15)

**Remark 2.1.2** *For the nonlinear systems, only the single-input single-output system in the form of (2.7)-(2.8) is considered in this thesis. For the multi-input multi-output case, a straightforward extension of most of the theory developed in this section is available in [34].*

## Special Case $(r = n)$

Consider the nonlinear system (2.7)-(2.8) with relative degree $r = n$, i.e. the system relative degree is exactly equal to the dimension of the state space, at the point $x = x^\circ$. In this case, it follows from (2.9) that the change of coordinates required to obtain the normal form is given exactly by

$$z(t) = \Phi(x(t)) = \begin{bmatrix} \Phi_1(x) \\ \Phi_2(x) \\ \cdots \\ \Phi_n(x) \end{bmatrix} = \begin{bmatrix} h(x) \\ L_f h(x) \\ \cdots \\ L_f^{n-1} h(x) \end{bmatrix} \tag{2.16}$$

i.e. by the function $h(x)$ and its first $n-1$ Lie derivates along the vector field $f(x)$. No extra functions are needed in order to complete the transformation. Thus, in the new coordinates (2.16) with the following state feedback control law

$$u = \frac{1}{a(z)}(-b(z)+v), \tag{2.17}$$

the system (2.7)-(2.8) can be described by equations of the following form

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ &\cdots \\ \dot{z}_{n-1} &= z_n \\ \dot{z}_n &= v \end{aligned} \tag{2.18}$$

where $z = (z_1, ..., z_n)$. Recall also that at the point $z^\circ = \Phi(x^\circ)$ and thus at all $z$ in a neighbourhood of $z^\circ$, the function $a(z)$ is non-zero. The resulting closed-loop system governed by (2.18) is linear and controllable.

Thus, it is concluded that any nonlinear system (2.7)-(2.8) with relative degree $n$ at some point $x^\circ$ can be transformed into a system which, in a neighbourhood of the point $z^\circ = \Phi(x^\circ)$, is linear and controllable. It is important to stress that:

- The coordinate transformation (2.16) is defined locally around the point $x^\circ$.
- The state feedback transformation (2.17) is defined locally around the point $x^\circ$.

**Remark 2.1.3** *It is easily checked that the two transformations used in order to obtain the linear form can be interchanged. One can first use a feedback and then change the coordinates in the state space, and the result is the same. From the coordinate transformation (2.16) and feedback transformation (2.17), it is easy to see that the feedback needed in the x coordinates can be described by,*

$$u = \frac{1}{a(\Phi(x))}(-b(\Phi(x))+v) \tag{2.19}$$

## Zero Dynamics

In this section, we discuss a concept "zero dynamics", that in many instances plays an important role in nonlinear system analysis and design for the case when $r < n$, where $r$ is relative degree and $n$ is system dimension.

Consider a nonlinear system with $r$ strictly less than $n$ and look at its normal form. In order to write the equations in a slightly more compact manner, we introduce a suitable vector notation. In particular, since there is no specific need to track each one of the last $n-r$ components of the state vector individually, we represent all of them in a compact way

$$\eta = \begin{bmatrix} z_{r+1} \\ \cdots \\ z_n \end{bmatrix}. \tag{2.20}$$

Sometimes, whenever convenient and not otherwise required, the first $r$ components of $z$ are denoted by:

$$\xi = \begin{bmatrix} z_1 \\ \cdots \\ z_r \end{bmatrix}. \tag{2.21}$$

With the help of these notations, the normal form of a single-input single-output nonlinear system with $r < n$ (at some point of interest $x^\circ$, e.g. an equilibrium point) can be rewritten as:

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ &\cdots \\ \dot{z}_{r-1} &= z_r \\ \dot{z}_r &= b(\xi,\eta) + a(\xi,\eta)u \\ \dot{\eta} &= q(\xi,\eta). \end{aligned} \tag{2.22}$$

Recall that, if $x^\circ$ satisfies $f(x^\circ) = 0$ and $h(x^\circ) = 0$, then necessarily the first $r$ new coordinates $z_1, z_2, \ldots, z_r$ are 0 at $x^\circ$. Note that it is always possible to choose arbitrarily the value at $x^\circ$ of the last $n-r$ new coordinates, thus in particular being 0 at $x^\circ$. Therefore, without loss of generality, one can assume that $\xi = 0$ and $\eta = 0$ at $x^\circ$. Thus, if $x^\circ$ was an equilibrium for the system in the original coordinates, its corresponding point $(\xi,\eta) = (0,0)$ is an equilibrium for the system in the new coordinates and from this we deduce that

$$b(\xi,\eta) = 0 \quad at(\xi,\eta) = (0,0) \tag{2.23}$$

$$q(\xi,\eta) = 0 \quad at(\xi,\eta) = (0,0). \tag{2.24}$$

Suppose now we want to analysis the following problem called the "Problem of Zeroing the Output". Find, if any, pairs consisting of an initial state $x^\circ$ and of an input function $u^\circ(\cdot)$, defined for all $t$ in a neighbourhood of $t = 0$, such that the corresponding output $y(t)$ of the system is identically zero for all $t$ in a neighbourhood of $t = 0$. Apart from that, we perform this analysis on the normal form of the system.

$$\dot{y}(t) = \dot{z}_1(t) = \dot{z}_2(t) = \ldots = \dot{z}_r(t) = 0. \tag{2.25}$$

**Definition 2.1.2 (Zero Dynamics)** *For system (2.22), it is clear that $\xi(t)$ for all $t$ is identically zero and its behaviour is governed by the differential equation*

$$\dot{\eta}(t) = q(0, \eta(t)). \tag{2.26}$$

*The dynamics of (2.26) correspond to the dynamics describing the "internal" behaviour of the system when input and initial conditions have been chosen in such a way as to constrain the output to remain identically zero. These dynamics, are rather important in many of our developments, are called the "zero dynamics" of system (2.22).*

## 2.2 Sliding Mode Control (SMC) Methodology

Sliding mode control changes the system dynamics by specifically employing a discontinuous control signal. The control methodology consists of two steps:

Step 1: To design a sliding surface such that the system considered has the desired performance when restricted to the surface.

Step 2: To design a controller which drives the system trajectory to the designed sliding surface in finite time and maintains a sliding motion on it thereafter.

A concise description is available in [115, 14, 101]. In view of these two steps, the system motion can be separated into two phases: the reaching phase and the sliding phase. The former refers to the motion when the system trajectory moves towards the sliding surface and the latter concerns the motion when the system trajectory moves on the sliding surface.

### 2.2.1 Sliding Phase

Consider the linear system (2.1)-(2.2) with $D = 0$. In order to design a proper sliding function[3]

$$s = s(x) \tag{2.27}$$

[3] Sometimes, also called switching function or $s$ function directly in some books

such that the resulting sliding motion has the desired performance, one way is to find the dynamical equations which will govern the sliding motion, and then synthesise the sliding surface based on the characteristics of the sliding mode dynamics or sliding motion. It is assumed that the sliding motion exists. The following approach[4] called "regular form" is usually employed to find the sliding mode dynamics and in this way the stability of the sliding motion is transformed to the problem of stability of an unforced system.

Suppose that there exists a coordinate transformation $z(t) = Tx(t)$ where matrix $T \in \mathbb{R}^{n\times n}$ is nonsingular such that in the new coordinate system $z$, the sliding surface $s(x) = 0$ can be described in the form

$$z_2 = Kz_1 \tag{2.28}$$

where $K$ is a design parameter, $z_1 \in \mathbb{R}^{n-m}$, $z_2 \in \mathbb{R}^m$, $z := col(z_1, z_2)$ and system (2.1)-(2.2) can be described by

$$\dot{z}_1 = A_{11}z_1 + A_{12}z_2 \tag{2.29}$$

$$\dot{z}_2 = A_{21}z_1 + A_{22}z_2 + B_2u \tag{2.30}$$

$$y = C_1z_1 + C_2z_2 \tag{2.31}$$

where $A_{11} \in \mathbb{R}^{(n-m)\times(n-m)}$, $B_2 \in \mathbb{R}^{m\times m}$ is nonsingular, $u \in \mathbb{R}^m$ is the control, $C_1 \in \mathbb{R}^{m\times(n-m)}$ and

$$\bar{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \tag{2.32}$$

with the matrix triple $(\bar{A}, \bar{B}, \bar{C})$ is defined in (2.5)-(2.6). Noting that system (2.29) is independent of the control signal $u$ and the dimension of $z_2$ is the same as the dimension of the control $u$. System (2.29)-(2.30) is the so-called **"regular form"**.

Based on the regular form, it is clear to see that the corresponding sliding mode dynamics of system (2.1) is described by

$$\dot{z}_1 = (A_{11} + A_{12}K)z_1 \tag{2.33}$$

which is a reduced-order system when compared with system (2.1) and the term $(A_{11} + A_{12}K)$ is always chosen to be *Hurwitz* stable[5] by introducing a proper gain $K$ if the matrix pair $(A, B)$ in (2.1) is controllable.

[4]Equivalent control is another approach for finding the sliding mode dynamics.
[5]A square matrix is called "*Hurwitz*" stable, if every eigenvalue of it has strictly negative real part.

### 2.2.2 Reaching Phase

In order to guarantee that the system trajectory can be driven to the sliding surface $s(x) = 0$ in finite time and a sliding motion can be maintained on it thereafter, a proper discontinuous control

$$u = u(t,x) \tag{2.34}$$

needs to be designed such that the following condition is satisfied [14, 101]

$$s^\top(x)\dot{s}(x) \leq -\eta \parallel s(x) \parallel \tag{2.35}$$

for some constant $\eta > 0$. The inequality (2.35) is the so-called reachability condition and $\eta$ is called the reachability constant.

The following condition

$$s^\top(x)\dot{s}(x) < 0 \tag{2.36}$$

is also called a reachability condition but it cannot guarantee that a sliding motion takes place in finite time and thus a sliding motion may not occur in this case.

### 2.2.3 A Numerical Example

Consider a system

$$\dot{x}_1(t) = x_2(t) \tag{2.37}$$

$$\dot{x}_2(t) = u(t) \tag{2.38}$$

Choosing the switching function

$$s(t) = mx_1(t) + x_2(t) \tag{2.39}$$

it follows that the reduced-order sliding motion will be given by

$$\dot{x}_1(t) = -mx_1(t) \tag{2.40}$$

and so the positive design scalar $m$ can be chosen to specify the rate of decay. And an appropriate sliding mode control action is

$$u(t) = -mx_2(t) - \eta sgn(s) \tag{2.41}$$

Choosing the Lyapunov candidate function $V(s) = \frac{1}{2}s^2$, then the first derivate along the dynamic trajectory (2.37)-(2.38) is

$$\dot{V}(s) = s\dot{s} = -\eta|s| \tag{2.42}$$

where the proposed sliding mode controller (2.41) with parameters $m = 0.5$, $\eta = 2$ is used to establish the above conclusion. According to (2.42), it is easy to check that the reachability condition (2.35) is satisfied which means the sliding motion (2.40) will occurs.

Fig. 2.2 shows the phase plane portrait of system (2.37)-(2.38) with controller (2.41). From Fig. 2.2, the system states $(x_1, x_2)$ are driven to the sliding surface from the initial point $x_0 = (4,3)$ (reaching phase), and then move along the sliding surface to converge to the origin (sliding phase).

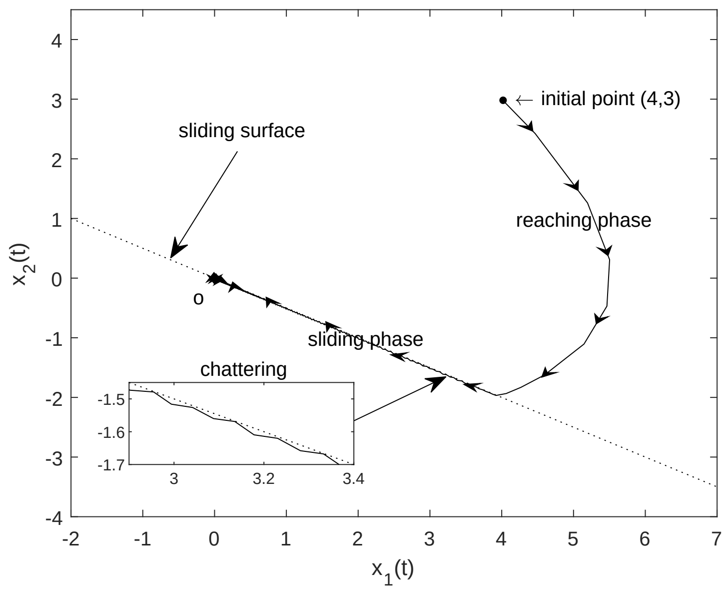


Fig. 2.2 The phase plane portrait of closed-loop system

### 2.2.4 Characteristics of Sliding Mode Control

It is essential to emphasis that sliding mode control has the following characteristics:

- *Sliding mode dynamics are a reduced-order system when compared with the original system dynamics.*

  For system (2.1) with sliding surface $s = s(x)$, the corresponding sliding mode dynamics can be described by (2.33). It is clear to see that the order of the system (2.33) is $n - m$ where $n$ is the dimension of the original system and $m$ is the dimension of the control. Therefore, during the sliding motion, the system exhibits reduced-order dynamics when compared with the original system.

- *Sliding motion is insensitive to matched uncertainty*[6].

  Suppose system (2.1) experiences an uncertainty/disturbance and from (2.33), it is straightforward to see that the dynamics governing the sliding motion are completely independent of the control and thus the sliding motion governed by sliding mode dynamics (2.33) is totally robust to matched uncertainty.

- *Uncertainties/disturbances will affect reachability.*

  In order to guarantee that the trajectory of the considered system is driven to the predesigned sliding surface, the reachability condition must be satisfied which involves the uncertainties or disturbances. Therefore, they may affect the reaching phase no matter whether they are matched or mismatched, but the effects of some uncertainties may be completely rejected by design of an appropriate control.

- *Chattering is inevitable in the evolution of control action.*

  The sliding mode control action (2.41) is given in Fig. 2.3. It can be seen that sliding takes place after 2.0 second when high frequency switching/chattering takes place due to the discontinuity in the control. Chattering is usually undesirable in practice, e.g. it may result in unnecessary wear and tear on the actuator components and result in unnecessary energy consumption. So how to reduce chattering is one of interesting topics in the area relating sliding mode control.

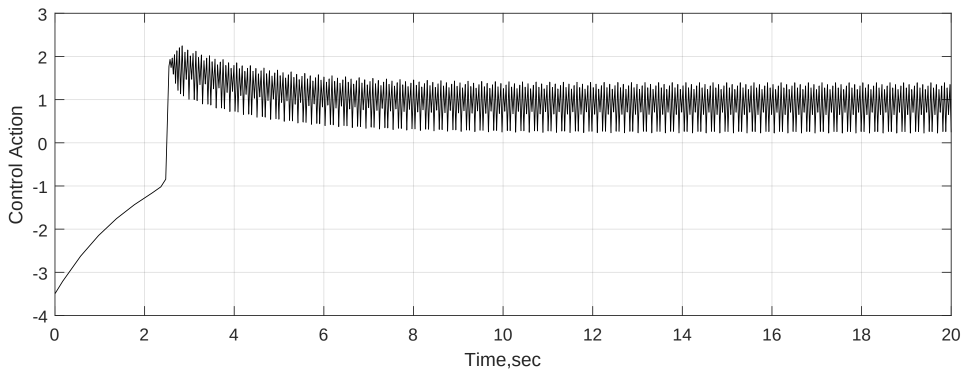


Fig. 2.3 Evolution of control action with respect to time

### 2.2.5 Output Sliding Surface Design

In order to form an output feedback sliding mode control scheme, it is usually required that the designed switching function is a function of the system outputs. Consider initially a linear

[6] If the uncertainty or disturbance acts in the input/control channel or the effects are equivalent to an uncertainty acting in the input channel, it is called matched uncertainty. Otherwise it is called mismatched uncertainty.

system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{2.43}$$

where $x \in R^n, u \in R^m$ and $y \in R^p$ are the states, inputs and outputs respectively. The triple $(A,B,C)$ are constant matrices of appropriate dimension with $B$ being of full column rank and $C$ being of full row rank.

Consider (2.43) in the case of $m = p$ which means the system (2.43) is a square system. Since $B$ has full column rank, there exists a coordinate transformation $\bar{x} = \bar{T}x$ such that in the new coordinate $\bar{x}$, the triple $(A,B,C)$ can be described by

$$\bar{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \tag{2.44}$$

where $A_{11} \in R^{(n-m)\times(n-m)}$, $B_2 \in R^{m\times m}$ is nonsingular and $C_1 \in R^{m\times(n-m)}$.

Assume that $\text{rank}(CB) = m$ and the invariant zeros of $(A,B,C)$ lie in the left half plane. From section 5.3 in [14], it follows that the matrix $C_2 \in R^{m\times m}$ in (2.44) is nonsingular because $m = \text{rank}(CB)$=rank($\bar{C}\bar{B}$)=rank($C_2B_2$) and $B_2$ is nonsingular. Then, a coordinate transformation $\hat{x} = \hat{T}\bar{x}$ with $\hat{T}$ defined by

$$\hat{T} = \begin{bmatrix} I & 0 \\ C_1 & C_2 \end{bmatrix} \tag{2.45}$$

is further introduced. Again from section 5.3 in [14], the triple $(\bar{A},\bar{B},\bar{C})$ in the new coordinates $\hat{x}$ has the following structure

$$\hat{A} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} 0 \\ \hat{B}_2 \end{bmatrix}, \quad \hat{C} = \begin{bmatrix} 0 & I \end{bmatrix} \tag{2.46}$$

where $\hat{A}_{11} \in R^{(n-m)\times(n-m)}$ is $Hurwitz$ stable, $\hat{B}_2 = C_2B_2$ is nonsingular.

**Remark 2.2.1** *It should be pointed out that the first transformation matrix $\bar{T}$ is used to change the original system $(A,B,C)$ into the regular form as in (2.44), and the second transformation matrix $\hat{T}$ is to make that the sub-matrix $\hat{A}_{11}$ of the triple in (2.46) is Hurwitz stable and the matrix $\hat{C}$ in (2.46) has the form of* $[0 \quad I]$.

## 2.3 Lyapunov Stability Analysis

### 2.3.1 Equilibrium Point

Consider a autonomous system

$$\dot{x}(t) = f(x) \tag{2.47}$$

where $f : D \mapsto \mathbb{R}^n$ is a locally Lipschitz map from a domain $D \subset \mathbb{R}^n$ into $\mathbb{R}^n$. $x_e \in D$ is called an "Equilibrium point" of (2.47), if

$$f(x_e) = 0$$

Our goal is to characterize and study stability of $x_e$. For convenience, we state all definitions and theorems for the case when the equilibrium point is at the origin of $\mathbb{R}^n$; that is, $x_e = 0$. There is no loss of generality in doing so because any equilibrium point can be shifted to the origin via a change of variables. Suppose $x_e \neq 0$, and consider the change of variables $y = x - x_e$. The derivative of $y$ is given by

$$\dot{y} = \dot{x} = f(x) = f(y + x_e) := g(y), \quad \text{where } g(0) = 0$$

In the new variable $y$, the system has equilibrium at the origin. Therefore, without loss of generality, we shall always assume that $f(x)$ satisfies $f(0) = 0$, and study stability of the origin $x = 0$.

**Remark 2.3.1** *Once the system's state reaches an equilibrium point $x_e$, it will remain at that point without further changes.*

**Remark 2.3.2** *If there are no other equilibrium points in the vicinity of an equilibrium point, it is referred to as an "isolated equilibrium point". If an equilibrium point is not isolated, there exists a continuum of equilibrium points at that location.*

**Remark 2.3.3** *Linear systems can only have one isolated equilibrium point, whereas nonlinear systems can have multiple isolated equilibrium points. This is why linear systems have a single steady-state operating point. And regardless of the initial state, the system's state will eventually converge to this stable operating point. In contrast, nonlinear systems have multiple stable operating points, and different initial states may converge to different stable operating points.*

Consider a following linear system

$$\dot{x}_1 = 3x_1 \tag{2.48}$$

$$\dot{x}_2 = x_1 + x_2 \tag{2.49}$$

It is easy to check that the equilibrium point of (2.48)-(2.49) is at its origins.

### 2.3.2 Stability in the Sense of Lyapunov

Now, we will discuss the stability analysis of equilibrium at the origin. Consider a spherical region of radius $R$ about an equilibrium point $x_e$ such that

$$\|x - x_e\| \leq R$$

where $\|x - x_e\|$ is defined as

$$\|x - x_e\| = [(x_1 - x_{1e})^2 + (x_2 - x_{2e})^2 + \ldots + (x_n - x_{ne})^2]^{1/2}$$

Let $S(\delta)$ consists of all points such that $\|x_0 - x_e\| \leq \delta$ and let $S(\varepsilon)$ consists of all points such that $\|x(t) - x_e\| \leq \varepsilon$ for all $t \geq t_0$.

**Definition 2.3.1 (Stable)** *An equilibrium point $x_e$ of the system (2.47) is said to be "stable" in the sense of Lyapunov. If corresponding to each $S(\varepsilon)$[7], there is $S(\delta)$ such that trajectories starting in $S(\delta)$ do not leave $S(\varepsilon)$ as time $t$ increases indefinitely. The real number $\delta$ depends on $\varepsilon$ and in general also depends on $t_0$. If $\delta$ does not depend on $t_0$, the equilibrium point is said to be "uniformly stable".*

**Definition 2.3.2 (Asymptotically Stable)** *An equilibrium point $x_e$ of the system (2.47) is said to be "asymptotically stable", if it is in the sense of Lyapunov and every solution starting within $S(\delta)$ converges without leaving $S(\varepsilon)$ to $x_e$ as $t$ increases indefinitely.*

*Also, an equilibrium point $x_e$ of the system (2.47) is said to be "globally asymptotically stable", if it is asymptotically stable for every initial states in the whole state space and every solution converges to $x_e$ as $t$ increases indefinitely.*

**Definition 2.3.3 (Unstable)** *An equilibrium point $x_e$ of the system (2.47) is said to be "unstable", if for some real number $\varepsilon > 0$ and any real number $\delta > 0$, irrespective of how small it is, there is always a state $x_0$ in $S(\delta)$ such that the trajectory starting at the states leaves $S(\varepsilon)$.*

[7] The region $S(\varepsilon)$ must be selected firstly and for each $S(\varepsilon)$, there must be a region $S(\delta)$ in such a way that trajectories starting within $S(\delta)$ do not leave $S(\varepsilon)$ as time $t$ progresses.

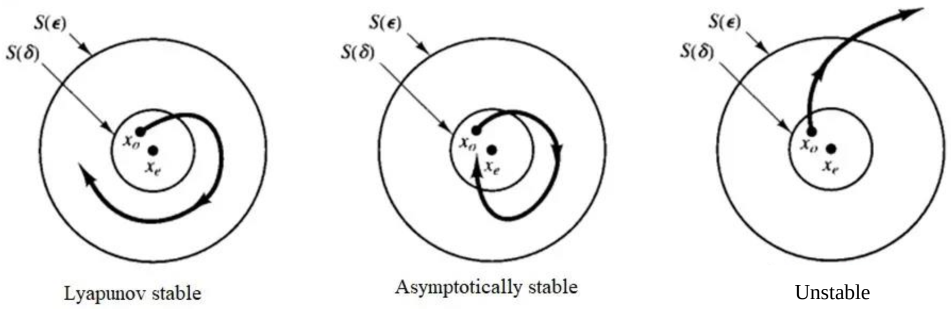


Fig. 2.4 Different dynamic situation of system

The graphical representation of stability, asymptotic stability and instability is shown in Fig. 2.4.

The region $S(\delta)$ is binding the initial state $x_0$ and the region $S(\varepsilon)$ corresponds to the boundary for the trajectory starting at $x_0$. The correct region of allowable initial conditions are not specified for the definitions of stability that we have seen previously. Thus these definitions apply to the neighbourhood of the equilibrium point only if $S(\varepsilon)$ is not corresponding to the entire state space.

## 2.3.3 Lyapunov Stability Theorems

We are now ready to state Lyapunov's stability theorem.

**Theorem 2.3.1** *Let $x = 0$ be an equilibrium point for (2.47) and $D \subset \mathbb{R}^n$ be a domain containing $x = 0$. Let $V : D \to \mathbb{R}$ be a continuously differentiable function, such that*

$$V(0) = 0 \; and \; V(x) > 0 \; in \; D - \{0\} \tag{2.50}$$

$$\dot{V}(x) \leq 0 \; in \; D \tag{2.51}$$

*Then, $x = 0$ is "stable". Moreover, if*

$$\dot{V}(x) < 0 \; in \; D - \{0\} \tag{2.52}$$

*then, $x = 0$ is "asymptotically stable".*

**Theorem 2.3.2** *Let $x = 0$ be an equilibrium point for (2.47). Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function, such that*

$$V(0) = 0 \text{ and } V(x) > 0, \quad \forall x \neq 0 \tag{2.53}$$

$$\|x\| \to \infty \Rightarrow V(x) \to \infty \tag{2.54}$$

$$\dot{V}(x) < 0, \quad \forall x \neq 0 \tag{2.55}$$

*then, $x = 0$ is "globally asymptotically stable".*

**Theorem 2.3.3** *Let $x = 0$ be an equilibrium point for (2.47). Let $V : D \to \mathbb{R}$ be a continuously differentiable function, such that $V(0) = 0$ and $V(x_0) > 0$ for some $x_0$ with arbitrarily small $\|x_0\|$. Define a set $U = \{x \in B_r | V(x) > 0\}$ with $B_r = \{x \in \mathbb{R}^n | \|x\| \leq r\}$[8] which is contained in $D$ and suppose that $\dot{V}(x) > 0$ in $U$. Then, $x = 0$ is "unstable".*

## 2.3.4 Converse Lyapunov Theorem

For readers' convenience, some basic concepts are introduced to help to understand converse Lyapunov theorem at first.

**Definition 2.3.4** *(see, [43, 115]) A continuous function $\alpha : [0, a) \mapsto \mathbb{R}^+$ is said to belong to class $\mathscr{K}$, if it is strictly increasing and $\alpha(0) = 0$. Also, it is said to belong to $\mathscr{K}_\infty$, if $a = \infty$ and $\lim_{r\to\infty} \alpha(r) = \infty$.*

**Definition 2.3.5** *(see, [43, 115]) A continuous function $\beta : [0, a) \times \mathbb{R}^+ \mapsto \mathbb{R}^+$ is said to belong to class $\mathscr{KL}$, if any given $s \in \mathbb{R}^+$, the mapping $\beta(r, s)$ belongs to class $\mathscr{K}$ with respect to variable $r$, and for any given $r \in [0, a)$, the mapping $\beta(r, s)$ is decreasing with respect to the variable $s$ and $\lim_{s\to\infty} \beta(r, s) = 0$.*

Consider a nonlinear non-autonomous system which is described by

$$\dot{x}(t) = f(t, x(t)) \tag{2.56}$$

where the function $f : \mathbb{R}^+ \times D \mapsto \mathbb{R}^n$ is continuous and $D \subset \mathbb{R}^n$ is a domain which contains the origin $x = 0$. It is assumed that

$$f(t, 0) = 0, \qquad t \in \mathbb{R}^+ \tag{2.57}$$

which implies that the origin is an equilibrium point of the system.

[8] For a given $\varepsilon$, choose $r \in (0, \varepsilon]$.

Consider system (2.56) in domain $D := \mathscr{B}_r = \{x \in \mathbb{R}^n | \parallel x \parallel < r\}$.[9] Let $\beta(\cdot)$ be a class $\mathscr{K}\mathscr{L}$ function and $r_0$ be a positive constant such that

$$\beta(r_0, 0) < r \quad \text{and} \quad \mathscr{B}_{r_0} := \{x| \parallel x \parallel < r_0\} \tag{2.58}$$

**Theorem 2.3.4 (Lyapunov Converse Theorem)** *For system (2.56), assume that the Jacobian matrix $\frac{\partial f}{\partial x}$ is bounded[10] in domain $D$ uniformly for $t \in \mathbb{R}^+$, and that the trajectory of system (2.56) satisfies*

$$\parallel x(t) \parallel \leq \beta(\parallel x(t_0) \parallel, t - t_0), \quad x(t_0) \in \mathscr{B}_{r_0}, \quad t \geq t_0 \geq 0 \tag{2.59}$$

*Then, there exists a continuously differentiable function $V : \mathbb{R}^+ \times \mathscr{B}_{r_0} \mapsto \mathbb{R}^+$ such that*

$$\alpha_1(\parallel x \parallel) \leq V(t,x) \leq \alpha_2(\parallel x \parallel) \tag{2.60}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq \alpha_3(\parallel x \parallel) \tag{2.61}$$

$$\parallel \frac{\partial V}{\partial x} \parallel \leq \alpha_4(\parallel x \parallel) \tag{2.62}$$

*where $\alpha_i$ for $i = 1,2,3,4$ are class $\mathscr{K}$ functions defined on the interval $[0, r_0]$. The function $V(\cdot)$ can be chosen independent of time $t$ if $f(\cdot)$ in system (2.56) is independent of the time $t$.*

### 2.3.5 Uniformly Ultimate Boundedness

For a given System (2.56), if asymptotic stability is not possible, uniform ultimate bounded stability can be considered. This is very useful in practical cases.

If there exists a function $V : \mathbb{R}^+ \times D \mapsto \mathbb{R}^+$ be a continuously differentiable function such that in $t \in \mathbb{R}^+$ and $x \in \mathbb{R}^n$,

$$\alpha_1(\parallel x \parallel) \leq V(t,x) \leq \alpha_2(\parallel x \parallel) \tag{2.63}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -W_3(x), \quad \text{for any} \quad \parallel x \parallel \geq \mu > 0, \tag{2.64}$$

[9] $\mathscr{B}_r$, the ball $\{x| \parallel x \parallel < r\}$ with radius $r$ where $r \in (0, +\infty)$ and $\bar{\mathscr{B}}_r$ denotes the closure of $\mathscr{B}_r$.

[10] if the function $f(\cdot)$ in (2.56) is continuously differentiable in the ball $\bar{\mathscr{B}}_r$, then $\frac{\partial f}{\partial x}$ is bounded in the domain $D = \mathscr{B}_r$.

where $\alpha_1(\cdot)$ and $\alpha_2(\cdot)$ are class $\mathscr{K}$ functions and $W_3(\cdot)$ is a continuous positive definite function in domain $D$. Then $x = 0$ is uniformly ultimately bounded.[11]

Further, $x = 0$ is globally uniformly ultimately bounded, if $D = \mathbb{R}^n$ and $\alpha_1(\cdot)$ belongs to class $\mathscr{K}_\infty$. (See theorem 4.18, p. 172 in [43] for the detailed proof.)

### 2.3.6 Lyapunov Functions for Linear Time-Invariant Systems

Given a linear system of the form $\dot{x} = Ax$, let us consider a quadratic Lyapunov function candidate

$$V = x^\top P x$$

where $P$ is a given symmetric positive definite matrix. Differentiating the positive definite function $V$ along the trajectory of system (2.56) yields

$$\dot{V} = \dot{x}^\top P x + x^\top P \dot{x} = -x^\top Q x \tag{2.65}$$

where

$$A^\top P + PA = -Q \tag{2.66}$$

**Theorem 2.3.5** *A necessary and sufficient condition for a LTI system $\dot{x} = Ax$ to be strictly stable is that, for any symmetric[12] positive definite matrix $Q$, the Lyapunov equation (2.66) has an unique solution $P$ which is symmetric positive definite. (See section 3.5.1 in [91] for proof.)*

**Example 2.3.1** *Consider a second-order linear system $\dot{x} = Ax$ where $A$ is given by*

$$A = \begin{bmatrix} 0 & 4 \\ -8 & -12 \end{bmatrix}$$

*Let us take $Q = I$ and denote $P$ by*

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

[11] The ultimate bound depends on the parameter $\mu$, which can be estimated using the result given in Theorem 4.18 in [43].

[12] A symmetric matrix is a matrix that is equal to its transpose.

*where, due to the symmetry of P,* $p_{12} = p_{21}$. *Then, the Lyapunov equation is*

$$\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 4 \\ -8 & -12 \end{bmatrix} + \begin{bmatrix} 0 & -8 \\ 4 & -12 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -Q$$

*whose solution is* $p_{11} = 5/16, p_{12} = p_{21} = p_{22} = 1/16$. *The corresponding matrix*

$$P = \frac{1}{16} \begin{bmatrix} 5 & 1 \\ 1 & 1 \end{bmatrix}$$

*is positive definite*[13]*, and therefore the linear system is globally asymptotically stable.*

## 2.4 Tracking Problem

Generally speaking, the tasks of control systems can be divided into two categories: stabilisation (or regulation) and tracking (or servo). In stabilisation problems, a control system is to be designed so that the state of the closed-loop system is stable in a domain around an equilibrium point. In tracking control problems which is mainly focused in this thesis, the design objective is to construct a controller so that the system tracks given time-varying signals. Problems such as making an aircraft fly along a specified path or making a robot hand draw straight lines or circles are typical tracking control tasks.

The task of asymptotic tracking consists of trajectory tracking and output tracking. Given a nonlinear dynamics system described by

$$\dot{x} = f(x,u,t) \tag{2.67}$$

$$y = h(x) \tag{2.68}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are state, input and output of the system.

**Definition 2.4.1 (Trajectory tracking)** *For system (2.67) with a desired signal* $x_d(t) \in \mathbb{R}^n$, *find a control law* $u(t)$ *such that the state variable* $x(t)$ *of the controlled system can track the desired signal* $x_d(t)$ *when time* $t$ *goes to infinity, i.e.*

$$\lim_{t \to \infty} \|x(t) - x_d(t)\| = 0 \tag{2.69}$$

**Definition 2.4.2 (Output tracking)** *For system (2.67)-(2.68) with a desired signal* $y_d(t) \in \mathbb{R}^p$, *find a control law* $u(t)$ *such that the controlled system states are bounded and the controlled*

[13] Matlab commands for solving the solution are available in Appendix **??**.

*system output $y(t)$ can track the desired output signal $y_d(t)$ when time $t$ goes to infinity, i.e.*

$$\lim_{t\to\infty} \|y(t) - y_d(t)\| = 0 \tag{2.70}$$

### 2.4.1 A Numerical Example of Output Tracking

Consider a simple system

$$\dot{x}_1(t) = x_2(t) \tag{2.71}$$

$$\dot{x}_2(t) = u(t) \tag{2.72}$$

$$y(t) = x_1(t) \tag{2.73}$$

Our goal is: for a given desired signal $y_d(t)$, proposed a sliding mode based controller such that the output $y(t)$ in (2.73) can asymptotically track the desired signal $y_d(t)$[14] in finite time.

Define a error $e(t) = y(t) - y_d(t)$, and choose a proper sliding surface

$$s(t) = \dot{e}(t) + a * e(t) \tag{2.74}$$

with the designed parameter $a$ is positive.

Propose a sliding mode-based controller

$$u(t) = \ddot{y}_d(t) - a * x_2(t) + a * \dot{y}_d(t) - \eta * sgn(s) \tag{2.75}$$

with a positive design parameter $\eta$. According to Lyapunov candidate function $V(s) = \frac{1}{2}s^2$, it is easy to get a negative result $\dot{V}(s) = -\eta|s|$, which the sliding motion $s(t)$ in (2.74) trends to zero in finite time.

Then, the condition in (2.70) is satisfied according to the $s(t) = 0$ with a positive parameter $a$ in (2.74). Thus, the output tracking is achieved where the desired signal is set as $sin(t)$ and initial state $x_1(0) = 1.5$.

[14] The desired signal $y_d(t)$ and its derivatives are assumed to be smooth in the considered domain.

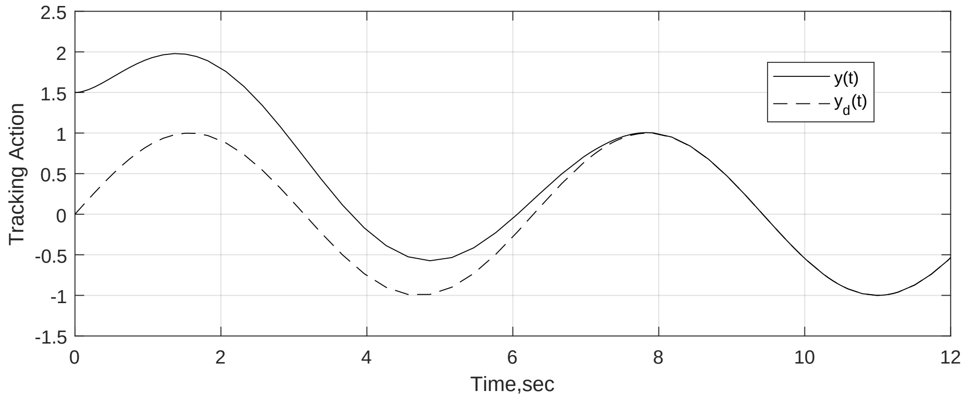


Fig. 2.5 Evolution of tracking action with respect to time

## 2.5 Large-Scale Interconnected Systems

In the real world, there are a number of important systems which can be modelled as dynamical equations composed of interconnections between a collection of lower-dimensional subsystems. Such classes of systems are called large-scale interconnected systems, which are often widely distributed in space [59, 63, 83]. A fundamental property of an interconnected system is that a perturbation of one subsystem may affect the other subsystems as well as the overall performance of the entire network. Decentralised control has been recognised as an effective method to control such a class of systems.

From the mathematical point of view, a nonlinear large-scale interconnected system composed of $N$ $n_i$-th order subsystems can be described by

$$\dot{x}_i = f_i(x_i) + g_i(x_i)(u_i + \Delta g_i(x_i)) + \Delta f_i(x_i) + \sum_{j=1 j\neq i}^{N} \zeta_{ij}(x_j) \tag{2.76}$$

$$y_i = h_i(x_i), \qquad i = 1,2,...,N \tag{2.77}$$

where $x_i \in \Omega_i \subseteq \mathbb{R}^{n_i}$ ($\Omega_i$ is a neighbourhood of the origin,) $u_i \in \mathbb{R}^{m_i}$ and $y_i \in \mathbb{R}^{p_i}$ are the states, inputs and outputs of the $i$-th subsystem respectively for $i = 1,2,...,N$. All the matrix functions $g_i(\cdot) \in \mathbb{R}^{n_i \times m_i}$ and the nonlinear vectors $f_i(\cdot) \in \mathbb{R}^{n_i}$ and $h_i(\cdot) \in \mathbb{R}^{p_i}$ are known. The terms $\Delta g_i(\cdot)$ and $\Delta f_i(\cdot)$ represent the matched and the mismatched uncertainties respectively. The term

$$\sum_{j=1 j\neq i}^{N} \zeta_{ij}(x_j)$$

represents the interconnection of the $i$-th subsystem. It is assumed that all the nonlinear functions are smooth enough such that the unforced systems have unique continuous solutions.

**Definition 2.5.1** *Consider system (2.76)-(2.77). The system*

$$\dot{x}_i = f_i(x_i) + g_i(x_i)(u_i + \Delta g_i(x_i)) + \Delta f_i(x_i) \tag{2.78}$$
$$y_i = h_i(x_i), \qquad i = 1, 2, ..., N \tag{2.79}$$

*is called the i-th isolated subsystem of system (2.76)-(2.77), and the system*

$$\dot{x}_i = f_i(x_i) + g_i(x_i)u_i \tag{2.80}$$
$$y_i = h_i(x_i), \qquad i = 1, 2, ..., N \tag{2.81}$$

*is called the i-th nominal isolated subsystem of system (2.76)-(2.77).*

It is well known that one of the main problems for interconnected systems is to develop a set of conditions under which the interconnected system (2.76)-(2.77) exhibits the desired performance if all the isolated subsystems (2.78)-(2.79) or all the nominal isolated subsystems (2.80)-(2.81) exhibit the required performance. Therefore, how to deal with interconnections is a key problem of interest in decentralised control.

### 2.5.1 Decentralised Control

**Definition 2.5.2** *Consider system (2.76)-(2.77). If the designed controllers $u_i$ for the i-th subsystems depend on the time $t$ and the states $x_i$ of the i-th subsystem only, i.e.,*

$$u_i = u_i(t, x_i), \qquad i = 1, 2, ..., N \tag{2.82}$$

*then (2.82) is called decentralised static state feedback control. If the controllers in (2.82) have the form*

$$u_i = u_i(t, y_i), \qquad i = 1, 2, ..., N \tag{2.83}$$

*that is, each local controller depends upon the time $t$ and the outputs $y_i(t)$ of the local subsystem only, then they are called decentralised static output feedback control.*

*Furthermore, if the designed controllers consist of the dynamical systems*

$$\dot{\hat{x}}_i = \phi_i(t, \hat{x}_i, u_i, y_i), \qquad i = 1, 2, ..., N \tag{2.84}$$

*and controllers*

$$u_i = u_i(t, \hat{x}, y_i), \qquad i = 1, 2, ..., N \tag{2.85}$$

*then (2.84)-(2.85) is called decentralised dynamical output feedback control. Specifically, if (2.84) is an observer of the system (2.76)-(2.77), then it is called decentralised observer-based feedback control.*

### 2.5.2 Distributed Control

There is a kind of special interconnected system whose subsystems are located in a chain structure such as a train. And the corresponding dynamical equation according to system (2.76)-(2.77) of this type of interconnected systems can be described by

$$\dot{x}_1 = f_1(x_1) + g_1(x_1)(u_1 + \Delta g_1(x_1)) + \Delta f_1(x_1) + \zeta_1(x_1, x_2) \tag{2.86}$$

$$\cdots$$

$$\dot{x}_i = f_i(x_i) + g_i(x_i)(u_i + \Delta g_i(x_i)) + \Delta f_i(x_i) + \zeta_i(x_{i-1}, x_i, x_{i+1}) \tag{2.87}$$

$$\cdots$$

$$\dot{x}_n = f_n(x_n) + g_n(x_n)(u_n + \Delta g_n(x_n)) + \Delta f_n(x_n) + \zeta_n(x_{n-1}, x_n) \tag{2.88}$$

where $i = 2, 3, \ldots, n-1$. It should be noted that in such a class of interconnected system, each subsystem are only directly affected by its adjacent subsystem through interconnection terms.

**Definition 2.5.3** *Consider system (2.86)-(2.88). If the designed controller $u_i$ for the i-th subsystems depends on the information of its own and its adjacent subsystems, i.e.,*

$$u_i = u_i(t, x_i, x_{i-1}, x_{i+1}) \tag{2.89}$$

*then (2.89) is called distributed control.*

Decentralised and distributed control are mentioned above which are usually used to deal with the interconnected systems.

### Centralised Control

Centralised control describes a system in which all the information of its subsystems (if it has) are transferred to and computed by a single controller whose structure is often in the form

$$u_i = u_i(t, x_1, x_2, \ldots, x_N). \tag{2.90}$$

This type of system architecture was common for power plants and other facilities using single-loop controllers or early digital controls in the past, but it has now been largely supplanted by distributed control because of the high cost associated with large data transformation, routing and installing all control system wiring to a central location.

Centralised control systems should only be considered for small industrial facilities and if used, must have fully redundant processors. Where redundancy is provided in a centralised control system segregated wiring pathways must be provided to assure that control signals to and from equipment or systems that are redundant are not subject to common failure from electrical fault, physical or environmental threats.

The Fig. 2.6 shows the basic structural diagrams of the controllers mentioned above.

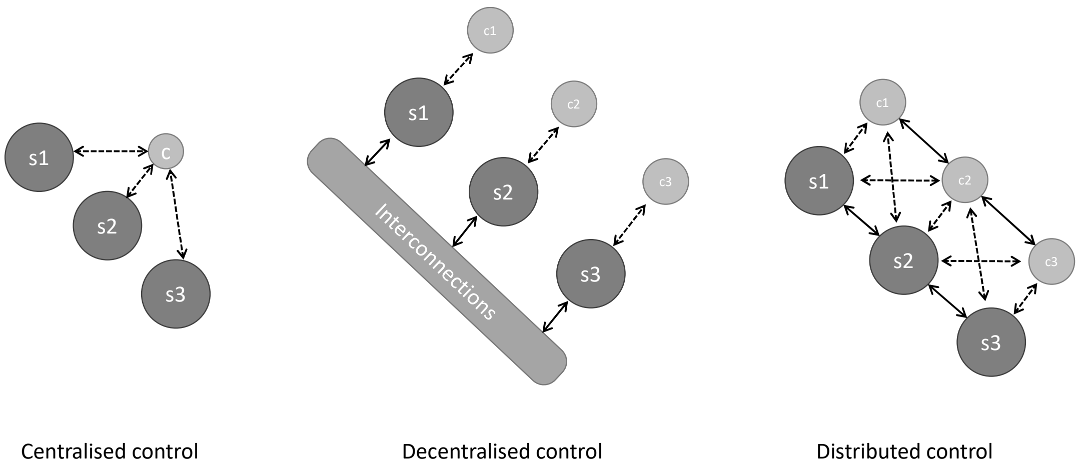


Fig. 2.6 Three different systems consist of identical subsystems (the bigger circles) and the corresponding controllers (the smaller circles). The solid lines represent interactions between members of the same system, while the dashed lines are input/output flow between plant and controller.

From the structure of decentralised control defined in (2.82), distributed control in (2.89) and centralised control in (2.90), it is straightforward to see that

- Decentralised Control: decision-making authority is distributed among multiple control subsystems or units. Each subsystem has its own autonomy and makes local decisions based on local information. There is limited or no communication between the subsystems, and they operate independently. Decentralised control is often used in systems where local decision-making is efficient and there is a need for fault tolerance or scalability.

- Distributed Control involves multiple control subsystems or units that communicate and coordinate with each other to achieve a common goal. These subsystems collect information from adjacent subsystems and collaborate in decision-making processes. Distributed control systems typically involve a networked architecture where control units share information and work together to optimize system performance. Distributed control is commonly used in complex systems that require coordination and cooperation among multiple entities.

- Centralised Control involves a central authority or control unit that has complete decision-making power and control over the system. The central controller receives information from various sources, processes it, and determines the control actions to be executed. All decisions and actions are centralised and executed by the central controller. Centralised control is often used in systems where a single authority can effectively coordinate and optimize the system's performance.

In summary, decentralised control distributes decision-making authority, distributed control involves collaboration among multiple entities, and centralised control concentrates decision-making power in a central authority. The choice of control approach depends on factors such as system complexity, communication capabilities, scalability requirements, and the need for coordination among control units.

### 2.5.3 State-of-art Control Methods

State-of-the-art control methods continue to evolve rapidly in various fields. Some noteworthy trends and approaches in control methods include:

- **Deep Learning-Based Control:** Integration of neural networks and deep learning for adaptive and robust control in complex systems.
- **Reinforcement Learning:** Application of reinforcement learning algorithms for optimizing control policies in dynamic and uncertain environments.
- **Data-Driven Control:** Increased emphasis on data-driven and model-free control strategies, leveraging large datasets for system identification and control.
- **Predictive Control:** Growing use of model predictive control (MPC) for systems with constraints, enabling real-time optimization and decision-making.
- **Adaptive and Robust Control:** Development of adaptive control methods to handle uncertainties and variations in system dynamics.
- **Quantum Control:** Exploration of control methods in quantum systems, particularly in quantum computing and quantum information processing.

- **Human-in-the-Loop Control:** Integration of control systems with human inputs, considering human factors and interactions for improved system performance.
- **Distributed and Networked Control:** Advancements in control methods for large-scale systems, including distributed control and networked control systems.
- **Cyber-Physical Systems (CPS):** Integration of control systems with information technologies in CPS, emphasizing connectivity, communication, and coordination.

In conclusion, the current state-of-the-art in control methods show a diverse and interdisciplinary landscape. The integration of AI, focus on adaptability, and the exploration of quantum technologies are key trends. Continued research and innovation are essential to addressing challenges and pushing the boundaries of control theory and applications of large-scale systems.

# Chapter 3

# Diffeomorphism-Based Decentralised Sliding Mode Output Tracking Control for Nonlinear Interconnected Systems

In this chapter, a decentralised tracking control scheme is proposed for a class of nonlinear interconnected systems with uncertainties using the sliding mode technique. The scheme considers both matched nonlinear uncertainty in isolated subsystems and mismatched known nonlinear interconnections. To facilitate system analysis and control design, a geometric transformation is applied to transfer the interconnected system into a new nonlinear interconnected system with a special structure. This transformation is performed under the condition that the nominal isolated subsystems have relative degrees. Next, a composite sliding surface is designed in terms of tracking errors, and decentralised controllers are proposed to drive the system states to the designed sliding surface in finite time and maintain a sliding motion on it thereafter. A set of conditions is developed to ensure that the output tracking errors converge to zero asymptotically while keeping all system state variables bounded. Compared with the existing works where the desired signals are mostly assumed to be constant, the closed-loop system can track the time-varying desired signals, even though the existence of uncertainties. Finally, the effectiveness of the proposed method is demonstrated through simulations on a coupled-pendulums system, showcasing its practical applicability and performance.

## 3.1 Introduction

Over the past few decades, significant progress has been made in the study of interconnected systems [30]. In [49], a fuzzy controller based on a reduced-order observer is designed for interconnected systems using integral sliding mode technique. The work presented by Mahmoud in [64] proposes a decentralised control strategy for linear interconnected time-delay systems. In [28], a finite-time control strategy is introduced for nonlinear interconnected systems with dead-zone input. Additionally, robust controllers are designed for an interconnected multimachine power system using output feedback sliding mode techniques in [114]. A decentralised control scheme is also proposed in [113] for fully nonlinear interconnected systems with time delay. More recently, a decentralised predictor method-based controller is developed for large-scale systems in [136], where the interconnected systems considered are linear, and there are strong limitations on the interconnections between subsystems. It is important to note that most of the existing results in the literature focus on stabilisation rather than tracking control, employing either state feedback or output feedback control. Compared to stabilisation, tracking control is more challenging, and thus the available results for tracking control are relatively limited, especially for large-scale nonlinear interconnected systems using decentralised control.

The tracking control problem is widely recognised as an interesting and important topic in engineering control. However, most existing work on tracking control focuses on centralised control approaches (see, e.g., [96], [1], [107]). Although decentralised tracking control for interconnected systems is studied in [77, 62], and distributed tracking control for interconnected systems with communication constraints is considered in [78], these studies require the isolated subsystems of the interconnected systems to be linear. Narendra and Zhang investigate linear interconnected systems in [75], focusing on model reference tracking control. Tracking control for interconnected systems is also explored in [6] using integral reinforcement learning, but it requires that interconnected terms are matched. Recently, Han and Yan propose an observer-based adaptive tracking control method for large-scale stochastic nonlinear systems in [22], which increases the dimension of the closed-loop system and consequently increases the computational load required for implementation. It should be noted that most existing results on tracking control for interconnected systems are not decentralised, requiring communication between each controller of one subsystem and all other subsystems through unobstructed channels for data transfer. This is obviously inconvenient for practical implementation. In [52], Li, Tong, and Yang propose a decentralised event-triggered control scheme using observer-based feedback control, which guarantees both tracking performance and stability of the closed-loop interconnected system. However, this approach may significantly increase the computational load. Decentralised event-triggered tracking control is also designed for nonlinear interconnected systems with unknown interconnections in [10]. Nevertheless, both [52] and [10] assume that all isolated subsystems have a triangular structure. Sliding mode control, known for its high robustness [115, 102], has been widely applied to address tracking problems (see, e.g., [116], [103], [24], and [126]). However, there are very few results on decentralised tracking control using sliding mode techniques for nonlinear interconnected systems, particularly when the desired signal is time-varying and the tracking errors are required to converge to zero asymptotically. In [16], an adaptive fuzzy control based on dynamic surface sliding mode technique is designed for prescribed output tracking. However, this approach can only be applied to specific multi-machine power systems, and unfortunately, the designed controller is not decentralised.

In this chapter, we focus on a class of nonlinear interconnected systems that incorporate both matched uncertainty in the isolated subsystems[1] and mismatched interconnections. To analyse and design the control for these systems, a nonlinear coordinate transformation is introduced to reveal the structure of the nominal isolated subsystems. This transformation effectively converts the interconnected systems into a desired form, enabling easier system analysis and control design using system structure. A composite sliding surface is designed in terms of the tracking errors to achieve sliding mode stability for the system. Subsequently, a decentralised sliding mode control scheme is proposed to drive the nonlinear interconnected systems towards the designed sliding surface within finite time. Compared to adaptive control approaches, sliding mode control imposes fewer restrictions to the bounds on the uncertainties and the system structure, allowing for a more general form of uncertainty representation. The main contributions of this chapter can be summarised as follows:

- The designed controller is decentralised, allowing for the control of interconnected systems without the need for information from other subsystems. Furthermore, the desired output signals are time-varying, which extends the applicability of the proposed method compared to existing approaches that assume constant desired signals.

- The developed control scheme guarantees asymptotic tracking of the desired outputs while ensuring boundedness of all the system states. This provides a desirable performance for the interconnected systems under consideration.

- The considered interconnected systems are nonlinear with nonlinear disturbances bounded by nonlinear functions. The proposed approach does not require the nominal isolated subsystems to be linearisable, making it applicable to a broader range of systems.

[1]The concept "isolated subsystems" has been defined in Section 2.5, Chapter 2.

- The interconnection terms are unmatched. Despite this, the developed control scheme exhibits high robustness, enabling effective control even in the presence of matched uncertainties and mismatched interconnections.

## 3.2 Problem Statement

Consider a nonlinear large-scale interconnected system formed by $N$ subsystems as follows

$$\begin{aligned} \dot{x}_i &= f_i(x_i) + g_i(x_i)(u_i + \varphi_i(x_i)) + p_i(x_i)\psi_i(x) \\ y_i &= h_i(x_i) \qquad i = 1,2,...,N \end{aligned} \tag{3.1}$$

where $x_i \in \Pi_i \subset R^{n_i}$, $u_i \in R$ and $y_i \in R$ are the states, input and output of the $i$th subsystem respectively, $\Pi_i$ are neighbourhoods of the origin, $x = col(x_1, x_2, ..., x_N) \in \Pi$, and $\Pi := \Pi_1 \times ... \times \Pi_N \in R^{\sum_{i=1}^{N} n_i}$. The terms $\varphi_i(x_i) \in R$ are matched uncertainties. The terms $p_i(x_i)\psi_i(x) \in R^{n_i}$ represent the interconnection of the $i$th subsystem where $p_i(x_i) \in R^{n_i}$ are known and used to describe the structure of the interconnections, and the terms $\psi_i(x) \in R$ defined in $\Pi$ are interconnections for $i = 1,2,\dots,N$. All of the nonlinear terms are assumed to be smooth enough in their arguments to guarantee the existence and uniqueness of the system solutions[2].

**Remark 3.2.1** *The interconnection term $\psi_i(x)$ in (3.1) is a function of all the states $x$, which represents the dynamic influence caused by and leads to the other subsystems.*

In this chapter, the local case will be considered, and the considered domain may not be specified in the subsequence unless it is necessary. It should be noted that each subsystem in system (3.1) is assumed to be single-input and single-output for simplifying the analysis. However, for the theory of nonlinear systems having many inputs and many outputs, it may needs some further research.

The objective of this chapter is, for a given desired output signal $y_{id}(t)$, to design a decentralised control such that the output $y_i(t)$ can track the desired signal $y_{id}(t)$ asymptotically, i.e.,

$$\lim_{t\to\infty} |y_i(t) - y_{id}(t)| = 0 \tag{3.2}$$

for $i = 1,2,...,N$, while all the state variables of the interconnected system (3.1) are bounded. To deal with the tracking problem stated above, some assumptions on the considered system (3.1) are introduced at first.

[2]For further details regarding the existence and uniqueness of solutions of nonlinear systems, please refer to Appendix A.3.

**Assumption 3.2.1** *There exist known continuous functions* $\rho_i(x_i)$ *in domain* $\Pi_i$ *such that* $|\varphi_i(x_i)| \leq \rho_i(x_i)$*, where* $x_i \in \Pi_i$ *with* $i = 1, 2, ..., N$.

**Remark 3.2.2** *Assumption 3.2.1 implies that the uncertainties* $\varphi_i(x_i)$ *in the system (3.1) have known upper bounds which will be used in the control design to cancel the effects of the corresponding uncertainties and enhance the robustness of system. For the case of unknown upper bounds, it's worthy some further research.*

**Assumption 3.2.2** *For system (3.1), the triples* $(f_i, g_i, h_i)$ *and* $(f_i, p_i, h_i)$ *have uniform relative degrees*[3] $r_i^a$ *and* $r_i^b$ *respectively in the domain* $\Pi_i$*, and satisfying* $r_i^a = r_i^b$ *for* $i = 1, 2, \ldots, N$.

**Remark 3.2.3** *Assumption 3.2.2 is the limitation to both the structure of nominal isolated subsystems and corresponding interconnections. The methodology developed in this chapter can be directly extended to the case* $r_i^a < r_i^b$*. Here, the condition* $r_i^a = r_i^b$ *imposed on system (3.1) is just for simplification of the later analysis. Similar limitation to the system relative degree is employed in [131].*

**Assumption 3.2.3** *The desired output signals* $y_{id}(t)$ *and their time derivatives up to the* $r_i^a$*th order are smooth, known and bounded for all* $t \in [0, \infty)$.

**Remark 3.2.4** *Assumption 3.2.3 is commonly introduced in tracking problem and usually can be satisfied in reality in most cases. However, if the desired signal* $y_{id}(t)$ *is not continuous in reality due to some engineering limitation, the work in this chapter may not be applied.*

## 3.3 A Local Diffeomorphism

In this section, the theory of local diffeomorphism is extended to deal with the linearisation of a large-scale system (3.1) with the uncertainties is considered. And the local diffeomorphism $z_i = T_i(x_i)$ is introduced as

$$\begin{bmatrix} x_{i,1} \\ x_{i,2} \\ \vdots \\ x_{i,(r_i^a-1)} \\ x_{i,r_i^a} \\ x_{i,(r_i^a+1)} \\ \vdots \\ x_{i,n_i} \end{bmatrix} \xrightarrow{z_i = T_i(x_i)} \begin{bmatrix} z_{i,1} \\ z_{i,2} \\ \vdots \\ z_{i,(r_i^a-1)} \\ z_{i,r_i^a} \\ z_{i,(r_i^a+1)} \\ \vdots \\ z_{i,n_i} \end{bmatrix} =: \begin{bmatrix} \xi_{i,1} \\ \xi_{i,2} \\ \vdots \\ \xi_{i,(r_i^a-1)} \\ \xi_{i,r_i^a} \\ \eta_{i,(r_i^a+1)} \\ \vdots \\ \eta_{i,n_i} \end{bmatrix} \tag{3.3}$$

[3]The definition of relative degree and more details are available in [34] and section 2.1.2.

and a feedback transformation is given by

$$u_i = \varpi_i^{-1}(x_i)(-\varsigma_i(x_i) + v_i) \tag{3.4}$$

where $\varsigma_i(x_i)$ and $\varpi_i(x_i)$ are defined by

$$\varsigma_i(x_i) = L_{f_i}^{r_i^a} h_i(x_i) \tag{3.5}$$

$$\varpi_i(x_i) = L_{g_i} L_{f_i}^{r_i^a - 1} h_i(x_i) \tag{3.6}$$

where $v_i$ is the new controller to be designed later, and the notation $L_{g_i} L_{f_i}^{r_i^a - 1} h_i(x_i)$ denotes Lie derivative (see e.g. [34] and section 2.1.2).

In the new coordinates $z_i$, the terms $\varsigma_i(x_i)$ and $\varpi_i(x_i)$ in (3.5) and (3.6) are, respectively, denoted by

$$\alpha_i(z_i) = \varsigma_i(x_i)|_{x_i = T_i^{-1}(z_i)}$$

$$\beta_i(z_i) = \varpi_i(x_i)|_{x_i = T_i^{-1}(z_i)}.$$

Then, under the diffeomorphism (3.3) and the feedback transformation (3.4), it follows from [34] that in the new coordinates $z_i$, the system (3.1) can be described by

$$\dot{\xi}_{i,1} = \xi_{i,2} \tag{3.7}$$

$$\dot{\xi}_{i,2} = \xi_{i,3} \tag{3.8}$$

$$\vdots$$

$$\dot{\xi}_{i,(r_i^a - 1)} = \xi_{i,r_i^a} \tag{3.9}$$

$$\dot{\xi}_{i,r_i^a} = v_i(t) + \beta_i(z_i)\tau_i(z_i) + \gamma_i(z_i)\delta_i(z) \tag{3.10}$$

$$\dot{\eta}_{i,(r_i^a + 1)} = q_{i,(r_i^a + 1)}(z_i) + \Gamma_{i,(r_i^a + 1)}\delta_i(z) \tag{3.11}$$

$$\vdots$$

$$\dot{\eta}_{i,n_i} = q_{i,n_i}(z_i) + \Gamma_{i,n_i}\delta_i(z) \tag{3.12}$$

$$y_i = \xi_{i,1} \tag{3.13}$$

where $z_i := col(\xi_i, \eta_i)$ with $\xi_i := col(\xi_{i,1}, \xi_{i,2}, \cdots, \xi_{i,r_i^a})$ and $\eta_i := col(\eta_{i,(r_i^a+1)}, \cdots, \eta_{i,n_i})$, $z = col(z_1, z_2, \cdots, z_N)$, and

$$\tau_i(z_i) = \varphi_i(x_i)|_{x_i=T_i^{-1}(z_i)} \tag{3.14}$$

$$\gamma_i(z_i) = L_{p_i} L_{f_i}^{r_i^b-1} h_i(x_i)|_{x_i=T_i^{-1}(z_i)} \tag{3.15}$$

$$\delta_i(z) = \psi_i(x)|_{x=T^{-1}(z)}. \tag{3.16}$$

The system (3.7)-(3.13) can be expressed in a compact form as

$$\dot{\xi}_i = A_i\xi_i + B_i[v_i + \beta_i(z_i)\tau_i(z_i) + \gamma_i(z_i)\delta_i(z)] \tag{3.17}$$

$$\dot{\eta}_i = q_i(\xi_i, \eta_i) + \Gamma_i(\xi_i, \eta_i)\delta_i(\xi_1, \eta_1, ..., \xi_N, \eta_N) \tag{3.18}$$

$$y_i = C_i\xi_i, \qquad i = 1, 2, ..., N \tag{3.19}$$

where $z_i = col(\xi_i, \eta_i)$ with $\xi_i \in R^{r_i^a}$ and $\eta_i \in R^{n_i - r_i^a}$. The triple $(A_i, B_i, C_i)$ with appropriate dimensions has a standard *Brunovsky* form [58] as follows

$$A_i = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}, \quad B_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{3.20}$$

$$C_i = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \end{bmatrix} \tag{3.21}$$

$q_i(\xi_i, \eta_i)$ and $\Gamma_i(\xi_i, \eta_i)$ are the last $n_i - r_i^a$ rows of the vectors

$$\left[\frac{\partial T_i}{\partial x_i} f_i(x_i)\right]_{x_i=T_i^{-1}(z_i)} \quad \text{and} \quad \left[\frac{\partial T_i}{\partial x_i} p_i(x_i)\right]_{x_i=T_i^{-1}(z_i)}$$

respectively, for $i = 1, 2, ..., N$.

**Remark 3.3.1** *It should be pointed out that the diffeomorphism $z_i = T_i(x_i) = col(\xi_i, \eta_i) = col(\xi_{i,1}, \xi_{i,2}, \cdots, \xi_{i,r_i^a}, \eta_{i,(r_i^a+1)}, \cdots, \eta_{i,n_i})$ given in (3.3) is not unique. According to [34], a way to choose the diffeomorphism can be given as follows: $\xi_i = col(h_i(x_i), L_{f_i}h_i(x_i), ..., L_{f_i}^{r_i^a} h_i(x_i))$, for $i = 1, 2, ..., N$. $\eta_i = col(\eta_{i,(r_i^a+1)}, \cdots, \eta_{i,n_i})$ where $\eta_{ij}$ can be obtained by solving the equations $L_{g_i}\eta_{ij} = 0$ for $j = r_i^a + 1, ..., n_i$ and $i = 1, 2, ..., N$.*

**Remark 3.3.2** *However, from (3.17)-(3.19), it is clear to see that in this chapter, it is not required that the nominal subsystems of system (3.1) are feedback linearisable. If the relative degree $r_i^a = n_i$, then (3.7)-(3.13) will have the following form*

$$\dot{\xi}_{i,1} = \xi_{i,2} \tag{3.22}$$
$$\dot{\xi}_{i,2} = \xi_{i,3} \tag{3.23}$$
$$\vdots$$
$$\dot{\xi}_{i,(n_i-1)} = \xi_{i,n_i} \tag{3.24}$$
$$\dot{\xi}_{i,n_i} = v_i(t) + \beta_i(z_i)\tau_i(z_i) + \gamma_i(z_i)\delta_i(z) \tag{3.25}$$
$$y_i = \xi_{i,1}. \tag{3.26}$$

*where $\gamma_i(z_i) \neq 0$ in $\Pi_i$ for $i = 1,2,...,N$. In this case, the nominal isolated subsystem of interconnected system (3.17) is completely feedback linearisable and thus the nonlinear part relating to the dynamics of variables $\eta_i$ in system (3.18) disappears.*

## 3.4 Synthesis of a Decentralised Control

In the subsequence, the nonlinear interconnected systems (3.17)-(3.19) are focused. Firstly, a sliding surface in terms of tracking errors will be proposed. Then, a decentralised controller based on sliding mode technique will be designed to implement the output tracking, and the boundedness of the considered interconnected system will be discussed.

### 3.4.1 Properties of the Sliding Motion

It is assumed that the desired output signals $y_{id}(t)$ satisfy Assumption 3.2.3. For (3.19), the output tracking errors $e_i$ are defined by

$$e_i = y_i(t) - y_{id}(t). \qquad i = 1,2,...,N \tag{3.27}$$

The sliding functions are introduced as follows

$$S_i(\cdot) = e_i^{(r_i^a-1)} + a_{i,1}e_i^{(r_i^a-2)} + ... + a_{i,(r_i^a-2)}e_i^{(1)} + a_{i,(r_i^a-1)}e_i^{(0)} \tag{3.28}$$

where $e_i^{(r_i^a-1)}$, $e_i^{(r_i^a-2)}$, $\cdots$, and $e_i^{(1)}$ denote the $(r_i^a-1)$th order, $(r_i^a-2)$th order, $\cdots$, and the 1st order derivatives of $e_i(t)$ respectively, $e_i^{(0)} := e_i(t)$, and $a_{i,1}, a_{i,2}, ..., a_{i,(r_i^a-1)}$ are a set of design parameters, which are chosen such that the following polynomials

$$\lambda^{r_i^a-1} + a_{i,1}\lambda^{r_i^a-2} + \ldots + a_{i,(r_i^a-2)}\lambda + a_{i,(r_i^a-1)} \tag{3.29}$$

are *Hurwitz* stable[4] for $i = 1, 2, \ldots, N$. Then, the composite sliding surface $\mathcal{S}$ for interconnected system (3.17)-(3.19) can be described by

$$\mathcal{S} = \{col(S_1, S_2, \cdots, S_N) \mid \; S_i = 0, \; i = 1, 2, \ldots, N\} \tag{3.30}$$

where $S_i$ are defined in (3.28). From the design above, it is clear to see that when $S_i = 0$,

$$\lim_{t\to\infty} |e_i(t)| = 0.$$

This implies that when sliding motion occurs,

$$\lim_{t\to\infty} |y_i(t) - y_{id}(t)| = \lim_{t\to\infty} |e_i(t)| = 0 \tag{3.31}$$

i.e., the outputs $y_i(t)$ of system (3.1) can track the ideal signal $y_{id}(t)$ asymptotically for $i = 1, 2, \ldots, N$. The following result is now ready to be presented:

**Theorem 3.4.1** *Consider the interconnected system (3.17)-(3.19). Under Assumption 3.2.3, when the system (3.17)-(3.19) is limited to moving on the sliding surface (3.30), the following results hold:*

*i).* $\lim_{t\to\infty} |y_i(t) - y_{id}(t)| = \lim_{t\to\infty} |e_i(t)| = 0$ *for* $i = 1, 2, \ldots, N$.

*ii). The state variables* $\xi_i$ *in (3.17) are bounded for* $i = 1, 2, \ldots, N$.

**Proof** *The result in i) has been shown above (see (3.31)). The remains are to prove that the result in ii) holds.*

*When system (3.17)-(3.19) is constrained to the sliding surface (3.28), it follows that*

$$S_i = e_i^{(r_i^a-1)} + a_{i,1}e_i^{(r_i^a-2)} + \ldots + a_{i,(r_i^a-2)}e_i^{(1)} + a_{i,(r_i^a-1)}e_i^{(0)} = 0.$$

*Then,*

$$e_i^{(r_i^a-1)} = -a_{i,1}e_i^{(r_i^a-2)} - \ldots - a_{i,(r_i^a-2)}e_i^{(1)} - a_{i,(r_i^a-1)}e_i^{(0)}.$$

[4] The Hurwitz polynomial is a polynomial whose roots (zeros) are located in the left half of the complex plane, that is, the real part of every root is negative.

*Let*

$$e_{i,1} \triangleq e_i^{(0)} = e_i.$$

*Then, the following error dynamics are obtained*

$$\begin{aligned}
\dot{e}_{i,1} &= e_i^{(1)} \triangleq e_{i,2} \\
\dot{e}_{i,2} &= e_i^{(2)} \triangleq e_{i,3} \\
&\vdots \\
\dot{e}_{i,(r_i^a-2)} &= e_i^{(r_i^a-2)} \triangleq e_{i,(r_i^a-1)} \\
\dot{e}_{i,(r_i^a-1)} &= -a_{i,1}e_{i,(r_i^a-1)} - \ldots - a_{i,(r_i^a-2)}e_{i,2} - a_{i,(r_i^a-1)}e_{i,1}.
\end{aligned}$$

*Therefore, the sliding mode dynamics of system (3.17)-(3.18) are given by the following equation by rewriting the system above in a compact form as*

$$\dot{\varepsilon}_i(t) = \underbrace{\begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \ldots & 1 \\ -a_{i,(r_i^a-1)} & -a_{i,(r_i^a-2)} & -a_{i,(r_i^a-3)} & \ldots & -a_{i,1} \end{bmatrix}}_{E_i} \varepsilon_i(t) \tag{3.32}$$

*where* $\varepsilon_i(t) = col(e_{i,1}(t), e_{i,2}(t), \cdots, e_{i,(r_i^a-1)}(t))$. *It should be noted that the entries of the last row of matrix* $E_i$*:* $a_{i,1}, a_{i,2}, \ldots, a_{i,(r_i^a-1)}$ *forms the Hurwitz stable polynomial (3.29). Thus, system (3.32) is Hurwitz stable which implies that*

$$\lim_{t\to\infty} |\varepsilon_i(t)| = 0. \tag{3.33}$$

*Further, from (3.31) and (3.33)*

$$\lim_{t\to\infty} \left\| \begin{matrix} \xi_{i,1}(t) - y_{id}^{(0)}(t) \\ \xi_{i,2}(t) - y_{id}^{(1)}(t) \\ \vdots \\ \xi_{i,(r_i^a-1)}(t) - y_{id}^{(r_i^a-2)}(t) \\ \xi_{i,r_i^a}(t) - y_{id}^{(r_i^a-1)}(t) \end{matrix} \right\| = 0.$$

*From Assumption 3.2.3, the desired output signal $y_{id}(t)$ and its derivatives: $y_{id}^{(1)}, y_{id}^{(2)}, \ldots, y_{id}^{(r_i^a-1)}$ are bounded in $t \in [0, \infty]$. It follows that the state variables $\xi_{i,1}, \xi_{i,2}, \ldots, \xi_{i,r_i^a}$ are bounded and thus the states $\xi_i$ in (3.17) are bounded when the states of the system are limited to the sliding surface (3.30). Hence, the result holds.*

It is well known that the sliding mode is a reduced-order system. Section 3.4.1 shows that for system (3.17)-(3.19) with the sliding surface given in (3.30), the corresponding sliding mode dynamics are system (3.32). Theorem 3.4.1 shows that the sliding mode (3.32) is asymptotically stable and the partial of state variable $\xi_i$ is bounded. Next, the decentralised controllers are to be designed to guarantee the reachability, and the boundedness of the partial states $\eta_i$ will be discussed as well.

### 3.4.2 The Reachability Problem

Now, the objective is to design a decentralised state-feedback controller based on sliding mode technique such that the states of the controlled system (3.17)-(3.19) can be driven to the designed sliding surface (3.28) in finite time.

Since $z_i = T_i(x_i)$ is a diffeomorphism, from Assumption 3.2.1 and definitions of $\tau_i(z_i)$ and $\delta_i(z)$ in (3.14) and (3.16) respectively, it follows that there are continuous function $\rho_i'(z_i)$ such that in the considered neighbourhood of the origin

$$|\tau_i(z_i)| \leq \rho_i'(z_i) \tag{3.34}$$

where $\rho_i'(\cdot)$ depends on the transformation $z_i = T_i(x_i)$ and $\rho_i(\cdot)$ in Assumption 3.2.1. Since $\rho_i(\cdot)$ is known, the bound $\rho_i'(\cdot)$ can be obtained from $z_i = T_i(x_i)$.

For system (3.17)-(3.19), the following control law is proposed

$$v_i = -\dot{S}_i + y_i^{(r_i^a)} - \left(K_i(z_i) + |\beta_i(z_i)|\rho_i'(z_i) + \frac{1}{2}|\gamma_i(z_i)|^2\right) sgn(S_i), \quad i = 1, 2, \ldots, N \tag{3.35}$$

where the function $K_i(z_i)$ is the feedback gain to be designed later. $S_i(\cdot)$ is given in (3.28) and $sgn(\cdot)$ is the signum function. It is clear to check that the controllers $v_i$ in (3.35) are decentralised.

**Remark 3.4.1** *From the structure of the control (3.35), it follows that only the variables $z_i, y_i, y_i^{(r_i^a)}$ and $y_{id}(t)$ are used in the i-th control $v_i$, which are available locally. Specially from (3.7)-(3.13), $y_i^{(r_i^a)}$ is actually the first order derivative of the state $x_{i,r_i^a}$, which totally depends on the local state $x_{i,r_i^a}$. Therefore, from the coordinate transformation $z_i = T_i(x_i)$ and the* *relationship between $u_i$ and $v_i$ in (3.4), it is straightforward to see that the proposed controllers are decentralised.*

**Theorem 3.4.2** *Under Assumptions 3.2.1 and 3.2.3, the nonlinear interconnected system (3.17)-(3.19) is driven to the sliding surface (3.28) in finite time by the controller (3.35) if the control gain $K_i(z_i)$ satisfies*

$$\sum_{i=1}^{N} K_i(z_i) > \frac{1}{2}\sum_{i=1}^{N} |\delta_i(z)|^2 + \sigma_i \tag{3.36}$$

*where $\sigma_i$ is a positive constant.*

**Proof** *The closed-loop system obtained by applying control law (3.35) into system (3.17)-(3.19) can be described by*

$$\begin{aligned} \dot{\xi}_i &= A_i\xi_i + B_i[-\dot{S}_i + y_i^{(r_i^a)} - (K_i(z_i) + |\beta_i(z_i)|\rho_i'(z_i) + \frac{1}{2}|\gamma_i(z_i)|^2)sgn(S_i) \\ &\qquad + \beta_i(z_i)\tau_i(z_i) + \gamma_i(z_i)\delta_i(z)] \qquad (3.37) \\ \dot{\eta}_i &= q_i(\xi_i,\eta_i) + \Gamma_i(\xi_i,\eta_i)\delta_i(\xi_1,\eta_1,\ldots,\xi_N,\eta_N) \qquad (3.38) \\ y_i &= C_i\xi_i, \qquad i = 1,2,\ldots,N \qquad (3.39) \end{aligned}$$

*With the special structure of the triple $(A_i, B_i, C_i)$ in (3.17)-(3.19), it follows that*

$$\begin{aligned} y_i &= \xi_{i,1} \qquad (3.40) \\ y_i^{(1)} &= \xi_{i,2} \qquad (3.41) \\ &\vdots \\ y_i^{(r_i^a-1)} &= \xi_{i,r_i^a} \qquad (3.42) \\ y_i^{(r_i^a)} &= \dot{\xi}_{i,r_i^a} = -\dot{S}_i + y_i^{(r_i^a)} - (K_i(z_i) + |\beta_i(z_i)|\rho_i'(z_i) + \frac{1}{2}|\gamma_i(z_i)|^2)sgn(S_i) \\ &\qquad + \beta_i(z_i)\tau_i(z_i) + \gamma_i(z_i)\delta_i(z). \qquad (3.43) \end{aligned}$$

*From (3.43),*

$$\dot{S}_i = -\left(K_i(z_i) + |\beta_i(z_i)|\rho_i'(z_i) + \frac{1}{2}|\gamma_i(z_i)|^2\right)sgn(S_i) + \beta_i(z_i)\tau_i(z_i) + \gamma_i(z_i)\delta_i(z). \tag{3.44}$$

*Then, from (3.34) - (3.44) and according to basic inequality:*[5] $ab \leq \frac{1}{2}(a^2+b^2)$,

$$\begin{aligned}
S^\top \dot{S} &= \sum_{i=1}^{N} S_i \dot{S}_i \\
&= \sum_{i=1}^{N} \Big( -\Big(K_i(z_i) + |\beta_i(z_i)|\rho_i'(z_i) + \frac{1}{2}|\gamma_i(z_i)|^2\Big)|S_i| + \beta_i(z_i)\tau_i(z_i)S_i + \gamma_i(z_i)\delta_i(z)S_i\Big) \\
&\leq \sum_{i=1}^{N} \Big( -K_i(z_i)|S_i| - \frac{1}{2}|\gamma_i(z_i)|^2|S_i| + \frac{1}{2}(|\gamma_i(z_i)|^2 + |\delta_i(z)|^2)|S_i|\Big) \\
&= \sum_{i=1}^{N} \Big( -K_i(z_i) + \frac{1}{2}|\delta_i(z)|^2\Big)|S_i|. \qquad (3.45)
\end{aligned}$$

*It follows from (3.45), (3.36) and the basic inequality:*[6] $\left(\sum_{i=1}^{N}|S_i|\right)^2 \geq \sum_{i=1}^{N}|S_i|^2$ *that*

$$S^T\dot{S} < -\sigma \sum_{i=1}^{N} |S_i| \leq -\sigma \|S\| \qquad (3.46)$$

*where* $\sigma := min_i\{\sigma_i\} > 0$ *due to* $\sigma_i > 0$ *for* $i = 1,2,\ldots,N$*, meaning that the reachability condition [14] holds for the closed-loop interconnected system (3.37)-(3.38). Hence, the result holds.*

**Remark 3.4.2** *Based on the analysis above and from the feedback transformation (3.35), it follows that the decentralised controller*

$$\begin{aligned}
u_i =& \varpi_i^{-1}(x_i)\Big[ -\varsigma_i(x_i) - \dot{S}_i + y_i^{(r_i^a)} - \Big(K_i(T_i(x_i)) + |\varpi_i(x_i)|\rho_i(x_i) \\
&+ \frac{1}{2}|L_{p_i}L_{f_i}^{r_i^b-1}h_i(x_i)|^2\Big) sgn(S_i)\Big] \qquad (3.47)
\end{aligned}$$

*can drive the system (3.1) to the corresponding sliding surface in finite time, where* $p_i = p_i(x_i)$, $f_i = f_i(x_i)$ *and* $S_i$ *is defined in (3.28) for* $i = 1,2,\ldots,N$.

### 3.4.3 The Boundedness of System States

In this subsection, due to the assumption of the relative degree is introduced as Assumption 3.2.2 which means the system (3.1) can not be fully linearised, the stability of zero dynamic $\eta_i$ for $i = 1,2,\ldots,N$ in (3.38) need to be checked. For this, the boundedness of the closed-loop system (3.37)-(3.38) is analysed. The following assumptions are introduced.

[5] More details of the Young's inequality are available in Appendix A.1.
[6] More details of the inequality are available in "Norm properties" in Appendix A.2.

**Assumption 3.4.1** *The functions $q_i(\xi_i, \eta_i)$ in system (3.37)-(3.38) satisfy the Lipschitz condition with the Lipschitz constants $L_{q_i}$ uniformly for $\eta_i$ in the considered domain. Moreover, there exists a Lyapunov function $V_{i0}(\eta_i)$ such that*

$$\begin{aligned}\chi_{i1} \parallel \eta_i \parallel^2 \leq V_{i0}(\eta_i) \leq \chi_{i2} \parallel \eta_i \parallel^2 \\ \frac{\partial V_{i0}}{\partial \eta_i} q_i(0, \eta_i) \leq -\chi_{i3} \parallel \eta_i \parallel^2 \\ \left\| \frac{\partial V_{i0}}{\partial \eta_i} \right\| \leq \chi_{i4} \parallel \eta_i \parallel\end{aligned} \tag{3.48}$$

*where $\chi_{i1}$, $\chi_{i2}$, $\chi_{i3}$, and $\chi_{i4}$ are positive constants for $i = 1, 2, \cdots, N$.*

**Remark 3.4.3** *The Assumption 3.4.1 implies that there exists a constant $L_{q_i}$ such that*

$$\parallel q_i(\xi_i, \eta_i) - q_i(0, \eta_i) \parallel \leq L_{q_i} \parallel \xi_i \parallel . \tag{3.49}$$

*Assumption 3.4.1 is the limitation to the nonlinear term $q_i(\xi_i, \eta_i)$ in (3.37)-(3.38). It also implies that the zero dynamics $\dot{\eta}_i = q_i(0, \eta_i)$ of the nominal system of system (3.37)-(3.38) is asymptotically stable.*

**Assumption 3.4.2** *There exist positive constants $\kappa_{1j}$ and $\kappa_{2j}$ such that*

$$\parallel \Gamma_i(\xi_i, \eta_i)\delta_i(\xi_1, \eta_1, ..., \xi_N, \eta_N) \parallel \leq \sum_{j=1}^{N} (\kappa_{1j} \parallel \xi_j \parallel + \kappa_{2j} \parallel \eta_j \parallel) \tag{3.50}$$

*for $i = 1, 2, \cdots, N$.*

**Theorem 3.4.3** *Under Assumptions 3.2.3 and 3.4.1-3.4.2, the states of the closed-loop system (3.37)-(3.38) are bounded, if the matrix $W^T + W$ is positive definite with the matrix $W$ defined by*

$$W := \begin{bmatrix} \chi_{13} - \chi_{14}\kappa_{21} & -\chi_{14}\kappa_{22} & \cdots & -\chi_{14}\kappa_{2N} \\ -\chi_{24}\kappa_{21} & \chi_{23} - \chi_{24}\kappa_{22} & \cdots & -\chi_{24}\kappa_{2N} \\ \vdots & \vdots & \ddots & \\ -\chi_{N4}\kappa_{21} & -\chi_{N4}\kappa_{22} & \cdots & \chi_{N3} - \chi_{N4}\kappa_{2N} \end{bmatrix} \tag{3.51}$$

*where $\chi_{ij}$ and $\kappa_{lj}$ satisfy the Assumptions 3.4.1 and 3.4.2 for $i = 1, 2, \cdots, N$, $j = 1, 2, 3, 4$ and $l = 1, 2$.*

**Proof** *From Theorem 3.4.1, it follows that the variables $\xi_i = col(\xi_{i,1}, \xi_{i,2}, ..., \xi_{i,r_i^a})$ with $i = 1, 2, \ldots, N$ are bounded when the sliding motion occurs. Theorem 3.4.2 shows that the interconnected system can be driven to the sliding surface in finite time. From Theorems 3.4.1 and*

*3.4.2, it follows that the variables $\xi_i = col(\xi_{i,1}, \xi_{i,2}, ..., \xi_{i,r_i^a})$ with $i = 1, 2, \dots, N$ are bounded. Therefore, there exist constants $\mathscr{C}_i > 0$ such that in the considered domain,*

$$\| \xi_i \| \leq \mathscr{C}_i. \qquad i = 1, 2, ..., N \tag{3.52}$$

*The remain is to prove that the variables $\eta_i$ in the closed-loop system (3.37)-(3.38) are bounded for $i = 1, 2, ..., N$.*

*It should be noted that from (3.52), the variables $\xi_i$ in the system (3.38) are bounded and can be considered as parameters defined in a compact set. For this system, consider the following Lyapunov candidate function*

$$V(\eta_1, \eta_2, ..., \eta_N) = \sum_{i=1}^{N} V_{i0}(\eta_i)$$

*where $V_{i0}(\eta_i)$ is defined in Assumption 3.4.1. Then, the time derivative of the Lyapunov function $V(\cdot)$ along the trajectories of system (3.37)-(3.38) is given by*

$$\begin{aligned}
&\dot{V}(\eta_1, \eta_2, ..., \eta_N) \\
&= \sum_{i=1}^{N} \frac{\partial V_{i0}(\eta_i)}{\partial \eta_i} \Big[ q_i(\xi_i, \eta_i) + \Gamma_i(\xi_i, \eta_i) \delta_i(\xi_1, \eta_1, ..., \xi_N, \eta_N) \Big] \\
&= \sum_{i=1}^{N} \Big[ \frac{\partial V_{i0}(\eta_i)}{\partial \eta_i} q_i(0, \eta_i) + \frac{\partial V_{i0}(\eta_i)}{\partial \eta_i} (q_i(\xi_i, \eta_i) - q_i(0, \eta_i)) \Big] \\
&\quad + \sum_{i=1}^{N} \frac{\partial V_{i0}(\eta_i)}{\partial \eta_i} \Big[ \Gamma_i(\xi_i, \eta_i) \delta_i(\xi_1, \eta_1, ..., \xi_N, \eta_N) \Big].
\end{aligned} \tag{3.53}$$

*Further, from (3.49) and Assumptions 3.4.1 and 3.4.2, it follows*

$$
\begin{aligned}
&\dot{V}(\eta_1,\eta_2,...,\eta_N)\\
&\le \sum_{i=1}^{N}(-\chi_{i3}\parallel \eta_i \parallel^2 + \chi_{i4}L_{qi}\mathscr{C}_i \parallel \eta_i \parallel + \left|\left|\frac{\partial V_{i0}(\eta_i)}{\partial \eta_i}\right|\right| \parallel \Gamma_i(\xi_i,\eta_i)\delta_i(\xi_1,\eta_1,...,\xi_N,\eta_N) \parallel)\\
&\le \sum_{i=1}^{N}(-\chi_{i3}\parallel \eta_i \parallel^2 + \chi_{i4}L_{qi}\mathscr{C}_i \parallel \eta_i \parallel + \chi_{i4}\parallel \eta_i \parallel \sum_{j=1}^{N}(\kappa_{1j}\parallel \xi_j \parallel + \kappa_{2j}\parallel \eta_j \parallel))\\
&\le \sum_{i=1}^{N}(-\chi_{i3}\parallel \eta_i \parallel^2 + \chi_{i4}L_{qi}\mathscr{C}_i \parallel \eta_i \parallel + \sum_{j=1}^{N}\chi_{i4}\kappa_{1j}\mathscr{C}_i \parallel \eta_i \parallel + \sum_{j=1}^{N}\chi_{i4}\kappa_{2j}\parallel \eta_i \parallel\parallel \eta_j \parallel)\\
&= -(\sum_{i=1}^{N}\chi_{i3}\parallel \eta_i \parallel^2 - \sum_{i=1}^{N}\sum_{j=1}^{N}\chi_{i4}\kappa_{2j}\parallel \eta_i \parallel\parallel \eta_j \parallel - \sum_{i=1}^{N}\sum_{j=1}^{N}\chi_{i4}\mathscr{C}_i(L_{qi}+\kappa_{1j})\parallel \eta_i \parallel)\\
&= -\frac{1}{2}\begin{pmatrix}\parallel \eta_1 \parallel,...,\parallel \eta_N \parallel\end{pmatrix}(W+W^T)\begin{pmatrix}\parallel \eta_1 \parallel\\ \parallel \eta_2 \parallel\\ \vdots\\ \parallel \eta_N \parallel\end{pmatrix} + \sum_{i=1}^{N}\sum_{j=1}^{N}\chi_{i4}\mathscr{C}_i(L_{qi}+\kappa_{1j})\parallel \eta_i \parallel\\
&\le -\frac{1}{2}\lambda_{min}(W+W^T)\|\eta\|^2 + \sum_{i=1}^{N}\sum_{j=1}^{N}\chi_{i4}\mathscr{C}_i(L_{qi}+\kappa_{1j})\parallel \eta_i \parallel\\
&= -\frac{1}{2}\lambda_{min}(W+W^T)\sum_{i=1}^{N}\|\eta_i\|^2 + \sum_{i=1}^{N}\sum_{j=1}^{N}\chi_{i4}\mathscr{C}_i(L_{qi}+\kappa_{1j})\|\eta_i\|\\
&= -\frac{1}{2}\sum_{i=1}^{N}\left\{\lambda_{min}(W+W^T)\|\eta_i\| - \sum_{j=1}^{N}\chi_{i4}\mathscr{C}_i(L_{qi}+\kappa_{1j})\right\}\|\eta_i\|\\
&\le 0
\end{aligned}
\tag{3.54}
$$

*where* $\|\eta\| := \|(\parallel \eta_1 \parallel, \parallel \eta_2 \parallel \;,...,\; \parallel \eta_N \parallel)^T\|$, *if*

$$
\|\eta_i\| \ge \frac{\sum_{j=1}^{N}\chi_{i4}\mathscr{C}_i(L_{qi}+\kappa_{1j})}{\lambda_{min}(W)}, \qquad i=1,2,...,N
$$

*Then, from Theorem 4.18 in [43], the variables* $\eta_i$ *are bounded for* $i=1,2,\dots,N$*. Hence, the result holds.*

**Remark 3.4.4** *From Remark 3.3.2, if* $r_i^a = n_i$ *for* $i=1,2,...,N$*, the considered system can be fully linearised and thus the dynamical equation (3.38) disappears. In this case, Assumptions 3.4.1-3.4.2 are unnecessary, and the interconnection terms are completely matched. This can be regarded as a special case of the results developed in this chapter.*

## 3.5 Example: Inverted Pendulum

Consider two inverted pendulums connected by a spring as shown in Fig.3.1. Each pendulum

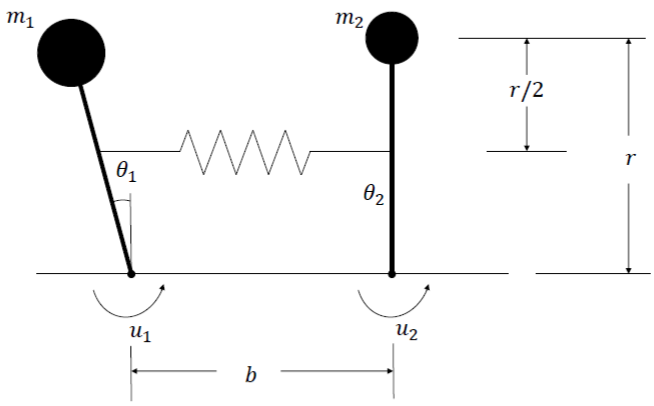


Fig. 3.1 Two inverted pendulums connected by a spring

is controlled by a servomotor which provides a torque input $u_i$ at the pivot. It is assumed that $\theta_i$ and $\dot{\theta}_i$ represent the angular position and angular velocity of the pendulums respectively for $i = 1,2$.The model which describes the motion of the pendulums is given by (see, [94])

$$\begin{aligned} \dot{x}_{1,1} &= x_{1,2} \\ \dot{x}_{1,2} &= \frac{u_1}{J_1} + \beta_1(x_1)\tau_1(x_1) + \gamma_1(x_1)\delta_1(x) + \frac{kr}{2J_1}(l-b) \\ y_1 &= x_{1,1} \end{aligned} \tag{3.55}$$

and

$$\begin{aligned} \dot{x}_{2,1} &= x_{2,2} \\ \dot{x}_{2,2} &= \frac{u_2}{J_2} + \beta_2(x_2)\tau_2(x_2) + \gamma_2(x_2)\delta_2(x) - \frac{kr}{2J_2}(l-b) \\ y_2 &= x_{2,1} \end{aligned} \tag{3.56}$$

where $x_{1,1} = \theta_1$, $x_{2,1} = \theta_2$, $x_{1,2} = \dot{\theta}_1$ and $x_{2,2} = \dot{\theta}_2$ are system states. It is assumed that $x_{1,1}$ and $x_{2,1}$ are measurable, which are taken as system outputs.

It should be pointed out that system (3.55)-(3.56) above has already been in the form of system (3.17) where

$$\beta_1 = \frac{m_1 gr}{J_1} - \frac{kr^2}{4J_1}, \qquad \beta_2 = \frac{m_2 gr}{J_2} - \frac{kr^2}{4J_2}$$

$$\tau_1(x_1) = sin(x_{1,1}), \qquad \tau_2(x_2) = sin(x_{2,1})$$
$$\gamma_1(x_1) = \frac{kr^2}{4J_1}, \qquad \gamma_2(x_2) = \frac{kr^2}{4J_2}$$
$$\delta_1(x) = sin(x_{2,1}), \qquad \delta_2(x) = sin(x_{1,1}).$$

From [94], the parameters are chosen as $m_1$ =2 kg and $m_2$ = 2.5 kg representing the end masses of the pendulum. $J_1$ = 0.5 kg· m$^2$ and $J_2$ = 0.625 kg· m$^2$ are the moments of inertia. $g$ = 9.81 m/s$^2$ is the gravitational acceleration. $k$ = 100 N/m is the spring constant of the connecting spring. $r$ = 0.5m is the pendulum height and $l$ = 0.5m is the natural length of the spring. The distance between the pendulum hinges is $b = 0.5$m.

By direct calculation

$$|\tau_1(x_1)| = |sin(x_{1,1})| \leq 1 = \rho_1(x_1),$$

$$|\tau_2(x_2)| = |sin(x_{2,1})| \leq 1 = \rho_2(x_2).$$

Here, both the value of $\sigma_i$ for $i = 1,2$ are designed as 0.1. It can be verified that the relative degree $r_i^a = r_i^b = 2$ for $i = 1,2$. The nominal subsystems can be feedback linearised. For simulation purposes, the desired output signals $y_{id}(t)$ are chosen as

$$y_{1d} = 0.5\sin(t), \qquad y_{2d} = 5e^{-t}. \tag{3.57}$$

It is clear that Assumption 3.2.3 is satisfied. Let

$$\begin{aligned} e_1 &= y_1 - y_{1d}, & e_2 &= y_2 - y_{2d} \\ \dot{e}_1 &= \dot{y}_1 - \dot{y}_{1d}, & \dot{e}_2 &= \dot{y}_2 - \dot{y}_{2d} \\ S_1 &= \dot{e}_1 + a_1 \cdot e_1, & S_2 &= \dot{e}_2 + a_2 \cdot e_2 \end{aligned} \tag{3.58}$$

where the sliding function parameters are chosen as $a_1 = 2$ and $a_2 = 3$. Then from (3.35), the control laws[7] can be described by

$$u_1 = J_1 \left( -\dot{S}_1 + y_1^{(2)} - K_1(x_1)\text{sgn}(S_1) \right) \tag{3.59}$$

and

$$u_2 = J_2 \left( -\dot{S}_2 + y_2^{(2)} - K_2(x_2)\text{sgn}(S_2) \right) \tag{3.60}$$

[7] In the control design, the signum function $sgn(\cdot)$ here is replaced by a sigmoid-like function in order to avoid a singularity in the solution. For more details, see Appendix B.1.

where, based on (3.36), the values of the control gain $K_i(\cdot)$ are chosen as 19.72 for $i = 1, 2$. By direct calculation, Assumptions 3.4.1-3.4.2 as well as the conditions of Theorems 3.4.1-3.4.3 are satisfied. Therefore, the outputs of the closed-loop system formed by applying controllers (3.59)-(3.60) to the system (3.55)-(3.56) can track the desired signals in (3.57) asymptotically.

The tracking results are shown in Fig.3.2 with a good tracking performance as expected. Each angular position $y_i$ of the subsystem can track the ideal reference $y_{id}$ for $i = 1, 2$, at around 2 seconds despite the interactions between the subsystems. The time responses of the states of the system (3.55)-(3.56) are presented in Fig.3.3 where it is clear to see that the system states are bounded. The simulation demonstrates that the results developed in this chapter are effective and in consistence with the theoretical results.

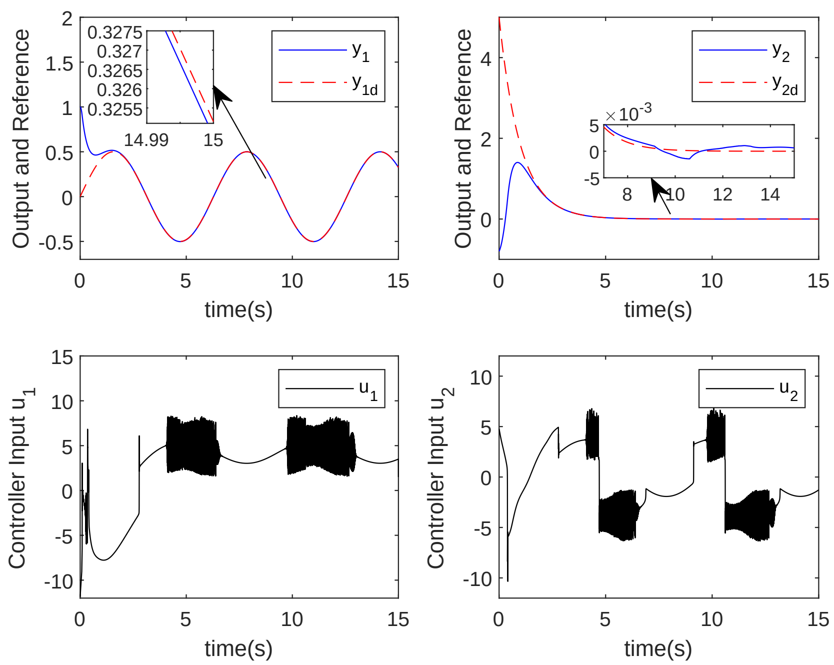


Fig. 3.2 Time responses of system's output, and the desired output (upper), and controller inputs (bottom) of system (3.55)-(3.56).

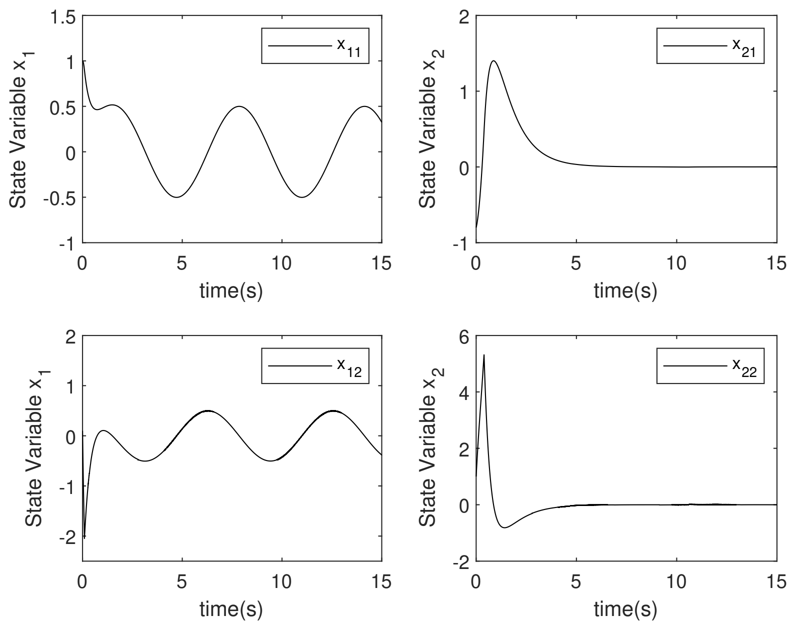


Fig. 3.3 Evolution of state variables of system (3.55)-(3.56).

## 3.6 Summary

In this chapter, a decentralised sliding mode control scheme has been proposed for the output tracking of a specific class of nonlinear interconnected systems. The developed control approach ensures asymptotic output-tracking performance while maintaining all system state variables bounded. One key advantage of the proposed scheme is that it is decentralised, meaning each subsystem can be controlled independently without requiring information from other subsystems. Additionally, the desired reference signals are allowed to be time-varying, making the approach more flexible and applicable to dynamic scenarios. Importantly, the proposed method does not assume linearity for either the interconnected system or the isolated subsystems. This allows for the control of a broader range of nonlinear interconnected systems, providing practical relevance in real-world applications. Furthermore, the developed results can be extended to cases where the isolated subsystems have multiple inputs and multiple outputs, enhancing the applicability of the method to complex large-scale systems. In summary, the method presented in this chapter offers a suitable control solution for a wide class of large-scale nonlinear interconnected systems. Its decentralised nature, ability to handle time-varying reference signals, and flexibility in dealing with nonlinearities make it a valuable contribution to the field.

# Chapter 4

# Decentralised Output Tracking of Interconnected Systems with Unknown Interconnections Using Sliding Mode Control

In this chapter, a class of nonlinear interconnected systems is considered, which involves both matched and unmatched uncertainties. The matched uncertainties and unmatched interconnection terms are assumed to be bounded by known nonlinear functions. To address the tracking problem, a state feedback decentralised control scheme is proposed using sliding mode techniques. The objective is to ensure that the system outputs can track the desired signals uniformly ultimately, even if the desired reference signals are time-varying. To facilitate the design of the sliding surface and decentralised control, appropriate transformations are introduced to reshape the considered system into a new interconnected system with a suitable structure. We then establish a set of conditions to ensure that the designed controller drives the tracking errors onto the sliding surface. Moreover, the sliding motion exhibited by the error dynamics is proven to be uniformly ultimately bounded. To demonstrate the effectiveness and feasibility of the proposed decentralised control strategy, a simulation example on a river quality control problem is presented. The simulation results confirm that the developed control approach yields satisfactory performance and achieves the desired tracking objectives.

## 4.1 Introduction

In recent years, significant progress has been made in the study of large-scale systems with interconnected terms, leading to several interesting findings. For instance, in [44], a large-scale fuzzy system with unknown interconnections is investigated, specifically focusing on systems without matched uncertainties or disturbances. Additionally, other studies have examined interconnected systems, such as [92], [6], [132], [31], and [79], where the emphasis is on matched interconnections without considering unmatched interconnections or uncertainties. Furthermore, certain work has approached large-scale systems by considering simplified or idealized dynamic models, as demonstrated in [105], [112], [97], and [22]. However, these works tend to focus on specific system structures and lack generality in their scope. In [117], decentralised sliding mode control is developed for a fully nonlinear system with a more general structure. Nonetheless, the emphasis of this study is on stabilisation problems rather than addressing tracking control. Specifically, there is a need to address the limitations of the existing studies by considering more comprehensive system structures and addressing tracking control objectives.

Trajectory tracking and output tracking are fundamental aspects of control theory and control engineering. Several research studies have addressed tracking control in different contexts. For instance, in [132], [7], and [6], tracking control results have been obtained. However, many of these studies focus on systems with specific or special structures, as highlighted in [112], [97], [50], and [22]. In the realm of large-scale systems, decentralised tracking control has been investigated in [77], which specifically explores model reference control. Furthermore, adaptive fuzzy techniques are utilized to address tracking control for interconnected systems in [78]. It is worth noting that both [77] and [78] assume linearity for the isolated subsystems. Therefore, it is significant to develop tracking control strategies that can handle a broader range of system structures and address non-linearities and uncertainties commonly encountered in large-scale interconnected systems.

Sliding mode control has gained significant popularity in addressing complex systems with uncertainties, thanks to its unique characteristics ([93], [120], [122], [126]). One of the advantages of sliding mode control is that the dynamics of the sliding mode are often described by a reduced-order system compared to the original system ([115], [14]), simplifying system analysis and design. Additionally, sliding mode control exhibits robustness against matched uncertainties and disturbances. As a result, sliding mode control has found widespread application in tracking problems, leading to numerous achievements in the field. For specific vehicles, trajectory tracking control schemes based on sliding mode techniques are proposed in ([133], [111]). In [80], an output tracking sliding mode control is designed for a linear system. While [129] considers tracking control for nonlinear systems with uncertainties using event-triggered tracking, it only accounts for matched disturbances. [135] explores a tracking problem for a class of large-scale systems with interconnections using sliding mode control, although it requires constant reference signals. Notably, there are very few results addressing output tracking for large-scale nonlinear interconnected systems with unknown interconnections, particularly when the reference signals are time-varying.

In this chapter, we investigate a class of nonlinear interconnected systems that incorporate both unknown matched uncertainty and unknown unmatched nonlinear interconnections. To analyse and design control strategies for output tracking, we introduce suitable coordinate transformations that modify the nominal subsystems of the interconnected system, resulting in systems with special structures. This transformation allows us to separate each subsystem of the transformed system into two distinct parts. Based on the transformed system, we develop the dynamic models for the tracking errors and design a sliding surface that governs the behaviour of the tracking error system. We establish a set of conditions to ensure the uniform ultimate boundedness of the sliding motion corresponding to the tracking errors. To achieve the desired output tracking performance, we propose a decentralised sliding mode control scheme. This control scheme is designed to drive the nonlinear interconnected systems towards the predetermined sliding surface. By utilizing sliding mode control techniques, we aim to achieve robustness against uncertainties and disturbances. The main contributions of this chapter can be summarized as follows:

- The proposed control scheme is decentralised.
- The nominal subsystem of the interconnected systems are nonlinear, and the interconnections are unknown and unmatched.
- The developed results can guarantee that the system states are uniformly ultimately bounded while all the uncertainties and interconnections are bounded.
- The developed results have high robustness against uncertainties and unknown interconnections. Both the bounds on uncertainties and the unknown interconnections have more general nonlinear forms.

In summary, the proposed decentralised control scheme offers a robust and practical solution for large-scale interconnected systems with nonlinear subsystems, unknown interconnections, and uncertainties. The application to a river quality control problem further illustrates the effectiveness and applicability of the developed approach in real-world scenarios.

## 4.2 Problem Formulation

Consider a nonlinear large-scale system formed by $N$ interconnected subsystems

$$\begin{aligned} \dot{x}_i &= A_i x_i + f_i(x_i) + B_i(u_i + \Delta g_i(x_i)) + h_i(x) \\ y_i &= C_i x_i \qquad i = 1, 2, \ldots, N \end{aligned} \tag{4.1}$$

where $x = col(x_1, x_2, \ldots, x_N)$, $x_i \in R^{n_i}$, $u_i \in R^{m_i}$ and $y_i \in R^{m_i}$ represent the states, inputs and outputs of the $i$th subsystem respectively with $m_i < n_i$. The triple $(A_i, B_i, C_i)$ represents constant matrices of appropriate dimension where $B_i$ is of full column rank and $C_i$ is of full row rank. The function $f_i(x_i)$ represents known nonlinear term in the $i$th subsystem which is used to model the nonlinear part of the $i$th subsystem, and the matched uncertainty of the $i$th isolated subsystem is denoted by $\Delta g_i(x_i)$ which is acting in the input channel. The unknown term $h_i(x)$ represents the system interconnection including all unmatched uncertainties. All the nonlinear functions in (4.1) are assumed to be continuous in their arguments to guarantee the existence of solutions of the controlled system (4.1).

The objective of this chapter is, for a given desired signal $y_{id}(t)$, to design a decentralised sliding mode control such that the system output $y_i(t)$ of the controlled system (4.1) can track the desired signal $y_{id}(t)$, i.e. the tracking errors $y_i(t) - y_{id}(t)$ are uniformly ultimately bounded for $i = 1, 2, \ldots, N$ while all the state variables of system (4.1) are bounded.

**Remark 4.2.1** *It should be noted that in this chapter, it is required that system (4.1) is square for simplification of statement, that is, the dimension of each subsystem output is equal to the dimension of the corresponding subsystem input. However, the developed results can be extended to the case when the dimension of subsystem output is greater than the dimension of the subsystem input by slightly modification.*

In order to deal with the tracking problem stated above, some assumptions need to be imposed on the considered interconnected system (4.1).

**Assumption 4.2.1** *All the invariant zeros of the triple $(A_i, B_i, C_i)$ lie in the left half of the complex plane, and $rank(C_i B_i) = m_i$ for $i = 1, 2, \ldots, N$.*

It follows from the section 2.2.5 in chapter 2. Under Assumption 4.2.1, there exists a nonsingular coordinate transformation $z_i = T_i x_i$ such that the triple $(\hat{A}_i, \hat{B}_i, \hat{C}_i)$ with respect to the new coordinates $z_i$ has the following structure

$$\begin{bmatrix} \hat{A}_{i11} & \hat{A}_{i12} \\ \hat{A}_{i21} & \hat{A}_{i22} \end{bmatrix}, \quad \begin{bmatrix} 0 \\ \hat{B}_{i2} \end{bmatrix}, \quad \begin{bmatrix} 0 & I_{i2} \end{bmatrix} \tag{4.2}$$

where $\hat{A}_{i11} \in R^{(n_i-m_i)\times(n_i-m_i)}$ is Hurwitz stable, the square matrices $\hat{B}_{i2} \in R^{m_i\times m_i}$ and $I_{i2} \in R^{m_i\times m_i}$ are nonsingular for $i = 1,2,...,N$.

**Assumption 4.2.2** *Suppose that $f_i(x_i)$ has the decomposition $f_i(x_i) = \Gamma_i(x_i)x_i$, where $\Gamma_i \in R^{n_i\times n_i}$ is a continuous function matrix for $i = 1,2,...,N$.*

**Remark 4.2.2** *If $f_i(0) = 0$ and $f_i$ is sufficiently smooth, then the decomposition $f_i(x_i) = \Gamma_i(x_i)x_i$ is guaranteed. Therefore, the limitation to $f_i(x_i)$ in Assumption 4.2.2 is not strict.*

**Assumption 4.2.3** *There exist known continuous functions $\rho_i(x_i)$ and $\eta_i(x)$ with $\eta_i(0) = 0$, and $\eta_i(\cdot)$ is differentiable at the origin, such that $\| \Delta g_i(x_i) \| \leq \rho_i(x_i)$ and $\| h_i(x) \| \leq \eta_i(\|x\|)$ for $i = 1,2,...,N$.*

**Remark 4.2.3** *If the interconnection $h_i(x)$ in system (4.1) satisfies the condition in Assumption 4.2.3, then from ([118]) and ([121]), it follows that there exists a continuous function $\gamma_i(\cdot)$ such that*

$$\eta_i(\|x\|) = \gamma_i(\|x\|)\|x\|. \tag{4.3}$$

**Remark 4.2.4** *Assumption 4.2.3 requires that the bounds on all uncertainties in system (4.1) are known but they are allowed to be nonlinear. Moreover, the unknown interconnections are allowed to have a more general nonlinear form.*

**Assumption 4.2.4** *The desired output signal $y_{id}(t)$ is differentiable and satisfies*

*(i). $\| y_{id}(t) \| \leq L_{i1}$; (ii). $\| \dot{y}_{id}(t) \| \leq L_{i2}$*

*for $t \in [0,\infty)$, where $L_{i1}$ and $L_{i2}$ are known constants for $i = 1,2,...,N$.*

**Remark 4.2.5** *Assumption 4.2.4 is a limitation on the desired output signals $y_{id}(t)$. It is required that the desired output signal $y_{id}(t)$ and its derivative $\dot{y}_{id}(t)$ are bounded. This assumption is quite standard and can be satisfied in most practical cases.*

## 4.3 A Coordinate Transformation

Consider the nonlinear interconnected system in (4.1). Under Assumption 4.2.1 and from (4.2), there exists a linear nonsingular coordinate transformation $z_i = T_i x_i$ such that in the new coordinate $z = col(z_1, z_2, ..., z_N)$, system (4.1) has the following form

$$\begin{aligned} \dot{z}_i &= \begin{bmatrix} \hat{A}_{i11} & \hat{A}_{i12} \\ \hat{A}_{i21} & \hat{A}_{i22} \end{bmatrix} z_i + \begin{bmatrix} F_{i1}(z_i) \\ F_{i2}(z_i) \end{bmatrix} + \begin{bmatrix} 0 \\ \hat{B}_{i2} \end{bmatrix} (u_i + \Delta g_i(T_i^{-1} z_i)) + \begin{bmatrix} H_{i1}(z) \\ H_{i2}(z) \end{bmatrix} \\ y_i &= \begin{bmatrix} 0 & I_{i2} \end{bmatrix} z_i, \qquad i = 1,2,...,N \end{aligned} \tag{4.4}$$

where $\hat{A}_{i11}$ is stable, the square sub-matrices $\hat{B}_{i2} \in R^{m_i \times m_i}$ are nonsingular. $I_{i2} \in R^{m_i \times m_i}$ is an identity matrix, $col(F_{i1}, F_{i2}) = T_i f_i(x_i)|_{x_i = T_i^{-1} z_i}$ and $F_{i1}(z_i) \in R^{n_i - m_i}$, $F_{i2}(z_i) \in R^{m_i}$. $col(H_{i1}(z), H_{i2}(z)) = T_i h_i(x)|_{x = T^{-1} z}$ and $H_{i1}(z) \in R^{n_i - m_i}$, $H_{i2}(z) \in R^{m_i}$. The entire coordinate transformation matrix $T := diag\{T_1, T_2, ..., T_N\}$.

Since $\hat{A}_{i11}$ is stable for $i = 1, 2, ..., N$, for any $Q_i > 0$, the following *Lyapunov* equation has a unique solution $P_i > 0$

$$\hat{A}_{i11}^{\mathsf{T}} P_i + P_i \hat{A}_{i11} = -Q_i, \qquad i = 1, 2, ..., N. \tag{4.5}$$

Now, in order to fully exploit the structural characteristics, partition $z_i = col(z_{i1}, z_{i2})$ with $z_{i1} \in R^{n_i - m_i}$ and $z_{i2} \in R^{m_i}$. It follows that (4.4) can be described by

$$\dot{z}_{i1} = \hat{A}_{i11} z_{i1} + \hat{A}_{i12} y_i + F_{i1}(z_{i1}, y_i) + H_{i1}(z_{11}, y_1, ..., z_{N1}, y_N) \tag{4.6}$$

$$\begin{aligned} \dot{y}_i = {} & \hat{A}_{i21} z_{i1} + \hat{A}_{i22} y_i + F_{i2}(z_{i1}, y_i) \\ & + \hat{B}_{i2}(u_i + \Delta g_i(T_i^{-1} z_i)) + H_{i2}(z_{11}, y_1, ..., z_{N1}, y_N). \end{aligned} \tag{4.7}$$

From system (4.4) and Assumption 4.2.2,

$$col(F_{i1}, F_{i2}) = T_i \Gamma_i(x_i)|_{x_i = T_i^{-1} z_i} T_i^{-1} col(z_{1i}, y_i). \tag{4.8}$$

In order to reduce conservatism in the later analysis, the functions $F_{i1}(z_{i1}, y_i)$ in system (4.6) are described by

$$F_{i1}(z_{i1}, y_i) = \Gamma_{i11}(z_{i1}, y_i) z_{i1} + \Gamma_{i12}(z_{i1}, y_i) y_i \tag{4.9}$$

where $\Gamma_{i11}(\cdot)$ and $\Gamma_{i12}(\cdot)$ are defined by

$$\begin{bmatrix} \Gamma_{i11}(\cdot) & \Gamma_{i12}(\cdot) \\ \star & \star \end{bmatrix} = T_i \Gamma_i(x_i)|_{x_i = T_i^{-1} z_i} T_i^{-1}$$

and the $\star$s are function matrices which are not necessary to specify. Therefore, (4.6) can be described by

$$\dot{z}_{i1} = \hat{A}_{i11} z_{i1} + \hat{A}_{i12} y_i + \Gamma_{i11}(z_{i1}, y_i) z_{i1} + \Gamma_{i12}(z_{i1}, y_i) y_i + H_{i1}(z_{11}, y_1, ..., z_{N1}, y_N) \tag{4.10}$$

where $\Gamma_{i11}(\cdot)$ and $\Gamma_{i12}(\cdot)$ satisfy (4.9).

## 4.4 Sliding Mode Tracking Control Design

In this section, a sliding surface in terms of output tracking errors will be designed based on the system structure analysis in the previous section. Then sliding mode controllers will be designed to implement the output tracking.

### 4.4.1 Hyperplane Design

Consider the situation when the desired output signal $y_{id}(t)$ satisfies Assumption 4.2.4. For system (4.1), the output tracking errors $e_i$ are defined by

$$e_i(t) = y_i(t) - y_{id}(t), \qquad i = 1,2,...,N. \tag{4.11}$$

Then, it follows that

$$\dot{e}_i(t) = \dot{y}_i(t) - \dot{y}_{id}(t), \qquad i = 1,2,...,N. \tag{4.12}$$

Combining with (4.7), (4.10), and (4.12), a new system comprising $z_{i1}$ and $e_i$ can be developed by

$$\dot{z}_{i1} = \hat{A}_{i11}z_{i1} + \hat{A}_{i12}y_i + \Gamma_{i11}(z_{i1},y_i)z_{i1} + \Gamma_{i12}(z_{i1},y_i)y_i + H_{i1}(z_{11},y_1,...,z_{N1},y_N) \tag{4.13}$$

$$\begin{aligned}\dot{e}_i = {} & \hat{A}_{i21}z_{i1} + \hat{A}_{i22}(e_i + y_{id}) + F_{i2}(z_{i1},y_i) \\ & + \hat{B}_{i2}(u_i + \Delta g_i(T_i^{-1}col(z_{i1},y_i))) + H_{i2}(z_{11},y_1,...,z_{N1},y_N) - \dot{y}_{id}(t)\end{aligned} \tag{4.14}$$

for $i = 1,2,...,N$.

From Assumption 4.2.3 and (4.3), it is easy to find functions $\gamma_{i1}(\cdot)$ and $\gamma_{i2}(\cdot)$ depending on $\eta_i(\cdot)$ and $T$ such that the following inequalities

$$\|H_{i1}(z_{11},y_1,...,z_{N1},y_N)\| \leq \gamma_{i1}(\|T^{-1}col(z_{11},y_1,...,z_{N1},y_N)\|)(\sum_{j=1}^{N}\|z_{j1}\| + \sum_{j=1}^{N}\|y_j\|) \tag{4.15}$$

$$\|H_{i2}(z_{11},y_1,...,z_{N1},y_N)\| \leq \gamma_{i2}(\|T^{-1}col(z_{11},y_1,...,z_{N1},y_N)\|)(\sum_{j=1}^{N}\|z_{j1}\| + \sum_{j=1}^{N}\|y_j\|) \tag{4.16}$$

hold for $i = 1, 2, \dots, N$. For the system (4.13)-(4.14), the following sliding surface can be defined as

$$\begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_N \end{bmatrix} = 0. \tag{4.17}$$

Then, the sliding mode dynamics have the following form according to the structure of (4.13)-(4.14)

$$\dot{z}_{i1} = \hat{A}_{i11} z_{i1} + \hat{A}_{i12} y_{id} + \Gamma_{i11}(z_{i1}, y_{id}) z_{i1} + \Gamma_{i12}(z_{i1}, y_{id}) y_{id} + H_{i1}(z_{11}, y_{1d}, \dots, z_{N1}, y_{Nd}) \tag{4.18}$$

for $i = 1, 2, \dots, N$.

**Remark 4.4.1** *When the sliding motion occurs, the equation (4.17) holds. From (4.11) and (4.15), it follows that on the sliding surface (4.17), the inequalities*

$$\begin{aligned} &\| H_{i1}(z_{11}, y_{1d}, \dots, z_{N1}, y_{Nd}) \| \\ &\qquad \le \gamma_{i1}(\|T^{-1} col(z_{11}, y_{1d}, \dots, z_{N1}, y_{Nd}\|))(\sum_{j=1}^{N} \| z_{j1} \| + \sum_{j=1}^{N} \| y_{jd} \|) \end{aligned} \tag{4.19}$$

*hold for $i = 1, 2, \dots, N$.*

Obviously, the sliding mode dynamic (4.18) is a reduced-order interconnected system composed of $N$ subsystems whose dimension is $n_i - m_i$.

Next, a stability result will be presented for the interconnected system (4.18).

**Theorem 4.4.1** *Consider the sliding mode dynamic given in (4.18). Under Assumptions 4.2.1-4.2.4, the sliding mode is uniformly ultimately bounded if there exists a domain $\Omega$ of the origin such that $M^\top + M > 0$ in $\Omega \backslash \{0\}$ where $M := (m_{ij})_{N \times N}$ and for $i, j = 1, 2, \dots, N$.*

$$m_{ij} = \begin{cases} \lambda_{min}(Q_i) - \| R_i(\cdot) \| - 2 \| P_i \| \gamma_{i1}(\cdot), & i = j \\ -2 \| P_i \| \gamma_{i1}(\cdot), & i \neq j \end{cases} \tag{4.20}$$

*with $P_i$ and $Q_i$ satisfying (4.5), and*

$$R_i(\cdot) := \Gamma_{i11}(z_{i1}, y_{id})^\top P_i + P_i \Gamma_{i11}(z_{i1}, y_{id})$$

*where $\Gamma_{i11}(z_{i1}, y_i)$ is given by (4.4) and $\gamma_{i1}(\cdot)$ is determined by (4.19).*

**Proof** *From the analysis above, it only needs to prove that system (4.18) is uniformly ultimately bounded. For system (4.18), consider the following Lyapunov function candidate*

$$V(z_{11}, z_{21}, \ldots, z_{N1}) = \sum_{i=1}^{N} (z_{i1})^{\top} P_i z_{i1} \tag{4.21}$$

*where $P_i$ satisfies (4.5).*

*Then, the time derivative of $V(z_{11}, z_{21}, \ldots, z_{N1})$ along the trajectories of system (4.18) is given by*

$$\begin{aligned}
&\dot{V}(z_{11}, z_{21}, \ldots, z_{N1}) \\
=&\sum_{i=1}^{N} [(\dot{z}_{i1})^{\top} P_i z_{i1} + z_{i1}^{\top} P_i \dot{z}_{i1}] \\
=&\sum_{i=1}^{N} [(\hat{A}_{i11} z_{i1} + \hat{A}_{i12} y_{id} + \Gamma_{i11}(z_{i1}, y_{id}) z_{i1} + \Gamma_{i12}(z_{i1}, y_{id}) y_{id} \\
&+ H_{i1}(z_{11}, y_{1d}, \ldots, z_{N1}, y_{Nd}))^{\top} P_i z_{i1} + z_{i1}^{\top} P_i (\hat{A}_{i11} z_{i1} + \hat{A}_{i12} y_{id} + \Gamma_{i11}(z_{i1}, y_{id}) z_{i1} \\
&+ \Gamma_{i12}(z_{i1}, y_{id}) y_{id} + H_{i1}(z_{11}, y_{1d}, \ldots, z_{N1}, y_{Nd}))] \\
=&\sum_{i=1}^{N} [z_{i1}^{\top} \hat{A}_{i11}^{\top} P_i z_{i1} + y_{id}^{\top} \hat{A}_{i12}^{\top} P_i z_{i1} + z_{i1}^{\top} \Gamma_{i11}(z_{i1}, y_{id})^{\top} P_i z_{i1} + y_{id}^{\top} \Gamma_{i12}(z_{i1}, y_{id})^{\top} P_i z_{i1} \\
&+ H_{i1}(z_{11}, y_{1d}, \ldots, z_{N1}, y_{Nd})^{\top} P_i z_{i1} + z_{i1}^{\top} P_i \hat{A}_{i11} z_{i1} + z_{i1}^{\top} P_i \hat{A}_{i12} y_{id} \\
&+ z_{i1}^{\top} P_i \Gamma_{i11}(z_{i1}, y_{id}) z_{i1} + z_{i1}^{\top} P_i \Gamma_{i12}(z_{i1}, y_{id}) y_{id} + z_{i1}^{\top} P_i H_{i1}(z_{11}, y_{1d}, \ldots, z_{N1}, y_{Nd})] \\
=&\sum_{i=1}^{N} \{-z_{i1}^{\top} Q_i z_{i1} + z_{i1}^{\top} [\Gamma_{i11}(z_{i1}, y_{id})^{\top} P_i + P_i \Gamma_{i11}(z_{i1}, y_{id})] z_{i1} + 2 z_{i1}^{\top} P_i \hat{A}_{i12} y_{id} \\
&+ 2 z_{i1}^{\top} P_i \Gamma_{i12}(z_{i1}, y_{id}) y_{id} + 2 z_{i1}^{\top} P_i H_{i1}(z_{11}, y_{1d}, \ldots, z_{N1}, y_{Nd})\}
\end{aligned} \tag{4.22}$$

*where (4.5) is used to establish the last equality above. By (4.19) and $(i)$ in Assumption 4.2.4, it follows that*

$$
\begin{aligned}
&\dot{V}(z_{11}, z_{21}, \ldots, z_{N1}) \\
&\leq \sum_{i=1}^{N} \{-\lambda_{min}(Q_i) \parallel z_{i1} \parallel^2 + \parallel \Gamma_{i11}(z_{i1}, y_{id})^{\mathsf{T}} P_i + P_i \Gamma_{i11}(z_{i1}, y_{id}) \parallel \parallel z_{i1} \parallel^2 \\
&\quad + 2 \parallel z_{i1} \parallel \parallel P_i \parallel \parallel \hat{A}_{i12} y_{id} \parallel + 2 \parallel z_{i1} \parallel \parallel P_i \parallel \parallel \Gamma_{i12}(z_{i1}, y_{id}) y_{id} \parallel \\
&\quad + 2 \parallel z_{i1} \parallel \parallel P_i \parallel \parallel H_{i1}(z_{11}, y_{1d}, \ldots, z_{N1}, y_{Nd}) \parallel \} \\
&= -\sum_{i=1}^{N} \{\lambda_{min}(Q_i) - \parallel R_i(\cdot) \parallel - 2 \parallel P_i \parallel \gamma_{i1}(\cdot)\} \parallel z_{i1} \parallel^2 \\
&\quad + 2 \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \parallel P_i \parallel \parallel z_{i1} \parallel \gamma_{i1}(\cdot)(\parallel z_{j1} \parallel + L_{i1}) \\
&\quad + 2 \sum_{i=1}^{N} (\parallel \hat{A}_{i12} y_{id} \parallel + \parallel \Gamma_{i12}(z_{i1}, y_{id}) y_{id} \parallel) \cdot \parallel P_i \parallel \parallel z_{i1} \parallel \\
&\leq -\frac{1}{2} \lambda_{min}(M^{\mathsf{T}} + M) \sum_{i=1}^{N} \parallel z_{i1} \parallel^2 \\
&\quad + 2 \sum_{i=1}^{N} (\parallel \hat{A}_{i12} y_{id} \parallel + \parallel \Gamma_{i12}(z_{i1}, y_{id}) y_{id} \parallel + \gamma_{i1}(\cdot) L_{i1}) \cdot \parallel P_i \parallel \parallel z_{i1} \parallel \\
&= -\frac{1}{2} \sum_{i=1}^{N} \{\lambda_{min}(M^{\mathsf{T}} + M) \parallel z_{i1} \parallel \\
&\quad - 4(\parallel \hat{A}_{i12} y_{id} \parallel + \parallel \Gamma_{i12}(z_{i1}, y_{id}) y_{id} \parallel + \gamma_{i1}(\cdot) L_{i1}) \parallel P_i \parallel\} \parallel z_{i1} \parallel
\end{aligned}
\tag{4.23}
$$

*where the matrix $M$ is defined in (4.20). Under the Assumption 4.2.4, $\parallel y_{id}(t) \parallel \leq L_{i1}$. It is clear to check $\dot{V} \leq 0$, if*

$$
\parallel z_{i1} \parallel \geq \frac{4(\parallel \hat{A}_{i12} L_{i1} \parallel + \parallel \Gamma_{i12}(z_{i1}, y_i) L_{i1} \parallel + \gamma_{i1}(\cdot) L_{i1}) \parallel P_i \parallel}{\lambda_{min}(M^{\mathsf{T}} + M)} \tag{4.24}
$$

*for $i = 1, 2, .., N$. Hence, the conclusion holds.*

**Remark 4.4.2** *From (4.24), it is clear to see that the ultimate bound of the sliding mode dynamics is affected by the upper bound of the desired output signal $y_{id}(t)$, the system submatrix $\hat{A}_{i12}$, the nonlinearity of the system $\Gamma_{i12}$ and the bound of the interconnections $\gamma_{i1}$.*

### 4.4.2 Control Law Construction

The objective is now to design a feedback sliding mode control such that the system state is driven to the sliding surface (4.17).

For the interconnected system (4.13)-(4.14), the reachability condition ([115, 117]) is described by

$$\sum_{i=1}^{N} \frac{e_i^\top(t)\dot{e}_i(t)}{\| e_i(t) \|} < 0. \tag{4.25}$$

Then, the following control law is proposed

$$\begin{aligned} u_i = &- B_{i2}^{-1} sgn(e_i)\{\| \hat{A}_{i21}z_{i1} \| + \| \hat{A}_{i22}y_i \| + \| F_{i2}(z_{i1}, y_i) \| \\ &+ \| \hat{B}_{i2} \| \rho_i(z_{i1}, y_i) + k_i(z_{i1}, y_i) + L_{i2}\} \end{aligned} \tag{4.26}$$

for $i = 1, 2, \ldots, N$, where $e_i$ and $L_{i2}$ are defined by (4.11) and Assumption 4.2.4, respectively. $k_i(z_{i1}, y_i)$ is the control gain to be designed later.

**Theorem 4.4.2** *Consider the nonlinear interconnected system (4.13)–(4.14) and Assumptions 4.2.2-4.2.4. The controller (4.26) drives the system (4.13)–(4.14) to the composite sliding surface (4.17) and maintains a sliding motion on it if the controller gains $k_i(z_{i1}, y_i)$ satisfy*

$$\sum_{i=1}^{N} k_i(z_{i1}, y_i) > \sum_{i=1}^{N} \left(\gamma_{i2}(\cdot) \sum_{j=1}^{N} (\| z_{j1} \| + \| y_i \|)\right) \tag{4.27}$$

*where $\gamma_{i2}$ are defined in (4.16) for $i = 1, 2, \ldots, N$.*

**Proof** *It is necessary to prove that the reachability condition (4.25) is satisfied. From (4.14) and Assumption 4.2.2,*

$$\begin{aligned} \dot{e}_i = &\hat{A}_{i21}z_{i1} + \hat{A}_{i22}y_i + F_{i2}(z_{i1}, y_i) \\ &+ \hat{B}_{i2}(u_i + \Delta g_i(T_i^{-1}col(z_{i1}, y_i))) + H_{i2}(z_{11}, y_1, \ldots, z_{N1}, y_N) - \dot{y}_{id} \end{aligned} \tag{4.28}$$

*for $i = 1, 2, \ldots, N$. Substituting (4.26) into (4.28), it follows*

$$\begin{aligned} \frac{e_i^\top \dot{e}_i}{\| e_i \|} = &\frac{e_i^\top}{\| e_i \|}[\hat{A}_{i21}z_{i1} + \hat{A}_{i22}y_i + F_{i2}(z_{i1}, y_i) + \hat{B}_{i2}\Delta g_i(T_i^{-1}col(z_{i1}, y_i)) + H_{i2}(\cdot) \\ &- \dot{y}_{id}] - \| \hat{A}_{i21}z_{i1} \| - \| \hat{A}_{i22}y_i \| - \| F_{i2}(z_{i1}, y_i) \| - \| \hat{B}_{i2} \| \rho_i(z_{i1}, y_i) \\ &- k_i(z_{i1}, y_i) - L_{i2}. \end{aligned} \tag{4.29}$$

*It is clear to see*

$$\| \hat{A}_{i21}z_{i1} + \hat{A}_{i22}y_i + F_{i2}(z_{i1}, y_i) \| \leq \| \hat{A}_{i21}z_{i1} \| + \| \hat{A}_{i22}y_i \| + \| F_{i2}(z_{i1}, y_i) \| . \tag{4.30}$$

*From Assumptions 4.2.3-4.2.4,*

$$\| \hat{B}_{i2}\Delta g_i(T_i^{-1}col(z_{i1}, y_i)) \| \leq \| \hat{B}_{i2} \| \rho_i(z_{i1}, y_i) \tag{4.31}$$

$$\| H_{i2}(z_{11}, y_1, \ldots, z_{N1}, y_N) \| \leq \gamma_{i2}(\cdot) \sum_{j=1}^{N} (\| z_{j1} \| + \| y_i \|) \tag{4.32}$$

$$\| \dot{y}_{id} \| \leq L_{i2}. \tag{4.33}$$

*Substituting the above four inequalities (4.30)-(4.33) into (4.29), it follows*

$$\sum_{i=1}^{N} \frac{e_i^{\top}(t)\dot{e}_i(t)}{\| e_i(t) \|} < -\sum_{i=1}^{N} k_i(z_{i1}, y_i) + \sum_{i=1}^{N} \Big(\gamma_{i2}(\cdot) \sum_{j=1}^{N} (\| z_{j1} \| + \| y_i \|)\Big).$$

*From (4.27), the reachability condition (4.25) is satisfied. Hence, the result holds.*

**Remark 4.4.3** *Theorem 4.4.1 shows that the sliding mode dynamic (4.18) which is an interconnected system, is uniformly ultimately bounded. Theorem 4.4.2 shows that the reachability condition is satisfied. According to the sliding mode theory, Theorems 4.4.1 and 4.4.2 combined together show that the closed-loop system is uniformly ultimately bounded.*

Remark 4.4.3 shows that the closed-loop systems formed by applying the control (4.26) to the systems (4.13)–(4.14) are uniformly ultimately bounded, which implies that the variables $\|z_{i1}(t)\|$ and $\|e_i(t)\|$ are bounded for $i = 1, 2, \ldots, N$. Further, from $e_i(t) = y_i(t) - y_{id}(t)$ and the Assumption 4.2.4 which guarantees that $y_{id}(t)$ is bounded, it is straightforward to see that $y_i(t)$ are bounded due to

$$y_i(t) = e_i(t) + y_{id}(t)$$

for $i = 1, 2, \ldots, N$. Therefore, all the state variables of the system (4.6)–(4.7) are bounded. Further, from $x_i = T_i^{-1}z_i$, the state variables $x_i$ of system (4.1) are bounded. This shows that the designed decentralised control (4.26) can not only makes the system outputs to track the desired reference signals but also keep all the system state variables bounded.

## 4.5 Example: Water Quality Management

In this section, the decentralised control scheme developed in this chapter will be applied to a river pollution problem [60] as shown in the Fig. 4.1. The water quality of a river is mainly dependent upon the concentrations of oxygen and pollutants. In a simplified manner, this problem can be stated as the task of controlling the pollutants discharged at different places along the river in such a way that the river pollution remains within a given tolerance.

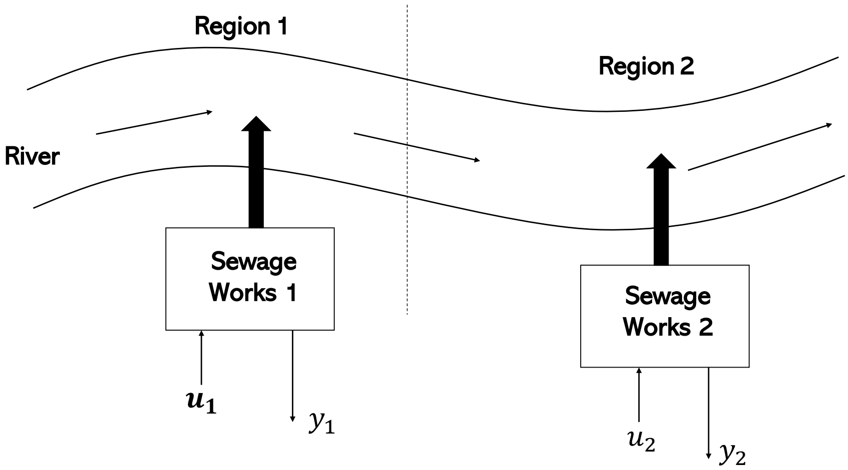


Fig. 4.1 River with sewage

Assume that the river has two regions and each region has a sewage station. Then, the river pollution system can be described by a nonlinear interconnected systems as follows (see, [115] for no delay case)

$$\dot{x}_1 = \underbrace{\begin{bmatrix} -1.32 & 0 \\ -0.32 & -1.2 \end{bmatrix}}_{A_1} x_1 + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B_1} (u_1 + \Delta g_1(\cdot)) + h_1(x) \tag{4.34}$$

$$y_1 = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C_1} x_1 \tag{4.35}$$

$$\dot{x}_2 = \underbrace{\begin{bmatrix} -1.32 & 0 \\ -0.32 & -1.2 \end{bmatrix}}_{A_2} x_2 + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B_2} (u_2 + \Delta g_2(\cdot)) + h_2(x) \tag{4.36}$$

$$y_2 = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C_2} x_2 \tag{4.37}$$

where $x_1 = col(x_{11}, x_{12})$, $x_2 = col(x_{21}, x_{22})$ and $x = col(x_1, x_2)$. The variables $x_{i1}$ and $x_{i2}$ for $i = 1, 2$ represent the concentration of biochemical oxygen demand (BOD) and the concentration of dissolved oxygen respectively, the controllers $u_i$ are the BOD of the effluent discharge into the river, $\Delta g_i$ represent any matched uncertainties and $h_i$ represent interconnections respectively for $i = 1,2$. It is assumed that the concentrations of BOD for the two regions are measurable.

In this example, according to (4.1) the nonlinear term $f_1(x_1) = f_2(x_2) = 0$, so the Assumption 4.2.2 is not required. Moreover, it can be verified that $rank(C_iB_i) = 1 = m_i$ for $i = 1,2$. So the Assumption 1 is satisfied. Some suitable coordinate transformation matrices $T_i$ are introduced as below ($z_i = T_ix_i$)

$$T_1 = T_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then, the system (4.34)-(4.37) in $z$ coordinates can be given by

$$\dot{z}_1 = \underbrace{\begin{bmatrix} -1.2 & -0.32 \\ 0 & -1.32 \end{bmatrix}}_{\hat{A}_1} z_1 + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\hat{B}_1} (u_1 + \Delta G_1(z_1)) + H_1(z) \tag{4.38}$$

$$y_1 = \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{\hat{C}_1} z_1 \tag{4.39}$$

$$\dot{z}_2 = \underbrace{\begin{bmatrix} -1.2 & -0.32 \\ 0 & -1.32 \end{bmatrix}}_{\hat{A}_2} z_2 + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\hat{B}_2} (u_2 + \Delta G_1(z_1)) + H_2(z) \tag{4.40}$$

$$y_2 = \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{\hat{C}_2} z_2 \tag{4.41}$$

For simulation purpose, the matched uncertainties $\Delta G_1(\cdot)$ and $\Delta G_2(\cdot)$ are chosen to satisfy

$$|\Delta G_1(\cdot)| \leq |-13.2z_{12}|, \quad |\Delta G_2(\cdot)| \leq |\cos^2(z_{22})| \tag{4.42}$$

and the interconnected terms are set to satisfy

$$\| H_1 \| \leq |z_{22}|, \quad \| H_2 \| \leq |0.9z_{12}|. \tag{4.43}$$

Combining (4.42)-(4.43), it is clear that the Assumption 4.2.3 is satisfied. And the sliding surfaces $S_i$ are

$$\dot{z}_{i1} = -1.2z_{i1} - 0.32z_{i2}. \qquad i = 1,2$$

The desired output signals $y_{id}$ are set as

$$y_{1d} = 2e^{-t}, \qquad y_{2d} = \sin(0.5t) + 1.$$

It is clear that the Assumption 4.2.4 is satisfied. Let

$$L_{12} = 2, \qquad L_{22} = 0.5.$$

From (4.26), the proposed sliding mode controllers are as follows

$$u_1 = -sgn(y_1 - y_{1d})(|1.32z_{12}| + |13.2z_{12}| + 3) \tag{4.44}$$
$$u_2 = -sgn(y_2 - y_{2d})(|1.32z_{22}| + |\cos^2(z_{22})| + 2.3). \tag{4.45}$$

According to (4.5), choose $Q_1 = Q_2 = 1$. Combining (4.34)-(4.36), $A_{i11} = -1.2$ for $i = 1, 2$. Then

$$P_1 = P_2 = 0.416.$$

By direct calculation, it follows from (4.20) that

$$M^\top + M = \begin{bmatrix} -1.664\gamma_{11} + 2 & -0.832(\gamma_{11} + \gamma_{21}) \\ -0.832(\gamma_{11} + \gamma_{21}) & -1.664\gamma_{21} + 2 \end{bmatrix}.$$

According to (4.19), (4.38) and (4.40),

$$\gamma_{11} = 6 \cdot \sin^2(z_{11}), \quad \gamma_{21} = 2 \cdot \cos(z_{21}) + 3.$$

By direct verification, it is straightforward to check that $M^\top + M > 0$ in the domain $\Omega$ of the origin satisfying $\Omega = \{(z_{11}, z_{21}) |\ |z_{11}| \leq 5.2,\ |z_{21}| \leq 3.9\}$. According to (4.23) for this example

$$\dot{V}(z_{11}, z_{21}) \leq 0 \tag{4.46}$$

if $|z_{11}| \geq 0.3$ and $|z_{21}| \geq 0.25$. Therefore, system (4.34)-(4.37) is uniformly ultimately bounded.

The tracking results are shown in Fig. 4.2, which offers a high tracking performance. The concentration of biochemical oxygen demand (BOD) of each subsystem $y_i$ can track the ideal reference $y_{id}$ using the controller from (4.44)-(4.45), even in the presence of uncertainties. The time response of the states and tracking errors of the system (4.34)-(4.37) are shown in Figs. 4.3-4.4. which indicates that the system states are bounded. Simulation results demonstrate that the method developed in this chapter is effective. It is worth noting that when the subsystems track the ideal signals within the first approximately 3 seconds, it indicates that the system state has reached the sliding mode surface. Subsequently, the system exhibits minor chattering in the vicinity of the sliding mode surface, although these chattering are not clearly visible in Fig. 4.4, they do exist in reality.

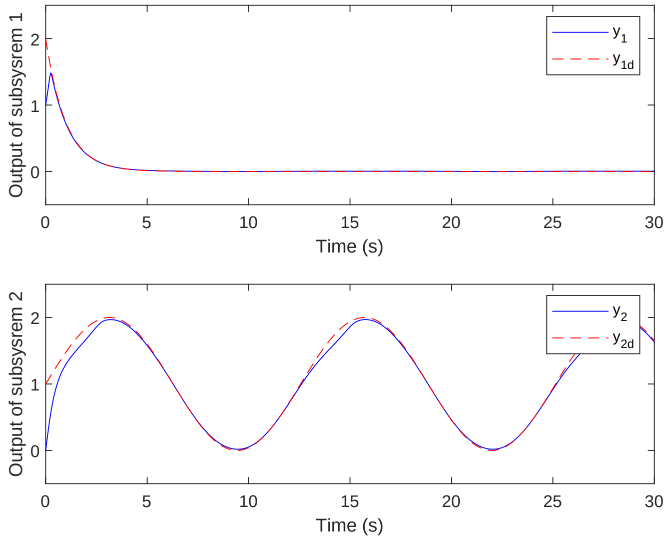


Fig. 4.2 Time responses of system outputs and desired outputs.

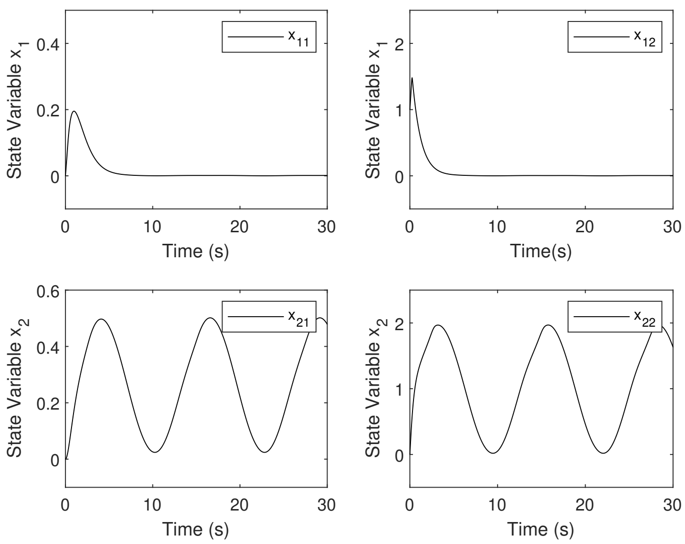


Fig. 4.3 Time response of system state variables.

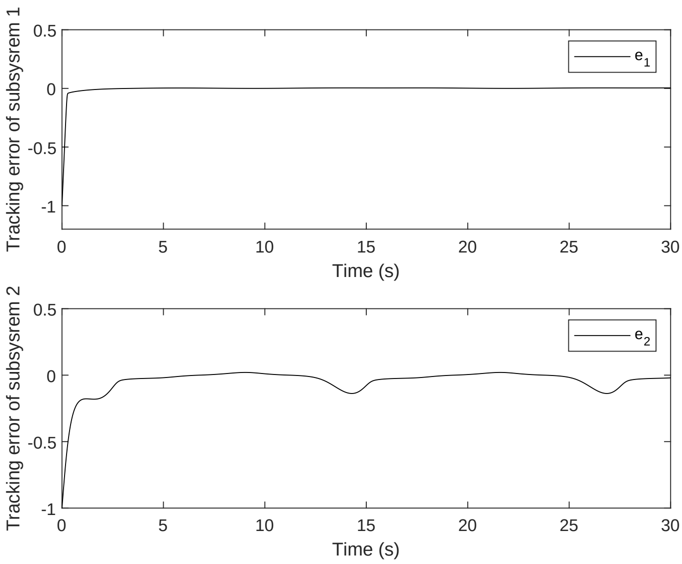


Fig. 4.4 Time response of the tracking errors.

## 4.6 Summary

This chapter has presented a sliding mode control strategy to deal with the output tracking problem of a class of large-scale systems with unmatched nonlinear interconnections. The desired reference signals are allowed to be time-varying. A decentralised sliding mode control scheme has been proposed to satisfy the reachability condition. This drives the interconnected system onto the pre-designed sliding surface. A set of conditions is introduced to guarantee that the output tracking errors are uniformly ultimately bounded while all the state variables of the interconnected system are bounded. The application of the developed results to a river pollution control system has demonstrated that the proposed approach is effective and practicable.

# Chapter 5

# Speed Tracking Control of Underactuated High-Speed Trains Using Sliding Mode Techniques

This chapter focuses on dealing with the speed-tracking problem of underactuated high-speed train systems. Proper coordinate transformations are introduced to explore systems structure to facilitate system analysis and control design. Then, system structure-based analysis is presented to decompose the high-speed train models into large-scale interconnected systems with special structure where both internal and external uncertainties are considered to reflect the practical situation. Decentralised sliding mode controllers are designed such that the resulting closed-loop systems are uniformly ultimately bounded to guarantee that the output tracking errors are uniformly bounded while all the system states are bounded. A simulation is carried out on a high-speed train with eight carriages to demonstrate that the proposed controller is effective and has high robustness against disturbances.

## 5.1 Introduction

Trains have emerged as a revolutionary mode of transportation in recent decades, transforming the way people travel and significantly impacting the global transportation landscape. With their unparalleled speed, efficiency, and numerous benefits, trains have become a vital component of modern societies. Among the trains in operation, the underactuated high-speed trains (see, e.g., [67, 127]) stand out as one of the most typical examples with the physical distribution of power units at the midpoint of the entire train length. These “units" can be individual units or multiple groups [45, 86], remotely controlled by a guiding locomotive.

Underactuated high-speed trains are special, not only because of their train structure, but also because they are a type of transportation vehicle that runs on rails. Since all trains run on finite rail lines, there are strict requirements for speed control between trains to ensure safety. Typically, a train line is divided into many small intervals, and the speed of the train is calculated automatically for each interval to avoid potential traffic accidents caused by train collisions. Therefore, speed tracking is essential in control of high-speed trains (see [7],[56],[66]). An optimization approach for the speed trajectory of high-speed train is studied in [88] where the high-speed train is considered as a whole system, and the research does not focus on the dynamic performance between the carriages of the running train. The railway system (see, e.g., [29], [35]), as a very special large-scale system, has played a very important role in human development. A train usually consists of many carriages where each carriage only receives dynamic influences from its adjacent carriages directly. The enthusiasm for research on such special large-scale systems continues to grow. For example, tracking and braking control schemes for high-speed trains subject to traction and braking failures are studied in [108], where the train model studied is based on a rather special chain structure which can be considered as one of typical large-scale interconnected systems. However, the research on the robustness and speed tracking of high-speed trains from a decentralised perspective is still limited.

When studying large-scale systems, it is required to consider disturbances in the system modelling, in order to make the model more realistic. Sliding mode control is always an optimal choice (see, e.g., [70], [69]) for dealing with disturbances. By using some coordinate transformations [14], a large-scale system can be transformed into a sliding mode regular form. At this point, the matched disturbance is brought into the zero space of the system in the new coordinate and thus does not affect the output performance of the system. Like in [74], a nonlinear sliding mode controller is studied to deal with an automated highway system with some certain disturbance, under which every vehicle is controlled within a safe distance. Another example of designing a controller based on sliding mode techniques is the interconnected power system. The significant penetration of renewable energy resources and ever-rising load demand often affect the smooth operation of power systems. Therefore, in the control field of power systems, anti-interference and robust control are particularly prominent. [124] takes advantage of the inherent advantages of sliding mode control and combines it with the design of an observer to supplement the shortcomings of sliding mode techniques, in which the proposed controller ultimately enables the closed-loop power system to have a good robustness.

It should be noted that tracking control of large-scale systems have received great attention and great achievement has been made recently. In [78], adaptive fuzzy control is employed to solve the tracking problem of large-scale systems. Additionally, [77] also studied tracking of large-scale systems, but the ideal signal in [77] is generated and referenced by another dynamic model with a similar structure. What's more, in [78, 77] strict limitations on the structure of large-scale systems are required. A four-body vehicle system is considered in [68] and the tracking problem is considered, but a centralised control approach is used which may not work in reality when information transfer between subsystems are blocked. It should be noted that the research work using decentralised sliding mode control to solve the high-speed train tracking control problem are very little.

Starting from establishing a mathematical model of a high-speed train based on Newton's second law and some existed knowledge, this chapter takes into account air resistance for certain carriages due to the high speed of the train, in order to make the model more general. Unlike some other studies that treat the entire train as a single entity which neglect the interaction between carriages, this chapter mainly focuses on the stability research between each carriage of the train. By cleverly transforming the mathematical model, the train model is divided into smaller subsystems, allowing mature large-scale system theory to be used to control the complex internal dynamics of the train. A decentralised control based on sliding mode technique is proposed to address the speed tracking problem of high-speed trains. Even with external and internal disturbances, the system can still exhibit a good robustness. Finally, a high-speed train system with eight carriages is used to demonstrate the obtained theoretical results. The simulation result shows that the control theory proposed in this chapter is consistent with expectations.

## 5.2 Multi-Body System

### 5.2.1 Description of a Carriage Motion

Consider the $i$th carriage in a high-speed train as shown in Fig. 5.1 where the carriage is connected by springs and dampers with its adjacent carriages.

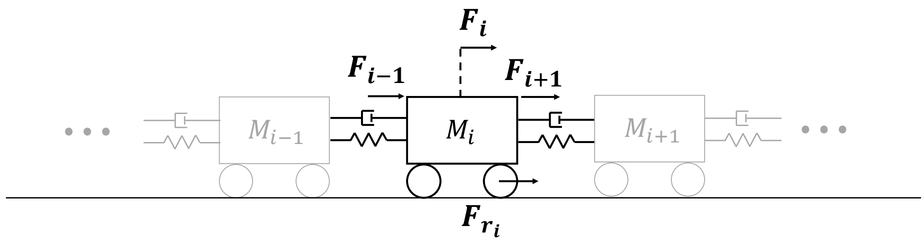


Fig. 5.1 Mechanical analysis of the $i$th carriage

According to the Newton's second law of motion and [68], its dynamical equation is given by

$$M_i\ddot{z}_i(t) = F_i(t) + F_{i-1}(t) - F_{i+1}(t) - F_{r_i}(t) \tag{5.1}$$

where $M_i$ is the mass of the $i$th carriage. $z_i$ denotes the corresponding distance, and $F_i$ is the engine traction force. $F_{i-1}$ and $F_{i+1}$ are the restoring forces caused by the connection of the adjacent carriages. $F_{r_i}$ is the resistive force.

Due to the relatively small displacement, the restoring force is mathematically modelled as a linear function which is given by

$$F_{i+1}(t) = k_i(z_i - z_{i+1}) + d_i(\dot{z}_i - \dot{z}_{i+1}) \tag{5.2}$$

where the known constants $k_i$ and $d_i$ denote the spring and damping parameters respectively. $z_i$, $z_{i+1}$ and $\dot{z}_i$, $\dot{z}_{i+1}$ are the $i$th and $(i+1)$th distances and the corresponding speeds respectively.

**Assumption 5.2.1** *The resistance force $F_{r_i}(t)$ satisfies*

$$F_{r_i}(t) = b_{io} + b_{iv}v_i(t) + b_{ia}v_i^2(t) \tag{5.3}$$

*where $v_i(t)$ is the ith carriage speed and the parameters $b_{io}, b_{iv}$ and $b_{ia}$ are positive constants.*

**Remark 5.2.1** *The first two terms are considered as rolling mechanical resistance, and the last term $b_{ia}v_i^2(t)$ denotes aerodynamic drag. Similar to the work (see, e.g. [11, 123]), aerodynamic drag is included in the first carriage, which is a motor carriage (locomotive) in the model of this chapter. Note that only the rolling resistance is considered for other carriages (except the locomotive).*

Consider a class of high-speed train whose engines are geographically distributed. Its carriages are shown in Fig. 5.2 where the trailer carriages do not have a motor, while the motor carriages do.

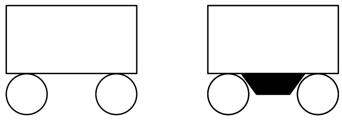

Fig. 5.2 Trailer carriage (Left) & motor carriage (Right)

For convenience of description in modelling the train system, the first carriage is assumed to be a motor carriage. The train is then divided into a number of subsystems. Each subsystem contains only one motor carriage and any other carriages within the subsystem should be trailer carriages situated after the motor carriage. Based on these rules, it is straightforward to see that the number of subsystems depends on the number of motor carriages which the train has.

Assume that there are $\bar{n}_i$ carriages in the $i$th subsystem for $i = 1, 2, ..., N$ which is shown as

$$M_{i,1}, M_{i,2}, ..., M_{i,\bar{n}_i}$$

where $M_{i,j_i}$ denotes the mass of the $j_i$th carriage in the $i$th subsystem, $j_i = 1, 2, ..., \bar{n}_i$.

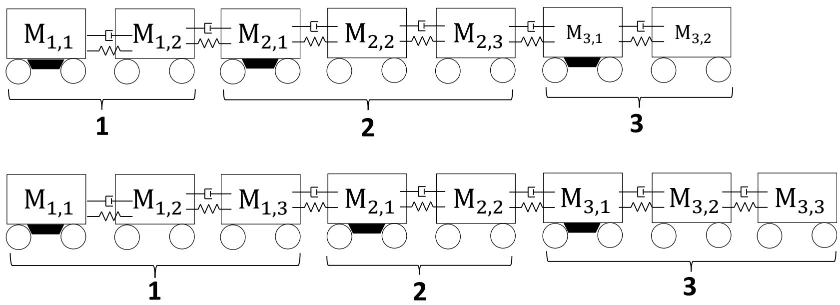


Fig. 5.3 Trains with 3 subsystems

In the upper graph of Fig. 5.3, the train has seven carriages which are separated into three subsystems: the first two carriages ($M_{1,1}$ and $M_{1,2}$) forms subsystem 1, the middle three carriages ($M_{2,1}$, $M_{2,2}$ and $M_{2,3}$) forms subsystem 2 while the last two carriages ($M_{3,1}$ and $M_{3,2}$) forms subsystem 3. In the lower graph of Fig. 5.3, the train has eight carriages which are divided into three subsystems as well: the first three carriages ($M_{1,1}$, $M_{1,2}$ and $M_{1,3}$) forms subsystem 1, the middle two carriages ($M_{2,1}$ and $M_{2,2}$) forms subsystem 2 while the last three carriages ($M_{3,1}$, $M_{3,2}$ and $M_{3,3}$) forms subsystem 3. Two examples are listed in Fig. 5.3 to illustrate potential subsystem groupings.

### 5.2.2 Mathematical Modelling

Compared with the original model in [68] and from the analysis above, a new dynamic equation of the high-speed train system including aerodynamic drag is given by

$$\begin{aligned} M_{i,j_i}\ddot{z}_{i,j_i} = F_{i,j_i}(t) - k_{i,j_i}(z_{i,j_i} - z_{i,j_i+1}) - k_{i,j_i-1}(z_{i,j_i} - z_{i,j_i-1}) - d_{i,j_i}(\dot{z}_{i,j_i} - \dot{z}_{i,j_i+1}) \\ - d_{i,j_i-1}(\dot{z}_{i,j_i} - \dot{z}_{i,j_i-1}) - (b_{i,j_io} + b_{i,j_iv}\dot{z}_{i,j_i} + b_{i,j_ia}\dot{z}^2_{i,j_i}) \end{aligned} \tag{5.4}$$

where $M_{i,j_i}$ is the mass of the $j_i$th carriage in the $i$th subsystem. $k_{i,j_i}$ and $d_{i,j_i}$ are spring and damping parameters of the corresponding carriage for $i = 1,2,...,N$ and $j_i = 1,2,...,\bar{n}_i$. $b_{i,j_io}$, $b_{i,j_iv}$ and $b_{i,j_ia}$ denote the corresponding friction parameters. $z_{i,j_i}$ and $\dot{z}_{i,j_i}$ are the distance and the speed. The traction force $F_{i,j_i}$ and the parameter $b_{i,j_ia}$ are given by

$$F_{i,j_i}\begin{cases} \neq 0, & j_i = 1 \\ = 0, & j_i = 2,...,\bar{n}_i. \end{cases} \qquad b_{i,j_ia}\begin{cases} \neq 0, & i = j_i = 1 \\ = 0, & \text{others}. \end{cases} \tag{5.5}$$

Moreover,

$$\begin{aligned} &k_{1,0} = d_{1,0} = 0, && z_{1,0} = \dot{z}_{1,0} = 0, \\ &k_{N,\bar{n}_N} = d_{N,\bar{n}_N} = 0, && z_{N,\bar{n}_N+1} = \dot{z}_{N,\bar{n}_N+1} = 0, \\ &k_{i,0} = k_{i-1,\bar{n}_{(i-1)}}(1 < i \leqslant N), && d_{i,0} = d_{i-1,\bar{n}_{(i-1)}}(1 < i \leqslant N), \\ &z_{i,0} = z_{i-1,\bar{n}_{(i-1)}}(1 < i \leqslant N), && \dot{z}_{i,0} = \dot{z}_{i-1,\bar{n}_{(i-1)}}(1 < i \leqslant N), \\ &z_{i,\bar{n}_i+1} = z_{i+1,1}(1 \leqslant i < N), && \dot{z}_{i,\bar{n}_i+1} = \dot{z}_{i+1,1}(1 \leqslant i < N). \end{aligned}$$

**Remark 5.2.2** *The description above is introduced to guarantee that all the parameters involved in the system (5.4) are well defined for all $i = 1,2,...N$ and $j_i = 1,2,...,\bar{n}_i$. For example, some parameters of the 0th and $(\bar{n}_N + 1)$th carriages exist mathematically which do not exist in practice due to the special physical location of the first and the last carriages in some certain subsystems.*

## 5.3 System Structure-Based Analysis

Some uncertainties in the isolated subsystems and interconnections are added into the modelling of the system (5.4) for practical reason.

### 5.3.1 A General Framework

For (5.4), the following coordinate transformation is introduced:

$$\begin{bmatrix} x_{i,(2j_i-1)} \\ x_{i,(2j_i)} \end{bmatrix} := \begin{bmatrix} z_{i,j_i} \\ \dot{z}_{i,j_i} \end{bmatrix} \tag{5.6}$$

where $i = 2, ..., N$ and $j_i = 1, 2, .., \bar{n}_i$. From (5.6), $x_{i,(2j_i-1)}$ represents the distance of the carriage, while $x_{i,(2j_i)}$ is the corresponding speed of the $j_i$th carriage in the $i$th subsystem.

**Remark 5.3.1** *It should be noted that the subscript $2j_i$ in (5.6) represents $2 \times j_i$, and this will be applied in similar expressions throughout this chapter. The multiplication sign is just omitted to make the expression more concise.*

It is assumed that the speed of the first carriages $\dot{z}_{i,1}$ are measurable and taken as the outputs. Then, (5.4) can be described by an interconnected system with N subsystems as follows:

$$\dot{x}_{i,(2j_i-1)} = x_{i,(2j_i)}, \tag{5.7}$$

$$\begin{aligned} \dot{x}_{i,(2j_i)} = {} & \frac{k_{i,(j_i-1)}}{M_{i,j_i}} x_{i,(2j_i-3)} + \frac{d_{i,(j_i-1)}}{M_{i,j_i}} x_{i,(2j_i-2)} - \frac{k_{i,j_i} + k_{i,(j_i-1)}}{M_{i,j_i}} x_{i,(2j_i-1)} \\ & - \frac{d_{i,j_i} + d_{i,(j_i-1)} + b_{i,j_i v}}{M_{i,j_i}} x_{i,(2j_i)} + \frac{k_{i,j_i}}{M_{i,j_i}} x_{i,(2j_i+1)} + \frac{d_{i,j_i}}{M_{i,j_i}} x_{i,(2j_i+2)} \\ & + \frac{F_{i,j_i}}{M_{i,j_i}} - \frac{(b_{i,j_i o} + b_{i,j_i a} x^2_{i,(2j_i)})}{M_{i,j_i}} + \Delta_{i,j_i}, \end{aligned} \tag{5.8}$$

$$y_i = x_{i,2}, \tag{5.9}$$

where

$$\Delta_{i,j_i} = \begin{cases} \Delta\Phi_i(x_i), & j_i = 1 \\ \Delta\Psi_{i,(2j_i)}(x), & j_i = 2, ..., \bar{n}_i \end{cases}$$

$F_{i,j_i}$ is the input. $x_i = col(x_{i,1}, x_{i,2}, ..., x_{i,2\bar{n}_i})$ and $x = col(x_1, ..., x_N)$ are system states. $\Delta\Phi_i(x_i)$ represent the uncertainties of the corresponding subsystem and $\Delta\Psi_{i,(2j_i)}(x)$ denote the uncertainties in the interconnection for $i = 1, 2, ..., N$ and $j_i = 1, 2, ..., \bar{n}_i$.

The objective of this chapter is to design a kind of decentralised controller such that the speed of each motor carriage $y_i(t)$ can track the given desired signal $y_d(t)$, i.e.,

$$\lim_{t\to\infty} |y_i(t) - y_d(t)| = 0 \tag{5.10}$$

for $i = 1, 2, ..., N$, while both the distance error $z_{i,(j_i+1)}(t) - z_{i,j_i}(t)$ and the speed of the $i$th trailer carriage $\dot{z}_{i,j_i}(t)$ are bounded for $j_i \neq 1$.

The following assumptions are applied to system (5.7)-(5.9).

**Assumption 5.3.1** *The desired output signal $y_d(t)$ and its first time derivative $\dot{y}_d(t)$ are smooth, known and bounded for all $t \in [0,\infty)$.*

**Assumption 5.3.2** *There exist known smooth mapping $\rho_i(x_i)$ and $\mu_{i,(2j_i)}(x)$ for $i = 1,2,...,N$ such that*

*(i)* $|\Delta\Phi_i(x_i)| \leq \rho_i(x_i)\|x_i\|$,

*(ii)* $|\Delta\Psi_{i,(2j_i)}(x)| \leq \mu_{i,(2j_i)}(x)\|x\|$ $(j_i = 2,...,\bar{n}_i)$.

For (5.8), the following feedback transformation is introduced

$$F_{i,1} = -k_{i-1,\bar{n}_{(i-1)}}x_{i-1,(2\bar{n}_{(i-1)}-1)} - d_{i-1,\bar{n}_{(i-1)}}x_{i-1,2\bar{n}_{(i-1)}} + k_{i,1}x_{i,1} + (d_{i,1} + b_{i,1v})x_{i,2} - k_{i,1}x_{i,3} - d_{i,1}x_{i,4} + b_{i,1o} + b_{i,1a}x_{i,2}^2 + M_{i,1}v_i \tag{5.11}$$

where $b_{i,1a}$ satisfies (5.5) and $v_i$ is the new input to be designed later. The transformed system is given by

$$\dot{x}_{i,(2j_i-1)} = x_{i,(2j_i)}, \tag{5.12}$$

$$\dot{x}_{i,(2j_i)} = \begin{cases} v_i + \Delta\Phi_i(x_i), & j_i = 1 \\ \frac{k_{i,(j_i-1)}}{M_{i,j_i}}x_{i,(2j_i-3)} + \frac{d_{i,(j_i-1)}}{M_{i,j_i}}x_{i,(2j_i-2)} - \frac{k_{i,j_i}+k_{i,(j_i-1)}}{M_{i,j_i}}x_{i,(2j_i-1)} \\ \quad -\frac{d_{i,j_i}+d_{i,(j_i-1)}+b_{i,j_iv}}{M_{i,j_i}}x_{i,(2j_i)} + \frac{k_{i,j_i}}{M_{i,j_i}}x_{i,(2j_i+1)} \\ \quad +\frac{d_{i,j_i}}{M_{i,j_i}}x_{i,(2j_i+2)} - \frac{b_{i,j_io}}{M_{i,j_i}} \\ \quad +\Delta\Psi_{i,(2j_i)}(x), & j_i = 2,...,\bar{n}_i \end{cases}$$

$$y_i = x_{i,2}. \tag{5.13}$$

where $v_i$ is the system input. It is easy to see that the dynamic equation (5.12)-(5.13) is in a regular form which will facilitate the following sliding mode control design.

An additional coordinate transformation $T_i$, for $i = 1,2,...,N$, is introduced as follows

$$T_i : \left[\begin{array}{c} \xi_i \\ \hline e_i \end{array}\right] = \left[\begin{array}{c} \xi_{i,1} \\ \xi_{i,\iota_i} \\ \xi_{i,\hbar_i} \\ \hline e_i \end{array}\right] = \left[\begin{array}{c} x_{(i-1),1} - x_{i,1} \\ x_{i,1} \cdot \vec{1}_i - x_{i,(2\ell_i-1)} \\ x_{i,(2\ell_i)} \\ \hline x_{i,2} - y_d \end{array}\right] \tag{5.14}$$

where the partial states $\xi_i = col(\xi_{i,1}, \xi_{i,2}, ..., \xi_{i,2\bar{n}_i-1}) \in \mathbb{R}^{2\bar{n}_i-1}$, the variables $e_i$, $x_{(i-1),1}$, $x_{i,1}$, $x_{i,2}$ and $y_d$ are scalars, $\vec{1}_i = col(1,1,...,1) \in \mathbb{R}^{\bar{n}_i-1}$, $\iota_i = 2,...,\bar{n}_i$, $\hbar_i = \bar{n}_i+1, \bar{n}_i+2, ..., 2\bar{n}_i-1$

and $\ell_i = 2, \dots, \bar{n}_i$. The system (5.12)-(5.13) after (5.14) is given by

$$\dot{\xi}_{i,1} = e_{i-1} - e_i, \tag{5.15}$$
$$\dot{\xi}_{i,2} = e_i + y_d - \xi_{i,(\bar{n}_i+1)}, \tag{5.16}$$
$$\vdots$$
$$\dot{\xi}_{i,\bar{n}_i} = e_i + y_d - \xi_{i,(2\bar{n}_i-1)}, \tag{5.17}$$
$$\begin{aligned}\dot{\xi}_{i,(\bar{n}_i+1)} = &\frac{k_{i,1}+k_{i,2}}{M_{i,2}}\xi_{i,2} - \frac{k_{i,2}}{M_{i,2}}\xi_{i,3} - \frac{d_{i,1}+d_{i,2}+b_{i,2v}}{M_{i,2}}\xi_{i,(\bar{n}_i+1)} \\ &+ \frac{d_{i,2}}{M_{i,2}}\xi_{i,(\bar{n}_i+2)} + \frac{d_{i,1}}{M_{i,2}}e_i + \frac{d_{i,1}}{M_{i,2}}y_d - \frac{b_{i,2o}}{M_{i,2}} + \Pi_{i,2}(\xi, e),\end{aligned} \tag{5.18}$$
$$\vdots$$
$$\begin{aligned}\dot{\xi}_{i,(2\bar{n}_i-1)} = &-\frac{k_{i,(\bar{n}_i-1)}}{M_{i,\bar{n}_i}}\xi_{i,(\bar{n}_i-1)} + \frac{k_{i,(\bar{n}_i-1)}+k_{i,\bar{n}_i}}{M_{i,\bar{n}_i}}\xi_{i,\bar{n}_i} - \frac{k_{i,\bar{n}_i}}{M_{i,\bar{n}_i}}\xi_{(i+1),1} \\ &+ \frac{d_{i,(\bar{n}_i-1)}}{M_{i,\bar{n}_i}}\xi_{i,(2\bar{n}_i-2)} - \frac{d_{i,(\bar{n}_i-1)}+d_{i,\bar{n}_i}+b_{i,\bar{n}_i v}}{M_{i,\bar{n}_i}}\xi_{i,(2\bar{n}_i-1)} \\ &+ \frac{d_{i,\bar{n}_i}}{M_{i,\bar{n}_i}}e_{i+1} + \frac{d_{i,\bar{n}_i}}{M_{i,\bar{n}_i}}y_d - \frac{b_{i,\bar{n}_i o}}{M_{i,\bar{n}_i}} + \Pi_{i,\bar{n}_i}(\xi, e),\end{aligned} \tag{5.19}$$
$$\dot{e}_i = v_i + \Gamma_i(\xi_i, e_i) - \dot{y}_d, \tag{5.20}$$

where

$$\Gamma_i(\xi_i, e_i) = T_i \Delta\Phi_i|_{col(\xi_i, e_i)=T_i x_i}, \tag{5.21}$$
$$\Pi_{i,j_i}(\xi, e) = T_i \Delta\Psi_{i,2j_i}|_{col(\xi,e)=Tx}, \quad (j_i = 2, \dots, \bar{n}_i) \tag{5.22}$$

where $\xi = col(\xi_1, \xi_2, \dots, \xi_N)$ and $e = col(e_1, e_2, \dots, e_N)$. $\xi_i, e_i$ and $T_i$ are defined in (5.14) and $T = diag(T_1, T_2, \dots, T_N)$.

**Remark 5.3.2** *In (5.14), the scalar $\xi_{i,1}$ is the distance difference between the $(i-1)$th and the ith motor carriages. The vector $\xi_{i,\iota_i}$ denotes the distance difference between the motor carriage and other trailer carriages in the ith subsystem while the vector $\xi_{i,\hbar_i}$ denotes the speed of the trailer carriages in the ith subsystem. The scalar $e_i$ is the speed difference between the ith motor carriage and the desired signal $y_d$. By introducing (5.14), system (5.12)-(5.13) will be transferred to (5.15)-(5.20). The analysis will now focus on (5.15)-(5.20) with state variables $(\xi_i, e_i)$ which avoid the unbounded case of state variables $x_i$ if system (5.12)-(5.13) is studied. This is consistent with real high-speed systems.*

### 5.3.2 Model when $\bar{n}_i = 2$

In order to simplify the statement, the train with $N$ subsystems where each subsystem with $\bar{n}_i = 2$ carriages is to be considered in this chapter.

In this case, the transformed interconnected system (5.15)-(5.20) can be described in a compact form by

$$\dot{\xi}_i = A_{i11}\xi_i + A_{i12}e_i + H_{i1}(\xi, e, y_d), \tag{5.23}$$

$$\dot{e}_i = B_i v_i + H_{i2}(\xi, e, \dot{y}_d), \tag{5.24}$$

where $\xi_i = col(\xi_{i1}, \xi_{i2}, \xi_{i3})$, for $i = 1, 2, ..., N$,

$$A_{i11} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & \frac{k_{i,1}+k_{i,2}}{M_{i,2}} & -\frac{d_{i,2}+d_{i,1}+b_{i,2v}}{M_{i,2}} \end{bmatrix}, \quad A_{i12} = \begin{bmatrix} -1 \\ 1 \\ \frac{d_{i,1}}{M_{i,2}} \end{bmatrix},$$

$$H_{i1} = \begin{bmatrix} e_{i-1} \\ y_d \\ \frac{d_{i,1}+d_{i,2}}{M_{i,2}} y_d + \frac{d_{i,2}}{M_{i,2}} e_{i+1} - \frac{k_{i,2}}{M_{i,2}} \xi_{(i+1),1} - \frac{b_{i,2o}}{M_{i,2}} + \Pi_i(\xi, e) \end{bmatrix}, \tag{5.25}$$

$$B_i = 1, \qquad H_{i2} = \Gamma_i(\xi_i, e_i) - \dot{y}_d,$$

the uncertainties $\Gamma_i(\xi_i, e_i)$ and $\Pi_i(\xi, e)$ are

$$\Gamma_i(\xi_i, e_i) = T_i \Delta\Phi_i|_{col(\xi_i, e_i) = T_i x_i}, \tag{5.26}$$

$$\Pi_i(\xi, e) = T_i \Delta\Psi_{i,2j_i}|_{col(\xi, e) = Tx}, \quad (j_i = 2, ..., \bar{n}_i), \tag{5.27}$$

where $\xi$ and $e$ are defined after (5.22).

## 5.4 Sliding Mode Stability Analysis

This section mainly focuses on designing a specific sliding surface which depends on the output tracking errors. Then the fundamental stability analysis of the sliding dynamics is discussed.

### 5.4.1 Sliding Surface Design

For (5.13), introduce the tracking errors $e_i$ as

$$e_i = y_i(t) - y_d(t), \quad i = 1, 2, ..., N. \tag{5.28}$$

Define an error dependent hyperplane as

$$col(e_1, e_2, ..., e_N) = 0. \tag{5.29}$$

When the system (5.23) attains the sliding surface, the sliding dynamic $\xi_i^s$ is given by

$$\dot{\xi}_i^s = A_{i11}\xi_i^s + H_{i1}^*(\xi^s, y_d) \tag{5.30}$$

for $i = 1, 2, ..., N$, $\xi^s = col(\xi_1^s, \xi_2^s, ..., \xi_N^s)$, $\xi_i^s = col(\xi_{i1}^s, \xi_{i2}^s, \xi_{i3}^s)$ and

$$H_{i1}^*(\xi^s, y_d) = \begin{bmatrix} 0 \\ y_d \\ \frac{d_{i,1}+d_{i,2}}{M_{i,2}} y_d - \frac{k_{i,2}}{M_{i,2}} \xi_{(i+1),1}^s - \frac{b_{i,2o}}{M_{i,2}} + \Pi_i(\xi^s) \end{bmatrix}. \tag{5.31}$$

The objective now is to study the stability of (5.30) which is rewritten in the following compact way

$$\dot{\xi}^s = A_{11}^*\xi^s + H_1^*(\xi^s, y_d) \tag{5.32}$$

where $\xi^s = col(\xi_1^s, \xi_2^s, ..., \xi_N^s)$ is the system state and $H_1^*(\cdot) = col(H_{11}^*, H_{21}^*, ..., H_{N1}^*)$ with $H_{i1}^*(\cdot)$ is defined in (5.31),

$$A_{11}^* = \begin{bmatrix} A_{111} & a_{111} & & & \\ & A_{211} & a_{211} & & \\ & & \ddots & \ddots & \\ & & & A_{(N-1),11} & a_{(N-1),11} \\ & & & & A_{N,11} \end{bmatrix}, \tag{5.33}$$

where the matrix $A_{i11}$ for $i = 1, 2, ..., N$ is defined in (5.25) and the additional matrices $a_{j11}$ are given by

$$a_{j11} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{k_{j,2}}{M_{j,2}} & 0 & 0 \end{bmatrix}. \tag{5.34}$$

where $j = 1, 2, ..., N-1$.

**Remark 5.4.1** *It is assumed that the interconnected uncertainties $\Delta\Psi_{i,(2j_i)}(x)$ satisfy the Assumption 5.3.2. Combining with the inequality $\|x\| \le \sum_{i=1}^N \sum_{j=1}^4 |x_{i,j}|$, the partial term $\Pi_i(\xi^s)$*

*in (5.31) after the transformation (5.22) becomes*

$$\Pi_i(\xi^s) \leq \mu_i(T^{-1}(\xi^s))(\sum_{i=1}^{N}\sum_{j^*=1}^{3}|\xi_{i,j^*}|) \tag{5.35}$$

where $i = 1,2,...,N$, and $\xi^s$ is defined after (5.30).

### 5.4.2 Properties of the Sliding Motion

From system (5.32) with $A^*_{11}$ defined by (5.33) and (5.34), it follows that

$$\dot{\xi}^s_{11} = y_d, \ (i.e., \ \dot{z}_{1,1}(t) = y_d) \tag{5.36}$$

$$\dot{\xi}^s_{i1} = 0, \ (i.e., \ \dot{z}_{(i-1),1}(t) - \dot{z}_{i,1}(t) = 0) \tag{5.37}$$

where $i = 2,3,...,N$.

**Remark 5.4.2** *Consider (5.36) and from (5.6), $\dot{\xi}^s_{11} = \dot{x}_{1,1} = x_{1,2} = y_d$. Given that $x_{1,2}$ and $x_{1,1}$ represent the speed and distance of the motor carriage in the first subsystem respectively, the speed $x_{1,2}$ tracks the desired speed $y_d$ which is consistent with the objective. Therefore, in the following analysis, the state variable $\xi^s_{11}$ can be separated from the rest of the states in (5.32).*

**Remark 5.4.3** *Consider (5.37) and from (5.6), the states $\xi^s_{i1} = z_{(i-1),1}(t) - z_{i,1}(t) = z_{(i-1),1}(0) - z_{i,1}(0)$, for $i = 2,3,...,N$ which means the distance between the adjacent motor carriages is invariant, the same as the initial value of the distance. Similarly, in the analysis of the simplified system stability, the state variables $\xi^s_{21},...,\xi^s_{N1}$ are independent as well. It follows that only the partial states $\bar{\xi}^s = col(\bar{\xi}^s_1, \bar{\xi}^s_2, ..., \bar{\xi}^s_N)$ where $\bar{\xi}^s_i = col(\xi^s_{i2}, \xi^s_{i3})$ for $i = 1,2,...,N$ need to be considered.*

According to Remarks 5.4.2-5.4.3, the new distribution matrix $\bar{A}^*_{11}$ are as follows

$$\dot{\bar{\xi}}^s = \bar{A}^*_{11}\bar{\xi}^s + \bar{H}^*_1(\bar{\xi}^s, y_d) \tag{5.38}$$

where $\bar{\xi}^s = col(\bar{\xi}^s_1, \bar{\xi}^s_2, ..., \bar{\xi}^s_N)$ with $\bar{\xi}^s_i = col(\xi^s_{i2}, \xi^s_{i3})$ for $i = 1,2,...,N$.

$$\bar{A}^*_{11} = \begin{bmatrix} \bar{A}_{111} & \bar{a}_{111} & & & \\ & \bar{A}_{211} & \bar{a}_{211} & & \\ & & \ddots & \ddots & \\ & & & \bar{A}_{(N-1),11} & \bar{a}_{(N-1),11} \\ & & & & \bar{A}_{N,11} \end{bmatrix}, \tag{5.39}$$

and

$$\bar{A}_{i11} = \begin{bmatrix} 0 & -1 \\ \frac{k_{i,1}+k_{i,2}}{M_{i,2}} & -\frac{d_{i,2}+d_{i,1}+b_{i,2v}}{M_{i,2}} \end{bmatrix}, \bar{a}_{j11} = \begin{bmatrix} 0 & 0 \\ -\frac{k_{j,2}}{M_{j,2}} & 0 \end{bmatrix}. \tag{5.40}$$

Moreover, $\bar{H}_1^*(\bar{\xi}^s, y_d) = col(\bar{H}_{11}^*, \bar{H}_{21}^*, \ldots, \bar{H}_{N1}^*)$, where

$$\bar{H}_{i1}^* = \begin{bmatrix} y_d \\ \frac{d_{i,1}+d_{i,2}}{M_{i,2}} y_d - \frac{b_{i,2o}}{M_{i,2}} + \Pi_i(\bar{\xi}^s) \end{bmatrix}, \tag{5.41}$$

for $i = 1, 2, \ldots, N$, and $j = 1, 2, \ldots, N-1$ in (5.40).

According to the conclusion in (5.35), $\bar{H}_1^*(\cdot)$ in (5.38) satisfies

$$\| \bar{H}_1^*(\bar{\xi}^s, y_d) \| \leq \| \bar{H}_1^{*s}(\bar{\xi}^s) \| + \| \bar{H}_1^{*y}(y_d) \| \tag{5.42}$$

where the terms

$$\bar{H}_1^{*s}(\bar{\xi}^s) = col(\bar{H}_{11}^{*s}(\bar{\xi}^s), \bar{H}_{21}^{*s}(\bar{\xi}^s), \ldots, \bar{H}_{N1}^{*s}(\bar{\xi}^s)),$$

$$\bar{H}_1^{*y}(y_d) = col(\bar{H}_{11}^{*y}(y_d), \bar{H}_{21}^{*y}(y_d), \ldots, \bar{H}_{N1}^{*y}(y_d))$$

and

$$\bar{H}_{i1}^{*s}(\bar{\xi}^s) = \begin{bmatrix} 0 \\ \mu_i(\cdot)(\sum_{i=1}^{N} \sum_{j_i^*=1}^{3} |\xi_{i,j_i^*}|) \end{bmatrix}, \tag{5.43}$$

$$\bar{H}_{i1}^{*y}(y_d) = \begin{bmatrix} y_d \\ \frac{d_{i,1}+d_{i,2}}{M_{i,2}} y_d - \frac{b_{i,2o}}{M_{i,2}} \end{bmatrix}. \tag{5.44}$$

The element $\bar{H}_1^{*s}(\bar{\xi}^s)$ only depends on the state $\bar{\xi}^s$ while $\bar{H}_1^{*y}(y_d)$ only depends on $y_d$.

**Assumption 5.4.1** *The matrix $\bar{A}_{11}^*$ in (5.38) is assumed to be a Hurwitz matrix.*

Based on the Assumption 5.4.1, for any positive definite matrix $Q$, the following *Lyapunov* equation has a corresponding unique solution $P > 0$

$$\bar{A}_{11}^{*\top} P + P \bar{A}_{11}^* = -Q. \tag{5.45}$$

**Theorem 5.4.1** *Under the Assumptions 5.3.1, 5.3.2 and 5.4.1, the sliding mode dynamics (5.38) are uniformly ultimately bounded if there is a domain $\Omega \backslash \{0\}$ such that $M^\top + M > 0$ in*

$\Omega\backslash\{0\}$, *where the matrix* $M(\cdot)$ *is defined by*

$$M = \begin{bmatrix} q & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ p_1 & q+p_1 & \cdots & p_1 & p_1 & \cdots & p_1 & p_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & q & 0 & \cdots & 0 & 0 \\ p_i & p_i & \cdots & p_i & q+p_i & \cdots & p_i & p_i \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & q & 0 \\ p_N & p_N & \cdots & p_N & p_N & \cdots & p_N & q+p_N \end{bmatrix} \tag{5.46}$$

*with* $q = \lambda_{min}(Q)$ *and* $p_i = -2\|P\|\mu_i(\cdot)$ *for* $i = 1,2,3,\ldots,N$. $\lambda_{min}(Q)$ *denotes the minimum eigenvalue of the matrix* $Q$.

**Proof** *For the modified sliding mode dynamics (5.38), consider the following Lyapunov candidate function*

$$V(\bar{\xi}^s) = \bar{\xi}^{s\top} P \bar{\xi}^s \tag{5.47}$$

*where* $P$ *satisfies (5.45). Then, the first derivate of* $V(\bar{\xi}^s)$ *is given by*

$$\begin{aligned} &\dot{V}(\bar{\xi}^s) \\ &= \dot{\bar{\xi}}^{s\top} P \bar{\xi}^s + \bar{\xi}^{s\top} P \dot{\bar{\xi}}^s \\ &= \Big(\bar{A}_{11}^* \bar{\xi}^s + \bar{H}_1^*(\bar{\xi}^s, y_d)\Big)^\top P \bar{\xi}^s + \bar{\xi}^{s\top} P \Big(\bar{A}_{11}^* \bar{\xi}^s + \bar{H}_1^*(\bar{\xi}^s, y_d)\Big) \\ &= \bar{\xi}^{s\top} \bar{A}_{11}^{*\top} P \bar{\xi}^s + \bar{\xi}^{s\top} P \bar{A}_{11}^* \bar{\xi}^s + \bar{H}_1^*(\bar{\xi}^s, y_d)^\top P \bar{\xi}^s + \bar{\xi}^{s\top} P \bar{H}_1^*(\bar{\xi}^s, y_d). \end{aligned} \tag{5.48}$$

*Combining with (5.42), it follows from (5.48) that*

$$\begin{aligned} &\dot{V}(\bar{\xi}^s) \\ &\le -\bar{\xi}^{s\top} Q \bar{\xi}^s + 2\bar{\xi}^{s\top} P \bar{H}_1^*(\bar{\xi}^s, y_d) \\ &\le -\lambda_{min}(Q)\|\bar{\xi}^s\|^2 + 2\|\bar{\xi}^s\|\|P\| \ \| \bar{H}_1^*(\bar{\xi}^s, y_d) \ \| \\ &\le -\frac{1}{2}\lambda_{min}(M^\top + M)\|\bar{\xi}^s\|^2 + 2\|\bar{\xi}^s\|\|P\|\|\bar{H}_1^{*y}(y_d)\| \\ &\le -\frac{1}{2}\Big(\lambda_{min}(M^\top + M)\|\bar{\xi}^s\| - 4\|P\|\|\bar{H}_1^{*y}(y_d)\|\Big)\|\bar{\xi}^s\| \end{aligned} \tag{5.49}$$

*where (5.45) is used to establish the result above. It is clear to see that $\dot{V}(\cdot) \leq 0$, if*

$$\|\bar{\xi}^s\| \geq \frac{4\|P\|\|\bar{H}_1^{*y}(y_d)\|}{\lambda_{min}(M^\top + M)}. \tag{5.50}$$

*Hence, the conclusion follows.*

## 5.5 Control Law Synthesis

For the simplified interconnected system (5.23)-(5.24), the corresponding reachability condition is given by

$$\sum_{i=1}^{N} \frac{e_i(t)\dot{e}_i(t)}{|e_i(t)|} < 0. \tag{5.51}$$

The control signal applied to the plant is designed as

$$v_i = -\frac{e_i}{|e_i|}\Big(K_i(\xi_i, e_i) + |\dot{y}_d|\Big) \tag{5.52}$$

where $K_i(\cdot)$ is to be designed later.

**Theorem 5.5.1** *Under Assumptions 5.3.1 and 5.3.2, the control signal (5.52) drives the states of the closed-loop system (5.23)-(5.24) to the composite sliding surface (5.29) and maintains a sliding motion on it thereafter if the controller gain $K_i(\xi_i, e_i)$ in (5.52) satisfies*

$$\sum_{i=1}^{N} K_i(\xi_i, e_i) > \rho_i(T^{-1}(\xi_i, e_i))(\sum_{m=1}^{3} |\xi_{i,m}| + |e_i|) \tag{5.53}$$

*where $\rho_i(\cdot)$ is defined in Assumption 5.3.2.*

**Proof** *From the error dynamics (5.24) for $\bar{n}_i = 2$, it follows*

$$\dot{e}_i = v_i + \Gamma_i(\xi_i, e_i) - \dot{y}_d \tag{5.54}$$

*for $i = 1, 2, \ldots, N$. According to (5.52)-(5.54),*

$$\frac{e_i\dot{e}_i}{|e_i|} = \frac{e_i}{|e_i|}\Big(\Gamma_i(\xi_i, e_i) - \dot{y}_d\Big) - K_i(\xi_i, e_i) - |\dot{y}_d|. \tag{5.55}$$

*According to (5.26), it follows from Assumption 5.3.2 that*

$$|\Gamma_i(\cdot)| \leq \rho_i(T^{-1}(\xi_i, e_i))(\sum_{m=1}^{3} |\xi_{i,m}| + |e_i|). \tag{5.56}$$

*Substituting the inequality (5.56) into (5.55), it follows that*

$$\sum_{i=1}^{N} \frac{e_i(t)\dot{e}_i(t)}{|e_i(t)|} < -\sum_{i=1}^{N} K_i(\xi_i, e_i) + \rho_i(T^{-1}(\xi_i, e_i))(\sum_{m=1}^{3} |\xi_{i,m}| + |e_i|). \tag{5.57}$$

*Then, the reachability condition (5.51) is satisfied with the condition in (5.53).*

**Remark 5.5.1** *Recall the feedback transformation (5.11), the decentralised controller in the original coordinate is given by*

$$\begin{aligned} F_{i,1} = & -k_{i-1,\bar{n}_{(i-1)}} x_{i-1,(2\bar{n}_{(i-1)}-1)} - d_{i-1,\bar{n}_{(i-1)}} x_{i-1,2\bar{n}_{(i-1)}} + k_{i,1}x_{i,1} + (d_{i,1} + b_{i,1v})x_{i,2} \\ & - k_{i,1}x_{i,3} - d_{i,1}x_{i,4} + b_{i,1o} + b_{i,1a}x_{i,2}^2 - \frac{M_{i,1}(y_i - y_d)}{|y_i - y_d|}(K_i(\xi_i, y_i, y_d) + |\dot{y}_d|) \end{aligned} \tag{5.58}$$

*can drive the system (5.7)-(5.9) when $\bar{n}_i = 2$ to the corresponding sliding surface.*

According to the Theorems 5.4.1 and 5.5.1, the system (5.23)-(5.24) with the proposed controller (5.52) is proved to be uniformly ultimately bounded (i.e., $\|\xi\|$ is uniformly ultimately bounded). And, the asymptotic stable errors $e_i$ with the limitation to the desired signal cause the bounded system output $y_i$, due to $y_i = e_i + y_d$ for $i = 1, 2, ..., N$.

## 5.6 Example: A High-Speed Train

Consider the following desired signal

$$y_d(t) = \tanh(t), \; t \geq 0 \tag{5.59}$$

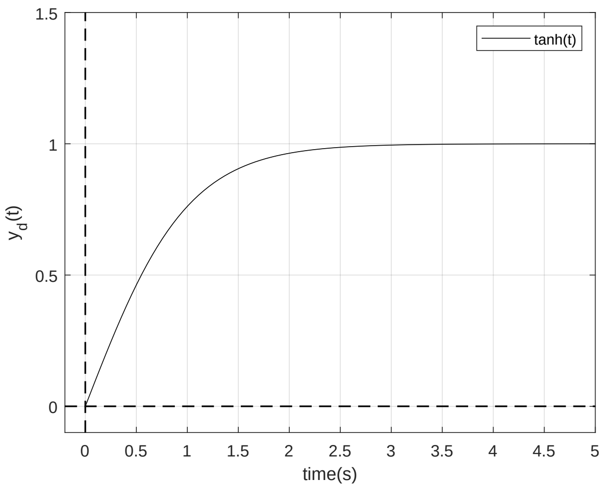


Fig. 5.4 Time response of $y_d(t)$

The selected desired signal shown in Fig. 5.4 reflects the controlled speed of a train system in reality when departing from a station. Therefore, it is reasonable to set the desired tracking signal as a hyperbolic function in (5.59). Such signals are continuous and differentiable across the considered domain which satisfies the Assumption 5.3.1.

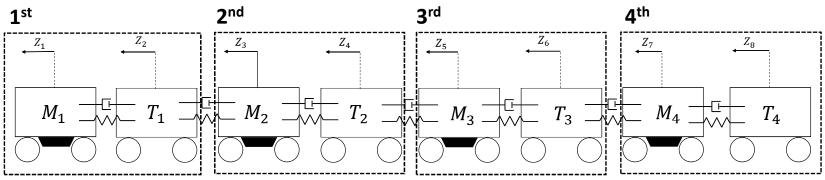


Fig. 5.5 Model of a high-speed train ($N = 4$)

For simulation purpose, consider the eight carriages connected by the springs and dampers as shown in Fig. 5.5 where four carriages $M_1, M_2, M_3$ and $M_4$ are with motors. The parameters for $i = 2, 3, 4$ are chosen as [68]

$$
\begin{aligned}
&M_1 = M_2 = M_3 = M_4 = 126000\text{kg},\\
&T_1 = T_2 = T_3 = T_4 = 101090\text{kg}, \quad k_{11} = k_{12} = k_{41} = 30 \times 10^4\text{N/m},\\
&k_{21} = 30 \times 10^9\text{N/m}, \quad k_{22} = 30 \times 10^8\text{N/m}, \quad k_{31} = 40 \times 10^{11}\text{N/m},\\
&k_{32} = k_{42} = 40 \times 10^{10}\text{N/m}, \quad d_{11} = d_{12} = 40 \times 10^4\text{Ns/m}, \quad d_{21} = d_{22} = 40 \times 10^5\text{Ns/m},\\
&d_{31} = 50 \times 10^4\text{Ns/m}, \quad d_{32} = d_{42} = 50 \times 10^5\text{Ns/m}, \quad d_{41} = 30 \times 10^4\text{Ns/m},\\
&b_{1,1o} = 7.665 \times 10^{-3}\text{N/kg}, \quad b_{1,1a} = 2.06 \times 10^{-5} Ns^2/(m^2 kg),\\
&b_{1,2o} = b_{i,1o} = b_{i,2o} = 6.362 \times 10^{-3}\text{N/kg},\\
&b_{1,1v} = b_{1,2v} = b_{i,1v} = b_{i,2v} = 1.08 \times 10^{-4}\text{Ns/(mkg)},
\end{aligned}
$$

The eigenvalues of the matrix $\bar{A}_{11}^*$ are $-0.8, -7.1, -39.6 \pm 570i, -27.2 \pm 6597.3i$ and $-26.2 \pm 1989.0i$, which satisfies the Assumption 5.4.1. For simulation purposes, the uncertainties satisfy

$$
\begin{aligned}
&|\Delta\Phi_i| \le \alpha_{i1}|x_{i2}| + \alpha_{i2}|x_{i4}|,\\
&|\Delta\Psi_1| \le sin^2(x_{14} + x_{34})\|x\|, \ |\Delta\Psi_2| \le 0.5cos^2(x_{11} + x_{44} - x_{13})\|x\|,\\
&|\Delta\Psi_3| \le \frac{3}{2}cos^2(x_{23} - x_{11} + x_{44})\|x\|, \ |\Delta\Psi_4| \le \frac{1}{3}sin^2(x_{31} - x_{33})\|x\|,
\end{aligned}
$$

where $\alpha_{i1}$ and $\alpha_{i2}$ are known constants for $i = 1, 2, 3, 4$.

According to the Assumption 5.3.2 and (5.53), the controller gain $K_i(\cdot)$ is given by

$$
K_i(\xi_i, e_i, t) > \alpha_{i1}|x_{i2}| + \alpha_{i2}|x_{i4}| + \sigma, \tag{5.60}
$$

for $i = 1,2,3,4$ and $\sigma = 1$, $\alpha_{11} = 1, \alpha_{12} = 2.1, \alpha_{21} = 1, \alpha_{22} = 2, \alpha_{31} = 1, \alpha_{32} = 0.5, \alpha_{41} = 2$ and $\alpha_{42} = 0.7$.

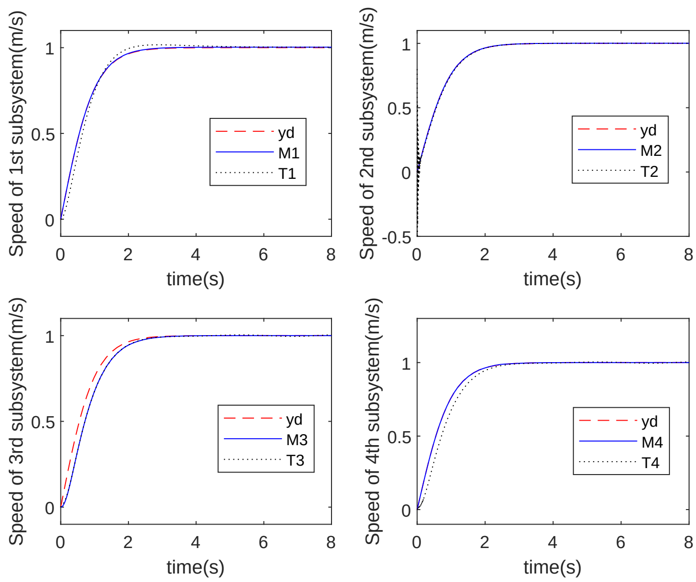


Fig. 5.6 Time response of the speed

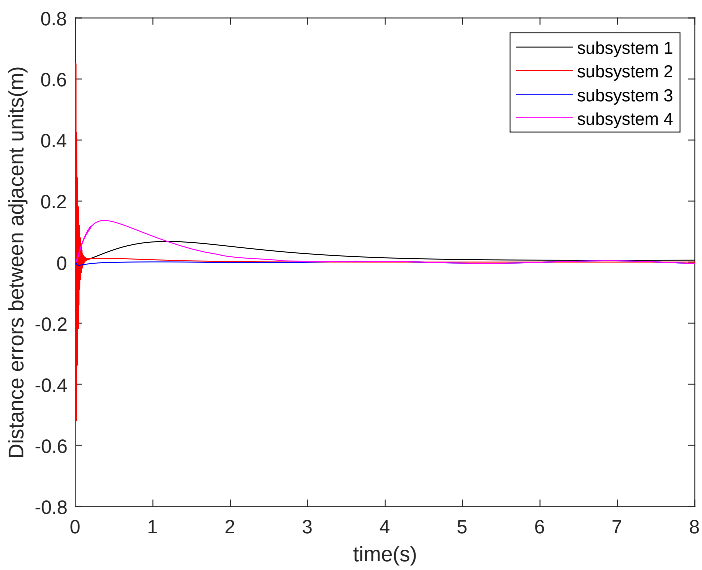


Fig. 5.7 Time response of the distance errors between adjacent carriages

The tracking results are shown in Fig. 5.6 where we could find the motor carriages $M_1, M_2, M_3$ and $M_4$ can track the desired signal $y_d(t)$, though there exist uncertainties in the system. The trailer carriages $T_1, T_2, T_3$ and $T_4$ also have similar dynamic performance, and the bounded distance errors are shown in Fig. 5.7. This example demonstrates the effective result developed in this chapter.

## 5.7 Summary

A kind of system structural-based partition has been applied to a high-speed train model which allows the train system to be treated as a large-scale system interconnected with a number of subsystems. By using a decentralised sliding mode-based tracking control, a closed-loop high-speed train system shows good tracking performance and high robustness, even in the presence of external and internal uncertainties. A numerical example of a high-speed train with eight carriages demonstrates the effective result in this chapter.

# Chapter 6

# Conclusions and Future Work

## 6.1 Summary and Conclusions

In this thesis, a research of sliding mode-based tracking control for large-scale interconnected systems has been presented. System structural analysis has been considered to formulate a standard sliding mode form and has achieved robustness against the internal disturbance by introducing some local diffeomorphisms. Some novel approaches have been developed to tackle tracking problem, to boost the system performance and to estimate the relevant unknown parameters of the systems.

Basic concepts like control theory, sliding mode methodology and thesis objectives have been introduced in Chapter 2. Relating mathematical knowledge and other Supplementary notes are given in Appendix sections.

In Chapters 3 and 4, decentralised controllers are proposed to deal with the output tracking of nonlinear large-scale interconnected systems with disturbances. A local diffeomorphism is introduced in Chapter 3 to explore system structure where some assumptions are introduced like, the interconnections are assumed to be known, the uniform relative degree of relevant system structural triples are the same ($r_i^a = r_i^b$) and the designed controller does not strictly adhere to a decentralized structure. And in Chapter 4, a sliding mode controller using output information is designed. By introducing a non-singular coordinate transformation containing output information, the original interconnected system is transformed into a kind of systems with a special structure where the output is merged with partial states. In the end, the output of the systems in Chapters 3 and 4 can track desired time-varying signals with the corresponding control schemes.

A new kind of interconnected systems, i.e., underactuated high-speed train, is considered in Chapter 5. For this system, not all the subsystems have input which means the system does not have enough actuators. In this case, a sliding mode-based decentralised control is proposed to successfully tackle each system that can still track the same desired signal, even if some of the subsystems are driven and powered by the interconnections. At the end of each chapter (from Chapter 3 to 5), simulations are performed to show that with the proposed control strategies in this thesis, the closed-loop systems achieve satisfactory tracking performance.

## 6.2 Future Work

There are several potential research directions for the future work in tracking control of large-scale systems related to the topics in this thesis.

**More general boundedness of Interconnections:** For the control of large-scale systems, dealing with the dynamic effects caused by interconnections is an inevitable topic. Interconnections are normally formed by the interactions between subsystems, which implies the presence of uncertainties, disturbances, or unmodelled dynamic components in the interconnections. In this thesis, the approach to handling interconnections assumes that the unknown interconnections are bounded by a known function related to the system states. This known function is incorporated into the controller design to compensate for the dynamic effects of the interconnections on individual subsystems. However, in reality, not all system states are fully measurable, and the assumption above is strong. Therefore, in future research, it is a promising topic to design decentralised control schemes considering more general boundaries for interconnections. Furthermore, in the main chapters of this thesis, the controllers proposed assume that all states of the system are measurable as well. Investigating output feedback control may be more meaningful than state feedback control in the following research.

**Utilisation of Distributed Control:** One key aspect of controlling large-scale systems is to ensure that even if one subsystem experiences a fault or issue, the other subsystems can continue to operate normally. Decentralised control naturally possesses an advantage in addressing such challenges due to its unique control structure. However, the existence of interconnections poses a challenge that decentralised control must confront. In this regard, distributed control offers similar advantages to decentralised control, but its controller design allows for a broader collection of system information while ensuring the robustness of individual subsystems. Therefore, in future designs, developing a class of distributed control algorithms to address control problems in large systems is also a potential research direction.

**Reinforcement with intelligent control:** Apart from that, the current advanced control theories, design methods, and stability analyses are mostly based on well-defined system models. However, these knowledge face significant limitations and drawbacks when applied to the analysis of large-scale systems in practical applications. For instance, the mathematical modelling of the system may be incomplete, leading to the neglect of certain dynamics and thereby reducing control performance. Alternatively, the designed controllers may require strong assumptions and conditions to be applicable, limiting their practicality. These issues gradually fail to meet the rapidly developing control demands. Given this situation, exploring the integration of machine learning techniques, such as deep learning or reinforcement learning, into tracking control algorithms for large-scale systems becomes very interesting. This can enhance the system's adaptability, robustness, and performance in handling uncertainties and complex dynamics.

**Focus on intelligence:** In the era of modern intelligence, many intelligent agents possess complex design structures with highly nonlinear models. Additionally, with the advancements in sensors, obtaining precise and abundant data has become easier. Therefore, when it comes to controlling complex large-scale systems, a shift from model-based control to data-driven control can be considered. Data-driven control strategies do not rely on the knowledge of model parameters or structures but instead directly utilize data to derive control laws. For instance, the widely used PID control is a form of data-driven control as its control input design is based on feedback of errors without considering model information. Therefore, utilizing real-time data collected from sensors and actuators to improve the system's tracking performance, optimize control parameters, and handle uncertainties and disturbances, is also worth trying in the future.

**Others:** With the advancements in technology, the research on large-scale systems is evolving to encompass the realm of discrete time systems. This expansion is particularly evident in domains such as network systems and digital communication systems. The integration of discrete time considerations into the study of large-scale systems is imperative, given the prevalence of digital technologies in contemporary applications. Whether addressing network dynamics or enhancing the efficiency of communication systems, the adaptation of research methodologies to discrete time frameworks becomes essential. This paradigm shift reflects a growing recognition of the need to align large-system studies with the intricacies of digital and discrete processes, thereby ensuring the relevance and applicability of research outcomes in the ever-evolving technological landscape.

# References


[1] A. Ferrara, G. P. I. and Sangiovanni, B. (2019). Tracking control via switched integral sliding mode with application to robot manipulators. *Control Engineering Practice*, 90:257–266.

[2] Alessandri, A. and Parisini, T. (1997). Nonlinear modeling of complex large-scale plants using neural networks and stochastic approximation. *IEEE Transactions on Systems, Man, and Cybernetics - Part A: Systems and Humans*, 27(6):750–757.

[3] Antsaklis, P. J. and Michel, A. N. (1997). *Linear Systems*. McGraw-Hill, Englewood Cliffs, NJ.

[4] Axelby, G. (1956). Synthesis of feedback control systems with a minimum lead for a specified performance. *IRE Transactions on Automatic Control*, 1(1):56–73.

[5] Bertsekas, D. (2019). *Reinforcement Learning and Optimal Control*. Athena Scientific.

[6] C. Liu, H. G. Zhang, G. Y. X. and Sun, S. X. (2019). Integral reinforcement learning based decentralized optimal tracking control of unknown nonlinear large-scale interconnected systems with constrained-input. *Neurocomputing*, 323:1–11.

[7] Cai, H. and Hu, G. (2017). Distributed tracking control of an interconnected leader-follower multiagent system. *IEEE Transactions on Automatic Control*, 62(7):3494–3501.

[8] Chen, P.-H. and Chang, H.-C. (1995). Large-scale economic dispatch by genetic algorithm. *IEEE Transactions on Power Systems*, 10(4):1919–1926.

[9] Ching-Yei Chung, S. and Lin, C.-L. (1999). A transformed lure problem for sliding mode control and chattering reduction. *IEEE Transactions on Automatic Control*, 44(3):563–568.

[10] Choi, Y. H. and Yoo, S. J. (2021). Decentralized event-triggered tracking of a class of uncertain interconnected nonlinear systems using minimal function approximators. *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, 51(3):1766–1778.

[11] Chou, M., Xia, X., and Kayser, C. (2007). Modelling and model validation of heavy-haul trains equipped with electronically controlled pneumatic brake systems. *Control Engineering Practice*, 15(4):501–509.

[12] Chung, S. C. and Lin, C.-L. (1998). A general class of sliding surface for sliding mode control. *IEEE Transactions on Automatic Control*, 43(1):115–119.

[13] Dong, L., Yu, D., and Nguang, S. K. (2020). Novel nonsingular terminal sliding mode control for multi-agent tracking systems with application to jerk circuit. *IEEE Transactions on Circuits and Systems II: Express Briefs*, 67(8):1429–1433.

[14] Edwards, C. and Spurgeon, S. (1998). *Sliding Mode Control: Theory and Applications*. Taylor and Francis Ltd, CRC Press, 1st edition.

[15] Ferreira, P. (2000). Robust tracking: relation between the numerators, mode readability, and inverse internal model. *IEEE Transactions on Automatic Control*, 45(7):1401–1405.

[16] G. Q. Zhu, L. L. Nie, Z. L. L. F. S. X. Y. Z. and Wang, C. L. (2020). Adaptive fuzzy dynamic surface sliding mode control of large-scale power systems with prescribe output tracking performance. *ISA Transactions*, 99:305–321.

[17] Guan, Z.-H., Liu, Y.-Q., and Wen, X.-C. (1995). Decentralized stabilization of singular and time-delay large-scale control systems with impulsive solutions. *IEEE Transactions on Automatic Control*, 40(8):1437–1441.

[18] Gundes, A. N. and Ozguler, A. B. (2007). Pid stabilization of mimo plants. *IEEE Transactions on Automatic Control*, 52(8):1502–1508.

[19] Guo, J., Zhang, J.-F., and Zhao, Y. (2011). Adaptive tracking control of a class of first-order systems with binary-valued observations and time-varying thresholds. *IEEE Transactions on Automatic Control*, 56(12):2991–2996.

[20] Hamamci, S. E. (2007). An algorithm for stabilization of fractional-order time delay systems using fractional-order pid controllers. *IEEE Transactions on Automatic Control*, 52(10):1964–1969.

[21] Han, Y., Cheng, Y., and Xu, G. (2019). Trajectory tracking control of agv based on sliding mode control with the improved reaching law. *IEEE Access*, 7:20748–20755.

[22] Han, Y. Q. and Yan, H. S. (2020). Observer-based multi-dimensional taylor network decentralised adaptive tracking control of large-scale stochastic nonlinear systems. *International Journal of Control*, 93(7):1605–1618.

[23] Horowitz, I. (1959). Fundamental theory of automatic linear feedback control systems. *IRE Transactions on Automatic Control*, 4(3):5–19.

[24] Hou, H., Yu, X., Xu, L., Rsetam, K., and Cao, Z. (2020). Finite-time continuous terminal sliding mode control of servo motor systems. *IEEE Transactions on Industrial Electronics*, 67(7):5647–5656.

[25] Hou, Z.-G., Gupta, M. M., Nikiforuk, P. N., Tan, M., and Cheng, L. (2007). A recurrent neural network for hierarchical control of interconnected dynamic systems. *IEEE Transactions on Neural Networks*, 18(2):466–481.

[26] Hsiao, F.-H., Xu, S.-D., Lin, C.-Y., and Tsai, Z.-R. (2008). Robustness design of fuzzy control for nonlinear multiple time-delay large-scale systems via neural-network-based approach. *IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics)*, 38(1):244–251.

[27] Hu, Z. (1994). Decentralized stabilization of large scale interconnected systems with delays. *IEEE Transactions on Automatic Control*, 39(1):180–182.

[28] Hua, C., Li, Y., and Guan, X. (2017). Finite/fixed-time stabilization for nonlinear interconnected systems with dead-zone input. *IEEE Transactions on Automatic Control*, 62(5):2554–2560.

[29] Huang, J., Wen, C., Wang, W., and Song, Y.-D. (2015). Global stable tracking control of underactuated ships with input saturation. *Systems and Control Letters*, 85:1–7.

[30] Huang, Y.-S. and Wang, Z.-Y. (2014). Decentralized adaptive fuzzy control for a class of large-scale mimo nonlinear systems with strong interconnection and its application to automated highway systems. *Information Sciences*, 274:210–224.

[31] Huerta, H., Loukianov, A. G., and Canedo, J. M. (2019). Passivity sliding mode control of large-scale power systems. *IEEE Transactions on Control Systems Technology*, 27(3):1219–1227.

[32] Hyland, D. and Bernstein, D. (1987). The majorant lyapunov equation: A nonnegative matrix equation for robust stability and performance of large scale systems. *IEEE Transactions on Automatic Control*, 32(11):1005–1013.

[33] Ioannou, P. (1986). Decentralized adaptive control of interconnected systems. *IEEE Transactions on Automatic Control*, 31(4):291–298.

[34] Isidori, A. (August, 1995). *Nonlinear Control Systems*. Springer London, 2nd edition.

[35] Jafari, R. and Mukherjee, R. (2013). Enlarging the region of attraction for underactuated systems using impulsive inputs. In *2013 American Control Conference*, pages 5613–5618.

[36] Jain, S. and Khorrami, F. (1997). Decentralized adaptive output feedback design for large-scale nonlinear systems. *IEEE Transactions on Automatic Control*, 42(5):729–735.

[37] Jeannin-Girardon, A., Ballet, P., and Rodin, V. (2015). Large scale tissue morphogenesis simulation on heterogenous systems based on a flexible biomechanical cell model. *IEEE/ACM Transactions on Computational Biology and Bioinformatics*, 12(5):1021–1033.

[38] Jones, A. and Porter, B. (1987). Design of adaptive digital set-point tracking pid controllers incorporating recursive step-response matrix identifiers for multivariable plants. *IEEE Transactions on Automatic Control*, 32(5):459–462.

[39] Kabouris, J. and Kanellos, F. D. (2010). Impacts of large-scale wind penetration on designing and operation of electric power systems. *IEEE Transactions on Sustainable Energy*, 1(2):107–114.

[40] Kachroo, P. (1999). Existence of solutions to a class of nonlinear convergent chattering-free sliding mode control systems. *IEEE Transactions on Automatic Control*, 44(8):1620–1624.

[41] Kailath, T. (1980). *Linear Systems*. Information and System Sciences Series. Prentice-Hall.

[42] Kandel, A. (1980). On the control and evaluation of uncertain processes. *IEEE Transactions on Automatic Control*, 25(6):1182–1187.

[43] Khalil, H. (2002). *Nonlinear Systems*. Prentice Hall Inc, New Jersey, 3rd edition.

[44] Kim, H. S., Park, J. B., and Joo, Y. H. (2017). Decentralized sampled-data tracking control of large-scale fuzzy systems: An exact discretization approach. *IEEE Access*, 5:12668–12681.

[45] Kovacik, M. and Pospisil, M. (2020). Design of electric multiple unit for tatra electric railway. In *2020 ELEKTRO*, pages 1–5.

[46] Le, T. L., Negnevitsky, M., and Piekutowski, M. (1997). Network equivalents and expert system application for voltage and var control in large-scale power systems. *IEEE Transactions on Power Systems*, 12(4):1440–1445.

[47] Lee, K., Ortiz, J., Mohtadi, M., and Park, Y. (1988). Optimal operation of large-scale power systems. *IEEE Transactions on Power Systems*, 3(2):413–420.

[48] Levant, A. and Alelishvili, L. (2007). Integral high-order sliding modes. *IEEE Transactions on Automatic Control*, 52(7):1278–1282.

[49] Li, J. and Zhang, Q. (2019). Fuzzy reduced-order compensator-based stabilization for interconnected descriptor systems via integral sliding modes. *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, 49(4):752–765.

[50] Li, X. and Liu, X. (2018). Backstepping-based decentralized adaptive neural h-infinity tracking control for a class of large-scale nonlinear interconnected systems. *Journal of the Franklin Institute*, 355(11):4533–4552.

[51] Li, Y., Yang, L., and Yang, G. (2007). Network-based coordinated motion control of large-scale transportation vehicles. *IEEE/ASME Transactions on Mechatronics*, 12(2):208–215.

[52] Li, Y.-X., Tong, S., and Yang, G.-H. (2020). Observer-based adaptive fuzzy decentralized event-triggered control of interconnected nonlinear system. *IEEE Transactions on Cybernetics*, 50(7):3104–3112.

[53] Liao, H., Zhang, X., and Ma, Z. (2021). Robust dichotomy solution-based model predictive control for the grid-connected inverters with disturbance observer. *CES Transactions on Electrical Machines and Systems*, 5(2):81–89.

[54] Liao, Z. and Dexter, A. L. (2010). An inferential model-based predictive control scheme for optimizing the operation of boilers in building space-heating systems. *IEEE Transactions on Control Systems Technology*, 18(5):1092–1102.

[55] Liaw, H. C., Shirinzadeh, B., and Smith, J. (2008). Sliding-mode enhanced adaptive motion tracking control of piezoelectric actuation systems for micro/nano manipulation. *IEEE Transactions on Control Systems Technology*, 16(4):826–833.

[56] Liu, C., Zhang, H., Xiao, G., and Sun, S. (2019). Integral reinforcement learning based decentralized optimal tracking control of unknown nonlinear large-scale interconnected systems with constrained-input. *Neurocomputing*, 323:1–11.

[57] Liu, L. and Fu, L. (2017). Minimum breakpoint set determination for directional overcurrent relay coordination in large-scale power networks via matrix computations. *IEEE Transactions on Power Delivery*, 32(4):1784–1789.

[58] Luenberger, D. (1967). Canonical forms for linear multivariable systems. *IEEE Transactions on Automatic Control*, 12(3):290–293.

[59] Lunze, J. (1992). *Feedback Control of Large Scale Systems*. PrenticeHall International, HemelHempstead.

[60] Lunze, J. (2020). *Feedback Control of Large-Scale Systems*. Bookmundo Direct.

[61] Ma, M., Wang, T., Qiu, J., and Karimi, H. R. (2021). Adaptive fuzzy decentralized tracking control for large-scale interconnected nonlinear networked control systems. *IEEE Transactions on Fuzzy Systems*, 29(10):3186–3191.

[62] Ma, T. (2020). Decentralized filtering adaptive constrained tracking control for interconnected nonlinear systems. *International Journal of Robust and Nonlinear Control*, 30:4652–4675.

[63] Mahmoud, M. (2011). *Decentralized Systems with Design Constraints*. Springer, London.

[64] Mahmoud, M. S. and Almutairi, N. B. (2009). Decentralized stabilization of interconnected systems with time-varying delays. *European Journal of Control*, 15(6):624 – 633.

[65] Malki, H., Misir, D., Feigenspan, D., and Chen, G. (1997). Fuzzy pid control of a flexible-joint robot arm with uncertainties from time-varying loads. *IEEE Transactions on Control Systems Technology*, 5(3):371–378.

[66] Mao, Z., Tao, G., Jiang, B., and Yan, X.-G. (2018). Adaptive actuator compensation of position tracking for high-speed trains with disturbances. *IEEE Transactions on Vehicular Technology*, 67(7):5706–5717.

[67] Mao, Z., Tao, G., Jiang, B., and Yan, X.-G. (2021a). Adaptive control design and evaluation for multibody high-speed train dynamic models. *IEEE Transactions on Control Systems Technology*, 29(3):1061–1074.

[68] Mao, Z., Tao, G., Jiang, B., and Yan, X.-G. (2021b). Zero dynamics analysis and adaptive tracking control of underactuated multibody systems with flexible links. *International Journal of Control*, 94(7):1931–1943.

[69] Mao, Z., Yan, X.-G., Jiang, B., and Chen, M. (2020). Adaptive fault-tolerant sliding-mode control for high-speed trains with actuator faults and uncertainties. *IEEE Transactions on Intelligent Transportation Systems*, 21(6):2449–2460.

[70] Mao, Z., Yan, X.-G., Jiang, B., and Spurgeon, S. K. (2022). Sliding mode control of nonlinear systems with input distribution uncertainties. *IEEE Transactions on Automatic Control*, pages 1–8.

[71] Marino, R. and Tomei, P. (1999). Nonlinear output feedback tracking with almost disturbance decoupling. *IEEE Transactions on Automatic Control*, 44(1):18–28.

[72] Mariscotti, A., Pozzobon, P., and Vanti, M. (2007). Simplified modeling of $2\times$ 25-kv at railway system for the solution of low frequency and large-scale problems. *IEEE Transactions on Power Delivery*, 22(1):296–301.

[73] Meluso, J., Austin-Breneman, J., Bagrow, J. P., and Hebert-Dufresne, L. (2022). A review and framework for modeling complex engineered system development processes. *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, 52(12):7679–7691.

[74] Mu, J., Yan, X.-G., Spurgeon, S. K., and Zhao, D. (2018). Nonlinear sliding mode control for interconnected systems with application to automated highway systems. *IEEE Transactions on Control of Network Systems*, 5(1):664–674.

[75] Narendra, K. and Oleng, N. (2002). Exact output tracking in decentralized adaptive control systems. *IEEE Transactions on Automatic Control*, 47(2):390–395.

[76] Oucheriah, S. (1999). Robust tracking and model following of uncertain dynamic delay systems by memoryless linear controllers. *IEEE Transactions on Automatic Control*, 44(7):1473–1477.

[77] Pagilla, P. R., Dwivedula, R. V., and Siraskar, N. B. (2007). A decentralized model reference adaptive controller for large-scale systems. *IEEE/ASME Transactions on Mechatronics*, 12(2):154–163.

[78] Ren, X.-X., Yang, G.-H., and Li, X.-J. (2020). Global adaptive fuzzy distributed tracking control for interconnected nonlinear systems with communication constraints. *IEEE Transactions on Fuzzy Systems*, 28(2):333–345.

[79] Rinaldi, G., Menon, P. P., Edwards, C., and Ferrara, A. (2019). Variable gains decentralized super-twisting sliding mode controllers for large-scale modular systems. In *2019 18th European Control Conference (ECC)*, pages 3577–3582.

[80] Ruiz-Duarte, J. E. and Loukianov, A. G. (2020). Approximate causal output tracking for linear perturbed systems via sliding mode control. *International Journal of Control*, 93(4):993–1004.

[81] Sabanovic, A. and Izosimov, D. B. (1981). Application of sliding modes to induction motor control. *IEEE Transactions on Industry Applications*, IA-17(1):41–49.

[82] Saeed, S., Zhao, W., Wang, H., Tao, T., and Khan, F. (2021). Fault-tolerant deadbeat model predictive current control for a five-phase pmsm with improved svpwm. *Chinese Journal of Electrical Engineering*, 7(3):111–123.

[83] Sandell, N., Varaiya, P., Athans, M., and Safonov, M. (1978). Survey of decentralized control methods for large scale systems. *IEEE Transactions on Automatic Control*, 23(2):108–128.

[84] Santos, A. C., Delbem, A. C. B., London, J. B. A., and Bretas, N. G. (2010). Node-depth encoding and multiobjective evolutionary algorithm applied to large-scale distribution system reconfiguration. *IEEE Transactions on Power Systems*, 25(3):1254–1265.

[85] Sarpturk, S., Istefanopulos, Y., and Kaynak, O. (1987). On the stability of discrete-time sliding mode control systems. *IEEE Transactions on Automatic Control*, 32(10):930–932.

[86] Sato, K., Yoshizawa, M., and Fukushima, T. (2010). Traction systems using power electronics for shinkansen high-speed electric multiple units. In *The 2010 International Power Electronics Conference - ECCE ASIA -*, pages 2859–2866.

[87] Shabbir, M. N. S. K., Liang, X., Li, W., Imtiaz, S., and Quaicoe, J. (2022). A novel model predictive controller for distributed generation in isolated microgrids part ii: Model predictive controller implementation. *IEEE Transactions on Industry Applications*, 58(5):5860–5870.

[88] ShangGuan, W., Yan, X.-H., Cai, B.-G., and Wang, J. (2015). Multiobjective optimization for train speed trajectory in ctcs high-speed railway with hybrid evolutionary algorithm. *IEEE Transactions on Intelligent Transportation Systems*, 16(4):2215–2225.

[89] Simon, D. (2006). *Optimal State Estimation: Kalman, H-infinity, and Nonlinear Approaches*. John Wiley & Sons Inc.

[90] Sira-Ramirez, H. (1989). Nonlinear variable structure systems in sliding mode: the general case. *IEEE Transactions on Automatic Control*, 34(11):1186–1188.

[91] Slotine, J.-J. E. and Li, W. (1991). *Applied nonlinear control*. Prentice Hall Englewood Cliffs, New Jersey, 1st edition edition.

[92] Song, J., Huang, L.-Y., Karimi, H. R., Niu, Y., and Zhou, J. (2020). Adp-based security decentralized sliding mode control for partially unknown large-scale systems under injection attacks. *IEEE Transactions on Circuits and Systems I: Regular Papers*, 67(12):5290–5301.

[93] Song, J., Wang, Y.-K., Niu, Y., Lam, H.-K., He, S., and Liu, H. (2022). Periodic event-triggered terminal sliding mode speed control for networked pmsm system: A ga-optimized extended state observer approach. *IEEE/ASME Transactions on Mechatronics*, 27(5):4153–4164.

[94] Spooner, J. and Passino, K. (1999). Decentralized adaptive control of nonlinear systems using radial basis neural networks. *IEEE Transactions on Automatic Control*, 44(11):2050–2057.

[95] Su, R. and Hunt, L. (1986). A canonical expansion for nonlinear systems. *IEEE Transactions on Automatic Control*, 31(7):670–673.

[96] T. L. Liao, L. C. F. and Hsu, C. F. (1990). Output tracking control of nonlinear systems with mismatched uncertainties. *Systems and Control Letters*, 18(1):39 – 47.

[97] Tan, L. N. (2020). Distributed h-infinity optimal tracking control for strict-feedback nonlinear large-scale systems with disturbances and saturating actuators. *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, 50(11):4719–4731.

[98] Tong, S., Liu, C., and Li, Y. (2010). Fuzzy-adaptive decentralized output-feedback control for large-scale nonlinear systems with dynamical uncertainties. *IEEE Transactions on Fuzzy Systems*, 18(5):845–861.

[99] Tong, S., Sun, K., and Sui, S. (2018). Observer-based adaptive fuzzy decentralized optimal control design for strict-feedback nonlinear large-scale systems. *IEEE Transactions on Fuzzy Systems*, 26(2):569–584.

[100] Utkin, V. (1977). Variable structure systems with sliding modes. *IEEE Transactions on Automatic Control*, 22(2):212–222.

[101] Utkin, V. (1978). *Sliding Modes and Their Application to Variable Structure Systems*. MIR Publication House, Moscow.

[102] V. Utkin, J. G. and Shijun, M. (1999). *Sliding mode control in electro-mechanical systems*. London, UK: Taylor & Francis.

[103] Van, M. (2019). Adaptive neural integral sliding-mode control for tracking control of fully actuated uncertain surface vessels. *International Journal of Robust and Nonlinear Control*, 29:1537–1557.

[104] Villani, L., De Wit, C., and Brogliato, B. (1999). An exponentially stable adaptive control for force and position tracking of robot manipulators. *IEEE Transactions on Automatic Control*, 44(4):798–802.

[105] Wan, M. and Yin, Y. (2020). Adaptive decentralized output feedback tracking control for large-scale interconnected systems with time-varying output constraints. *Mathematical Problems in Engineering*, pages 1–14.

[106] Wang, W.-J. and Mau, L.-G. (1997). Stabilization and estimation for perturbed discrete time-delay large-scale systems. *IEEE Transactions on Automatic Control*, 42(9):1277–1282.

[107] Wang, Y. and Liu, Y. (2021). Global practical tracking via adaptive output feedback for uncertain nonlinear systems without polynomial constraint. *IEEE Transactions on Automatic Control*, 66(4):1848–1855.

[108] Wang, Y., Song, Y., Gao, H., and Lewis, F. L. (2016). Distributed fault-tolerant control of virtually and physically interconnected systems with application to high-speed trains under traction/braking failures. *IEEE Transactions on Intelligent Transportation Systems*, 17(2):535–545.

[109] Wang, Y. and Xiong, J. (2022). A noise history decomposition approach for decentralized optimal control of large-scale systems defined over a weakly connected graph. *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, 52(5):3337–3348.

[110] Wang, Y., Xiong, J., and Ho, D. W. C. (2019). Globally optimal state-feedback lqg control for large-scale systems with communication delays and correlated subsystem process noises. *IEEE Transactions on Automatic Control*, 64(10):4196–4201.

[111] Wei, Y., Zheng, Z., Li, Q., Jiang, Z., and Yang, P. (2020). Robust tracking control of an underwater vehicle and manipulator system based on double closed-loop integral sliding mode. *International Journal of Advanced Robotic Systems*, 17(4):1729881420941778.

[112] Wu, L.-B., Wang, H., He, X.-Q., and Zhang, D.-Q. (2018). Decentralized adaptive fuzzy tracking control for a class of uncertain large-scale systems with actuator nonlinearities. *Applied Mathematics and Computation*, 332:390–405.

[113] X. G. Yan, S. K. S. and Edwards, C. (2012). Decentralised stabilisation for nonlinear time delay interconnected systems using static output feedback. *Automatica*, 49(2):633–641.

[114] X. G. Yan, C. Edwards, S. K. S. and Bleijs, J. A. M. (2004). Decentralised sliding-mode control for multimachine power systems using only output information. *IEE Proceedings - Control Theory and Applications*, 151(5):627–635.

[115] Xing-Gang Yan, S. K. S. and Edwards, C. (2016). *Variable Structure Control of Complex Systems: Analysis and Design*. Springer Cham, 1st edition.

[116] Y. Yan, S. H. Y. and Sun, C. Y. (2020). Quantization-based event-triggered sliding mode tracking control of mechanical systems. *Information Sciences*, 523:296–306.

[117] Yan, X.-G., Edwards, C., and Spurgeon, S. K. (2004). Decentralised robust sliding mode control for a class of nonlinear interconnected systems by static output feedback. *Automatica*, 40(4):613–620.

[118] Yan, X.-G., Lam, J., and Dai, G.-Z. (1999). Decentralized robust control for nonlinear large-scale systems with similarity. *Computers & Electrical Engineering*, 25(3):169–179.

[119] Yan, X.-G., Spurgeon, S. K., and Edwards, C. (2014a). Memoryless static output feedback sliding mode control for nonlinear systems with delayed disturbances. *IEEE Transactions on Automatic Control*, 59(7):1906–1912.

[120] Yan, X.-G., Spurgeon, S. K., and Edwards, C. (2014b). Memoryless static output feedback sliding mode control for nonlinear systems with delayed disturbances. *IEEE Transactions on Automatic Control*, 59(7):1906–1912.

[121] Yan, X.-G., Wang, J.-J., Lu, X.-Y., and Zhang, S.-Y. (1998). Decentralized output feedback robust stabilization for a class of nonlinear interconnected systems with similarity. *IEEE Transactions on Automatic Control*, 43(2):294–299.

[122] Yan, Y., Yu, S., and Sun, C. (2020). Quantization-based event-triggered sliding mode tracking control of mechanical systems. *Information Sciences*, 523:296–306.

[123] Yang, C.-D. and Sun, Y.-P. (2001). Mixed h2/h cruise controller design for high speed train. *International Journal of Control*, 74(9):905–920.

[124] Yang, F., Shao, X., Muyeen, S. M., Li, D., Lin, S., and Fang, C. (2021). Disturbance observer based fractional-order integral sliding mode frequency control strategy for interconnected power system. *IEEE Transactions on Power Systems*, 36(6):5922–5932.

[125] Yao, D., Li, H., Lu, R., and Shi, Y. (2020a). Distributed sliding-mode tracking control of second-order nonlinear multiagent systems: An event-triggered approach. *IEEE Transactions on Cybernetics*, 50(9):3892–3902.

[126] Yao, D., Li, H., Lu, R., and Shi, Y. (2020b). Distributed sliding-mode tracking control of second-order nonlinear multiagent systems: An event-triggered approach. *IEEE Transactions on Cybernetics*, 50(9):3892–3902.

[127] Ye, C., Zhigang, L., Ke, H., and Hongyi, Z. (2017). Modeling analysis of electric multiple units passing insulated rail joints in high-speed railway station. In *2017 IEEE Transportation Electrification Conference and Expo, Asia-Pacific (ITEC Asia-Pacific)*, pages 1–6.

[128] Yuechen, H., Haiyang, L., and Hongxin, S. (2018). Optimal feedback based control for mars entry trajectory tracking. *Journal of Systems Engineering and Electronics*, 29(1):134–141.

[129] Zarei, F. and Shafiei, M. H. (2018). On event-triggered tracking for non-linear SISO systems via sliding mode control. *IMA Journal of Mathematical Control and Information*, 37(1):105–119.

[130] Zhang, F., Dawson, D., de Queiroz, M., and Dixon, W. (2000). Global adaptive output feedback tracking control of robot manipulators. *IEEE Transactions on Automatic Control*, 45(6):1203–1208.

[131] Zhang, K., Jiang, B., Yan, X., Mao, Z., and Polycarpou, M. M. (2021). Fault-tolerant control for systems with unmatched actuator faults and disturbances. *IEEE Transactions on Automatic Control*, 66(4):1725–1732.

[132] Zhao, B., Liu, D., Yang, X., and Li, Y. (2017). Observer-critic structure-based adaptive dynamic programming for decentralised tracking control of unknown large-scale nonlinear systems. *International Journal of Systems Science*, 48(9):1978–1989.

[133] Zhao, Y., Sun, X., Wang, G., and Fan, Y. (2021). Adaptive backstepping sliding mode tracking control for underactuated unmanned surface vehicle with disturbances and input saturation. *IEEE Access*, 9:1304–1312.

[134] Zhou, Q., Shi, P., Liu, H., and Xu, S. (2012). Neural-network-based decentralized adaptive output-feedback control for large-scale stochastic nonlinear systems. *IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics)*, 42(6):1608–1619.

[135] Zhu, G., Nie, L., Lv, Z., Sun, L., Zhang, X., and Wang, C. (2020). Adaptive fuzzy dynamic surface sliding mode control of large-scale power systems with prescribe output tracking performance. *ISA Transactions*, 99:305–321.

[136] Zhu, Y. and Fridman, E. (2020). Predictor methods for decentralized control of large-scale systems with input delays. *Automatica*, 116:108903.

# Appendix A

# Mathematical Preliminaries

## A.1 Young's inequality for products

In mathematics, Young's inequality for products is a mathematical inequality about the product of two numbers. The inequality is named after William Henry Young.

For non-negative real numbers $a, b$ and real numbers $p, q$ such that $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{A.1}$$

The inequality (A.1) is called Young's inequality.

### Elementary case

An elementary case of Young's inequality is the inequality (A.1) when $p = q = 2$, that is,

$$ab \leq \frac{a^2}{2} + \frac{b^2}{2}, \tag{A.2}$$

which also gives rise to the so-called Young's inequality with $\varepsilon$ (valid for every $\varepsilon > 0$).

$$ab \leq \frac{a^2}{2\varepsilon} + \frac{\varepsilon b^2}{2}. \tag{A.3}$$

Sometimes it is called the Peter-Paul inequality. This name refers to the fact that tighter control of the second term is achieved at the cost of losing some control of the first term.

## A.2 Euclidean Space

The one-dimensional Euclidean space consists of all real numbers and is denoted by $\mathbb{R}$. The set of all $n$-dimensional vectors $x = [x_1, ..., x_n]^\top$, where $x_1, ..., x_n$ are real numbers, defines the $n$-dimensional Euclidean space denoted by $\mathbb{R}^n$. Vectors in $\mathbb{R}^n$ can be added by adding their corresponding components. The inner product of two vectors $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ and $y = (y_1, y_2, ..., y_n)^\top \in \mathbb{R}^n$ is $x^\top y = \sum_{i=1}^{n} x_i y_i$.

## Vector and Matrix Norms

The length of a vector is a non-negative number that describes the extent of the vector in space, and is sometimes referred to as the vector's magnitude or the norm. The norm $\| x \|$ of a vector $x$ is a real-valued function with the following properties

- $\| x \| \geq 0$ for all $x \in \mathbb{R}^n$, with $\| x \| = 0$ if and only if $x = 0$.
- $\| x + y \| \leq \| x \| + \| y \|$, for all $x, y \in \mathbb{R}^n$.
- $\| \alpha x \| = |\alpha| \, \| x \|$, for all $\alpha \in \mathbb{R}$ and $x \in \mathbb{R}^n$.

The second property is the triangle inequality. Consider the class of $p$-norms defined by

$$\| x \|_p = (|x_1|^p + ... + |x_n|^p)^{1/p}, \qquad 1 \leq p \leq \infty$$

and

$$\| x \|_\infty = \max_i |x_i|.$$

The three most commonly used norms are $\| x \|_1$, $\| x \|_\infty$ and the Euclidean norm

$$\| x \|_2 = (|x_1|^2 + ... + |x_n|^2)^{1/2} = (x^\top x)^{1/2}.$$

Quite often when we use norms, we only use properties deduced from the three basic properties satisfied by any norm. In those cases, the subscript $p$ is dropped, indicating that the norm can be any $p$-norm.

An $m \times n$ matrix $A$ of real elements defines a linear mapping $y = Ax$ from $\mathbb{R}^n$ into $\mathbb{R}^m$. The induced $p$-norm of $A$ is defined by[1]

$$\| A \|_p = \sup_{x \neq 0} \frac{\| Ax \|_p}{\| x \|_p} = \max_{\|x\|_p = 1} \| Ax \|_p$$

[1] $sup$ denotes supremum, the least upper bound; $inf$ denotes Infimum, the greatest lower bound.

which for $p = 1, 2$ and $\infty$ is given by

$$\| A \|_1 = \max_j \sum_{i=1}^{m} |a_{ij}|, \quad \| A \|_2 = [\lambda_{max}(A^\top A)]^{1/2}, \quad \| A \|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}|$$

where $\lambda_{max}(A^\top A)$ is the maximum eigenvalue of $A^\top A$.

## A.3 Ordinary differential equations and its properties

This section states some fundamental properties of the solutions of ordinary differential equations, like existence and uniqueness which are essential for the state equation $\dot{x} = f(t, x)$ to be a useful mathematical model of a physical system. In experimenting with a physical system such as the pendulum, we expect that starting the experiment from a given initial state at time $t_0$, the system will move and its state will be defined in the (at least immediate) future time $t > t_0$. Moreover, with a deterministic system, we expect that if we could repeat the experiment exactly, we could get exactly the same motion and the same state at the same time $t > t_0$. For the mathematical model to predict the future state of the system from its current state at $t_0$, the initial-value problem

$$\dot{x} = f(t, x), \quad x(t_0) = x_0 \tag{A.4}$$

must have a unique solution. This is the question of existence and uniqueness which can be ensured by imposing some constraints on the right-hand side function $f(t, x)$ of equation (A.4). The key constraint here is the Lipschitz condition, whereby $f(t, x)$ satisfies the inequality[2]

$$\| f(t, x) - f(t, y) \| \leq L \| x - y \| \tag{A.5}$$

for all $(t, x)$ and $(t, y)$ in the considered neighbourhood of $(t_0, x_0)$.

An essential factor in the validity of any mathematical model is the continuous dependence of its solutions on the date of the problem. The least we should expect from a mathematical model is that arbitrarily small errors in the data will not result in large errors in the solution obtained by the model.

[2] $\| \cdot \|$ denotes any $p$-norm, as defined in Appendix A.2.

## Existence and Uniqueness

In this section, we present sufficient conditions for the existence and uniqueness of the solution of the initial-value problem (A.4). By a solution of (A.4) over an interval $[t_0,t_1]$, it means that a continuous function $x:[t_0,t_1]\to\mathbb{R}^n$ such that $\dot{x}(t)$ is defined and $\dot{x}(t)=f(t,x)$ for all $t\in[t_0,t_1]$. If $f(t,x)$ is continuous in $t$ and $x$, then the solution $x(t)$ will be continuously differentiable. It is assumed that $f(t,x)$ is continuous in $x$, but only piecewise continuous in $t$, in which case, a solution $x(t)$ could only be piecewise continuously differentiable. The assumption that $f(t,x)$ be piecewise continuous in $t$ allows us to include the case when $f(t,x)$ depends on a time-varying input that may experience step changes with time. We introduce a theorem that employs the Lipschitz condition to show existence and uniqueness.

**Theorem A.3.1 (Local Existence and Uniqueness)** *If there exists a function $f(t,x)$ be piecewise continuous in t and satisfy the Lipschitz condition*

$$\| f(t,x)-f(t,y) \| \leq L \| x-y \|$$

$\forall x,y\in B=\{x\in\mathbb{R}^n \mid \| x-x_0 \| \leq r\}, \forall t\in[t_0,t_1]$. *Then, there exists some $\delta>0$ such that the state equation $\dot{x}=f(t,x)$ with $x(t_0)=x_0$ has a unique solution over $[t_0,t_0+\delta]$.*

The key assumption in Theorem A.3.1 is the Lipschitz condition (A.5). A function satisfying (A.5) is said to be Lipschitz in $x$, and the positive constant $L$ is called a Lipschitz constant.

**Lemma A.3.1** *Let $f:[a,b]\times D\to\mathbb{R}^m$ be continuous for some domain $D\subset\mathbb{R}^n$. Suppose that $[\partial f/\partial x]$ exists and is continuous on $[a,b]\times D$. If, for a convex subset $W\subset D$, there is a constant $L\geq 0$ such that*

$$\| \frac{\partial f}{\partial x}(t,x) \| \leq L$$

*on $[a,b]\times W$, then*

$$\| f(t,x)-f(t,y) \| \leq L \| x-y \|$$

*for all $t\in[a,b], x\in W$, and $y\in W$.*

The lemma shows how a Lipschitz constant can be calculated using knowledge of $[\partial f/\partial x]$. The Lipschitz property of a function is stronger than continuity. It can be easily seen that if $f(x)$ is Lipschitz on $W$, then it is uniformly continuous on $W$. The converse is not true, as seen from function $f(x)=x^{1/3}$, which is continuous, but not locally Lipschitz at $x=0$. The Lipschitz property is weaker than continuous differentiability, as stated in the next lemma.

**Lemma A.3.2** *If $f(t,x)$ and $[\partial f/\partial x](t,x)$ are continuous on $[a,b]\times D$, for some domain $D\subset\mathbb{R}^n$, then $f$ is locally Lipschitz in $x$ on $[a,b]\times D$.*

**Lemma A.3.3** *If $f(t,x)$ and $[\partial f/\partial x](t,x)$ are continuous on $[a,b]\times\mathbb{R}^n$, then $f$ is globally Lipschitz in $x$ on $[a,b]\times\mathbb{R}^n$ if and only if $[\partial f/\partial x]$ is uniformly bounded on $[a,b]\times\mathbb{R}^n$.*

**Theorem A.3.2 (Global Existence and Uniqueness)** *Suppose $f(t,x)$ be piecewise continuous in $t$ and satisfy the Lipschitz condition*

$$\| f(t,x)-f(t,y) \| \leq L \| x-y \|$$

$\forall x,y\in\mathbb{R}^n, \forall t\in[t_0,t_1]$. *Then, the state equation $\dot{x}=f(t,x)$ with $x(t_0)=x_0$ has a unique solution over $[t_0,t_1]$.*

## A.4 Gradient and Lie derivatives

The objective of this section is to formalize and generalize the previous intuitive concepts for a broad class of nonlinear systems. To this effect, we first introduce some mathematical tools from differential geometry and topology. To limit the conceptual and notational complexity, we discuss these tools directly in the context of nonlinear dynamic systems (instead of general topological spaces).

In describing these mathematical tools, we shall call a vector function $f:\mathbb{R}^n\to\mathbb{R}^n$ a vector field in $\mathbb{R}^n$, to be consistent with the terminology used in differential geometry. The intuitive reason for this term is that to every vector function $f$ corresponds a field of vectors in an $n$-dimensional space (one can think of a vector $f(x)$ emanating from every point $x$.) In the following, we shall only be interested in smooth vector fields. By smoothness of a vector field, we mean that function $f(x)$ has continuous partial derivatives of any required order.

Given a smooth scalar function $h(x)$ of the state $x$, the gradient of $h$ is denoted by $\nabla h$

$$\nabla h=\frac{\partial h}{\partial x}$$

The gradient is represented by a row-vector of elements $(\nabla h)_j=\partial h/\partial x_j$. Similarly, given a vector field $f(x)$, the Jacobian of $f$ is denoted by $\nabla f$

$$\nabla f=\frac{\partial f}{\partial x}$$

It is represented by an $n\times n$ matrix of elements $(\nabla f)_{ij}=\partial f_i/\partial x_j$.

Now, given a scalar function $h(x)$ and a vector field $f(x)$, we define a new scalar function $L_f h$, called the Lie derivative of $h$ with respect to $f$.

**Definition A.4.1** *If there exists a function $h : \mathbb{R}^n \to \mathbb{R}$ be a smooth scalar function, and $f : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth vector field on $\mathbb{R}^n$, then the Lie derivative of h with respect to $f$ is a scalar function defined by $L_f h = \nabla h f$.*

Thus, the Lie derivative $L_f h$ is simply the directional derivative of $h$ along the direction of the vector $f$.

Repeated Lie derivatives can be defined recursively

$$
\begin{aligned}
&L_f^0 h = h \\
&L_f^i h = L_f(L_f^{i-1} h) = \nabla(L_f^{i-1} h) f \qquad i = 1, 2, ...
\end{aligned}
$$

Similarly, if $g$ is another vector field, then the scalar function $L_g L_f h(x)$ is

$$L_g L_f h(x) = \nabla(L_f h) g$$

One can easily see the relevance of Lie derivatives to dynamic systems by considering the following single-output system

$$
\begin{aligned}
\dot{x} &= f(x) \\
y &= h(x)
\end{aligned}
$$

The derivatives of the output are

$$
\begin{aligned}
\dot{y} &= \frac{\partial h}{\partial x}\dot{x} = L_f h \\
\ddot{y} &= \frac{\partial [L_f h]}{\partial x} = L_f^2 h
\end{aligned}
$$

and so on. Similarly, if $V$ is a Lyapunov function candidate for the system, its derivative $\dot{V}$ can be written as $L_f V$.

## A.5 Diffeomorphisms and State transformations

The concept of diffeomorphism can be viewed as a generalization of the familiar concept coordinate transformation. It is formally defined as follows:

**Definition A.5.1** *A function $\Phi : \mathbb{R}^n \to \mathbb{R}^n$, defined in a region $\Omega$, is called a diffeomorphism if it is smooth, and if its inverse $\Phi^{-1}$ exists and is smooth.*

If the region $\Omega$ is the whole space $\mathbb{R}^n$, then $\Phi(x)$ is called a global diffeomorphism. Global diffeomorphisms are rare, and therefore one often looks for local diffeomorphisms, i.e., for transformations defined only in a finite neighbourhood of a given point. Given a nonlinear function $\Phi(x)$, it is easy to check whether it is a local diffeomorphism by using the following lemma, which is a straightforward consequence of the well-known implicit function theorem.

**Lemma A.5.1** *Let $\Phi(x)$ be a smooth function defined in a region $\Omega$ in $\mathbb{R}^n$. If the Jacobian matrix $\nabla\Phi$ is non-singular at a point $x = x_0$ of $\Omega$, then $\Phi(x)$ defines a local diffeomorphism in a subregion of $\Omega$.*

A diffeomorphism can be used to transform a nonlinear system into another nonlinear system in terms of a new set of states, similarly to what is commonly done in the analysis of linear systems. Consider the dynamic system described by

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned}$$

and let a new set of states be defined by $z = \Phi(x)$. Differentiation of $z$ yields

$$\dot{z} = \frac{\partial \Phi}{\partial x}\dot{x} = \frac{\partial \Phi}{\partial x}(f(x) + g(x)u)$$

One can easily write the new state-space representation as

$$\begin{aligned} \dot{z} &= f^*(z) + g^*(z)u \\ y &= h^*(z) \end{aligned} \tag{A.6}$$

where $x = \Phi^{-1}(z)$ has been used, and the functions $f^*$, $g^*$ and $h^*$ are defined obviously.

# Appendix B

# Assorted mfiles & simulink

In this thesis, MATLAB/SIMULINK software are mainly used to do the simulation. To illustrate how the simulation was carried out, the example in Chapter 3 is used to show how the Simulink blocks were built in this thesis. The section B.2.1 shows the MATLAB codes. The section B.3 shows the simulink screenshots[1] of the two-inverted pendulums example which is studied in section 3.5. Apart from that, some other relating materials are attached as an explanation in this section.

## B.1 Pseudo-sliding with a smooth control action

In certain problems, such as control of electric motors and power converters, the control action is naturally discontinuous and sliding mode ideas can be used to obtain extremely high performance. Although the control signal obtained from some theoretical analysis is preferable to the earlier designs in terms of its chattering behaviour, in many situations (e.g. digital simulation in computers) such a control signal would still not be considered acceptable. A natural solution is to attempt to smooth the discontinuity in the signum function to obtain an arbitrarily close but continuous approximation. One possible approximation is the sigmoid-like function

$$v_\delta(s) = \frac{s}{(|s| + \delta)} \tag{B.1}$$

where $\delta$ is a small positive scalar, which is shown in Fig. B.1. It can be visualised that as $\delta \to 0$, the continuous function $v_\delta(s)$ can approach the signum function in any accuracy. The

[1]The simulation demonstrated in section B.3 is based on graphical Simulink and does not include the process of coordinate transformation, due to the original system (3.55)-(3.56) in Chapter 3 is already in the required form.

variable $\delta$ can be used to trade off the requirement of maintaining ideal performance with that of ensuring a smooth control action.

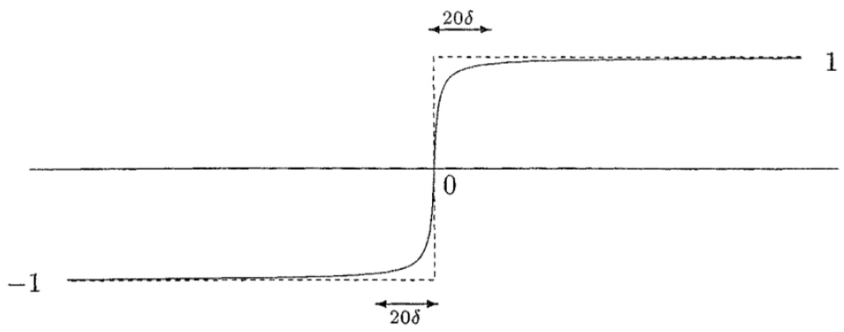


Fig. B.1 A differentiable approximation of the signum function

Fig. B.2 shows an ideal sliding mode control action with signum function while Fig. B.3 shows a sliding mode control action where the signum function is replaced by a sigmoid-like function as shown in Fig. B.1.

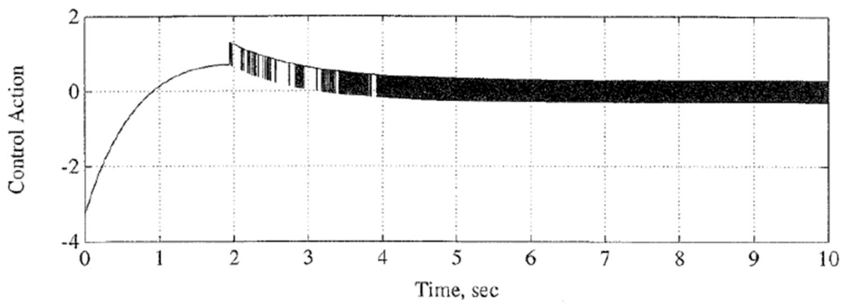


Fig. B.2 Evolution of control action with the signum function

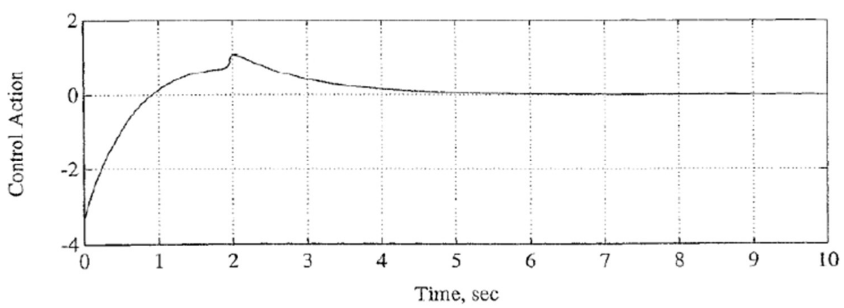


Fig. B.3 Evolution of control action with the sigmoid-like function

Such continuous approximations enable ‘sliding mode’ controllers to be utilised in situations where high frequency chattering effects would be unacceptable. It should be stressed that ideal sliding no longer takes place: the continuous control action only drives the states to a neighbourhood of the switching surface. However, arbitrarily close approximation to ideal sliding can be obtained by making $\delta$ small. In the literature, this is often referred to as pseudo-sliding. And in the simulink code of the examples in this thesis, the signum function is replace by the sigmoid-like function with a small $\delta$.

## B.2 Matlab code

### B.2.1 mfile: Product of the simulation figure

```
clear
clc

figure(1) % Figure of state variables of the system
subplot(2,2,1)
plot(out.t,out.y1,'k')
legend('x_{11}')
xlim([0,15])
ylim([-1,1.5])
xlabel('time(s)')
ylabel('State Variable x_1')

subplot(2,2,2)
plot(out.t,out.y2,'k')
legend('x_{21}')
xlim([0,15])
ylim([-1,2])
xlabel('time(s)')
ylabel('State Variable x_2')

subplot(2,2,3)
plot(out.t,out.x12,'k')
legend('x_{12}')
xlim([0,15])
ylim([-2.5,2])
xlabel('time(s)')
ylabel('State Variable x_1')
```

```
subplot(2,2,4)
plot(out.t,out.x22,'k')
legend('x_{22}')
xlim([0,15])
ylim([-2,6])
xlabel('time(s)')
ylabel('State Variable x_2')

 figure(2) %Figure of system's outputs, desired signals and controller inputs
 subplot(2,2,1)
 plot(out.t,out.y1,'k')
 hold on;
 plot(out.t,out.y1d,'--k')
 legend('y_1','y_{1d}')
 xlim([0,15])
 ylim([-0.7,2])
 xlabel('time(s)')
 ylabel('Output and Reference');
 ax = axes('Position',[0.2 0.777 0.07 0.13]);
 plot(ax,out.t,out.y1d,'--k');
 hold on;
 plot(ax,out.t,out.y1,'k');
 axis([14.99 15 0.3251 0.3275]);

 x = [0.325 0.27];
 y = [0.7 0.83];
 annotation('textarrow',x,y);

 subplot(2,2,2)
 plot(out.t,out.y2,'K')
 hold on;
 plot(out.t,out.y2d,'--K')
 legend('y_2','y_{2d}')
 xlim([0,15])
 ylim([-1,5])
```

```
xlabel('time(s)')
ylabel('Output and Reference')
ax = axes('Position',[0.68 0.7 0.2 0.075]);
plot(ax,out.t,out.y2d,'--K');
hold on;
plot(ax,out.t,out.y2,'K');
axis([7 15 -0.005 0.005]);

x = [0.75 0.73];
y = [0.65 0.7];
annotation('textarrow',x,y);

subplot(2,2,3)
plot(out.t,out.u1,'k')
legend('u_1')
xlim([0,15])
ylim([-12,15])
xlabel('time(s)')
ylabel('Controller Input u_1')

subplot(2,2,4)
plot(out.t,out.u2,'k')
legend('u_2')
xlim([0,15])
ylim([-12,12])
xlabel('time(s)')
ylabel('Controller Input u_2')
```

## B.3 Simulation screenshots

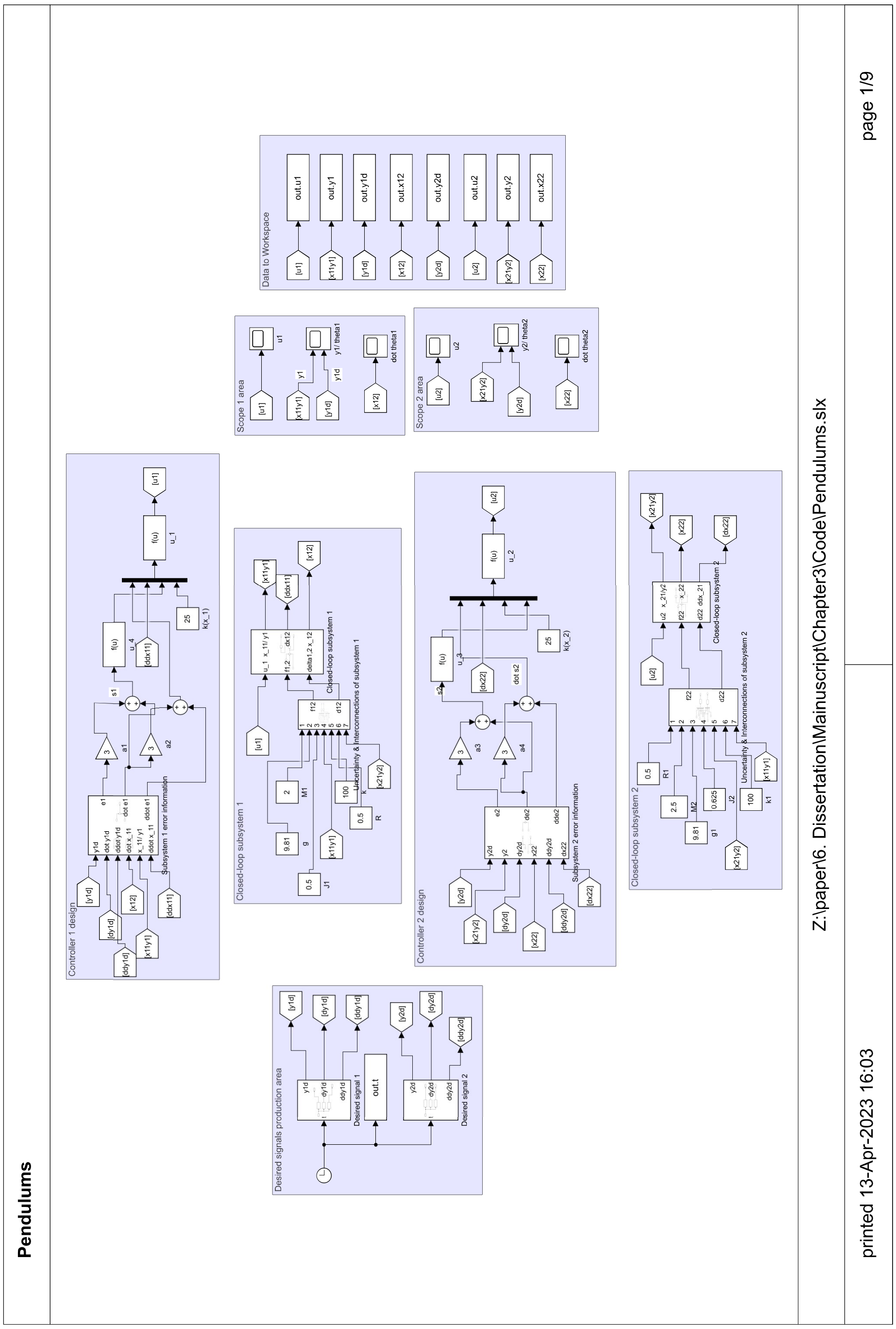


Fig. B.4 Block diagram of the inverted-pendulums system.

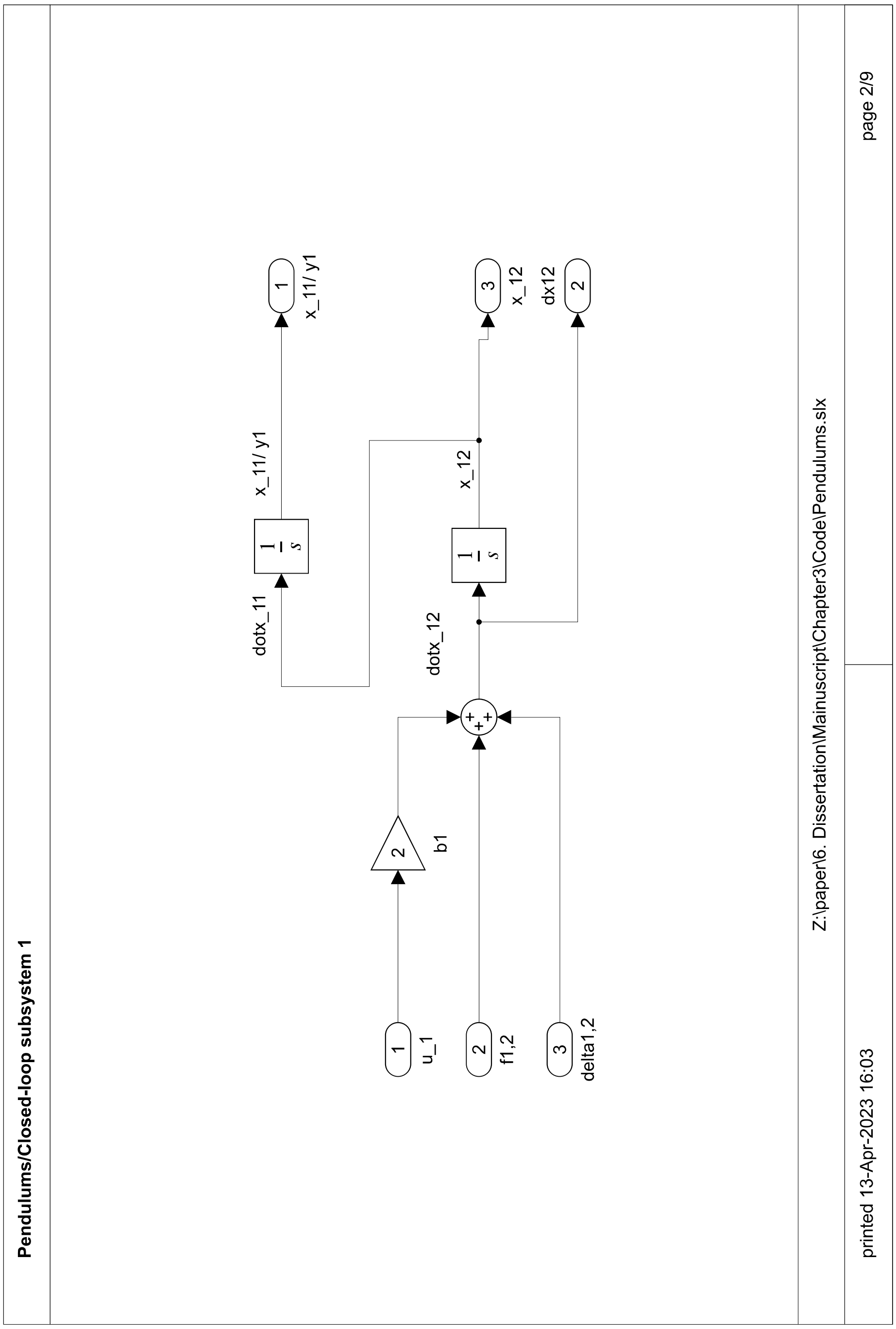


Fig. B.5 Sub-block diagram of the closed-loop subsystem 1.

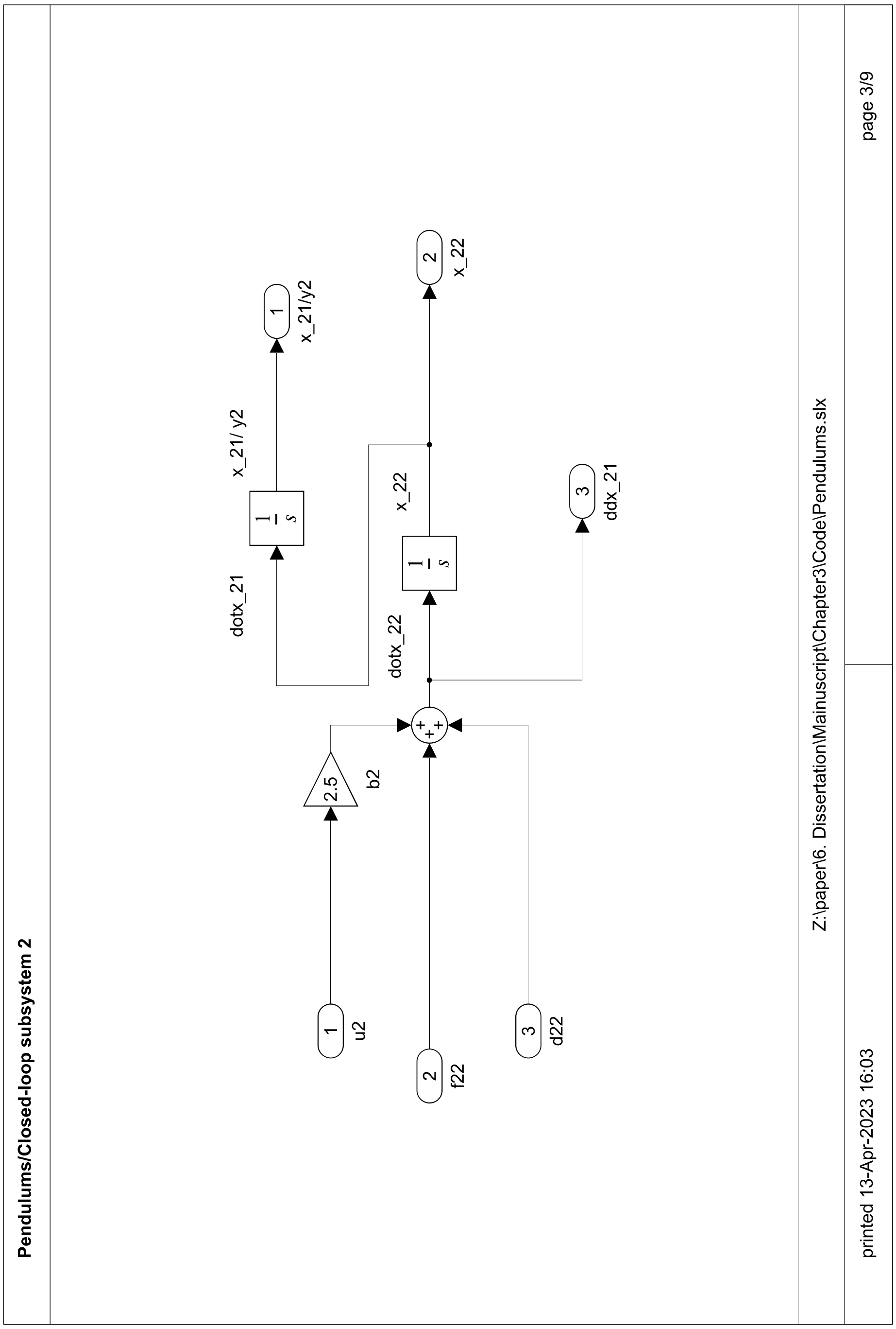


Fig. B.6 Sub-block diagram of the closed-loop subsystem 2.

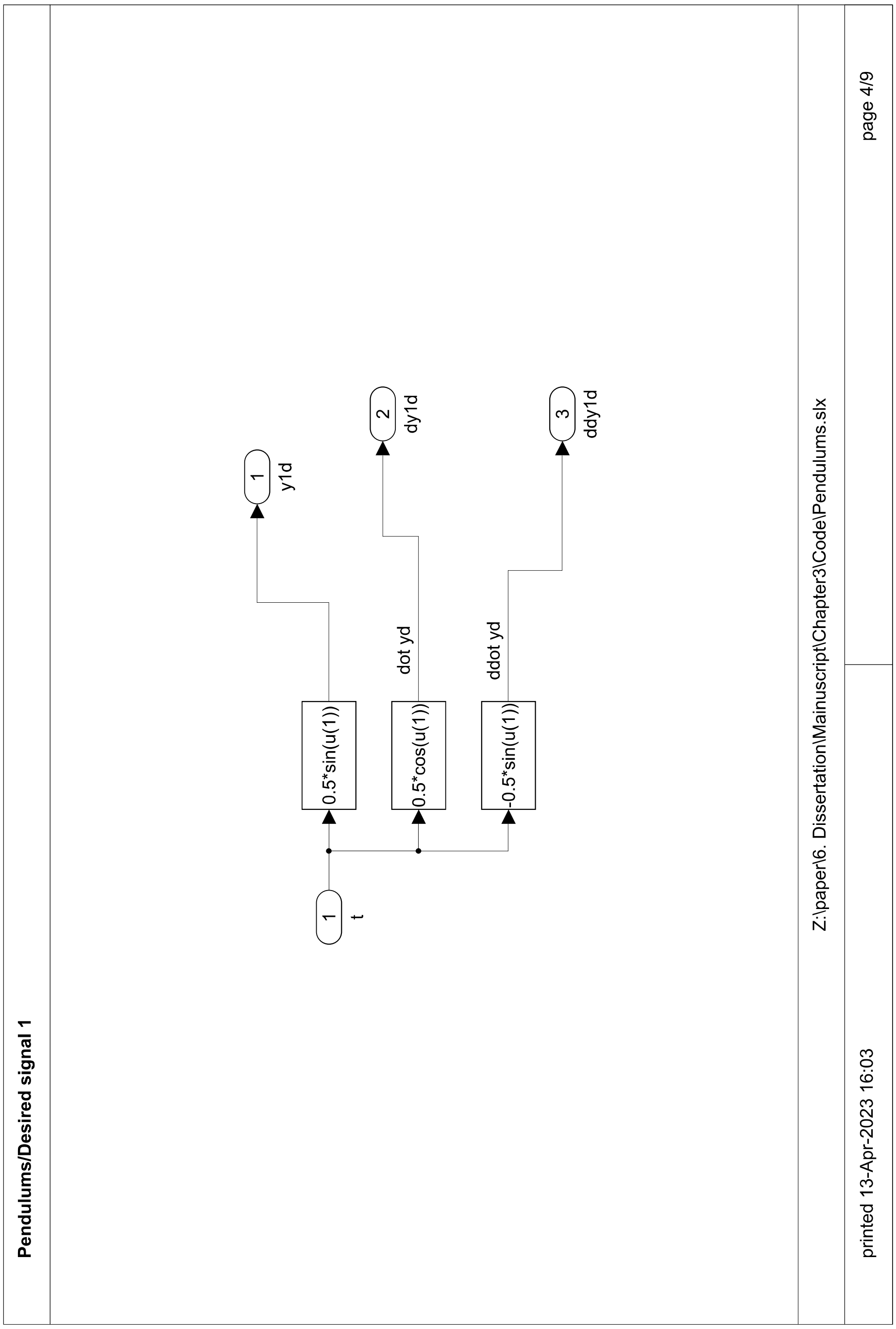


Fig. B.7 Production of the desired signal 1.

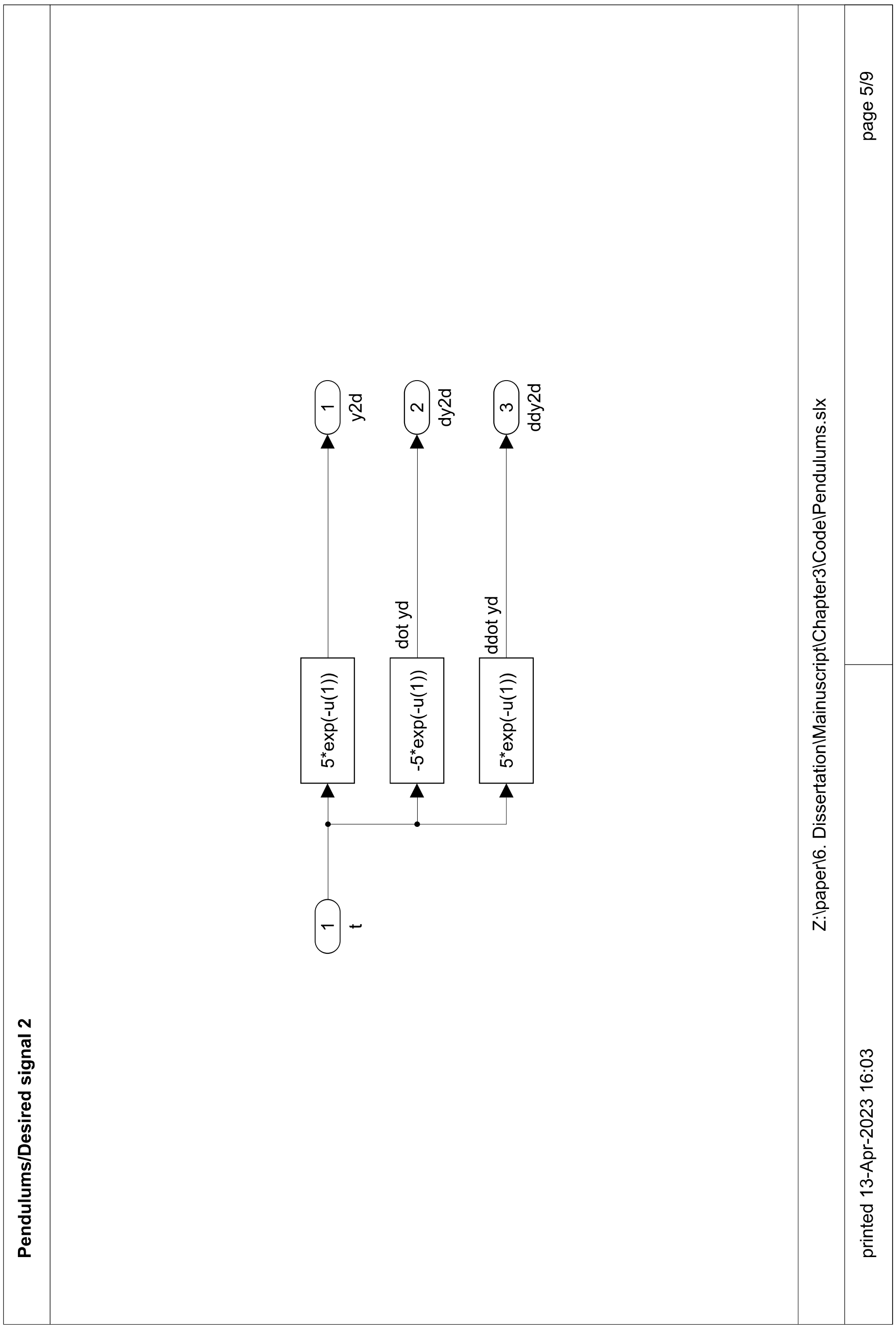


Fig. B.8 Production of the desired signal 2.

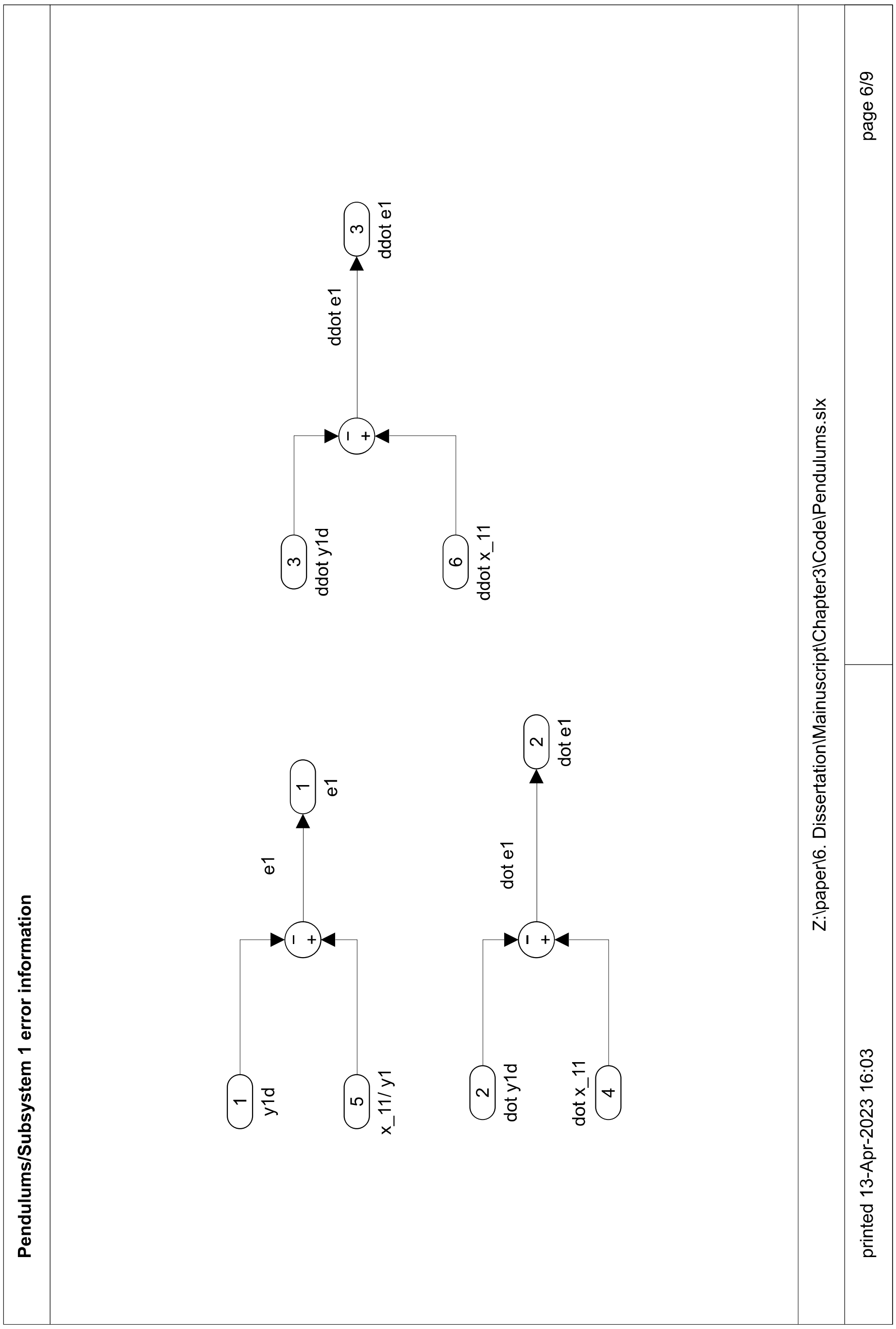


Fig. B.9 Error information of subsystem 1.

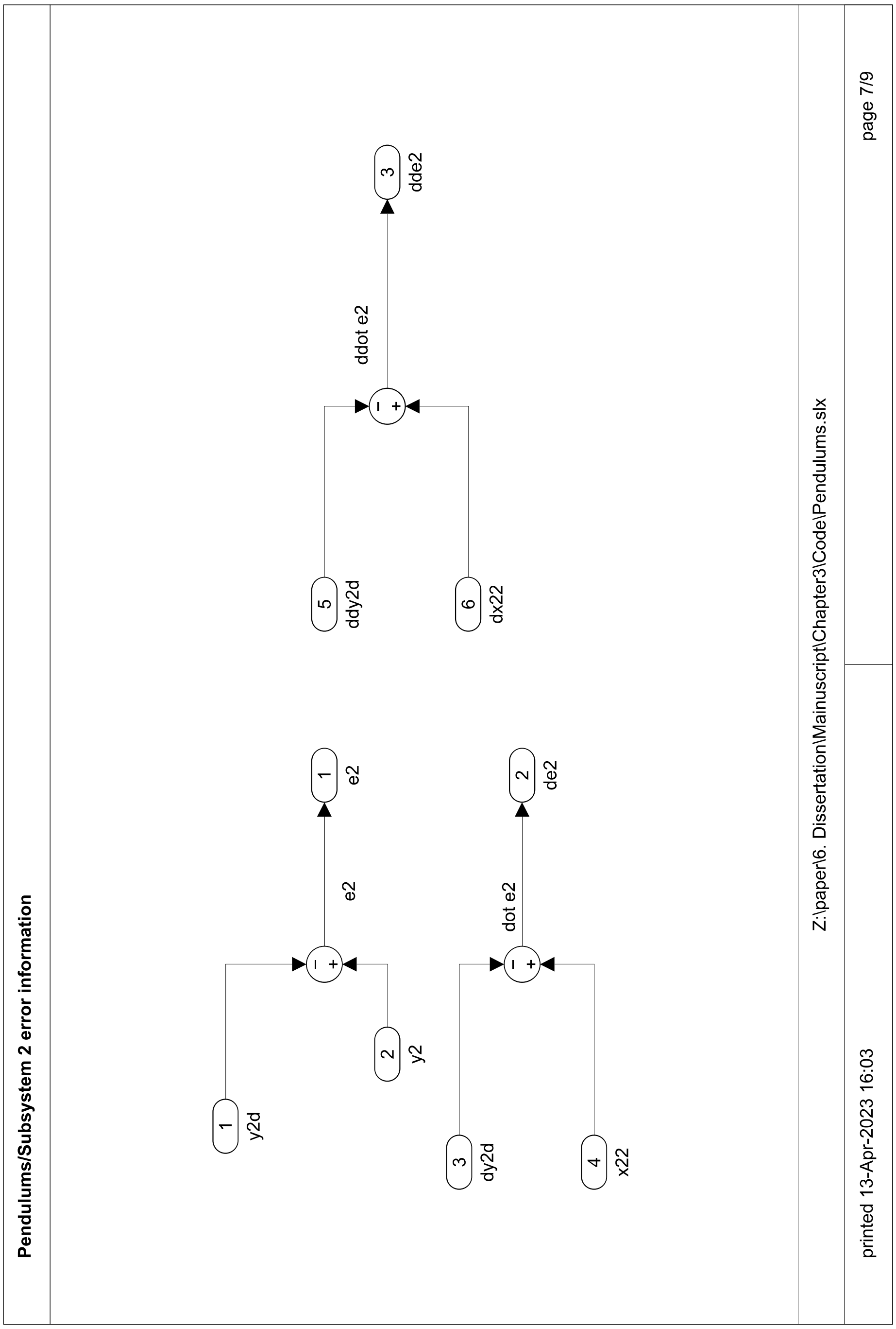


Fig. B.10 Error information of subsystem 2.

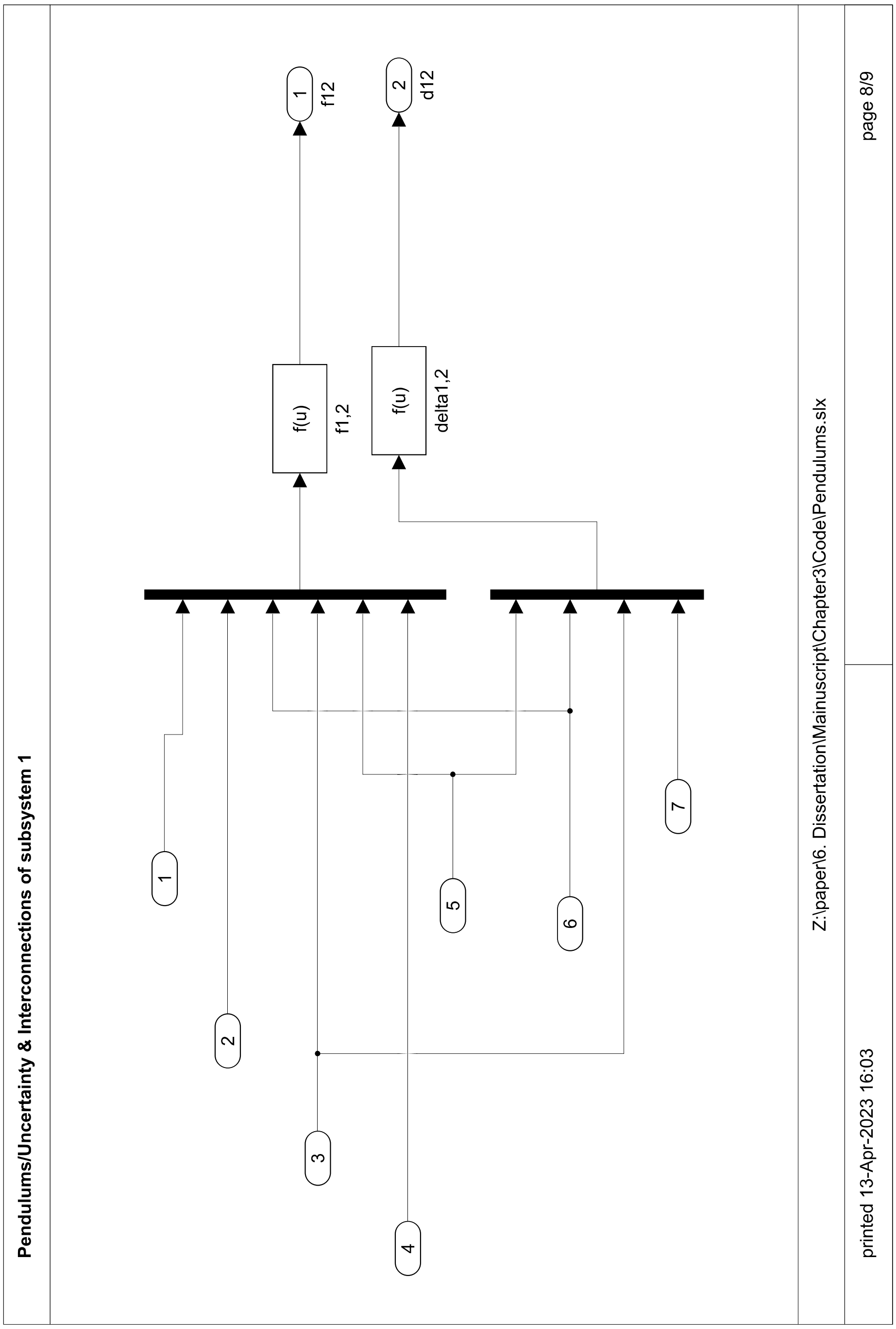


Fig. B.11 Uncertainty & Interconnections of subsystem 1.

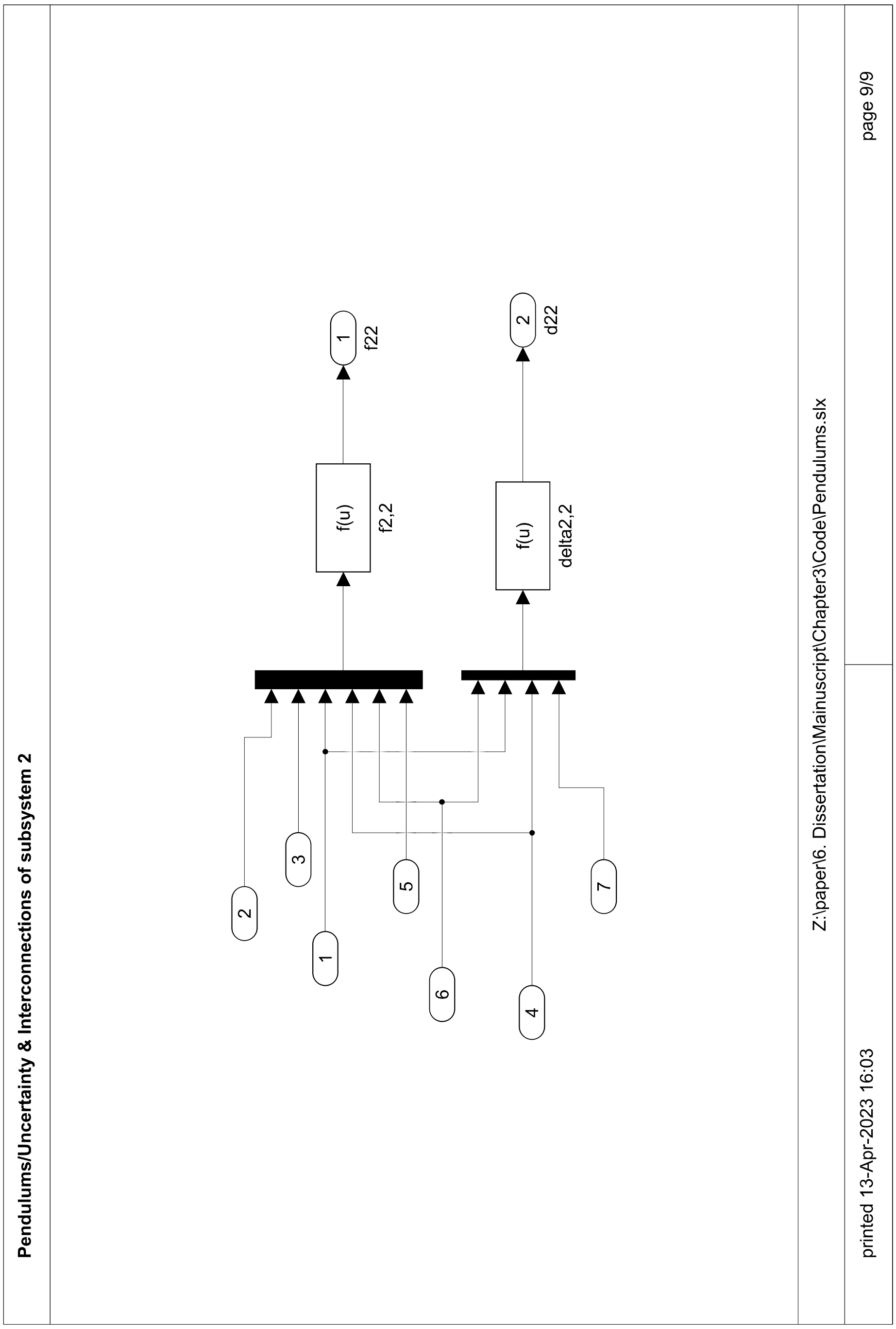


Fig. B.12 Uncertainty & Interconnections of subsystem 2.

## B.4 List of Publication

### Journal Papers

[1] Y. Ding, X. Yan, Z. Mao, S. K. Spurgeon and B. Jiang, "System structure based decentralized sliding mode output tracking control for nonlinear interconnected systems," in *International Journal of Robust and Nonlinear Control*, 33(3): 1704-1719, 2023.

[2] Y. Ding, X. Yan, Z. Mao, B. Jiang and S. K. Spurgeon, "Decentralised output tracking of interconnected systems with unknown interconnections using sliding mode control," in *International Journal of Systems Science*, 54:2, 283-294, 2022.

[3] Y. Ding, X. Yan, Z. Mao, B. Jiang and S. K. Spurgeon, "Structural Decomposition-Based Speed Tracking of Underactuated High-speed Train Using Sliding Mode Techniques," submitted to *IEEE Transaction on Intelligent Transportation Systems*.

### Conference Papers

[1] Y. Ding, X. Yan, Z. Mao and S. K. Spurgeon, "Decentralised Sliding Mode Tracking Control for a Class of Nonlinear Interconnected Systems," in *2021 American Control Conference (ACC)*, New Orleans, LA, USA, pp. 2157-2162, 2021.

[2] Y. Ding, X. Yan, Z. Mao, B. Jiang and S. K. Spurgeon, "Decentralized Sliding Mode Control for Output Tracking of Large-Scale Interconnected Systems," in *2021 60th IEEE Conference on Decision and Control (CDC)*, Austin, TX, USA, pp. 7094-7099, 2021.

[3] Y. Ding, X. Yan, Z. Mao, B. Jiang and S. K. Spurgeon, "Sliding Mode Based Decentralized Tracking Control of Underactuated Four-Body Systems," in *2022 13th Asian Control Conference (ASCC)*, Jeju, Korea, Republic of, pp. 1765-1770, 2022.

# Index