

Horofunctions and metric compactification of noncompact Hermitian symmetric spaces

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Abstract

Given a Hermitian symmetric space M of noncompact type, we show, among other things, that the metric compactification of M with respect to its Carathéodory distance is homeomorphic to a closed ball in its tangent space. We first give a complete description of the horofunctions in the compactification of M via the realisation of M as the open unit ball D of a Banach space $(V, \|\cdot\|)$ equipped with a particular Jordan structure, called a JB^* -triple. We identify the horofunctions in the metric compactification of $(V, \|\cdot\|)$ and relate its geometry and global topology, via a homeomorphism, to the closed unit ball of the dual space V^* . Finally, we show that the exponential map $\exp_0: V \rightarrow D$ at $0 \in D$ extends to a homeomorphism between the metric compactifications of $(V, \|\cdot\|)$ and (D, ρ) , preserving the geometric structure, where ρ is the Carathéodory distance on D . Consequently, the metric compactification of M admits a concrete realisation as the closed dual unit ball of $(V, \|\cdot\|)$.

Keywords Hermitian symmetric space, bounded symmetric domain, horofunction, metric compactification, JB^* -triple.

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1 Introduction

Compactifications of symmetric spaces is a particularly rich subject, which has been studied extensively [4, 15]. A variety of compactifications of symmetric spaces have been introduced with different applications in mind. For instance, Satake [32] introduced his compactifications in his study of automorphic forms, and the Martin and Furstenberg compactifications [12, 28] were introduced to analyse harmonic functions.

Recently, metric (or horofunction) compactifications with respect to invariant Finsler metrics have been used to investigate different types of compactifications of symmetric spaces. In particular, it was shown in [16, 33] that generalised Satake compactifications and Martin compactifications of symmetric spaces can be realised as metric compactifications under suitable invariant Finsler metrics. In [21] an explicit invariant Finsler metric was constructed on symmetric spaces whose metric compactification gives

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the maximal Satake compactification, and in [13] the minimal Satake compactification of $\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n$ was realised as a metric compactification.

In this paper we determine completely the metric compactification of noncompact type Hermitian symmetric spaces with respect to the Carathéodory distance and provide a detailed analysis of its geometry and global topology. The Carathéodory distance plays an important role in the geometry and analysis of Hermitian symmetric spaces.

Given a Hermitian symmetric space M of noncompact type, the Harish-Chandra embedding,

$$M \approx \mathcal{D} \subset \mathfrak{p}^+ \hookrightarrow M_c,$$

identifies M bihomorphically with a bounded symmetric domain \mathcal{D} in a complex Euclidean space \mathfrak{p}^+ , which is biholomorphic to an open dense subset of the compact dual M_c of M . By the seminal works of Loos [27] and Kaup [22], one can equip \mathfrak{p}^+ with a Jordan algebraic structure and a norm $\|\cdot\|$ so that \mathcal{D} is biholomorphic to the open unit ball D of $(\mathfrak{p}^+, \|\cdot\|)$, which is called a JB^* -triple (see also [34]). Since the Carathéodory distance is invariant under biholomorphisms, one can transfer from M to the open unit ball D in the JB^* -triple $V = (\mathfrak{p}^+, \|\cdot\|)$ to study the metric compactification and exploit Jordan theory and functional analysis. This is our task in the paper.

For symmetric spaces of noncompact type, the metric compactification with respect to the Riemannian distance can be identified with its geodesic compactification ([2] and [18, Proposition 12.6]), and is known to be homeomorphic to a Euclidean ball, see [5, Chapter II.8]. It has also been observed for various classes of finite dimensional normed spaces that the geometry and global topology of the metric compactification is closely related to the closed dual unit ball of the norm. In [19, 20] this connection was established for normed spaces with polyhedral unit balls, see also [10]. At present, however, it is unknown [21, Question 6.18] if this duality phenomenon holds for general finite dimensional normed spaces.

In [23, 25] it was shown that the duality phenomenon does not only appear in finite dimensional normed spaces, but also occurs in metric compactifications of certain symmetric spaces with invariant Finsler norms. More precisely, it was shown in [23, 25] for symmetric cones equipped with the Thompson and Hilbert distances that the geometry and global topology of the metric compactification coincides with the geometry of the closed dual unit ball of the Finsler norm in the tangent space at the basepoint of the metric compactification.

In these symmetric cones N the connection between the geometry of the metric compactification and the dual unit ball manifests itself in the following way. The horofunction boundary $N(\infty)$ in the metric compactification of the symmetric space N carries an equivalence relation where two horofunctions g and h are equivalent if $\sup_{x \in N} |g(x) - h(x)| < \infty$. This relation yields a natural partition of the horofunction boundary into equivalence classes. On the other hand, the boundary of the closed dual unit ball B^* of the Finsler norm on the tangent space $T_b N$ at the basepoint $b \in N$, is partitioned by the relative interiors of its boundary faces. For the symmetric spaces N considered in [23, 25] it was shown that there exists a homeomorphism φ from the metric compactification $N \cup N(\infty)$ onto B^* , which maps each equivalence class of $N(\infty)$ onto the relative interior of a boundary face of B^* . In this sense the dual ball captures the geometry of the metric compactification.

Moreover, in [23] it was observed that the metric compactification of N is closely related to the metric compactification of the normed space $(T_b N, \|\cdot\|_b)$, where $\|\cdot\|_b$ is the Finsler norm on the tangent space $T_b N$ at the basepoint $b \in N$. More explicitly, it was shown that the exponential map $\exp_b: T_b N \rightarrow N$ extends to a homeomorphism between the metric compactifications of the normed space $(T_b N, \|\cdot\|_b)$ and N under the invariant Finsler distance. Furthermore, the extension preserves the equivalence classes in the horofunction boundaries.

For a noncompact type Hermitian symmetric space M identified as the open unit ball D of a JB^* -triple $V = (\mathfrak{p}^+, \|\cdot\|)$, with Carathéodory distance ρ , the JB^* -triple norm $\|\cdot\|$ is the corresponding Finsler norm [22, (4.5)]. We show in this paper that the analogues of the results in [23, 25] hold for the metric compactification of (D, ρ) .

More specifically, the following results are established. We provide a complete description of the horofunctions of (D, ρ) in Theorem 4.4. We also determine the horofunctions of the JB^* -triple $(V, \|\cdot\|)$ in Theorem 5.2, and establish in Theorem 6.5 an explicit homeomorphism φ between the horofunction

compactification of $(V, \|\cdot\|)$ and its closed dual unit ball. Further, we show in Section 7 that the homeomorphism φ maps each equivalence class in the horofunction boundary of $(V, \|\cdot\|)$ onto the relative interior of a boundary face of the closed dual unit ball. Finally we prove in Theorem 8.1 that the exponential map $\exp_0: V \rightarrow D$ extends to a homeomorphism between the metric compactifications $V \cup V(\infty)$ and $D \cup D(\infty)$, which maps equivalence classes onto equivalence classes in the horofunction boundaries. Combining the results we see that the geometry and global topology of the metric compactification of (D, ρ) coincides with the geometry of the closed dual unit ball of $(V, \|\cdot\|)$.

We start by recalling the essential background on metric compactifications and the theory of JB*-triples in Sections 2 and 3, respectively.

2 Horofunctions

The origins of the idea of the metric (or horofunction) compactification go back to Gromov [2]. It has proven to be a valuable tool in numerous fields such as, geometric group theory, complex and real dynamics, Riemannian geometry, and geometric analysis. It captures asymptotic geometric properties of metric spaces and provides ways to analyse mappings or groups acting on them. Here we mostly follow the set up as in [30].

Let (X, d) be a metric space and let \mathbb{R}^X be the space of all real functions on X equipped with the topology of pointwise convergence. Fix a point $b \in X$, called the *basepoint*, and write $\text{Lip}_b^1(X)$ to denote the set of all functions $h \in \mathbb{R}^X$ such that $h(b) = 0$ and h is 1-Lipschitz, i.e., $|h(x) - h(y)| \leq d(x, y)$ for all $x, y \in X$.

The set $\text{Lip}_b^1(X)$ is a compact subset of \mathbb{R}^X . Indeed, it is easy to verify that the complement of $\text{Lip}_b^1(X)$ is open. Moreover, as $|h(x)| = |h(x) - h(b)| \leq d(x, b)$ for all $x \in X$ and $h \in \text{Lip}_b^1(X)$, we have that $\text{Lip}_b^1(X) \subseteq [-d(x, b), d(x, b)]^X$. The set $[-d(x, b), d(x, b)]^X$ is compact by Tychonoff's theorem, and hence $\text{Lip}_b^1(X)$ is a compact subset of \mathbb{R}^X .

For $y \in X$ define the real valued function,

$$h_y(z) = d(z, y) - d(b, y) \quad (z \in X). \quad (2.1)$$

Note that $h_y(b) = 0$ and $|h_y(z) - h_y(w)| = |d(z, y) - d(w, y)| \leq d(z, w)$ for all $z, w \in X$, and hence $h_y \in \text{Lip}_b^1(X)$.

Definition 2.1. Denote the closure of $\{h_y: y \in X\}$ in \mathbb{R}^X by \overline{X} , which is compact. The set $X(\infty) = \overline{X} \setminus \{h_y: y \in X\}$ is called the *horofunction boundary* of (X, d) . The elements of $X(\infty)$ are called *horofunctions*, and the set $X \cup X(\infty)$ is called the *metric (or horofunction) compactification* of (X, d) . The elements of \overline{X} are called *metric functionals*, and the metric functionals h_y in (2.1) are called *internal points*.

The topology of pointwise convergence on $\text{Lip}_b^1(X)$ coincides with the topology of uniform convergence on compact sets, see [29, p.291]. In general the topology of pointwise convergence on $\text{Lip}_b^1(X)$ is not metrizable, and hence horofunctions are limits of nets rather than sequences. However, if the metric space is separable, then the topology is metrizable and each horofunction is the limit of a sequence. In fact, one can verify that given a countable dense subset $\{y_k: k \in \mathbb{N}\}$ of (X, d) , the function σ on $\text{Lip}_b^1(X) \times \text{Lip}_b^1(X)$ given by,

$$\sigma(f, g) = \sum_k 2^{-k} \min\{1, |f(y_k) - g(y_k)|\} \quad \text{for } f, g \in \text{Lip}_b^1(X),$$

is a metric whose topology coincides with the pointwise convergence topology on $\text{Lip}_b^1(X)$, [29, p.289, Ex.10*].

The metric compactification may not be a compactification in the usual topological sense, as the embedding $\iota: y \in X \mapsto h_y \in \text{Lip}_b^1(X)$ may fail to have the necessary properties. However, the embedding $\iota: X \rightarrow \iota(X)$ is always a continuous bijection. Indeed, if $x, y \in X$, $\varepsilon > 0$, and we consider a neighbourhood $U_y = \{h \in \text{Lip}_b^1(X): |h(y) - h_x(y)| < \varepsilon\}$ of h_x , then for $z \in X$ with $d(x, z) < \varepsilon/2$ we have that $|h_z(y) - h_x(y)| \leq |d(y, z) - d(y, x)| + |d(b, z) - d(b, x)| \leq 2d(z, x) < \varepsilon$, and hence $\iota(z) \in U$, which shows that

ι is continuous. Moreover, if $x, z \in X$ and $h_x = h_z$, then $0 = (h_z(x) - h_x(x)) + (h_x(z) - h_z(z)) = 2d(x, z)$, which gives injectivity.

It can happen that $\iota: X \rightarrow \iota(X)$ does not have a continuous inverse. If, however, (X, d) is proper and geodesic, then the metric compactification will be a compactification in the usual topological sense. We provide some details of this fact below. Recall that a metric space (X, d) is *proper* if all its closed balls are compact. Note that a proper metric space is separable, as each closed ball is compact and hence separable.

A map γ from a, possibly unbounded, interval $I \subseteq \mathbb{R}$ into a metric space (X, d) is called a *geodesic path* if

$$d(\gamma(s), \gamma(t)) = |s - t| \quad \text{for all } s, t \in I.$$

The image, $\gamma(I)$, is called a *geodesic*. A metric space (X, d) is said to be *geodesic* if for each $x, y \in X$ there exists a geodesic path $\gamma: [a, b] \rightarrow X$ connecting x and y , i.e. $\gamma(a) = x$ and $\gamma(b) = y$.

Hermitian symmetric spaces with Carathéodory distance are proper geodesic metric spaces (cf. [14]). In the discussion below we will focus on the metric compactification of such metric spaces.

The horofunctions of a proper geodesic metric space (X, d) are precisely the limits of converging sequences (h_{x_k}) such that $d(b, x_k) \rightarrow \infty$. A slightly stronger assertion was shown in [30, Theorem 4.7], but for our purposes the following statement will suffice, see also [24, Lemma 2.1].

Lemma 2.2. *If (X, d) is a proper geodesic metric space, then $h \in X(\infty)$ if and only if there exists a sequence (x_k) in X with $d(b, x_k) \rightarrow \infty$ such that (h_{x_k}) converges to $h \in \overline{X}$ as $k \rightarrow \infty$.*

We use this lemma to show that, in this case, the embedding $\iota: X \rightarrow \iota(X)$ has a continuous inverse.

Lemma 2.3. *If (X, d) is a proper geodesic metric space, then $\iota: X \rightarrow \iota(X)$ is a homeomorphism.*

Proof. From the previous observations it remains to show that $\iota: X \rightarrow \iota(X)$ has a continuous inverse. Let $h_{z_0} = \iota(z_0)$ where $z_0 \in X$. Note that as (X, d) is proper it is also separable. So, to prove continuity of ι^{-1} at h_{z_0} , we can use sequences, as the topology of pointwise convergence on $\text{Lip}_b^1(X)$ is metrizable. Let (z_k) be a sequence in X with $h_{z_k} \rightarrow h_{z_0}$. By Lemma 2.2 we know that (z_k) is bounded, and hence after taking a subsequence we may assume that $z_k \rightarrow z$. It follows that $h_z = h_{z_0}$, and hence the injectivity of ι implies that $z = z_0$, which completes the proof. \square

Thus, the metric compactification is a compactification in the usual topological sense if (X, d) is a proper geodesic space.

Special horofunctions come from so-called almost geodesics sequences. They were introduced by Rieffel [30] and further developed by Walsh and co-workers in [1, 26, 35, 36]. A sequence (x_k) in (X, d) is called an *almost geodesic* if for each $\varepsilon > 0$ there exists an $N \geq 0$ such that

$$d(x_n, x_m) + d(x_m, x_0) - d(x_n, x_0) < \varepsilon \quad \text{for all } n \geq m \geq N.$$

In particular, every unbounded almost geodesic sequence yields a horofunction for a proper geodesic metric space, see [30].

Lemma 2.4. *Let (X, d) be a proper geodesic metric space. If (x_k) is an unbounded almost geodesic in (X, d) , then*

$$h(z) = \lim_k d(z, x_k) - d(b, x_k)$$

exists for all $z \in X$ and $h \in X(\infty)$.

Given a proper geodesic metric space (X, d) , a horofunction h is called a *Busemann point* if there exists an almost geodesic (x_k) in X such that $h(z) = \lim_k d(z, x_k) - d(b, x_k)$ for all $z \in X$. The collection of all Busemann points is denoted by \mathcal{B}_X .

The set of Busemann points can be equipped with a metric known as the detour distance, which was introduced in [1], and is defined as follows. Suppose (X, d) is a proper geodesic metric space and

$h, h' \in X(\infty)$ are horofunctions. Let W_h be the collection of neighbourhoods of h in \overline{X} . Then the *detour cost* is given by

$$H(h, h') = \sup_{W \in W_h} \left(\inf_{x: \iota(x) \in W} d(b, x) + h'(x) \right),$$

and the *detour distance* is defined by

$$\delta(h, h') = H(h, h') + H(h', h).$$

It is known [26, 36] that if (x_k) is an almost geodesic converging to a horofunction h , then

$$H(h, h') = \lim_k d(b, x_k) + h'(x_k) \quad (2.2)$$

for all horofunctions h' . Moreover, on the set of Busemann points \mathcal{B}_X the detour distance is a metric where points can be at infinite distance from each other, see [26, 36].

The detour distance induces a partition of \mathcal{B}_X into equivalence classes, called *parts*, where h and h' in \mathcal{B}_X are equivalent if $\delta(h, h') < \infty$. In particular, if all horofunctions are Busemann points, so $\mathcal{B}_X = X(\infty)$, then $X(\infty)$ is the disjoint union of parts, each of which is a metric space under the detour distance.

The horofunction boundary $X(\infty)$ has a natural partition induced by the equivalence relation:

$$h \sim h' \quad \text{if} \quad \sup_{x \in X} |h(x) - h'(x)| < \infty.$$

It was shown in [36, Proposition 4.5] that two Busemann points h and h' in $X(\infty)$ have finite detour distance if, and only if, $h \sim h'$. Thus, if all horofunctions are Busemann points, then the partition of $X(\infty)$ into parts coincides with the partition into equivalence classes $X(\infty)/\sim$.

We also like to note that each (surjective) isometry $\psi: X \rightarrow X$ extends as a homeomorphism to $X(\infty)$ by

$$\psi(h)(x) = h(\psi^{-1}(x)) - h(\psi^{-1}(b)) \quad (x \in X, h \in X(\infty)).$$

It is known that on the Busemann points in $X(\infty)$ the extension ψ is an isometry under the detour distance, see e.g., [26]. Also if $h \sim g$ in $X(\infty)$, then $\psi(h) \sim \psi(g)$.

3 Jordan algebraic structures

In this section, we recall some necessary definitions and results concerning Jordan algebraic structures associated with Hermitian symmetric spaces. We refer to [6, 34] for more details of symmetric manifolds and JB*-triples.

Throughout, M will denote an arbitrary Hermitian symmetric space of noncompact type. To determine the horofunctions of M and analyse the geometry and global topology of its metric compactification with respect to the Carathéodory distance (and base point $b \in M$), we make use of the fact that M can be realised as the open unit ball in a JB*-triple. Indeed, there exists a biholomorphism,

$$\psi: M \xrightarrow{\psi_1} \mathcal{D} \xrightarrow{\psi_2} D \subset V, \quad (3.1)$$

onto the open unit ball D of a finite dimensional JB*-triple V , with $\psi(b) = 0$, where ψ_1 is the Harish-Chandra embedding and ψ_2 is the Kaup Riemann mapping, which is unique up to a linear isometry [22, Theorem 4.9].

Since the biholomorphism ψ preserves the Carathéodory distance, it induces a homeomorphism between the metric compactifications of M and D . Hence we can, and will, work in the setting

$$D \subset V, \quad \text{where } D \text{ is the open unit ball of a finite dimensional JB*-triple } V.$$

The results can be transferred to the corresponding ones for M via ψ .

3.1 JB*-triples

A JB*-triple is a complex Banach space V equipped with a continuous triple product

$$\{\cdot, \cdot, \cdot\}: V \times V \times V \longrightarrow V,$$

called a *Jordan triple product*, which is linear and symmetric in the outer variables, and conjugate linear in the middle variable, and satisfies the following axioms:

- (i) $\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\}$, (JP1)
- (ii) $a \square a$ is Hermitian, that is, $\|\exp it(a \square a)\| = 1$ for all $t \in \mathbb{R}$;
- (iii) $a \square a$ has nonnegative spectrum $\sigma(a \square a)$;
- (iv) $\|a \square a\| = \|a\|^2$,

for $a, b, x, y, z \in V$, where $a \square b : V \longrightarrow V$ is a bounded linear map, called a *box operator*, defined by

$$a \square b(x) = \{a, b, x\} \quad (x \in V) \quad (3.2)$$

and condition (iv) can be replaced by

$$\|\{a, a, a\}\| = \|a\|^3 \quad (a \in V). \quad (3.3)$$

We note that the box operator in (3.2) satisfies $\|a \square b\| \leq \|a\|\|b\|$.

Remark 3.1. By definition, a Hermitian operator $T: V \longrightarrow V$ has real numerical range, which is the closed convex hull of its spectrum $\sigma(T)$ and $\|T\| = \sup\{|\lambda|: \lambda \in \sigma(T)\}$ [3, pp. 46–54]. In particular, given a, b in a JB*-triple, (i) and (ii) above implies

$$\|a \square a\| = \sup\{\lambda: \lambda \in \sigma(a \square a)\}$$

and $a \square b + b \square a$ is Hermitian. Further, if $\|a \square b + b \square a\| \leq 1$, then we have $\sigma(a \square b + b \square a) \subset [-1, 1]$.

Example 3.2. A prime example of a JB*-triple is the space of $p \times q$ complex matrices $M_{p,q}(\mathbb{C})$ with Jordan triple product,

$$\{A, B, C\} = \frac{1}{2}(AB^*C + CB^*A) \quad (A, B, C \in M_{p,q}(\mathbb{C})),$$

which has open unit ball $D = \{A \in M_{p,q}(\mathbb{C}): I - AA^* \text{ positive definite}\}$. In particular if $q = 1$, we get the complex Euclidean space \mathbb{C}^p with Jordan triple product

$$\{x, y, z\} = \frac{1}{2}(\langle x, y \rangle z + \langle z, y \rangle x) \quad (x, y, z \in \mathbb{C}^p)$$

and $D = \{z \in \mathbb{C}^p: \langle z, z \rangle < 1\}$ is the Euclidean ball.

Example 3.3. Given JB*-triples V_1, \dots, V_d , the direct sum $V_1 \oplus \dots \oplus V_d$, with the ℓ_∞ -norm,

$$\|(a_1, \dots, a_d)\|_\infty = \max\{\|a_i\|: i = 1, \dots, d\}, \quad (a_i \in V_i)$$

is a JB*-triple with the coordinatewise triple product.

A finite dimensional JB*-triple V decomposes into a finite direct sum $V_1 \oplus \dots \oplus V_d$ of so-called Cartan factors V_j ($j = 1, \dots, d$) with ℓ_∞ -norm. There are six different types of (finite dimensional) Cartan factors:

(1) $M_{p,q}(\mathbb{C})$ (2) $S_q(\mathbb{C})$ (3) $H_q(\mathbb{C})$ (4) $Sp_n(\mathbb{C})$ (5) $M_{1,2}(\mathcal{O})$ (6) $H_3(\mathcal{O})$,

where $S_q(\mathbb{C})$ and $H_q(\mathbb{C})$ are norm closed subspaces of $M_{q,q}(\mathbb{C})$ consisting of $q \times q$ skew-symmetric and

symmetric matrices, respectively; and $Sp_n(\mathbb{C})$ is a spin factor of dimension $n > 2$. The Cartan factors of types 5 and 6 are *exceptional* Cartan factors (cf. [6, Theorem 2.5.9]).

There are various operators that play an important role in the theory of JB*-triples. Besides the box operators, we will use the *Bergman operator* $B(b, c): V \rightarrow V$ and the *Möbius transformation* $g_a: D \rightarrow D$, where $a \in D$ and $b, c \in V$, which are defined as follows:

$$B(b, c)(x) = x - 2(b \square c)(x) + \{b, \{c, x, c\}, b\} \quad (x \in V), \quad (3.4)$$

$$g_a(x) = a + B(a, a)^{1/2}(I + x \square a)^{-1}(x) \quad (x \in D). \quad (3.5)$$

Here I denotes the identity operator on V , and the inverse $(I + x \square a)^{-1}: V \rightarrow V$ exists, as $\|x \square a\| \leq \|x\| \|a\| < 1$.

We note that $B(a, b)$ is invertible for $\|a\| \|b\| < 1$. The proof of the following two identities can be found in [6, Proposition 3.2.13. Lemma 3.2.17].

$$\|B(z, z)^{-1/2}\| = \frac{1}{1 - \|z\|^2} \quad (\|z\| < 1), \quad (3.6)$$

$$1 - \|g_{-y}(z)\|^2 = \frac{1}{\|B(z, z)^{-1/2}B(z, y)B(y, y)^{-1/2}\|} \quad (y, z \in D). \quad (3.7)$$

For the Euclidean ball $D \subset \mathbb{C}^d$ with inner product $\langle \cdot, \cdot \rangle$, we have from [7, Example 3.2.29] the formula

$$1 - \|g_{-y}(z)\|^2 = \frac{(1 - \|y\|^2)(1 - \|z\|^2)}{|1 - \langle y, z \rangle|^2} \quad (y, z \in D). \quad (3.8)$$

Given $a \in V$, the *quadratic operator* $Q_a: V \rightarrow V$ is defined by

$$Q_a(x) = \{a, x, a\} \quad (x \in V).$$

An element e in a JB*-triple V is called a *tripotent* if $\{e, e, e\} = e$. Although 0 is a tripotent in a JB*-triple, we are only interested in the nonzero ones, of which the norm is always 1. Tripotents in C*-algebras are exactly the partial isometries.

Any tripotent e in V induces an eigenspace decomposition of V , called the *Peirce decomposition associated with e* . The eigenvalues of the box operator $e \square e: V \rightarrow V$ are in the set $\{0, 1/2, 1\}$. Let

$$V_k(e) = \{x \in V : (e \square e)(x) = \frac{k}{2}x\} \quad (k = 0, 1, 2)$$

be the corresponding eigenspaces, called the *Peirce k -space* of e . We have the algebraic direct sum

$$V = V_0(e) \oplus V_1(e) \oplus V_2(e).$$

where the Peirce k -spaces satisfy

$$\{V_i(e), V_j(e), V_k(e)\} \subseteq V_{i-j+k}(e) \quad (3.9)$$

if $i - j + k$ belongs to the set $\{0, 1, 2\}$, and $\{V_i(e), V_j(e), V_k(e)\} = \{0\}$ otherwise. Further, we have

$$\{V_2(e), V_0(e), V\} = \{V_0(e), V_2(e), V\} = 0. \quad (3.10)$$

The Peirce k -space $V_k(e)$ is the range of the *Peirce k -projection* $P_k(e): V \rightarrow V$, which are contractive and given by

$$P_2(e) = Q_e^2, \quad P_1(e) = 2(e \square e - Q_e^2), \quad P_0(e) = B(e, e).$$

A nonzero tripotent e in a JB*-triple V is called *minimal* if $Q_e(V) = \mathbb{C}e$, or equivalently, $V_2(e) = \mathbb{C}e$. It is called *maximal* if $V_0(e) = \{0\}$. In fact, the maximal tripotents in V coincide with the extreme points of the closed unit ball V (cf. [6, Theorem 3.2.3]).

We note that, with the inherited norm from V , the Peirce 2-space $V_2(e)$ is a JB^* -algebra with identity e , where the Jordan product \circ and involution $*$ are given by

$$x \circ y = \{x, e, y\}, \quad x^* = \{e, x, e\} = Q_e x, \quad (x, y \in V_2(e)) \quad (3.11)$$

respectively [7, Example 2.4.18]. In particular, we have

$$\|x\| = \|x^*\| = \|Q_e x\| \quad (x \in V_2(e)). \quad (3.12)$$

We refer to [7, Definition 2.4.16] for the definition of a JB^* -algebra, which are examples of JB^* -triples [6, Lemma 3.1.6].

Let

$$A(e) = \{x \in V_2(e) : x^* = x\} = \{x \in V_2(e) : \{e, x, e\} = x\} \quad (3.13)$$

be the *self-adjoint part* of $V_2(e)$. Then it is a closed real subalgebra of $(V_2(e), \circ)$ satisfying

$$\|a\|^2 = \|a^2\| \leq \|a^2 + b^2\|, \quad (a, b \in A(e))$$

where $a^2 = a \circ a$, in other words, it is a so-called *JB-algebra* [17, 3.1.4].

There is a natural partial ordering \leq on $A(e)$ defined by the closed cone

$$A(e)_+ = \{x^2 : x \in A(e)\}$$

where $x \leq y$ if and only if $y - x \in A(e)_+$. We will make use of the fact that $\{a, A(e)_+, a\} \subset A(e)_+$ for all $a \in A(e)$, and

$$\|a\| \leq 1 \text{ if and only if } -e \leq a \leq e \quad (3.14)$$

(cf. [17, Proposition 3.3.6; 3.1.5]). An element $a \in V_2(e)$ is called *invertible* if there is a (unique) element a^{-1} , called the *inverse* of a , such that $a \circ a^{-1} = e$ and $a^2 \circ a^{-1} = a$. If $a \in A(e)$, then $a^{-1} \in A(e)$.

Given $a, b \in V$, we say that a is *orthogonal* to b if $a \square b = 0$. It is known that a is orthogonal to b if and only if $\{a, a, b\} = 0$. Moreover, a orthogonal to b implies b orthogonal to a , in which case we have

$$\|a + b\| = \max\{\|a\|, \|b\|\}$$

from [6, Corollary 3.1.21].

3.2 Spectral decomposition

A linear subspace $W \subset V$ of a JB^* -triple V is called a JB^* -subtriple if $x, y, z \in W$ implies $\{x, y, z\} \in W$, in the inherited Jordan triple product.

The rank r of a finite dimensional JB^* -triple V is defined by

$$r = \sup\{\dim V(a) : a \in V\},$$

where $V(a)$ denotes the smallest closed subtriple of V containing $a \in V$. It can be shown that r is the maximal number of mutually orthogonal minimal tripotents in V [7, Example 3.3.3].

In a finite dimensional JB^* -triple V of rank r , each tripotent can be decomposed as a sum of pairwise orthogonal minimal tripotents. Moreover, each element $a \in V$ admits a decomposition

$$a = \lambda_1 e_1 + \lambda_2 e_2 + \cdots + \lambda_r e_r \quad (\|a\| = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0) \quad (3.15)$$

where e_1, \dots, e_r are mutually orthogonal minimal tripotents. This is called a *spectral decomposition* of a .

Remark 3.4. The eigenvalues $\lambda_1, \dots, \lambda_r$ in the spectral decomposition (3.15) are unique, but the minimal tripotents e_i need not be unique. We can however collect terms with equal non-zero eigenvalue in the sum and write $x = \sum_{i=1}^s \mu_i c_i$, where $\mu_1 > \dots > \mu_s > 0$ and the c_i 's are (not necessarily minimal) pairwise orthogonal tripotents. In this case both the μ_i 's and c_i 's are unique, see [6, Theorem 1.2.34]. For clarity we refer to this decomposition of a , as the *unique spectral decomposition*.

Example 3.5. The nonzero tripotents in the Euclidean space \mathbb{C}^d are exactly the boundary points of the closed unit ball $\overline{D} = \{(z_1, \dots, z_d) \in \mathbb{C}^d : |z_1|^2 + \dots + |z_d|^2 \leq 1\}$ and there are no mutually orthogonal nonzero tripotents. Each element $a \in \mathbb{C}^d$ has a unique spectral decomposition $a = \|a\|(\frac{a}{\|a\|})$.

Example 3.6. In the JB*-triple $\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$ (ℓ_∞ -sum), with the standard orthonormal basis $\{e_1, e_2, e_3\}$, each element $a = (\alpha, \beta, \gamma)$ can be written as

$$a = \alpha e_1 + \beta e_2 + \gamma e_3 = |\alpha|e^{i\theta_\alpha}e_1 + |\beta|e^{i\theta_\beta}e_2 + |\gamma|e^{i\theta_\gamma}e_3 \quad (\theta_\alpha, \theta_\beta, \theta_\gamma \in \mathbb{R})$$

where $e^{i\theta_\alpha}e_1, e^{i\theta_\beta}e_2, e^{i\theta_\gamma}e_3$ are pairwise orthogonal minimal tripotents, and the sum can be written in such a way that the coefficients are in decreasing order. Trivially, if $\gamma = 0$, we can write

$$a = |\alpha|e^{i\theta_\alpha}e_1 + |\beta|e^{i\theta_\beta}e_2 + 0e_3 = |\alpha|e^{i\theta_\alpha}e_1 + |\beta|e^{i\theta_\beta}e_2 + 0e'_3 \quad (e'_3 = e^{2i}e_3).$$

Example 3.7. In the JB*-triple $H_q(\mathbb{C})$ of $q \times q$ symmetric matrices, the spectral decomposition can be derived from the usual spectral theorem for matrices. In $H_3(\mathbb{C})$, we have the following two spectral decompositions of the identity matrix:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

3.3 Peirce decomposition

Let $\{e_1, \dots, e_n\}$ a family of mutually orthogonal tripotents in a JB*-triple V . For $i, j \in \{0, 1, \dots, n\}$, the *joint Peirce space* V_{ij} is defined by

$$V_{ij} = V_{ij}(e_1, \dots, e_n) = \{z \in V : 2\{e_k, e_k, z\} = (\delta_{ik} + \delta_{jk})z \text{ for } k = 1, \dots, n\}, \quad (3.16)$$

where δ_{ij} is the Kronecker delta and $V_{ij} = V_{ji}$.

The decomposition

$$V = \bigoplus_{0 \leq i \leq j \leq n} V_{ij}$$

is called a *joint Peirce decomposition*.

The Peirce multiplication rules

$$\{V_{ij}, V_{jk}, V_{kl}\} \subset V_{il} \quad \text{and} \quad V_{ij} \square V_{pq} = \{0\} \quad \text{for } i, j \notin \{p, q\}$$

hold. The contractive projection $P_{ij}(e_1, \dots, e_n)$ from V onto $V_{ij}(e_1, \dots, e_n)$ is called a *joint Peirce projection* which satisfies

$$P_{ij}(e_1, \dots, e_n)(e_k) = \begin{cases} 0 & (i \neq j) \\ \delta_{ik}e_k & (i = j). \end{cases} \quad (3.17)$$

We shall simplify the notation $P_{ij}(e_1, \dots, e_n)$ to P_{ij} if the tripotents e_1, \dots, e_n are understood.

Let $M = \{0, 1, \dots, n\}$ and $N \subset \{1, \dots, n\}$. The Peirce k -spaces of the tripotent $e_N = \sum_{i \in N} e_i$ are given by

$$\begin{aligned} V_2(e_N) &= \bigoplus_{i, j \in N} V_{ij}, \\ V_1(e_N) &= \bigoplus_{\substack{i \in N \\ j \in M \setminus N}} V_{ij}, \\ V_0(e_N) &= \bigoplus_{i, j \in M \setminus N} V_{ij}. \end{aligned} \quad (3.18)$$

The Peirce projections provide a very useful formulation of the Bergman operators. Let e_1, \dots, e_n be mutually orthogonal tripotents in a JB^* -triple V and let $x = \sum_{i=1}^n \lambda_i e_i$ with $\lambda_i \in \mathbb{C}$. Then the Bergman operator $B(x, x)$ satisfies

$$B(x, x) = \sum_{0 \leq i \leq j \leq n} (1 - |\lambda_i|^2)(1 - |\lambda_j|^2)P_{ij},$$

where we set $\lambda_0 = 0$ and $P_{ij} = P_{ij}(e_1, \dots, e_n)$. This gives the following formulae for the square roots

$$B(x, x)^{1/2} = \sum_{0 \leq i \leq j \leq n} (1 - |\lambda_i|^2)^{1/2}(1 - |\lambda_j|^2)^{1/2}P_{ij} \quad (\|x\| < 1), \quad (3.19)$$

$$B(x, x)^{-1/2} = \sum_{0 \leq i \leq j \leq n} (1 - |\lambda_i|^2)^{-1/2}(1 - |\lambda_j|^2)^{-1/2}P_{ij} \quad (\|x\| < 1). \quad (3.20)$$

The following lemma will be useful later for computing the horofunctions in a noncompact Hermitian symmetric space.

Lemma 3.8. *Let D be the open unit ball of a JB^* -triple V . Given a sequence (y_k) in D such that $y_k \rightarrow \xi \in \partial D$, we have*

$$\lim_k \|g_{-y_k}(z)\| = 1 \quad (z \in D).$$

Proof. By (3.6) and (3.7), we have

$$\begin{aligned} 0 < 1 - \|g_{-y_k}(z)\|^2 &= \frac{1}{\|B(z, z)^{-1/2}B(z, y_k)B(y_k, y_k)^{-1/2}\|} \\ &\leq \frac{\|B(z, y_k)^{-1}B(z, z)^{1/2}\|}{\|B(y_k, y_k)^{-1/2}\|} \\ &= \|B(z, y_k)^{-1}B(z, z)^{1/2}\|(1 - \|y_k\|^2) \rightarrow 0 \quad (z \in D) \end{aligned}$$

as $k \rightarrow \infty$, since $\lim_k \|B(z, y_k)^{-1}B(z, z)^{1/2}\| = \|B(z, \xi)^{-1}B(z, z)^{1/2}\|$. \square

3.4 Identities in JB^* -triple

The identity (JP1) in the definition of a JB^* -triple is called the *main triple identity*. In the sequel, we will make use of several identities, listed below, which have been derived in [6, §1.2] and can be found in [27, Appendix]. Let V be a JB^* -triple. Then all elements $a, b \in V$ satisfy

$$(JP2) \quad (a \square b)Q_a = Q_a(b \square a),$$

$$(JP3) \quad Q_{Q_a(b)} = Q_a Q_b Q_a,$$

$$(JP4) \quad (a \square b)Q_a = \{a, \cdot, Q_a(b)\}.$$

4 Horofunctions of Hermitian symmetric spaces

Given the open unit ball D in a finite dimensional JB^* -triple V , we now determine the horofunctions of D under the Carathéodory distance ρ , with the origin $0 \in D$ as a basepoint. Recall that the horofunctions of the corresponding Hermitian symmetric space M with basepoint $b \in M$ can be obtained via ψ in (3.1), with $\psi(b) = 0$, where ψ preserves the Carathéodory distance. In fact, we have

$$M(\infty) = \{h \circ \psi : h \in D(\infty)\}.$$

The Carathéodory distance ρ on D is given by

$$\rho(x, y) = \sup\{\omega(f(x), f(y)) : f \in H(D, \mathbb{D})\} \quad (x, y \in D),$$

where $H(D, \mathbb{D})$ is the set of all holomorphic functions $f: D \rightarrow \mathbb{D}$ and ω is the Poincaré distance of the unit disc $\mathbb{D} = \{z \in \mathbb{C}: |z| < 1\}$. We will make use of the formula (cf. [7, Theorem 3.5.9]):

$$\rho(x, y) = \tanh^{-1} \|g_{-y}(x)\|.$$

For each $y, z \in D$ we have

$$h_y(z) = \rho(z, y) - \rho(0, y) = \frac{1}{2} \log \frac{1 + \|g_{-y}(z)\|}{1 - \|g_{-y}(z)\|} - \frac{1}{2} \log \frac{1 + \|y\|}{1 - \|y\|} = \frac{1}{2} \log \left(\frac{1 - \|y\|^2}{1 - \|g_{-y}(z)\|^2} \left(\frac{1 + \|g_{-y}(z)\|}{1 + \|y\|} \right)^2 \right), \quad (4.1)$$

which can also be written as

$$h_y(z) = \frac{1}{2} \log \left(\frac{1 - \|y\|^2}{1 - \|g_{-z}(y)\|^2} \left(\frac{1 + \|g_{-z}(y)\|}{1 + \|y\|} \right)^2 \right). \quad (4.2)$$

Lemma 4.1. *Let D be the open unit in a finite dimensional JB^* -triple V , and ρ the Carathéodory distance on D . Then $h \in D(\infty)$ if and only if there exists a sequence (y_k) in D with $y_k \rightarrow \xi \in \partial D$ such that*

$$h(z) = \lim_k \frac{1}{2} \log \left(\frac{1 - \|y_k\|^2}{1 - \|g_{-y_k}(z)\|^2} \right) = \lim_k \frac{1}{2} \log \left(\frac{1 - \|y_k\|^2}{1 - \|g_{-z}(y_k)\|^2} \right) \quad (4.3)$$

for all $z \in D$.

Proof. Suppose that $h \in D(\infty)$. Then by Lemma 2.2 we know that there exists a sequence (y_k) in D with $\rho(0, y_k) \rightarrow \infty$ and $\lim_k h_{y_k}(z)$ exists for all $z \in D$. By taking a subsequence we may assume that $y_k \rightarrow \xi \in \partial D$. The implication now follows from (4.1) and Lemma 3.8. On the other hand, if there exists a sequence (y_k) in D with $y_k \rightarrow \xi \in \partial D$ such that (4.3) holds for all $z \in D$, then $\rho(0, y_k) = \tanh^{-1} \|y_k\| \rightarrow \infty$, as $\|y_k\| \rightarrow 1$. So, we deduce from (4.1) and Lemma 2.2 that $h \in D(\infty)$. \square

Remark 4.2. The notion of a horofunction h on D is essentially the same as the function F introduced in [9, Lemma 4.1]. Indeed, the formula (4.3) in Lemma 4.1 for h is related to F by

$$h(x) = \frac{1}{2} \log F(x) \quad (x \in D).$$

In case D is the open Euclidean ball in \mathbb{C}^d and $y_k \rightarrow \xi$ with $\|\xi\|_2 = 1$, we find that

$$h(z) = \lim_k \frac{1}{2} \log \left(\frac{1 - \|y_k\|^2}{1 - \|g_{-y_k}(z)\|^2} \right)$$

is a horofunction, and from (3.8) we have

$$h(z) = \lim_k \frac{1}{2} \log \left(\frac{|1 - \langle z, y_k \rangle|^2}{1 - \|z\|^2} \right) = \frac{1}{2} \log \left(\frac{|1 - \langle z, \xi \rangle|^2}{1 - \|z\|^2} \right).$$

In particular, for the disc $\mathbb{D} \subset \mathbb{C}$ we find that $h: \mathbb{D} \rightarrow \mathbb{R}$ is given by the well known expression,

$$h(z) = \frac{1}{2} \log \frac{|1 - z\bar{\xi}|^2}{1 - |z|^2} = \frac{1}{2} \log \frac{|\xi - z|^2}{1 - |z|^2} \quad (z \in \mathbb{D}).$$

We now compute the limit in (4.3) for general D . By [7, Lemma 3.2.28], one can express the limit $h = \lim_k h_{y_k}$ in (4.3) in terms of the Bergman operators as

$$h(z) = \lim_{k \rightarrow \infty} \frac{1}{2} \log(1 - \|y_k\|^2) \|B(z, z)^{-1/2} B(z, y_k) B(y_k, y_k)^{-1/2}\| \quad (z \in D).$$

Let r be the rank of V . Each $y_k \in D$ has a spectral decomposition

$$y_k = \alpha_{1k} e_{1k} + \cdots + \alpha_{rk} e_{rk} \quad (4.4)$$

where e_{1k}, \dots, e_{rk} are mutually orthogonal minimal tripotents in V and

$$1 > \|y_k\| = \alpha_{1k} \geq \alpha_{2k} \geq \dots \geq \alpha_{rk} \geq 0.$$

Choosing a subsequence, we may assume for each i that the sequence (α_{ik}) converges to some $\alpha_i \in [0, 1]$ and the minimal tripotent e_{ik} converges to a minimal tripotent e_i , as the set of minimal tripotents is a closed subset of ∂D . Note that $\alpha_1 = \lim_k \|y_k\| = 1$, as $\|y_k\| \rightarrow \|\xi\| = 1$.

By [9, Lemma 5.8], we have

$$\xi = \lim_k y_k = \alpha_1 e_1 + \dots + \alpha_r e_r$$

and there exists $r_0 \in \{1, \dots, r\}$ such that

- (i) $\alpha_s > 0$ for each $1 \leq s \leq r_0$,
- (ii) $\alpha_s = 0$ for $s > r_0$,
- (iii) $\{e_1, \dots, e_{r_0}\}$ is family of mutually orthogonal minimal tripotents.

Remark 4.3. The minimal tripotents e_{1k}, \dots, e_{rk} in the spectral decomposition of y_k induce a joint Peirce decomposition of V , with joint Peirce projections $P_{ij}^k = P_{ij}(e_{1k}, \dots, e_{rk})$ and $0 \leq i \leq j \leq r$. By (3.20), the Bergman operator $B(y_k, y_k)^{-1/2}$ is of the form

$$B(y_k, y_k)^{-1/2} = \sum_{0 \leq i \leq j \leq r} (1 - \alpha_{ik}^2)^{-1/2} (1 - \alpha_{jk}^2)^{-1/2} P_{ij}^k \quad (\alpha_{0k} = 0).$$

As $(e_{sk})_k$ converges to a minimal tripotent e_s for $1 \leq s \leq r_0$, with r_0 as above, and the e_s 's are pairwise orthogonal, we have the following norm convergence,

$$\lim_{k \rightarrow \infty} P_{ij}(e_{1k}, \dots, e_{r_0k}) = P_{ij}(e_1, \dots, e_{r_0})$$

of Peirce projections (cf. [9, Remark 5.9]).

Furthermore, if e_1, \dots, e_m are mutually orthogonal tripotents and $1 \leq q < m$, then $P_{ij}(e_1, \dots, e_q) = P_{ij}(e_1, \dots, e_m)$ for all $1 \leq i \leq j \leq q$, see [9, Lemma 2.1(i)].

We will use the observations in the previous remark to prove the following theorem.

Theorem 4.4. *Let D be the open unit ball of a finite dimensional JB*-triple V , with rank r . Then the horofunction functions in $D(\infty)$ are exactly the functions of the form*

$$h(z) = \frac{1}{2} \log \left\| \sum_{1 \leq i \leq j \leq p} \lambda_i \lambda_j B(z, z)^{-1/2} B(z, e) P_{ij} \right\| \quad (z \in D),$$

where $p \in \{1, \dots, r\}$, $\lambda_i \in (0, 1]$ ($i = 1, \dots, p$) with $\max_i \lambda_i = 1$, $e = e_1 + e_2 + \dots + e_p \in \partial D$, and $P_{ij}: V \rightarrow V$ are the Peirce projections induced by the mutually orthogonal minimal tripotents e_1, \dots, e_p .

Proof. Suppose that h is a horofunction. Then by Lemma 4.1 there exists a sequence (y_k) in D converging to $\xi \in \partial D$ such that $h(z) = \lim_{k \rightarrow \infty} h_{y_k}(z)$ for all $z \in D$.

Let

$$y_k = \alpha_{1k} e_{1k} + \dots + \alpha_{rk} e_{rk}$$

be the spectral decomposition. From Remark 4.3 we know there is an $r_0 \in \{1, \dots, r\}$ such that

$$\xi = \lim_k y_k = \alpha_1 e_1 + \dots + \alpha_r e_r$$

with $\alpha_1 = 1$, $\alpha_s > 0$ for $1 \leq s \leq r_0$ and $\alpha_s = 0$ for $s > r_0$. Moreover,

$$B(y_k, y_k)^{-1/2} = \sum_{0 \leq i \leq j \leq r} (1 - \alpha_{ik}^2)^{-1/2} (1 - \alpha_{jk}^2)^{-1/2} P_{ij}^k \quad (\alpha_{0k} = 0).$$

Since $0 < 1 - \alpha_{1k}^2 \leq 1 - \alpha_{ik}^2$ for $i \in \{1, \dots, r\}$, we may assume, by choosing subsequence if necessary, for each i that

$$\frac{1 - \alpha_{1k}^2}{1 - \alpha_{ik}^2} \text{ converges to some } \lambda_i \in [0, 1].$$

Note that $\lambda_1 = 1 > \|y_k\| = \alpha_{1k}$ and $\lambda_i = 0$ for $i > r_0$. Combining this with Remark 4.3 we get that

$$\begin{aligned} \lim_{k \rightarrow \infty} (1 - \|y_k\|^2)B(y_k, y_k)^{-1/2} &= \lim_{k \rightarrow \infty} \sum_{0 \leq i \leq j \leq r} \sqrt{\frac{1 - \alpha_{1k}^2}{1 - \alpha_{ik}^2}} \sqrt{\frac{1 - \alpha_{1k}^2}{1 - \alpha_{jk}^2}} P_{ij}^k \\ &= \lim_{k \rightarrow \infty} \sum_{0 \leq i \leq j \leq r_0} \sqrt{\frac{1 - \alpha_{1k}^2}{1 - \alpha_{ik}^2}} \sqrt{\frac{1 - \alpha_{1k}^2}{1 - \alpha_{jk}^2}} P_{ij}(e_{1k}, \dots, e_{r_0k}) \\ &= \sum_{0 \leq i \leq j \leq r_0} \lambda_i \lambda_j P_{ij}, \end{aligned}$$

where P_{ij} are the Peirce projections induced by the orthogonal minimal tripotents e_1, \dots, e_{r_0} . So,

$$\begin{aligned} h(z) &= \lim_{k \rightarrow \infty} \frac{1}{2} \log(1 - \|y_k\|^2) \|B(z, z)^{-1/2} B(z, y_k) B(y_k, y_k)^{-1/2}\| \\ &= \frac{1}{2} \log \left\| \sum_{1 \leq i \leq j \leq r_0} \lambda_i \lambda_j B(z, z)^{-1/2} B(z, \xi) P_{ij}(e_1, \dots, e_{r_0}) \right\| \quad (z \in D). \end{aligned} \quad (4.5)$$

Let $p \in \{1, \dots, r_0\}$ be such that $\alpha_i = 1$ for $i \leq p$, and $\alpha_i < 1$ otherwise. Since $\lambda_i = 0$ when $\alpha_i < 1$, the horofunction h in (4.5) reduces to

$$h(z) = \frac{1}{2} \log \left\| \sum_{1 \leq i \leq j \leq p} \lambda_i \lambda_j B(z, z)^{-1/2} B(z, \xi) P_{ij}(e_1, \dots, e_p) \right\|, \quad (4.6)$$

as $P_{ij}(e_1, \dots, e_p) = P_{ij}(e_1, \dots, e_{r_0})$ for $1 \leq i \leq j \leq p$ by [9, Lemma 2.1].

Let $e = e_1 + \dots + e_p$. For $k \notin \{i, j\}$ and $w \in V_{ij}(e_1, \dots, e_{r_0})$ we have that $(e_k \square w)(V) = \{0\}$ by the Peirce multiplication rules, as $e_k \in V_{kk}(e_1, \dots, e_{r_0})$. Therefore, for $1 \leq i \leq j \leq p$,

$$\xi \square P_{ij}(e_1, \dots, e_p)(\cdot) = \xi \square P_{ij}(e_1, \dots, e_{r_0})(\cdot) = e \square P_{ij}(e_1, \dots, e_{r_0})(\cdot) = e \square P_{ij}(e_1, \dots, e_p)(\cdot),$$

and likewise

$$\begin{aligned} Q_\xi P_{ij}(e_1, \dots, e_p)(\cdot) &= \{e_1 + \alpha_2 e_2 + \dots + \alpha_{r_0} e_{r_0}, P_{ij}(e_1, \dots, e_p)(\cdot), e_1 + \alpha_2 e_2 + \dots + \alpha_{r_0} e_{r_0}\} \\ &= \{e, P_{ij}(e_1, \dots, e_p)(\cdot), e\}. \end{aligned}$$

Thus, $Q_\xi P_{ij}(e_1, \dots, e_p) = Q_e P_{ij}(e_1, \dots, e_p)$. It now follows for $1 \leq i \leq j \leq p$ that

$$B(z, \xi) P_{ij}(e_1, \dots, e_p) = P_{ij}(e_1, \dots, e_p) - 2(z \square e) P_{ij}(e_1, \dots, e_p) + Q_z Q_e P_{ij}(e_1, \dots, e_p) = B(z, e) P_{ij}(e_1, \dots, e_p).$$

Thus, the horofunction h in (4.6) can be expressed as

$$h(z) = \frac{1}{2} \log \left\| \sum_{1 \leq i \leq j \leq p} \lambda_i \lambda_j B(z, z)^{-1/2} B(z, e) P_{ij}(e_1, \dots, e_p) \right\| \quad (z \in D).$$

To prove that each function of the form (4.4) is a horofunction we let

$$h(z) = \frac{1}{2} \log \left\| \sum_{1 \leq i \leq j \leq p} \lambda_i \lambda_j B(z, z)^{-1/2} B(z, e) P_{ij} \right\| \quad (z \in D)$$

where $p \in \{1, \dots, r\}$, $\lambda_i \in (0, 1]$ ($i = 1, \dots, p$) with $\max_i \lambda_i = 1$, and $e = e_1 + e_2 + \dots + e_p \in \partial D$, with Peirce projections $P_{ij}: V \rightarrow V$ induced by the mutually orthogonal minimal tripotents e_1, \dots, e_p .

For all $k \in \mathbb{N}$ sufficiently large (depends on $\min_i \lambda_i$) we can define

$$\alpha_{ik} = \sqrt{1 - \frac{2k-1}{k^2 \lambda_i^2}}$$

for $1 \leq i \leq p$. For those k set $y_k = (1 - 1/k)e_1 + \alpha_{2k}e_2 + \dots + \alpha_{pk}e_p$ and note that $y_k \in D$.

Then the sequence (y_k) norm converges to e and

$$\lim_{k \rightarrow \infty} (1 - \|y_k\|^2)B(y_k, y_k)^{-1/2} = \lim_{k \rightarrow \infty} \sum_{0 \leq i \leq j \leq p} \sqrt{\frac{1 - (1 - 1/k)^2}{1 - \alpha_{ik}^2}} \sqrt{\frac{1 - (1 - 1/k)^2}{1 - \alpha_{jk}^2}} P_{ij} = \sum_{0 \leq i \leq j \leq p} \lambda_i \lambda_j P_{ij}.$$

Hence

$$\begin{aligned} \lim_{k \rightarrow \infty} h_{y_k}(z) &= \lim_{k \rightarrow \infty} \frac{1}{2} \log \left(\frac{1 - \|y_k\|^2}{1 - \|g_{-y_k}(z)\|^2} \right) \\ &= \lim_{k \rightarrow \infty} \frac{1}{2} \log(1 - \|y_k\|^2) \|B(z, z)^{-1/2} B(z, y_k) B(y_k, y_k)^{-1/2}\| \\ &= \frac{1}{2} \log \left\| \sum_{1 \leq i \leq j \leq p} \lambda_i \lambda_j B(z, z)^{-1/2} B(z, e) P_{ij} \right\|, \end{aligned}$$

which completes the proof. \square

We see from the proof of Theorem 4.4 that it can happen that two sequences (y_k) and (z_k) in D converging to distinct points in ∂D can give the same horofunction. Indeed, if we let y_k be as in the proof Theorem 4.4 and set $z_k = y_k + (1 - 1/\sqrt{k})(e_{p+1} + \dots + e_r)$, then both h_{y_k} and h_{z_k} converge to h given by (4.4).

We also note that the horofunction h given by (4.4) can be obtained by taking the limit of an appropriate sequence in the flat $\mathbb{R}e_1 \oplus \dots \oplus \mathbb{R}e_p$. This is consistent with the observation in [16, Lemma 4.4]. Later in Lemma 8.3 we shall show that one can obtain the horofunctions in $D(\infty)$ by taking limits along geodesics in the flats.

5 Horofunctions of finite dimensional JB*-triples

We now determine the horofunctions of finite dimensional JB*-triples $(V, \|\cdot\|)$ as normed spaces, with basepoint 0. Throughout we let r be the rank of V . As in (4.4), each element $a \in V$ has a spectral decomposition,

$$a = \lambda_1 e_1 + \lambda_2 e_2 + \dots + \lambda_r e_r, \quad (\|a\| = \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r \geq 0),$$

where e_1, \dots, e_r are mutually orthogonal minimal tripotents in V .

Given a sequence (a_k) in V with $h_{a_k} \rightarrow h \in V(\infty)$, we have $r_k = \|a_k\| \rightarrow \infty$ (by Lemma 2.2) and

$$h_{a_k}(x) = \|x - a_k\| - \|a_k\| = \frac{\|x - a_k\|^2 - \|a_k\|^2}{\|x - a_k\| + \|a_k\|} = \frac{(2r_k)^{-1}(\|(x - a_k) \square (x - a_k)\| - r_k^2)}{2^{-1}(\|r_k^{-1}(x - a_k)\| + 1)}.$$

As the denominator goes to 1 when $k \rightarrow \infty$, we need to analyse the limit

$$\lim_{k \rightarrow \infty} h_{a_k}(x) = \lim_{k \rightarrow \infty} (2r_k)^{-1}(\|(x - a_k) \square (x - a_k)\| - r_k^2) \quad (5.1)$$

to determine the form of the horofunctions.

First we note that, for each $y \in V$, the spectrum $\sigma(y \square y)$ is the set of eigenvalues in $[0, \infty)$ since $\dim V < \infty$, and by Remark 3.1, $\|y \square y\| = \sup \sigma(y \square y)$. On the other hand, V is a finite dimensional Hilbert space with inner-product

$$\langle x, y \rangle = \text{Tr}(x \square y) \quad (x, y \in V). \quad (5.2)$$

For each self-adjoint operator T on $(V, \langle \cdot, \cdot \rangle)$, we use the notation

$$\Lambda(T) = \sup\{\langle Tz, z \rangle : z \in V, \langle z, z \rangle = 1\}$$

to denote the maximum eigenvalue of T . In particular, we have

$$\Lambda(y \square y) = \sup \sigma(y \square y),$$

as $y \square y$ is a positive self-adjoint operator on the Hilbert space V (cf. [6, Lemma 1.2.22]).

Since $h_{a_k}(x) = \|x - a_k\| - \|a_k\| \geq -\|x\|$ for all $x \in V$, we have

$$(2r_k)^{-1}(\|(x - a_k) \square (x - a_k)\| - r_k^2) \geq -2\|x\| \quad (5.3)$$

for sufficiently large k , which will be useful later.

To determine the limit (5.1), and hence the form of the horofunctions, we need the following preliminary lemma.

Lemma 5.1. *Let (a_k) be a sequence in V such that $r_k = \|a_k\| \rightarrow \infty$. Let $a_k = \sum_{i=1}^r \lambda_{ik} e_{ik}$ be a spectral decomposition of a_k , with $r_k = \lambda_{1k} \geq \lambda_{2k} \geq \dots \geq \lambda_{rk} \geq 0$ and mutually orthogonal minimal tripotents e_{1k}, \dots, e_{rk} .*

If $\alpha_{ik} = r_k - \lambda_{ik} \rightarrow \alpha_i \in [0, \infty]$ and $e_{ik} \rightarrow e_i$ for all i , then

$$\begin{aligned} \lim_{k \rightarrow \infty} h_{a_k}(x) &= \lim_{k \rightarrow \infty} (2r_k)^{-1}(\|(x - a_k) \square (x - a_k)\| - r_k^2) \\ &= \sup_{u \in V_2(e_I): \langle u, u \rangle = 1} \left\langle \left(-\frac{1}{2}(e_I \square x + x \square e_I) - \sum_{i \in I} \alpha_i (e_i \square e_i) \right) u, u \right\rangle, \end{aligned} \quad (5.4)$$

where $I = \{i : \alpha_i < \infty\}$ and $e_I = \sum_{i \in I} e_i$.

Proof. We will show that every subsequence of $((2r_k)^{-1}(\|(x - a_k) \square (x - a_k)\| - r_k^2))_k$ has a convergent subsequence whose limit is the right-hand side of (5.4). Let $((2r_m)^{-1}(\|(x - a_m) \square (x - a_m)\| - r_m^2))_m$ be a subsequence. Note that

$$(2r_m)^{-1}(\|(x - a_m) \square (x - a_m)\| - r_m^2) = \Lambda \left(\frac{x \square x}{2r_m} - \frac{1}{2} \left(\frac{a_m}{r_m} \square x + x \square \frac{a_m}{r_m} \right) + \frac{1}{2r_m} (a_m \square a_m - r_m^2 I) \right), \quad (5.5)$$

where $I: V \rightarrow V$ is the identity operator.

For each m there exists $w^m \in V$ with $\langle w^m, w^m \rangle = 1$ such that

$$\begin{aligned} \Lambda \left(\frac{1}{2r_m} x \square x - \frac{1}{2} \left(\frac{a_m}{r_m} \square x + x \square \frac{a_m}{r_m} \right) + \frac{1}{2r_m} (a_m \square a_m - r_m^2 I) \right) \\ = \sup_{v \in V: \langle v, v \rangle = 1} \left\langle \left(\frac{x \square x}{2r_m} - \frac{1}{2} \left(\frac{a_m}{r_m} \square x + x \square \frac{a_m}{r_m} \right) + \frac{1}{2r_m} (a_m \square a_m - r_m^2 I) \right) v, v \right\rangle \\ = \left\langle \left(\frac{x \square x}{2r_m} - \frac{1}{2} \left(\frac{a_m}{r_m} \square x + x \square \frac{a_m}{r_m} \right) + \frac{1}{2r_m} (a_m \square a_m - r_m^2 I) \right) w^m, w^m \right\rangle. \end{aligned}$$

After taking a further subsequence, we may assume that $w^m \rightarrow w$ and $\lambda_{im}/r_m \rightarrow \mu_i \in [0, 1]$ for all i . So, $a_m/r_m \rightarrow a = \sum_{i=1}^r \mu_i e_i$. Also note that $\mu_i = 1$ for all $i \in I$, as

$$\mu_i = \lim_m \frac{\lambda_{im}}{r_m} = \lim_m \frac{r_m - \alpha_{im}}{r_m} = 1 \quad \text{for all } i \in I.$$

Using (5.5) we will prove that

$$\begin{aligned}
\lim_{m \rightarrow \infty} (2r_m)^{-1} (\|(x - a_m) \square (x - a_m)\| - r_m^2) &= \langle -\frac{1}{2}(a \square x + x \square a)w, w \rangle - \sum_{s \in I} \alpha_s \langle w_{ss}, w_{ss} \rangle \\
&\quad - \sum_{s, t \in I: s < t} \frac{1}{2} (\alpha_s + \alpha_t) \langle w_{st}, w_{st} \rangle \\
&= \langle -\frac{1}{2}(e_I \square x + x \square e_I)w, w \rangle - \sum_{s \in I} \alpha_s \langle (e_s \square e_s)w, w \rangle. \quad (5.6)
\end{aligned}$$

To establish this identity we first show that $w \in V_2(e_I)$, where $e_I = \sum_{i \in I} e_i$. Consider the Peirce decomposition $V = \bigoplus_{0 \leq s \leq t \leq r} V_{st}^m$ with respect to e_{1m}, \dots, e_{rm} and write

$$w^m = \sum_{0 \leq s \leq t \leq r} w_{st}^m.$$

Set $\lambda_{0m} = 0$ and $\alpha_{0m} = r_m$ for all m . From (3.16), we have

$$(e_{im} \square e_{im})w_{st}^m = \begin{cases} w_{st}^m & (i = s = t) \\ \frac{1}{2}w_{st}^m & (i = s \neq t \text{ or } i = t \neq s) \\ 0 & \text{otherwise.} \end{cases}$$

Therefore

$$\begin{aligned}
\langle \frac{1}{2r_m}(a_m \square a_m - r_m^2 I)w^m, w^m \rangle &= \langle \sum_{i=1}^r \frac{1}{2r_m} (\lambda_{im}^2 (e_{im} \square e_{im})w_m - r_m^2 w^m), w^m \rangle \\
&= \sum_{0 \leq s \leq r} \frac{\lambda_{sm}^2 - r_m^2}{2r_m} \langle w_{ss}^m, w_{ss}^m \rangle + \sum_{0 \leq s < t \leq r} \frac{(\lambda_{sm}^2 + \lambda_{tm}^2)/2 - r_m^2}{2r_m} \langle w_{st}^m, w_{st}^m \rangle \\
&= - \sum_{0 \leq s \leq r} \frac{\alpha_{sm}(2r_m - \alpha_{sm})}{2r_m} \langle w_{ss}^m, w_{ss}^m \rangle \\
&\quad - \sum_{0 \leq s < t \leq r} \left(\frac{\alpha_{sm}(2r_m - \alpha_{sm})}{4r_m} + \frac{\alpha_{tm}(2r_m - \alpha_{tm})}{4r_m} \right) \langle w_{st}^m, w_{st}^m \rangle. \quad (5.7)
\end{aligned}$$

Note that, as the set of minimal tripotents is compact and e_{1k}, \dots, e_{rk} are mutually orthogonal tripotents, e_1, \dots, e_r are mutually orthogonal minimal tripotents. Let $V = \bigoplus_{0 \leq s \leq t \leq r} V_{st}$ and $w = \sum_{0 \leq s \leq t \leq r} w_{st}$ be the Peirce decompositions with respect to e_1, \dots, e_r .

Then $w_{st} = 0$ if $\{s, t\} \not\subset I$. Indeed, $w_{st}^m \rightarrow w_{st}$ for all $0 \leq s \leq t \leq r$, and if $w_{st} \neq 0$ for some $\{s, t\} \not\subset I$, then the right-hand side of (5.7) goes to $-\infty$ as $m \rightarrow \infty$, since

$$\frac{\alpha_{sm}(2r_m - \alpha_{sm})}{2r_m} \geq \frac{\alpha_{sm}}{2} \rightarrow \infty \quad \text{or} \quad \frac{\alpha_{tm}(2r_m - \alpha_{tm})}{2r_m} \geq \frac{\alpha_{tm}}{2} \rightarrow \infty.$$

As $\langle \frac{x \square x}{2r_m} w^m, w^m \rangle \rightarrow 0$ and

$$\langle -\frac{1}{2} \left(\frac{a_m}{r_m} \square x + x \square \frac{a_m}{r_m} \right) w^m, w^m \rangle \rightarrow \langle -\frac{1}{2}(a \square x + x \square a)w, w \rangle,$$

we find that $(2r_k)^{-1} (\|(x - a_k) \square (x - a_k)\| - r_k^2) \rightarrow -\infty$, which contradicts (5.3). Hence

$$w \in \bigoplus_{s, t \in I: s \leq t} V_{st} = V_2(e_I).$$

Let us now show (5.6). Using the Peirce decomposition $V = V_2(e_I) \oplus V_1(e_I) \oplus V_0(e_I)$ with respect to the tripotent e_I and the Peirce multiplication rules, we find for each $z \in V_2(e_I)$ (and in particular for w) that

$$\langle (a \square x)z, z \rangle = \langle \{e_I, x, z\} + \left\{ \sum_{i \notin I} \mu_i e_i, x, z \right\}, z \rangle = \langle (e_I \square x)z, z \rangle$$

and

$$\langle (x \square a)z, z \rangle = \langle \{x, e_I, z\} + \{x, \sum_{i \notin I} \mu_i e_I, z\}, z \rangle = \langle (x \square e_I)z, z \rangle.$$

Hence we deduce from (5.7) that (5.6) holds.

From (5.6) we see that the left-hand side of (5.4) does not exceed the right-hand side. To prove equality in (5.4) pick $v \in V_2(e_I)$ with $\langle v, v \rangle = 1$ such that

$$\langle (-\frac{1}{2}(e_I \square x + x \square e_I) - \sum_{i \in I} \alpha_i (e_i \square e_i))v, v \rangle = \sup_{u \in V_2(e_I): \langle u, u \rangle = 1} \langle (-\frac{1}{2}(e_I \square x + x \square e_I) - \sum_{i \in I} \alpha_i (e_i \square e_i))u, u \rangle.$$

Let $v^m = P_2(e_I^m)v$, where $e_I^m = \sum_{i \in I} e_{im}$. By definition of w^m , we have for large m ,

$$\begin{aligned} & \langle (\frac{x \square x}{2r_m} - \frac{1}{2}(\frac{a_m}{r_m} \square x + x \square \frac{a_m}{r_m}) + \frac{1}{2r_m}(a_m \square a_m - r_m^2 I))w^m, w^m \rangle \\ & \geq \langle (\frac{x \square x}{2r_m} - \frac{1}{2}(\frac{a_m}{r_m} \square x + x \square \frac{a_m}{r_m}) + \frac{1}{2r_m}(a_m \square a_m - r_m^2 I))v^m, v^m \rangle \langle v^m, v^m \rangle^{-1}, \end{aligned}$$

and $v^m = P_2(e_I^m)v \rightarrow P_2(e_I)v = v$.

Write $v^m = \sum_{s, t \in I: s \leq t} v_{st}^m \in \bigoplus_{0 \leq s \leq t \leq r} V_{st}^m$. Then $\langle \frac{x \square x}{2r_m} v^m, v^m \rangle \rightarrow 0$ and

$$\langle -\frac{1}{2}(\frac{a_m}{r_m} \square x + x \square \frac{a_m}{r_m})v^m, v^m \rangle \rightarrow \langle -\frac{1}{2}(a \square x + x \square a)v, v \rangle = \langle -\frac{1}{2}(e_I \square x + x \square e_I)v, v \rangle.$$

As before,

$$\begin{aligned} \langle \frac{1}{2r_m}(a_m \square a_m - r_m^2 I)v^m, v^m \rangle &= - \sum_{s \in I} \frac{\alpha_{sm}(2r_m - \alpha_{sm})}{2r_m} \langle v_{ss}^m, v_{ss}^m \rangle \\ &\quad - \sum_{s, t \in I: s < t} \left(\frac{\alpha_{sm}(2r_m - \alpha_{sm})}{4r_m} + \frac{\alpha_{tm}(2r_m - \alpha_{tm})}{4r_m} \right) \langle v_{st}^m, v_{st}^m \rangle \\ &\rightarrow - \sum_{s \in I} \alpha_s \langle (e_s \square e_s)v, v \rangle. \end{aligned}$$

We conclude that

$$\begin{aligned} & \lim_{m \rightarrow \infty} (2r_m)^{-1} (\|(x - a_m) \square (x - a_m)\| - r_m^2) \\ &= \lim_{m \rightarrow \infty} \langle (\frac{x \square x}{2r_m} - \frac{1}{2}(\frac{a_m}{r_m} \square x + x \square \frac{a_m}{r_m}) + \frac{1}{2r_m}(a_m \square a_m - r_m^2 I))w^m, w^m \rangle \\ & \geq \langle -\frac{1}{2}(e_I \square x + x \square e_I)v - \sum_{s \in I} \alpha_s (e_s \square e_s)v, v \rangle, \end{aligned}$$

which completes the proof. \square

Remark. In (5.4), we have

$$\begin{aligned} & \sup_{u \in V_2(e_I): \langle u, u \rangle = 1} \langle (-\frac{1}{2}(e_I \square x + x \square e_I) - \sum_{i \in I} \alpha_i (e_i \square e_i))u, u \rangle \\ &= \sup_{u \in V_2(e_I): \langle u, u \rangle = 1} \langle (-\frac{1}{2}(e_I \square P_2(e_I)x + P_2(e_I)x \square e_I) - \sum_{i \in I} \alpha_i (e_i \square e_i))u, u \rangle \\ &= \Lambda_{V_2(e_I)}(-\frac{1}{2}(e_I \square P_2(e_I)x + P_2(e_I)x \square e_I) - \sum_{i \in I} \alpha_i (e_i \square e_i)), \quad (5.8) \end{aligned}$$

where the latter denotes the maximum eigenvalue of the operator

$$-\frac{1}{2}(e_I \square P_2(e_I)x + P_2(e_I)x \square e_I) - \sum_{i \in I} \alpha_i (e_i \square e_i)$$

restricted to the subspace $V_2(e_I)$, which it leaves invariant.

We are now ready to describe the horofunction boundary $V(\infty)$.

Theorem 5.2. *Let h be a horofunction in $V(\infty)$. Then there exist $I \subset \{1, \dots, r\}$ nonempty, mutually orthogonal minimal tripotents $e_i \in V$ and $\alpha_i \geq 0$ for $i \in I$, with $\min_{i \in I} \alpha_i = 0$, such that*

$$\begin{aligned} h(x) &= \sup \left\{ \left\langle \left(-\frac{1}{2}(e_I \square x + x \square e_I) - \sum_{i \in I} \alpha_i (e_i \square e_i) \right) u, u \right\rangle : u \in V_2(e_I), \langle u, u \rangle = 1 \right\} \\ &= \Lambda_{V_2(e_I)} \left(-\frac{1}{2}(e_I \square P_2(e_I)x + P_2(e_I)x \square e_I) - \sum_{i \in I} \alpha_i (e_i \square e_i) \right) \quad (x \in V), \end{aligned} \quad (5.9)$$

where $e_I = \sum_{i \in I} e_i$ is a tripotent with Peirce 2-space $V_2(e_I)$ and Peirce projection $P_2(e_I)$.

Conversely, each function $h: V \rightarrow \mathbb{R}$ of the form in (5.9) is a horofunction of V .

Proof. Let $h = \lim_k h_{a_k}$ be a horofunction, where $a_k \in V$ and $r_k = \|a_k\| \rightarrow \infty$. For each k , let $a_k = \sum_{i=1}^r \lambda_{ik} e_{ik}$ be a spectral decomposition, with pairwise orthogonal minimal tripotents e_{ik} and $\|a_k\| = \lambda_{1k} \geq \dots \geq \lambda_{rk} \geq 0$.

After taking a subsequence, we may assume $\alpha_{ik} = r_k - \lambda_{ik} \rightarrow \alpha_i \in [0, \infty]$ and $e_{ik} \rightarrow e_i$ for all $i = 1, \dots, r$, where e_1, \dots, e_r are mutually orthogonal minimal tripotents. Let $I = \{i: \alpha_i < \infty\}$ and $e_I = \sum_{i \in I} e_i$, and note that $I \neq \emptyset$, since $\alpha_1 = 0$.

We have

$$h_{a_k}(x) = \frac{\|x - a_k\|^2 - \|a_k\|^2}{\|x - a_k\| + \|a_k\|} = \frac{(2r_k)^{-1}(\|(x - a_k) \square (x - a_k)\| - r_k^2)}{2^{-1}(\|r_k^{-1}(x - a_k)\| + 1)}$$

and $2^{-1}(\|r_k^{-1}(x - a_k)\| + 1) \rightarrow 1$. Hence (5.9) follows readily from Lemma 5.1 and the preceding remark.

Conversely, let $h: V \rightarrow \mathbb{R}$ be of the form (5.9), where I is a nonempty subset of $\{1, \dots, r\}$, $\min_{i \in I} \alpha_i = 0$ and $e_I = \sum_{i \in I} e_i$ is the sum of mutually orthogonal minimal tripotents e_i .

We show $h \in V(\infty)$. For $k = 1, 2, \dots$, define

$$a_k = k e_I - \sum_{i \in I} \alpha_i e_i.$$

For $k \geq \max_{i \in I} \alpha_i$, we have $r_k = \|a_k\| = k$ and, in the notation of Lemma 5.1,

$$\alpha_{ik} = \begin{cases} \alpha_i & (i \in I) \\ k & (\text{otherwise}) \end{cases}$$

It follows from Lemma 5.1 that

$$\lim_{k \rightarrow \infty} (2r_k)^{-1}(\|(x - a_k) \square (x - a_k)\| - r_k^2) = \sup_{u \in V_2(e_I): \langle u, u \rangle = 1} \left\langle \left(-\frac{1}{2}(e_I \square x + x \square e_I) - \sum_{i \in I} \alpha_i (e_i \square e_i) \right) u, u \right\rangle$$

and

$$\begin{aligned} \lim_{k \rightarrow \infty} h_{a_k}(x) &= \lim_{k \rightarrow \infty} \frac{(2r_k)^{-1}(\|(x - a_k) \square (x - a_k)\| - r_k^2)}{2^{-1}(\|r_k^{-1}(x - a_k)\| + 1)} \\ &= \sup_{u \in V_2(e_I): \langle u, u \rangle = 1} \left\langle \left(-\frac{1}{2}(e_I \square x + x \square e_I) - \sum_{i \in I} \alpha_i (e_i \square e_i) \right) u, u \right\rangle = h(x) \end{aligned}$$

for all $x \in V$. Hence $h \in V(\infty)$. □

If I is a singleton in the preceding theorem, i.e., $e_I = e$ with e a minimal tripotent, then the last term of (5.9) vanishes and h is a real continuous linear functional of V . Indeed, $P_2(e)(V) = V_2(e) = \mathbb{C}e$ implies $P_2(e)x = \ell(x)e$ for some functional $\ell \in V^*$ and

$$\begin{aligned} h(x) &= \langle -\frac{1}{2}(e \square P_2(e)x + P_2(e)x \square e)e, e \rangle \langle e, e \rangle^{-1} = -\frac{1}{2}(\langle e, (\ell(x)e \square e)e \rangle + \langle (\ell(x)e \square e)e, e \rangle) \langle e, e \rangle^{-1} \\ &= -\frac{1}{2}(\overline{\ell(x)} + \ell(x)) = -\operatorname{Re} \ell(x). \end{aligned}$$

To illustrate the theorem let us consider an example. Recall that \mathbb{C}^n is a JB*-triple with triple product,

$$\{x, y, z\} = \frac{1}{2}(\langle x, y \rangle z + \langle z, y \rangle x) \quad \text{for } x, y, z \in \mathbb{C}^n, \quad (5.10)$$

with the Euclidean norm, $\|z\|_2^2 = |z_1|^2 + \dots + |z_n|^2$, as the triple norm. Using this triple product we can equip $V = \mathbb{C}^{n_1} \times \dots \times \mathbb{C}^{n_r}$ with a triple product by defining it componentwise as follows:

$$\{x, y, z\}_j = \frac{1}{2}(\langle x_j, y_j \rangle z_j + \langle z_j, y_j \rangle x_j) \quad \text{for } x_j, y_j, z_j \in \mathbb{C}^{n_j} \quad (5.11)$$

for $j = 1, \dots, r$. In this case the triple norm on V is given by $\|z\| = \max_j \|z_j\|_2$ for $z_j \in \mathbb{C}^{n_j}$, and hence its open unit ball is a product of Euclidean balls, $D = B^{n_1} \times \dots \times B^{n_r}$. In V we have that $e = (\xi_1, \dots, \xi_r)$ ($\xi_j \in \mathbb{C}^{n_j}$) is a tripotent if, and only if, for each $j = 1, \dots, r$ we have that $\xi_j = 0$ or $\|\xi_j\|_2 = 1$. Moreover, e is a minimal tripotent precisely when there exists a k with $\|\xi_k\|_2 = 1$ and $\xi_j = 0$ for all $j \neq k$.

Let us determine the form of the horofunction given by (5.9) where the tripotent $e_I = (\eta_1, \dots, \eta_r)$ satisfies $\eta_i \neq 0$ if and only if $i \in I$. So $e_I = \sum_{i \in I} e_i$, where $e_i \in V$ is the minimal tripotent whose i -th component is η_i . We have that $(P_2(e_I)x)_j = \{e_I, \{e_I, x, e_I\}, e_I\}_j = \langle x_j, \eta_j \rangle \eta_j$ for $j = 1, \dots, r$, hence

$$\begin{aligned} h(x) &= \Lambda_{V_2(e_I)}(-\frac{1}{2}(e_I \square P_2(e_I)x + P_2(e_I)x \square e_I) - \sum_{i \in I} \alpha_i (e_i \square e_i)) \\ &= \Lambda_{V_2(e_I)}(\sum_{i \in I} (-\operatorname{Re} \langle x_i, \eta_i \rangle - \alpha_i)(e_i \square e_i)) \\ &= \max_{i \in I} -\operatorname{Re} \langle x_i, \eta_i \rangle - \alpha_i. \end{aligned}$$

Remark 5.3. In the course of proving the preceding theorem, we observe that each horofunction $h \in V(\infty)$ can actually be constructed from a sequence (a_k) going to infinity along a straight line, which is a geodesic in the normed space V . In fact, the sequence $a_k = ke_I - \sum_{i \in I} \alpha_i e_i$ used in the proof lies on the straight line, $t \mapsto te_I - \sum_{i \in I} \alpha_i e_i$ in the flat $\oplus_{i \in I} \mathbb{R}e_i$. Also note that if $k \geq m$ with $k \geq \max_{i \in I} \alpha_i$, then $\|a_k\| = k$ and $h_{a_k}(a_m) = \|a_k - a_m\| - \|a_k\| = (k - m) - k = -m$, so that $h(a_m) = -m$ for all m .

By the remark we have the following corollary.

Corollary 5.4. *Each horofunction in $V(\infty)$ is a Busemann point.*

For general finite dimensional normed vector spaces it need not be true that all horofunctions are Busemann points, see [35].

6 Homeomorphism onto the dual unit ball

In this section we give a homeomorphism of the metric compactification of V onto the closed dual unit ball B^* of $(V, \|\cdot\|)$. We subsequently show in the next section that this homeomorphism maps each equivalence class in $V(\infty)/\sim$ onto the relative interior of a boundary face of D^* . So, the dual ball D^* captures the geometry of the metric compactification of V .

To prove these results we need a lemma concerning the partial ordering \leq on the set of tripotents of a JB*-triple. Given two tripotents d and e in V , we write $d \leq e$ if $e - d$ is a tripotent in V orthogonal

to d , or equivalently, if $P_2(d)e = d$ [11, (2.4)]. We also have $d \leq e$ if and only if d is a projection in the JB*-algebra $V_2(e)$, in which case

$$d = \{e, d, e\} \quad (6.1)$$

(cf. [6, pp.34-36]).

Lemma 6.1. *Given two tripotents $d, e \in V$, the following conditions are equivalent:*

- (i) $d \leq e$,
- (ii) $\{d, e, d\} + \{d, d, e\} = 2d$,
- (iii) $\{d, P_2(d)e, d\} + \{d, d, P_2(d)e\} = 2d$,
- (iv) $B(e, d)w = 0$ for all $w \in V_2(d)$.

Proof. (i) \Rightarrow (ii). If $d \leq e$, then $e = d + (e - d)$ where $e - d$ is a tripotent orthogonal to d . Hence we have $\{d, e, d\} = d = \{d, d, e\}$, which proves (ii).

(ii) \Rightarrow (iii). By (JP2), we have $\{d, P_2(d)e, d\} = \{d, e, d\} = P_2(d)(\{d, e, d\})$, the latter implies $\{d, e, d\} \in V_2(d)$. Hence (ii) implies $\{d, d, e\} = 2d - \{d, e, d\} \in V_2(d)$. Applying the box operator $d \square d$ to the Peirce decomposition

$$e = P_2(d)e + P_1(d)e + P_0(d)e$$

gives $\{d, d, e\} = P_2(d)e + \frac{1}{2}P_1(d)e$, which implies $P_1(d)e = 0$ since $P_1(d)e = 2(\{d, d, e\} - P_2(d)e) \in V_2(d) \cap V_1(d)$. It follows that

$$\{d, P_2(d)e, d\} + \{d, d, P_2(d)e\} = \{d, e, d\} + P_2(d)e = \{d, e, d\} + \{d, d, e\} = 2d.$$

(iii) \Rightarrow (i). We have noted in (3.11) that $V_2(d)$ is a JB*-algebra with identity d , which is an extreme point of the closed unit ball of $V_2(d)$. By (iii), we have

$$d = \frac{1}{2}\{d, e, d\} + \frac{1}{2}P_2(d)e$$

which implies $P_2(d)e = d$ as $\|\{d, e, d\}\|, \|P_2(d)e\| \leq 1$.

(i) \Rightarrow (iv). Since $e = d + (e - d)$ and $d \square (e - d) = 0$, we have $e \square d = d \square d$. Let $w \in V_2(d)$. Using (6.1) and the Jordan triple identity (JP1), we deduce

$$\begin{aligned} B(e, d)w &= w - 2\{e, d, w\} + \{e, \{d, w, d\}, e\} \\ &= w - 2\{e, d, w\} + 2\{e, d, \{e, d, w\}\} - \{w, d, \{e, d, e\}\} \quad (\text{by (JP1)}) \\ &= w - 2\{d, d, w\} + 2\{d, d, \{d, d, w\}\} - \{w, d, d\} \quad (\text{by (6.1)}) \\ &= w - 2w + 2w - w = 0. \end{aligned}$$

(iv) \Rightarrow (i). Let $a = P_2(d)e \in V_2(d)$. Then $\{d, d, a\} = a$ and $\{d, a, a\} \in V_2(d)$ implies $\{d, d, \{d, a, a\}\} = \{d, a, a\}$. We show $a = d$.

Observe that $Q_d(a) = Q_d(e)$. Hence (JP3) implies

$$Q_d Q_a Q_d = Q_{Q_d(a)} = Q_{Q_d(e)} = Q_d Q_e Q_d. \quad (6.2)$$

By (JP2) and (JP4), we have

$$Q_d(a \square d) = (d \square a)Q_d = \{d, \cdot, Q_d(a)\} = \{d, \cdot, Q_d(e)\} = (d \square e)Q_d = Q_d(e \square d). \quad (6.3)$$

Using (6.2) and (6.3), one deduces that

$$Q(d)B(a, d)w = (Q(d) - 2Q_d(a \square d) + Q_d Q_a Q_d)(w) = Q(d)B(e, d)w = 0, \quad (w \in V_2(d)).$$

Now take $w = d - \{d, a, d\} \in V_2(d)$. Then we have

$$\begin{aligned}
B(a, d)w &= B(a, d)d - B(a, d)\{d, a, d\} \\
&= d - 2a + \{a, d, a\} - (\{d, a, d\} - 2\{a, d, \{d, a, d\}\} + \{a, \{d, \{d, a, d\}, d\}, a\}) \\
&= d - 2a + \{a, d, a\} - \{d, a, d\} + 2\{d, a, a\} - \{a, a, a\} \\
&= \{d - a, d - a, d - a\},
\end{aligned}$$

where $\{d, a, a\} = \{d, a, \{d, d, a\}\} = \{\{d, a, d\}, d, a\} - \{d, \{a, d, d\}, a\} + \{d, d, \{d, a, a\}\} = \{\{d, a, d\}, d, a\}$. As $\{d - a, d - a, d - a\} \in V_2(d)$, it follows from (3.12) that

$$\|\{d - a, d - a, d - a\}\| = \|Q_d\{d - a, d - a, d - a\}\| = \|Q_d B(a, d)w\| = 0,$$

so that $d - a = 0$ by (3.3). \square

Remark 6.2. Given tripotents $d, e \in V$ and identity map $I: V \rightarrow V$, we have $\|d \square P_2(d)e + P_2(d)e \square d\| \leq 2$ and $2I - (d \square P_2(d)e + P_2(d)e \square d)$ is a positive operator on the Hilbert space $(V_2(d), \langle \cdot, \cdot \rangle)$ by Remark 3.1. Hence $\langle (2I - (d \square P_2(d)e + P_2(d)e \square d))v, v \rangle \geq 0$ for all $v \in V_2(d)$. Moreover, $\langle (2I - (d \square P_2(d)e + P_2(d)e \square d))v, v \rangle = 0$ for all $v \in V_2(d)$ is equivalent to $2I = d \square P_2(d)e + P_2(d)e \square d$ on $V_2(d)$.

The previous lemma has the following useful corollary.

Corollary 6.3. *Suppose that h and h' are horofunctions given, respectively, by (5.9) and*

$$h'(x) = \Lambda_{V_2(c_J)}(-\frac{1}{2}(c_J \square P_2(c_J)x + P_2(c_J)x \square c_J) - \sum_{j \in J} \beta_j(c_j \square c_j)).$$

Let $a = \sum_{i \in I} \alpha_i e_i$ and $b = \sum_{j \in J} \beta_j c_j$. Then $h = h'$ if and only if $e_I = c_J$ and $a = b$.

Proof. The sufficiency follows from the observation that $e_I = c_J$ and $a = b$ implies that

$$\sum_{i \in I} \alpha_i (e_i \square e_i) = (\sum_{i \in I} \alpha_i e_i) \square e_I = (\sum_{j \in J} \beta_j c_j) \square c_J = \sum_{j \in J} \beta_j (c_j \square c_j).$$

Conversely, let $\lim_k h_{a_k} = h = h' = \lim_k h_{b_k}$, where we can choose $a_k = ke_I - a$ and $b_k = kc_J - b$ by Remark 5.3. As $h(a_k) = -k$, we have $k + h'(a_k) = 0$ and

$$\begin{aligned}
k + h'(a_k) &= k + \sup_{v \in V_2(c_J): \langle v, v \rangle = 1} \langle (-\frac{1}{2}(c_J \square (ke_I - a) + (ke_I - a) \square c_J) - \sum_{j \in J} \beta_j (c_j \square c_j))v, v \rangle \\
&= k + \sup_{v \in V_2(c_J): \langle v, v \rangle = 1} (\langle -\frac{k}{2}((c_J \square P_2(c_J)e_I) + (P_2(c_J)e_I \square c_J))v, v \rangle \\
&\quad + \langle (\frac{1}{2}(c_J \square a + a \square c_J) - \sum_{j \in J} \beta_j (c_j \square c_j))v, v \rangle) \\
&= k(\sup_{v \in V_2(c_J): \langle v, v \rangle = 1} (\langle (I - \frac{1}{2}((c_J \square P_2(c_J)e_I) + (P_2(c_J)e_I \square c_J)))v, v \rangle \\
&\quad + \langle (\frac{1}{2k}(c_J \square a + a \square c_J) - \frac{1}{k} \sum_{j \in J} \beta_j (c_j \square c_j))v, v \rangle))
\end{aligned}$$

for $k = 1, 2, \dots$. So we find that $\langle (I - \frac{1}{2}(c_J \square P_2(c_J)e_I + P_2(c_J)e_I \square c_J))v, v \rangle = 0$ for all $v \in V_2(c_J)$, and hence $2I = c_J \square P_2(c_J)e_I + P_2(c_J)e_I \square c_J$ by Remark 6.2 on $V_2(c_J)$. Moreover,

$$\sup_{v \in V_2(c_J): \langle v, v \rangle = 1} \langle (\frac{1}{2}(c_J \square a + a \square c_J) - \sum_{j \in J} \beta_j (c_j \square c_j))v, v \rangle = 0. \quad (6.4)$$

In particular, we have $2c_J = (c_J \square P_2(c_J)e_I)(c_J) + (P_2(c_J)e_I \square c_J)(c_J)$, so $c_J \leq e_I$ by Lemma 6.1.

Analogously, $0 = k + h'(b_k) = k + h(b_k)$ implies $e_I \leq c_J$ and

$$\sup_{v \in V_2(e_I): \langle v, v \rangle = 1} \left\langle \left(\frac{1}{2}(e_I \square b + b \square e_I) - \sum_{j \in J} \alpha_i(e_i \square e_i) \right) v, v \right\rangle = 0. \quad (6.5)$$

We conclude that $e_I = c_J$ and note that this implies that

$$\frac{1}{2}(c_J \square a + a \square c_J) = \frac{1}{2}(e_I \square a + a \square e_I) = \sum_{i \in I} \alpha_i(e_i \square e_i)$$

and

$$\frac{1}{2}(e_I \square b + b \square e_I) = \frac{1}{2}(c_J \square b + b \square c_J) = \sum_{j \in J} \beta_j(c_j \square c_j).$$

It now follows from (6.4) and (6.5) that $\sum_{i \in I} \alpha_i(e_i \square e_i) = \sum_{j \in J} \beta_j(c_j \square c_j)$ on $V_2(e_I) = V_2(c_J)$, as both operators are self-adjoint. In particular,

$$a = \sum_{i \in I} \alpha_i e_i = \sum_{i \in I} \alpha_i(e_i \square e_i)(e_I) = \sum_{j \in J} \beta_j(c_j \square c_j)(c_J) = \sum_{j \in J} \beta_j c_j = b.$$

□

To show that the metric compactification $V \cup V(\infty)$ is homeomorphic to closed dual unit ball of $(V, \|\cdot\|)$, we identify V with its (algebraic) dual space V^* by using an inner-product on V , which we can do as V is finite dimensional. For convenience we adjust the inner-product $\langle \cdot, \cdot \rangle$ on V to define a new inner-product $[\cdot, \cdot]$ such that $[c, c] = 1$ for each minimal tripotent $c \in V$.

To realise this we note that a finite dimensional JB*-algebra V decomposes into a finite ℓ_∞ -sum, $V = V_1 \oplus \cdots \oplus V_d$, of *Cartan factors*, each of which contains a minimal tripotent (cf. [7, Theorem 3.3.5, Theorem 3.8.17]). In a finite dimensional Cartan factor V_j , the tripotents form a compact submanifold M_j of V_j , in which the minimal tripotents form a connected component N_j [27, §5]. As shown in [8, Proposition 2.2], the tangent space $T_e(N_j)$ at each $e \in N_j$ identifies with $iA(e) \oplus V_1(e)$, where $A(e)$ is defined in (3.13).

In particular, $\dim(iA(e) \oplus V_1(e)) = 1 + \dim V_1(e)$ is constant for all $e \in N_j$. On V_j we define, for a fixed $e \in N_j$, a normalised trace form

$$\langle x, y \rangle_j = \frac{1}{1 + \dim V_1(e)/2} \text{Trace}(x \square y) \quad (x, y \in V_j),$$

so that $\langle c, c \rangle_j = 1$ for all $c \in N_j$, and $(V_j, \langle \cdot, \cdot \rangle_j)$ is a Hilbert space.

Henceforth, we denote by $[\cdot, \cdot]$ the inner-product of the Hilbert space direct sum $V = V_1 \oplus \cdots \oplus V_d$. Then each minimal tripotent $c \in V$ lies in some V_j and $[c, c] = 1$. Moreover, the inner-product is associative, i.e.,

$$[\{a, b, y\}, z] = [y, \{b, a, z\}] \quad (a, b, y, z \in V)$$

(cf. [6, (2.31)]). In particular, we have $[P_2(c)y, z] = [y, P_2(c)z]$ for each tripotent $c \in V$. We note that if $a, b \in V$ are triple orthogonal, i.e., $a \square b = 0$, then $[a, b] = 0$.

By the Riesz representation theorem, the map

$$x \in V \mapsto \tilde{x} = [\cdot, x] \in V^* \quad (6.6)$$

is a conjugate linear isomorphism.

Let $D^* = \{\tilde{x} \in V^*: \|\tilde{x}\|_* \leq 1\}$ be the closed dual unit ball in dual space of the JB*-triple $(V, \|\cdot\|)$. Then for each $\tilde{x} = [\cdot, x] \in V^*$, where x has spectral decomposition $x = \sum_{j=1}^r \lambda_j c_j \in V$, we have $\|\tilde{x}\|_* = \sum_{j=1}^r \lambda_j$. Indeed,

$$\|\tilde{x}\|_* = \sup_{\|y\|=1} |[y, x]| = \sup_{\|y\|=1} \sum_{j=1}^r \lambda_j |[y, c_j]| = \sup_{\|y\|=1} \sum_{j=1}^r \lambda_j |[P_2(c_j)y, c_j]| \leq \sup_{\|y\|=1} \sum_{j=1}^r \lambda_j [c_j, c_j] = \sum_{j=1}^r \lambda_j,$$

as $P_2(c_j)y = \mu c_j$ for some $\mu \in \mathbb{C}$ and $|\mu| = \|P_2(c_j)y\| \leq \|y\| = 1$. On the other hand, $y = \sum_{j=1}^r c_j$ satisfies $\|y\| \leq 1$ and hence $\|\tilde{x}\|_* \geq |\sum_{j=1}^r [c_j, x]| = \sum_{j=1}^r \lambda_j$.

We define

$$D_0 = \{x \in V : x = \sum_{j=1}^r \lambda_j c_j \text{ spectral decomposition with } \sum_{j=1}^r \lambda_j \leq 1\}. \quad (6.7)$$

It follows from the previous observations that the conjugate linear isomorphism $x \in V \mapsto \tilde{x} \in V^*$ in (6.6) maps D_0 onto D^* , and hence D_0 is the closed unit ball of a norm on V .

Thus, to prove that $V \cup V(\infty)$ is homeomorphic to D^* , it suffices to show that there exists a homeomorphism φ from $V \cup V(\infty)$ onto D_0 , which is what we will do.

Remark 6.4. Given $x \in V$ with spectral decomposition $x = \sum_{i=1}^r \lambda_i e_i$, so r is the rank of V , we have that the eigenvalues $\lambda_1 \geq \dots \geq \lambda_r \geq 0$ are unique, but the pairwise orthogonal minimal tripotents e_i need not be unique. We can however collect terms with equal non-zero eigenvalue in the sum and write $x = \sum_{i=1}^s \mu_i d_i$, where $\mu_1 > \dots > \mu_s > 0$ and the d_i 's are (not necessarily minimal) pairwise orthogonal tripotents. In this case both the μ_i 's and d_i 's are unique, see [27, Corollary 3.12]. For clarity we refer to this decomposition of x , as the *unique spectral decomposition*.

We define the map $\varphi: V \cup V(\infty) \longrightarrow D_0$ by

$$\varphi(x) = \frac{\sum_{i=1}^r (\exp \lambda_i - \exp(-\lambda_i)) c_i}{\sum_{i=1}^r (\exp \lambda_i + \exp(-\lambda_i))} \quad (6.8)$$

for each $x \in V$ with spectral decomposition $x = \sum_{i=1}^r \lambda_i c_i \in V$, and

$$\varphi(h) = \frac{\sum_{i \in I} \exp(-\alpha_i) e_i}{\sum_{i \in I} \exp(-\alpha_i)} \quad (6.9)$$

for $h \in V(\infty)$ of the form,

$$h(x) = \Lambda_{V_2(e_I)} \left(-\frac{1}{2} (e_I \square P_2(e_I)x + P_2(e_I)x \square e_I) - \sum_{i \in I} \alpha_i (e_i \square e_i) \right) \quad (x \in V).$$

To see that $\varphi(x)$ is well-defined note that the right-hand side of (6.8) is the spectral decomposition of $\varphi(x)$. Moreover, if $\lambda_i = 0$, then the corresponding coefficient in $\varphi(x)$ is also 0. So by switching to the unique spectral decomposition we find that $\varphi(x)$ is well-defined by Remark 6.4. Also $\varphi(h)$ is well defined. Indeed, if h was expressed as

$$h(x) = \Lambda_{V_2(c_J)} \left(-\frac{1}{2} (c_J \square P_2(c_J)x + P_2(c_J)x \square c_J) - \sum_{j \in J} \beta_j (c_j \square c_j) \right) \quad (x \in V),$$

then $e_I = c_J$ and $a = \sum_{i \in I} \alpha_i e_i = \sum_{j \in J} \beta_j c_j = b$ by Corollary 6.3. Viewing a and b in the JB*-subtriple $V_2(e_I) = V_2(c_J)$ we may assume after relabelling that $I = J$ and $\alpha_i = \beta_i$ for all $i \in I = J$. Now using the unique spectral decomposition and the fact that $e_I = c_J$, we find that $\varphi(h)$ is well-defined.

Theorem 6.5. *The map $\varphi: V \cup V(\infty) \longrightarrow D_0$ is a homeomorphism.*

The proof of the theorem will be split up in several lemmas. Note that the interior of D_0 , denoted $\text{int } D_0$, consists of those $x = \sum_{j=1}^r \lambda_j c_j$ with $\sum_{j=1}^r \lambda_j < 1$, and the boundary, ∂D_0 , of D_0 are precisely those $x = \sum_{j=1}^r \lambda_j c_j$ with $\sum_{j=1}^r \lambda_j = 1$. This follows from the fact that $x \mapsto \tilde{x} = [\cdot, x]$ is a conjugate linear isomorphism mapping D_0 onto D^* .

Lemma 6.6. *We have $\varphi(V) \subset \text{int } D_0$ and $\varphi(V(\infty)) \subset \partial D_0$.*

Proof. The right-hand side of (6.8) is the spectral decomposition of $\varphi(x)$ and

$$0 \leq \left(\sum_{i=1}^r e^{\lambda_i} + e^{-\lambda_i} \right)^{-1} \left(\sum_{i=1}^r e^{\lambda_i} - e^{-\lambda_i} \right) < 1,$$

so $\varphi(x) \in \text{int } D_0$. Clearly, $\varphi(h) \in \partial D_0$. □

Lemma 6.7. *The map $\varphi: V \cup V(\infty) \rightarrow D_0$ is continuous on V .*

Proof. Let (v_k) be a sequence in V converging to $v \in V$. To show $\varphi(v_k) \rightarrow \varphi(v)$ as $k \rightarrow \infty$, we show that each subsequence of (v_k) contains a subsequence (v_n) satisfying $\varphi(v_n) \rightarrow \varphi(v)$ as $n \rightarrow \infty$.

For each k , let $v_k = \sum_{i=1}^r \mu_{ik} c_{ik}$ be a spectral decomposition. By convergence, (v_k) is bounded. Hence each subsequence of (v_k) contains a subsequence

$$v_n = \sum_{i=1}^r \mu_{in} c_{in}$$

such that $\mu_{in} \rightarrow \mu_i \geq 0$ and $c_{in} \rightarrow c_i$ as $n \rightarrow \infty$, where c_1, \dots, c_r are mutually orthogonal minimal tripotents.

It follows that $v = \lim_n v_n = \sum_{i=1}^r \mu_i c_i$ and

$$\varphi(v_n) = \frac{\sum_{i=1}^r (e^{\mu_{in}} - e^{-\mu_{in}}) c_{in}}{\sum_{i=1}^r e^{\mu_{in}} + e^{-\mu_{in}}} \rightarrow \frac{\sum_{i=1}^r (e^{\mu_i} - e^{-\mu_i}) c_i}{\sum_{i=1}^r e^{\mu_i} + e^{-\mu_i}} = \varphi(v)$$

as $n \rightarrow \infty$, which completes the proof. □

Lemma 6.8. *We have $\varphi(V) = \text{int } D_0$.*

Proof. As φ is continuous on V and maps V into $\text{int } D_0$, we know from the Brouwer invariance of domain theorem that $\varphi(V)$ is open in $\text{int } D_0$. Suppose, for the sake of contradiction, that $\varphi(V) \neq \text{int } D_0$. Then there exists $w \in \partial \varphi(V) \cap \text{int } D_0$. Let (v_n) in V be such that $\varphi(v_n) \rightarrow w$. As φ is continuous on V , we must have that $\|v_n\| \rightarrow \infty$.

Consider the spectral decomposition $v_n = \sum_{i=1}^r \lambda_{in} c_{in}$. After taking a subsequence we may assume that (1) $c_{in} \rightarrow c_i$, and (2) $\alpha_{in} = \lambda_{1n} - \lambda_{in} \rightarrow \alpha_i \in [0, \infty]$, for all $i = 1, \dots, r$. Note that $\lambda_{1n} = \|v_n\| \rightarrow \infty$.

Let $I = \{i: \alpha_i < \infty\}$. Then $1 \in I$ and

$$\varphi(v_n) = \frac{\sum_{i=1}^r (e^{\lambda_{in}} - e^{-\lambda_{in}}) c_{in}}{\sum_{i=1}^r e^{\lambda_{in}} + e^{-\lambda_{in}}} = \frac{\sum_{i=1}^r (e^{-\alpha_{in}} - e^{-\lambda_{1n} - \lambda_{in}}) c_{in}}{\sum_{i=1}^r e^{-\alpha_{in}} + e^{-\lambda_{1n} - \lambda_{in}}} \rightarrow \frac{\sum_{i \in I} e^{-\alpha_i} c_i}{\sum_{i \in I} e^{-\alpha_i}}.$$

This implies that $w = \lim_{n \rightarrow \infty} \varphi(v_n) \in \partial D_0$, which contradicts the assumption that $w \in \text{int } D_0$. □

Lemma 6.9. *The map φ satisfies $\varphi(V(\infty)) = \partial D_0$.*

Proof. Let $x \in \partial D_0$. Then x has spectral decomposition $x = \sum_{i=1}^r \lambda_i e_i$ with $\sum_{i=1}^r \lambda_i = 1$ and there exists a $p \in \{1, \dots, r\}$ such that $\lambda_1 \geq \dots \geq \lambda_p > 0$ and $\lambda_s = 0$ for $p < s \leq r$.

For $i = 1, \dots, p$ put $\mu_i = -\log \lambda_i$ and $\alpha_i = \mu_i - \mu_1$. Then $\alpha_i \geq 0$ and $\alpha_1 = 0$. Now consider the horofunction $h \in V(\infty)$ given by

$$h(x) = \Lambda_{V_2(e)} \left(-\frac{1}{2} (e \square P_2(e)x + P_2(e)x \square e) - \sum_{i=1}^p \alpha_i (e_i \square e_i) \right) \quad (x \in V).$$

Then we have

$$\varphi(h) = \frac{\sum_{i=1}^p \exp(-\alpha_i) e_i}{\sum_{i=1}^p \exp(-\alpha_i)} = \frac{\sum_{i=1}^p \lambda_i e_i}{\sum_{i=1}^p \lambda_i} = x.$$

□

Lemma 6.10. *The map φ is injective on $V \cup V(\infty)$.*

Proof. Suppose first that $\varphi(x) = \varphi(y)$. Let $x = \sum_{i=1}^r \lambda_i c_i$ and $y = \sum_{i=1}^r \mu_i d_i$ be the spectral decompositions. Then we have

$$\varphi(x) = \frac{\sum_{i=1}^r (e^{\lambda_i} - e^{-\lambda_i}) c_i}{\sum_{i=1}^r e^{\lambda_i} + e^{-\lambda_i}} = \frac{\sum_{i=1}^r (e^{\mu_i} - e^{-\mu_i}) d_i}{\sum_{i=1}^r e^{\mu_i} + e^{-\mu_i}} = \varphi(y)$$

where the coefficients of the minimal tripotents of both sides are decreasing through the order of the indices. If we let for $j = 1, \dots, r$,

$$\alpha_j = \left(\sum_{i=1}^r e^{\lambda_i} + e^{-\lambda_i} \right)^{-1} (e^{\lambda_j} - e^{-\lambda_j}) \quad \text{and} \quad \beta_j = \left(\sum_{i=1}^r e^{\mu_i} + e^{-\mu_i} \right)^{-1} (e^{\mu_j} - e^{-\mu_j}),$$

then $\alpha_j = \beta_j$ by Remark 6.4. It now follows from [25, Lemma 3.7] that $\lambda_j = \mu_j$ for $j = 1, \dots, r$.

Note that $\alpha_i = 0$ if and only if $\lambda_i = 0$, and similarly $\beta_i = 0$ if and only if $\mu_i = 0$. So by considering the unique spectral decompositions of $\varphi(x)$ and $\varphi(y)$ and using Remark 6.4 we find that $x = y$.

Now suppose $h, h' \in V(\infty)$ with $\varphi(h) = \varphi(h')$. Let h be of the form (5.9) and h' of the form

$$h'(x) = \Lambda_{V_2(c_J)} \left(-\frac{1}{2} (c_J \square P_2(c_J)x + P_2(c_J)x \square c_J) - \sum_{j \in J} \beta_j (c_j \square c_j) \right).$$

Then we have

$$\frac{\sum_{i \in I} \exp(-\alpha_i) e_i}{\sum_{i \in I} \exp(-\alpha_i)} = \frac{\sum_{j \in J} \exp(-\beta_j) c_j}{\sum_{j \in J} \exp(-\beta_j)}$$

where the coefficients of the minimal tripotents on both sides are strictly positive. By relabelling the indices, we may assume that $I = \{1, \dots, p\}$ and $0 = \alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_p$. Likewise we can assume that $J = \{1, \dots, q\}$ and $0 = \beta_1 \leq \beta_2 \leq \dots \leq \beta_q$. Since the norm of both sides above in V are equal, we have

$$\sum_{i \in I} \exp(-\alpha_i) = \sum_{j \in J} \exp(-\beta_j).$$

By Remark 6.4, we have $p = q$, $e^{-\alpha_i} = e^{-\beta_i}$, $e_I = e_1 + \dots + e_p = c_1 + \dots + c_q = c_J$ and $\sum_i \alpha_i e_i = \sum \beta_i c_i$. Hence $h = h'$ by Corollary 6.3. \square

Lemma 6.11. *If (a_k) in V is such that $h_{a_k} \rightarrow h \in V(\infty)$, then $\varphi(a_k) \rightarrow \varphi(h)$.*

Proof. Let h be given by (5.9). To show that $\varphi(a_k) \rightarrow \varphi(h)$, we show that each subsequence of $(\varphi(a_k))$ has a convergent subsequence with limit $\varphi(h)$. So let $(\varphi(a_m))$ be a subsequence. Using the spectral decomposition we write $a_m = \sum_{i=1}^r \mu_{im} c_{im}$ with $\mu_{1m} \geq \dots \geq \mu_{rm} \geq 0$. As h is a horofunction, $\mu_{1m} = \|a_m\| \rightarrow \infty$ by Lemma 2.2.

After taking a subsequence we may assume that $\beta_{im} = \mu_{1m} - \mu_{im} \rightarrow \beta_i \in [0, \infty]$ and $c_{im} \rightarrow c_i$ for all i . Let $J = \{i: \beta_i < \infty\} = \{1, \dots, q\}$ and note that $q \geq 1$, as $\beta_1 = 0$.

It follows from Lemma 5.1 that $h_{a_m} \rightarrow h'$ where

$$h'(x) = \Lambda_{V_2(c_J)} \left(-\frac{1}{2} (c_J \square P_2(c_J)x + P_2(c_J)x \square c_J) - \sum_{j \in J} \beta_j (c_j \square c_j) \right) \quad (x \in V),$$

and $h' \in V(\infty)$ by Theorem 5.2. As $h_{a_k} \rightarrow h$, we know that $h' = h$, and hence $e_I = c_J$ and

$$\sum_{i \in I} \alpha_i e_i = \sum_{j \in J} \beta_j c_j$$

by Corollary 6.3.

We can relabel the α_i 's such that $I = \{1, \dots, p\}$ and $0 = \alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_p$. It follows from Remark 6.4 that $p = q$ and $\alpha_i = \beta_i$ for all $i \in \{1, \dots, p\}$. Moreover, $\sum_{i=1}^p e^{-\alpha_i} e_i = \sum_{i=1}^p e^{-\beta_i} c_i$.

As

$$\varphi(a_m) = \frac{\sum_{i=1}^r (e^{\mu_{im}} - e^{-\mu_{im}}) c_{im}}{\sum_{i=1}^r e^{\mu_{im}} - e^{-\mu_{im}}} = \frac{\sum_{i=1}^r (e^{-\beta_{im}} - e^{-\mu_{1m} - \mu_{im}}) c_{im}}{\sum_{i=1}^r e^{-\beta_{im}} - e^{-\mu_{1m} - \mu_{im}}},$$

we find that

$$\lim_m \varphi(a_m) = \left(\sum_{i=1}^p e^{-\beta_i} \right)^{-1} \sum_{i=1}^p e^{-\beta_i} c_i = \left(\sum_{i=1}^p e^{-\alpha_i} \right)^{-1} \sum_{i=1}^p e^{-\alpha_i} e_i.$$

Thus, $\varphi(a_m) \rightarrow \varphi(h)$, which completes the proof. \square

To prove continuity of φ on the boundary $V(\infty)$, we need the following technical lemma.

Lemma 6.12. *Let $J \subset \{1, \dots, r\}$ be nonempty, and for each n , let $\{c_{jn} : j \in J\}$ be a collection of mutually orthogonal minimal tripotents in V such that $c_{jn} \rightarrow c_j$ for all $j \in J$. For $j \in J$ let $(\beta_{jn})_n$ be a sequence in $[0, \infty)$ converging to $\beta_j \in [0, \infty]$, with $\min_{j \in J} \beta_{jn} = 0$ for each n . If we let $J' = \{j \in J : \beta_j < \infty\} \neq \emptyset$ and consider the horofunctions,*

$$h_n(x) = \Lambda_{V_2(c_{Jn})} \left(-\frac{1}{2} (c_{Jn} \square P_2(c_{Jn})x + P_2(c_{Jn})x \square c_{Jn}) - \sum_{j \in J} \beta_{jn} (c_{jn} \square c_{jn}) \right) \quad (x \in V),$$

where $c_{Jn} = \sum_{i \in J} c_{in}$, then

$$\lim_{n \rightarrow \infty} h_n(x) = \Lambda_{V_2(c_{J'})} \left(-\frac{1}{2} (c_{J'} \square P_2(c_{J'})x + P_2(c_{J'})x \square c_{J'}) - \sum_{j \in J'} \beta_j (c_j \square c_j) \right) \quad (6.10)$$

with $c_{J'} = \sum_{j \in J'} c_j$.

Proof. We show that each subsequence of $(h_n(x))$ has a convergent subsequence with limit the right-hand side of (6.10). Pick a subsequence $(h_k(x))$. As $\{u \in V : \langle u, u \rangle = 1\}$ is compact, there exists $w^k \in V_2(c_{Jk})$ with $\langle w^k, w^k \rangle = 1$ and

$$\begin{aligned} h_k(x) = \sup_{v \in V_2(c_{Jn})} \langle & \left(-\frac{1}{2} (c_{Jn} \square P_2(c_{Jn})x + P_2(c_{Jn})x \square c_{Jn}) - \sum_{j \in J} \beta_{jn} (c_{jn} \square c_{jn}) \right) v, v \rangle \\ & \langle \left(-\frac{1}{2} (c_{Jk} \square P_2(c_{Jk})x + P_2(c_{Jk})x \square c_{Jk}) - \sum_{j \in J} \beta_{jk} (c_{jk} \square c_{jk}) \right) w^k, w^k \rangle. \end{aligned} \quad (6.11)$$

Taking a subsequence, we may assume that $w^k \rightarrow w$. For each k , let

$$V = \bigoplus_{0 \leq s \leq t; s, t \in J} V_{st}^k \quad (6.12)$$

be the Peirce decomposition of V with respect to the tripotents $\{c_{jk} : j \in J\}$, and let

$$V = \bigoplus_{0 \leq s \leq t; s, t \in J} V_{st}$$

be the decomposition with respect to the tripotents $\{c_j : j \in J\}$. We show

$$w \in \bigoplus_{s, t \in J' : s \leq t} V_{st} = V_2(c_{J'}). \quad (6.13)$$

Let $w_{st}^k \in V_{st}^k$ be the (s, t) -component of w^k in the Peirce decomposition in (6.12). Then we have $w_{st}^k \rightarrow w_{st} \in V_{st}$ and

$$-\frac{1}{2} (c_{Jk} \square P_2(c_{Jk})x + P_2(c_{Jk})x \square c_{Jk}) w_{st}^k \rightarrow -\frac{1}{2} (c_J \square P_2(c_J)x + P_2(c_J)x \square c_J) w_{st}.$$

Moreover,

$$\sum_{j \in J} \beta_{jk}(c_{jk} \square c_{jk})w_{st}^k = \sum_{j \in J} \left(\frac{\delta_{sj}\beta_{jk} + \delta_{tj}\beta_{jk}}{2} \right) w_{st}^k.$$

Recall from the proof of Lemma 5.1 that if $s, t \in J$ with $\{s, t\} \not\subseteq J'$, then $\beta_{sk} \rightarrow \infty$ or $\beta_{tk} \rightarrow \infty$. As $h_k(x) \geq -\|x\|$ for all k and $\beta_{jk} \geq 0$ for all $j \in J$, we find that $w_{st}^k \rightarrow 0$ for all $s, t \in J$ with $\{s, t\} \not\subseteq J'$. This implies that (6.13) holds.

Next, we make use of the following fact. Let e be a tripotent. If $v \in V_2(e) \cup V_0(e)$, $y \in V$ and $z \in V_2(e)$, then $P_2(e)\{v, y, z\} = \{P_2(e)v, P_2(e)y, z\}$ and $P_2(e)\{y, v, z\} = \{P_2(e)y, P_2(e)v, z\}$. In particular,

$$\begin{aligned} \langle -\frac{1}{2}(c_J \square P_2(c_J)x + P_2(c_J)x \square c_J)w, w \rangle &= \langle -\frac{1}{2}(c_{J'} \square P_2(c_{J'})P_2(c_J)x + P_2(c_{J'})P_2(c_J)x \square c_{J'})w, w \rangle \\ &= \langle -\frac{1}{2}(c_{J'} \square P_2(c_{J'})x + P_2(c_{J'})x \square c_{J'})w, w \rangle, \end{aligned}$$

as $P_2(c_{J'})P_2(c_J) = P_2(c_{J'})$.

It follows that

$$\begin{aligned} \lim_{k \rightarrow \infty} h_k(x) &= \lim_{k \rightarrow \infty} \langle (-\frac{1}{2}(c_{Jk} \square P_2(c_{Jk})x + P_2(c_{Jk})x \square c_{Jk}) - \sum_{j \in J'} \beta_{jk}(c_{jk} \square c_{jk}))w^k, w^k \rangle \\ &= \langle (-\frac{1}{2}(c_{J'} \square P_2(c_{J'})x + P_2(c_{J'})x \square c_{J'}) - \sum_{j \in J'} \beta_j(c_j \square c_j))w, w \rangle. \end{aligned}$$

Since $w \in V_2(c_{J'})$, we find that

$$\lim_{k \rightarrow \infty} h_k(x) \leq \Lambda_{V_2(c_{J'})}(-\frac{1}{2}(c_{J'} \square P_2(c_{J'})x + P_2(c_{J'})x \square c_{J'}) - \sum_{j \in J'} \beta_j(c_j \square c_j)).$$

To show that this is an equality, let $u \in V_2(c_{J'})$ be such that

$$\begin{aligned} \Lambda_{V_2(c_{J'})}(-\frac{1}{2}(c_{J'} \square P_2(c_{J'})x + P_2(c_{J'})x \square c_{J'}) - \sum_{j \in J'} \beta_j(c_j \square c_j)) \\ = \langle (-\frac{1}{2}(c_{J'} \square P_2(c_{J'})x + P_2(c_{J'})x \square c_{J'}) - \sum_{j \in J'} \beta_j(c_j \square c_j))u, u \rangle. \end{aligned}$$

As $P_2(c_{J'k})u \in V_2(c_{Jk})$, we have that

$$\begin{aligned} \langle (-\frac{1}{2}(c_{Jk} \square P_2(c_{Jk})x + P_2(c_{Jk})x \square c_{Jk}) - \sum_{j \in J} \beta_{jk}(c_{jk} \square c_{jk}))w^k, w^k \rangle \\ \geq \langle (-\frac{1}{2}(c_{Jk} \square P_2(c_{Jk})x + P_2(c_{Jk})x \square c_{Jk}) \\ - \sum_{j \in J} \beta_{jk}(c_{jk} \square c_{jk}))P_2(c_{J'k})u, P_2(c_{J'k})u \rangle \langle P_2(c_{J'k})u, P_2(c_{J'k})u \rangle^{-1}. \end{aligned}$$

Note that, as $\langle P_2(c_{J'k})u, P_2(c_{J'k})u \rangle \rightarrow \langle u, u \rangle = 1$, the right-hand side is defined for all k large.

So,

$$\begin{aligned} \langle (-\frac{1}{2}(c_{Jk} \square P_2(c_{Jk})x + P_2(c_{Jk})x \square c_{Jk}) - \sum_{j \in J} \beta_{jk}(c_{jk} \square c_{jk}))P_2(c_{J'k})u, P_2(c_{J'k})u \rangle \\ = \langle (-\frac{1}{2}(c_{J'k} \square P_2(c_{J'k})x + P_2(c_{J'k})x \square c_{J'k}) - \sum_{j \in J'} \beta_{jk}(c_{jk} \square c_{jk}))P_2(c_{J'k})u, P_2(c_{J'k})u \rangle \\ \rightarrow \langle (-\frac{1}{2}(c_{J'} \square P_2(c_{J'})x + P_2(c_{J'})x \square c_{J'}) - \sum_{j \in J'} \beta_j(c_j \square c_j))u, u \rangle, \end{aligned}$$

which proves (6.10). \square

Lemma 6.13. *The map $\varphi: V \cup V(\infty) \rightarrow D_0$ is continuous on $V(\infty)$.*

Proof. Let $h_n \rightarrow h$ in $V(\infty)$. We show that each subsequence $(\varphi(h_k))$ of $(\varphi(h_n))$ has a convergent subsequence with limit $\varphi(h)$. As h_n is horofunction, we can express it as

$$h_n(x) = \Lambda_{V_2(c_{J_n n})} \left(-\frac{1}{2}(c_{J_n n} \square P_2(c_{J_n n})x + P_2(c_{J_n n})x \square c_{J_n n}) - \sum_{j \in J_n} \beta_{jn}(c_{jn} \square c_{jn}) \right) \quad (x \in V).$$

After taking successive subsequences we may assume for each k that $J_k = J \subseteq \{1, \dots, r\}$, $c_{jk} \rightarrow c_j$, and $\beta_{jk} \rightarrow \beta_j \in [0, \infty]$ for all $j \in J$. Note that $\min_{j \in J} \beta_j = 0$, as $\min_{j \in J} \beta_{jk} = 0$ for all k .

Now let $J' = \{j \in J: \beta_j < \infty\}$. By Lemma 6.12, we find that $h_k(x) \rightarrow h'(x)$ for each $x \in V$, where

$$h'(x) = \Lambda_{V_2(c_{J'})} \left(-\frac{1}{2}(c_{J'} \square P_2(c_{J'})x + P_2(c_{J'})x \square c_{J'}) - \sum_{j \in J'} \beta_j(c_j \square c_j) \right).$$

By Theorem 5.2 we know that h' is a horofunction. As $h_n \rightarrow h$, we conclude that $h = h'$, and hence $e_I = c_{J'}$ and $\sum_{i \in I} \alpha_i e_i = \sum_{j \in J'} \beta_j c_j$ by Corollary 6.3. This implies, by Remark 6.4, that

$$\lim_{k \rightarrow \infty} h_k(x) = \lim_{k \rightarrow \infty} \frac{\sum_{j \in J} e^{-\beta_{jk}} c_{jk}}{\sum_{j \in J} e^{-\beta_{jk}}} = \frac{\sum_{j \in J'} e^{-\beta_j} c_j}{\sum_{j \in J'} e^{-\beta_j}} = \frac{\sum_{i \in I} e^{-\alpha_i} e_i}{\sum_{i \in I} e^{-\alpha_i}} = \varphi(h)$$

and hence we are done. \square

Collecting the results it is now easy to show that φ is a homeomorphism.

Proof of Theorem 6.5. Note that φ is continuous on $V \cup V(\infty)$ by Lemmas 6.7 and 6.13. Moreover, φ is a bijection by Lemmas 6.8, 6.9, and 6.10. As $V \cup V(\infty)$ is compact and D_0 is Hausdorff, we conclude that φ is a homeomorphism. \square

7 Geometry of $V \cup V(\infty)$

We now analyse the geometry of the metric compactification of V . Recall that on $V(\infty)$ there is a natural equivalence relation, $h \sim g$ if $\sup_{x \in V} |h(x) - g(x)| < \infty$. In this section, we show that the partition of $V(\infty)$ into equivalence classes is closely related to the geometry of D_0 (and hence also to D^*). In fact, we prove that the homeomorphism $\varphi: V \cup V(\infty) \rightarrow D_0$ given in (6.8) and (6.9) maps each equivalence class onto the relative interior of a boundary face of D_0 .

For the basic terminology from convex analysis we follow [31]. If $C \subseteq V$ is convex, then $F \subseteq C$ is called a *face* if $\lambda x + (1 - \lambda)y \in F$ for some $0 < \lambda < 1$ and $x, y \in C$ implies that $x, y \in F$. Note that the empty set and C are both faces of C , and each face is convex. The *relative interior* of a face F , denoted $\text{ri } F$, is the interior of F regarded as a subset of the affine hull of F . It is well known that each nonempty convex set C is partitioned by the relative interiors of its nonempty faces, see [31, Theorem 18.2].

To analyse the equivalence classes it is useful to recall that for Busemann points $h, g \in V(\infty)$ one has that $h \sim g$ if and only if $\delta(h, g) < \infty$, see [36, Proposition 4.5]. As each horofunction is a Busemann point by Corollary 5.4, we see that the equivalence classes coincide with the parts of $V(\infty)$. Therefore we start by analysing the parts.

Using (2.2) and Remark 5.3 we find for $h, h' \in V(\infty)$, where

$$h(x) = \Lambda_{V_2(e_I)} \left(-\frac{1}{2}(e_I \square P_2(e_I)x + P_2(e_I)x \square e_I) - \sum_{i \in I} \alpha_i(e_i \square e_i) \right) \quad (x \in V) \quad (7.1)$$

and

$$h'(x) = \Lambda_{V_2(c_J)} \left(-\frac{1}{2}(c_J \square P_2(c_J)x + P_2(c_J)x \square c_J) - \sum_{j \in J} \beta_j(c_j \square c_j) \right) \quad (x \in V), \quad (7.2)$$

that

$$H(h, h') = \lim_{t \rightarrow \infty} \|\psi(t)\| + h'(\psi(t)), \quad (7.3)$$

with $\psi(t) = te_I - \sum_{i \in I} \alpha_i e_i$. Likewise, $H(h', h) = \lim_{t \rightarrow \infty} \|\psi'(t)\| + h(\psi'(t))$, with $\psi'(t) = tc_J - \sum_{j \in J} \beta_j c_j$.

Lemma 7.1. *Let h and h' be given by (7.1) and (7.2). If $e_I = c_J$, then*

$$H(h, h') = \sup_{u \in V_2(e_I): \langle u, u \rangle = 1} \langle ((a - b) \square e_I)u, u \rangle$$

and

$$H(h', h) = \sup_{u \in V_2(e_I): \langle u, u \rangle = 1} \langle ((b - a) \square e_I)u, u \rangle.$$

where $a = \sum_{i \in I} \alpha_i e_i$ and $b = \sum_{j \in J} \beta_j c_j$.

Proof. Let $\psi(t) = te_I - \sum_{i \in I} \alpha_i e_i$. For $t \geq 0$ large, $\|\psi(t)\| = t$, and hence

$$\begin{aligned} \|\psi(t)\| + h'(\psi(t)) &= t + \sup_{u \in V_2(c_J): \langle u, u \rangle = 1} \langle -\frac{1}{2}(c_J \square \psi(t) + \psi(t) \square c_J)u - \sum_{j \in J} \beta_j (c_j \square c_j)u, u \rangle \\ &= \sup_{u \in V_2(c_J): \langle u, u \rangle = 1} \langle t(c_J \square c_J)u - \frac{1}{2}(c_J \square (te_I - a) + (te_I - a) \square c_J)u - \sum_{j \in J} \beta_j (c_j \square c_j)u, u \rangle \\ &= \sup_{u \in V_2(c_J): \langle u, u \rangle = 1} \langle \frac{1}{2}(e_I \square a + a \square e_I)u - \sum_{j \in J} \beta_j (c_j \square c_j)u, u \rangle \\ &= \sup_{u \in V_2(c_J): \langle u, u \rangle = 1} \langle (a \square e_I)u - (b \square c_J)u, u \rangle \\ &= \sup_{u \in V_2(e_I): \langle u, u \rangle = 1} \langle ((a - b) \square e_I)u, u \rangle. \end{aligned}$$

So, by (7.3) the first equality holds. The second one is obtained by changing the roles of h and h' . \square

We use this lemma to give a simple criterion for two horofunctions to be in the same part of $V(\infty)$.

Theorem 7.2. *Two horofunctions h and h' given by (7.1) and (7.2), respectively, are in the same part of $V(\infty)$ if and only if $e_I = c_J$,*

Proof. Let $e_I = c_J$. Then by Lemma 7.1, we have that

$$\delta(h, h') = \sup_{u \in V_2(e_I): \langle u, u \rangle = 1} \langle ((a - b) \square e_I)u, u \rangle + \sup_{u \in V_2(e_I): \langle u, u \rangle = 1} \langle ((b - a) \square e_I)u, u \rangle < \infty$$

where $a = \sum_{i \in I} \alpha_i e_i$ and $b = \sum_{j \in J} \beta_j c_j$. Hence h and h' are in the same part.

Conversely, given $e_I \neq c_J$, we need to show that h and h' are in different parts, that is, $\delta(h, h') = \infty$. We have either $c_J \not\leq e_I$ or $e_I \not\leq c_J$. Assume the former. Note that it suffices to show $H(h, h') = \infty$, since the detour cost is nonnegative.

By Lemma 6.1 and Remark 6.2 we have

$$\langle c_J - \frac{1}{2}(c_J \square P_2(c_J)e_I + P_2(c_J)e_I \square c_J)(c_J), c_J \rangle > 0. \quad (7.4)$$

As before, for large t , we have $\|\psi(t)\| = t$, so that

$$\begin{aligned} \|\psi(t)\| + h'(\psi(t)) &= t + \sup_{u \in V_2(c_J): \langle u, u \rangle = 1} \langle -\frac{1}{2}(c_J \square \psi(t) + \psi(t) \square c_J)u - \sum_{j \in J} \beta_j (c_j \square c_j)u, u \rangle \\ &= \sup_{u \in V_2(c_J): \langle u, u \rangle = 1} \langle tu - \frac{t}{2}(c_J \square P_2(c_J)e_I + P_2(c_J)e_I \square c_J)u + \frac{1}{2}(c_J \square a + a \square c_J)u - \sum_{j \in J} \beta_j (c_j \square c_j)u, u \rangle. \end{aligned}$$

Hence (7.4) implies

$$H(h, h') \geq \lim_{t \rightarrow \infty} \langle tc_J - \frac{t}{2}(c_J \square P_2(c_J)e_I + P_2(c_J)e_I \square c_J)(c_J) + \frac{1}{2}(c_J \square a + a \square c_J)c_J - \sum_{j \in J} \beta_j c_j, c_J \rangle = \infty.$$

Analogously, $e_I \not\leq c_J$ implies $H(h', h) = \infty$. \square

Let us now recall the facial structure of D^* . By [11, Theorem 4.4], the closed boundary faces of D^* are exactly the sets of the form $F_e^* = \{\tilde{x} \in D^* : \tilde{x}(e) = [e, x] = 1\}$, where e is a tripotent in V . So, the boundary faces of D_0 are precisely the sets of the form

$$F_e = \{x \in D_0 : [e, x] = 1\} \quad (e \in V \text{ tripotent}).$$

Note that $F_e \subset \partial D_0$, as $F_e^* \subset \partial D^*$.

The next lemma gives an alternative description of F_e , which will be useful in our discussion.

Lemma 7.3. *Suppose that $e \in V$ is a tripotent. Then*

$$F_e = \{\sum_{i=1}^p \lambda_i e_i : \sum_{i=1}^p \lambda_i = 1, \lambda_i > 0, \text{ and } e_i \text{'s mutually orthogonal minimal tripotents with } e_i \leq e\}.$$

Proof. Suppose that $x \in F_e$. Using the spectral decomposition we can write x as $x = \sum_{i=1}^p \lambda_i e_i$, where $\lambda_p > 0$ and $p \leq r$ (so we ignore the zero eigenvalues). Then $\sum_{i=1}^p \lambda_i = 1$, since $x \in \partial D_0$. As F_e is a face and $e_i \in D_0$, we know that $e_i \in F_e$, and hence $1 = [e, e_i] = [P_2(e_i)e, e_i]$. Combining this with the fact that $P_2(e_i)e \in \mathbb{C}e_i$ and $[e_i, e_i] = 1$ gives $P_2(e_i)e = e_i$. Thus, $e_i \leq e$ for $i = 1, \dots, p$.

On the other hand, given $x = \sum_{i=1}^p \lambda_i e_i$ such that $\sum_{i=1}^p \lambda_i = 1$, $\lambda_i > 0$, and $e_1, \dots, e_p \leq e$ mutually orthogonal minimal tripotents in V , we have that $x \in D_0$ and

$$[e, x] = \sum_{i=1}^p \lambda_i [e, e_i] = \sum_{i=1}^p \lambda_i [e_i, e_i] = 1.$$

Hence $x \in F_e$. □

We like to point out that $F_e = \partial D_0 \cap A(e)_+$, where $A(e)_+$ is the closed positive cone in the JB-algebra $A(e)$ in $V_2(e)$, cf. [27, Theorem 6.12].

Theorem 7.4. *If $h \in V(\infty)$ is given by (7.1), then $\varphi(h) \in \text{ri } F_{e_I}$. Moreover, φ maps each equivalence class in $V(\infty)/\sim$ onto the relative interior of a boundary face of D_0 .*

Proof. Let $|I| = q$ and set $w = q^{-1}e_I$, where $e_I = \sum_{i \in I} e_i$. Note that $[e_I, w] = 1$ and $w \in F_{e_I}$. We claim that w is in the relative interior of F_{e_I} . Let $x \in F_{e_I}$. To prove the claim it suffices to show that for each $\varepsilon > 0$ small, $w_\varepsilon = w + \varepsilon(w - x)$ is in F_{e_I} , see [31, Theorem 6.4].

By Lemma 7.3 we know that we can write $x = \sum_{i=1}^p \lambda_i d_i$, where $\sum_{i=1}^p \lambda_i = 1$, $\lambda_i > 0$ and $d_i \leq e_I$ for all i . We have that $e_I - \sum_{i=1}^p d_i$ is a (possibly 0) tripotent orthogonal to the tripotent $\sum_{i=1}^p d_i$, say with spectral decomposition $e_I - \sum_{i=1}^p d_i = \sum_{i=p+1}^q d_i$. So,

$$w_\varepsilon = \sum_{i=1}^p (q^{-1}(1 + \varepsilon) - \varepsilon \lambda_i) d_i + \sum_{i=p+1}^q q^{-1}(1 + \varepsilon) d_i \quad (7.5)$$

and $[e_I, w_\varepsilon] = (1 + \varepsilon)[e_I, w] - \varepsilon[e_I, x] = 1$. As $q^{-1}(1 + \varepsilon) - \varepsilon \lambda_i > 0$ for all $\varepsilon > 0$ small, the right hand-side of (7.5) is a spectral decomposition of w_ε for all $\varepsilon > 0$ small. (We have ignored terms with zero eigenvalues.) Thus, $w_\varepsilon \in D_0$ for all $\varepsilon > 0$ small, and hence $w_\varepsilon \in F_{e_I} = \{x \in D_0 : [e_I, x] = 1\}$.

To complete the proof of the first assertion we assume by way of contradiction that $\varphi(h) \notin \text{ri } F_{e_I}$. As $\varphi(h) \in F_{e_I}$, we know that $\varphi(h)$ is in the relative boundary of F_{e_I} . This implies that $z_\varepsilon = (1 + \varepsilon)\varphi(h) - \varepsilon w \notin F_{e_I}$ for all $\varepsilon > 0$. Here we use the fact that w is in $\text{ri } F_{e_I}$ and F_{e_I} is a convex set.

Note that

$$z_\varepsilon = \sum_{i=1}^q \left(\frac{(1 + \varepsilon)e^{-\alpha_i}}{\sum_{j=1}^q e^{-\alpha_j}} - \frac{\varepsilon}{q} \right) d_i. \quad (7.6)$$

Let μ_i^ε be the coefficient of the d_i in the sum (7.6). Then $\mu_i^\varepsilon > 0$ for all i when $\varepsilon > 0$ is sufficiently small. Moreover, $\sum_{i=1}^q \mu_i^\varepsilon = [e_I, z_\varepsilon] = 1$, since $e_I = d_1 + \dots + d_q$. But this implies that $z_\varepsilon \in F_{e_I}$ for all $\varepsilon > 0$ small, which is impossible.

To show the second statement we note that for $h, h' \in V(\infty)$ given by (7.1) and (7.2), respectively, we have that $h \sim h'$ if and only if they are in the same part, as all horofunctions are Busemann points. This is equivalent to saying that $e_I = c_J$ by Theorem 7.2. So by the first assertion, we get that h and h' are both mapped into $\text{ri } F_{e_I}$ under φ . As φ maps $V(\infty)$ onto ∂D_0 we conclude that φ maps each equivalence class in $V(\infty)/\sim$ onto the relative interior of a boundary face of D_0 . \square

In the remained of this section we show that each part in $V(\infty)$ with the detour distance is isometric to a normed space. More specifically, we shall see that the part of $h \in V(\infty)$, where h given by (7.1), equipped with the detour distance is isometric to a quotient space of the JB-algebra $A(e_I) = \{x \in V : x = x^*\} = \{x \in V_2(e_I) : \{e_I, x, e_I\} = x\}$ under a suitable norm. To introduce the norm let $e \in V$ be a tripotent and define for $x \in A(e)$,

$$\begin{aligned} \|x\|_{\text{var}} &= \sup_{u \in V_2(e) : \langle u, u \rangle = 1} \langle (x \square e)u, u \rangle + \sup_{u \in V_2(e) : \langle u, u \rangle = 1} \langle (-x \square e)u, u \rangle \\ &= \sup_{u \in V_2(e) : \langle u, u \rangle = 1} \langle (x \square e)u, u \rangle - \inf_{u \in V_2(e) : \langle u, u \rangle = 1} \langle (x \square e)u, u \rangle. \end{aligned} \quad (7.7)$$

Lemma 7.5. *The function $\|\cdot\|_{\text{var}}$ is a semi-norm on the real vector space $A(e)$, with $\|x\|_{\text{var}} = 0$ if and only if $x = \lambda e$ for some $\lambda \in \mathbb{R}$.*

Proof. From the definition, we have $\|x\|_{\text{var}} \geq 0$ for all $x \in A(e)$. If $\alpha \geq 0$, then $\|\alpha x\|_{\text{var}} = \alpha \|x\|_{\text{var}}$ by (7.7). For $\alpha < 0$ we have

$$\begin{aligned} \|\alpha x\|_{\text{var}} &= \sup_{u \in V_2(e) : \langle u, u \rangle = 1} \langle (\alpha x \square e)u, u \rangle - \inf_{u \in V_2(e) : \langle u, u \rangle = 1} \langle (\alpha x \square e)u, u \rangle \\ &= - \inf_{u \in V_2(e) : \langle u, u \rangle = 1} \langle (-\alpha x \square e)u, u \rangle + \sup_{u \in V_2(e) : \langle u, u \rangle = 1} \langle (-\alpha x \square e)u, u \rangle \\ &= -\alpha \|x\|_{\text{var}}, \end{aligned}$$

and hence $\|\alpha x\|_{\text{var}} = |\alpha| \|x\|_{\text{var}}$ for all $\alpha \in \mathbb{R}$ and $x \in A(e)$.

It follows directly from the definition that $\|x + y\|_{\text{var}} \leq \|x\|_{\text{var}} + \|y\|_{\text{var}}$ for all $x, y \in A(e)$. Given $\lambda \in \mathbb{R}$ and $x \in A(e)$, we have

$$\begin{aligned} \|x + \lambda e\|_{\text{var}} &= \sup_{u \in V_2(e) : \langle u, u \rangle = 1} \langle ((x + \lambda e) \square e)u, u \rangle - \inf_{u \in V_2(e) : \langle u, u \rangle = 1} \langle ((x + \lambda e) \square e)u, u \rangle \\ &= \sup_{u \in V_2(e) : \langle u, u \rangle = 1} (\langle (x \square e)u, u \rangle + \lambda \langle (e \square e)u, u \rangle) - \inf_{u \in V_2(e) : \langle u, u \rangle = 1} (\langle (x \square e)u, u \rangle + \lambda \langle (e \square e)u, u \rangle) \\ &= \sup_{u \in V_2(e) : \langle u, u \rangle = 1} (\langle (x \square e)u, u \rangle + \lambda) - \inf_{u \in V_2(e) : \langle u, u \rangle = 1} (\langle (x \square e)u, u \rangle + \lambda) \\ &= \sup_{u \in V_2(e) : \langle u, u \rangle = 1} \langle (x \square e)u, u \rangle - \inf_{u \in V_2(e) : \langle u, u \rangle = 1} \langle (x \square e)u, u \rangle, \end{aligned}$$

and hence $\|x + \lambda e\|_{\text{var}} = \|x\|_{\text{var}}$.

On the other hand, if $\|x\|_{\text{var}} = 0$ with a spectral decomposition $x = \sum_{i=1}^p \alpha_i e_i$ in the JB-algebra $A(e)$, where $\alpha_1 \geq \dots \geq \alpha_p$ and $e_1 + \dots + e_p = e$, then we show $x = \lambda e$ for some $\lambda \in \mathbb{R}$.

Take $\mu = \inf_{u \in V_2(e) : \langle u, u \rangle = 1} \langle (x \square e)u, u \rangle$ and set $y = x - \mu e$. So, $\|y\|_{\text{var}} = \|x\|_{\text{var}} = 0$ and

$$\|y\|_{\text{var}} = \sup_{u \in V_2(e) : \langle u, u \rangle = 1} \langle (y \square e)u, u \rangle - \inf_{u \in V_2(e) : \langle u, u \rangle = 1} \langle (y \square e)u, u \rangle = \sup_{u \in V_2(e) : \langle u, u \rangle = 1} \langle (y \square e)u, u \rangle = 0.$$

But $y = \sum_{i=1}^p (\alpha_i - \mu) e_i$, and hence for each $1 \leq k \leq p$ we get that

$$0 \geq \langle (y \square e)e_k, e_k \rangle = (\alpha_k - \mu) \langle (e_k \square e_k)e_k, e_k \rangle = (\alpha_k - \mu) \langle e_k, e_k \rangle,$$

as $e_k \in A(e) \subseteq V_2(e)$.

This implies that $\alpha_k - \mu \leq 0$ for all $1 \leq k \leq p$ and hence $-y$ is in the closed cone of the JB-algebra $A(e)$, so that $x \leq \mu e$ in $A(e)$. Likewise

$$\sup_{u \in V_2(e) : \langle u, u \rangle = 1} \langle (-y \square e)u, u \rangle = - \inf_{u \in V_2(e) : \langle u, u \rangle = 1} \langle (y \square e)u, u \rangle = 0,$$

gives $y \geq 0$ and hence $\mu e \leq x$. We conclude that $x = \mu e$, which completes the proof. \square

The preceding result shows that $\|\cdot\|_{\text{var}}$ is genuine norm on the quotient space $A(e)/\mathbb{R}e$ of the JB-algebra $A(e)$. Further, we have the following corollary.

Corollary 7.6. *For $h \in V(\infty)$ we have that $([h], \delta)$ is isometric to $(A(e_I)/\mathbb{R}e_I, \|\cdot\|_{\text{var}})$.*

Proof. Let $h \in V(\infty)$ be given by (7.1). We define a map $\tau: [h] \rightarrow A(e_I)/\mathbb{R}e_I$ by

$$\tau(h') = \sum_{j \in J} \beta_j c_j + \mathbb{R}e_I \in A(e_I)/\mathbb{R}e_I \quad (h' \in [h] \text{ given by (7.2)}).$$

This is a bijection, as $\min_{j \in J} \beta_j = 0$ and $e_I = c_J$ for all horofunctions $h' \in [h]$ by Theorem 7.2. It is an isometry by Lemmas 7.1 and 7.5. \square

8 Extension of the exponential map

The exponential map $\exp_0: V \rightarrow D$ at $0 \in D$ of the Bergman metric g is a real analytic homeomorphism, where $g_0(z, w) = 2\text{Trace}(z \square w)$ for $z, w \in V$ (cf. [7, Example 3.5.13]) and

$$\exp_0(x) = \tanh(x) = \sum_{i=1}^r \tanh(\lambda_i) e_i \quad (8.1)$$

for each $x \in V$ with spectral decomposition $x = \sum_{i=1}^r \lambda_i e_i$ [27, Lemma 4.3 and Corollary 4.8].

In this final section we show that \exp_0 extends as a homeomorphism $\widetilde{\exp}_0: V \cup V(\infty) \rightarrow D \cup D(\infty)$ such that $\widetilde{\exp}_0$ maps each equivalence class in $V(\infty)/\sim$ onto an equivalence class of $D(\infty)/\sim$. In particular, we find that the metric compactification of a Hermitian symmetric space $M \approx D \subset V$ can be realised as the closed dual unit ball D^* , by Theorem 6.5, and its geometry coincides with the facial structure of D^* by Theorem 7.4.

Given $h \in V(\infty)$ with

$$h(x) = \Lambda_{V_2(e)} \left(-\frac{1}{2}(e \square P_2(e)x + P_2(e)x \square e) - \sum_{i=1}^p \alpha_i (e_i \square e_i) \right) \quad (x \in V),$$

we define $\widetilde{\exp}_0(h) = g$, where the function $g: D \rightarrow \mathbb{R}$ is given by

$$g(z) = \frac{1}{2} \log \left\| \sum_{1 \leq i \leq j \leq p} e^{-\alpha_i} e^{-\alpha_j} B(z, z)^{-1/2} B(z, e) P_{ij} \right\| \quad (z \in D), \quad (8.2)$$

which is a horofunction by Theorem 4.4, as $\min_i \alpha_i = 0$ implies $\max_i e^{-\alpha_i} = 1$.

We will prove that the extension $\widetilde{\exp}_0$ is a homeomorphism in following theorem.

Theorem 8.1. *The extension $\widetilde{\exp}_0: V \cup V(\infty) \rightarrow D \cup D(\infty)$ of the exponential map is a well-defined homeomorphism that maps each equivalence class of $V(\infty)/\sim$ onto an equivalence class of $D(\infty)/\sim$.*

It follows from this theorem that the geometry and global topology of the metric compactifications of (D, ρ) and of the JB*-triple $(V, \|\cdot\|)$ with open unit ball D coincide. So have the following consequence by Theorems 6.5 and 7.4.

Corollary 8.2. *There exists a homeomorphism $\psi: D \cup D(\infty) \rightarrow D_0$ that maps each equivalence class in $D(\infty)/\sim$ onto the relative interior of a boundary face of D_0 .*

To prove Theorem 8.1, we will need the fact that all horofunctions of $D \cup D(\infty)$ are Busemann points and exploit the detour distance on the parts in $D(\infty)$.

Let h be horofunction functions in $D(\infty)$ of the form,

$$h(z) = \frac{1}{2} \log \left\| \sum_{1 \leq i \leq j \leq p} \lambda_i \lambda_j B(z, z)^{-1/2} B(z, e) P_{ij} \right\| \quad (z \in D) \quad (8.3)$$

for some $p \in \{1, \dots, r\}$, $\lambda_i \in (0, 1]$ with $\max_i \lambda_i = 1$ and $e = e_1 + e_2 + \dots + e_p$ a tripotent. Using $V_2(e) = \bigoplus_{1 \leq i \leq j \leq p} V_{ij}$ and (3.18), the norm in (8.3) can be computed over $V_2(e)$, that is,

$$\left\| \sum_{1 \leq i \leq j \leq p} \lambda_i \lambda_j B(z, z)^{-1/2} B(z, e) P_{ij} \right\| = \left\| \sum_{1 \leq i \leq j \leq p} \lambda_i \lambda_j B(z, z)^{-1/2} B(z, e) P_{ij} P_2(e) \right\|. \quad (8.4)$$

For $i = 1, \dots, p$, let $\alpha_i = -\log \lambda_i \geq 0$, so $\min_i \alpha_i = 0$. Let $\gamma: [0, \infty) \rightarrow D$ be the path

$$\gamma(t) = \exp_0(te - \sum_i \alpha_i e_i) = \tanh(te - \sum_i \alpha_i e_i) = \sum_i \tanh(t - \alpha_i) e_i \quad (t \geq 0). \quad (8.5)$$

Note that $\gamma(t) \rightarrow e \in \partial D$ as $t \rightarrow \infty$, in other words, $\gamma(t)$ goes to infinity in the metric space (D, ρ) . We show below that γ is a *geodesic* in the metric space (D, ρ) , in the sense that $d(\gamma(s), \gamma(t)) = |s - t|$ for all $s, t \in [0, \infty)$. We will see that h in (8.3) is a horofunction obtained by taking a sequence $(\gamma(t_k))$ along γ . For simplicity we say that $\gamma(t)$ *converges to* h , viewing $h \in D(\infty)$ as an ideal boundary point of (D, ρ) .

Lemma 8.3. *The path γ in (8.5) is a geodesic in (D, ρ) converging to the horofunction h in (8.3), and h is a Busemann point.*

Proof. Recall that

$$g_{-\gamma(s)}(\gamma(t)) = -\gamma(s) + B(\gamma(s), \gamma(s))^{1/2} (I - \gamma(t) \square \gamma(s))^{-1} \gamma(t).$$

We note that

$$\begin{aligned} (I - \gamma(t) \square \gamma(s)) \left(\sum_i \frac{\tanh(t - \alpha_i)}{1 - \tanh(t - \alpha_i) \tanh(s - \alpha_i)} e_i \right) \\ = \sum_i \frac{\tanh(t - \alpha_i) - \tanh^2(t - \alpha_i) \tanh(s - \alpha_i)}{1 - \tanh(t - \alpha_i) \tanh(s - \alpha_i)} e_i = \sum_i \tanh(t - \alpha_i) e_i. \end{aligned}$$

So,

$$(I - \gamma(t) \square \gamma(s))^{-1} \gamma(t) = \sum_i \frac{\tanh(t - \alpha_i)}{1 - \tanh(t - \alpha_i) \tanh(s - \alpha_i)} e_i.$$

Now using (3.19) we find that

$$B(\gamma(s), \gamma(s))^{1/2} (I - \gamma(t) \square \gamma(s))^{-1} \gamma(t) = \sum_i \frac{(1 - \tanh^2(s - \alpha_i)) \tanh(t - \alpha_i)}{1 - \tanh(t - \alpha_i) \tanh(s - \alpha_i)} e_i,$$

which implies that

$$-\gamma(s) + B(\gamma(s), \gamma(s))^{1/2} (I - \gamma(t) \square \gamma(s))^{-1} \gamma(t) = \sum_i \frac{\tanh(t - \alpha_i) - \tanh(s - \alpha_i)}{1 - \tanh(t - \alpha_i) \tanh(s - \alpha_i)} e_i = \sum_i \tanh(t - s) e_i.$$

Thus, for $t \geq s$ we have that

$$\|g_{-\gamma(s)}(\gamma(t))\| = \left\| \sum_{i=1}^p \tanh(t - s) e_i \right\| = \tanh(t - s)$$

and hence $\rho(\gamma(t), \gamma(s)) = t - s$, which shows that γ is a geodesic.

Observe that for sufficiently large $t \geq 0$, we have $\|\gamma(t)\| = \tanh(t)$, as $\min_i \alpha_i = 0$. It follows that $1 - \|\gamma(t)\|^2 = 1 - \tanh^2(t)$ for large t .

Set $\beta_{it} = \tanh(t - \alpha_i)$ for $i = 1, \dots, p$, and put $\beta_{it} = 0$ for $i = 0$. From (3.20) we know that

$$B(\gamma(t), \gamma(t))^{-1/2} = \sum_{0 \leq i \leq j \leq p} (1 - \beta_{it}^2)^{-1/2} (1 - \beta_{jt}^2)^{-1/2} P_{ij}.$$

Using the identity $e^{2x} = (1 + \tanh(x))/(1 - \tanh(x))$, we get

$$\left(\frac{1 - \tanh^2(t)}{1 - \beta_{it}^2}\right)^{1/2} = \frac{e^{-t}(1 + \tanh(t))}{e^{-t+\alpha_i}(1 + \tanh(t - \alpha_i))} \longrightarrow e^{-\alpha_i} \quad \text{as } t \longrightarrow \infty$$

for $i = 1, \dots, p$. For $i = 0$, we have $(1 - \tanh^2(t))/(1 - \beta_{it}^2) = 1 - \tanh^2(t) \longrightarrow 0$.

Recall that by Lemma 3.8 and equations (3.7) and (4.2), we have for each $z \in D$ that

$$\begin{aligned} \lim_t h_{\gamma(t)}(z) &= \lim_t \frac{1}{2} \log \left\| (1 - \tanh^2(t))B(z, z)^{-1/2}B(z, \gamma(t))B(\gamma(t), \gamma(t))^{-1/2} \right\| \\ &= \lim_t \frac{1}{2} \log \left\| \sum_{0 \leq i \leq j \leq p} \left(\frac{1 - \tanh^2(t)}{1 - \beta_{it}^2}\right)^{1/2} \left(\frac{1 - \tanh^2(t)}{1 - \beta_{jt}^2}\right)^{1/2} B(z, z)^{-1/2}B(z, \gamma(t))P_{ij} \right\|. \end{aligned}$$

This implies that

$$\lim_t h_{\gamma(t)}(z) = \frac{1}{2} \log \left\| \sum_{1 \leq i \leq j \leq p} e^{-\alpha_i} e^{-\alpha_j} B(z, z)^{-1/2}B(z, e)P_{ij} \right\| = h(z) \quad (z \in D)$$

and shows that h is a Busemann point. □

Let us now analyse the parts of horofunction boundary $D(\infty)$.

Proposition 8.4. *Let $h, h' \in D(\infty)$ with h given as in (8.3) and h' given by*

$$h'(z) = \frac{1}{2} \log \left\| \sum_{1 \leq i \leq j \leq q} \mu_i \mu_j B(z, z)^{-1/2}B(z, c)P'_{ij} \right\|, \quad (8.6)$$

where $c = c_1 + \dots + c_q$. If h and h' are in the same part, then $e = c$.

Proof. If h and h' are in the same part, then $H(h, h') < \infty$. Let $\gamma(t)$ be the geodesic converging to h given in (8.5). As h and h' are Busemann, $H(h, h') = \lim_t \rho(0, \gamma(t)) + h'(\gamma(t)) = \lim_t t + h'(\gamma(t))$ by (2.2), so that

$$H(h, h') = \lim_t t + \frac{1}{2} \log \left\| \sum_{1 \leq i \leq j \leq q} \mu_i \mu_j B(\gamma(t), \gamma(t))^{-1/2}B(\gamma(t), c)P'_{ij} \right\|.$$

If we let $v = \sum_{i=1}^q \mu_i^{1/2} c_i$, then

$$\sum_{1 \leq i \leq j \leq q} \mu_i \mu_j B(\gamma(t), \gamma(t))^{-1/2}B(\gamma(t), c)P'_{ij} = B(\gamma(t), \gamma(t))^{-1/2}B(\gamma(t), c)Q_v^2$$

by [27, Corollary 3.15]. So, $H(h, h') < \infty$ implies that

$$\|B(\gamma(t), \gamma(t))^{-1/2}B(\gamma(t), c)Q_v^2\| \longrightarrow 0. \quad (8.7)$$

We claim that this implies that $B(e, c)w = 0$ for all $w \in V_2(c)$. To show this we set $\beta_{it} = \tanh(t - \alpha_i)$ for $i = 1, \dots, q$, and set $\beta_{it} = 0$ for $i = 0$. Then

$$B(\gamma(t), \gamma(t))^{-1} = \sum_{0 \leq i \leq j \leq q} (1 - \beta_{it}^2)^{-1}(1 - \beta_{jt}^2)^{-1}P_{ij}.$$

We note that $B(\gamma(t), \gamma(t))$ is a self-adjoint invertible operator on the Hilbert space $(V, \langle \cdot, \cdot \rangle)$ in (5.2), by [6, Lemma 1.2.22]. Hence the preceding equation implies

$$\langle B(\gamma(t), \gamma(t))^{-1}v, v \rangle \geq \langle v, v \rangle \quad (v \in V).$$

Suppose that there exists a $w \in V_2(c)$ with $z = B(e, c)w \neq 0$. Now letting $z_t = B(\gamma(t), c)w$, we find that

$$\langle B(\gamma(t), \gamma(t))^{-1/2} z_t, B(\gamma(t), \gamma(t))^{-1/2} z_t \rangle = \langle B(\gamma(t), \gamma(t))^{-1} z_t, z_t \rangle \geq \langle z_t, z_t \rangle \longrightarrow \langle z, z \rangle > 0. \quad (8.8)$$

From [27, Corollary 3.15] we know that Q_v^2 is invertible on $V_2(c)$, with inverse $Q_{v^{-1}}^2$ and $v^{-1} = \sum_{i=1}^q \mu_i^{-1/2} c_i$. So, if we let $u = Q_{v^{-1}}^2 w$, then there exists $\delta > 0$ such that for all large $t > 0$,

$$\|B(\gamma(t), \gamma(t))^{-1/2} B(\gamma(t), c) Q_v^2\| \geq \|B(\gamma(t), \gamma(t))^{-1/2} B(\gamma(t), c) Q_v^2 u\| \|u\|^{-1} = \|B(\gamma(t), \gamma(t))^{-1/2} z_t\| \|u\|^{-1} \geq \delta$$

by (8.8), which contradicts (8.7).

It now follows from Lemma 6.1 that $c \leq e$. As $H(h', h) < \infty$ as well, we can interchange the roles of h and h' and deduce $e \leq c$, hence concluding the proof of $e = c$. \square

In the next proposition we show that $e = c$ is also a sufficient condition for h and h' to be in the same part.

Proposition 8.5. *Let $h, h' \in D(\infty)$ be given by (8.3) and (8.6), respectively. If $e = c$, then h and h' are in the same part. Moreover, $h = h'$ if and only if $e = c$ and $\sum_{i=1}^p \lambda_i e_i = \sum_{i=1}^q \mu_i c_i$.*

Proof. Let $e = c$. Then $p = q$. Since h and h' are Busemann points, we have by (2.2) that

$$H(h, h') = \lim_t t + h'(\gamma(t)) = \lim_t \frac{1}{2} \log \left\| \exp(2t) \sum_{1 \leq i \leq j \leq p} \mu_i \mu_j B(\gamma(t), \gamma(t))^{-1/2} B(\gamma(t), e) P'_{ij} \right\|,$$

where $\gamma(t)$ is the geodesic converging to h in (8.5), and we have used $e = c$.

Let $V = \bigoplus_{0 \leq k \leq l \leq p} V_{kl}$ be the Peirce decomposition with respect to e_1, \dots, e_p . For $w_{kl} \in V_{kl}$, we have

$$B(\gamma(t), \gamma(t))^{-1/2} B(\gamma(t), e) w_{kl} = \left(\frac{(1 - \beta_{kt})(1 - \beta_{lt})}{(1 - \beta_{kt}^2)^{1/2} (1 - \beta_{lt}^2)^{1/2}} \right) w_{kl} = \exp(-(t - \alpha_k)) \exp(-(t - \alpha_l)) w_{kl},$$

where $\beta_{it} = \tanh(t - \alpha_i)$ for $1 \leq i \leq p$ and $\beta_{0t} = 0$. So,

$$\lim_t \exp(2t) B(\gamma(t), \gamma(t))^{-1/2} B(\gamma(t), e) w_{kl} = \exp(\alpha_k) \exp(\alpha_l) w_{kl} = \lambda_k^{-1} \lambda_l^{-1} w_{kl},$$

where $\alpha_i = -\log \lambda_i$ for $i = 1, \dots, p$ and $\alpha_0 = 1$.

For $w \in V_2(e) = \bigoplus_{1 \leq k \leq l \leq p} V_{kl}$ we have that

$$\begin{aligned} \lim_t \exp(2t) B(\gamma(t), \gamma(t))^{-1/2} B(\gamma(t), e) w &= \lim_t \exp(2t) \sum_{1 \leq k \leq l \leq p} B(\gamma(t), \gamma(t))^{-1/2} B(\gamma(t), e) w_{kl} \\ &= \sum_{1 \leq k \leq l \leq p} \lambda_k^{-1} \lambda_l^{-1} w_{kl} = \sum_{1 \leq k \leq l \leq p} \lambda_k^{-1} \lambda_l^{-1} Q_e^2 w_{kl} = Q_{a^{-1}} Q_e w, \end{aligned}$$

where $a^{-1} = \sum_{k=1}^p \lambda_k^{-1} e_k$ by [27, Corollary 3.15]. Recall that by (3.11), $V_2(e)$ carries the structure of a JB*-algebra, in which the self-adjoint part $A(e)$ is partially ordered by the cone $A(e)_+ = \{x^2 : x \in A(e)\}$. The tripotents $e_1, \dots, e_p, c_1, \dots, c_q$ reside in $A(e)_+$ and are idempotents in the JB-algebra $A(e)$. Let $a = \sum_{k=1}^p \lambda_k e_k$ and $b = \sum_{i=1}^q \mu_i c_i$. Then a and b are invertible elements in $A(e)_+$, with inverses $a^{-1} = \sum_{k=1}^p \lambda_k^{-1} e_k$ and $b^{-1} = \sum_{i=1}^q \mu_i^{-1} c_i$, respectively.

So, as $e = c$, we now find for $v \in V_2(e) = V_2(c)$ that

$$\lim_t \exp(2t) \sum_{1 \leq i \leq j \leq q} \mu_i \mu_j B(\gamma(t), \gamma(t))^{-1/2} B(\gamma(t), e) P'_{ij} v = \sum_{1 \leq i \leq j \leq q} \mu_i \mu_j Q_{a^{-1}} Q_e P'_{ij} v = Q_{a^{-1}} Q_e Q_b Q_e v.$$

Hence, by (8.4),

$$H(h, h') = \frac{1}{2} \log \|Q_{a^{-1}} Q_e Q_b Q_e P_2(e)\| = \frac{1}{2} \log \|Q_{a^{-1}} Q_e Q_b Q_e\| < \infty. \quad (8.9)$$

Interchanging the roles of h and h' , we conclude

$$\delta(h, h') = H(h, h') + H(h', h) < \infty.$$

Finally, given $e = c$ and $a = b$, we can use the identity (JP3) to get

$$Q_{a^{-1}}Q_eQ_bQ_e = \{b^{-1}, \{b, \cdot, b\}, b^{-1}\} = P_2(e),$$

which is the identity operator on $V_2(e)$, and therefore $H(h, h') = 0$ by (8.9). Likewise, $H(h', h) = 0$, so that $\delta(h, h') = 0$ and hence $h = h'$.

Conversely, if $h = h'$, then they are in the same part. By Proposition 8.4, we have $e = c$, and (8.9) implies $\|Q_{a^{-1}}Q_eQ_bQ_e\| = 1$. In particular,

$$\|\{a^{-1}, b^2, a^{-1}\}\| = \|Q_{a^{-1}}Q_eQ_bQ_e\| \leq 1 \quad \text{and} \quad \|\{b^{-1}, a^2, b^{-1}\}\| = \|Q_{b^{-1}}Q_eQ_aQ_e\| \leq 1.$$

In $A(e)$, the first inequality implies $\{a^{-1}, b^2, a^{-1}\} \leq e$, by (3.14), and hence

$$b^2 = \{a, \{a^{-1}, b^2, a^{-1}\}, a\} \leq \{a, e, a\} = a^2,$$

whereas the second implies $a^2 \leq b^2$. It follows that $a^2 = b^2$ and $a = b$, since $a, b \in A(e)_+$. \square

We now begin the proof of Theorem 8.1, which will be split into several lemmas. Let

$$\widetilde{\text{exp}}_0: V \cup V(\infty) \longrightarrow D \cup D(\infty)$$

be as defined in (8.1) and (8.2).

Lemma 8.6. *The map $\widetilde{\text{exp}}_0$ is a well-defined bijection which maps V onto D and $V(\infty)$ onto $D(\infty)$. Further, it maps each equivalence class in $V(\infty)/\sim$ onto an equivalence class of $D(\infty)/\sim$.*

Proof. To see that $\widetilde{\text{exp}}_0$ is well-defined, pick $h \in V(\infty)$ with two formulations

$$h(x) = \Lambda_{V_2(e)}\left(-\frac{1}{2}(e \square P_2(e)x + P_2(e)x \square e) - \sum_{i=1}^p \alpha_i(e_i \square e_i)\right) \quad (x \in V)$$

and

$$h(x) = \Lambda_{V_2(e')}\left(-\frac{1}{2}(e' \square P_2(e')x + P_2(e')x \square e') - \sum_{i=1}^q \gamma_i(e'_i \square e'_i)\right) \quad (x \in V).$$

Note that by [27, Corollary 3.15(2)],

$$\frac{1}{2} \log \left\| \sum_{1 \leq i \leq j \leq p} e^{-\alpha_i} e^{-\alpha_j} B(z, z)^{-1/2} B(z, e) P_{ij} \right\| = \frac{1}{2} \log \left\| B(z, z)^{-1/2} B(z, e) Q_a Q_e P_2(e) \right\| \quad (z \in D),$$

where $a = \sum_{i=1}^p e^{-\alpha_i} e_i$. Likewise,

$$\frac{1}{2} \log \left\| \sum_{1 \leq i \leq j \leq q} e^{-\gamma_i} e^{-\gamma_j} B(z, z)^{-1/2} B(z, e') P'_{ij} \right\| = \frac{1}{2} \log \left\| B(z, z)^{-1/2} B(z, e') Q_b Q_{e'} P_2(e') \right\| \quad (z \in D),$$

where $b = \sum_{i=1}^q e^{-\gamma_i} e'_i$.

It follows from Corollary 6.3 that $e = e'$, $p = q$ and $\sum_{i=1}^p \alpha_i e_i = \sum_{i=1}^q \gamma_i e'_i$. By relabelling we may as well assume that $\alpha_1 \geq \dots \geq \alpha_p = 0$ and $\gamma_1 \geq \dots \geq \gamma_p = 0$. As the eigenvalues in the spectral decomposition in V are unique, we conclude that $a = b$ by Remark 6.4, and hence $\widetilde{\text{exp}}_0$ is well-defined.

Note that it follows from Theorems 4.4 and 5.2 that $\widetilde{\text{exp}}_0$ maps $V(\infty)$ onto $D(\infty)$. Moreover, given h' is in the same part as h , with

$$h'(x) = \Lambda_{V_2(c)}\left(-\frac{1}{2}(c \square P_2(c)x + P_2(c)x \square c) - \sum_{i=1}^q \beta_i(c_i \square c_i)\right) \quad (x \in V),$$

Theorem 7.2 implies $e = c$, and hence $\widetilde{\text{exp}}_0(h)$ and $\widetilde{\text{exp}}_0(h')$ are in the same part in $D(\infty)$ as well, by Proposition 8.4. Thus, the extension maps each equivalence class in $V(\infty)/\sim$ onto an equivalence class on $D(\infty)/\sim$, as all horofunctions are Busemann.

To complete the proof, we need to show that $\widetilde{\text{exp}}_0$ is injective on $V(\infty)$. Let $h, h' \in V(\infty)$ be given by

$$h(x) = \Lambda_{V_2(e)}\left(-\frac{1}{2}(e \square P_2(e)x + P_2(e)x \square e) - \sum_{i=1}^p \alpha_i(e_i \square e_i)\right) \quad (x \in V)$$

and

$$h'(x) = \Lambda_{V_2(c)}\left(-\frac{1}{2}(c \square P_2(c)x + P_2(c)x \square c) - \sum_{i=1}^q \beta_i(c_i \square c_i)\right) \quad (x \in V).$$

Set $g = \widetilde{\text{exp}}_0(h)$ and $g' = \widetilde{\text{exp}}_0(h')$, and suppose $g = g'$. Then $e = c$, $p = q$ and $\sum_{i=1}^p e^{-\alpha_i} e_i = \sum_{i=1}^q e^{-\beta_i} c_i$ by Proposition 8.5. By Remark 6.4, we have $\sum_{i=1}^p \alpha_i e_i = \sum_{i=1}^q \beta_i c_i$, and hence $h = h'$ by Corollary 6.3, which concludes the proof. \square

Lemma 8.7. *Let (v_n) be a sequence in V converging to $h \in V(\infty)$. Then $(\widetilde{\text{exp}}_0(v_n))$ converges to $\widetilde{\text{exp}}_0(h)$.*

Proof. Let $h \in V(\infty)$ with

$$h(x) = \Lambda_{V_2(e)}\left(-\frac{1}{2}(e \square P_2(e)x + P_2(e)x \square e) - \sum_{i=1}^p \alpha_i(e_i \square e_i)\right) \quad (x \in V),$$

and

$$g(z) = \widetilde{\text{exp}}_0(h) = \frac{1}{2} \log \left\| \sum_{1 \leq i \leq j \leq p} e^{-\alpha_i} e^{-\alpha_j} B(z, z)^{-1/2} B(z, e) P_{ij} \right\| \quad (z \in D).$$

It suffices to show that each subsequence $(\widetilde{\text{exp}}_0(v_k))$ has a convergent subsequence with limit g .

Since $h \in V(\infty)$, we know $\|v_k\| \rightarrow \infty$. Let $r_k = \|v_k\|$ and denote by

$$v_k = \mu_{1k} c_{1k} + \mu_{2k} c_{2k} + \cdots + \mu_{rk} c_{rk}$$

the spectral decomposition of v_k .

Taking subsequences, we may assume that

$$(1) \quad \beta_{ik} = r_k - \mu_{ik} \rightarrow \beta_i \in [0, \infty],$$

$$(2) \quad c_{ik} \rightarrow c_i,$$

for all $i = 1, \dots, r$, where $0 = \beta_1 \leq \beta_2 \leq \dots \leq \beta_r$. Let $J = \{i: \beta_i < \infty\} = \{1, \dots, q\}$ and set $c = c_1 + \cdots + c_q$.

Observe that

$$h_{v_k}(x) = \frac{\|x - v_k\|^2 - \|v_k\|^2}{\|x - v_k\| + \|v_k\|} = \frac{(2r_k)^{-1}(\|(x - v_k) \square (x - v_k)\| - r_k^2)}{2^{-1}(\|r_k^{-1}(x - v_k)\| + 1)}$$

and $2^{-1}(\|r_k^{-1}(x - v_k)\| + 1) \rightarrow 1$. Hence it follows from Lemma 5.1 that (h_{v_k}) converges to

$$h'(x) = \Lambda_{V_2(c)}\left(-\frac{1}{2}(c \square P_2(c)x + P_2(c)x \square c) - \sum_{i=1}^q \beta_i(c_i \square c_i)\right) \quad (x \in V)$$

and, it follows from Theorem 5.2 that $h' \in V(\infty)$.

Therefore $h = h'$, and it follows from Corollary 6.3 that $e = c$, $p = q$ and $\sum_{i=1}^p \alpha_i e_i = \sum_{i=1}^q \beta_i c_i$. By Remark 6.4, we get

$$\sum_{i=1}^p e^{-\alpha_i} e_i = \sum_{i=1}^q e^{-\beta_i} c_i.$$

Let $w_k = \widetilde{\exp}_0(v_k)$ and $f_{w_k}(z) = \rho(z, w_k) - \rho(0, w_k)$ for $z \in D$. By taking a subsequence, we may assume that $w_k \rightarrow \xi \in \partial D$. Note that ξ has a spectral decomposition $\xi = \sum_{i=1}^r \mu_i c_i$, where $\mu_i = 1$ for $i = 1, \dots, q$. Let $g_{-w_k}: D \rightarrow D$ be the Möbius transformation that maps w_k to 0. Then

$$\begin{aligned} f_{w_k}(z) &= \rho(z, w_k) - \frac{1}{2} \log \frac{1 + \tanh r_k}{1 - \tanh r_k} \\ &= \frac{1}{2} \log \frac{1 + \|g_{-w_k}(z)\|}{1 - \|g_{-w_k}(z)\|} - \frac{1}{2} \log \frac{1 + \tanh r_k}{1 - \tanh r_k} \\ &= \frac{1}{2} \log \left(\frac{1 - \tanh^2 r_k}{1 - \|g_{-w_k}(z)\|^2} \right) \left(\frac{1 + \|g_{-w_k}(z)\|}{1 + \tanh r_k} \right)^2. \end{aligned}$$

By Lemma 3.8, we have $\lim_k \|g_{-w_k}(z)\| = 1$, so that

$$\left(\frac{1 + \|g_{-w_k}(z)\|}{1 + \tanh r_k} \right)^2 \rightarrow 1.$$

As before, we set $\mu_{0k} = 0$ and let P_{ij}^k be the Peirce projections with respect to the tripotents c_{1k}, \dots, c_{rk} . Then

$$\begin{aligned} \frac{1 - \tanh^2 r_k}{1 - \|g_{-w_k}(z)\|^2} &= \|(1 - \tanh^2 r_k)B(z, z)^{-1/2}B(z, w_k)B(w_k, w_k)^{-1/2}\| \\ &= \left\| \sum_{0 \leq i \leq j \leq r} \left(\frac{1 - \tanh^2 r_k}{1 - \tanh^2 \mu_{ik}} \right)^{1/2} \left(\frac{1 - \tanh^2 r_k}{1 - \tanh^2 \mu_{jk}} \right)^{1/2} B(z, z)^{-1/2}B(z, w_k)P_{ij}^k \right\| \\ &= \left\| \sum_{0 \leq i \leq j \leq r} e^{-r_k + \mu_{ik}} e^{-r_k + \mu_{jk}} \left(\frac{1 + \tanh r_k}{1 + \tanh \mu_{ik}} \right) \left(\frac{1 + \tanh r_k}{1 + \tanh \mu_{jk}} \right) B(z, z)^{-1/2}B(z, w_k)P_{ij}^k \right\| \\ &= \left\| \sum_{0 \leq i \leq j \leq r} e^{-\beta_{ik}} e^{-\beta_{jk}} \left(\frac{1 + \tanh r_k}{1 + \tanh(r_k - \beta_{ik})} \right) \left(\frac{1 + \tanh r_k}{1 + \tanh(r_k - \beta_{jk})} \right) B(z, z)^{-1/2}B(z, w_k)P_{ij}^k \right\|, \end{aligned}$$

where $\beta_{0k} = r_k$.

For each $i = 1, \dots, q$, we have

$$\frac{1 + \tanh r_k}{1 + \tanh(r_k - \beta_{ik})} \rightarrow 1.$$

By [9, Remark 5.9], the Peirce projections P_{ij}^k converge to the Peirce projections P'_{ij} of the minimal tripotents c_1, \dots, c_r , as $c_{ik} \rightarrow c_i$ for all i . Using the fact that $w_k \rightarrow \xi = \sum_{i=1}^r \mu_i c_i$ and $\mu_i = 1$ for $i = 1, \dots, q$, we find that

$$\begin{aligned} &\left\| \sum_{0 \leq i \leq j \leq r} e^{-\beta_{ik}} e^{-\beta_{jk}} \left(\frac{1 + \tanh r_k}{1 + \tanh(r_k - \beta_{ik})} \right) \left(\frac{1 + \tanh r_k}{1 + \tanh(r_k - \beta_{jk})} \right) B(z, z)^{-1/2}B(z, w_k)P_{ij}^k \right\| \\ &\rightarrow \left\| \sum_{1 \leq i \leq j \leq q} e^{-\beta_i} e^{-\beta_j} B(z, z)^{-1/2}B(z, \xi)P'_{ij} \right\| = \left\| \sum_{1 \leq i \leq j \leq q} e^{-\beta_i} e^{-\beta_j} B(z, z)^{-1/2}B(z, c)P'_{ij} \right\| \end{aligned}$$

and hence

$$f_{w_k}(z) \rightarrow \frac{1}{2} \log \left\| \sum_{1 \leq i \leq j \leq q} e^{-\beta_i} e^{-\beta_j} B(z, z)^{-1/2}B(z, c)P'_{ij} \right\|.$$

The right-hand side is a horofunction, say f , in $D(\infty)$ by Theorem 4.4. As $e = c$ and $\sum_{i=1}^p e^{-\alpha_i} e_i = \sum_{i=1}^q e^{-\beta_i} c_i$, we obtain $g = f$ from Proposition 8.5. This shows that $(\widetilde{\exp}_0(v_k))$ has a subsequence converging to g . \square

Finally, we prove the following lemma which, together with the preceding ones, complete the proof of Theorem 8.1.

Lemma 8.8. *Let (h_n) be a sequence in $V(\infty)$ converging to $h \in V(\infty)$. Then $(\widetilde{\text{exp}}_0(h_n))$ converges to $\widetilde{\text{exp}}_0(h)$.*

Proof. Let $h_n \in V(\infty)$ be given by

$$h_n(x) = \Lambda_{V_2(c^n)} \left(-\frac{1}{2} (c^n \square P_2(c^n)x + P_2(c^n)x \square c^n) - \sum_{i=1}^{q_n} \beta_{in} (c_{in} \square c_{in}) \right) \quad (x \in V).$$

and let $h \in V(\infty)$ be given by

$$h(x) = \Lambda_{V_2(e)} \left(-\frac{1}{2} (e \square P_2(e)x + P_2(e)x \square e) - \sum_{i=1}^p \alpha_i (e_i \square e_i) \right) \quad (x \in V).$$

We show that each subsequence $(\widetilde{\text{exp}}_0(h_k))$ has a convergent subsequence converging to $\widetilde{\text{exp}}_0(h) = g$, where

$$g(z) = \frac{1}{2} \log \left\| \sum_{1 \leq i \leq j \leq p} e^{-\alpha_i} e^{-\alpha_j} B(z, z)^{-1/2} B(z, e) P_{ij} \right\| \quad (z \in D).$$

Taking further subsequences, we may assume that $q_k = q_0$ for all k , $\beta_{ik} \rightarrow \beta_i \in [0, \infty]$ and $c_{ik} \rightarrow c_i$ for all $i = 1, \dots, q_0$. As $\min\{\beta_{ik} : i = 1, \dots, q_0\} = 0$, we have $\min\{\beta_i : i = 1, \dots, q_0\} = 0$. Let $J = \{i : \beta_i < \infty\}$. After relabelling we may assume that $J = \{1, \dots, q\}$. Let $c = c_1 + \dots + c_q$. Then Lemma 6.12 implies

$$\lim_k h_k(x) = \Lambda_{V_2(c)} \left(-\frac{1}{2} (c \square P_2(c)x + P_2(c)x \square c) - \sum_{i=1}^q \beta_i (c_i \square c_i) \right) \quad (x \in V).$$

The right-hand side is horofunction by Theorem 5.2, say $h' \in V(\infty)$. As $h_k \rightarrow h$, we conclude that $h = h'$ and hence Corollary 6.3 gives $e = c$, $p = q$, and $\sum_{i=1}^p \alpha_i e_i = \sum_{i=1}^q \beta_i c_i$. By Remark 6.4, we have

$$\sum_{i=1}^p e^{-\alpha_i} e_i = \sum_{i=1}^q e^{-\beta_i} c_i.$$

The right-hand side of the following limit is a horofunction g' in $D(\infty)$, by Theorem 4.4.

$$\begin{aligned} \widetilde{\text{exp}}_0(h_k) &= \frac{1}{2} \log \left\| \sum_{1 \leq i \leq j \leq q_0} e^{-\beta_{ik}} e^{-\beta_{jk}} B(z, z)^{-1/2} B(z, c^k) P_{ij}^k \right\| \\ &\rightarrow \frac{1}{2} \log \left\| \sum_{1 \leq i \leq j \leq q} e^{-\beta_i} e^{-\beta_j} B(z, z)^{-1/2} B(z, c) P_{ij}' \right\|, \end{aligned}$$

where P_{ij}^k are the Peirce projections for the tripotents c_{1k}, \dots, c_{q_0k} , and P_{ij}' the Peirce projections for the tripotents c_1, \dots, c_q . As $e = c$ and $\sum_{i=1}^p e^{-\alpha_i} e_i = \sum_{i=1}^q e^{-\beta_i} c_i$, Proposition 8.5 gives $g = g'$. We conclude that $(\widetilde{\text{exp}}_0(h_k))$ has a subsequence converging to g , which completes the proof. \square

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Declarations

Conflict of Interest The authors declare that they have no conflict of interest.

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