A copula-based approach to modelling the failure process of items under two-dimensional warranty and applications

Shaomin Wu*

Kent Business School, University of Kent, Canterbury, Kent CT2 7PE, United Kingdom

Abstract

Hundreds of scholarly papers on optimisation of preventive maintenance policies for items under warranty have been published in the reliability related literature. They typically have two limitations: they make a simplified assumption on the relationship between age and usage for items under two-dimensional warranty and they assume that it is cost-effective to conduct preventive maintenance (PM) on each sold product item. These assumptions may not reflect the reality. This paper therefore proposes a copula-based approach to modelling the relationship between age and usage of items under two-dimensional (2D) warranty. It investigates the probabilistic properties of the bivariate failure rate functions of age and usage and derives optimal PM policies for users who bought multiple product items. It also proposes approaches to incorporating maintenance effectiveness in bivariate copulas for depicting the dependence between age and usage. A case study and numerical examples are provided to illustrate the findings.

Keywords: (T) maintenance, warranty, cost-effectiveness, asymmetric copula, bivariate failure rate function.

1 Introduction

Warranty is a contract agreed by the warrantor and the buyer when the latter purchases product items from the former. It defines conditions and obligations in case the items fail to perform their intended function. From a warrantor's perspective, they wish to save budget on warranty claims, for which one of the means is to reduce the number of failures of the items during the warranty period. To this end, it inspires researchers to develop numerous methods, including optimisation of preventive maintenance (PM) policies, as discussed in this paper.

In the literature, there are many scholarly papers on the development of PM policies for product items under warranty. Among them, two-dimensional (2D) warranty is one of the central topics. A

*Email: s.m.wu@kent.ac.uk. Telephone: 0044 1227 827 940.

Suggested citation: Shaomin Wu, A copula-based approach to modelling the failure process of items under two-dimensional warranty and applications, European Journal of Operational Research, 2023, DOI: 10.1016/j.ejor.2023.10.043.

2D warranty covers age and usage. For example, the warranty of a car covers its age and mileage and the warranty of a copying machine covers its age and the number of papers it has copied. The existing research, nevertheless, has two limitations: They assume that relationship between age and usage is linear; and (2) they do not validate whether it is cost-effective to conduct PM or not. To overcome these two limitations, this paper aims to propose a new approach to modelling the relationship between age and usage and then a new method to optimise the PM policy.

1.1 Related work

To model the relationship between age and usage of a product item, three approaches have been developed (Wu, 2012): marginal approach, bivariate approach, and composite scale approach. The marginal approach uses the multiplication law of probability to obtain the joint distribution between age and usage, the bivariate approach directly estimates a joint distribution, and the composite scale approach seeks a linear combination of age and usage to form a new variable. The composite scale approach assumes a linear combination and such an assumption may be too restrictive. Many authors assume a linear relationship between age and usage to simplify their calculation processes. However, this simplification can fail to capture important information on the dependence between age and usage. As such, a joint distribution should be estimated, for which copulas can be used (Wu, 2014; Anderson, Chukova, & Hirose, 2017; Gupta, De, & Chatterjee, 2017). A copula is a multivariate cumulative distribution function whose one-dimensional margins are uniform on the interval [0,1] (Nelsen, 1999). In the reliability related literature, copulas have been used to depict (1) the dependence between different levels of a degradation process of a system under condition based maintenance (Xu, Liang, Li, & Wang, 2021; Andersen, Andersen, Kulahci, & Nielsen, 2022), (2) the dependence between components in a multi-component system (Ahmadi, 2019, 2020; Torrado, 2022; H. Li, Zhu, Dieulle, & Deloux, 2022), or (3) the association between two adjacent working periods before and after a repair for items whose failure processes are measured by age (Wu & Wang, 2018).

Preventive maintenance for product items under warranty has been widely investigated. Shafiee and Chukova (2013) provide a literature review on maintenance for items under warranty in 2013. We therefore review the research published after 2013. In the literature, there are many papers on PM policies for products under 2D warranty. Here are some examples. For instance, Shahanaghi, Noorossana, Jalali-Naini, and Heydari (2013) propose a method to optimise the number and degrees of preventive repairs for items under 2D warranty to minimize the extended warranty provider's servicing cost. Huang, Chen, and Ho (2013) optimise a non-periodic PM policy to optimise a two-dimensional warranty term with a constraint on reliability. In their paper, they assume the relationship between the usage and age is linear. Y. Wang, Liu, and Liu (2015) investigate a periodic and imperfect PM policy for an item covered by a fixed and combined 2D base warranty and extended warranty region from a warrantor's perspective. Su and Wang (2016) propose a method under which the item is preventively maintained according to a specified age interval or usage interval, whichever occurs first. Huang, Huang, and Ho (2017); Dai, Wei, Zhang, and He (2020) propose a customized 2D extended warranty policy, in which customers are classified into three categories based on their usage intensity, and optimise different PM policies accordingly. J. Wang, Zhou, and Peng (2017) optimise preventive maintenance policies by considering the cooperative and non-cooperative interactions between the manufacturer and the consumer. X. Li, Liu, Wang, and Li (2019) jointly optimise a burn-in policy and a PM policy for a product under warranty. Zhao, Liu, Xu, and Wang (2022) optimises imperfect maintenance policies for warranted products subject to stochastic performance degradation. Dai et al. (2020) present a PM policy for a product with their users being grouped into two groups according to their usage rates. Peng, Jiang, and Zhao (2021) develop a PM policy for a product under warranty and consider the impact of random and dynamic usage rates on the PM decisions. J. Wang, Zhu, and Du (2021) develop a mathematical model to optimise a PM policy for a product under a 2D warranty. To the best of our knowledge, a common implicit assumption in the existing literature is that preventive maintenance is performed on every product item and few authors have validate this assumption.

1.2 Gaps between research assumptions and practice

From our observations, there are at least the following two gaps between the reality and the academic research.

Gap1 **Relationship between age and usage.** Some existing literature assumes a linear relationship between age t and usage u with $u = \beta t$, where β is a constant parameter. It is logical to assume that this relationship is estimated by building a linear regression model from warranty claim data. Based on the analysis in Section 3.1 in this paper, such a linear relationship does not hold on the warranty claim data collected from a car manufacturer.

Gap2 **Necessity for PM.** Existing literature predominately assumes that preventive maintenance can be performed on every product item under warranty. According to Davidson and Drake (2019), however, from an analysis of the approximately 340 claims made on more than 2,450 representations and warranties insurance policies placed by Aon in North America between 2013 and 2019, the warranty claim frequency is 6.2% in 2019. With such low warranty claim frequency (or defective rates), it is hard to imagine that performing PM on each product item is cost effective. That is, there is a need to investigate under what conditions preventive maintenance should be performed on items under warranty.

1.3 Novelty and contributions

This paper aims to tackle the two aforementioned gaps, which creates novelty and makes contributions to the literature of the reliability and maintenance research area. The main contributions of the paper are listed below.

(a) It investigates the dependence between age and usage based on asymmetry copulas from different perspectives, including the bivariate hazard function and the conditional survival function, both of which are derived from the asymmetric copula. It also provides the bounds of the bivariate failure rate function, investigates probabilistic properties of the conditional survival function, and suggests some approaches to using the copula to model the effectiveness of preventive maintenance.

(b) It proposes two necessary conditions, under which PM can be executed in a cost effective manner, and then proposes a novel method to optimise PM policies, which minimise the expected cost, subject to the constraints.

It is noted there a vast number of papers on preventive maintenance optimisation for product items under warranty. However, little has been reported on the justification of the applications of preventive maintenance policies to product items under warranty. For example, if there are only less than 5% of the failed items of a product, which are then claimed warranty, conducting preventive maintenance policies on each item may not cost-effective. From this perspective, the significance of this current paper lies in its consideration of the claim rate and further indirectly challenging the validity of the assumption—which is the preventive maintenance can be conducted on all product items— made in the existing research.

1.4 Overview

The remainder of this paper is structured as follows. Section 2 lists assumptions. Section 3 derives failure rates and conditional survival probability. Section 4 introduces methods to assess maintenance effectiveness. Section 5 studies planned PM, constrained by the cost-effectiveness. Section 6 provides a case study to illustrate the applicability of the proposed approaches to assessing maintenance effectiveness. Section 7 provides numerical examples. Section 8 concludes the paper.

2 Assumptions

This paper makes the following assumptions.

A1 Suppose all the items start operating at time $t = 0$, and the warranty duration is w units of time and L units of usage.

A2 The product has two types of possible failures: type I failure and type II failure, with initial failure rates $r(t, u)$ (or $r_{\mathrm{I}}(t, u)$) and $r_{\mathrm{II}}(t, u)$ if the product is covered by 2D warranty, respectively. $r(t, u)$ is a decreasing function of t and u and $r_{\mathrm{II}}(t, u)$ is an increasing function of t and u.

A3 Both failure types I and II are repairable. Repair on type I failures is always minimal, that is, a repair will restore the item to the status just before it failed. The failure rate function of the type II failure mode after the $(k-1)$-th repair is $r_{\mathrm{II},k}(t, u)$. $r_{\mathrm{II}}(t, u)$ is the failure rate function before the first failure, respectively.

A4 The two failure types are statistically independent.

A5 The warranty policy is the NFRW (non-renewing free repair warranty) policy, under which the warrantor guarantees a satisfactory service during the base warranty period and the failed item is repaired by the warrantor at no cost to the user.

A6 It is the warrantor's responsibility to perform preventive maintenance (PM) and the associated costs will be borne by the warrantor.

A7 Time on repair or PM is so short that it is neglectable.

Remark 1 (Justification of the assumptions of the two failure types) *Theoretically, PM only takes effect on items when their failure rate function increases in time t. However, it is often stated that base warranty is designated for items with a decreasing failure rate, or items with teething problems. As such, there is a need to distinguish items with decreasing and increasing failure rates and treat them differently.*

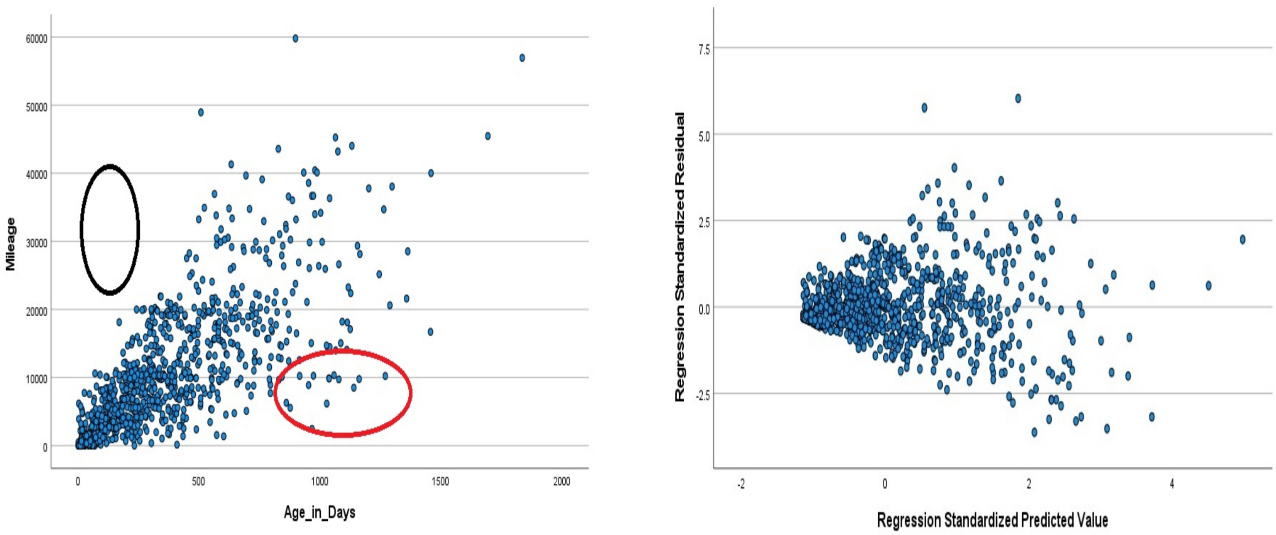
3 Modelling the dependence between age and usage

This section investigates some assumptions made on the dependence between age and usage in the existing literature and identifies their irrationality. It recalls the asymmetric copula proposed in Wu (2014) for measuring such a dependence and provides the relationship between this copula and Spearman's rho. For this copula, we derives the lower and upper bounds of the associated bivariate failure rate and obtain some properties of the related conditional survival functions.

3.1 Comments on the existing approaches

Regarding 2-D warranty, many papers assume there is a linear relationship between age t and usage u, that is, $u = \beta t$, where β is a constant parameter. Apparently, this relationship is assumed based on a linear regression model built from warranty claim data. To validate it, we have collected some warranty claim data. Figure 1a, on which the X-axis represents age and the Y-axis represents mileage, shows the distribution of 1,000 warranty claims collected from a UK auto manufacturer. Each dot in the figure represents a warranty claim. From the figure, the variability of the mileage increases in respect of age. As such, it is statistically inappropriate to fit a simple linear regression model between the age and mileage because there exists heteroscedasticity (i.e., unequal variances) in the residual of the model. Fig 1b, on which the X-axis represents the standardized predict values and the Y-axis represents standardized residuals of the model, is the residual plot of the model. The figure shows a <-shaped pattern, which suggests that the variability of the mileage increases in age and that there exists heteroscedasticity. Some methods that can be used to cure the heteroscedasticity problem include:

Method (i): A weighted linear regression model is developed to explore the relationship between age and usage.



(a) Age against mileage of 1,000 cars during their warranty duration.

(b) Residual plot of the linear model built on the data shown in Fig 1a.

Figure 1: Warranty claim data collected from an auto manufacturer in the UK.

Method (ii): A transformation such as the logarithm transformation is used on the dependent variable u and then re-build a linear regression model $\ln(u) = \beta_0 + \beta_1 t + \epsilon$, or equivalently, $u = e^{\beta_0+\beta_1 t+\epsilon}$, where ϵ is the residual of the model. This method will end up with a nonlinear relationship between age and usage, instead of a linear relationship.

Method (iii): A bivariate probability distribution is used to fit the observations, as shown in Wu (2014).

It should be noted that if Method (i) is used in development of PM policies, the weight matrix must be incorporated into the PM policy optimisation. To the best of the author's knowledge, however, such a consideration has never been reported in the literature; and (2) that the non-linear regression model developed in Method (ii) is unable to capture the asymmetric phenomenon. The asymmetric phenomenon can be observed from Fig 1a: there is no warranty claims in the oval close to the Y-axis—which represents the area occupied by warranty claims for items with small age and large mileage, and there are some warranty claims within the oval close to the X-axis—which represents the area occupied by warranty claims for items with large age and small mileage. One may presume that such an asymmetric phenomenon may exist in other applications.

It should also be noted that for widely studied systems such as cars, research is often ill-conducted because they assume that the operating time is available for analysis, which is not the case as automotive vehicles normally do not record such data. If data on operating age is available, there is no doubt that they can find the relationship between the operating age and the usage is approximately linear.

Consequently, Method (iii) should be used to capture the relationship between age and usage and is the method that will be investigated in this paper. It should be noted that Wu (2014) only investigates the relationship between age and usage for non-repairable systems, whereas this paper concentrates on the scenarios of repairable systems.

3.2 An asymmetric copula approach

In probability and statistics, for a given collection of random variables, if each variable has a marginal probability distribution, then one can construct their joint distribution using a copula. That is, a copula is a joint probability distribution that represents a multivariate uniform distribution of multiple random variables and it depicts the dependence between the random variables. For example, the joint distribution, $F(t, u)$, of age (t) and usage (u) can be expressed by a copula, $C(v_1, v_2)$. That is, we can find an expression for $F(t, u)$ by noting that

$$\begin{aligned} F(t,u) &= P(X_1 < t, X_2 < u) \\ &= P(F_1(X_1) < F_1(t), F_2(X_2) < F_2(u)) \\ &= P(F_1(X_1) < v_1, F_2(X_2) < v_2), \end{aligned} \tag{1}$$

where $v_1 := F_1(t)$ and $v_2 := F_2(u)$. v_1 and v_2 are uniformly distributed random variables. Then Eq. (1) yields

$$F(t,u) = C(F_1(t), F_2(u)) = C(v_1, v_2), \tag{2}$$

The reader is referred to Nelsen (1999) for a detailed introduction to copulas.

A copula can be symmetric or asymmetric, as defined below.

Definition 1 *(Nelsen, 1999) We say a copula is symmetric if $C(v_1, v_2) = C(v_2, v_1)$, and asymmetric if $C(v_1, v_2) \neq C(v_2, v_1)$.*

According to Wu (2014), the dependence between age and usage can be depicted by two characteristics: *positive dependence* and *tail dependence*, which are elaborated below.

The dependence between age and usage is positive, as shown in Fig 1a. The strength of the association can be measured by Spearman's correlation coefficient, or Spearman's rho for short. Hence, Spearman's rho between age and usage must be positive, which is referred to as the *positive dependence* in what follows.

Furthermore, as explained in Gap1, there exists an asymmetry phenomenon between age and usage. That is, the joint distribution between age and usage should be an asymmetric copula. Although Definition 1 clearly defines the meaning of an asymmetric copula, a better way to detect the asymmetric phenomenon existing in age and usage is to use the *tail dependence*, which is a concept borrowed from probability theory. The tail dependence of a pair of random variables measures the dependence between the variables in the upper-right quadrant and in the lower-left quadrant of $[0, 1]^2$ (Nelsen, 1999), i.e., it is a measure of their co-movements in the tails of the distributions. Accordingly, the lower-upper and upper-lower tail dependence coefficients of copula $C(v_1, v_2)$ are given by

$$\lambda^{l,u}(C) = 1 - \lim_{v \to 0+} \frac{C(v, 1-v)}{v}, \tag{3}$$

and

$$\lambda^{u,l}(C) = 1 - \lim_{v \to 0+} \frac{C(1-v, v)}{v}, \tag{4}$$

where $\lambda^{u,l}(C)$ depicts the dependency between large age and small usage and $\lambda^{l,u}(C)$ depicts the association between small age and large usage. Then the asymmetric phenomenon between age and usage can be described by $\lambda^{u,l}(C) \geq \lambda^{l,u}(C)$.

To construct a copula that satisfies the above two conditions, i.e., positive dependence and tail dependence, Wu (2014) constructs the following asymmetric copula.

$$C(v_1, v_2) = p_0 C^*(v_1, v_2) + (1 - p_0)(v_2 - C^*(1 - v_1, v_2)) \tag{5}$$

where $p_0 \in [0, 1]$ and $C^*(v_1, v_2)$ is a base, symmetric copula.

Wu (2014) compares the performance of $C(v_1, v_2)$ in Eq (5) with two other models on a set of warranty claims collected from an auto manufacturer in the UK and concludes that the copula in (5) outperforms the two other models in terms of the AIC (Akaike information criterion). Nevertheless, the author did not discuss other properties of the copula, such as its Spearman's rho and its bivariate failure rate function, which will be investigated below.

In statistics, Spearman's rho is a non-parametric statistic measure of the strength of the association between two random variables. Denote ρ^* as Spearman's rho of the copula $C^*(v_1, v_2)$. Then we can obtain Spearman's rho of the copula $C(v_1, v_2)$ as follows.

Then we obtain the following proposition.

Proposition 1 *Spearman's rho of $C(v_1, v_2)$ shown in Eq. (5) is $(2p_0 - 1)\rho^*$.*

The proofs of the above and other propositions can be found in Appendix.

Hence, Spearman's rho of $C(v_1, v_2)$ is positive if $\rho^* > 0$ and $p_0 > 0.5$, or $\rho^* < 0$ and $0 < p_0 < 0.5$. There are many copulas that have been proposed (Nelsen, 1999). The following copula, the survival Gumbel copula, is a symmetric copula that is widely used in the literature of reliability:

$$C^*(v_1, v_2) = v_1 + v_2 - 1 + C_0(1 - v_1, 1 - v_2). \tag{6}$$

where $C_0(v_1, v_2) = \exp\left\{-\left[(-\ln(v_1)^\theta + (-\ln(v_2)^\theta\right]^{1/\theta}\right\}$ is the Gumbel copula.

Plugging $C^*(v_1, v_2)$ in Eq. (6) into Eq. (5), we obtain

$$\begin{aligned} C(v_1, v_2) &= p_0(v_1 + v_2 - 1 + C_0(1 - v_1, 1 - v_2)) + (1 - p_0)(v_1 - C_0(v_1, 1 - v_2)) \\ &= v_1 - p_0(1 - v_2) + p_0 \exp\left\{-\left[(-\ln(1 - v_1))^\theta + (-\ln(1 - v_2))^\theta\right]^{1/\theta}\right\} \\ &\quad -(1 - p_0)\exp\left\{-\left[(-\ln(v_1))^\theta + (-\ln(1 - v_2))^\theta\right]^{1/\theta}\right\}. \end{aligned} \tag{7}$$

If $p_0 = 1$, $C(v_1, v_2)$ in Eq. (7) is a symmetric copula. According to Wu (2014), $\lambda^{l,u}(C) = 0$ and $\lambda^{u,l}(C) = (1 - p_0)(2 - 2^{1/\theta})$. That is, the copula $C(v_1, v_2)$ has the tail dependence if $p_0 \neq 1$ and $\theta > 1$. As such, in what follows, we impose $p_0 \neq 1$ and $\theta > 1$.

3.3 An asymmetric-copula-derived hazard function

To optimise a PM policy on product items, we need to know the expected number of failures, which is derived from the failure rate function of the items. When the lifetime distribution function of the items is a univariate function of time t: $F(t)$, the failure rate function is defined by $\frac{dF(t)/dt}{1-F(t)}$. Nevertheless, if the lifetime distribution function of the items is a bivariate function of time t and usage u, there are different definitions of failure rate functions. Below we adopt the one introduced by Basu (1971).

Given a bivariate cumulative distribution function $F(t,u)$ and its associated density function $f(t,u)$, then its corresponding bivariate hazard rate can be defined by

$$r(t,u) = \lim_{\max\{\Delta_1,\Delta_2\}\to 0} \frac{P\{t \le X_1 < t+\Delta_1, u \le X_2 < u+\Delta_2 | t \le X_1, u \le X_2\}}{\Delta_1\Delta_2}, \tag{8}$$

or equivalently:

$$r(t,u) = \frac{f(t,u)}{\bar{F}(t,u)}. \tag{9}$$

For a given copula $C(v_1,v_2)$, one can obtain its associated failure rate function as follows.

Denote $c(v_1,v_2) = \frac{\partial^2 C(v_1,v_2)}{\partial v_1 \partial v_2}$, then

$$c(v_1,v_2) = \frac{\partial^2 F(F_1^{-1}(v_1), F_2^{-1}(v_2))}{\partial v_1 \partial v_2} = \frac{f(F_1^{-1}(v_1), F_2^{-1}(v_2))}{f_1(F_1^{-1}(v_1)) f_2(F_2^{-1}(v_2))} = \frac{f(t,u)}{f_1(t) f_2(u)}, \tag{10}$$

where $f(t,u) = \partial^2 F(t,u)/\partial t \partial u$, $f_1(t) = \int_0^\infty f(t,u)du$, and $f_2(u) = \int_0^\infty f(t,u)dt$.

Suppose F_i is differentiable ($i = 1,2$), since $\bar{F}(t,u) = 1 - F_1(t) - F_2(u) + F(t,u)$, the bivariate failure rate function can be defined with the copula as follows:

$$r(t,u) = \frac{f(t,u)}{\bar{F}(t,u)} = \frac{c(v_1,v_2) f_1(F_1^{-1}(v_1)) f_2(F_2^{-1}(v_2))}{1 - v_1 - v_2 + C(v_1,v_2)}. \tag{11}$$

Denote $g_{t,u}(v_1,v_2) = \ln(C_0(v_1,v_2)) = \left[(-\ln(1-v_1))^\theta + (-\ln(1-v_2))^\theta\right]^{1/\theta}$. Then, for example, for the copula given in Eq. (7), its density is obtained by

$$\begin{aligned} & c(v_1,v_2) \\ =& \frac{[-\ln(1-v_2)]^{\theta-1}}{1-v_2} \Bigg\{ p_0[g_{t,u}(v_1,v_2)+\theta-1][g_{t,u}(v_1,v_2)]^{1-2\theta} \frac{[-\ln(1-v_1)]^{\theta-1}}{1-v_1} \exp(-g_{t,u}(v_1,v_2)) \\ & + (1-p_0)[g_{t,u}(1-v_1,v_2)+\theta-1][g_{t,u}(1-v_1,v_2)]^{1-2\theta} \frac{[-\ln(v_1)]^{\theta-1}}{v_1} \exp(-g_{t,u}(1-v_1,v_2)) \Bigg\}. \end{aligned} \tag{12}$$

It can be seen that one can plug $C(v_1,v_2)$ in Eq. (5) and $c(v_1,v_2)$ in Eq. (12) into Eq. (11) to obtain $r(t,u)$, respectively. However, the expression of $r(t,u)$ can be very complex. When we calculate the failure rate $\int_0^\infty \int_0^\infty r(t,u)dtdu$, the result will become even more complex, numerical methods may therefore be pursued. This may also be the case when we calculate the failure rate for the case that the joint bi-variate distribution is symmetric (i.e., for the case when $p_0 = 1$).

Following on the above discussion, we have the following result.

Denote

$$C^+(v_1, v_2) = v_2 - C^*(1 - v_1, v_2) = 1 - v_1 - C_0(v_1, 1 - v_2), \tag{13}$$

$F^+(t,u) = C^+(v_1, v_2)$, $r^+(t,u) = \frac{f^+(t,u)}{\bar{F}^+(t,u)}$, and $F^*(t,u) = C^*(v_1, v_2)$, $r^*(t,u) = \frac{f^*(t,u)}{\bar{F}^*(t,u)}$, where $f^*(t,u)$ and $f^+(t,u)$ are the density functions of $F^*(t,u)$ and $F^+(t,u)$, respectively.

Then

$$C(v_1, v_2) = p_0 C^*(v_1, v_2) + (1 - p_0) C^+(v_1, v_2). \tag{14}$$

That is, $C(v_1, v_2)$ is a mix of $C^*(v_1, v_2)$ and $C^+(v_1, v_2)$.

As aforementioned, the expression of $r(t,u)$ is complex. However, we can find lower and upper bounds of $r(t,u)$, as shown below.

Proposition 2 $\min\{r^*(t,u), r^+(t,u)\} \le r(t,u) \le \max\{r^*(t,u), r^+(t,u)\}$.

3.4 Conditional survival functions under a special case

Another interesting quantity is the conditional survival probability, which provides an alternative measure of the dependence between age and usage. It is the probability of future survival of one dimension by accounting for the length elapsed of the other dimension.

For simplicity, let's consider a special case: Suppose $F_1(t) = 1 - \exp\left\{-\left(\frac{t}{\gamma_1}\right)^{\beta_1}\right\} = v_1$ and $F_2(u) = 1 - \exp\left\{-\left(\frac{u}{\gamma_2}\right)^{\beta_2}\right\} = v_2$. That is, we assume that the distributions of X_1 and X_2 are Weibull distributions, respectively. Then, the dependence between age and usage can also be shown by the following proposition.

Proposition 3 *The conditional survival functions satisfy the following statements.*

- $P(X_1 \ge t | X_2 \ge u)$ *decreases in* t *for all* u *and* $P(X_2 \ge u | X_1 \ge t)$ *decreases in* u *for all* t*;*
- $p_0 \bar{F}_1(t) + (1 - p_0)\frac{\bar{F}^+(t,u)}{\bar{F}_2(u)} \le P(X_1 \ge t | X_2 \ge u) \le (1 - p_0)\bar{F}_1(t) + p_0 \frac{\bar{F}^+(t,u)}{\bar{F}_2(u)}$*; and*
- $P(X_2 \ge u | X_1 \ge t) \ge p_0 \bar{F}_2(u) + (1 - p_0)\frac{\bar{F}^+(t,u)}{\bar{F}_1(t)}$.

Proposition 3 can provide the dealers of second hand products with useful information in reliability assessment. It is noted that warranty analysis for second-hand products is a popular research topic.

4 Modelling the effectiveness of maintenance

The preceding section mainly focused on methods of modelling the dependence between age and usage and investigating relevant probabilistic properties for product items under 2D warranty, which are useful for non-repairable systems or describing the dependence between age and use for repairable systems during the time before the first failure (or warranty claims. But those investigations cannot be applied to systems with multiple repairs or preventive maintenance activities. To overcome this limitation, this section therefore aims at investigation of the effectiveness of maintenance for repairable systems under 2-D warranty.

When scheduling a PM policy, we need to understand the effectiveness of each maintenance, including PM and CM (corrective maintenance). The effectiveness of maintenance can be perfect, imperfect, or minimal. A perfect repair brings the item under maintenance to the status that is as good as new (AGAN); a minimal repair restores the item under maintenance to the status just before the item failed, or as bad as old (ABAO); and an imperfect repair brings the item to a status between AGAN and ABAO.

There are methods that have been developed to model the effectiveness of PM (see Nakagawa (1988); Wu and Zuo (2010), for example) and many models have been proposed to model the effectiveness of repair (or corrective maintenance). The reader is referred to Wu (2019) for a brief review of some models and a comparison of the performance of more than ten models on fifteen real-world datasets, where the performance is measured by AIC, corrected AIC, and BIC (Bayesian information criterion).

Nevertheless, methods on modelling the effectiveness of maintenance on items with their reliability being measured by both age and usage is sparse. Below we investigate possible methods.

Based on failure rate functions. In this case, $r(t,u)$ is regarded as the failure rate function of the time to the first failure, based on which the failure intensity function is developed. Similar to the approaches proposed for items whose reliability is measured by one variable (Wu & Zuo, 2010), for items with $r_k(t,u)$ as the failure intensity function of the item after the $(k-1)$th PM, one can assume the following three scenarios (with $a_1,\ldots,a_4,b_1$ and b_2 being parameters): (i) *Linear relationship*: $r_k(t,u) = a_1 r_{k-1}(t,u) + a_2$; (ii) *Nonlinear relationship*: $r_k(t,u) = r_{k-1}(a_3 t + a_4, b_1 u + b_2)$; and (iii) *Hybrid relationship*: $r_k(t,u) = a_1 r_{k-1}(a_3 t + a_4, b_1 u + b_2) + a_2$.

Based on copulas. Let $Y_{1,k}$ and $Y_{2,k}$ $(k \geq 1)$ denote the working time to the kth PM and the associated usage, respectively. $U_{1,k} = Y_{1,k} - Y_{1,k-1}$ is the time duration between the kth and $(k-1)$th failures and $U_{2,k} = Y_{2,k} - Y_{2,k-1}$ is the associated usage, where $Y_{1,0} = 0, Y_{2,0} = 0$. Let $G_{1,k}(x)$ and $G_{2,k}(x)$ be the cdf (cumulative distribution function) of $U_{1,k}$ and $U_{2,k}$, respectively, $H_k(y_1,y_2)$ be the joint probability distribution of $U_{1,k}$ and $U_{2,k}$, and $C_k(u_{1,k},u_{2,k}\colon \theta_k)$ be the associated copula of $H_k(y_1,y_2)$, where θ_k is the parameter in the copula. For example,

$$C_k(u_{1,k},u_{2,k}\colon \theta_k) = p_0 C^*(u_{1,k},u_{2,k}\colon \theta_k) + (1-p_0) C^+(u_{1,k},u_{2,k}\colon \theta_k). \quad (15)$$

where $C^*(u_{1,k},u_{2,k}\colon \theta_k)$ is defined as the copula $C^*(u_{1,k},u_{2,k})$ in Eq. (6), θ_k is the parameter of the copula. $C^+(u_{1,k},u_{2,k}\colon \theta_k) = u_{2,k} - C^*(1-u_{1,k},u_{2,k}) = 1 - u_{1,k} - C_0(u_{1,k},1-u_{2,k})$ as the copula $C^+(v_1,v_2)$ defined in Eq. (13). The parameter θ_k in $C_k(u_{1,k},u_{2,k}\colon \theta_k)$ controls the strength of the dependence between v_1 and v_2. θ_k may depend on $\theta_{k-1},\theta_{k-2},\ldots$. For example, θ_k may follow the autoregressive process, that is, $\theta_k = \alpha_0 + \sum_{j=1}^{p} \alpha_j \theta_{k-j}$. This provides a unique strength that other approaches may not have as it assumes the dependences between age and usage after repairs evolve over time.

If we investigate the failure process based on the copula after each PM, there are following two scenarios.

Scenario A: $U_{i,k}$ and $U_{i,k-1}$ are independent: For example, if the geometric process (Lam, 1988) is used[i], $U_{i,k} \sim G_{i,k}(a_i^{k-1}x)$ with $a_i > 0$ if $\{U_{i,k}, k = 1, 2, \ldots\}$ follows the geometric process. Then $(U_{1,k}, U_{2,k})$ follows a two-dimensional geometric process. Rizwan and Yunus (2015) obtain the expected number of failures up to (t, u) as follows

$$M(t,u) = F(t,u) + \int_0^t \int_0^u M(a_1(t-u_1), a_2(u-u_2)) f(u_1,u_2) du_1 du_2, \tag{16}$$

where $F(t,u)$ is the copula with the form shown in Eq. (15).

Scenario B: $U_{i,k}$ and $U_{i,k-1}$ are dependent: Models such as the virtual age models (Kijima, 1989) and the failure process model with the exponential smoothing of intensity functions (Wu, 2019) can be used. For example, in case the virtual age model (Kijima, 1989) is used, we assume $V_{i,k} = V_{i,k-1} + \rho_i U_{i,k}$, where $V_{i,k}$ is the virtual age/usage after the kth repair. Then

$$G_{i,k+1}(x) = \int_0^\infty \frac{\bar{G}_{i,1}(y+x)}{\bar{G}_{i,1}(y)} dH_{i,k}(y), \tag{17}$$

where

$$H_{i,k}(y) = H_{i,k-1}(y) - \int_0^x \frac{\bar{G}_{i,1}(y + \frac{x-y}{\rho_i})}{\bar{G}_{i,1}(y)} dH_{i,k-1}(y),$$

and $i = 1, 2$, $\rho_i \in [0, 1]$: (a) the repair is a harmful repair if $\rho_i > 0$; (b) the repair is a minimal repair if $\rho_i = 1$; (c) the repair is an good-as-new repair (or replacement) if $\rho_i = 0$; and (d) the repair is efficient if $\rho_i \in (0, 1)$.

In this case, there is no closed form of calculating the expected number of failures $M(t, u)$. One may therefore use a numerical method to estimate it. One can then estimate a copula $C(H_{i,1}(y), H_{i,2}(y))$ based on historical data. It should be noted that in this case, $H_{i,k}(y)$ is a conditional probability distribution.

Apparently, the above assumptions, on the dependence between $U_{i,k}$ and $U_{i,k-1}$ and the dependence between θ_k and θ_{k-j}, must be tested and validated on failure data such as lab experiment data or warranty claim data and repair data.

[i]The definition of the geometric process is: Given a sequence of non-negative random variables $\{X_k, k = 1, 2, \ldots\}$, if they are independent and the cdf of X_k is given by $F(a^{k-1}x)$ for $k = 1, 2, \ldots$, where a is a positive constant, then $\{X_k, k = 1, 2, \cdots\}$ is called a geometric process (GP) (Lam, 1988).

5 Optimisation of PM policies

The above discussion suggests that there is a need to use the failure rate/intensity function derived from an asymmetric copula in describing the failure process of items under 2D warranty. This section aims at development of PM policies that incorporate the methods to tackle both knowledge gaps Gap1 and Gap2.

It is not cost effective to conduct PM on each sold product item if the failure rate of the product is very low. A special case is that the warrantor possesses historical data about the usage rate of different users, select those users with high usage rates, i.e., high warranty claim rates, and then conduct PM on those product items. For example, a car manufacturer may perform PM on taxies as their usage rate is very high. However, in most cases, at the time when a product item is purchased and a decision on whether PM should be conducted on it, the warrantor will need to know customers' sensitive information such as the usage rate. Such information is often not available and needs protecting according to the relevant data protection laws. A feasible method is to model the relationship between age and usage based on warranty claim data and then apply such a model in the future (Mitra, 2021; Zhao, Chen, Lv, & He, 2022).

Let w denote the warranty period and L denote the limit of the warranty usage. We assume that PM is only performed on the age basis. That is, PM is carried out at time $T, 2T, \ldots, (N-1)T$ with $NT = w$. To ensure PM is conducted in a cost-effective manner, it can be conducted on each sold product item under 2D warranty only if the following condition is satisfied.

Condition 1 *It is cost effective for the warrantor to perform PM for a product under 2D warranty if the following condition is satisfied.*

$$c_r \sum_{k=1}^{N} \int_{(k-1)T}^{kT} \int_0^L r_{\mathrm{II},k-1}(t,u)dudt + (c_a + c_p)(N-1) < c_r \int_0^w \int_0^L r_{II}(t,u)dudt. \tag{18}$$

In the above inequality, the term $c_r \sum_{k=1}^{N} \int_{(k-1)T}^{kT} \int_0^L r_{\mathrm{II},k-1}(t,u)dudt$ is the cost of repairing possible failures after the $N-1$ PM activities are performed, $(c_a + c_p)(N-1)$ is the cost associated with the $N-1$ PM activities, and $c_r \int_0^w \int_0^L r_{\mathrm{II}}(t,u)dudt$ is the cost of repairing possible failures of the items when no PM is performed. Then Condition 1 means: the expected cost of failure and maintenance if PM is performed is lower than the expected cost of failure and maintenance if PM is not performed. Under this condition, PM should be performed.

An alternative condition may be derived from a perspective of reputation protection, which aims to ensures the failure rate of a product is less than a pre-specified threshold.

Condition 2 *For a product under 2D warranty, the warrantor may perform PM if the following condition is satisfied.*

$$\int_0^w \int_0^L r_I(t,u)dudt + \sum_{k=1}^{N} \int_{(k-1)T}^{kT} \int_0^L r_{\mathrm{II},k-1}(t,u)dudt \le r_0, \tag{19}$$

where $r_0 < \int_0^w \int_0^L r_I(t,u)dudt + \int_0^w \int_0^L r_{\mathrm{II}}(t,u)dudt.$

Similar conditions can be proposed for items under 1D warranty.

Another case is that, if a customer purchases multiple product items, it may be reasonable for the warrantor to perform PM from both economic and reputation protection perspective. For example, the warrantor may send a repairman to conduct PM on the lot, which can save the cost on administration and travel, as discussed below.

Consider two scenarios: in Scenario 1, M product items are sold to a customer and they are installed at the same location; in Scenario 2, M product items are sold to M different customers and the items are installed at different locations. In case a PM activity on a product needs carrying out, from the warrantor's perspective, the main differences between these two scenarios include: (i) on each planned PM, from an administration perspective, that the warrantor needs to handle one PM activity in Scenario 1 and M PM activities in Scenario 2; (ii) that the repairman needs to travel to only one location to conduct PM in Scenario 1 and M different locations in Scenario 2. These incur different travel cost; and (iii) that the warrantor may want to retain its bigger customer by keeping the number of failures of the product items low, that is, they may prefer to serve the customer in Scenario 1 over those in Scenario 2.

In Scenario 1, the expected cost of repairing the failures of the M items during the warranty period is $c_rM\left(\int_0^w\int_0^L r(t,u)dtdu+\sum_{k=1}^N\int_{(k-1)\frac{w}{N}}^{k\frac{w}{N}}\int_0^L r_{\mathrm{II},k-1}(t,u)dudt\right)$ if no PM is performed, while the expected cost of performing PM on the M items is $c_a(N-1)+c_pM(N-1)$ if $N-1$ PM actions are performed, where c_a is the cost of administration and the travel on the repairman's visit to the place where the items are located. In Scenario 2, the expected cost of repairing the failures of the M items during the warranty period is $c_rM\left(\int_0^w\int_0^L r(t,u)dtdu+\sum_{k=1}^N\int_{(k-1)\frac{w}{N}}^{k\frac{w}{N}}\int_0^L r_{\mathrm{II},k-1}(t,u)dudt\right)$, which is the same as that in Scenario 1, if no PM is performed, while the cost of performing PM on the M items is $c_aM(N-1)+c_pM(N-1)$ if $N-1$ PM actions are performed. Assuming the PM takes effect (i.e., the number of failures becomes less if more PM actions are taken), that is,

$$\sum_{k=1}^{N+1}\int_{(k-1)\frac{w}{N+1}}^{k\frac{w}{N+1}}\int_0^L r_{\mathrm{II},k-1}(t,u)dudt<\sum_{k=1}^{N}\int_{(k-1)\frac{w}{N}}^{k\frac{w}{N}}\int_0^L r_{\mathrm{II},k-1}(t,u)dudt, \tag{20}$$

that is, more PM actions will result in few failures.

Then the following condition must be satisfied to ensure that performing the $N-1$ PM activities is cost effective.

$$\begin{aligned}c_a(N-1)+c_rM&\left(\int_0^w\int_0^L r(t,u)dtdu+\sum_{k=1}^N\int_{(k-1)\frac{w}{N}}^{k\frac{w}{N}}\int_0^L r_{\mathrm{II},k-1}(t,u)dudt\right)+c_p(N-1)M\\&<c_rM\left(\int_0^w\int_0^L r(t,u)dtdu+\int_0^w\int_0^L r_{\mathrm{II}}(t,u)dudt\right),\end{aligned} \tag{21}$$

The left hand side of inequality (21) is the cost incurred if PM is performed on each product item and the right hand side is the cost incurred if PM is not conducted. That is, the inequality gives the condition that PM is needed only if it is cost-effective.

Inequality (21) can be re-written as,

$$M > \frac{c_a(N-1)}{c_r \int_0^w \int_0^L r_{\mathrm{II}}(t,u)dudt - c_r \sum_{k=1}^{N} \int_{(k-1)\frac{w}{N}}^{k\frac{w}{N}} \int_0^L r_{\mathrm{II},k-1}(t,u)dudt - c_p(N-1)}. \quad (22)$$

One may simply set $N = 2$, which implies at least one PM to be performed, to check the above condition.

Then, the optimal number N of PM activities that should be performed can be obtained by minimising the following objective function (23), which aims to minimise the cost incurred if PM is performed, subject to the two constraints shown in inequalities (24) and (25).

$$\min_N \quad c_a(N-1) + c_r M \left(\int_0^w \int_0^L r(t,u)dtdu + \sum_{k=1}^{N} \int_{(k-1)\frac{w}{N}}^{k\frac{w}{N}} \int_0^L r_{\mathrm{II},k-1}(t,u)dudt \right) + c_p M(N-1), \quad (23)$$

$$\text{s.t.} \quad \frac{c_a(N-1)}{c_r \int_0^w \int_0^L r_{\mathrm{II}}(t,u)dtdu - c_r \sum_{k=1}^{N} \int_{(k-1)\frac{w}{N}}^{k\frac{w}{N}} \int_0^L r_{\mathrm{II},k-1}(t,u)dtdu - c_p(N-1)} < M, \quad (24)$$

$$\sum_{k=1}^{N+1} \int_{(k-1)\frac{w}{N+1}}^{k\frac{w}{N+1}} \int_0^L r_{\mathrm{II},k-1}(t,u)dudt < \sum_{k=1}^{N} \int_{(k-1)\frac{w}{N}}^{k\frac{w}{N}} \int_0^L r_{\mathrm{II},k-1}(t,u)dudt, \quad (25)$$

where Inequality (24) and Inequality (25) are taken from Inequality (22) and Inequality (20), respectively.

Proposition 4 *There exists a finite value N^* that satisfies Eq.* (23), *constraints* (24), *and* (25).

Based on the above discussion, we can create the following flowchart (i.e. Figure 2) to guide practitioners to use the proposed PM optimisation

Remark 2 *For the case of PM policies on items protected under 1D warranty, a similar objective function and constraint can be obtained. We aim to minimise the following objective function to find the optimum N, one can obtain an optimal periodic PM policy, where N must be an integer.*

$$\min_N \quad c_a(N-1) + c_r M \left(\int_0^w \lambda_{\mathrm{I}}(t)dt + \sum_{k=1}^{N} \int_{(k-1)\frac{w}{N}}^{k\frac{w}{N}} \lambda_{\mathrm{II},k-1}(t)dt \right) + c_p M(N-1), \quad (26)$$

$$s.t. \quad \frac{c_a(N-1)}{c_r \int_0^w \lambda_{\mathrm{II}}(t)dt - c_r \sum_{k=1}^{N} \int_{(k-1)\frac{w}{N}}^{k\frac{w}{N}} \lambda_{\mathrm{II},k-1}(t)dt - c_p(N-1)} < M, \quad (27)$$

$$\sum_{k=1}^{N+1} \int_{(k-1)\frac{w}{N+1}}^{k\frac{w}{N+1}} \lambda_{\mathrm{II},k-1}(t)dt < \sum_{k=1}^{N} \int_{(k-1)\frac{w}{N}}^{k\frac{w}{N}} \lambda_{\mathrm{II},k-1}(t)dt, \quad (28)$$

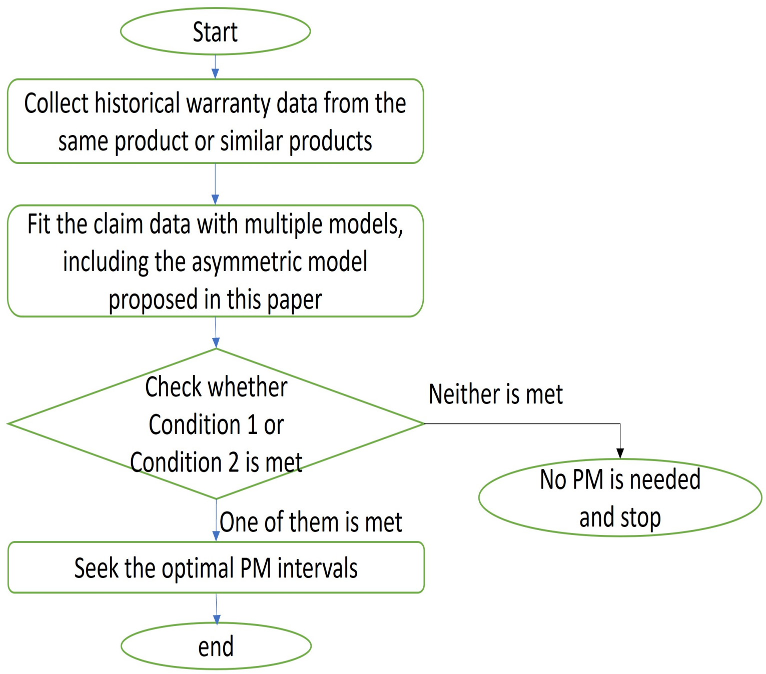


Figure 2: Flowchart for PM.

where $\lambda_{\mathrm{I}}(t)$ *is the failure intensity of the type I failure mode and* $\lambda_{\mathrm{II},k-1}(t)$ *is that of the type II failure mode.*

The above integer nonlinear programming can be easily solved as in the real problem the integer N *should be small, for example, it is smaller than 25. One can therefore simply use the brute-force search to find the optimal solution.*

6 Case studies

Suppose that the time-between-claims and usage-between-claims of M product items are observed and N_m warranty claims have been received on item m. Denote $t_{m,k}$ and $u_{m,k}$ as the time and mileage between the $(k-1)$-th and k claims, respectively.

6.1 Relationship between $t_{m,k}$ and $u_{m,k}$ for $k = 1, 2, \ldots, N_m$ and $m = 1, 2, \ldots, M$.

For all time-between-claims and usage-between-claims of M product items, the log-likelihood function is given by

$$\begin{aligned} l(\boldsymbol{\theta}) &= \sum_{m=1}^{M}\sum_{k=1}^{N_m} \log f_k(t_{m,k}, u_{m,k}) \\ &= \sum_{m=1}^{M}\sum_{k=1}^{N_m} \log[f_{1,k}(t_{m,k}) f_{2,k}(u_{m,k}) c_k(F_{1,k}(t_{m,k}), F_{2,k}(u_{m,k}))] \\ &= \sum_{m=1}^{M}\sum_{k=1}^{N_m} [\log f_{1,k}(t_{m,k}) + \log f_{2,k}(u_{m,k}) + \log c_k(F_{1,k}(t_{m,k}), F_{2,k}(u_{m,k}))], \end{aligned} \tag{29}$$

where $c_k(F_{1,k}(t_{m,k}), F_{2,k}(u_{m,k}))$ is the density of copula $C_k(F_{1,k}(t_{m,k}), F_{2,k}(u_{m,k}))$ and $\boldsymbol{\theta}$ is the set of parameters in $f_k(t_{m,k}, u_{m,k})$.

The warranty claim data were collected from an automotive manufacturer in the UK and contains the warranty claims from 1257 cars with different numbers of claims. That is, $M = 1257$ and N_m is the number of warranty claims of car m with $m = 1, 2, \ldots, 1257$. The maximum number of claims in those 1257 cars is 29 and the minimum number of claims is 3. Each claim includes the age of the claimed car and the accumulated miles. However, there are not many cars that are claimed for many times. To ensure a sufficient sample size for estimating the parameters in the models shown in Eqs. (30), (31), and (32), we choose $\max_m\{N_m\} = 12$ and $\min_m\{N_m\} = 3$. In total, there are 7219 claims that are used in the following analysis.

We adopt the method proposed illustrated in Scenario A in Section 4. The Weibull distribution is used as the marginals here. That is, we assume the marginal distribution of age and that of usage between two adjacent warranty claims follow geometric processes, respectively. Denote $v_{1,k} = F_{1,k}(x_1) = 1 - \exp\{-\left(\frac{a_1^{k-1}x}{\beta_1}\right)^{\alpha_1}\}$ and $v_{2,k} = F_{2,k}(x) = 1 - \exp\{-\left(\frac{a_2^{k-1}x}{\beta_2}\right)^{\alpha_2}\}$, in which a_1 and a_2 are the parameters of the two geometric processes, respectively, and $k = 1, 2, \ldots, 12$. We then apply the three following models to fit the claim data.

The model in Asymmetric-A. The copula is

$$\begin{aligned} C_k(u_{1,k}, u_{2,k}) =& p_0 \left\{ v_{1,k} + v_{2,k} - 1 + \exp\left\{ - \left[(-\ln(1 - v_{1,k}))^{\theta_{1,k}} + (-\ln(1 - v_{2,k}))^{\theta_{1,k}} \right]^{1/\theta_{1,k}} \right\} \right\} \\ &+ p_1 \left\{ v_{1,k} - \exp\left\{ - \left[(-\ln(v_{1,k}))^{\theta_{2,k}} + (-\ln(1 - v_{2,k}))^{\theta_{2,k}} \right]^{1/\theta_{2,k}} \right\} \right\}. \end{aligned} \tag{30}$$

The model in Symmetric-A. The copula is

$$\begin{aligned} C_k(u_{1,k}, u_{2,k}) =& v_{1,k} + v_{2,k} - 1 + p_0 \left\{ \exp\left\{ - \left[(-\ln(1 - v_{1,k}))^{\theta_{1,k}} + (-\ln(1 - v_{2,k}))^{\theta_{1,k}} \right]^{1/\theta_{1,k}} \right\} \right\} \\ &+ p_1 \left\{ \exp\left\{ - \left[(-\ln(1 - v_{1,k}))^{\theta_{2,k}} + (-\ln(1 - v_{2,k}))^{\theta_{2,k}} \right]^{1/\theta_{2,k}} \right\} \right\}. \end{aligned} \tag{31}$$

Table 1: Results of the model estimation.

	α_1	β_1	α_2	β_2	a_1	a_2	p_0	$\theta_1 \sim \theta_{24}$	$-l(\boldsymbol{\theta})$	AIC
Asymmetric-A	311.31	0.75	9177.98	0.72	1.23	1.20	0.93	see Table 2	108678.9	217419.8
Symmetric-A	298.55	0.74	8387.83	0.68	1.23	1.21	0.32	see Table 2	108914.6	217891.2
Single-A	272.05	0.73	7857.13	0.67	1.23	1.18		see Table 2	109070.8	218177.6

Table 2: Parameters of the copulas

	k	1	2	3	4	5	6	7	8	9	10	11	12
Asymmetric-A	$\theta_{1,k}$	3.03	3.01	3.09	3.08	2.99	2.87	2.96	3.03	3.01	2.72	3.07	2.98
	$\theta_{2,k}$	0.38	0.66	0.86	0.85	0.62	0.36	0.92	0.47	0.47	0.39	0.37	0.44
Symmetric-A	$\theta_{1,k}$	1.58	1.97	2.14	1.89	3.45	2.00	1.74	2.09	3.34	1.23	3.41	1.90
	$\theta_{2,k}$	3.67	3.56	3.35	3.67	2.65	3.58	3.59	3.70	2.56	3.41	2.88	3.62
Single-A	θ_k	2.81	3.09	2.96	3.11	3.06	3.04	2.78	3.12	2.54	2.71	2.57	2.32

The model in Single-A. The copula is

$$C_k(u_{1,k}, u_{2,k}) = v_{1,k} + v_{2,k} - 1 + \exp\{-[(-\ln(1-v_{1,k}))^{\theta_k} + (-\ln(1-v_{2,k}))^{\theta_k}]^{1/\theta_k}\}, \quad (32)$$

where $p_0 + p_1 = 1$.

In the above models, the model in Asymmetric-A (i.e., Eq. (30)) is the model proposed in Eq. (15), the model in Symmetric-A (i.e., Eq. (31)) is a weighted linear combination of two symmetric copulas, and the model in Single-A (i.e., Eq. (32)) is a bivariate distribution.

Table 1 shows the parameters of the three models estimated on the claim data. Table 2 shows the copula parameters $\theta_{1,k}$.

From the last column in Table 1, we can find that the AIC of the proposed method is the smallest, which suggests that the proposed method outperforms the two other methods. Another observation is that $a_1, a_2 > 1$, which implies that the times between warranty claims increase with respect of the number k of claims.

To visually understand the dynamics of the copula parameters θ_1 and θ_2, the three figures in Figure 3 show their values against k.

If we do not apply the model $\theta_k = \alpha_0 + \sum_{j=1}^{p} \alpha_j \theta_{k-j}$ on the copula parameters and simply impose all $\theta_{1,k}$ for $k = 1, 2, \ldots, 12$ are equal and $\theta_{2,k}$ for $k = 1, 2, \ldots, 12$ are equal, based on which

Table 3: Results of the model estimation.

	α_1	β_1	α_2	β_2	a_1	a_2	p_0	θ_1	θ_2	$-l(\boldsymbol{\theta})$	AIC
Asymmetric-B	318.404	0.754	9322.931	0.662	1.170	1.133	0.921	3.441	0.823	108875.4	217768.8
Symmetric-B	338.164	0.763	8447.803	0.709	1.185	1.106	0.820	3.512	1.486	108979.6	217977.2
Single-B	289.444	0.739	7927.174	0.646	1.119	1.064		2.932		109228.3	218470.6

the parameters of models (30), (31) and (32) are shown in Table 3. The three models are renamed to Asymmetric-B, Symmetric-B, and Single-B, respectively. It can be seen that the AIC values in the last column in Table 3 are similar to those in Table 1, which suggests that we can impose all $\theta_{1,k}$ for $k = 1, 2, \ldots, 12$ are equal and $\theta_{2,k}$ for $k = 1, 2, \ldots, 12$ are equal.

(a) $\theta_{1,k}$ and $\theta_{2,k}$ in Asymmetric-A.

(b) $\theta_{1,k}$ and $\theta_{2,k}$ in Symmetric-A.

(c) θ_k in Single-A.

Figure 3: Copula parameters in k's.

In summary, from the above analyses, we can find that the AIC values of the proposed method are the smallest for both the copulas with changing parameters and those with fixed parameters, which shows the superiority of the proposed method.

6.2 Relationship between $t_{m,k}$ and $u_{m,k}$ for a given k

If we estimate the probability $F_k(t_{m,k}, u_{m,k})$ for a specific k, we can maximise the following likelihood.

$$
\begin{aligned}
l_k(\boldsymbol{\theta}) &= \sum_{m=1}^{M} \log f_k(t_{m,k}, u_{m,k}) \\
&= \sum_{m=1}^{M} \left[\log f_{1,k}(t_{m,k}) + \log f_{2,k}(u_{m,k}) + \log c_k(F_{1,k}(t_{m,k}), F_{2,k}(u_{m,k}))\right]. \qquad (33)
\end{aligned}
$$

That is, l_k is the likelihood of the gap-time and gap-mileage between the $(k-1)-$th and kth claims. The minimum values of $-l_k$ are shown in Table 4, from which we can see that the asymmetric model has the smallest $-l_k$ for $k = 1, 2, \ldots, 12$. The last row shows the total number of the k warranty claims, for instance, $N_5 = 720$ (located on the 6th column and the 5th row in the table) means that 720 items have received five claims.

The result shows that the model shown in Eq. (30) has the smallest value of the maximum likelihood if we fit a bivariate probability distribution between two adjacent claims.

Table 4: $-l_k$ on specific gap times and gap mileage.

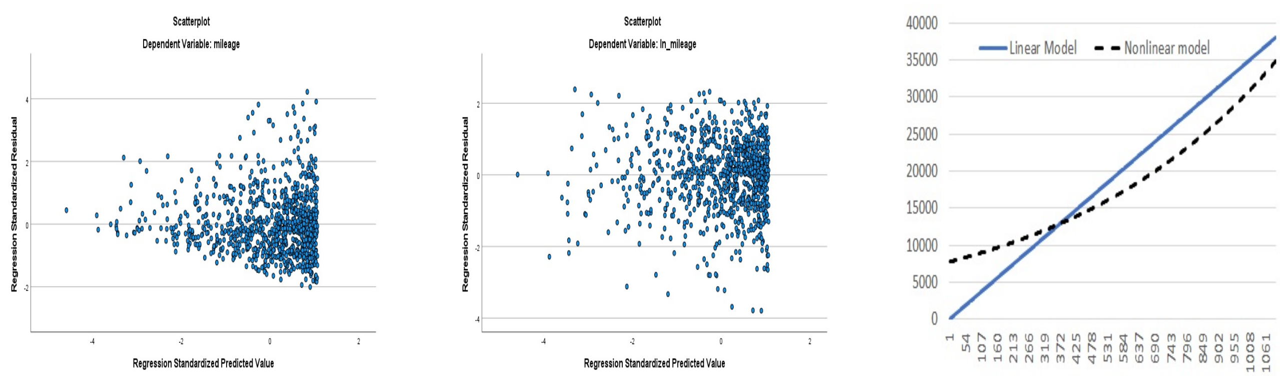
	$-l_1$	$-l_2$	$-l_3$	$-l_4$	$-l_5$	$-l_6$	$-l_7$	$-l_8$	$-l_9$	$-l_{10}$	$-l_{11}$	$-l_{12}$
Asymmetric	19886.2	19612.3	19365.1	14417.7	10584.0	7945.7	5653.2	3899.5	2665.9	1821.4	1285.4	787.0
Symmetric	19941.5	19626.5	19363.6	14435.7	10590.0	7974.3	5654.6	3914.5	2685.6	1841.8	1289.6	799.2
Single	20006.1	19643.5	19389.0	14456.5	10593.7	7993.2	5671.3	3918.6	2685.6	1847.6	1291.1	800.4
N_m	1257	1257	1257	972	720	540	385	272	187	129	91	55

6.3 Difference between the assumptions of linear and nonlinear relationship between age and usage

Now we analyse the data of the times to the last claims and the mileage to the last claims. After removing some outliers and only concentrate on the claims within 1095 days (i.e., three years), 950 observations are left and used for analysis. If we build a linear regression model, we obtain the following model equation,

$$\text{mileage} = 34.766 \times \text{age}, \tag{34}$$

with its R^2 being 78.7% and the residual plot shown in Figure 4a. From Figure 4a, we can see all of the residuals are distributed with a <-shaped region. That is, the residual variance increases from left to right, which suggests the presence of heteroscedasticity and a logarithm transformation can therefore be applied on the values of mileage. We perform a natural logarithm transformation on



(a) The residual plot of the linear model (34).

(b) The residual plot of the nonlinear model (35).

(c) The cumulative intensity functions of the linear and nonlinear models.

the dependent variable *mileage* and then build a linear regression model. The residual plot of the new model is shown in 4b, from which one can see that the variance is stabilized. The result of the new model equation is shown in Eq. (35).

$$\text{mileage} = e^{8.956+0.001373\times\text{age}}. \tag{35}$$

The model has a $R^2 = 82.2\%$, which shows an increase from 78.7% from the last model. The residual plot of the new model is shown in Figure 4b, from which heteroscedasticity disappears.

In both the above models, we did not remove observations with standardized residuals greater than 3.0.

We then fit the data with models (30), (31) and (32), respectively, and obtain their likelihood values 23787.28, 23800.52, and 23801.18, respectively. This shows that model (30) outperforms the two other models. The parameters of the fitted model (30) are $\alpha_1 = 1077.521, \beta_1 = 1.00912, \alpha_2 = 32983.540, \beta_2 = 0.955, p = 0.988, \theta_1 = 3.152, \theta_2 = 0.548$.

We can use obtain the bivariate failure rate from (8) based on the copula proposed in Eq. (30). Nevertheless, the resulting expression of $r(t, u$ will be extremely complex. For the sake of simplicity, we use the following cumulative bivariate failure intensity to illustrate the impact of using linear and nonlinear relationship between age and mileage, as discussed in Gap1.

$$H(t, u) = \delta_0 t^{\beta_1} u^{\beta_2}. \tag{36}$$

Figure 4c illustrates the cumulative intensity function $H(t, u)$ for the models (34) and (35), respectively, which shows the difference resulted from using linear and nonlinear relationships between age and usage.

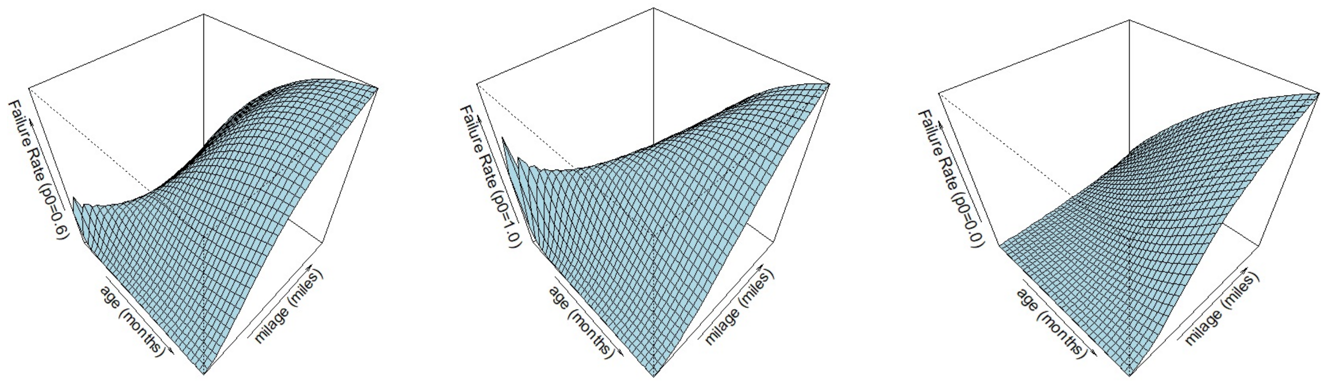
Let $\delta_0 = \frac{1}{\alpha_1^{\beta_1} \alpha_2^{\beta_2}} = \frac{1}{1077.521^{1.00912} \times 32983.540^{0.955}}$, $\beta_1 = 1.00912$ and $\beta_2 = 0.955$. If we assume the relationship between age and mileage shown in Eq. (34), then within 3 years, i.e., $t = 1095$ days, the expected number of failures is estimated as $H(t, u) = \delta_0 t^{\beta_1} u^{\beta_2} = \delta_0 t^{\beta_1} (34.766t)^{\beta_2} = \frac{29.635}{1077.521^{1.00912} \times 32983.540^{0.955}} t^{1.96412} = 1.1655$. On the other hand, if we assume the relationship between age and mileage shown in Eq. (35), then within 3 years, the expected number of failures is estimated as $H(t, u) = \delta_0 t^{\beta_1} u^{\beta_2} = \delta_0 t^{\beta_1} (e^{8.956+0.001373t})^{\beta_2} = \frac{t^{1.00912} e^{8.956+0.001373t}}{1077.521^{1.00912} \times 32983.54^{0.955}} = 1.7161$. That is, if we take the assumption that the relationship between age and mileage is linear, within a 3 year period, each product item underestimates 0.5506 (= 1.7161-1.1655) claims.

7 Numerical examples

The preceding sections provided some conditions of conducting PM for items under warranty and investigated some properties of the failure rate functions of a bivariate asymmetric copula. This section offers some numerical examples to help boost the understanding of those content.

7.1 Failure rate functions of items under 2D warranty

Set $\theta = 0.6, \beta_1 = 1.3, \beta_2 = 1.5, \gamma_1 = 30$(months)$, \gamma_2 = 30000$(miles) and set $p_0 = 0.6$, $p_0 = 1.0$ and $p_0 = 0.0$, respectively. We obtain the bivariate failure rate functions with (11) and (14), as shown in Fig 4a, Fig 4b, and Fig 4c, respectively. In those figures, the X-axis, the Y-axis, and the Z-axis represent age, mileage, and the failure rate, respectively. It is interesting to observe that the failure rate function decreases in both age and mileage in Fig 4a and Fig 4b at an early stage and then increase later, while the failure rate function increase in both age and mileage in Fig 4c, compared to the fact that the failure rate functions of the marginal distributions, $F_1(t)$ and $F_2(u)$, increase as $\beta_1 > 1$ and $\beta_2 > 1$.



(a) Failure rate function for $p_0 = 0.6$.

(b) Failure rate function for $p_0 = 1.0$.

(c) Failure rate function for $p_0 = 0.0$.

Figure 4: Failure rate functions with different p_0's.

7.2 PM policies for items under 2D warranty

Via a PM activity, the failure intensity of the type II failure mode for both age and usage is assumed to change from $\lambda(t)$ before the PM to $a_0\lambda(t)$ after the PM. Suppose that $M = 100, C_a = 100, C_r = 300, C_p = 20, \gamma_{1,1} = 20$(months)$, \beta_{1,1} = 0.9, \gamma_{1,2} = 200$(100miles)$, \beta_{1,2} = 0.95, \gamma_{2,1} = 30$(months)$, \beta_{2,1} = 1.5, \gamma_{2,2} = 300$(months)$, \beta_{2,2} = 1.8, a_0 = 1.1$, $w = 36$(months). The failure rate functions are assumed by $\lambda_{\mathrm{I},i}(t) = \beta_{1,i}t^{\beta_{1,i}-1}/\gamma_{1,i}^{\beta_{1,i}}$ (for $i = 1, 2$), and $\lambda_{\mathrm{II},i}(t) = \beta_{2,i}t^{\beta_{2,i}-1}/\gamma_{2,i}^{\beta_{2,i}}$ (for $i = 1, 2$), where $i = 1$ stands for age and $i = 2$ for usage. As shown in Fig 5 (where the X-axis represents N and the Y-axis represents the expected cost), the objective function in Eq. (23) decreases when N increases from 1 to 4 and then they increase when $N(> 4)$ increases, where $N = 1$ means no PM is performed.

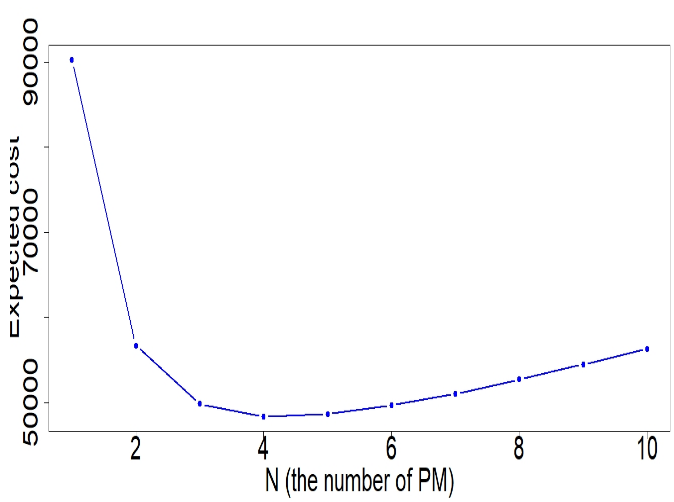


Figure 5: The expected cost for $N \in \{1, 2, \ldots, 10\}$

7.3 PM policies for items under one-dimensional (1D) warranty

Although the focus of this paper is on dependence and maintenance policy optimisation of 2D products, Remark 2 provides a method of maintenance policy optimisation for 1D products. Thus, this subsection provides a numerical example for maintenance policy optimisation for items under 1D warranty.

On a PM activity, the failure intensity is assumed to change from $\lambda(t)$ before the PM to $a_0\lambda(t)$ after it and the time after a PM is reset to 0 (Nakagawa, 1988). Let $M = 100, C_a = 100, C_r = 200, C_p = 20, \gamma_1 = 25(\text{months}), \beta_1 = 0.8, \gamma_2 = 30(\text{months}), \beta_2 = 1.5, a_0 = 1.1,\ w = 36(\text{months})$, $\lambda_{\mathrm{I}}(t) = \beta_1 t^{\beta_1 - 1}/\gamma_1^{\beta_1}$, and $\lambda_{\mathrm{II}}(t) = \beta_2 t^{\beta_2 - 1}/\gamma_2^{\beta_2}$. As shown in Fig 6 (where the X-axis represents N and the Y-axis represents the expected cost), the objective function in Eq. (26) decreases when N increases from 1 to 3 and then they increase when $N(> 3)$ increases, where $N = 1$ means no PM is performed.

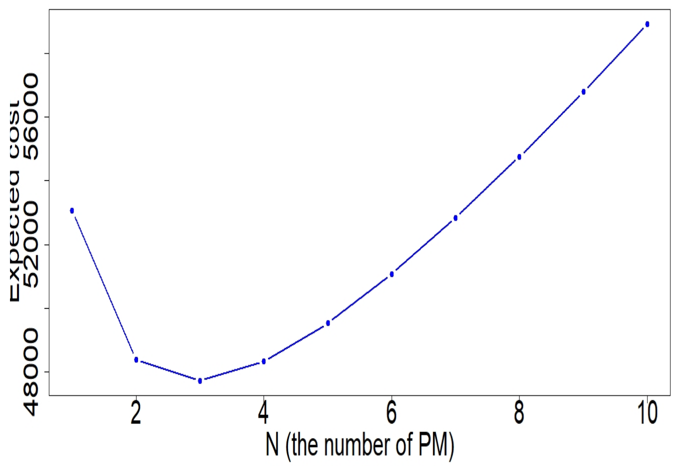


Figure 6: The expected cost for $N \in \{1, 2, \ldots, 10\}$

8 Conclusions and future work

This paper proposed a copula-based approach to model the failure process of items under two-dimensional warranty. It derived a bivariate failure rate function, some probabilistic properties of the conditional survival probabilities, and methods to model the effectiveness of maintenance. The paper then proposed the necessary conditions under which preventive maintenance can be performed in a cost-effective way and proposed a method to optimise preventive maintenance policies.

The strength of using the copula-based approach is its capability of modelling the asymmetric phenomenon in the field failure data. From this perspective, it is able to capture more information that the other approaches such as those methods obtained based on the assumption of the linear relationship between age and usage. The drawback is its complex expression of the bivariate failure rate function derived from the asymmetric copula and therefore an analytical solution may not exist. As a result, numerical solutions may need perusing in applications of the bivariate failure rate function. However, this will also provide a research topic on the derivation of elegant expressions of the bivariate failure rate functions.

Other future research topics include: optimisation of preventive maintenance policies considering the number of product items that have been sold and optimisation of preventive maintenance policies for items with multiple repair modes.

Acknowledgement

The author is grateful to the two anonymous reviewers for their comments, which leads to an improvement in the presentation of the paper.

References

Ahmadi, R. (2019). A copula-based maintenance modeling for parallel systems with non-self announcing failures and dependent components. *International Journal of Supply and Operations Management*, *6*(4), 282–295.

Ahmadi, R. (2020). Reliability and maintenance modeling for a load-sharing k-out-of-n system subject to hidden failures. *Computers & Industrial Engineering*, *150*, 106894.

Andersen, J. F., Andersen, A. R., Kulahci, M., & Nielsen, B. F. (2022). A numerical study of markov decision process algorithms for multi-component replacement problems. *European Journal of Operational Research*, *299*(3), 898–909.

Anderson, B., Chukova, S., & Hirose, Y. (2017). Using copulas to model and simulate two-dimensional warranty data. *Quality and Reliability Engineering International*, *33*(3), 595–605.

Basu, A. (1971). Bivariate failure rate. *Journal of the American Statistical Association*, *66*(333), 103–104.

Block, H., & Joe, H. (1997). Tail behavior of the failure rate functions of mixtures. *Lifetime Data Analysis*, *3*(3), 269–288.

Dai, A., Wei, G., Zhang, Z., & He, S. (2020). Design of a flexible preventive maintenance strategy for two-dimensional warranted products. *Proceedings of the Institution of Mechanical Engineers, Part O: Journal of Risk and Reliability*, *234*(1), 74–87.

Davidson, D., & Drake, J. a. (2019). *Representations and warranties insurance claim study—an analysis of claim trends, data, and recoveries* (Tech. Rep.). AON.

Gupta, S. K., De, S., & Chatterjee, A. (2017). Some reliability issues for incomplete two-dimensional warranty claims data. *Reliability Engineering & System Safety*, *157*, 64–77.

Huang, Y.-S., Chen, E., & Ho, J.-W. (2013). Two-dimensional warranty with reliability-based preventive maintenance. *IEEE Transactions on Reliability*, *62*(4), 898–907.

Huang, Y.-S., Huang, C.-D., & Ho, J.-W. (2017). A customized two-dimensional extended warranty with preventive maintenance. *European Journal of Operational Research*, *257*(3), 971–978.

Kijima, M. (1989). Some results for repairable systems with general repair. *Journal of Applied probability*, *26*(1), 89–102.

Lam, Y. (1988). Geometric processes and replacement problem. *Acta Mathematicae Applicatae Sinica*, *4*(4), 366–377.

Li, H., Zhu, W., Dieulle, L., & Deloux, E. (2022). Condition-based maintenance strategies for stochastically dependent systems using nested lévy copulas. *Reliability Engineering & System Safety*, *217*, 108038.

Li, X., Liu, Z., Wang, Y., & Li, M. (2019). Optimal burn-in strategy for two-dimensional warranted products considering preventive maintenance. *International Journal of Production Research*, *57*(17), 5414–5431.

Mitra, A. (2021). Warranty parameters for extended two-dimensional warranties incorporating consumer preferences. *European Journal of Operational Research*, *291*(2), 525–535.

Nakagawa, T. (1988). Sequential imperfect preventive maintenance policies. *IEEE Transactions on Reliability*, *37*(3), 295–298.

Nelsen, R. (1999). *An introduction to copulas*. New York: Springer.

Peng, S., Jiang, W., & Zhao, W. (2021). A preventive maintenance policy with usage-dependent failure rate thresholds under two-dimensional warranties. *IISE Transactions*, *53*(11), 1231–1243.

Rizwan, U., & Yunus, M. M. (2015). Geometric processes in two dimensions: Basic results and applications. *SCIENCE AND HUMANITIES*, *1*(2), 747-774.

Shafiee, M., & Chukova, S. (2013). Maintenance models in warranty: A literature review. *European Journal of Operational Research*, *229*(3), 561–572.

Shahanaghi, K., Noorossana, R., Jalali-Naini, S. G., & Heydari, M. (2013). Failure modeling and optimizing preventive maintenance strategy during two-dimensional extended warranty contracts. *Engineering Failure Analysis*, *28*, 90–102.

Su, C., & Wang, X. (2016). A two-stage preventive maintenance optimization model incorporating two-dimensional extended warranty. *Reliability Engineering & System Safety*, *155*, 169–178.

Torrado, N. (2022). Optimal component-type allocation and replacement time policies for parallel systems having multi-types dependent components. *Reliability Engineering & System Safety*, *224*, 108502.

Wang, J., Zhou, Z., & Peng, H. (2017). Flexible decision models for a two-dimensional warranty policy with periodic preventive maintenance. *Reliability Engineering & System Safety*, *162*, 14–27.

Wang, J., Zhu, S., & Du, S. (2021). Analysis of a two-dimensional stair-case warranty policy with preventive maintenance. *IMA Journal of Management Mathematics*, *32*(1), 51–67.

Wang, Y., Liu, Z., & Liu, Y. (2015). Optimal preventive maintenance strategy for repairable items under two-dimensional warranty. *Reliability Engineering & System Safety*, *142*, 326–333.

Wu, S. (2012). Warranty data analysis: A review. *Quality and Reliability Engineering International*, *28*(8), 795–805.

Wu, S. (2014). Construction of asymmetric copulas and its application in two-dimensional reliability modelling. *European Journal of Operational Research*, *238*(2), 476–485.
Wu, S. (2019). A failure process model with the exponential smoothing of intensity functions. *European Journal of Operational Research*, *275*(2), 502–513.
Wu, S., & Wang, G. (2018). The semi-geometric process and some properties. *IMA Journal of Management Mathematics*, *29*(2), 229–245.
Wu, S., & Zuo, M. J. (2010). Linear and nonlinear preventive maintenance models. *IEEE Transactions on Reliability*, *59*(1), 242–249.
Xu, J., Liang, Z., Li, Y.-F., & Wang, K. (2021). Generalized condition-based maintenance optimization for multi-component systems considering stochastic dependency and imperfect maintenance. *Reliability Engineering & System Safety*, *211*, 107592.
Zhao, X., Chen, P., Lv, S., & He, Z. (2022). Reliability testing for product return prediction. *European Journal of Operational Research*.
Zhao, X., Liu, B., Xu, J., & Wang, X. (2022). Imperfect maintenance policies for warranted products under stochastic performance degradation. *European Journal of Operational Research*. doi: 10.1016/j.ejor.2022.11.001

Appendix: Proofs of the propositions

Proof of Proposition 1

Proof. It is known that Spearman's rho can be obtained from a copula with the formula $\rho = 12\int_0^1\int_0^1 C(v_1, v_2)dv_1dv_2 - 3$. Hence, we can express Spearman's rho with ρ^* as follows:

$$\begin{aligned}\rho =&12\int_0^1\int_0^1 C(v_1, v_2)dv_1dv_2 - 3\\ =&12p_0\int_0^1\int_0^1 C^*(v_1, v_2)dv_1dv_2 + 12(1-p_0)\int_0^1\int_0^1 v_2dv_1dv_2\\ &- 12(1-p_0)\int_0^1\int_0^1 C^*(1-v_1, v_2)dv_1dv_2 - 3\\ =&12p_0\int_0^1\int_0^1 C^*(v_1, v_2)dv_1dv_2 + 6(1-p_0)\\ &- 12(1-p_0)\int_0^1\int_0^1 C^*(v_1, v_2)dv_1dv_2 - 3\\ =&(2p_0-1)\rho^*. \end{aligned} \tag{37}$$

This establishes Proposition 1. □

Proof of Proposition 2

Proof. If $p_0 = 1$ and $\beta_i > 1$, then $C(t, u) = C^*(t, u)$. Hence, $r(t, u) = r^*(t, u)$.

Further,

$$
\begin{aligned}
r(t,u) &= \frac{f(t,u)}{\bar{F}(t,u)} \\
&= \frac{p_0 f^*(t,u) + (1-p_0) f^+(t,u)}{p_0 F^*(t,u) + (1-p_0) F^+(t,u)} \\
&= p_0(t) r^*(t,u) + (1-p_0(t)) r^+(t,u), \qquad (38)
\end{aligned}
$$

where

$$p_0(t) = \frac{p_0 F^*(t,u)}{p_0 F^*(t,u) + (1-p_0) F^+(t,u)}.$$

It is easy to obtain that

$$\min\{r^*(t,u), r^+(t,u)\} \le r(t,u) \le \max\{r^*(t,u), r^+(t,u)\}. \qquad (39)$$

It should be noted that Block and Joe (1997) provide a derivation for the case of the univariate failure rate function as Eq. (39).

This establishes the proof of Proposition 2. □

Proof of Proposition 3

Proof. Since $F_1(t) = 1 - \exp\left\{-\left(\frac{t}{\gamma_1}\right)^{\beta_1}\right\} = v_1$ and $F_2(u) = 1 - \exp\left\{-\left(\frac{u}{\gamma_2}\right)^{\beta_2}\right\} = v_2$, then

$$g_{t,u}(v_1, v_2) = \left[(-\ln(1-v_1))^\theta + (-\ln(1-v_2))^\theta\right]^{1/\theta} = \left[\left(\frac{t}{\gamma_1}\right)^{\beta_1\theta} + \left(\frac{u}{\gamma_2}\right)^{\beta_2\theta}\right]^{1/\theta},$$

and

$$g_{t,u}(1-v_1, v_2) = \left[(-\ln(v_1)^\theta + (-\ln(1-v_2))^\theta\right]^{1/\theta} = \left[\left(-\ln\left(1 - \exp\left\{-\left(\frac{t}{\gamma_1}\right)^{\beta_1}\right\}\right)\right)^\theta + \left(\frac{u}{\gamma_2}\right)^{\beta_2\theta}\right]^{1/\theta}.$$

It is noted that $g_{t,u}(v_1, v_2)$ is an increasing function of t and u, respectively, while $g_{t,u}(1-v_1, v_2)$ increases in u but decreasing in t.

Then, $\bar{F}^*(t,u) = \exp(-g_{t,u}(v_1, v_2))$ and $\bar{F}^+(t,u) = 1 - v_1 - \exp(-g_{t,u}(v_1, 1-v_2))$, from which we obtain

$$c^*(v_1, v_2) = \frac{[\ln(1-v_1)\ln(1-v_2)]^{\theta-1}}{(1-v_1)(1-v_2)}[g_{t,u}(v_1,v_2) + \theta - 1][g_{t,u}(v_1,v_2)]^{1-2\theta}\exp(-g_{t,u}(v_1,v_2)). \qquad (40)$$

Thus, $r^*(t,u) = \frac{\beta_1\beta_2}{\gamma_1\gamma_2}\left(\frac{t}{\gamma_1}\right)^{\beta_1\theta-1}\left(\frac{u}{\gamma_2}\right)^{\beta_2\theta-1}[g_{t,u}(v_1,v_2)+\theta-1][g_{t,u}(v_1,v_2)]^{1-2\theta}$, and

$$c^+(v_1,v_2,:\theta) = \frac{[\ln(v_1)\ln(1-v_2)]^{\theta-1}}{v_1(1-v_2)}[g_{t,u}(1-v_1,v_2)+\theta-1][g_{t,u}(1-v_1,v_2)]^{1-2\theta}\exp(-g_{t,u}(1-v_1,v_2)).$$

Thus, we can obtain $r^+(t,u)$ as follows.

$$r^+(t,u)$$
$$= \frac{\beta_1\beta_2 t^{\beta_1-1}t^{\beta_1\theta-1}}{\gamma_1^{\beta_1}\gamma_2^{\beta_2\theta-1}} \frac{\left[-\ln(1-\exp\left\{-\left(\frac{t}{\gamma_1}\right)^{\beta_1}\right\})\right]^{\theta-1}}{1-\exp\left\{-\left(\frac{t}{\gamma_1}\right)^{\beta_1}\right\}} \frac{[g_{t,u}(1-v_1,v_2)+\theta-1][g_{t,u}(1-v_1,v_2)]^{1-2\theta})}{\exp\left\{g_{t,u}(1-v_1,v_2)-\left(\frac{t}{\gamma_1}\right)^{\beta_1}\right\}-1}. \tag{41}$$

Then it is easy to obtain the following survival function.

$$\begin{aligned}
P(X_1 \geq t, X_2 \geq u) =& \bar{F}(t,u) \\
=& (1-p_0)(1-v_2) + p_0 \exp\left\{-\left[(-\ln(1-v_1))^\theta + (-\ln(1-v_2))^\theta\right]^{1/\theta}\right\} \\
&- (1-p_0)\exp\left\{-\left[(-\ln(v_1))^\theta + (-\ln(1-v_2))^\theta\right]^{1/\theta}\right\} \\
=& (1-p_0)\exp\left\{-\left(\frac{u}{\gamma_2}\right)^{\beta_2}\right\} + p_0\exp\left\{-g_{t,u}(v_1,v_2)\right\} \\
&- (1-p_0)\exp\left\{-g_{t,u}(1-v_1,v_2)\right\}.
\end{aligned} \tag{42}$$

From Eq. (42), we can obtain the conditional survival functions, for example, the probability of X_1 given $X_2 \geq u$, i.e., $P(X_1 \geq t | X_2 \geq u) = \frac{P(X_1 \geq t, X_2 \geq u)}{P(X_2 \geq u)}$.

Take the first part of the proposition for example.

$$\begin{aligned}
P(X_1 \geq t | X_2 \geq u) =& \frac{P(X_1 \geq t, X_2 \geq u)}{P(X_2 \geq u)} \\
=& (1-p_0) + p_0 \exp\left\{-g_{t,u}(v_1,v_2) + \left(\frac{u}{\gamma_2}\right)^{\beta_2}\right\} \\
&- (1-p_0)\exp\left\{-g_{t,u}(1-v_1,v_2) + \left(\frac{u}{\gamma_2}\right)^{\beta_2}\right\}.
\end{aligned} \tag{43}$$

It is easy to see that $P(X_1 \geq t | X_2 \geq u)$ decreases in t for all u.

Since $\theta > 1$, for positive d_1 and d_2, we have $(d_1 + d_2)^{1/\theta} \leq d_1^{1/\theta} + d_2^{1/\theta}$. Hence,

$$\begin{aligned} P(X_1 \geq t | X_2 \geq u) \geq & (1 - p_0) + p_0 \exp\left\{ - \left(\frac{t}{\gamma_1} \right)^{\beta_1} \right\} \\ & - (1 - p_0) \exp\left\{ -g_{t,u}(1 - v_1, v_2) + \left(\frac{u}{\gamma_2} \right)^{\beta_2} \right\} \\ & = p_0 \bar{F}_1(t) + (1 - p_0) \frac{\bar{F}^+(t, u)}{\bar{F}_2(u)}, \end{aligned} \tag{44}$$

and

$$\begin{aligned} P(X_1 \geq t | X_2 \geq u) \leq & (1 - p_0) + p_0 \exp\left\{ -g_{t,u}(v_1, v_2) + \left(\frac{u}{\gamma_2} \right)^{\beta_2} \right\} \\ & - (1 - p_0) \left(1 - \exp\left\{ - \left(\frac{t}{\gamma_1} \right)^{\beta_1} \right\} \right) \\ & = (1 - p_0) \bar{F}_1(t) + p_0 \frac{\bar{F}^+(t, u)}{\bar{F}_2(u)}. \end{aligned} \tag{45}$$

Similarly,

$$\begin{aligned} P(X_2 \geq u | X_1 \geq t) = & \frac{P(X_1 \geq t, X_2 \geq u)}{P(X_2 \geq t)} \\ = & (1 - p_0) \exp\left\{ \left(\frac{t}{\gamma_1} \right)^{\beta_1} - \left(\frac{u}{\gamma_2} \right)^{\beta_2} \right\} + p_0 \exp\left\{ \left(\frac{t}{\gamma_1} \right)^{\beta_1} - g_{t,u}(v_1, v_2) \right\} \\ & - (1 - p_0) \exp\left\{ \left(\frac{t}{\gamma_1} \right)^{\beta_1} - g_{t,u}(1 - v_1, v_2) \right\}. \end{aligned} \tag{46}$$

It is easy to see that $P(X_2 \geq u | X_1 \geq t)$ decreases in u for all t.

Since $\theta > 1$, for positive d_1 and d_2, we have $(d_1 + d_2)^{1/\theta} \leq d_1^{1/\theta} + d_2^{1/\theta}$. Hence,

$$\begin{aligned} P(X_2 \geq u | X_1 \geq t) \geq & (1 - p_0) \exp\left\{ \left(\frac{t}{\gamma_1} \right)^{\beta_1} - \left(\frac{u}{\gamma_2} \right)^{\beta_2} \right\} + p_0 \exp\left\{ - \left(\frac{u}{\gamma_2} \right)^{\beta_2} \right\} \\ & - (1 - p_0) \exp\left\{ \left(\frac{t}{\gamma_1} \right)^{\beta_1} - g_{t,u}(1 - v_1, v_2) \right\} \\ & = p_0 \bar{F}_2(u) + (1 - p_0) \frac{\bar{F}^+(t, u)}{\bar{F}_1(t)}. \end{aligned} \tag{47}$$

This establishes Proposition 3. □

Proof of Proposition 4

Proof. Our purpose is to seek the optimal number N^* that satisfies Eq. (23), constraints (24), and (25). That is, we need to find an integer N^* that satisfies $L(N-1)-L(N)>0$ and $L(N+1)-L(N)>0$, where $L(N)=c_a(N-1)+c_rM\left(\int_0^w\int_0^L r(t,u)dtdu+\sum_{k=1}^N\int_{(k-1)\frac{w}{N}}^{k\frac{w}{N}}\int_0^L r_{\mathrm{II},k-1}(t,u)dudt\right)+c_pM(N-1)$.

It is easy to find from constraint (24) that $L(1)-L(2)>0$.

From constraint (25), $\sum_{k=1}^N\int_{(k-1)\frac{w}{N}}^{k\frac{w}{N}}\int_0^L r_{\mathrm{II},k-1}(t,u)dudt$ decreases in N. Hence, there exists an N^* entails that $\sum_{k=1}^N\int_{(k-1)\frac{w}{N}}^{k\frac{w}{N}}\int_0^L r_{\mathrm{II},k-1}(t,u)dudt-\sum_{k=1}^{N+1}\int_{(k-1)\frac{w}{N+1}}^{k\frac{w}{N+1}}\int_0^L r_{\mathrm{II},k-1}(t,u)dudt$ becomes smaller . Hence, we can find an integer that satisfies

$$\begin{aligned}&L(N+1)-L(N)\\&\quad=c_a+c_pM-c_rM\left(\sum_{k=1}^{N}\int_{(k-1)\frac{w}{N}}^{k\frac{w}{N}}\int_0^L r_{\mathrm{II},k-1}(t,u)dudt-\sum_{k=1}^{N+1}\int_{(k-1)\frac{w}{N+1}}^{k\frac{w}{N+1}}\int_0^L r_{\mathrm{II},k-1}(t,u)dudt\right)>0\end{aligned} \tag{48}$$

That is, there exists a finite value N^* that satisfies Eq. (23), constraints (24), and (25).

This establishes Proposition 4. □